A SHARP CONDITION FOR EXISTENCE OF AN INERTIAL MANIFOLD

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1. Introduction. In recent years it has been shown, see [1-9] for some references, that solutions of many important PDE's approach exponentially to a flow on a smooth invariant finite dimensional manifold. The crucial part often lies in finding an invariant attracting manifold for the flow in a Hilbert space X generated by

$$u' + Au = F(u) \tag{1.1}$$

where A is a sectorial operator [6] in X and F is such that for some $\alpha \in [0,1), B \in \mathcal{B}(X^\alpha, X)$

$$||F(x) - F(y)|| \leqq ||B(x-y)|| \quad \text{for all } x, y \in X^\alpha;$$

here $\mathcal{B}(X^\alpha, X)$ denotes the space of bounded operators from X^α [6] into X. Various conditions that ensure existence of an inertial (invariant and attracting) manifold for (1.1) are known [1-9], however, since the problem seems to be a fundamental one it would be nice to have optimal conditions. Here a condition that is sharp in a sense is presented.

Let $\lambda > 0$ be such that

$$\lambda + i\omega \text{ is in the resolvent set of } A,\ \rho(A)\text{, for all } \omega \in \mathbb{R}.$$

It is well known [6] that A has an invariant subspace X_1 which is the range of the projection associated with the spectral set in the half plane $Re\, z < \lambda$. X_1 is an inertial manifold for $u' + Au = 0$. If $\lambda + i\omega \in \rho(A - B)$ for all $\omega \in \mathbb{R}$ the same could be said for the equation $u' + Au = Bu$ and one way to ensure this is by requiring that

$$||B(A - \lambda - i\omega)^{-1}|| < 1 \qquad \text{for all } \omega \in \mathbb{R} \tag{1.2}$$

since

$$(A - B - \lambda - i\omega)^{-1} = (A - \lambda - i\omega)^{-1}(1 - B(A - \lambda - i\omega)^{-1})^{-1}.$$

In this paper it is proven that (1.2) is actually also sufficient for existence of an inertial manifold for the nonlinear equation (1.1) - no additional assumptions are needed. In spite of weaker and much simpler assumptions the exponential attractivity result presented here (Theorem 4.1) is actually stronger then in [1,3,4,6].

The paper is organized as follows. Assumptions, notation and some well known facts are presented in Section 2. Existence and some properties of the invariant manifold are derived in Section 3. In Section 4 exponential tracking is proven. Sections 3 and 4 are almost completely independent. In Section 5 it is shown how to modify assumptions so that the results of Sections 3, 4 become applicable also to hyperbolic problems. A comparison of various assumptions is made in Section 6.

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2. Assumptions and Preliminaries. The following is the list of all assumptions that will be in effect in Sections 3, 4:

(H1) *X is a complex Banach space.*

(H2) *There exists $M_0 \in (0,\infty)$ such that if $f \in C(\mathbb{R}\backslash\{0\}, X)$ and $\|f(\cdot)\| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} \|f(t)\|^2 dt \le M_0 \int_{-\infty}^{\infty} \|\hat{f}(\omega)\|^2 d\omega \le M_0^2 \int_{-\infty}^{\infty} \|f(t)\|^2 dt$$

where

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \qquad \text{for } \omega \in \mathbb{R} .$$

(H3) *A is a sectorial operator [6] in X, $\alpha \in [0,1)$ and X^α is defined as usually [6, p.29].*

(H4) *$\lambda \in \mathbb{R}$ is such that $\lambda + i\omega$ is in the resolvent set of A for all $\omega \in \mathbb{R}$.*

(H5) *$F : \mathbb{R} \times X^\alpha \to X$ is continuous and such that for some $B_1, \ldots, B_m \in \mathcal{B}(X^\alpha, X)$ we have that*

$$\|F(t,x) - F(t,y)\| \le \sum_{j=1}^{m} \|B_j(x-y)\| \qquad \text{for } t \in \mathbb{R},\ x,y \in X^\alpha .$$

(H6) $\int_{-\infty}^{0} \|e^{\mu t} F(t,0)\|^2 dt < \infty$ *for some $\mu < \lambda$.*

(H7) $M_0 \sum_{j=1}^{m} \sup_{\omega\in\mathbb{R}} \|B_j(A - \lambda - i\omega)^{-1}\| < 1$.

Observe that if X is any Hilbert space then (H2) holds with $M_0 = 1$. If $M_0 = 1$ then X has to be a Hilbert space [10]. A slightly strengthened version of (H2) would imply that X would have to be homeomorphic to a Hilbert space [10]. However, the intuitive argument presented in the introduction suggests that (H2) is probably not needed. (H2) is used only in the proofs of Lemmas 3.2, 4.2.

Various well known consequences of the above assumptions and some definitions that will be used in Sections 3,4 will now be presented.

Let $\sigma(A)$ denotes the spectrum of A. Choose a $< \inf Re\, \sigma(A)$ and note that [6, p.29] X^α is equal to the domain of $(A-a)^\alpha$, $\|x\|_\alpha = \|(A-a)^\alpha x\|$ for $x \in X^\alpha$. Since [6, p.26]

$$\|(A-a)^\alpha (A - \lambda - i\omega)^{-1}\| \le \text{const}.\|(A-a)(A-\lambda-i\omega)^{-1}\|^\alpha \|(A-\lambda-i\omega)^{-1}\|^{1-\alpha}$$

we have that

$$\sup_{\omega\in\mathbb{R}} \|(A-a)^\alpha (A-\lambda-i\omega)^{-1}\| < \infty .$$

Thus, by choosing $B_0 = l(A-a)^\alpha$ with $l \in (0,\infty)$ small enough we may assume that

$$\rho(\lambda) \equiv M_0 \sum_{j=0}^{m} c_j(\lambda) < 1$$

where

$$c_j(\lambda) = \sup_{\omega\in\mathbb{R}} \|B_j(A-\lambda-i\omega)^{-1}\| \qquad \text{for} \quad j=0,1,\ldots,m \ .$$

Since A is sectorial we have that $\rho(\tilde\lambda) < 1$ whenever $|\lambda-\tilde\lambda|$ is small enough. Observe also that

$$\|F(t,x)-F(t,y)\| \le L\|x-y\|_\alpha \quad \text{for} \quad t\in\mathbb{R},\ x,y\in X^\alpha$$

where

$$L = \sum_{j=0}^{m} \|B_j(A-a)^{-\alpha}\| \ .$$

Let

$$\sigma_1 = \{z\in\sigma(A)|\, Re\, z < \lambda\}\ , \quad \sigma_2 = \{z\in\sigma(A)|\, Re\, z > \lambda\}\ .$$

Note that $\sigma(A) = \sigma_1\cup\sigma_2$ and that σ_1 is bounded. Let $P_1 \in \mathcal{B}(X)$ be the projection associated with σ_1, $P_2 = 1-P_1$, $X_i = P_iX$ for $i=1,2$. Choose $\lambda_1,\lambda_2\in\mathbb{R}$ so that $\sup Re\,\sigma_1 < \lambda_1 < \lambda < \lambda_2 < \inf Re\,\sigma_2$. From [6, p.30] one obtains that

$$\begin{aligned}
&X_1 \subset \mathcal{D}(A)\ ,\ AX_1\subset X_1\ ,\\
&A_1 \equiv A \text{ restricted to } X_1\ ,\ A_1\in\mathcal{B}(X_1),\\
&P_ie^{-At} = e^{-At}P_i \quad \text{for} \quad t\ge 0,\ i=1,2,\\
&e^{-A_1z} = \sum_{n=0}^{\infty}\frac{(-A_1z)^n}{n!} \quad \text{for} \quad z\in\mathbb{C},\\
&e^{-A_1t}x = e^{-At}x \quad \text{for} \quad x\in X_1,\ t\ge 0,
\end{aligned}$$

and that there exists $M<\infty$ such that for all $x\in X$

$$\text{(2.1)} \qquad \begin{aligned}
&\|e^{-At}\| \le Me^{-at} && t\ge 0,\\
&\|e^{-At}x\|_\alpha \le Mt^{-\alpha}e^{-at}\|x\| && t>0,\\
&\|e^{-A_1t}P_1x\| \le Me^{-\lambda_1t}\|x\| && t\le 0,\\
&\|e^{-A_1t}P_1x\|_\alpha \le Me^{-\lambda_1t}\|x\| && t\le 0,\\
&\|e^{-At}P_2x\| \le Me^{-\lambda_2t}\|x\| && t\ge 0,\\
&\|e^{-At}P_2x\|_\alpha \le Mt^{-\alpha}e^{-\lambda_2t}\|x\| && t>0.
\end{aligned}$$

3. Invariant Manifold. For $\tau \in \mathbb{R}$ define $\mathcal{M}(\tau) \subset X$ as follows: $x \in \mathcal{M}(\tau)$ if and only if there exists $v \in C((-\infty, 0], X^\alpha)$ such that

$$v(0) = x,$$

$$v(t) = e^{-A(t-T)}v(T) + \int_T^t e^{-A(t-s)}F(s+\tau, v(s))ds \quad \text{for} \quad -\infty < T \le t \le 0,$$

$$\int_{-\infty}^0 \|e^{\lambda t}v(t)\|_\alpha^p dt < \infty \quad \text{for} \quad p = 1, 2.$$

Note that for each $\tau \in \mathbb{R},\ x \in X^\alpha$ there exists a unique $u \in C([\tau, \infty), X^\alpha)$ such that

$$u(t) = e^{-A(t-\tau)}x + \int_\tau^t e^{-A(t-s)}F(s, u(s))ds \quad \text{for} \quad t \ge \tau.$$

Therefore if $x \in \mathcal{M}(\tau)$ then there exists $u \in C(\mathbb{R}, X^\alpha)$ such that $u(\tau) = x,\ u(t) \in \mathcal{M}(t)$ for all $t \in \mathbb{R}$ and

$$u(t) = e^{-A(t-T)}u(T) + \int_T^t e^{-A(t-s)}F(s, u(s))ds \quad \text{for} \quad -\infty < T \le t < \infty.$$

Thus, $\mathcal{M}$ is an invariant manifold. Obviously, if F is periodic in or independent of the first variable the same is true for $\mathcal{M}$. Some of the properties of $\mathcal{M}$ that are proved in the rest of this section are gathered in the following theorem.

THEOREM 3.1. *There exists a continuous* $h : \mathbb{R} \times X_1 \to X_2 \cap X^\alpha$ *such that*

$$\mathcal{M}(\tau) = \{x + h(\tau, x) | x \in X_1\} \quad \textit{for all } \tau \in \mathbb{R}.$$

Moreover, there exists $c < \infty$ *such that*

$$\|h(\tau, x) - h(\tau, y)\|_\alpha \le c\|x - y\| \quad \textit{for all } \tau \in \mathbb{R},\ x, y \in X_1.$$

Define a normed space Y by

$$Y = \{\, v \in C((-\infty, 0], X^\alpha) \mid \int_{-\infty}^0 \|e^{\lambda t}v(t)\|_\alpha^p < \infty \quad \text{for} \quad p = 1, 2\,\},$$

$$|v|_Y = \sum_{j=0}^m \left(\int_{-\infty}^0 \|e^{\lambda t}B_j v(t)\|^2 dt \right)^{1/2} \quad \text{for} \quad v \in Y\ .$$

Following [3] define $S : \mathbf{R} \times Y \times X_1 \to Y$ by

$$S(\tau, v, x)(t) = e^{-A_1 t}x + \int_0^t e^{-A_1(t-s)} P_1 F(s+\tau, v(s))ds + \int_{-\infty}^t e^{-A(t-s)} P_2 F(s+\tau, v(s))ds$$

for $\tau \in \mathbf{R}, v \in Y, x \in X_1, t \leq 0$. To see that $u = S(\tau, v, x) \in Y$ observe that

$$\|e^{\lambda t}u(t)\|_\alpha \leq Me^{(\lambda-\lambda_1)t}\|x\| + M \int_{-\infty}^0 K(t-s)\theta(s)ds$$

where

$$\theta(s) = e^{\lambda s}\|F(s+\tau, v(s))\| \leq L\|e^{\lambda s}v(s)\|_\alpha + e^{\lambda s}\|F(s+\tau, 0)\|$$
$$K(t) = \begin{cases} e^{(\lambda-\lambda_1)t} & \text{if } t \leq 0 \\ t^{-\alpha}e^{(\lambda-\lambda_2)t} & \text{if } t > 0 \end{cases}$$
$$\theta \in L^1(-\infty, 0) \cap L^2(-\infty, 0)\ ,\ K \in L^1(\mathbf{R}).$$

The following observation will come useful

$$u(t) = e^{-A(t-T)}u(T) + \int_T^t e^{-A(t-s)}F(s+\tau, v(s))ds \quad \text{for } -\infty < T \leq t \leq 0 \tag{3.1}$$

which follows from the following

$$P_1 u(t) = e^{-A_1 t}x + \int_0^t e^{-A_1(t-s)} P_1 F(s+\tau, v(s))ds$$

$$e^{A_1 t}P_1 u(t) = x + \int_0^t e^{A_1 s} P_1 F(s+\tau, v(s))ds$$

$$P_1 u(t) = e^{-A_1(t-T)}P_1 u(T) + \int_T^t e^{-A_1(t-s)} P_1 F(s+\tau, v(s))ds$$

$$P_2 u(t) = \int_{-\infty}^t e^{-A(t-s)} P_2 F(s+\tau, v(s))ds$$

$$P_2 u(t) = e^{-A(t-T)}P_2 u(T) + \int_T^t e^{-A(t-s)} P_2 F(s+\tau, v(s))ds.$$

LEMMA 3.2. $|S(\tau,u,x)-S(\tau,v,x)|_Y \le \rho(\lambda)|u-v|_Y$ *for* $\tau\in\mathbb{R},\ x\in X_1,\ u,v\in Y$.

Proof. Let $c=S(\tau,u,x)(0)-S(\tau,v,x)(0)$

$$g(t)=\begin{cases} e^{\lambda t}e^{-At}c & \text{if } t>0 \\ e^{\lambda t}(S(\tau,u,x)(t)-S(\tau,v,x)(t)) & \text{if } t\le 0 \end{cases}$$
$$f(t)=\begin{cases} 0 & \text{if } t>0 \\ e^{\lambda t}(F(\tau+t,u(t))-F(\tau+t,v(t))) & \text{if } t\le 0. \end{cases}$$

Note that $g\in C(\mathbb{R},X^\alpha)$

$$\|g(\cdot)\|_\alpha\ ,\ \|f(\cdot)\|\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$$

$$g(t)=\int_0^t e^{\lambda(t-s)}e^{-A_1(t-s)}P_1f(s)ds+\int_{-\infty}^t e^{\lambda(t-s)}e^{-A(t-s)}P_2f(s)ds\ ,\quad t\le 0$$

$$c=P_2c=\int_{-\infty}^0 e^{-\lambda s}e^{As}P_2f(s)ds\ .$$

For $\omega\in\mathbb{R}$ let

$$\hat g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}g(t)dt \qquad \text{(Bochner integral in } X^\alpha).$$

A straightforward calculation gives that for all $\omega\in\mathbb{R}$

$$\begin{aligned}\sqrt{2\pi}\ \hat g(\omega)-(A-\lambda+i\omega)^{-1}c &= \int_{-\infty}^0 e^{-i\omega t}g(t)dt \\ &= -\int_{-\infty}^0 ds\int_{-\infty}^s dt\, e^{(\lambda-i\omega)(t-s)}e^{-A_1(t-s)}P_1e^{-i\omega s}f(s)+ \\ &\quad+\int_{-\infty}^0 ds\int_s^0 dt\, e^{(\lambda-i\omega)(t-s)}e^{-A(t-s)}P_2e^{-i\omega s}f(s) \\ &= \int_{-\infty}^0 (A-\lambda+i\omega)^{-1}P_1e^{-i\omega s}f(s)ds+ \\ &\quad+\int_{-\infty}^0 (A-\lambda+i\omega)^{-1}(1-e^{As}e^{-s(\lambda-i\omega)})P_2e^{-i\omega s}f(s)ds\end{aligned}$$

$$\hat g(\omega)=(A-\lambda+i\omega)^{-1}\hat f(\omega)\ .$$

Since $B_j \in \mathcal{B}(X^\alpha, X)$ for $j = 0, 1, \ldots, m$ we have that $\widehat{B_j g} = B_j \hat{g}$ and

$$\int_{-\infty}^{0} \|B_j g(t)\|^2 dt \le \int_{-\infty}^{\infty} \|B_j g(t)\|^2 dt \le M_0 \int_{-\infty}^{\infty} \|B_j \hat{g}(\omega)\|^2 d\omega$$

$$\le M_0 c_j^2 \int_{-\infty}^{\infty} \|\hat{f}(\omega)\|^2 d\omega \le M_0^2 c_j^2 \int_{-\infty}^{0} \|f(t)\|^2 dt \le M_0^2 c_j^2 |u - v|_Y^2.$$

which implies the conclusion.

LEMMA 3.3. *For each $x \in X_1$, $\tau \in \mathbb{R}$ there exists a unique $v \in Y$ so that $S(\tau, v, x) = v$.*

Proof. Uniqueness follows from Lemma 3.2. Define

$$v_0 = 0, \quad v_{n+1} = S(\tau, v_n, x) \quad \text{for} \quad n \ge 0$$
$$r_n(s) = e^{\lambda s}(F(\tau + s, v_{n+1}(s)) - F(\tau + s, v_n(s))) \quad \text{for} \quad n \ge 0,\ s \le 0.$$

Note that for $n \ge 0,\ t \le 0$

$$\left(\int_{-\infty}^{0} \|r_n(s)\|^2 ds \right)^{1/2} \le |v_{n+1} - v_n|_Y \le \rho^n |v_1|_Y$$

$$e^{\lambda t}(v_{n+2}(t) - v_{n+1}(t)) = \int_0^t e^{\lambda(t-s)} e^{-A_1(t-s)} P_1 r_n(s) ds + \int_{-\infty}^{t} e^{\lambda(t-s)} e^{-A(t-s)} P_2 r_n(s) ds$$

$$\|e^{\lambda t}(v_{n+2}(t) - v_{n+1}(t))\| \le k_1 \rho^n \ , \ k_1 = M|v_1|_Y((\lambda - \lambda_1)^{-1/2} + (\lambda_2 - \lambda)^{-1/2}).$$

Choose $\varepsilon \in (0,1)$ so that $\varepsilon \ge \rho$ and fix $T \in (-\infty, 0)$. For $n \ge 0,\ t \in (T, 0]$ we obtain from (3.1)

$$v_{n+2}(t) - v_{n+1}(t) = e^{-A(t-T)}(v_{n+2}(T) - v_{n+1}(T)) +$$
$$+ \int_T^t e^{-A(t-s)}(F(s+\tau, v_{n+1}(s)) - F(s+\tau, v_n(s)))ds$$

$$\|v_{n+2}(t) - v_{n+1}(t)\|_\alpha \le M(t-T)^{-\alpha} e^{-a(t-T) - \lambda T} k_1 \varepsilon^n +$$
$$+ ML \int_T^t (t-s)^{-\alpha} e^{-a(t-s)} \|v_{n+1}(s) - v_n(s)\|_\alpha ds$$
$$\le k_2 (t-T)^{-\alpha} \varepsilon^n + \varepsilon k_3 \int_T^t (t-s)^{-\alpha} \|v_{n+1}(s) - v_n(s)\|_\alpha ds$$

where $k_2 = Me^{(a-\lambda)T}k_1(1+e^{-aT}),\ k_3 = ML(1+e^{aT})/\varepsilon$. Thus, for $n \geq 1,\ t \in (T,0]$

$$(3.2)\qquad \begin{aligned} \varepsilon^{-n+1}\|v_{n+1}(t)-v_n(t)\|_\alpha &\leq k_2\sum_{j=1}^{n} k_3^{j-1}\frac{\Gamma(1-\alpha)^j}{\Gamma(j-j\alpha)}\,(t-T)^{j-j\alpha-1}+ \\ &+ k_3^n\frac{\Gamma(1-\alpha)^n}{\Gamma(n-n\alpha)}\int_T^t (t-s)^{n-n\alpha-1}\varepsilon\|v_1(s)\|_\alpha ds \end{aligned}$$

and therefore there exists $k_4 < \infty$ such that

$$\|v_{n+1}(t)-v_n(t)\|_\alpha \leq \varepsilon^{n-1}(t-T)^{-\alpha}k_4 \qquad \text{for} \quad n\geq 1,\ t\in(T,0].$$

Therefore there exists $v \in C((-\infty,0],X^\alpha)$ such that for all $T \in (-\infty,0)$

$$\lim_{n\to\infty}\sup_{T\leq t\leq 0}\|v(t)-v_n(t)\|_\alpha = 0$$

and this implies

$$\int_{-\infty}^{0}\|e^{\lambda t}v(t)\|_\alpha^2 dt < \infty$$

$$\int_{-\infty}^{0}\|e^{\lambda t}(v(t)-v_n(t))\|_\alpha^2 dt \leq \left(\frac{\rho^n}{1-\rho}\,\frac{|v_1|_Y}{l}\right)^2 \qquad \text{for} \quad n\geq 0.$$

Since $\rho(\tilde{\lambda}) < 1$ and v_n do not depend on $\tilde{\lambda}$ for $\tilde{\lambda}$ close to λ we also have

$$\int_{-\infty}^{0}\|e^{\tilde{\lambda}t}v(t)\|_\alpha^2 dt < \infty$$

and hence $v \in Y,\ |v-v_n|_Y \to 0$ as $n\to\infty$ and therefore $S(\tau,v,x) = v$.

Define $h : \mathbb{R}\times X_1 \to X_2 \cup X^\alpha$ as follows: choose $x\in X_1, \tau\in\mathbb{R}$, let $v\in Y$ be such that $S(\tau,v,x)=v$ and define

$$h(\tau,x) = P_2v(0) = \int_{-\infty}^{0} e^{As}P_2F(s+\tau,v(s))ds = v(0)-x\ .$$

Note, if F is bounded in X and $\lambda_2 > 0$ then h is bounded in X^α.

Lemma 3.4. *$\mathcal{M}(\tau) = \{x + h(\tau, x) | x \in X_1\}$ for all $\tau \in \mathbb{R}$.*

Proof. If $x \in X_1, \tau \in \mathbb{R}$ and $v \in Y$ satisfies $S(\tau, v, x) = v$ then $x + h(\tau, x) = v(0)$ and in view of (3.1) we have that $x + h(\tau, x) \in \mathcal{M}(\tau)$.

If v is as in the definition of $\mathcal{M}(\tau)$ then for $t \leq 0$

$$P_1 v(0) = e^{A_1 t} P_1 v(t) + \int_t^0 e^{A_1 s} P_1 F(s + \tau, v(s)) ds$$

$$P_1 v(t) = e^{-A_1 t} P_1 v(0) + \int_0^t e^{-A_1(t-s)} P_1 F(s + \tau, v(s)) ds.$$

If $-\infty < T \leq t \leq 0$ then

$$P_2 v(t) = e^{-A(t-T)} P_2 v(T) + \int_T^t e^{-A(t-s)} P_2 F(s + \tau, v(s)) ds$$

and since the integral converges as $T \to -\infty$ the limit of

$$\|e^{-A(t-T)} P_2 v(T)\| \leq M e^{-\lambda_2 t} e^{\lambda T} \|v(T)\|$$

as $T \to -\infty$ has to exist and since $v \in Y$ it has to be 0. Therefore

$$P_2 v(t) = \int_{-\infty}^t e^{-A(t-s)} P_2 F(s + \tau, v(s)) ds \qquad \text{for} \quad t \leq 0$$

and hence $v = S(\tau, v, P_1 v(0))$, $v(0) = P_1 v(0) + h(\tau, P_1 v(0))$.

Lemma 3.5. *There exists $c < \infty$ such that*

$$\|h(\tau, x) - h(\tau, y)\|_\alpha \leq c\|x - y\| \qquad \textit{for all} \quad \tau \in \mathbb{R},\ x, y \in X_1.$$

Proof. Choose $x, y \in X_1, \tau \in \mathbb{R}$ and let $u, v \in Y$ be such that

$$S(\tau, u, x) = u\ ,\ S(\tau, v, y) = v.$$

$d_1, d_2, \ldots$ will denote various constants - independent of τ, x, y. Note

$$u - v = S(\tau, u, y) - S(\tau, v, y) + e^{-A_1 t}(x - y)$$
$$|u - v|_Y \leq \rho |u - v|_Y + d_1 \|x - y\|$$
$$|u - v|_Y \leq d_2 \|x - y\|$$

$$e^{\lambda t}(u(t) - v(t)) = e^{\lambda t} e^{-A_1 t}(x - y) + \int_0^t e^{\lambda(t-s)} e^{-A_1(t-s)} P_1 r(s) ds +$$

$$+ \int_{-\infty}^t e^{\lambda(t-s)} e^{-A(t-s)} P_2 r(s) ds$$

where $r(t) = e^{\lambda t}(F(t+\tau, u(t)) - F(t+\tau, v(t)))$ for $t \le 0$. Since $\int\limits_{-\infty}^{0} \|r(t)\|^2 dt \le |u-v|_Y^2$

$$e^{\lambda t}\|u(t)-v(t)\| \le M\|x-y\| + M\int\limits_t^0 e^{(\lambda-\lambda_1)(t-s)}\|r(s)\|ds + M\int\limits_{-\infty}^t e^{(\lambda-\lambda_2)(t-s)}\|r(s)\|ds$$
$$\le M\|x-y\| + d_3|u-v|_Y \le d_4\|x-y\| \quad \text{for } t \le 0.$$

(3.1) implies that for $-\infty < T < t \le 0$

$$u(t)-v(t) = e^{-A(t-T)}(u(T)-v(T)) + \int\limits_T^t e^{-A(t-s)}(F(s+\tau, u(s)) - F(s+\tau, v(s)))ds$$

$$\|u(t)-v(t)\|_\alpha \le M(t-T)^{-\alpha}e^{-a(t-T)-\lambda T}d_4\|x-y\| +$$
$$+ ML\int\limits_T^t (t-s)^{-\alpha}e^{-a(t-s)}\|u(s)-v(s)\|_\alpha ds$$

which implies, see (3.2), that for some d_5 we have

$$\|u(t)-v(t)\|_\alpha \le \|x-y\|(t-T)^{-\alpha}d_5 \qquad \text{for} \quad t \in (T, 0]$$

and since $h(\tau, x) - h(\tau, y) = u(0) - v(0) - x + y$ we are done.

LEMMA 3.6. *$h : \mathbb{R} \times X_1 \to X^\alpha$ is continuous.*

REMARK. *Inequality (3.3) below can sometimes imply more regularity of h.*

Proof of Lemma 3.6. Fix $\tau \in \mathbb{R}$, $x \in X_1$. Take $u \in C(\mathbb{R}, X^\alpha)$ such that $u(\tau) = x + h(\tau, x)$, $u(t) \in \mathcal{M}(t)$ for $t \in \mathbb{R}$ and for $-\infty < T \le t < \infty$

$$u(t) = e^{-A(t-T)}u(T) + \int\limits_T^t e^{-A(t-s)}F(s, u(s))ds.$$

For $\sigma \in \mathbb{R}$, $y \in X_1$ we have

$$h(\sigma, y) - h(\tau, x) = h(\sigma, y) - h(\sigma, x) + h(\sigma, P_1u(\tau)) - h(\sigma, P_1u(\sigma)) + P_2u(\sigma) - P_2u(\tau)$$

(3.3) $\|h(\sigma, y) - h(\tau, x)\|_\alpha \le c\|y-x\| + c\|P_1\|\ \|u(\tau) - u(\sigma)\| + \|P_2\|\ \|u(\sigma) - u(\tau)\|_\alpha$

Therefore $\|h(\sigma, y) - h(\tau, x)\|_\alpha \to 0$ as $\sigma \to \tau$, $y \to x$.

4. Exponential Tracking. Choose any $\tau \in \mathbf{R}$, $u \in C([\tau,\infty), X^\alpha)$ such that

$$u(t) = e^{-A(t-\tau)}u(\tau) + \int_\tau^t e^{-A(t-s)}F(s,u(s))ds \quad \text{for} \quad t \geq \tau.$$

The purpose of this section is to prove

THEOREM 4.1. *There exists a unique* $v \in C(\mathbf{R}, X^\alpha)$ *such that*

$$v(t) = e^{-A(t-T)}v(T) + \int_T^t e^{-A(t-s)}F(s,v(s))ds \quad \textit{for} \quad -\infty < T \leq t < \infty$$

$$\int_\tau^\infty \|e^{\lambda t}(u(t)-v(t))\|_\alpha^p dt + \int_{-\infty}^\tau \|e^{\lambda t}v(t)\|_\alpha^p dt < \infty \quad \textit{for} \quad p = 1, 2.$$

Moreover, $v(t) \in \mathcal{M}(t)$ *for all* $t \in \mathbf{R}$ *and there exists* $C \in [0,\infty)$ *which depends only on* $M, \lambda, \lambda_1, \lambda_2, \alpha, L, l, \rho(\lambda)$ *such that*

$$e^{\lambda t}\|v(t)-u(t)\|_\alpha \leq Ce^{\lambda T} \inf_{x \in \mathcal{M}(T)} \|x - u(T)\|_\alpha \qquad \textit{for all} \quad \tau \leq T \leq t < \infty.$$

Theorem 3.1 gives a bound for

$$P_2u(t) - h(t,P_1u(t)) = P_2(u(t)-v(t)) + h(t,P_1v(t)) - h(t,P_1u(t)).$$

Define $u(t) = e^{-\lambda_1(t-\tau)}u(\tau)$ for $t < \tau$ and let

$$Z = \{\, \phi \in C(\mathbf{R}, X^\alpha) \mid \int_{-\infty}^\infty \|e^{\lambda t}\phi(t)\|_\alpha^p dt < \infty \quad \text{for} \quad p = 1, 2 \,\}$$

$$|\phi|_z = \sum_{j=0}^m \left(\int_{-\infty}^\infty \|e^{\lambda t}B_j\phi(t)\|^2 dt \right)^{1/2} \qquad \text{for} \quad \phi \in Z$$

$$w(t) = -u(t) + e^{-A_1(t-\tau)}P_1u(\tau) + \int_{-\infty}^t e^{-A(t-s)}P_2F(s,u(s))ds -$$

$$- \int_t^\tau e^{-A_1(t-s)}P_1F(s,u(s))ds \qquad \text{for} \quad t \leq \tau$$

$$w(t) = e^{-A(t-\tau)}w(\tau) = e^{-A(t-\tau)}P_2w(\tau) \quad \text{for} \quad t > \tau.$$

Observe that $w \in Z$. Define $R : Z \to Z$ by

$$(R\phi)(t) = w(t) + \int_{-\infty}^{t} e^{-A(t-s)} P_2(F(s, \phi(s) + u(s)) - F(s, u(s)))ds -$$
$$- \int_{t}^{\infty} e^{-A_1(t-s)} P_1(F(s, \phi(s) + u(s)) - F(s, u(s)))ds.$$

LEMMA 4.2. $|R\phi - R\psi|_z \leq \rho(\lambda)|\phi - \psi|_z$ *for* $\phi, \psi \in Z$.

Proof. For $t \in \mathbb{R}$ let

$$g(t) = e^{\lambda t}((R\phi)(t) - (R\psi)(t))$$
$$f(t) = e^{\lambda t}(F(t, \phi(t) + u(t)) - F(t, \psi(t) + u(t)))\ .$$

As in the proof of Lemma 3.2

$$\hat{g}(\omega) = (A - \lambda + i\omega)^{-1}\hat{f}(\omega) \quad \text{for} \quad \omega \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} \|B_j g(t)\|^2 dt \leq M_0 \int_{-\infty}^{\infty} \|B_j \hat{g}(\omega)\|^2 d\omega \leq M_0^2 c_j^2 \int_{-\infty}^{\infty} \|f(t)\|^2 dt$$
$$\leq M_0^2 c_j^2 |\phi - \psi|_z^2.$$

LEMMA 4.3. *There exists a unique* $\Theta \in Z$ *such that* $R\Theta = \Theta$. *Moreover, there exists* $C \in [0, \infty)$ *which depends only on* $M, \lambda, \lambda_1, \lambda_2, \alpha, L, l, \rho(\lambda)$ *such that*

$$e^{\lambda t}\|\Theta(t)\|_\alpha \leq Ce^{\lambda T} \inf_{x \in \mathcal{M}(T)} \|x - u(T)\|_\alpha \qquad \textit{whenever} \quad \tau \leq T \leq t < \infty.$$

Proof. Uniqueness follows from Lemma 4.2. Let $\phi_0 = 0$, $\phi_{n+1} = R\phi_n$ for $n \geq 0$. Note that for $t \in \mathbb{R}, n \geq 0$

$$e^{\lambda t}(\phi_{n+2}(t) - \phi_{n+1}(t)) = \int_{-\infty}^{t} e^{\lambda(t-s)} e^{-A(t-s)} P_2 r_n(s) ds - \int_{t}^{\infty} e^{\lambda(t-s)} e^{-A_1(t-s)} P_1 r_n(s) ds\ ,$$
$$r_n(t) = e^{\lambda t}(F(t, \phi_{n+1}(t) + u(t)) - F(t, \phi_n(t) + u(t)))\ .$$

Let $f_n(t) = e^{\lambda t}\|\phi_{n+1}(t) - \phi_n(t)\|_\alpha$ and

$$K(t) = \begin{cases} MLe^{(\lambda - \lambda_1)t} & \text{if} \quad t \leq 0 \\ MLt^{-\alpha}e^{(\lambda - \lambda_2)t} & \text{if} \quad t > 0 \end{cases}$$

and note that

$$f_{n+1}(t) \leq \int_{-\infty}^{\infty} K(t-s) f_n(s) ds \quad \text{for} \quad n \geq 0, \ t \in \mathbb{R} \ .$$

Define $K_1 = K, \ K_{j+1} = K * K_j$ for $j \geq 1$ and note that

$$f_{n+j} \leq K_j * f_n \quad \text{for} \quad n \geq 0 \ , \ j \geq 1 \ .$$

Choose an integer $N \geq 1$ such that $2N(1-\alpha) > 1$. Young's inequality [6, p.34] gives

$$K_n \in L^{q_n}(\mathbb{R}) \, , \ q_n = 2N/(2N-n) \quad \text{for} \quad 1 \leq n \leq 2N.$$

Thus, $K_N \in L^2(\mathbb{R})$ and for $n \geq 0$

$$\|f_{n+N}\|_\infty \leq \|K_N\|_2 \|f_n\|_2 \leq \|K_N\|_2 l^{-1} |\phi_{n+1} - \phi_n|_z$$
$$\|f_{n+N}\|_\infty \leq \|K_N\|_2 l^{-1} |\phi_1|_z \rho^n \ .$$

Therefore, there exist $\Theta \in C(\mathbb{R}, X^\alpha)$, $d \in \mathbb{R}$ such that

$$e^{\lambda t} \|\Theta(t) - \phi_n(t)\|_\alpha \leq d\rho^{n-N} \quad \text{for} \quad n \geq N \ , \ t \in \mathbb{R}.$$

Using the facts that we can replace λ with $\tilde{\lambda}$ provided that $|\lambda - \tilde{\lambda}|$ is small enough and that this change does not effect ϕ_n and hence Θ we see that there exist $d_1 < \infty$, $\varepsilon < 1$, $\delta > 0$ such that

$$e^{\lambda t} \|\Theta(t) - \phi_n(t)\|_\alpha \leq d_1 \varepsilon^n e^{-\delta|t|} \quad \text{for} \quad t \in \mathbb{R}, \ n \geq N.$$

Thus $\Theta \in Z, \ |\Theta - \phi_n|_z \to 0$ as $n \to \infty$ and therefore $R\Theta = \Theta$.

To prove the moreover part choose any $T \in [\tau, \infty), \ x \in \mathcal{M}(T)$. Let $\psi \in Y$ be such that $S(T, \psi, P_1 x) = \psi$, hence $x = \psi(0)$. Define

$$\psi_0(t) = \begin{cases} \psi(t-T) - u(t) & \text{for} \quad t \leq T \\ 0 & \text{for} \quad t > T \ . \end{cases}$$

ψ_0 is in general not continuous, however, $\psi_1 \equiv R\psi_0$ can clearly be evaluated and a long but straightforward calculation gives

$$\psi_1(t) - \psi_0(t) = \begin{cases} e^{-A_1(t-T)} P_1(u(T) - x) & \text{if} \quad t \leq T \\ e^{-A(t-T)} P_2(x - u(T)) & \text{if} \quad t > T \end{cases}$$

and thus $\psi_1 \in Z$. Define $\psi_{n+1} = R\psi_n$ also for $n \geq 1$.

Since $|\phi_n - \Theta|_z \to 0$ as $n \to \infty$ Lemma 4.2 implies

$$|\Theta - \psi_n|_z \le \rho^{n-1}|\psi_2 - \psi_1|_z/(1-\rho) \qquad \text{for} \quad n \ge 1.$$

For $n \ge 0,\ t \in \mathbb{R}$ define $g_n(t) = e^{\lambda t}\|\psi_{n+1}(t) - \psi_n(t)\|_\alpha$. As above

$$\begin{gathered}
g_{n+j} \le K_j * g_n \qquad \text{for} \qquad n \ge 0,\ j \ge 1 \\
\|g_{n+N}\|_\infty \le \|K_N\|_2\|g_n\|_2 \le \|K_N\|_2 l^{-1}|\psi_{n+1} - \psi_n|_z \le \|K_N\|_2 l^{-1}|\psi_2 - \psi_1|_z\rho^{n-1} \quad \text{for } n \ge 1 \\
\|g_n\|_\infty \le \|K_N\|_2 L l^{-1}\|g_1\|_2\rho^{n-N-1} \le \|K_N\|_2\|K\|_1 L l^{-1}\|g_0\|_2\rho^{n-N-1} \quad \text{for } n \ge N+1 \\
e^{\lambda t}\|\psi_j(t) - \psi_n(t)\|_\alpha \le \|K_N\|_2\|K\|_1 L l^{-1}(1-\rho)^{-1}\|g_0\|_2\rho^{n-N-1} \quad \text{for } N+1 \le n \le j,\ t \in \mathbb{R} \\
e^{\lambda t}\|\Theta(t) - \psi_n(t)\|_\alpha \le \|K_N\|_2\|K\|_1 L l^{-1}(1-\rho)^{-1}\|g_0\|_2\rho^{n-N-1} \quad \text{for } n \ge N+1,\ t \in \mathbb{R}.
\end{gathered}$$

For $t > T$ we have

$$\begin{aligned}
e^{\lambda t}\|\Theta(t)\|_\alpha &\le e^{\lambda t}\|\Theta(t) - \psi_{N+1}(t)\|_\alpha + g_0(t) + g_1(t) + \cdots + g_N(t) \\
&\le \|K_N\|_2\|K\|_1 L l^{-1}(1-\rho)^{-1}\|g_0\|_2 + (1 + \|K\|_1 + \cdots + \|K\|_1^N)\|g_0\|_\infty .
\end{aligned}$$

Evaluation of $\|g_0\|_2, \|g_0\|_\infty$ gives

$$e^{\lambda t}\|\Theta(t)\|_\alpha \le Ce^{\lambda T}\|x - u(T)\|_\alpha \qquad \text{for} \quad t \ge T$$

where

$$C = \|K_N\|_2\|K\|_1 L l^{-1}(1-\rho)^{-1}M\sqrt{\frac{1}{2(\lambda - \lambda_1)} + \frac{1}{2(\lambda_2 - \lambda)}} + M(1 + \|K\|_1 + \cdots + \|K\|_1^N) .$$

Proof of Theorem 4.1. If $\Theta \in Z$ is such that $R\Theta = \Theta$ and $v = \Theta + u$ then a long but obvious calculation shows that this v has the desired properties.

Suppose that we have v as in Theorem 4.1. Obviously $v - u \in Z$. We will show that $v - u = R(v-u)$ and hence Lemma 4.3 will imply uniqueness.

For $-\infty < T \le t < \infty$ we have

$$P_2(v(t) - u(t)) = -P_2u(t) + e^{-A(t-T)}P_2v(T) + \int_T^t e^{-A(t-s)}P_2(F(s,v(s)) - F(s,u(s)))ds +$$
$$+ \int_T^t e^{-A(t-s)}P_2F(s,u(s))ds$$

letting $T \to -\infty$ we obtain (as in the proof of Lemma 3.4)

$$
(4.2)\qquad \begin{aligned} P_2(v(t)-u(t)) &= -P_2u(t) + \int_{-\infty}^{t} e^{-A(t-s)}P_2F(s,u(s))ds + \\ &+ \int_{-\infty}^{t} e^{-A(t-s)}P_2(F(s,v(s))-F(s,u(s)))ds. \end{aligned}
$$

If $\tau \le t \le T$ then

$$
P_1(v(T)-u(T)) = e^{-A_1(T-t)}P_1(v(t)-u(t)) + \int_t^T e^{-A_1(T-s)}P_1(F(s,v(s))-F(s,u(s)))ds
$$

$$
P_1(v(t)-u(t)) = e^{-A_1(t-T)}P_1(v(T)-u(T)) - \int_t^T e^{-A_1(t-s)}P_1(F(s,v(s))-F(s,u(s)))ds
$$

letting $T \to \infty$ we obtain that for $t \ge \tau$

$$
(4.3)\qquad P_1(v(t)-u(t)) = -\int_t^{\infty} e^{-A_1(t-s)}P_1(F(s,v(s))-F(s,u(s)))ds\ .
$$

If $t \le \tau$ then

$$
\begin{aligned} P_1(v(t)-u(t)) &= -P_1u(t) + e^{-A_1(t-\tau)}P_1(v(\tau)-u(\tau)) + e^{-A_1(t-\tau)}P_1u(\tau) - \\ &\quad - \int_t^{\tau} e^{-A_1(t-s)}P_1F(s,v(s))ds \quad \text{and (4.3) implies} \\ &= -P_1u(t) + e^{-A_1(t-\tau)}P_1u(\tau) - \int_t^{\tau} e^{-A_1(t-s)}P_1F(s,u(s))ds - \\ &\quad - \int_t^{\infty} e^{-A_1(t-s)}P_1(F(s,v(s))-F(s,u(s)))ds \\ &= P_1w(t) - \int_t^{\infty} e^{-A_1(t-s)}P_1(F(s,v(s))-F(s,u(s)))ds \end{aligned}
$$

this, (4.2) and (4.3) imply that $v-u = R(v-u)$.

5. Hyperbolic extension. The assumption used so far that A is a sectorial operator can be weakened by requiring that $-A$ is the generator of a strongly continuous semigroup and thus the theory becomes applicable to hyperbolic problems. In this case the condition that $\lambda + i\omega$ is in the resolvent set of A for all real ω does not guarantee existence of subspaces with bounds on the semigroup as presented in Section 2 – therefore we have to postulate them. With these changes and $\alpha = 0$ (hence $X^\alpha = X$, $\|\cdot\|_\alpha = \|\cdot\|$) the results of Sections 3 and 4 apply unchanged. For the sake of clarity let me state explicitly all assumptions needed in this case.

(V1) *X is a complex Banach space.*

(V2) *There exists $M_0 \in (0,\infty)$ such that if $f \in C(\mathbb{R}\backslash\{0\}, X)$ and $\|f(\cdot)\| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} \|f(t)\|^2 dt \le M_0 \int_{-\infty}^{\infty} \|\hat{f}(\omega)\|^2 d\omega \le M_0^2 \int_{-\infty}^{\infty} \|f(t)\|^2 dt$$

where

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \quad for \quad \omega \in \mathbb{R} .$$

(V3) *$-A$ is the generator of a strongly continuous semigroup $e^{-At}, t \ge 0$, on X.*

(V4) *$X = X_1 \oplus X_2$ with $e^{-At}X_1 = X_1$, $e^{-At}X_2 \subset X_2$ for $t \ge 0$ and there exist $M_1 < \infty$ and $-\infty < \lambda_1 < \lambda_2 < \infty$ such that*

$$\begin{aligned} \|e^{-At}x\| &\le M_1 e^{-\lambda_2 t}\|x\| \quad for \quad t \ge 0,\ x \in X_2, \\ \|x\| &\le M_1 e^{\lambda_1 t}\|e^{-At}x\| \quad for \quad t \ge 0,\ x \in X_1. \end{aligned}$$

(V5) *$F : \mathbb{R} \times X \to X$ is continuous and such that for some $B_1, \ldots, B_m \in \mathcal{B}(X)$ we have that*

$$\|F(t,x) - F(t,y)\| \le \sum_{j=1}^{m} \|B_j(x-y)\| \quad for \quad t \in \mathbb{R},\ x, y \in X.$$

(V6) *$\lambda \in (\lambda_1, \lambda_2)$ and $\int_{-\infty}^{0} \|e^{\mu t}F(t,0)\|^2 dt < \infty$ for some $\mu < \lambda$.*

(V7) *$M_0 \sum_{j=1}^{m} \sup_{\omega \in \mathbb{R}} \|B_j(A - \lambda - i\omega)^{-1}\| < 1$.*

Observe that (V4) implies that there exist projections $P_1, P_2 \in \mathcal{B}(X)$ such that $P_i X = X_i$, $P_i e^{-At} = e^{-At} P_i$ for $i = 1, 2$, $t \ge 0$ and $P_1 + P_2 = I$. (V4) also implies that e^{-At} is invertible on X_1, hence, it can be extended to a strongly continuous group defined by

$$\mathcal{B}(X_1) \ni e^{-A_1 t} = \begin{cases} e^{-At} & \text{for} \quad t \ge 0 \\ (e^{At})^{-1} & \text{for} \quad t < 0. \end{cases}$$

For $z \in \mathbb{C}$ with $\lambda_1 < Re\, z < \lambda_2$ one can easily show that z is in the resolvent set of A and that for all $x \in X$

$$(A-z)^{-1}P_1x = -\int_{-\infty}^{0} e^{-A_1t}e^{zt}P_1x\,dt$$

$$(A-z)^{-1}P_2x = \int_{0}^{\infty} e^{-At}e^{zt}P_2x\,dt$$

$$\|(A-z)^{-1}\| \le M_1\left(\frac{\|P_1\|}{Re\, z-\lambda_1}+\frac{\|P_2\|}{\lambda_2-Re\, z}\right) .$$

As in Section 2 let $B_0 = l \cdot I$ with $l \in (0,\infty)$ so small that

$$\rho(\lambda) \equiv M_0 \sum_{j=0}^{m} c_j(\lambda) < 1$$

where

$$c_j(\lambda) = \sup_{\omega\in\mathbb{R}} \|B_j(A-\lambda-i\omega)^{-1}\| \qquad \text{for} \quad j=0,1,\ldots,m.$$

(V6), (V7) and $\rho(\lambda) < 1$ also remain valid if λ is replaced with $\tilde\lambda$ provided that $|\lambda-\tilde\lambda|$ is small enough. Note

$$\|F(t,x)-F(t,y)\| \le L\|x-y\| \qquad \text{for} \quad t\in\mathbb{R},\ x,y\in X$$

where

$$L = \sum_{j=0}^{m}\|B_j\| .$$

Now $\alpha = 0$, $X^\alpha = X$, $\|\cdot\|_\alpha = \|\cdot\|$. Clearly, one can find $a < \lambda_1$ and $M < \infty$ so that the bounds (2.1) hold.

THEOREM 5.1. *In Sections 3 and 4 everything remains valid under the above assumptions (V1)-(V7) and with the above notation.*

Observe also that proofs of Lemmas 3.3, 3.5 and 4.3 can be simplified when $\alpha < 1/2$.

6. Examples. The following examples are presented for comparison purposes.

Example 6.1. Assume that A is a selfadjoint operator in a Hilbert space X and that

the spectrum of A is contained in $(a,\lambda_1]\cup[\lambda_2,\infty)$

for some $-\infty < a < \lambda_1 < \lambda_2 < \infty$. A can have continuous spectrum. Observe that if $\alpha \in [0,1)$ and $\lambda \in (\lambda_1, \lambda_2)$ then

$$\text{(6.1)} \qquad \sup_{\omega \in \mathbf{R}} \|(A-a)^\alpha (A - \lambda - i\omega)^{-1}\| \leq \max\left\{ \frac{(\lambda_1 - a)^\alpha}{\lambda - \lambda_1}, \frac{(\lambda_2 - a)^\alpha}{\lambda_2 - \lambda} \right\} .$$

Fix $\alpha \in [0,1)$ and assume that $F : \mathbf{R} \times X^\alpha \to X$ is continuous and that for some $L \in [0, \infty)$

$$\|F(t,x) - F(t,y)\| \leq L\|(A-a)^\alpha (x-y)\| \quad \text{for} \quad t \in \mathbf{R},\ x, y \in \mathcal{D}((A-a)^\alpha) = X^\alpha.$$

Using (6.1) with $\lambda \in (\lambda_1, \lambda_2)$ that minimizes the right hand side of (6.1) we see that all assumptions (H1)–(H7) are satisfied if

$$\int_{-\infty}^{0} \|e^{\lambda_1 t} F(t,0)\|^2 dt < \infty \qquad \text{and}$$

$$\text{(6.2)} \qquad L((\lambda_2 - a)^\alpha + (\lambda_1 - a)^\alpha) < \lambda_2 - \lambda_1.$$

In the literature the conditions corresponding to (6.2) are much more involved (see (5.1) in [3], (5.3) in [4], p.143-150 in [6], p.423 in [9]). Their expressions become singular as $\alpha \to 1$ ($\alpha \leq 1/2$ in [9]). However, it was known [8] that if $\alpha = 0$ then (6.2) is sufficient for existence of an inertial manifold when the spectrum of A consists of eigenvalues only - which is assumed also in [4,9]. No assumptions on the range of F are made here, however, if one has that $F : X^{\alpha+\beta} \to X^\beta$ for some $\beta > 0$ then one may want to use X^β instead of X for the basic space.

Example 6.2. Consider

$$\text{(6.3)} \qquad \begin{aligned} &u_t = u_{xx} + f(x,t,u,u_x) \qquad 0 < x < \pi,\ t > 0 \\ &u(0,t) = u(\pi,t) = 0 \end{aligned}$$

where $f : [0,\pi] \times \mathbf{R} \times \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is continuous and such that for some $L_1 < \infty, L_2 < \infty$ we have that for all values of arguments

$$|f(x,t,z_1,z_2) - f(x,t,s_1,s_2)| \leq L_1|z_1 - s_1| + L_2|z_2 - s_2|.$$

Let $X = L^2(0,\pi)$, $Au = -u''$ for $u \in H_0^1(0,\pi) \cap H^2(0,\pi)$, $B_1 u = L_1 u$, $B_2 u = L_2 u'$ for $u \in H^1(0,\pi)$, $\alpha \in [1/2, 1)$, $\lambda = (n^2 + (n+1)^2)/2$. By using (6.1) (H7) becomes

$$\frac{L_1}{n + 1/2} + L_2 \frac{n+1}{n+1/2} < 1 .$$

Thus if $L_2 < 1$ and

$$\int_{-\infty}^{0} dt \int_{0}^{\pi} dx\, |e^{\mu t} f(x,t,0,0)|^2 < \infty$$

for some real μ then all assumptions (H1)–(H7) can be satisfied by choosing n large enough. Existence of an invariant manifold for sufficiently small L_2 has been shown in [2].

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