NONLINEAR STABILITY OF ROTATING PSEUDO-RIGID BODIES

By

D. Lewis

and

J.C. Simo

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By D. Lewis* and J.C. Simo†

Mathematical Sciences Institute, Cornell University, Ithaca NY 14853, USA
and
Division of Applied Mechanics, Stanford University, Stanford, CA 94305, USA

Abstract

A rigorous nonlinear stability analysis of rotating homogeneous elastic bodies is presented, which exploits the Hamiltonian structure and symmetries inherent to homogeneous elasticity by means of the energy-momentum method. It is shown that stability of a relative equilibrium is implied by the definiteness of the second variation of a modified Hamiltonian restricted to an appropriate subspace. The analysis makes crucial use of a special parametrization of the constrained space of admissible variations, which results in a nearly diagonal second variation. The stability conditions obtained by this method include the conditions for stability of the equilibrium configuration as a rigid body and satisfaction of the Baker-Ericksen inequalities. As an application of our results, we obtain complete, explicit stability conditions for a particular form of relative equilibria for three classes of materials: for two of these, Ciarlet-Geymonat and St. Venant-Kirchhoff materials, these equilibria are always stable; for the third, a compressible Mooney-Rivlin material, both stable and unstable equilibria exist.

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CONTENTS

1. Introduction

The theory of pseudo-rigid bodies generalizes that of rigid bodies by allowing affine deformations of the body. We shall use pseudo-rigid bodies here to model homogeneous elasticity; i.e., the pseudo-rigid body is identified with a three dimensional elastic body whose deformation gradient is assumed to be constant throughout the body. While the assumption of homogeneity is obviously an extreme simplification of nonlinear elasticity, this model still captures many interesting features of the full infinite dimensional system; in particular, it encompasses constitutive equations with functional forms identical to those for three dimensional elasticity. The pseudo-rigid model is simple enough to enable us to make some fairly general statements concerning the nonlinear stability of relative equilibria, yet detailed enough to capture much of the qualitative behavior of elastic materials. We obtain sufficient conditions for the nonlinear stability of a rigidly rotating pseudo-rigid body with diagonal deformation gradient. These stability conditions are shown to have a direct physical interpretation.

1.a Pseudo-rigid bodies

The pseudo-rigid model can be viewed from several different perspectives. The most mathematically oriented approach is to develop the pseudo-rigid theory as an extension of the theory of rigid bodies. (See Cohen (1981).) This perspective is particularly suggestive when considering the Lie group structure of the configuration space for pseudo-rigid bodies. The configuration space for a rigid body may be identified with the Lie group $SO(3)$, the group of orientation-preserving orthogonal transformations; for a pseudo-rigid body, the configuration space is $\mathcal{C} = GL(3)^+$, the group of orientation-preserving invertible linear transformations of $\mathbb{R}^3$. We shall see that the rigid body is, in some sense, contained in the pseudo-rigid body model.

A physically motivated approach is to view a pseudo-rigid body as a homogeneous elastic body. As noted by Ciarlet (1988), the pseudo-rigid model may be viewed as the lowest order term in a power series expansion for the deformations of a fully elastic body. Pseudo-rigid bodies have been proposed by Muncaster (1984a,b) and Cohen and Muncaster (1984, 1988) as an example of the development of a 'coarse' theory from the 'fine' theory of full three dimensional elasticity. In this light, pseudo-rigid bodies may be thought of as the most extreme case of director theories, such as the Cosserat rod and shell models; see e.g. Ericksen and Truesdell (1958) or Toupin (1965).

In the present work we attempt to maintain a balance between the various possible treatments of pseudo-rigid bodies. We make repeated use of the Lie group structure of the system, keeping the rigid body in mind as a guiding example, but derive our Hamiltonian from the classical total energy function for elasticity. In keeping with the mechanical origins of our problem, we attempt to interpret our results in terms of known elastic phenomena and, with the exception of the St. Venant-Kirchhoff material, select as our examples classes of materials which satisfy appropriate growth and polyconvexity conditions. We note in this context that some of the stability conditions we obtain are equivalent to a set of inequalities which are analogous to, but stronger than, the Baker-Ericksen inequalities.

1.b Nonlinear stability of pseudo-rigid bodies

Our primary goal is the derivation of general nonlinear stability criteria which are applicable to a variety of materials. We quote from Cohen and Muncaster (1988): "For pseudo-rigid bodies, an analysis of the general stability of steady spinning ... is a formidable task." We have been able to both extend the general treatment of stability of rigidly rotating equilibria and strengthen the results for specific configurations and materials obtained by Cohen and Muncaster. In light of the preceding quotation, we feel that our stability analysis illustrates the power and convenience of the energy-momentum method proposed in Simo et al. (1988) and applied here, among other examples. (See Posbergh et al. (1988) and Marsden et al. (1989).) Homogeneous elasticity has a natural Hamiltonian structure; effective exploitation of this structure, and of the symmetries associated to the material frame indifference of elasticity, is the key to our stability analysis.

We summarize here our results. In the most general case, to test for nonlinear orbital stability, it is sufficient to check that the equilibrium configuration is a local minimum of an effective potential; in particular, that the configuration is in steady rotation about its shortest principal axis. In the specific case that the configuration is an isotropic material with diagonal deformation gradient, it can be shown that a configuration is a local minimum of the effective potential if and only if it rotates about its shortest principal axis, satisfies three scalar inequalities, and is a local minimum with respect to the principal stretches alone. We note that the scalar inequalities are equivalent to a variant of the Baker-Ericksen inequalities; in particular, any stable equilibrium satisfies the Baker-Ericksen inequalities. We demonstrate stability of rigidly rotating equilibria with diagonal deformation gradient for two particular classes of materials: St. Venant-Kirchhoff materials and Ciarlet-Geymonat materials. We also show that a class of compressible Mooney-Rivlin materials exhibits both stable and unstable relative equilibria.

As noted above, our nonlinear stability results extend the linear stability results obtained by Cohen and Muncaster (1988). In addition to developing a general quadratic approximation to the Routhian, Cohen and Muncaster consider the linearized stability of a St. Venant-Kirchhoff pseudo-rigid body with respect to variations of angular velocity and variations of the principal stretches. They point out that these variations decouple when the constraint of constant angular momentum is imposed; the stability conditions obtained by considering purely rotational variations are simply the conditions of stable rigid body motion. This decoupling is a particular case of the block diagonal structure determined by the energy-momentum and block diagonalization theorems. The decoupling of rotational modes from an appropriately selected subspace representing the remaining 'internal' modes is a general characteristic of material frame invariant systems; the block diagonalization theorem provides a straightforward means of determining this space of internal modes. (The method guarantees only that the rotational modes decouple from the internal modes in the sense that the second variation block diagonalizes; the modes are generally dynamically coupled.)

The energy-momentum method states that an equilibrium is stable if it is a local extremum of a modified Hamiltonian functional restricted to an appropriately determined submanifold. The block diagonalization theorem specifies local coordinates for this submanifold such that the second variation of the Hamiltonian functional has a block diagonal structure. In our example, this method allows us to split the rotational variations from all 'internal vibrational' modes. In fact, the second variation has additional block structure; for an isotropic material, the second variation almost completely diagonalizes, with the exception of a single three by three block. The degree of structure we obtain enables us to consider the definiteness of the second variation with respect to all possible variations and hence obtain completely rigorous stability results. (Some of the additional stability conditions found by this method are, in fact, those which imply the Baker-Ericksen inequalities.)

2. Stability in the presence of symmetry

As alluded to above, our nonlinear stability analysis exploits in a crucial manner the underlying geometric and Hamiltonian structure of pseudo-rigid bodies. Analyses of this type date back to the pioneering work of Arnold (1966) on hydrodynamic stability. (See also, Arnold (1978), Appendix 2.) Similar techniques have been used by Benjamin (1972), in his analysis of the stability of solitary waves, among others. Extensions and further applications of Arnold's original work to a variety of Hamiltonian systems are considered in Holm et al. (1985).

We provide here an extremely brief description of the geometric structures to be used in the course of this paper. The reader who is primarily interested in pseudo-rigid bodies and has only peripheral interest in the method by which the stability results are obtained may safely skip over this section. On the other hand, the reader who desires a more thorough discussion of Hamiltonian mechanics is referred to Abraham and Marsden (1978) or Arnold (1978). The Hamiltonian structures of several mechanical systems are discussed in detail in, e.g., Arnold (1978) and Marsden and Weinstein (1983): the rigid body and the Euler equations; Benjamin and Olver (1982): water waves; Benjamin (1987): bubbles in inviscid fluid; Simo et al. (1988): elasticity, rigid bodies, rods, and plates.

2.a Relative equilibria

We take as our basic configuration space a manifold Q. The phase space is the cotangent bundle $P = \mathrm{T}^*Q$, elements of which may be viewed as position and momentum variables. The cotangent bundle T^*Q possesses the canonical symplectic structure

$$\omega(\mathbf{q},\mathbf{p})((\delta\mathbf{q}_1,\delta\mathbf{p}_1),(\delta\mathbf{q}_2,\delta\mathbf{p}_2)) := \langle\delta\mathbf{p}_2,\delta\mathbf{q}_1\rangle - \langle\delta\mathbf{p}_1,\delta\mathbf{q}_2\rangle\,.$$

The dynamics of the system are determined by the symplectic structure and the Hamiltonian function H by means of the Hamiltonian vector field X_H defined by

$$DH(\mathbf{q},\mathbf{p})\cdot V(\mathbf{q},\mathbf{p}) = \omega(\mathbf{q},\mathbf{p})(X_H(\mathbf{q},\mathbf{p}),V(\mathbf{q},\mathbf{p})),$$

for all vector fields V on P. The vector field X_H generates the canonical form of Hamilton's equations:

$$\dot{\mathbf{q}} = \frac{\delta H}{\delta\mathbf{p}} \qquad \text{and} \qquad \dot{\mathbf{p}} = -\frac{\delta H}{\delta\mathbf{q}}.$$

It is often the case that a group action plays a prominent role in the analysis of a particular system, either as a symmetry of the system or of a particular equilibrium, or as the generator of the trajectory under consideration. This group action interacts naturally with the Hamiltonian structure of the system. We are typically concerned not only with true equilibrium states, where the configuration is actually at rest, but also with configurations which evolve in time in a particularly simple fashion. An important class of such configurations are those whose trajectories follow the orbit of a group, e.g. configurations in steady translation or rigid rotation. We define a *relative equilibrium* as a point $\mathbf{z}_e \in P$ such that the trajectory through $\mathbf{z}_e$ is given by

$$\mathbf{z}_e(t) = \exp(t\xi) \cdot \mathbf{z}_e \tag{1}$$

for some $\xi \in \mathcal{G}$.

Two constructions associated to the action of a Lie group on a manifold which are central to the stability analysis are infinitesimal generators and momentum maps. Associated to a Lie group G are the Lie algebra $\mathcal{G} = \mathrm{T}_{\mathrm{Id}}G$ and its dual $\mathcal{G}^* = \mathrm{T}^*_{\mathrm{Id}}G$. Given a Lie group G acting on a manifold Q, the infinitesimal generator ξ_Q associated to an element $\xi \in \mathcal{G}$ is the vector field

$$\xi_Q(\mathbf{q}) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \exp(\epsilon\,\xi) \cdot \mathbf{q},$$

where $\cdot$ denotes the action of $g \in G$ on $\mathbf{q} \in Q$. The momentum map $\mathbf{J} : \mathrm{T}^*Q \to \mathcal{G}^*$ associated to a lifted action on a cotangent bundle is given by

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \xi \rangle := \langle \mathbf{p}, \xi_Q(\mathbf{q}) \rangle \qquad \forall \xi \in \mathcal{G}.$$

This definition generalizes the traditional concepts of linear and angular momentum, which are the momentum maps associated to the translation and rotation groups, respectively.

Just as true equilibria of a Hamiltonian system can be characterized as critical points of the Hamiltonian, relative equilibria are also critical points of an appropriate function. Relative equilibria of a Hamiltonian system can be characterized as critical points of the Hamiltonian subject to the constraint of constant momentum map. The implementation of this constrained variational principle via Lagrange multipliers leads to the following result, which plays a central role in our subsequent developments.

Theorem 1 (Relative Equilibrium Theorem) *A point $\mathbf{z}_e \in \mathrm{T}^*Q$ is a relative equilibrium of the dynamical system if and only if there exists a $\xi \in G$ such that $\mathbf{z}_e$ is a critical point of*

$$H_\xi = H - \langle \mathbf{J}, \xi \rangle + \text{constant}.$$

Proof: See Arnold (1978).

Note that the algebra element $\xi \in \mathcal{G}$ plays the role of a Lagrange multiplier. We shall refer to H_ξ as the modified Hamiltonian or energy-momentum functional.

2.b The energy-momentum method

When considering the stability of a (relative) equilibrium, it is important to take into account the symmetries of the system. If a system is equivariant with respect to the action of a continuous group G, any equilibrium on which G acts nontrivially cannot be stable in the strictest sense of the word; the equilibrium is neutrally stable to perturbations along the group orbit. A useful characterization of stability in such contexts is that of *orbital stability*: If a dynamical system is G-equivariant, under appropriate technical conditions it projects to a system on the quotient manifold P/G; a relative equilibrium is said to be orbitally stable if the corresponding equilibrium of the reduced system is stable. If the original system is Hamiltonian with a symplectic structure, it is often possible to consider either the Poisson manifold P/G or the reduced symplectic manifold $P_{\mu_e} = \mathbf{J}^{-1}(\mu_e)/(G_{\mu_e} \cdot \mathbf{z}_e)$. Here $\mu_e = \mathbf{J}(\mathbf{z}_e)$ and $G_{\mu_e} = \{\zeta \in \mathcal{G} : \mathrm{ad}^*_\zeta \mu_e = \mathbf{0}\}$. Provided that μ_e is a regular value (in the sense that the coadjoint orbits in a neighborhood of μ_e are of fixed dimension), stability on P/G is equivalent to stability on P_{μ_e} and hence orbital stability may be tested on either manifold.

When we use the word stability in this paper, we typically mean *nonlinear stability.* An equilibrium $\mathbf{z}_e$ is said to be nonlinearly stable if, given a neighborhood $\mathcal{U}$ of $\mathbf{z}_e$, there exists a neighborhood $\mathcal{V}$ of $\mathbf{z}_e$ such that any configuration $\mathbf{z}$ initially lying in $\mathcal{V}$ remains in $\mathcal{U}$ for all time. We shall verify the nonlinear stability of equilibria by testing for *formal stability.* An equilibrium $\mathbf{z}_e$ of a Hamiltonian system with modified Hamiltonian H_ξ is said to be formally stable if $D^2H_\xi(\mathbf{z}_e)$ is definite. If the phase space P is finite dimensional, formal stability implies nonlinear stability. (If the phase space P is infinite dimensional, additional estimates are required to establish nonlinear stability.) Thus, when we say that a relative equilibrium $\mathbf{z}_e$ is orbitally stable, we mean that the reduced Hamiltonian on P_{μ_e} or P/G has definite second variation at the equivalence class $[\mathbf{z}_e]$.

It is possible in some cases to directly check the stability of the equilibria of the reduced system, but in a number of situations this is computationally difficult, if not impossible. The energy-momentum method is designed to enable one to test for orbital stability directly on the unreduced manifold P by constructing a subspace $\mathcal{S} \subset \mathrm{T}_{\mathbf{z}_e}P$ such that $\mathcal{S}$ is isomorphic to $\mathrm{T}_{[\mathbf{z}_e]}P_{\mu_e}$. We present here a brief description of the energy-momentum method; for a detailed description of the method, see Simo et al. (1988). The constrained subspace $\mathcal{S}$ is implemented by enforcing the condition that $\mathcal{S}$ be tangent to the level set of constant angular momentum and eliminating the neutrally stable variations associated to the isotropy subgroup G_{μ_e}.

The space $\mathcal{S}$ is not uniquely defined; the equivalence classes in the quotient space may be characterized by elements of any subspace of $\mathrm{T}_{\mathbf{z}_e}\mathbf{J}^{-1}(\mu_e)$ complementary to $\mathrm{T}_{\mathbf{z}_e}(G_{\mu_e}\cdot\mathbf{z}_e)$. In actual computation of the second variation and determination of its definiteness, the selection of an appropriate $\mathcal{S}$ and a basis for that subspace is crucial to the success of the method for nontrivial examples. We apply here a construction proposed in Simo et al. (1988) which dictates a choice of $\mathcal{S}$ such that the second variation decouples into a block associated to infinitesimal group motions which do not preserve the modified Hamiltonian H_ξ and a block associated to variations complementary to infinitesimal group motions, i.e. variations which may be identified with variations of the equivalence classes in P_{μ_e}.

Assume that Q has a G-invariant metric g and a G-invariant potential V such that the modified Hamiltonian H_ξ is given by

$$H_\xi(\mathbf{z}) := \tfrac{1}{2}\,\|\mathbf{p}\|^2_{g^{-1}} + V(\mathbf{q}) - \langle \mathbf{J}(\mathbf{z}), \xi\rangle\,. \tag{2}$$

Here $\|\mathbf{p}\|^2_{g^{-1}} = \|\mathbf{FL}^{-1}(\mathbf{p})\|^2_g$, where the Legendre transform $\mathbf{FL} : \mathrm{T}Q \to \mathrm{T}^*Q$ is given by $\langle \mathbf{FL}(\mathbf{u}), \mathbf{v}\rangle := \langle \mathbf{u}, \mathbf{v}\rangle_g \;\; \forall\, \mathbf{v} \in \mathrm{T}_{\mathbf{q}}Q$. We decompose H_ξ as follows:

$$H_\xi(\mathbf{z}) = V_\xi(\mathbf{q}) + \Gamma_\xi(\mathbf{z}),$$

where

$$\Gamma_\xi(\mathbf{z}) := \tfrac{1}{2}\,\|\mathbf{p} - \mathbf{FL}(\xi_Q(\mathbf{q}))\|^2_{g^{-1}} \tag{3}$$

and the effective potential $V_\xi : Q \to \mathbb{R}$ is given by $V_\xi := V + L_\xi$ for

$$L_\xi(\mathbf{q}) := -\tfrac{1}{2}\,\|\xi_Q(\mathbf{q})\|^2_g\,. \tag{4}$$

Define the space $\mathcal{S} \subset \mathrm{T}_{\mathbf{z}_e}\mathrm{T}^*Q$ to be the intersection of $\ker \mathrm{T}_{\mathbf{z}_e}\mathbf{J}$ with the metric orthogonal complement to the tangent to the orbit of G_{μ_e}, i.e.

$$\mathcal{S} := \left\{ \delta\mathbf{z} \in \ker \mathrm{T}_{\mathbf{z}_e}\mathbf{J} : \langle \zeta_Q(\mathbf{q}_e), \mathrm{T}\pi(\delta\mathbf{z}) \rangle_g = 0 \qquad \forall \zeta \in \mathcal{G}_{\mu_e} \right\}, \tag{5}$$

where $\pi : T^*Q \to Q$ is the canonical projection $\pi(\mathbf{q}, \mathbf{p}) := \mathbf{q}$.

We now find subspaces $\mathcal{S}_{RIG}$ and $\mathcal{S}_{INT}$ such that $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$ and

$$D^2 H_\xi(\mathbf{z}_e)(\Delta\mathbf{z}, \delta\mathbf{z}) = 0,$$

for all $\Delta\mathbf{z} \in \mathcal{S}_{RIG}$ and $\delta\mathbf{z} \in \mathcal{S}_{INT}$. $\mathcal{S}_{RIG}$ is the subspace associated to infinitesimal group actions which do *not* preserve H_ξ; its complement $\mathcal{S}_{INT}$ is chosen such that $DH_\xi|_{\mathcal{S}_{INT}}$ is G-invariant. Specifically, letting $\mathcal{G}^\perp_{\mu_e}$ denote the orthogonal complement to $\mathcal{G}_{\mu_e}$, we define

$$\mathcal{S}_{RIG} := \left\{ \left.\tfrac{d}{d\epsilon}\right|_{\epsilon=0} \left(\exp(\epsilon\,\eta) \cdot \mathbf{q}_e, \mathbf{FL}\left(\left.\tfrac{d}{dt}\right|_{t=0} \exp(\epsilon\,(\eta + t\zeta)) \cdot \mathbf{q}_e(t) \right) \right) \in \mathcal{S} : \eta, \zeta \in \mathcal{G}^\perp_{\mu_e} \right\}. \tag{6}$$

The complement $\mathcal{S}_{INT}$ in $\mathcal{S}$ to $\mathcal{S}_{RIG}$ is chosen to be

$$\mathcal{S}_{INT} := \left\{ \delta\mathbf{z} \in \mathcal{S} : \left\langle \mathrm{T}\pi(\delta\mathbf{z}), \mathcal{L}_{\eta_Q(\mathbf{q}_e)} \tfrac{\delta L_\xi}{\delta\mathbf{q}}(\mathbf{q}_e) \right\rangle_g = 0 \qquad \forall \eta \in \mathcal{G}^\perp_{\mu_e} \right\}, \tag{7}$$

where $\mathcal{L}_{\mathbf{v}}\mathbf{u}$ denotes the Lie derivative of the vector field $\mathbf{u}$ with respect to $\mathbf{v}$.

Remark: In the energy-momentum method, one tests for orbital stability with respect to the reduced symplectic manifold P_{μ_e}. One can, however, focus instead on the manifold P/G. This manifold is typically not symplectic, but has a Poisson structure $\{\ ,\ \}$ induced by the symplectic structure on P. The Poisson bracket obtained on this space is typically degenerate; i.e. there exist nonconstant functions C such that $\{F, C\} = 0$ for all functions F on P. The analog of the energy-momentum method on the Poisson manifold P/G is the energy-Casimir method. This method has been highly successful in a variety of problems, including two dimensional fluid flow and RHMD (see Holm et al. (1985)). It is, however, limited by the need for sufficient Casimirs to obtain a critical point of the function $H + C$ at the relative equilibrium. It has been shown rigorously that there are noncanonical Poisson structures for which no Casimirs exist. It appears that this is the case for the convective representation of three dimensional elasticity studied by Simo et al. (1988). Even in cases where Casimirs are available (e.g. the convective representation of pseudo-rigid bodies), it may be advantageous to apply the energy-momentum method, due to the convenience of the block diagonalization procedure. (The analog for the energy-Casimir method of the block diagonalization theorem used in the energy-momentum method has not yet been developed.)

3. Hamiltonian structure of pseudo-rigid bodies

3.a Canonical phase space

We now present the configuration space for the pseudo-rigid body model. We consider systems which evolve by affine motions preserving some fixed point, i.e. configurations whose evolution is determined by an element of $\mathcal{C} := Gl^+(3)$. (In the case of incompressible elasticity, the group $Sl(3)$ of volume and orientation preserving matrices would be the appropriate group.) In the examples considered here, we can assume without loss of generality that the center of mass of the body is the fixed point. The configuration space $\mathcal{C}$ may, in fact, be obtained as the reduction of the full affine group by the symmetries associated to translation invariance, but we do not consider that reduction here.

The embedding ϕ_t of the reference manifold $\mathcal{B} \subset \mathbb{R}^3$ into $\mathbb{R}^3$ at time t is of the form

$$\phi_t(X) = F_t X,$$

where $F_t \in \mathcal{C}$ acts on $X \in \mathcal{B}$ by standard matrix multiplication. It follows that the matrix F_t is the deformation gradient of the embedding ϕ_t. A tangent vector to F is of the form (F, V) for some $V \in L(3)$, the space of general 3×3 matrices. The cotangent bundle $\mathrm{T}^*\mathcal{C}$ consists of pairs (F, P) such that $F \in \mathcal{C}$ and $P \in L(3)$. The pairing between $(F, P) \subset \mathrm{T}^*_F\mathcal{C}$ and $(F, V) \in \mathrm{T}_F\mathcal{C}$ is given by

$$\langle P, V \rangle := \operatorname{tr}(P^T V).$$

In general, given an invertible symmetric matrix S, we define the weighted inner product $\langle \, , \, \rangle_S$ by

$$\langle U, V \rangle_S := \operatorname{tr}(U^T V S),$$

and the associated weighted norm $\| \, \|_S$ by

$$\|V\|_S^2 := \langle V, V \rangle_S \,.$$

We consider here the treatment of the pseudo-rigid body in the canonical phase space $\mathrm{T}^*\mathcal{C}$. The coordinates (F, P) are referred to as material or Lagrangian coordinates. The cotangent bundle $\mathrm{T}^*\mathcal{C}$ possesses the canonical symplectic structure

$$\omega(F, P)((\delta F_1, \delta P_1), (\delta F_2, \delta P_2)) := \langle \delta P_2, \delta F_1 \rangle - \langle \delta P_1, \delta F_2 \rangle \,. \tag{8}$$

The Hamiltonian vector field X_H associated to a function $H : \mathrm{T}^*\mathcal{C} \to \mathbb{R}$ is determined by requiring that the relation

$$DH(F, P) \cdot V(F, P) = \omega(F, P)(X_H(F, P), V(F, P)), \tag{9}$$

hold for all vector fields V on $\mathrm{T}^*\mathcal{C}$. In this case, the general formula $\dot{\mathbf{x}} = X_H(\mathbf{x})$ generates the equations of motion in Hamilton's form:

$$\dot{F} = \frac{\delta H}{\delta P} \qquad \text{and} \qquad \dot{P} = -\frac{\delta H}{\delta F}. \tag{10}$$

3.b The Hamiltonian for homogeneous elasticity

The Hamiltonian for homogeneous elasticity in the absence of body and surface forces is

$$H(F, P) := \tfrac{1}{2} \langle P, P \rangle_{E^{-1}} + W(F^T F), \tag{11}$$

where

$$E := \int_{\mathcal{B}} \rho_{\mathrm{ref}}(X) X \otimes X \, dX,$$

and W is the stored energy function for the homogeneous material, expressed as a function of the right Cauchy-Green tensor $C = F^T F$. This Hamiltonian is determined as follows:

We define the velocity field V associated to the momentum P by $V = PE^{-1}$. The kinetic energy $K(V)$ is given by

$$
\begin{aligned}
K(V) &:= \tfrac{1}{2}\int_{\mathcal{B}} \rho_{\mathrm{ref}}(X)\langle VX, VX\rangle dX \\
&= \tfrac{1}{2}\int_{\mathcal{B}} \rho_{\mathrm{ref}}(X)\mathrm{tr}\left((VX)^T(VX)\right) dX \\
&= \tfrac{1}{2}\int_{\mathcal{B}} \rho_{\mathrm{ref}}(X)\mathrm{tr}\left((X\otimes X)V^TV\right) dX \\
&= \tfrac{1}{2}\mathrm{tr}\left((VE)^T V\right) \\
&= \tfrac{1}{2}\langle V, V\rangle_E \\
&= \tfrac{1}{2}\langle P, P\rangle_{E^{-1}}.
\end{aligned}
$$

Alternatively, one could start with the velocity field V and determine the appropriate definition for the momentum by considering the kinetic energy (which is the only velocity dependent component of the total energy for the system) and performing a Legendre transformation. The stored energy function W for the homogeneous material is related to the pointwise stored energy function $\widetilde{W}$ for nonhomogeneous elasticity by

$$
\begin{aligned}
W(C) &= \int_{\mathcal{B}} \rho_{\mathrm{ref}}(X)\widetilde{W}(C)dX \\
&= \int_{\mathcal{B}} \rho_{\mathrm{ref}}(X)dX\,\widetilde{W}(C).
\end{aligned}
$$

The Hamiltonian H has functional derivatives

$$
\begin{aligned}
\tfrac{\delta H}{\delta F} &= 2F\tfrac{\delta W}{\delta C} \\
\tfrac{\delta H}{\delta P} &= V,
\end{aligned}
$$

where $\frac{\delta W}{\delta C}$ is the symmetric matrix determined by

$$
\left\langle \tfrac{\delta W}{\delta C}, \delta C\right\rangle = DW\cdot\delta C,
$$

for all symmetric matrices δC. Thus the equations of motion for homogeneous elasticity in the material representation are

$$
\begin{aligned}
\dot{F} &= V \\
\dot{P} &= -2F\tfrac{\delta W}{\delta C}.
\end{aligned}
$$

The acceleration is given by

$$
\dot{V} = -2F\tfrac{\delta W}{\delta C}E^{-1}.
$$

If an external force T acts on the surface of the configuration, this adds a potential energy term of the form $\mathrm{tr}(F^T\widetilde{T})$ to the Hamiltonian, where

$$
\widetilde{T} = \int_{\partial\mathcal{B}} \rho_{\mathrm{ref}} T(X)\otimes X\,dA. \tag{12}
$$

Similarly, a body force B acting on the configuration induces a term of the form $\mathrm{tr}(F^T\widetilde{B})$ in the Hamiltonian, where

$$
\widetilde{B} = \int_{\mathcal{B}} \rho_{\mathrm{ref}} B(X)\otimes X\,dX. \tag{13}
$$

3.c Group actions and momentum maps

The Lie group $\mathcal{C} = Gl^{+}(3)$ acts on itself by multiplication on the left and right. These actions induce in turn lifted actions on the cotangent bundle $\mathrm{T}^*\mathcal{C}$. We identify $gl(3)^*$ with $gl(3)$ by means of the duality pairing $\langle \mu, \xi \rangle = \mathrm{tr}\,(\mu^T \xi)$ for $\xi \in gl(3)$ and $\mu \in gl(3)^*$.

We define the infinitesimal generators $\xi^L_{\mathcal{C}} : \mathcal{C} \to \mathrm{T}\mathcal{C}$ and $\xi^R_{\mathcal{C}} : \mathcal{C} \to \mathrm{T}\mathcal{C}$ by

$$\begin{aligned} \xi^L_{\mathcal{C}}(F) &:= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \exp(\epsilon\,\xi)F \\ &= \xi F \end{aligned}$$

and

$$\begin{aligned} \xi^R_{\mathcal{C}}(F) &:= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F\exp(\epsilon\,\xi) \\ &= F\xi. \end{aligned}$$

The associated equivariant momentum maps $\mathbf{J}_{L(R)} : \mathrm{T}^*\mathcal{C} \to gl(3)^*$ are determined by the formulae

$$\langle \mathbf{J}_L(F,P), \xi \rangle := \left\langle P, \xi^L_{\mathcal{C}}(F) \right\rangle,$$

and

$$\langle \mathbf{J}_R(F,P), \xi \rangle := \left\langle P, \xi^R_{\mathcal{C}}(F) \right\rangle.$$

Specifically, the momentum map associated to the right action of $\mathcal{C}$ on $\mathrm{T}^*\mathcal{C}$ is

$$\mathbf{J}_R(F,P) = F^T P, \tag{14}$$

and the momentum map associated to the left action of $\mathcal{C}$ is

$$\mathbf{J}_L(F,P) = PF^T. \tag{15}$$

We shall be particularly concerned with the left action of the rotation group $SO(3) \subset \mathcal{C}$ on $\mathrm{T}^*\mathcal{C}$. The Lie algebra $so(3)$ is the algebra of skew symmetric matrices; we identify $so(3)$ with $\mathbb{R}^3$ as follows: we identify the vector $\eta \in \mathbb{R}^3$ with the skew-symmetric matrix $\hat{\eta}$ such that $\hat{\eta}\,\mathbf{x} = \eta \times \mathbf{x}\ \ \forall\, \mathbf{x} \in \mathbb{R}^3$. Analogously, we identify $so(3)^*$ with $\mathbb{R}^3$ by associating to the vector μ the skew-symmetric matrix $\hat{\mu}$ such that $\hat{\mu}\,\mathbf{x} = \frac{1}{2}\mu \times \mathbf{x}\ \ \forall\, \mathbf{x} \in \mathbb{R}^3$. The factor of one half is chosen so that $\langle \hat{\mu}, \hat{\eta} \rangle = \mathrm{tr}(\hat{\mu}^T \hat{\eta}) = \mu \cdot \eta$, where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^3$. For the sake of notational simplicity, we denote by $\xi_{\mathcal{C}}$ the infinitesimal generator $\hat{\xi}_{\mathcal{C}}$ associated to an infinitesimal rotation about ξ. The momentum map associated to the left action of $SO(3)$ on $\mathrm{T}^*\mathcal{C}$ is the angular momentum, $\mathcal{J} : \mathrm{T}^*\mathcal{C} \to so(3)^* \approx \mathbb{R}^3$, given by

$$(\mathcal{J}(F,P))\hat{} = (PF^T)^{\mathcal{A}}, \tag{16}$$

where $M^{\mathcal{A}} = \frac{1}{2}(M - M^T)$ denotes the skew-symmetric component of M. (We denote the symmetric component of a matrix M by $M^{\mathcal{S}} = \frac{1}{2}(M + M^T)$.) We shall use the angular momentum to construct the modified Hamiltonian for which a given rigidly rotating equilibrium is a critical point.

4. Stability of relative equilibria

4.a First and second variations of the Hamiltonian H_ξ

In this section we compute the first and second variation of the Hamiltonian for a general rigidly rotating relative equilibrium. In the following section we shall specialize these expressions for a specific class of equilibria and determine explicit conditions for the stability of such configurations. We wish to find configurations in rigid rotation with angular velocity ξ. In this case, the evolution of the configuration is given by

$$F_e(t) = \exp(t\hat{\xi})F_e.$$

Thus a rigidly rotating configuration is a relative equilibrium with respect to the left action of the rotation group $SO(3)$. In accordance with the Relative Equilibrium Theorem, we define the modified Hamiltonian

$$H_\xi(F,P) := \tfrac{1}{2}\langle P,P\rangle_{E^{-1}} + W(F^TF) - \mathcal{J}(F,P)\cdot\xi \tag{17}$$

and search for critical points of H_ξ.

The first variation of the modified Hamiltonian H_ξ is

$$DH_\xi(F,P) = \left\langle \tfrac{\delta W}{\delta F} + \hat{\xi}P, \delta F\right\rangle + \left\langle PE^{-1} - \hat{\xi}F, \delta P\right\rangle. \tag{18}$$

It follows that the configuration (F_e, P_e) is a critical point of the modified Hamiltonian H_ξ if and only if

$$P_e = \hat{\xi}F_eE \tag{19}$$

and

$$\begin{aligned} \tfrac{\delta W}{\delta F} &= -\hat{\xi}P_e \\ &= -\hat{\xi}\hat{\xi}F_eE \\ &= |\xi|^2\,\mathbf{P}F_eE, \end{aligned} \tag{20}$$

where $\mathbf{P} = \mathrm{Id} - (\xi\otimes\xi)/|\xi|^2$ is the orthogonal projection parallel to ξ.

The righthand side of (20) can be interpreted as the centrifugal force acting on the configuration at a relative equilibrium. This force is conservative with potential $L_\xi : \mathcal{C}\to\mathbb{R}$, defined by

$$\begin{aligned} L_\xi(F) &:= -\tfrac{1}{2}\,\|\xi_{\mathcal{C}}(F)\|_E^2 \\ &= -\tfrac{1}{2}\left\|\hat{\xi}F\right\|_E^2 \\ &= -\tfrac{1}{2}\left\|\hat{\xi}\right\|_{\mathcal{E}}^2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{E} &:= FEF^T \\ &= \int_{\mathcal{B}_{\mathrm{ref}}}(FX)\otimes(FX)\,dX \end{aligned} \tag{21}$$

is the inertia dyadic of the current configuration. In fact, the first variation of L_ξ is

$$\begin{aligned} DL_\xi(F_e)\cdot\delta F &= \left\langle \hat{\xi}\hat{\xi}F_eE, \delta F\right\rangle \\ &= \left\langle \hat{\xi}P_e, \delta F\right\rangle, \end{aligned} \tag{22}$$

which agrees with (20). Consequently, (20) is the Euler-Lagrange equation of an effective potential $V_\xi : \mathcal{C} \to \mathbb{R}$, given by

$$V_\xi := W + L_\xi. \tag{23}$$

It follows from (22) and (18) that a relative equilibrium F_e is a critical point of V_ξ. At a relative equilibrium (F_e, P_e), V_ξ has second variation

$$\begin{aligned} D^2V_\xi(F_e)\cdot(\delta F_1, \delta F_2) &= D^2W(F_e)(\delta F_1, \delta F_2) + \left\langle \delta F_1, \hat{\xi}\hat{\xi}\delta F_2 \right\rangle_E & (24)\\ &= D^2W(F_e)(\delta F_1, \delta F_2) - \left\langle \hat{\xi}\delta F_1, \hat{\xi}\delta F_2 \right\rangle_E . & (25)\end{aligned}$$

The modified Hamiltonian H_ξ can be reformulated in terms of the effective potential V_ξ as follows. Define the function $\Gamma_\xi : \mathrm{T}^*\mathcal{C} \to \mathbb{R}$ by

$$\begin{aligned} \Gamma_\xi(F,P) &:= \tfrac{1}{2}\left\|P - \xi c(F)E\right\|^2_{E^{-1}} \\ &= \tfrac{1}{2}\left\|P - \hat{\xi}FE\right\|^2_{E^{-1}} . \end{aligned} \tag{26}$$

Since at equilibrium we have $P_e = \xi c(F_e)E$, it follows that $D\Gamma_\xi(F_e, P_e) = \mathbf{0}$ if (F_e, P_e) is a relative equilibrium. We note that Γ_ξ has second variation

$$D^2\Gamma_\xi((\delta F_1, \delta P_1),(\delta F_2, \delta P_2)) = \left\langle \delta P_1 - \hat{\xi}\delta F_1 E, \delta P_2 - \hat{\xi}\delta F_2 E \right\rangle_{E^{-1}} . \tag{27}$$

A direct computation now shows that $H_\xi = V_\xi + \Gamma_\xi$; in fact:

$$\begin{aligned} &\Gamma_\xi(F,P) + V_\xi(F) \\ &\quad= \tfrac{1}{2}\left\|P - \xi c(F)E\right\|^2_{E^{-1}} - \tfrac{1}{2}\left\|\xi c(F)\right\|^2_E + W(F^TF) \\ &\quad= \tfrac{1}{2}\left\|P\right\|^2_{E^{-1}} - \langle P, \xi c(F)\rangle + W(F^TF) \\ &\quad= H_\xi(F,P). \end{aligned}$$

It follows from (25) and (27) that the second variation of the modified Hamiltonian H_ξ is given by

$$\begin{aligned} &D^2H_\xi(F_e,P_e)((\delta F_1, \delta P_1),(\delta F_2, \delta P_2)) \\ &\quad= D^2\Gamma_\xi(F_e)(\delta F_1, \delta F_2) + D^2V_\xi(F_e)(\delta F_1, \delta F_2) \\ &\quad= \left\langle \delta P_1 - \hat{\xi}\delta F_1 E, \delta P_2 E^{-1} - \hat{\xi}\delta F_2 \right\rangle \\ &\qquad\quad - \left\langle \hat{\xi}\delta F_1, \hat{\xi}\delta F_2 \right\rangle_E + D^2W(F_e)(\delta F_1, \delta F_2). \end{aligned}$$

4.b The momentum map at equilibrium

In view of the relative equilibrium conditions (19) and (20), the momentum map associated to the left action of $SO(3)$ and given by (16) takes the following value at equilibrium:

$$\begin{aligned} \hat{\mu}_e &= \left(P_e F_e^T\right)^{\mathcal{A}} \\ &= \left(\hat{\xi}F_e E F_e^T\right)^{\mathcal{A}} \\ &= \left(\hat{\xi}\mathcal{E}\right)^{\mathcal{A}} \\ &= \widehat{\mathbb{I}_\infty \xi}, \end{aligned}$$

where $\mathbb{I}_\infty = (\operatorname{tr}\mathcal{E})\mathrm{Id} - \mathcal{E}$ is the inertia dyadic of the current configuration. Thus $\mu_e = \mathbb{I}_\infty\xi$.

We remark that the momentum map associated to the left action of $SO(3)$ is equal to the usual angular momentum for a rotating body. This may be seen directly from the definition of angular momentum as follows:

$$
\begin{aligned}
\int_{\mathcal{B}_{\mathrm{ref}}} \varphi(X) \times \mathbf{v}_\varphi(X) dX &= \int_{\mathcal{B}_{\mathrm{ref}}} (FX) \times (\xi \times (FX)) dX \\
&= -\int_{\mathcal{B}_{\mathrm{ref}}} (\widehat{FX})(\widehat{FX})\xi \, dX \\
&= \left(\int_{\mathcal{B}_{\mathrm{ref}}} \left(|FX|^2 \,\mathrm{Id} - (FX) \otimes (FX) \right) dX \right) \xi \\
&= ((\operatorname{tr}\mathcal{E})\mathrm{Id} - \mathcal{E})\, \xi \\
&= \mathbb{I}_\infty \xi \\
&= \mu_e.
\end{aligned}
$$

The relative equilibria conditions (19) and (20) have a familiar mechanical interpretation. We claim that

$$\mu_e \times \xi = \mathbf{0}. \tag{28}$$

That is, at equilibrium, the total angular momentum is aligned with the axis of rotation ξ. Furthermore, we have

$$\mathbb{I}_\infty \xi = \lambda_\infty \xi \qquad \text{for some } \lambda_\infty > 0. \tag{29}$$

The proof of these conditions follows easily by direct computation. To prove (28), we observe that

$$
\begin{aligned}
(\widehat{\mu_e \times \xi}) &= \left[\hat{\mu}_e, \hat{\xi}\right] \\
&= \hat{\mu}_e \hat{\xi} - \hat{\xi}\hat{\mu}_e \\
&= \tfrac{1}{2}\left(\hat{\xi}\mathcal{E}\hat{\xi} + \mathcal{E}\hat{\xi}\hat{\xi} - \hat{\xi}\hat{\xi}\mathcal{E} - \hat{\xi}\mathcal{E}\hat{\xi} \right) \\
&= \tfrac{1}{2}\left(\mathcal{E}\hat{\xi}\hat{\xi} - \hat{\xi}\hat{\xi}\mathcal{E} \right) \\
&= -\left(\hat{\xi}\hat{\xi}\mathcal{E} \right)^{\mathcal{A}}.
\end{aligned}
$$

Hence $\mu_e \times \xi = \mathbf{0}$ if and only if $\hat{\xi}\hat{\xi}\mathcal{E}$ is symmetric. However, the equilibrium condition (18) implies that

$$
\begin{aligned}
\hat{\xi}\hat{\xi}\mathcal{E} &= -\tfrac{\delta W}{\delta F}(F_e)F_e^T \\
&= -\tau_e,
\end{aligned}
$$

where τ_e is the Kirchhoff stress tensor at equilibrium, which is symmetric. Thus $\mu_e \times \xi = \mathbf{0}$. Since μ_e is parallel to ξ, we must have $\mathbb{I}_\infty\xi = \mu_e = \lambda_\infty\xi$ for some $\lambda_\infty \in \mathbb{R}$. Finally, the fact that $\mathbb{I}_\infty$ is a positive definite matrix implies $\lambda_\infty > 0$.

We conclude this subsection by providing an explicit characterization of the Lie algebra $\mathcal{G}_{\mu_e}$ and the orthogonal complement $\mathcal{G}_{\mu_e}^\perp$. In the present context, $\mathcal{G} = so(3)$. Recall that the subalgebra $so(3)_{\mu_e}$ associated to the isotropy subgroup $SO(3)_{\mu_e}$ consists of the elements

$\eta \in so(3)$ such that for all $\zeta \in so(3)$

$$\begin{aligned} 0 &= \left(\mathrm{ad}^*_\eta \mu_e\right) \cdot \zeta \\ &= \mu_e \cdot [\eta, \zeta] \\ &= \lambda_\infty \xi \cdot (\eta \times \zeta) \\ &= \lambda_\infty \zeta \cdot (\xi \times \eta), \end{aligned}$$

where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^3$. Thus, we conclude that $so(3)_{\mu_e}$ is simply the one dimensional space spanned by ξ. The orthogonal complement $so(3)^\perp$ to $so(3)_{\mu_e}$ is

$$so(3)^\perp := \{\eta \in so(3) : \eta \cdot \xi = 0\}. \tag{30}$$

Note that $so(3)/so(3)_{\mu_e} \approx so(3)^\perp$.

4.c The block diagonalization theorem

We now determine the stability of the relative equilibrium using the block diagonal decomposition procedure recently proposed in Simo et al. (1988). According to the general procedure for the energy-momentum method, we need only consider the definiteness of the second variation of H_ξ on a subspace $\mathcal{S}$ corresponding to the tangent space at the equilibrium to the reduced manifold P_{μ_e}. We shall now define a series of subspaces to be used in the determination of $\mathcal{S}$.

The steps of the decomposition method can be organized as follows:

Step 1:
Determine the space $V \subset \mathrm{T}_{F_e}\mathcal{C}$ which is orthogonal with respect to the kinetic energy inner product to the space of momentum preserving rigid body configuration variations. By a rigid body configuration variation, we mean a tangent vector $\Delta F \in \mathrm{T}_{F_e}\mathcal{C}$ such that $\Delta F = \eta_{\mathcal{C}}(F_e) = \hat{\eta} F_e$ for some $\eta \in so(3)$. Since $so(3)_{\mu_e} = \mathrm{span}\,\{\xi\}$, V is given by

$$V := \{\delta F \in \mathrm{T}_{F_e}\mathcal{C} : \langle \delta F, \xi_{\mathcal{C}}(F_e)\rangle_E = 0\}. \tag{31}$$

Step 2:
Define the subspace $V_{RIG} \subset V$ of rigid body configuration variations associated to $so(3)^\perp$ (i.e., rigid body variations which do *not* preserve angular momentum):

$$V_{RIG} := \left\{\eta_{\mathcal{C}}(F_e) : \eta \in so(3)^\perp\right\}. \tag{32}$$

That $V_{RIG} \subset V$ may be seen as follows:

$$\begin{aligned} \left\langle \Delta F, \hat{\xi} F_e \right\rangle_E &= \left\langle \hat{\eta} F_e, \hat{\xi} F_e \right\rangle_E \\ &= \left\langle \hat{\eta}, \hat{\xi} \right\rangle_{\mathcal{E}} \\ &= \eta \cdot (\mathbb{I}_\infty \xi) \\ &= \lambda_\infty (\eta \cdot \xi) \\ &= 0 \end{aligned}$$

if and only if $\eta \in so(3)^\perp$. It follows that $\dim V_{RIG} = \dim so(3)^\perp = 2$.

Step 3:
Although the original potential W is material frame indifferent, i.e. invariant under the left action of $SO(3)$, the effective potential V_ξ fails to be frame indifferent due to the addition of the term L_ξ, which is invariant only with respect to the isotropy subgroup $SO(3)_{\mu_e}$. We now determine the subspace $V_{INT} \subset V$ such that $DL_\xi|_{V_{INT}}$ is equivariant under the left $SO(3)$ action:

$$V_{INT} := \left\{ \delta F \in V : \left\langle \delta F, \mathcal{L}_{\eta c(F_e)} \frac{\delta L_\xi}{\delta F}(F_e) \right\rangle_E = 0 \qquad \forall \eta \in so(3)^\perp \right\}, \tag{33}$$

where the functional derivative $\frac{\delta L_\xi}{\delta F}$ is determined by

$$\left\langle \frac{\delta L_\xi}{\delta F}(F), \delta F \right\rangle_E = DL_\xi(F) \cdot \delta F \qquad \forall\, \delta F \in \mathrm{T}_F\mathcal{C}. \tag{34}$$

DL_ξ is automatically equivariant under the action of $so(3)_{\mu_e}$; hence to guarantee full left equivariance of DL_ξ, it is sufficient to require equivariance with respect to $so(3)^\perp$. It follows from the $SO(3)$ invariance of the inner product $\langle\ ,\ \rangle_E$ that for $\delta F \in V_{INT}$

$$\begin{aligned} \left(\frac{d}{d\epsilon}\Big|_{\epsilon=0} \exp(\epsilon\hat{\eta})^* DL_\xi \right)(F_e) \cdot \delta F &= \mathcal{L}_{\eta c(F_e)} DL_\xi(F_e) \cdot \delta F \\ &= \left\langle \delta F, \mathcal{L}_{\eta c(F_e)} \frac{\delta L_\xi}{\delta F}(F_e) \right\rangle_E \\ &= 0. \end{aligned}$$

Hence $DL_\xi|_{V_{INT}}$ is left equivariant.

An alternate formulation of $\left\langle \delta F, \mathcal{L}_{\eta c(F_e)} \frac{\delta L_\xi}{\delta F}(F_e) \right\rangle_E$, which appears to be computationally convenient, can be derived by means of the following manipulation:

$$\begin{aligned} \left\langle \delta F, \mathcal{L}_{\eta c(F_e)} \frac{\delta L_\xi}{\delta F} \right\rangle &= \left\langle \delta F, \mathcal{L}_{\eta c(F_e)} \frac{\delta V_\xi}{\delta F} \right\rangle \\ &= D^2 V_\xi(F_e)(\eta c(F_e), \delta F) \\ &= \left\langle \eta c(F_e), \mathcal{L}_{\delta F} \frac{\delta V_\xi}{\delta F} \right\rangle \\ &= \mathcal{L}_{\delta F}\left(\left\langle \eta c(F_e), \frac{\delta V_\xi}{\delta F} \right\rangle \right) \\ &= \mathcal{L}_{\delta F}\left(\mathcal{L}_{\eta c(F_e)} V_\xi \right) \\ &= -\mathcal{L}_{\delta F}\left(\left\langle \xi c(F_e), \mathcal{L}_{\eta c(F_e)} \xi c(F_e) \right\rangle_E \right) \\ &= \mathcal{L}_{\delta F}\left(\langle \xi c(F_e), [\xi, \eta]_{\mathcal{C}}\,(F_e) \rangle \right) \\ &= \mathcal{L}_{\delta F}\left(\langle \xi c(F_e), (\xi \times \eta) c(F_e) \rangle \right). \end{aligned}$$

The preceding calculation makes use of the following facts: the stored energy function W is invariant under the left action of $SO(3)$, and hence satisfies $\mathcal{L}_{\eta c(F_e)} \frac{\delta W}{\delta F} = \mathbf{0}$, and $\frac{\delta V_\xi}{\delta F} = \mathbf{0}$ at a relative equilibrium. Since $\eta \in so(3)^\perp$ implies $\xi \times \eta \in so(3)^\perp$ and $\mathcal{L}_{\delta F} F$ is simply the derivative of F in the direction of δF, we can reformulate (33) as

$$V_{INT} = \left\{ (\delta F, \delta P) \in V : 0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \langle \eta c(F_\epsilon), \xi c(F_\epsilon) \rangle_E \qquad \begin{array}{c} \forall\, \eta \in so(3)^\perp \ \text{ and} \\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} F_\epsilon = \delta F. \end{array} \right\} \tag{35}$$

In the case of pseudo-rigid bodies, $\frac{d}{d\epsilon}\Big|_{\epsilon=0}\langle \eta_{\mathcal{C}}(F_\epsilon), \xi_{\mathcal{C}}(F_\epsilon)\rangle_E$ can be rewritten as

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}\left\langle \hat{\eta}F_\epsilon, \hat{\xi}F_\epsilon\right\rangle_E &= \left\langle \hat{\eta}\delta F, \hat{\xi}F_e\right\rangle_E + \left\langle \hat{\eta}F_e, \hat{\xi}\delta F\right\rangle_E \\
&= -\left\langle \delta F, (\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\xi})F_e\right\rangle_E \\
&= -\left\langle \delta F, 2(\hat{\xi}\hat{\eta})^{\mathcal{S}} F_e\right\rangle_E.
\end{aligned}$$

Thus, we see that the definition of V_{INT} reduces to

$$V_{INT} = \left\{(\delta F, \delta P) \in V : \left\langle \delta F, (\hat{\xi}\hat{\eta})^{\mathcal{S}} F_e\right\rangle_E = 0 \qquad \forall \eta \in so(3)^\perp\right\}. \tag{36}$$

We note that $V_{RIG} \cap V_{INT} = \mathbf{0}$ provided that the eigenvalue λ_∞ has multiplicity one. The subspace $V_{INT} \subset V$ is determined by two equations, hence $\operatorname{codim}_V V_{INT} \leq 2 = \dim V_{RIG}$. It follows that $V_{RIG} \oplus V_{INT} = V$.

Step 4:
Define the space $\mathcal{S} \subset \mathrm{T}_{(F_e,P_e)}\mathrm{T}^*\mathcal{C}$ by

$$\mathcal{S} := \left\{(\delta F, \delta P) \in \ker \mathrm{T}_{(F_e,P_e)}\mathcal{J} : \delta F \in V\right\}. \tag{37}$$

It follows from (16) and (19) that the tangent map of the momentum map at equilibrium is

$$\left(\mathrm{T}_{(F_e,P_e)}\mathcal{J}\cdot(\delta F, \delta P)\right)\hat{} = \left(\delta P F_e^T + P_e \delta F^T\right)^{\mathcal{A}}. \tag{38}$$

Thus, $\mathcal{S}$ is explicitly characterized as the constrained subspace

$$\mathcal{S} = \left\{(\delta F, \delta P) \in \mathrm{T}_{(F_e,P_e)}\mathrm{T}^*\mathcal{C} : \left(\delta P F_e^T + P_e \delta F^T\right)^{\mathcal{A}} = \mathbf{0} \quad \text{and} \quad \langle \xi_{\mathcal{C}}(F_e), \delta F\rangle_E = 0\right\}. \tag{39}$$

Step 5:
Define the subspace $\mathcal{S}_{RIG} \subset \mathcal{S}$ associated to superposed rigid body motions within the subgroup $SO(3)/SO(3)_{\mu_e}$. Specifically,

$$\mathcal{S}_{RIG} := \left\{(\Delta F, \Delta P) \in \mathcal{S} : (\Delta F, \Delta P) = (\hat{\eta}F_e, \hat{\eta}P_e + \hat{\zeta}F_e E) \text{ for } \eta \text{ and } \zeta \in so(3)^\perp\right\}. \tag{40}$$

This may be seen as follows: Let $\eta(t)$ be a curve in $so(3)^\perp$ with $\eta(0) = \eta$ and tangent $\dot{\eta} = \zeta$ and recall that the time evolution of the relative equilibrium is given by $F(t) = \exp(t\hat{\xi})F_e$. Let $\mathbf{FL} : \mathrm{T}\mathcal{C} \to \mathrm{T}^*\mathcal{C}$ denote the Legendre transform, which in this case is given by $\mathbf{FL}(M) = ME$. We then define

$$\begin{aligned}
(\Delta F, \Delta P) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}\left(\exp(\epsilon\hat{\eta})F_e, \mathbf{FL}\left(\frac{d}{dt}\Big|_{t=0}(\exp(\epsilon\hat{\eta}(t))F(t))\right)\right) \\
&= \left(\eta_{\mathcal{C}}(F_e), \mathrm{T}_{(F_e,\xi_{\mathcal{C}}(F_e))}\mathbf{FL}\left(\mathrm{T}_{F_e}\eta_{\mathcal{C}}(\xi_{\mathcal{C}}(F_e)) + \zeta_{\mathcal{C}}(F_e)\right)\right) \\
&= (\hat{\eta}F_e, \hat{\eta}P_e + \hat{\zeta}F_e E).
\end{aligned}$$

We compute that

$$\begin{aligned}
\left(\mathrm{T}_{(F_e,P_e)}\mathcal{J}(\hat{\eta}F_e, \hat{\eta}P_e + \hat{\zeta}F_e E)\right)\hat{} &= \left(\left(\hat{\eta}P_e + \hat{\zeta}F_e E\right)F_e^T + P_e(\hat{\eta}F_e)^T\right)^{\mathcal{A}} \\
&= \left(\hat{\zeta}\mathcal{E}\right)^{\mathcal{A}} - \left(\hat{\xi}\mathcal{E}\hat{\eta} - \hat{\eta}\hat{\xi}\mathcal{E}\right)^{\mathcal{A}} \\
&= \widehat{\mathbb{I}_\infty\zeta} - [\hat{\mu}_e, \hat{\eta}] \\
&= (\mathbb{I}_\infty\zeta - \mu_e \times \eta)\hat{}.
\end{aligned}$$

Thus $(\hat{\eta}F_e, \hat{\eta}P_e + \hat{\zeta}F_eE) \in \ker \mathrm{T}_{(F_e,P_e)}\mathcal{J}$ if and only if $\mathbb{I}_\infty\zeta = \mu_e \times \eta$. It follows that the space $\mathcal{S}_{RIG}$ is given by

$$\mathcal{S}_{RIG} = \left\{(\hat{\eta}F_e, \hat{\eta}P_e + \widehat{\zeta(\eta)}F_eE) : \eta \in so(3)^\perp\right\}, \tag{41}$$

where $\zeta(\eta) := \mathbb{I}_\infty^{-1}(\mu_e \times \eta)$. Note that $\dim \mathcal{S}_{RIG} = \dim V_{RIG} = 2$.

Step 6:
Define the subspace $\mathcal{S}_{INT} \subset \mathcal{S}$ by

$$\mathcal{S}_{INT} := \{(\delta F, \delta P) \in \mathcal{S} : \delta F \in V_{INT}\}. \tag{42}$$

Recall that $(\Delta F, \Delta P) \in \mathcal{S}_{RIG}$ implies $\Delta F \in V_{RIG}$ and hence $V_{RIG} \cap V_{INT} = \mathbf{0}$ implies $\mathcal{S}_{RIG} \cap \mathcal{S}_{INT} = \mathbf{0}$. Since two equations specify $\mathcal{S}_{INT}$ as a subspace of $\mathcal{S}$, we have $\mathrm{codim}_{\mathcal{S}}\mathcal{S}_{INT} \leq 2 = \dim \mathcal{S}_{RIG}$ and hence $\mathcal{S}_{RIG} \oplus \mathcal{S}_{INT} = \mathcal{S}$ provided that λ_∞ has multiplicity one.

The significance of the preceding construction lies in the following crucial block diagonalization result: We show below that

$$D^2H_\xi(F_e, P_e)((\Delta F, \Delta P), (\delta F, \delta P)) = 0 \tag{43}$$

for all $(\Delta F, \Delta P) \in \mathcal{S}_{RIG}$ and $(\delta F, \delta P) \in \mathcal{S}_{INT}$.

To prove (43), we consider first the potential V_ξ. Recall that $(\Delta F, \Delta P) \in V_{RIG}$ implies that $\Delta F = \eta c(F_e)$ for some $\eta \in so(3)^\perp$. For $\delta F \in V_{INT}$, using (33) and the fact that $\frac{\delta W}{\delta F}$ is frame invariant, and hence $\mathcal{L}_{\eta c(F_e)}\frac{\delta W}{\delta F} = \mathbf{0}$, we have

$$\begin{aligned} D^2V_\xi(F_e)(\Delta F, \delta F) &= D^2V_\xi(F_e)(\eta c(F_e), \delta F) \\ &= \left\langle \delta F, \mathcal{L}_{\eta c(F_e)}\frac{\delta V_\xi}{\delta F}\right\rangle \\ &= \left\langle \delta F, \mathcal{L}_{\eta c(F_e)}\frac{\delta L_\xi}{\delta F}\right\rangle \\ &= 0. \end{aligned} \tag{44}$$

Thus, D^2V_ξ block diagonalizes on $V_{RIG} \times V_{INT}$. Next, we show that $D^2\Gamma_\xi$ block diagonalizes on $\mathcal{S}_{RIG} \times \mathcal{S}_{INT}$; i.e.,

$$D^2\Gamma_\xi(F_e, P_e)\left((\Delta F, \Delta P), (\delta F, \delta P)\right) = 0. \tag{45}$$

To prove (45), we first note that for $(\Delta F, \Delta P) = (\hat{\eta}F_e, \hat{\eta}P_e + \hat{\zeta}F_eE) \in \mathcal{S}_{RIG}$ we have

$$\begin{aligned} \Delta P - \hat{\xi}\Delta FE &= \hat{\eta}P_e + \hat{\zeta}F_eE - \hat{\xi}\hat{\eta}F_eE \\ &= \left(\hat{\zeta} - \widehat{[\xi, \eta]}\right)F_eE \\ &= \hat{\nu}F_eE, \end{aligned}$$

for $\nu \in so(3)$ given by $\nu = \zeta - [\xi, \eta]$. In fact, $\nu \in so(3)^\perp$, a result which follows from the direct computation:

$$\begin{aligned} \xi \cdot \nu &= \xi \cdot (\zeta - [\xi, \eta]) \\ &= \xi \cdot (\zeta - \xi \times \eta) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_\infty}(\xi \cdot (\mathbb{I}_\infty \zeta)) \\
&= \frac{1}{\lambda_\infty}(\xi \cdot (\mu_e \times \eta)) \\
&= \xi \cdot (\xi \times \eta) \\
&= 0.
\end{aligned}
$$

Hence $\nu \in so(3)^\perp$. Next, we make use of the preceding relation to obtain

$$
\begin{aligned}
D^2\Gamma_\xi(F_e, P_e)((\Delta F, \Delta P), (\delta F, \delta P)) &= \left\langle \Delta P - \hat{\xi}\Delta F E, \delta P - \hat{\xi}\delta F E \right\rangle_{E^{-1}} \\
&= \left\langle \hat{\nu} F_e, \delta P - \hat{\xi}\delta F E \right\rangle \\
&= \langle \hat{\nu} F_e, \delta P \rangle + \left\langle \hat{\xi}\hat{\nu} F_e E, \delta F \right\rangle \\
&= \left\langle \hat{\nu}, \delta P F_e^T + P_e \delta F \right\rangle + \left\langle 2(\hat{\xi}\hat{\nu})^{\mathcal{S}} F_e, \delta F \right\rangle_E \\
&= 0,
\end{aligned}
$$

since $(\delta F, \delta P) \in \mathcal{S}_{INT}$ satisfies (38) and (36).

It follows from (44) and (45) that

$$
\begin{aligned}
D^2 H_\xi(F_e, P_e)((\Delta F, \Delta P), (\delta F, \delta P)) &= D^2 V_\xi(F_e)(\Delta F, \delta F) \\
&\quad + D^2\Gamma_\xi(F_e, P_e)((\Delta F, \Delta P), (\delta F, \delta P)) \\
&= 0
\end{aligned}
$$

for $(\Delta F, \Delta P) \in \mathcal{S}_{RIG}$ and $(\delta F, \delta P) \in \mathcal{S}_{INT}$. We note that definiteness of $D^2 H_\xi|_{\mathcal{S}_{RIG}}$ implies that λ_∞ has multiplicity one and hence that the decomposition $\mathcal{S} = \mathcal{S}_{RIG} \oplus \mathcal{S}_{INT}$ holds. Accordingly, because of the block diagonal structure of the second variation, the task of testing for definiteness of the second variation simplifies considerably: Definiteness of $D^2 H_\xi|_{\mathcal{S}_{RIG}}$ and $D^2 H_\xi|_{\mathcal{S}_{INT}}$ is equivalent to definiteness of $D^2 H_\xi|_{\mathcal{S}}$.

4.d Symmetric orthotropic equilibria

We now consider orthotropic materials in configurations for which the reference configuration has an axis of symmetry which coincides with the symmetry of the material. In particular, we assume that there exists $\theta \in so(3)$ such that $E = e^{\epsilon\hat{\theta}} E e^{-\epsilon\hat{\theta}}$ and $W(e^{\epsilon\hat{\theta}} C e^{-\epsilon\hat{\theta}}) = W(C)$ for all symmetric matrices C. The Hamiltonian for such configurations is invariant under the right action of the group S^1 generated by $e^{\epsilon\hat{\theta}}$, which can be seen as follows: By assumption, the stored energy function is invariant under the right action, hence we need consider only the kinetic energy. Letting $K(P) = \frac{1}{2}\langle P, P\rangle_{E^{-1}} = \frac{1}{2}\mathrm{tr}(P^T P E^{-1})$ denote the kinetic energy associated to the momentum P, we compute that

$$
\begin{aligned}
K(Pe^{\epsilon\hat{\theta}}) &= \tfrac{1}{2}\left\langle Pe^{\epsilon\hat{\theta}}, Pe^{\epsilon\hat{\theta}} \right\rangle_{E^{-1}} \\
&= \tfrac{1}{2}\mathrm{tr}\left(P^T P (e^{\epsilon\hat{\theta}} E e^{-\epsilon\hat{\theta}})^{-1}\right) \\
&= \tfrac{1}{2}\mathrm{tr}\left(P^T P E^{-1}\right) \\
&= K(P).
\end{aligned}
$$

Hence the Hamiltonian is invariant under the right S^1 action. This action has an associated momentum map J_R^θ. This momentum map is simply the $\hat{\theta}$ component of the right momentum map:

$$\begin{aligned} J_R^\theta(F,P) &= \left\langle \mathbf{J}_R(F,P), \hat{\theta} \right\rangle \\ &= \left\langle F^T P, \hat{\theta} \right\rangle. \end{aligned}$$

To obtain the strongest possible stability conditions it is necessary to include this momentum map in the Hamiltonian used in the energy-momentum method. We point out that the left (respectively right) momentum map is invariant under the right (respectively left) action of $\mathcal{C}$, hence the modified Hamiltonian $H - \mathcal{J} \cdot \xi$ determined for the asymmetric bodies is invariant under the right S^1 action.

We now specialize to the case of symmetric rigidly rotating bodies for which the axis of rotation coincides with the axis of symmetry. In particular, we consider relative equilibria such that the current configuration has the same symmetry as the reference configuration. (For many classes of materials, all relative equilibria which are in rotation about the axis of symmetry of the reference configuration inherit that symmetry.) For a configuration F such that $e^{\epsilon\hat{\theta}} F e^{-\epsilon\hat{\theta}} = F$, we have

$$\begin{aligned} J_L^\theta(F,P) &= \left\langle P, \hat{\theta} F \right\rangle \\ &= \left.\tfrac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle P, e^{\epsilon\hat{\theta}} F \right\rangle \\ &= \left.\tfrac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle P, F e^{\epsilon\hat{\theta}} \right\rangle \\ &= \left\langle P, F\hat{\theta} \right\rangle \\ &= J_R^\theta(F,P). \end{aligned}$$

Thus, if θ is parallel to ξ, where ξ is the vector determining the rotation of the relative equilibrium, then $J_L^\theta(F_e,P_e) = J_R^\theta(F_e,P_e)$. Since $P_e = \hat{\theta} F_e E$ commutes with $\hat{\theta}$, an analogous computation shows that $\mathrm{T}_{(F_e,P_e)} J_L^\theta = \mathrm{T}_{(F_e,P_e)} J_R^\theta$.

Given $\lambda \in \mathbb{R}$, define the Hamiltonian

$$\begin{aligned} H_{\xi,\lambda}(F,P) &= K(P) + W(F^T F) - (1-\lambda) J_L^\xi(F,P) - \lambda J_R^\xi(F,P) \\ &= \Gamma_{\xi,\lambda}(F,P) + V_{\xi,\lambda}(F), \end{aligned}$$

where

$$\Gamma_{\xi,\lambda}(F,P) = \tfrac{1}{2} \left\| P - \left((1-\lambda)\hat{\xi} F + \lambda F \hat{\xi} \right) E \right\|_{E^{-1}}^2$$

and

$$V_{\xi,\lambda}(F) = W(F^T F) - \tfrac{1}{2} \left\| (1-\lambda)\hat{\xi} F + \lambda F \hat{\xi} \right\|_E^2 .$$

The symmetric configuration (F_e, P_e) is a rigidly rotating relative equilibrium with rotation rate ξ if and only if (F_e, P_e) is a critical point of $H_{\xi,\lambda}$. By varying the parameter λ, it is possible to obtain a family of stability conditions by requiring that second variation of $H_{\xi,\lambda}$ be definite.

The block diagonalization method in the symmetric case proceeds exactly as in the previous section. In the symmetric case, the definition of the subspace $\mathcal{S}_{INT} \subset \mathcal{S}$ takes the form

$$\mathcal{S}_{INT} = \left\{ (\delta F, \delta P) \in \mathcal{S} : 0 = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle \hat{\eta} F_\epsilon, (1-\lambda)\hat{\xi} F_\epsilon + \lambda F_\epsilon \hat{\xi} \right\rangle \qquad \begin{array}{r} \forall\, \eta \in so(3)^\perp \text{ and} \\ \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F_\epsilon = \delta F \end{array} \right\}. \tag{46}$$

The first defining condition in (46) can be rephrased to read

$$\begin{aligned} &\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left\langle \hat{\eta} F_\epsilon, (1-\lambda)\hat{\xi} F_\epsilon + \lambda F_\epsilon \hat{\xi} \right\rangle_E \\ &= \left\langle \hat{\eta}\delta F, (1-\lambda)\hat{\xi} F_e + \lambda F_\epsilon \hat{\xi} \right\rangle_E + \left\langle \hat{\eta} F_e, (1-\lambda)\hat{\xi}\delta F + \lambda \delta F \hat{\xi} \right\rangle_E \\ &= -\left\langle \delta F, \left(\hat{\xi}\hat{\eta} + \hat{\eta}\hat{\xi} - \lambda\hat{\xi}\hat{\eta} + \lambda\hat{\eta}\hat{\xi} \right) F_e \right\rangle_E \\ &= -2\left\langle \delta F, \left((\hat{\xi}\hat{\eta})^{\mathcal{S}} - \lambda(\hat{\xi}\hat{\eta})^{\mathcal{A}} \right) F_e \right\rangle_E . \end{aligned}$$

Thus definition (42) takes the form

$$\mathcal{S}_{INT} = \left\{ (\delta F, \delta P) \in \mathcal{S} : \left\langle \delta F, \left((\hat{\xi}\hat{\eta})^{\mathcal{S}} - \lambda(\hat{\xi}\hat{\eta})^{\mathcal{A}} \right) F_e \right\rangle_E = 0 \qquad \forall \eta \in so(3)^\perp \right\}. \tag{47}$$

5. Relative equilibria with diagonal deformation gradient

In this section, we specialize the general setup outlined above to the particular class of relative equilibria possessing a diagonal deformation gradient. Our objective here is to derive complete, explicit, and rigorous nonlinear stability results for several classes of nonlinear elastic materials, which include

(i) Isotropic materials with symmetric and asymmetric reference configurations (§5.b and §5.d)

(ii) General nonlinear anisotropic materials (§5.c).

Finally, we relate the relative equilibrium conditions derived in this section to the Baker-Ericksen inequalities.

5.a The equilibrium configuration

We consider a reference configuration with diagonal inertia dyadic

$$\begin{aligned} E &= \int_{\mathcal{B}} \rho_{\mathrm{ref}}(X) X \otimes X \, dX \\ &= \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \end{aligned}$$

diagonal deformation gradient

$$F_e = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \tag{48}$$

and momentum given by

$$P_e = \hat{\xi} F_e E = \begin{pmatrix} 0 & -|\xi|\,\beta e_2 & 0 \\ |\xi|\,\alpha e_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\hat{\xi} = \begin{pmatrix} 0 & -|\xi| & 0 \\ |\xi| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This motion corresponds to rigid rotation about the $\hat{\mathbf{z}}$ axis with angular velocity $|\xi|$. We denote by $\mathcal{E}$ the inertia dyadic of the current configuration, $\mathcal{E} = FEF^T$. $\mathcal{E}$ is given by

$$\mathcal{E} = \begin{pmatrix} \ell_1 & 0 & 0 \\ 0 & \ell_2 & 0 \\ 0 & 0 & \ell_3 \end{pmatrix}. \tag{49}$$

The momentum P_e is chosen such that the momentum component of the first variation (18) equals zero. The orthogonal projection $\mathbf{P} = \mathrm{Id} - (\xi \otimes \xi)/\,|\xi|^2$ in (20) is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

Finally, we note that the relative equilibrium condition (20) can be rewritten as

$$\tfrac{\delta W}{\delta F} F_e^T = |\xi|^2 \, \mathbf{P}\mathcal{E}. \tag{51}$$

5.b *Asymmetric isotropic equilibria*

5.b.i *Conditions of equilibrium*

We first consider a rigidly rotating asymmetric equilibrium configuration made up of an arbitrary isotropic material. We take the constitutive law for the material to be of the form $W(C) = \Phi(\mathrm{I}, \mathrm{II}, \mathrm{III})$, where $\mathrm{I} = \operatorname{tr} C$, $\mathrm{II} = \frac{1}{2}((\operatorname{tr} C)^2 - \operatorname{tr} C^2)$, and $\mathrm{III} = \det C$ are the principal invariants of $C = F^T F$. We denote the derivatives of Φ as follows:

$$\Phi_1 = \tfrac{\partial \Phi}{\partial \mathrm{I}}(\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e), \qquad \Phi_{12} = \tfrac{\partial^2 \Phi}{\partial \mathrm{I} \partial \mathrm{II}}(\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e), \qquad \text{etc.}, \tag{52}$$

where I_e, II_e, and III_e are the principal invariants of $C_e = F_e^T F_e$. If we define $\lambda_1 = \alpha^2$, $\lambda_2 = \beta^2$, and $\lambda_3 = \gamma^2$, and let the vectors V_i, $i = 1, 2, 3$ be given by

$$V_i = 2\lambda_i(1, \Sigma_{j \neq i} \lambda_j, \Pi_{j \neq i} \lambda_j), \tag{53}$$

then the diagonal components of the system of equations (51) may be expressed as

$$\begin{aligned} D\Phi \cdot V_1 &= |\xi|^2 \, \ell_1 \\ D\Phi \cdot V_2 &= |\xi|^2 \, \ell_2 \\ D\Phi \cdot V_3 &= 0. \end{aligned} \tag{54}$$

The off-diagonal components of (51) are automatically satisfied.

Further simplifications in the expression for the second variation can be obtained by introducing the following parametrization of the space of admissible variations. Define the map

$$\begin{aligned} \mathcal{T} : \mathrm{T}^*\mathcal{C} \times gl(3) \times gl(3) &\rightarrow \mathrm{T}\,(\mathrm{T}^*\mathcal{C}) \\ \mathcal{T}((F,P),\delta f,\delta p) &= \left(\delta f F, \delta p F E - \delta f^T P\right). \end{aligned} \tag{55}$$

We then define the variation $(\delta F, \delta P) \in \mathcal{S} \subset \mathrm{T}_{(F_e,P_e)}(\mathrm{T}^*\mathcal{C})$ by

$$(\delta F, \delta P) = \mathcal{T}((F_e,P_e),\delta f,\delta p) \tag{56}$$

for

$$\delta f = \begin{pmatrix} \delta_F^1 & \ell_1\sigma_F^3 & -(\ell_1\sigma_F^2+\alpha_F^2) \\ \ell_2\sigma_F^3 & \delta_F^2 & \ell_2\sigma_F^1+\alpha_F^1 \\ \ell_3\sigma_F^2+\alpha_F^2 & -(\ell_3\sigma_F^1+\alpha_F^1) & \delta_F^3 \end{pmatrix} \tag{57}$$

and

$$\delta p = \begin{pmatrix} \delta_P^1 & \sigma_P^3+\alpha_P^3 & \sigma_P^2-\alpha_P^2 \\ \sigma_P^3-\alpha_P^3 & \delta_P^2 & \sigma_P^1+\alpha_P^1 \\ \sigma_P^2+\alpha_P^2 & \sigma_P^1-\alpha_P^1 & \delta_P^3 \end{pmatrix}, \tag{58}$$

where the components α_P^i are determined by the condition that $(\delta F,\delta P) \in \ker \mathrm{T}_{(F_e,P_e)}\mathcal{J}$.

If we identify $(\delta F,\delta P) \in \mathcal{S}$ with the vector

$$\mathbf{v} = (\alpha_F^1,\alpha_F^2,\sigma_F^1,\sigma_F^2,\sigma_F^3,\delta_F^1,\delta_F^2,\delta_F^3,\sigma_P^1,\sigma_P^2,\sigma_P^3,\delta_P^1,\delta_P^2,\delta_P^3), \tag{59}$$

then the second variation of D^2H_ξ is given by $D^2H_\xi(F_e,P_e)(\delta F,\delta P)^2 = \mathbf{v}^T M \mathbf{v}$, where

$$M = \begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Sigma_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta_{12} & \Delta_{22} & \Delta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta_{13} & \Delta_{23} & \Delta_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_3
\end{pmatrix}. \tag{60}$$

The further diagonalization of the second variation, which is the result of the choice of variables given in (55), should be noted. The symbols appearing in (60) are defined as

$$\rho_1 = \frac{|\xi|^2(\ell_1+\ell_2)(\ell_2-\ell_3)}{\ell_1+\ell_3}$$

$$
\begin{aligned}
\rho_2 &= \frac{|\xi|^2(\ell_1+\ell_2)(\ell_1-\ell_3)}{\ell_2+\ell_3} \\
\Sigma_1 &= \frac{|\xi|^2\lambda_2\lambda_3\ell_2(\ell_2-\ell_3)(e_2-e_3)}{\lambda_2-\lambda_3} \\
\Sigma_2 &= \frac{|\xi|^2\lambda_1\lambda_3\ell_1(\ell_1-\ell_3)(e_1-e_3)}{\lambda_1-\lambda_3} \\
\Sigma_3 &= \frac{|\xi|^2\lambda_1\lambda_2(\ell_1+\ell_2)^2(e_1-e_2)}{\lambda_1-\lambda_2} \\
\mu_i &= \frac{4\Pi_{j\neq i}\ell_j}{\Sigma_{j\neq i}\ell_j} \\
\Delta_{ij} &= D^2\Phi(\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e)(V_i, V_j) - |\xi|^2 R_{ij},
\end{aligned}
$$

where I_e, II_e, and III_e are the principal invariants of the equilibrium configuration and

$$
\begin{aligned}
R_{11} &= -\frac{4{\ell_1}^2}{\ell_1+\ell_2} \\
R_{12} &= \frac{2\lambda_1\lambda_2(e_1-e_2)}{\lambda_1-\lambda_2} - \frac{4\ell_1\ell_2}{\ell_1+\ell_2} \\
R_{13} &= \frac{2\lambda_3\ell_1}{\lambda_1-\lambda_3} \\
R_{22} &= -\frac{4{\ell_2}^2}{\ell_1+\ell_2} \\
R_{23} &= \frac{2\lambda_3\ell_2}{\lambda_2-\lambda_3} \\
R_{33} &= 0.
\end{aligned}
$$

The diagonal entries associated to the momentum variations are clearly positive, thus the matrix cannot be negative definite. It follows that if the configuration (F_e, P_e) is to be formally stable, $D^2H_\xi(F_e, P_e)$ must be positive definite. The conditions that ρ_1 and ρ_2 be positive imply that the configuration must be in rotation about the shortest axis of the current configuration. Given this assumption, the conditions $\Sigma_i > 0$, $i = 1, 2, 3$, imply that the ordering by magnitude of the principal stretches must match the ordering by magnitude of the axes of the initial configuration, i.e. the body must satisfy $(e_i - e_j)/(\lambda_i - \lambda_j) > 0$ for $i < j$. The only terms remaining to be checked for positivity are those associated to the matrix $\boldsymbol{\Delta} = \{\Delta_{ij}\}$. The entries of this matrix depend explicitly on the second derivatives of the stored energy function. Hence $\boldsymbol{\Delta}$ must be analysed on a case by case basis for each particular material. We shall determine the definiteness of this matrix for two specific classes of materials in a later section.

We summarize the results of this section as follows:

A rigidly rotating relative equilibrium of an isotropic material is nonlinearly stable if

(i) $\ell_1 > \ell_3$ and $\ell_2 > \ell_3$,

(ii) $\dfrac{e_i - e_j}{\lambda_i - \lambda_j} > 0$ *for* $i < j$,

(iii) $\boldsymbol{\Delta}$ *is positive definite.*

Remark: The matrix $\mathbf{\Delta}$ is the Jacobian with respect to diagonal variations of the effective potential V_ξ defined by (23). With the notation of §5.b.i, this Jacobian reduces to that of the function $\tilde{\Phi} : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\tilde{\Phi}(\mathbf{x}) = \Phi(\mathbf{x}_0 + \mathbf{V}\mathbf{x}) - |\xi|^2 \left(\mathbf{x}^T\mathbf{R}_1 + \tfrac{1}{2}\mathbf{x}^T\mathbf{R}_2\mathbf{x}\right), \tag{61}$$

where $\mathbf{x}_0 = (\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e)$, $\mathbf{V}$ is the matrix such that $\mathbf{V}_{ij} = (V_j)_i$, $\mathbf{R}_1 = (\ell_1, \ell_2, 0)$, and $\mathbf{R}_2 = \{R_{ij}\}$. In particular, (F_e, P_e) is a nonlinearly stable relative equilibrium (i.e. a critical point of H_ξ with definite second variation) if

(i) $D\tilde{\Phi}(\mathbf{0}) = \mathbf{0}$,

(ii) $\ell_1 > \ell_3$ and $\ell_2 > \ell_3$,

(iii) $\dfrac{e_i - e_j}{\lambda_i - \lambda_j} > 0$ for $i < j$,

(iv) $D^2\tilde{\Phi}(\mathbf{0})$ is positive definite.

5.c *Anisotropic equilibria*

In this section we briefly present the conditions for equilibrium and stability of rigidly rotating relative equilibria for an anisotropic material. In this case the Hamiltonian is given by

$$H(F,P) = \tfrac{1}{2}\langle P, P\rangle_{E^{-1}} + \Psi(c_{11}, c_{22}, c_{33}, c_{23}, c_{13}, c_{12}),$$

where Ψ is the stored energy function and the c_{ij} are the entries of the right Cauchy-Green tensor $C = F^TF$. We define $\mathbf{c}(F) := (c_{11}, c_{22}, c_{33}, c_{23}, c_{13}, c_{12})$ and let Ψ_i denote the partial derivative of Ψ with respect to the i-th component of $\mathbf{c}$. As before, a rigidly rotating relative equilibrium corresponds to a critical point of the modified Hamiltonian

$$H_\xi(F,P) = \tfrac{1}{2}\langle P, P\rangle_{E^{-1}} + \Psi(\mathbf{c}(F)) - \left\langle PF^T, \hat{\xi}\right\rangle .$$

The configuration (F_e, P_e) is a critical point of H_ξ if

$$\Psi_i v_i = |\xi|^2 r_i \qquad i = 1, \ldots, 6, \tag{62}$$

where

$$\begin{aligned}
v_1 &= 2\alpha^2 \\
v_2 &= 2\beta^2 \\
v_3 &= 2\gamma^2 \\
v_4 &= \beta\gamma(\ell_2 - \ell_3) \\
v_5 &= \alpha\gamma(\ell_3 - \ell_1) \\
v_6 &= \alpha\beta(\ell_1 + \ell_2) \\
r_1 &= \ell_1 \\
r_2 &= \ell_2 \\
r_i &= 0 \qquad i = 3, \ldots, 6.
\end{aligned}$$

This implies

$$\begin{aligned}
\Psi_1 &= \tfrac{1}{2}|\xi|^2 e_1\\
\Psi_2 &= \tfrac{1}{2}|\xi|^2 e_2\\
\Psi_i &= 0 \qquad i = 3, \ldots, 6.
\end{aligned}$$

In the computation of the second variation, we make the same choice of basis of $\mathcal{S}$ as in the previous section. (The stored energy function for anisotropic materials is still left invariant (i.e., it is materially frame indifferent), so the same decomposition is relevant.) At equilibrium the second variation is given by

$$D^2 H_\xi(F_e, P_e)(\delta F, \delta P)^2 = \mathbf{v}^T \begin{pmatrix} \rho & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \chi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu \end{pmatrix} \mathbf{v}, \tag{63}$$

where

(i) ρ is a two by two diagonal matrix with diagonal entries

$$\begin{aligned}
\rho_1 &= \frac{|\xi|^2 (\ell_1 + \ell_2)(\ell_2 - \ell_3)}{\ell_1 + \ell_3}\\
\rho_2 &= \frac{|\xi|^2 (\ell_1 + \ell_2)(\ell_1 - \ell_3)}{\ell_2 + \ell_3}
\end{aligned}$$

(ii) χ is a six by six symmetric matrix with entries

$$\chi_{ij} = \Psi_{ij} v_i v_j - |\xi|^2 r_{ij} \tag{64}$$

for v_i as above and

$$\begin{aligned}
r_{11} &= -\frac{4{\ell_1}^2}{\ell_1 + \ell_2}\\
r_{12} &= -\frac{4\ell_1\ell_2}{\ell_1 + \ell_2}\\
r_{22} &= -\frac{4{\ell_2}^2}{\ell_1 + \ell_2}\\
r_{44} &= \ell_2\ell_3(\ell_2 - \ell_3)\\
r_{55} &= \ell_1\ell_3(\ell_1 - \ell_3)\\
r_{ij} &= 0 \qquad \text{for all remaining } i \text{ and } j
\end{aligned}$$

(iii) μ is a six by six diagonal matrix with diagonal entries

$$\begin{aligned}
\mu_1 &= \frac{4\ell_2\ell_3}{\ell_2 + \ell_3}\\
\mu_2 &= \frac{4\ell_1\ell_3}{\ell_1 + \ell_3}\\
\mu_3 &= \frac{4\ell_1\ell_2}{\ell_1 + \ell_2}\\
\mu_4 &= \ell_1\\
\mu_5 &= \ell_2\\
\mu_6 &= \ell_3.
\end{aligned}$$

Thus, we have obtained the following result:

A rigidly rotating relative equilibrium is nonlinearly stable if

(i) $\ell_1 > \ell_3$ and $\ell_2 > \ell_3$

(ii) χ *is positive definite.*

Remark: As in the isotropic case, we can express the non-diagonal component of the second variation as the Jacobian of effective potential, restricted in this case to V_{INT}. This Jacobian corresponds to that of the function $\widetilde{\Psi} : \mathbb{R}^6 \to \mathbb{R}$ given by

$$\widetilde{\Psi}(\mathbf{x}) = \Psi(\mathbf{c}(F_e) + \mathbf{v}\mathbf{x}) - |\xi|^2 \left(\mathbf{x}^T\mathbf{r}_1 + \tfrac{1}{2}\mathbf{x}^T\mathbf{r}_2\mathbf{x}\right), \tag{65}$$

where $\mathbf{v}$ is the six by six diagonal matrix with diagonal entries v_i, $\mathbf{r}_1 = (\ell_1, \ell_2, 0, 0, 0, 0)$, and $\mathbf{r}_2 = \{r_{ij}\}$. In fact, (F_e, P_e) is an equilibrium configuration if $\nabla\widetilde{\Psi}(\mathbf{0}) = \mathbf{0}$ and is stable if $\ell_1 > \ell_3$, $\ell_2 > \ell_3$, and $D^2\widetilde{\Psi}(\mathbf{0})$ is positive definite.

Remark: In the case where Ψ is a function of the principal invariants of C, the expressions given above reduce to those for the isotropic case.

5.d Symmetric isotropic equilibria

We consider the same configuration as was discussed in §5.b, where we now assume that both the reference and current configurations are symmetric about the axis of rotation, i.e. that $e_1 = e_2$ and $\alpha = \beta$. The conditions for equilibrium in this case are

$$\begin{aligned} D\Phi \cdot V_1 &= 2\,|\xi|^2\,\ell_1 \\ D\Phi \cdot V_2 &= 0 \end{aligned} \tag{66}$$

for

$$\begin{aligned} V_1 &= 4\alpha^2(1, \alpha^2 + \gamma^2, \alpha^2\gamma^2) \\ V_2 &= 2\gamma^2(1, 2\alpha^2, \alpha^4). \end{aligned}$$

As in the asymmetric case, the block diagonalization theorem, supplemented by additional diagonalization considerations, determines an appropriate space $\mathcal{S}$ and basis with which to compute the second variation of the modified Hamiltonian. (See the Appendix for the derivation of this basis.) The second variation of the modified Hamiltonian is given by

$D^2 H_{\xi,\lambda}(F_e, P_e)(\delta F, \delta P)^2 = \mathbf{v}^T M \mathbf{v}$, where

$$M = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Sigma_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{11} & \Delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{12} & \Delta_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\ell_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\ell_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\ell_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_3 \end{pmatrix}, \tag{67}$$

for

$$\begin{aligned} \rho &= 2\,|\xi|^2\,\ell_1\left(\frac{\ell_1-\ell_3}{\ell_1+\ell_3}+\lambda\right) \\ \Sigma_1 &= |\xi|^2\,\ell_1\ell_3(\ell_1-\ell_3+\lambda(\ell_1+\ell_3))\left(\frac{\alpha^2\gamma^2\,(e_1-e_3+\lambda(e_1+e_3))}{(\alpha^2-\gamma^2)\,\ell_3}+\lambda^2(1-\lambda)\right) \\ \Sigma_2 &= 4\left(|\xi|^2\,\ell_1\left(\frac{\gamma^2}{\alpha^2-\gamma^2}+2\lambda(1-\lambda)\right)-2\alpha^2\Phi_2(\alpha^2-\gamma^2)\right) \\ \Delta_{11} &= \pi_{11}-\tau \\ \Delta_{12} &= \pi_{12}-\sigma \\ \Delta_{22} &= \pi_{22} \\ \tau &= 4\left(\frac{|\xi|^2\,\ell_1\,(3\gamma^2-2\alpha^2)}{\alpha^2-\gamma^2}-2\alpha^2\Phi_2\left(\alpha^2-\gamma^2\right)\right) \\ \sigma &= \frac{4\,|\xi|^2\,\gamma^2\ell_1}{\alpha^2-\gamma^2} \\ \mu &= \frac{4\ell_1\ell_3}{\ell_1+\ell_3} \end{aligned}$$

where $\pi_{ij} = D^2\Phi(\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e)(V_i, V_j)$.

It follows that the second variation is positive definite if and only if ρ, Σ_1, and Σ_2 are positive and the matrix $\mathbf{\Delta}$ is positive definite. $\mathbf{\Delta}$ has eigenvalues $\operatorname{tr}\mathbf{\Delta} \pm \sqrt{(\operatorname{tr}\mathbf{\Delta})^2 - \det\mathbf{\Delta}}$; thus $\mathbf{\Delta}$ is positive definite if and only if

$$\begin{aligned} \operatorname{tr}\mathbf{\Delta} &= \pi_{11}+\pi_{22}-\tau \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} \det\mathbf{\Delta} &= (\pi_{11}-\tau)\,\pi_{22}-(\pi_{12}-\sigma)^2 \\ &> 0. \end{aligned}$$

If the second variation $D^2H_{\xi,\lambda}(F_e, P_e)$ is positive definite for *any* value of λ, then the equilibrium configuration (F_e, P_e) is nonlinearly stable.

Remark: Note that if $\pi_{22} = 0$, then $\det \boldsymbol{\Delta}$ is negative and $\boldsymbol{\Delta}$ is indefinite. It follows that for a stable relative equilibrium, $D^2\Phi(\mathrm{I}_e, \mathrm{II}_e, \mathrm{III}_e) \neq \mathbf{0}$. This demonstrates that for a constitutive law which is linear in the invariants of $C = F^TF$, symmetric equilibria cannot be shown to be stable by these methods.

We now consider the stability conditions given by the choice $\lambda = 1$. For this value of λ,

$$\begin{aligned}
\rho &= \frac{4\,|\xi|^2\,\ell_1{}^2}{\ell_1 + \ell_3} \\
&> 0 \\
\Sigma_1 &= \frac{4\,|\xi|^2\,\gamma^2\ell_1{}^3}{\alpha^2 - \gamma^2} \\
\Sigma_2 &= 4\left(\frac{|\xi|^2\,\gamma^2\ell_1}{\alpha^2 - \gamma^2} - 2\alpha^2\Phi_2\left(\alpha^2 - \gamma^2\right)\right).
\end{aligned}$$

Since ρ is always positive, it follows that the second variation is positive definite if $\alpha^2 > \gamma^2$, $\Sigma_2 > 0$, $\operatorname{tr}\boldsymbol{\Delta} > 0$, and $\det\boldsymbol{\Delta} > 0$. This result may be summarized as follows:

A symmetric configuration of an isotropic material is nonlinearly stable if

(i) $\alpha^2 > \gamma^2$

(ii) $\Sigma_2 > 0$

(iii) $\pi_{11} + \pi_{22} > \tau$

(iv) $\pi_{11}\pi_{22} - \pi_{12}{}^2 + \sigma\,(2\pi_{12} - \sigma) > \pi_{22}\tau.$

5.e *Relative equilibrium conditions and the Baker-Ericksen inequalities*

In this section we consider the Baker-Ericksen inequalities and a related, somewhat more restrictive, set of inequalities. These sets of inequalities are closely related to the stability criteria presented in the previous sections. In particular, *a stably rotating relative equilibrium of an isotropic material satisfies the Baker-Ericksen inequalities.*

The equilibrium equations (54) may be regrouped to obtain

$$\begin{aligned}
\Phi_1 + \beta^2\Phi_2 &= \frac{|\xi|^2\,\ell_1}{2(\alpha^2 - \gamma^2)} \\
\Phi_1 + \alpha^2\Phi_2 &= \frac{|\xi|^2\,\ell_2}{2(\beta^2 - \gamma^2)} \\
\Phi_1 + \gamma^2\Phi_2 &= \frac{|\xi|^2\,(\ell_1 - \ell_2)}{2(\alpha^2 - \beta^2)}.
\end{aligned} \tag{68}$$

If we define

$$\tilde{\Phi}(\alpha, \beta, \gamma) = \Phi(\alpha^2 + \beta^2 + \gamma^2, \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2, \alpha^2\beta^2\gamma^2), \tag{69}$$

i.e. if we consider Φ as a function of the entries of the matrix F_e, then we can compute that

$$\begin{aligned}
\alpha\tilde{\Phi}_\alpha - \gamma\tilde{\Phi}_\gamma &= 2(\alpha^2-\gamma^2)(\Phi_1+\beta^2\Phi_2) \\
&= |\xi|^2\,\ell_1 \\
\beta\tilde{\Phi}_\beta - \gamma\tilde{\Phi}_\gamma &= 2(\beta^2-\gamma^2)(\Phi_1+\alpha^2\Phi_2) \\
&= |\xi|^2\,\ell_2 \\
\alpha\tilde{\Phi}_\alpha - \beta\tilde{\Phi}_\beta &= 2(\alpha^2-\beta^2)(\Phi_1+\gamma^2\Phi_2) \\
&= |\xi|^2\,(\ell_1-\ell_2).
\end{aligned} \tag{70}$$

The lefthand sides of these equations are clearly related to the Baker-Ericksen inequalities:

$$\begin{aligned}
\frac{\alpha\tilde{\Phi}_\alpha - \gamma\tilde{\Phi}_\gamma}{\alpha-\gamma} &> 0 \\
\frac{\beta\tilde{\Phi}_\beta - \gamma\tilde{\Phi}_\gamma}{\beta-\gamma} &> 0 \\
\frac{\alpha\tilde{\Phi}_\alpha - \beta\tilde{\Phi}_\beta}{\alpha-\beta} &> 0.
\end{aligned} \tag{71}$$

(See Marsden and Hughes (1983).) Hence, if the material determined by the stored energy function Φ is 'physically plausible', i.e. if the configuration satisfies the Baker-Ericksen inequalities, then $\alpha > \gamma$, $\beta > \gamma$, and $(\ell_1-\ell_2)/(\alpha-\beta) > 0$.

An alternate regrouping of the equations for equilibrium yields the equations

$$\begin{aligned}
\Phi_2+\beta^2\Phi_3 &= -\frac{|\xi|^2\,e_1}{2(\alpha^2-\gamma^2)} \\
\Phi_2+\alpha^2\Phi_3 &= -\frac{|\xi|^2\,e_2}{2(\beta^2-\gamma^2)} \\
\Phi_2+\gamma^2\Phi_3 &= -\frac{|\xi|^2\,(e_1-e_2)}{2(\alpha^2-\beta^2)}.
\end{aligned} \tag{72}$$

(73)

We note that

$$\begin{aligned}
\gamma\tilde{\Phi}_\alpha - \alpha\tilde{\Phi}_\gamma &= -2\alpha\gamma(\alpha^2-\gamma^2)(\Phi_2+\beta^2\Phi_3) \\
&= |\xi|^2\,\alpha\gamma e_1 \\
\gamma\tilde{\Phi}_\beta - \beta\tilde{\Phi}_\gamma &= -2\beta\gamma(\beta^2-\gamma^2)(\Phi_2+\alpha^2\Phi_3) \\
&= |\xi|^2\,\beta\gamma e_2 \\
\beta\tilde{\Phi}_\alpha - \alpha\tilde{\Phi}_\beta &= -2\alpha\beta(\alpha^2-\beta^2)(\Phi_2+\gamma^2\Phi_3) \\
&= |\xi|^2\,\alpha\beta(e_1-e_2);
\end{aligned} \tag{74}$$

the alternate inequalities

$$\begin{aligned}
\frac{\gamma\tilde{\Phi}_\alpha - \alpha\tilde{\Phi}_\gamma}{\alpha-\gamma} &> 0 \\
\frac{\gamma\tilde{\Phi}_\beta - \beta\tilde{\Phi}_\gamma}{\beta-\gamma} &> 0 \\
\frac{\beta\tilde{\Phi}_\alpha - \alpha\tilde{\Phi}_\beta}{\alpha-\beta} &> 0
\end{aligned} \tag{75}$$

imply $\alpha > \gamma$, $\beta > \gamma$, and $(e_1 - e_2)/(\alpha - \beta) > 0$. The inequalities (75) imply the Baker-Ericksen inequalities. Both the Baker-Ericksen inequalities and the alternate inequalities imply that the principal stretch along the axis of rotation is shorter than the stretches within the plane of rotation. In addition, the Baker-Ericksen inequalities imply that the principal stretch along the longer axis of the *current* configuration within the plane of rotation is greater than that along the shorter axis, while the alternate inequalities imply that the principal stretch along the longer axis of the *reference* configuration is greater.

The Baker-Ericksen inequalities (71) and the inequalities (75) are closely related to the stability conditions derived in §5.b. We note that the terms Σ_i can be rewritten using expressions involving the first derivatives of the stored energy function Φ by means of the equations (70) and (74). Using these expressions, we can see that for a stable equilibrium the conditions $\rho_i > 0$ and $\Sigma_i > 0$ imply that the alternative inequalities (75) and, hence, the Baker-Ericksen inequalities (71) hold at a stable relative equilibrium. On the other hand, if we assume that the alternative inequalities hold, then the conditions $\rho_i > 0$ and $\Sigma_i > 0$ can be replaced by the conditions $e_1 > e_3$ and $e_2 > e_3$. (The derivations of the preceding statements are given in the Appendix.) In summary,

A rigidly rotating relative equilibrium is nonlinearly stable if

(i) $e_1 > e_3$ *and* $e_2 > e_3$,

(ii) the inequalities (75) hold

(iii) $\boldsymbol{\Delta}$ *is positive definite.*

6. Illustrative Examples

6.a Introduction

In this section we consider the nonlinear stability of rigidly rotating relative equilibria for a few representative compressible isotropic materials: the first is the Ciarlet-Geymonat material; this material, described in Ciarlet (1988), is polyconvex and satisfies appropriate growth conditions. A closely related model is given in Simo and Pister (1984). We shall show that for this material an asymmetric body in rigid rotation about the shortest principal axis of the reference configuration and a symmetric body in rigid rotation about an arbitrary principal axis are nonlinearly stable relative equilibria. The Ciarlet-Geymonat material is a limiting case of a class of compressible Mooney-Rivlin materials obtained as a (penalty) regularization of the classical incompressible Mooney-Rivlin model. (See Simo and Taylor (1988) and Ogden (1984).) While the relative equilibria of the Ciarlet-Geymonat material are always stable, we show below that for this family of regularized Mooney-Rivlin materials in general both stable and unstable branches of equilibria can exist. We present both numerical results for a representative member of this family and analytic results concerning the asymptotic behavior of the equilibria. Finally, we show that for St. Venant-Kirchhoff materials, both asymmetric and symmetric rigidly rotating relative equilibria are nonlinearly stable, provided, as above, that the asymmetric body rotates about the shortest principal axis of the reference configuration. The St. Venant-Kirchhoff model severely violates well-established restrictions on the stored energy function of an elastic material, including polyconvexity. (See Ciarlet (1988).) However, we have included this example due to its prevalence in the literature.

We emphasize that within our framework the demonstration of nonlinear stability for specific materials involves nothing more than straightforward algebraic manipulation of the rational expressions obtained in the previous sections. The high degree of structure present in this system has made it possible for us to perform the geometrically and computationally intensive steps of the analysis in a very general setting. No linearizations or other approximations are required. (Compare to the linear stability analysis presented in Cohen and Muncaster (1988).) The manipulations required for the analysis of specific materials are readily carried out using a symbolic computation language; the computations required for the examples presented here could easily be performed by hand. (The numerical treatment of the Mooney-Rivlin materials is necessitated by the complexity of the equilibrium equations, rather than the stability conditions.)

In the case of an asymmetric equilibrium, the steps of the stability analysis are as follows: The equilibrium conditions are used to show that the inequalities (75) hold and to compute the entries of the matrix $\mathbf{\Delta}$. The subdeterminants of $\mathbf{\Delta}$ are factored into a convenient form and the equilibrium conditions are used to show that the subdeterminants are positive and hence $\mathbf{\Delta}$ is positive definite. It follows that the equilibrium is nonlinearly stable provided that it is in rotation about the shortest axis of the reference configuration. In the case of a symmetric equilibrium, we compute the trace and determinant of $\mathbf{\Delta}$, rather than the subdeterminants.

6.b Relative equilibria for a Ciarlet-Geymonat material

6.b.i Asymmetric equilibria

We now consider a Ciarlet-Geymonat material, with stored energy function

$$W(C) = \tfrac{\lambda}{2}(\det C - 1 - \log\det C) + \tfrac{\mu}{2}\left((\operatorname{tr} C - 3) - \log\det C\right),$$

where μ and λ are arbitrary positive constants. This function is polyconvex, since

(i) the λ term is linear in $\det C$ and hence convex in $\sqrt{\det C}$ for any $\lambda > 0$,

(ii) the trace term is linear with positive constant in the eigenvalues of C and hence convex and increasing, and

(iii) the log term has second derivative $\mu/(2(\det C)^2) > 0$ and hence is convex.

Furthermore, this material satisifes proper growth conditions; in particular, $W \to \infty$ as $\det F \to 0$ or $\det F \to \infty$. Finally, the quadratic approximation to $W(C)$ at the reference state yields the classical isotropic stored energy function of an isotropic linear elastic material. (The same property is shared by the more general family of so-called Ogden materials (Ogden 1984).) We remark that polyconvexity (which is closely related to Morrey's notion of quasi-convexity in the calculus of variations) leads to the only known existence theory for nonlinear elasticity, due to Ball (1977). For elastostatics, the solutions are characterized as minimizers of the energy in a suitable $W^{1,p}$-space, where p is dictated by growth conditions on the stored energy function. (See Ciarlet (1988) for a comprehensive account of the current status of the subject.) For elastodynamics the situation is far more complicated, and little is known beyond the results of Hughes, Kato and Marsden (1977).

We now consider a rigidly rotating asymmetric equilibrium made up of a Ciarlet-Geymonat material in the configuration described in §5.b. If we subtract the left momentum map J_L^ξ from the Hamiltonian, then (F_e, P_e) is a critical point of the resulting modified Hamiltonian if the equations

$$\begin{aligned} (1-\gamma^2)\mu &= \left(\alpha^2\beta^2\gamma^2 - 1\right)\lambda \\ (\alpha^2-\gamma^2)\mu &= |\xi|^2\,\ell_1 \\ (\beta^2-\gamma^2)\mu &= |\xi|^2\,\ell_2 \end{aligned} \tag{76}$$

are satisfied. It follows that at equilibrium $1 > \gamma^2$, $\alpha^2 > \gamma^2$, and $\beta^2 > \gamma^2$. By regrouping the last two equations, we obtain

$$\frac{e_1 - e_2}{\alpha^2 - \beta^2} = \mu\left(\frac{\gamma}{\alpha\beta\,|\xi|}\right)^2 > 0.$$

Hence the alternative inequalities (75) are satisfied by the equilibria of this class of materials.

The equations (76) can be solved in closed form for α, β and γ but, since the equations are cubic in these variables, the expressions for the solutions are very complicated and we do not present them here. Rather than directly substituting the equilibrium values into the second variation D^2H_ξ, we use the relations (76) to simplify the second variation to the extent that we can establish definiteness.

We now consider the stability of the relative equilibria. The function Φ is given by

$$\Phi(\mathrm{I},\mathrm{II},\mathrm{III}) = \tfrac{\lambda}{2}\left((\mathrm{III}-1) - \log\mathrm{III}\right) + \tfrac{\mu}{2}\left((\mathrm{I}-3) - \log\mathrm{III}\right), \tag{77}$$

with partial derivatives

$$\begin{aligned} \Phi_1 &= \frac{\mu}{2} \\ \Phi_2 &= 0 \\ \Phi_3 &= \frac{\lambda(\mathrm{III}_e - 1) - \mu}{2\mathrm{III}_e} \\ \Phi_{33} &= \frac{\lambda+\mu}{2\mathrm{III}_e^2} \\ \Phi_{ij} &= 0 \quad ij \neq 33. \end{aligned}$$

Thus

$$\begin{aligned} D^2\Phi(\mathrm{I}_e,\mathrm{II}_e,\mathrm{III}_e)(V_i,V_j) &= \frac{\lambda+\mu}{2{\mathrm{III}_e}^2}(2\alpha^2\beta^2\gamma^2)^2 \\ &= 2(\lambda+\mu) \end{aligned}$$

for all i and j. Substituting the equilibrium conditions (76) for α^2 and β^2 into the expressions for the R_{ij} yields

$$\begin{aligned} |\xi|^2 R_{12} &= \mu\gamma^2 - \frac{4\,|\xi|^2\,\ell_1\ell_2}{\ell_1+\ell_2} \\ |\xi|^2 R_{13} &= \mu\gamma^2 \\ |\xi|^2 R_{23} &= \mu\gamma^2. \end{aligned}$$

It follows that the matrix $\mathbf{\Delta}$ has subdeterminants

$$\begin{aligned}
\det{}_1 &= 2(\lambda+\mu) + \frac{4\,|\xi|^2\,{\ell_1}^2}{\ell_1+\ell_2} \\
&> 0 \\
\det{}_2 &= 8\left(\frac{|\xi|^2}{\ell_1+\ell_2}\left((\lambda+\mu)(\ell_1-\ell_2)^2 + 2\mu\gamma^2\ell_1\ell_2\right) + \mu\gamma^2\left(\lambda+\mu\left(1-\tfrac{1}{2}\gamma^2\right)\right)\right) \\
&> 0 \\
\det{}_3 &= 8\mu\gamma^2\Bigg(\frac{4\,|\xi|^2}{\ell_1+\ell_2}\left(\left(\lambda+\mu\left(1-\tfrac{1}{2}\gamma^2\right)\right)(\ell_1-\ell_2)^2 + (\lambda+\mu)\ell_1\ell_2\right) \\
&\qquad +3\mu\gamma^2\left(\lambda+\mu\left(1-\tfrac{2}{3}\gamma^2\right)\right)\Bigg) \\
&> 0,
\end{aligned}$$

since $1 > \gamma^2$. Hence $\mathbf{\Delta}$ is positive definite. Thus, we have shown the following:

> *A rigidly rotating pseudo-rigid body made up of a Ciarlet-Geymonat material is nonlinearly stable if it is in rotation about the shortest axis of the reference configuration.*

6.b.ii Symmetric equilibria

We now consider the case in which the reference configuration of the body is symmetric about the axis of rotation, i.e. $e_1 = e_2$. It follows from the equilibrium equations (76) that $\alpha = \beta$ and hence $\ell_1 = \ell_2$. In the computation of the stability conditions, we will make use of the following equilibrium conditions:

(i) $\dfrac{|\xi|^2\,\ell_1}{\alpha^2-\gamma^2} = \mu$

(ii) $1 > \gamma^2$

(iii) $\pi_{11} = 8(\lambda+\mu)$, $\pi_{12} = 4(\lambda+\mu)$, and $\pi_{22} = 2(\lambda+\mu)$.

To establish stability it is sufficient to show that conditions (i)–(iv) of §5.d hold. Since $\mu > 0$, it is clear that $\alpha > \gamma$. We compute that

$$\begin{aligned}
\Sigma_2 &= \frac{4\,|\xi|^2\,\gamma^2\ell_1}{\alpha^2-\gamma^2} \\
&= 4\gamma^2\mu \\
&> 0 \\
\tau &= \frac{4\,|\xi|^2\,\ell_1(3\gamma^2-2\alpha^2)}{\alpha^2-\gamma^2} \\
&= 4(3\gamma^2-2\alpha^2)\mu \\
\sigma &= \frac{4\,|\xi|^2\,\gamma^2\ell_1}{\alpha^2-\gamma^2} \\
&= 4\gamma^2\mu.
\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{tr}\boldsymbol{\Delta} &= \pi_{11}+\pi_{33}-\tau\\ &= 2\mu\left(4\left(\alpha^2-\gamma^2\right)+2\left(1-\gamma^2\right)+3\right)+10\lambda\\ &> 0,\end{aligned}$$

and

$$\begin{aligned}\det\,\boldsymbol{\Delta} &= \pi_{11}\pi_{22}-{\pi_{12}}^2+\sigma(2\pi_{12}-\sigma)-\pi_{22}\tau\\ &= 2\mu\left(\mu\left(2\left(\alpha^2-\gamma^4\right)+\gamma^2\right)+\lambda\left(2\alpha^2+\gamma^2\right)\right)\\ &> 0,\end{aligned}$$

since $1>\gamma^2$ and $\alpha^2>\gamma^2$. Hence the second variation is positive definite and thus:

The symmetric relative equilibria of Ciarlet-Geymonat materials in rigid rotation about their axis of symmetry are nonlinearly stable.

6.c *Relative equilibria for a class of Mooney-Rivlin materials*

To provide an example of a material for which relative equilibria need not be stable, we consider here a special class of regularized compressible Mooney-Rivlin materials. This class of materials is a generalization of that of the previous section; these materials are polyconvex, satisfy appropriate growth conditions, and may also be considered as a particular case of the Ogden materials; see Simo and Taylor (1989). The asymmetric relative equilibria for typical examples of this class of material can be shown to lose stability when rotated at a sufficiently rapid rate.

The stored energy function we consider is of the following form:

$$\Phi(\mathrm{I},\mathrm{II},\mathrm{III})=\tfrac{\mu}{2}\left((\tfrac{1}{2}+\beta)(\mathrm{I}-3-\log\mathrm{III})+(\tfrac{1}{2}-\beta)(\mathrm{II}-3-2\log\mathrm{III})\right)+\tfrac{\lambda}{2}\,(\mathrm{III}-1-\log\mathrm{III}) \quad (78)$$

for $\mu>0$, $\lambda>0$, and $-\frac{1}{2}\le\beta\le\frac{1}{2}$. In the case $\beta=\frac{1}{2}$, this material becomes the Ciarlet-Geymonat material of §6.b. Φ has partial derivatives

$$\begin{aligned}\Phi_1 &= \tfrac{\mu}{2}\left(\tfrac{1}{2}+\beta\right)\\ \Phi_2 &= \tfrac{\mu}{2}\left(\tfrac{1}{2}-\beta\right)\\ \Phi_3 &= \left(\tfrac{\mu}{2}\left(\beta-\tfrac{3}{2}\right)+\tfrac{\lambda}{2}(\mathrm{III}_e-1)\right)\frac{1}{\mathrm{III}_e}\\ \Phi_{33} &= \left(\tfrac{\mu}{2}\left(\tfrac{3}{2}-\beta\right)+\tfrac{\lambda}{2}\right)\frac{1}{\mathrm{III}_e^2}\\ \Phi_{ij} &= 0 \quad ij\neq 33.\end{aligned}$$

We now introduce the scaled parameters $k=\frac{\lambda}{\mu}$ and $\zeta=\frac{2|\xi|^2}{\mu}$. With respect to these parameters the equations of motion (54) take the form

$$\begin{aligned}U\cdot V_1 &= \zeta\ell_1\\ U\cdot V_2 &= \zeta\ell_2\\ U\cdot V_3 &= 0,\end{aligned}$$

where $U = \left(\frac{1}{2}+\beta, \frac{1}{2}-\beta, k-\left(\frac{3}{2}+k-\beta\right)\frac{1}{\mathrm{III}}\right)$.

Next, we specialize to the case $\beta = 0$, $k = 100$, $e_1 = \frac{11}{10}$, $e_2 = \frac{9}{10}$, and $\frac{9}{10} > e_3 > 0$ and numerically investigate the behavior of this set of equations. We find that in the range $0 \le \zeta \le 20$ there are two branches of solutions; a stable branch which exists only for ζ in the range $0 \le \zeta < \zeta_m = 1.2834$ and an unstable branch which exists only for $\zeta > \zeta_u = 1.1111$. In the range $.9091 = \zeta_s < \zeta < \zeta_m$ there are two possible stable equilibrium configurations; these configurations approach one another as ζ approaches the value ζ_s. The principal stretches for these branches of relative equilibria are plotted in figures 1 through 3. In figure 4 we plot the Euclidean distance from the principal stretches of the reference configuration, i.e. $\lambda_i = 1$, to those of the equilibrium configuration with respect to the angular momentum μ. When plotted with respect to momentum, the stable branch no longer appears to fold back upon itself, while the unstable branch takes on the shape of an isola. The distance from the reference stretches becomes approximately linear as a function of μ as $\mu \to \infty$.

The stability of the branches is determined as follows: We have chosen the reference configuration such that the conditions $e_1 > e_3$ and $e_2 > e_3$ are satisfied. Since $e_1 > e_2$, the conditions $\chi_i > 0$ will be satisfied if $\lambda_1 > \lambda_2 > \lambda_3$. For the stable branch, the principal stretches satisfy $\lambda_1 \ge \lambda_2 \ge \lambda_3$, with equality only at the reference configuration. Thus, to establish stability of the first branch, it is sufficient to show that the matrix $\boldsymbol{\Delta}$ is positive definite; this condition is checked numerically and holds throughout the branch. For the unstable branch, the principal stretches satisfy $\lambda_2 > \lambda_1 > \lambda_3$; hence $\chi_3 < 0$ and the stability conditions fail.

Some general analytic results can be obtained regarding the behavior of the equilibria as the parameters ζ and β are varied. We find that the 'blow up' of the principal stretches seen at $\zeta_s = .9091$ and $\zeta_u = 1.1111$ is common to all values of β in the range $-\frac{1}{2} < \beta < \frac{1}{2}$. In general, the asymptotes occur at

$$\zeta_s = \frac{1+2\beta}{e_1} \qquad \text{and} \qquad \zeta_u = \frac{1+2\beta}{e_2} \tag{79}$$

if $e_1 > e_2$. Since $\zeta_u - \zeta_s = (1+2\beta)(e_1 - e_2)/(e_1 e_2)$, it follows that the distance $\zeta_s - \zeta_u$ between the stable and unstable asymptotes shrinks to zero as the parameter β approaches the limiting value $-\frac{1}{2}$. As ζ approaches ζ_s (respectively ζ_u) and λ_1 (respectively λ_2) approaches ∞, the remaining principal stretches approach zero as follows:

$$\begin{aligned}
\lim_{\zeta \to \zeta_s} \lambda_1 \lambda_2 &= \lim_{\zeta \to \zeta_u} \lambda_1 \lambda_2 \\
&= \lim_{\zeta \to \zeta_s} \lambda_1 \lambda_3 \\
&= \lim_{\zeta \to \zeta_u} \lambda_2 \lambda_3 \\
&= \frac{3+2k-2\beta}{1-2\beta}.
\end{aligned}$$

This implies

$$\begin{aligned}
\lim_{\zeta \to \zeta_s} \mathrm{III} &= \lim_{\zeta \to \zeta_s} \lambda_1 \lambda_2 \lambda_3 \\
&= \lim_{\zeta \to \zeta_s} \frac{(3+2k-2\beta)^2}{(1-2\beta)^2 \lambda_1} \\
&= 0
\end{aligned}$$

and, similarly, $\lim_{\zeta \to \zeta_u} \mathrm{III} = 0$.

We can also consider the asymptotic behavior of the relative equilibria as $\zeta \to \infty$. We compute that

$$\begin{aligned}
\lim_{\zeta \to \infty} \frac{\lambda_1}{\zeta} &= \frac{e_2}{1-2\beta} \\
\lim_{\zeta \to \infty} \frac{\lambda_2}{\zeta} &= \frac{e_1}{1-2\beta} \\
\lim_{\zeta \to \infty} \lambda_3 \zeta^2 &= \frac{(1-2\beta)^2(3+2k-2\beta)}{2ke_1e_2},
\end{aligned}$$

which implies that

$$\lim_{\zeta \to \infty} \mathrm{III} = \frac{3+2k-2\beta}{2k}.$$

In particular, the body is symmetric about the axis of rotation in the limit $\zeta \to \infty$.

Remark: We have numerically investigated the behavior of this system for a few other sample values of β in the range $-\frac{1}{2} < \beta < \frac{1}{2}$. The behavior is qualitatively similar to that for $\beta = 0$.

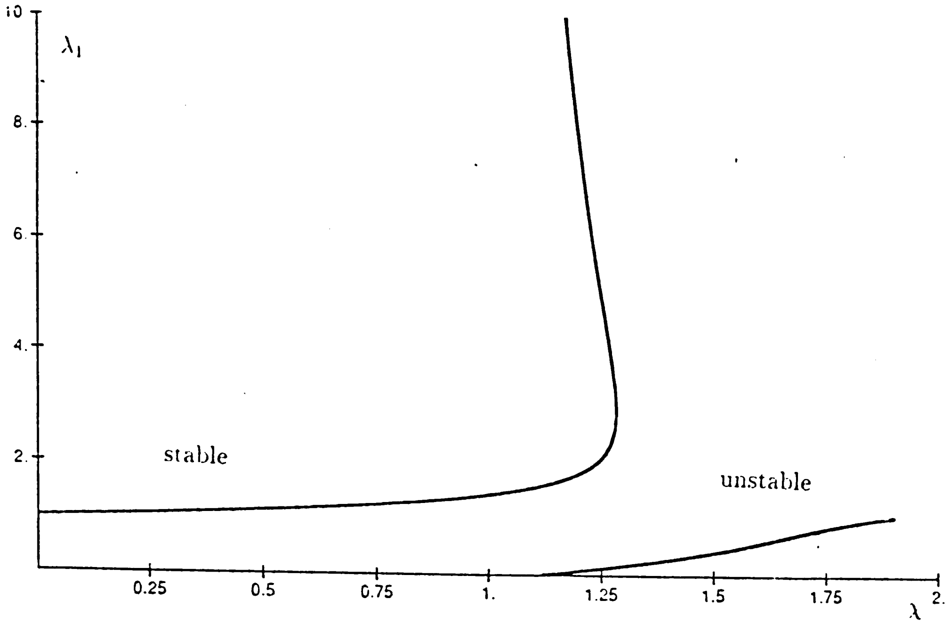

Figure 1: λ_1 plotted with respect to χ

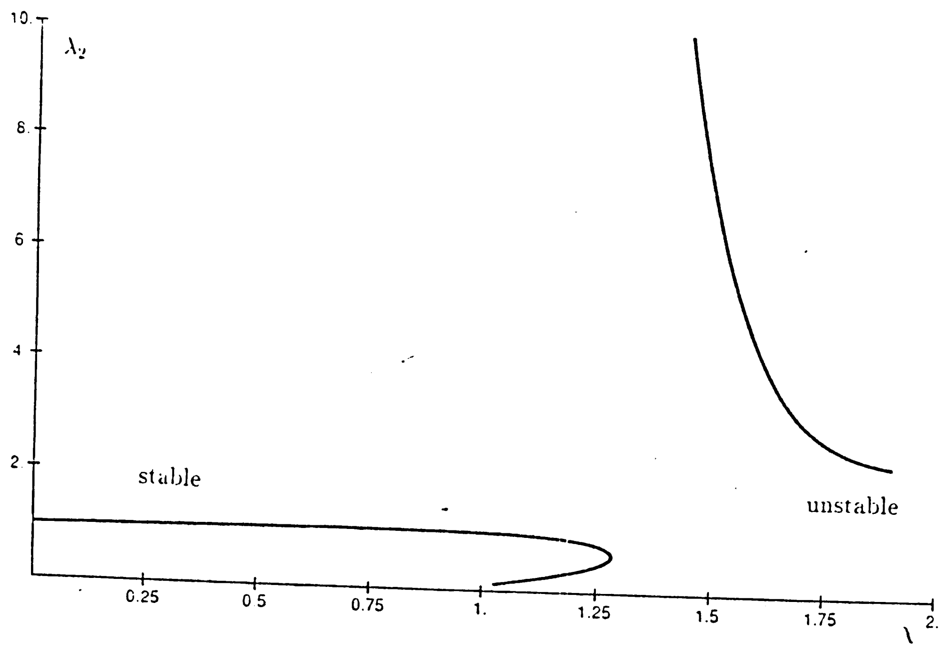

Figure 2: λ_2 plotted with respect to χ

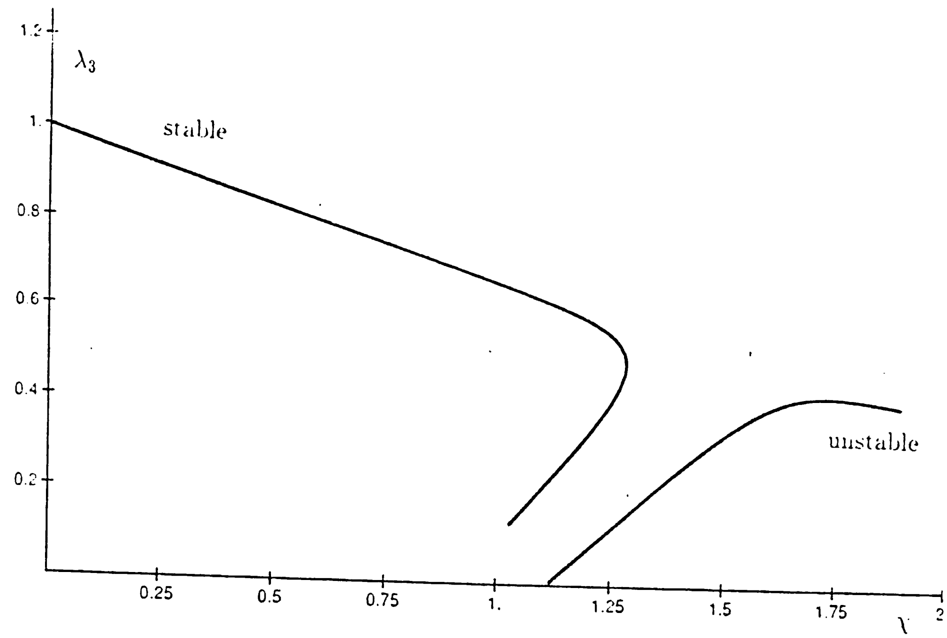

Figure 3: λ_3 plotted with respect to χ

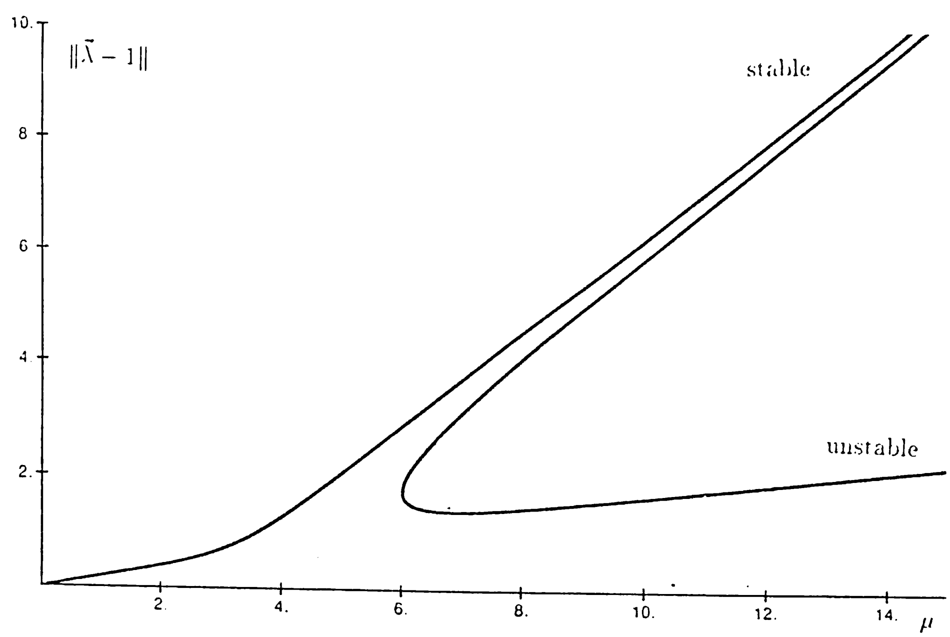

Figure 4: $||\vec{\lambda} - 1||$ plotted with respect to μ

6.d *Relative equilibria for a St. Venant-Kirchhoff material*

6.d.i *Asymmetric equilibria*

We consider here a St. Venant-Kirchhoff material, with stored energy function

$$W(F) = \tfrac{\lambda}{2}\left(\operatorname{tr}(C - \mathrm{Id})\right)^2 + \mu \operatorname{tr}\left((C - \mathrm{Id})^2\right), \tag{80}$$

where $C = F^T F$ and the Lamé constants λ and μ satisfy the conditions $\mu > 0$ and $3\lambda + 2\mu > 0$.

The configuration (F_e, P_e) is a relative equilibrium if and only if

$$\begin{aligned}
\alpha^2 &= 1 + \frac{|\xi|^2\left(2e_1(\mu+\lambda) - \lambda e_2\right)}{4\mu(3\lambda+2\mu)} \\
\beta^2 &= 1 + \frac{|\xi|^2\left(2e_2(\mu+\lambda) - \lambda e_1\right)}{4\mu(3\lambda+2\mu)} \\
\gamma^2 &= 1 - \frac{\lambda|\xi|^2(e_1+e_2)}{4\mu(3\lambda+2\mu)}.
\end{aligned}$$

We now check the stability of these equilibria by applying the results of §4.c. We first note that by rearranging the difference of the first two equations of equilibrium we obtain

$$\begin{aligned}
\frac{\alpha^2 - \beta^2}{e_1 - e_2} &= \frac{|\xi|^2}{4\mu} \\
\alpha^2 - \gamma^2 &= \frac{|\xi|^2 e_1}{4\mu} \\
\beta^2 - \gamma^2 &= \frac{|\xi|^2 e_2}{4\mu},
\end{aligned} \tag{81}$$

so the alternative inequalities (75) are satisfied. If we assume that the figure is in rotation about the shortest axis of the reference configuration, then the stability of the equilibrium configuration is determined by the definiteness of the matrix Δ. We compute that the function Φ, which expresses the stored energy function W as a function of the invariants of the Cauchy-Green tensor $C = F^T F$, is given by

$$\Phi(\mathrm{I}, \mathrm{II}, \mathrm{III}) = \left(\tfrac{\lambda}{2} + \mu\right)\mathrm{I}^2 - (3\lambda + 2\mu)\mathrm{I} - 2\mu\mathrm{II}. \tag{82}$$

Hence

$$\begin{aligned}
\Phi_1 &= (\lambda + 2\mu)\mathrm{I} - (3\lambda + 2\mu) \\
\Phi_2 &= -2\mu \\
\Phi_3 &= 0 \\
\Phi_{11} &= \lambda + 2\mu \\
\Phi_{ij} &= 0 \qquad \text{for all } ij \neq 11.
\end{aligned}$$

Using the equilibrium conditions (81), we find that

$$\begin{aligned}
|\xi|^2 R_{12} &= 4\mu\alpha^2\beta^2 - \frac{4|\xi|^2 \ell_1\ell_2}{\ell_1 + \ell_2} \\
|\xi|^2 R_{13} &= 4\mu\alpha^2\gamma^2 \\
|\xi|^2 R_{23} &= 4\mu\beta^2\gamma^2.
\end{aligned}$$

We now substitute the expressions given above into the definition of the matrix $\mathbf{\Delta}$. Straightforward factoring and regrouping of the subdeterminants of $\mathbf{\Delta}$ yields

$$\begin{aligned}
\det_1 &= 4\alpha^4\left(\frac{|\xi|^2 e_1^2}{\ell_1+\ell_2}+(\lambda+2\mu)\right)\\
&> 0\\
\det_2 &= 16\alpha^4\beta^4\left(4(\lambda+\mu)\mu+\frac{|\xi|^2}{\ell_1+\ell_2}\left((\lambda+2\mu)(e_1-e_2)^2+4\mu e_1e_2\right)\right)\\
&> 0\\
\det_3 &= 256\mu\alpha^4\beta^4\gamma^4\left((3\lambda+2\mu)\mu+\frac{|\xi|^2}{\ell_1+\ell_2}\left((\lambda+\mu)(e_1-e_2)^2+(\lambda+2\mu)e_1e_2\right)\right)\\
&> 0.
\end{aligned}$$

Thus $\mathbf{\Delta}$ is always positive definite and we have shown that:

The rigidly rotating asymmetric St. Venant-Kirchhoff ellipse is nonlinearly stable if the axis of rotation is the shortest axis of the reference configuration.

6.d.ii Symmetric equilibria

For a reference configuration which is symmetric about the axis of rotation, i.e. for which $e_1 = e_2$, the principal stretches of the relative equilibrium are given by

$$\begin{aligned}
\alpha^2 &= \beta^2\\
&= 1+\frac{|\xi|^2(\lambda+2\mu)e_1}{4\mu(3\lambda+2\mu)}\\
\gamma^2 &= 1-\frac{|\xi|^2\lambda e_1}{2\mu(3\lambda+2\mu)}.
\end{aligned}$$

Using the equilibrium condition $|\xi|^2 e_1/(\alpha^2-\gamma^2) = 4\mu$ and $\Phi_2 = -2\mu$, we compute that

$$\begin{aligned}
\Sigma_2 &= 16\mu\alpha^4\\
&> 0\\
\pi_{11} &= 16(\lambda+2\mu)\alpha^4\\
\pi_{12} &= 8(\lambda+2\mu)\alpha^2\gamma^2\\
\pi_{22} &= 4(\lambda+2\mu)\gamma^4\\
\tau &= 16\mu\alpha^2(2\gamma^2-\alpha^2)\\
\sigma &= 16\mu\alpha^2\gamma^2.
\end{aligned}$$

Since $\alpha^2 > \gamma^2$ and $\Sigma_2 > 0$, we need only check that the trace and determinant of $\mathbf{\Delta}$ are positive. Simple algebraic manipulation shows that

$$\begin{aligned}
\operatorname{tr}\mathbf{\Delta} &= \pi_{11}+\pi_{22}-\tau\\
&= 8\mu\left(\left(2\alpha^2-\gamma^2\right)^2+2\alpha^4\right)+4\lambda\left(4\alpha^2+\gamma^2\right)\\
&> 0\\
\det\mathbf{\Delta} &= \pi_{11}\pi_{22}-{\pi_{12}}^2+\sigma(2\pi_{12}-\sigma)-\pi_{22}\tau\\
&= 64\mu\alpha^2\gamma^4\left(2(\lambda+2\mu)(\alpha^2-\gamma^2)+(3\lambda+2\mu)\alpha^2\right)\\
&> 0.
\end{aligned}$$

Thus we have shown that:

Any symmetric St. Venant-Kirchhoff body which is in rigid rotation about its axis of symmetry is nonlinearly stable.

7. Appendix

7.a Application of the energy-momentum method

7.a.i Asymmetric diagonal equilibria

In §5.b.ii we defined the map

$$\begin{aligned} \mathcal{T} : \mathrm{T}^*\mathcal{C} \times gl(3) \times gl(3) &\rightarrow \mathrm{T}_{(F,P)}(\mathrm{T}^*\mathcal{C}) \\ \mathcal{T}((F,P),\delta f,\delta p) &= \delta f_{\mathrm{T}^*\mathcal{C}}(F,P) + \mathrm{T}_{\hat{\xi}}\Psi \cdot \delta p \\ &= \left(\delta f F, \delta p F E - \delta f^T P\right), \end{aligned}$$

where $\delta f_{\mathrm{T}^*\mathcal{C}}(F,P)$ is the infinitesimal generator of the Lie algebra element δf associated to the left action of $\mathcal{C}$ on $\mathrm{T}^*\mathcal{C}$ and $\Psi_F : \mathcal{C} \rightarrow \mathrm{T}^*_F\mathcal{C}$ is given by $\Psi_F(\eta) := \mathbf{FL}(\eta_{\mathcal{C}}(F))$. The definition of the map $\mathcal{T}$ is motivated as follows: Construct the curve $(F_\epsilon, P_\epsilon) \in \mathrm{T}^*\mathcal{C}$ given by

$$\begin{aligned} (F_\epsilon, P_\epsilon) &= \exp(\epsilon\, \delta f) \cdot (F, P + \epsilon \mathbf{FL}(\delta p_{\mathcal{C}}(F))) \\ &= (\exp(\epsilon\, \delta f) F, (\exp(\epsilon\, \delta f))^{-T}(P + \epsilon\, \delta p F E)). \end{aligned}$$

Then $\mathcal{T}((F,P),\delta f,\delta p) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (F_\epsilon, P_\epsilon)$.

The tangent vector with respect to which the second variation of H_ξ is computed in §5.b is given by

$$(\delta F, \delta P) = \mathcal{T}((F_e, P_e), \delta f, \delta p) \tag{83}$$

for

$$\delta f = \begin{pmatrix} \delta_F^1 & \ell_1 \sigma_F^3 & -(\ell_1 \sigma_F^2 + \alpha_F^2) \\ \ell_2 \sigma_F^3 & \delta_F^2 & \ell_2 \sigma_F^1 + \alpha_F^1 \\ \ell_3 \sigma_F^2 + \alpha_F^2 & -(\ell_3 \sigma_F^1 + \alpha_F^1) & \delta_F^3 \end{pmatrix} \tag{84}$$

and

$$\delta p = \begin{pmatrix} \delta_P^1 & \sigma_P^3 + \alpha_P^3 & \sigma_P^2 - \alpha_P^2 \\ \sigma_P^3 - \alpha_P^3 & \delta_P^2 & \sigma_P^1 + \alpha_P^1 \\ \sigma_P^2 + \alpha_P^2 & \sigma_P^1 - \alpha_P^1 & \delta_P^3 \end{pmatrix} \tag{85}$$

where

$$\begin{aligned} \alpha_P^1 &= \frac{(\ell_2 - \ell_3)\sigma_P^1 - |\xi| \left((\ell_1 + \ell_2)\, \alpha_F^2 + \ell_1(\ell_2 + \ell_3)\sigma_F^2\right)}{\ell_2 + \ell_3} \\ \alpha_P^2 &= -\frac{(\ell_1 - \ell_3)\sigma_P^2 - |\xi| \left((\ell_1 + \ell_2)\, \alpha_F^1 + \ell_2(\ell_1 + \ell_3)\sigma_F^1\right)}{\ell_1 + \ell_3} \\ \alpha_P^3 &= \frac{(\ell_1 - \ell_2)\left(\sigma_P^3 + |\xi| \left(\delta_F^1 - \delta_F^2\right)\right)}{\ell_1 + \ell_2}. \end{aligned} \tag{86}$$

This choice appears to result in optimal diagonalization of the second variation; other choices of translation from the algebra to the tangent space result in the presence of cross terms within the block $D^2 H_\xi(F_e, P_e)|_{\mathcal{S}_{INT}}$.

The Lie algebra elements δf and δp are determined by the block diagonalization theorem as follows:

Step 1:
The orthogonality condition (31) can be expressed as follows: $\delta F \in V$ implies that $\delta f = \delta F F_E^{-1}$ satisfies

$$0 = \left\langle \delta f, \hat{\xi} \right\rangle_{\mathcal{E}}, \tag{87}$$

which implies $\ell_1 \delta f_{21} = \ell_2 \delta f_{12}$.

Step 2:
Variations δF_{RIG} in V_{RIG} can be expressed as $\delta F_{RIG} = \widehat{\delta f}_{RIG} F_e$, where $\widehat{\delta f}_{RIG}$ is the skew-symmetric matrix associated with the vector $\delta f_{RIG} = (\alpha_F^1, \alpha_F^2, 0) \in so(3)^\perp$.

Step 3:
Condition (36) establishing that $\delta F_{INT} = \delta f_{INT} F_e$ belongs to V_{INT} now reduces to

$$\left\langle \delta f_{INT}, (\hat{\xi}\hat{\eta})^{\mathcal{S}} \right\rangle_{\mathcal{E}} = 0 \qquad \forall \eta \in so(3)^\perp. \tag{88}$$

Equation (88) implies $\ell_3 \delta f_{13} + \ell_1 \delta f_{31} = 0$ and $\ell_2 \delta f_{12} + \ell_1 \delta f_{21} = 0$. Solving these equations and imposing (87), we find that

$$\delta f_{INT} = \begin{pmatrix} \delta_F^1 & \ell_1 \sigma_F^3 & -\ell_1 \sigma_F^2 \\ \ell_2 \sigma_F^3 & \delta_F^2 & \ell_2 \sigma_F^1 \\ \ell_3 \sigma_F^2 & -\ell_3 \sigma_F^1 & \delta_F^3 \end{pmatrix}. \tag{89}$$

Step 4:
To impose the condition $\mathcal{T}((F_e, P_e), \delta f, \delta p) \in \mathcal{S} \subset \ker \mathrm{T}_{(F_e, P_e)} \mathcal{J}$, we solve (38) for the skew-symmetric component $\widehat{\alpha_P}$ of δp.

Step 5:
The variation $(\delta F_{RIG}, \delta P_{RIG}) \in \mathcal{S}_{RIG}$ is of the form

$$\begin{aligned} (\delta F_{RIG}, \delta P_{RIG}) &= \mathcal{T}((F_e, P_e), \widehat{\delta f}_{RIG}, \widehat{\delta p}_{RIG}) \\ &= \left(\widehat{\delta f}_{RIG} F_e, \widehat{\delta f}_{RIG} P_e + \widehat{\delta p}_{RIG} F_e E \right) \end{aligned}$$

for $\delta f_{RIG}, \delta p_{RIG} \in so(3)^\perp$. Solving (38), we find that $\delta p_{RIG} = (\alpha_P^1, \alpha_P^2, 0)$, where the entries α_P^i are given by

$$\begin{aligned} \alpha_P^1 &= \frac{|\xi| (\ell_1 + \ell_2) \alpha_F^2}{\ell_2 + \ell_3} \\ \alpha_P^2 &= \frac{|\xi| (\ell_1 + \ell_2) \alpha_F^1}{\ell_1 + \ell_3}. \end{aligned} \tag{90}$$

Step 6:
$(\delta F_{INT}, \delta P_{INT}) \in \mathcal{S}_{INT}$ is of the form $(\delta F_{INT}, \delta P_{INT}) = \mathcal{T}((F_e, P_e), \delta f_{INT}, \delta p_{INT})$ for

δf_{INT} as given in (89) and

$$\delta p_{INT} = \begin{pmatrix} \delta_P^1 & \sigma_P^3 + \alpha_P^3 & \sigma_P^2 - \alpha_P^2 \\ \sigma_P^3 - \alpha_P^3 & \delta_P^2 & \sigma_P^1 + \alpha_P^1 \\ \sigma_P^2 + \alpha_P^2 & \sigma_P^1 - \alpha_P^1 & \delta_P^3 \end{pmatrix}, \tag{91}$$

where

$$\begin{aligned} \alpha_P^1 &= \frac{\ell_2 - \ell_3}{\ell_2 + \ell_3}\sigma_P^1 - |\xi|\,\ell_1\sigma_F^2 \\ \alpha_P^2 &= \frac{\ell_1 - \ell_3}{\ell_1 + \ell_3}\sigma_P^2 - |\xi|\,\ell_2\sigma_F^1 \\ \alpha_P^3 &= \frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}\left(\sigma_P^3 + |\xi|\left(\delta_F^1 - \delta_F^2\right)\right) \end{aligned} \tag{92}$$

and σ_P^i and δ_P^i are arbitrary.

Letting $\delta f = \delta f_{RIG} + \delta f_{INT}$ and $\delta p = \delta p_{RIG} + \delta p_{INT}$, we see that $(\delta F, \delta P) \in \mathcal{S}$ is given by

$$(\delta F, \delta P) = \mathcal{T}((F_e, P_e), \delta f_{RIG} + \delta f_{INT}, \delta p_{RIG} + \delta p_{INT}). \tag{93}$$

Remark: The coefficients of the matrix δp result from the decomposition of the Lie algebra $gl(3)$ into skew-symmetric, diagonal, and symmetric off-diagonal components. The matrix δp may also be expressed in terms of the same decomposition of $gl(3)$ as δf, but the terms in D^2H_ξ associated to momentum variations are slightly simpler with the choice of basis given above.

7.a.ii Symmetric diagonal equilibria

As before, we wish to choose our variations so they not only satisfy the conditions of the block diagonalization lemma, which guarantees that the second variation will split into two blocks, but so that the blocks themselves are as nearly diagonal as possible. We choose our variations to be of the form

$$(\delta F, \delta P) = \mathcal{T}\left(\delta f, \delta p - \lambda\,|\xi|\,\widetilde{\delta f}\right), \tag{94}$$

where $\mathcal{T}$ is as defined in (55),

$$\delta f = \begin{pmatrix} \delta_F^2 + \delta_F^1 & \sigma_F^3 & -(\kappa_1\sigma_F^2 + \alpha_F^2) \\ \sigma_F^3 & \delta_F^2 - \delta_F^1 & \kappa_1\sigma_F^1 + \alpha_F^1 \\ \kappa_2\sigma_F^2 + \alpha_F^2 & -(\kappa_2\sigma_F^1 + \alpha_F^1) & \delta_F^3 \end{pmatrix} \tag{95}$$

for $\kappa_1 = (1+\lambda)\ell_1$, $\kappa_2 = (1-\lambda)\ell_3$,

$$\widetilde{\delta f} = \begin{pmatrix} -2\sigma_F^3 & 2\delta_F^1 & \tau\sigma_F^1 \\ 2\delta_F^1 & 2\sigma_F^3 & \tau\sigma_F^2 \\ \tau\sigma_F^1 & \tau\sigma_F^2 & 0 \end{pmatrix} \tag{96}$$

for $\tau = \frac{1}{2}(\kappa_2 - \kappa_1)$, and

$$\delta p = \begin{pmatrix} \delta_P^2 + \delta_P^1 & \sigma_P^3 & \sigma_P^2 - \alpha_P^2 \\ \sigma_P^3 & \delta_P^2 - \delta_P^1 & \sigma_P^1 + \alpha_P^1 \\ \sigma_P^2 + \alpha_P^2 & \sigma_P^1 - \alpha_P^1 & \delta_P^3 \end{pmatrix}, \tag{97}$$

for

$$\alpha_P^1 = \frac{(\ell_1 - \ell_3)\sigma_P^1 - 2\,|\xi|\,\ell_1\alpha_F^2}{\ell_1 + \ell_3} - |\xi|\left(\ell_1 + \tfrac{1}{2}\lambda(1-\lambda)(\ell_1 - \ell_3)\right)\sigma_F^2$$

$$\alpha_P^2 = -\frac{(\ell_1 - \ell_3)\sigma_P^2 - 2\,|\xi|\,\ell_1\alpha_F^1}{\ell_1 + \ell_3} + |\xi|\left(\ell_1 + \tfrac{1}{2}\lambda(1-\lambda)(\ell_1 - \ell_3)\right)\sigma_F^1.$$

The regrouping of the diagonal terms in δf and δp splits the diagonal variations into their symmetry preserving and symmetry breaking components. Aside from this, the determination of the variation δf proceeds essentially as for the asymmetric case. The rotational components α_P^1 and α_P^2 are determined by the condition that $V \subset \ker \mathrm{T}_{(F_e,P_e)}\mathcal{J}$. (The third rotational component α_P^3 of δp is identically equal to zero.)

The introduction of the correction term $\lambda\,|\xi|\,\widetilde{\delta f}$ in the momentum variation is necessary for the elimination of cross terms which would otherwise be introduced by the second variation of the momentum map associated to the right S^1 action. This term is determined by requiring that the second variation of $\Gamma_{\xi,\lambda}$ block diagonalize with respect to position and momentum variations. Specifically, given an as yet unspecified linear mapping $\chi : gl(3) \to gl(3)$, we define the function $\mathcal{Q} : gl(3) \times gl(3) \to \mathbb{R}$ by

$$\mathcal{Q}(\delta f, \delta p) = \Gamma_{\xi,\lambda}\left(\mathcal{T}((F_e, P_e), \delta f, \delta p + \chi(\delta f))\right). \tag{98}$$

The correction term $\widetilde{\delta f} = \chi(\delta f)$ is then found by solving the equation

$$D^2\mathcal{Q}(\mathbf{0},\mathbf{0})((\delta f, \mathbf{0}), (\mathbf{0}, \delta p)) = 0. \tag{99}$$

We note that $\chi|_{V_{RIG}} \equiv \mathbf{0}$. In the case $\lambda = 0$, the correction term vanishes and the variations are the same as those used for the stability analysis of asymmetric equilibria.

7.b Equilibrium conditions and the Baker-Ericksen inequalities

When computing the second variation D^2H_ξ, it is necessary to incorporate in some fashion the fact that the second variation is evaluated at a relative equilibrium, i.e. at a critical point of H_ξ. In most cases, it is infeasible to explicitly substitute the equilibrium values of the principal stretches λ_i, since the expressions for the stretches are typically very complicated, if available in closed form at all. As an alternative, we note that the equilibrium equations (51) may be solved for the derivatives Φ_i, with solutions

$$\Phi_1 = \frac{|\xi|^2\left(\lambda_1\ell_1(\lambda_2 - \lambda_3) - \lambda_2\ell_2(\lambda_1 - \lambda_3)\right)}{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$$

$$\Phi_2 = \frac{|\xi|^2\left(\ell_2(\lambda_1 - \lambda_3) - \ell_1(\lambda_2 - \lambda_3)\right)}{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$$

$$\Phi_3 = \frac{|\xi|^2\left(e_1(\lambda_2 - \lambda_3)) - e_2(\lambda_1 - \lambda_3)\right)}{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}$$

provided that $\lambda_i \neq \lambda_j$ for $i \neq j$. The matrix M given in (60) is obtained by substituting these equilibrium values into the general second variation. While this approach simplifies the expression of the second variation to the extent that the block diagonal structure is readily apparent, it need not provide the most computationally convenient form of the stability conditions in cases where the derivatives Φ_i have a particularly simple form.

We note that the terms Σ_i can be rewritten using expressions involving the first derivatives of the stored energy function Φ by means of the equations (68) and (72). The stability conditions obtained by these means can be directly related to the Baker-Ericksen inequalities (71) and the alternative inequalities (75). In particular, we have

$$\begin{aligned}
\Sigma_1 &= 2\lambda_2\lambda_3(\Phi_1+\lambda_1\Phi_2)(e_2-e_3)(\ell_2-\ell_3) \\
&= -2\lambda_1\lambda_2\lambda_3(\Phi_2+\lambda_1\Phi_3)(e_2-e_3)(\ell_2-\ell_3) \\
\Sigma_2 &= 2\lambda_1\lambda_3(\Phi_1+\lambda_2\Phi_2)(e_1-e_3)(\ell_1-\ell_3) \\
&= -2\lambda_1\lambda_2\lambda_3(\Phi_2+\lambda_2\Phi_3)(e_1-e_3)(\ell_1-\ell_3) \\
\Sigma_3 &= \frac{2\lambda_1\lambda_2(\ell_1+\ell_2)^2(\Phi_1+\lambda_3\Phi_2)(e_1-e_2)}{\ell_1-\ell_2} \\
&= -2\lambda_1\lambda_2(\ell_1+\ell_2)^2(\Phi_2+\lambda_3\Phi_3).
\end{aligned}$$

Using these expressions, we can see that for a stable equilibrium the conditions $\rho_i > 0$ and $\Sigma_i > 0$ imply that both the Baker-Ericksen inequalities (71) and the alternative inequalities (75) hold at a stable equilibrium. On the other hand, if we assume that the alternative inequalities hold, then the conditions $\rho_i > 0$ and $\Sigma_i > 0$ can be replaced by the conditions $e_1 > e_3$ and $e_2 > e_3$. This can be seen as follows: Using the equations (72) and (74), we can show that the inequalities (75) imply $\lambda_1 > \lambda_3$, $\lambda_2 > \lambda_3$, and $(e_1-e_2)/(\lambda_1-\lambda_2) > 0$. The preceding assumptions imply that $\ell_1 = \lambda_1 e_1 > \lambda_3 e_3 = \ell_3$ and, analogously, $\ell_2 > \ell_3$. Thus ρ_1, ρ_2, and Σ_i, $i = 1,2,3$, must be positive.

In addition, we point out that for $i \neq j$ the terms R_{ij} can be rewritten as follows:

$$\begin{aligned}
|\xi|^2 R_{12} &= -4\left(\lambda_1\lambda_2(\Phi_2+\lambda_3\Phi_3)+\frac{|\xi|^2\,\ell_1\ell_2}{\ell_1+\ell_2}\right) \\
|\xi|^2 R_{13} &= -4\lambda_1\lambda_3(\Phi_2+\lambda_2\Phi_3) \\
&= 4\lambda_3(\Phi_1+\lambda_2\Phi_2) \\
|\xi|^2 R_{23} &= -4\lambda_2\lambda_3(\Phi_2+\lambda_1\Phi_3) \\
&= 4\lambda_3(\Phi_1+\lambda_1\Phi_2).
\end{aligned}$$

The reformulations described above for asymmetric equilibria apply to symmetric equilibria as well. The equilibrium equations (66) may be rewritten as

$$\begin{aligned}
\Phi_1 &= \frac{|\xi|^2\,\ell_1}{2(\lambda_1-\lambda_3)} - \Phi_2\lambda_1 \\
\Phi_3 &= -\frac{|\xi|^2\,\ell_1}{2\lambda_1^2(\lambda_1-\lambda_3)} - \frac{\Phi_2}{\lambda_1}
\end{aligned}$$

provided that $\lambda_1 \neq \lambda_3$. We use this form of the equations to implement the condition that we are at a critical point of the modified Hamiltonian when computing the second variation of H_λ.

Using the equilibrium conditions (66), we can express Σ_1 and Σ_2 in the form

$$\Sigma_1 = 8\lambda_3\ell_1^2(\Phi_1+\lambda_1\Phi_2)$$

$$
\begin{aligned}
&= -8\lambda_1\lambda_3\ell_1^2(\Phi_2+\lambda_1\Phi_3)\\
\Sigma_2 &= 8(\lambda_3\Phi_1+\lambda_1\Phi_2(2\lambda_3-\lambda_1))\\
&= -8\lambda_1^2(\Phi_2+\lambda_3\Phi_3)
\end{aligned}
$$

when $\lambda = 1$. Thus, the stability conditions $\Sigma_1 > 0$ and $\Sigma_2 > 0$ follow from the conditions

(i) $0 < \Phi_1 + \lambda_1\Phi_2$

(ii) $0 < \Phi_1 + \lambda_3\Phi_2$

or

(i) $0 > \Phi_2 + \lambda_1\Phi_3$

(ii) $0 > \Phi_2 + \lambda_3\Phi_3$,

which are related to the Baker-Ericksen and alternative inequalities. These forms of the conditions are particularly convenient when considering materials for which the signs of the derivatives Φ_i are obvious (e.g. constant).

We can eliminate the rotation rate $|\xi|$ from the stability conditions by writing

$$
\begin{aligned}
\tau &= 8\left(\Phi_1(3\lambda_3-2\lambda_1)+\lambda_1\Phi_2(4\lambda_3-3\lambda_1)\right)\\
&= 8\lambda_1\left(\Phi_2(\lambda_1-2\lambda_3)+\lambda_1\Phi_3(2\lambda_1-3\lambda_3)\right)\\
\sigma &= 8\lambda_3(\Phi_1+\lambda_1\Phi_2)\\
&= -8\lambda_1\lambda_3(\Phi_2+\lambda_1\Phi_3).
\end{aligned}
$$

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