COMPLICATED DYNAMICS IN SCALAR SEMILINEAR PARABOLIC EQUATIONS IN HIGHER SPACE DIMENSION

By

Peter Poláčik

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PETER POLÁČIK*

Abstract. We study the dynamics of the boundary value problem

$$u_t - Lu = g(x, u, \nabla u), x \in \Omega, \tag{1}$$

$$u|_{\partial\Omega} = 0, \tag{2}$$

where L is a second order uniformly elliptic operator and $\Omega \subset \mathbb{R}^N$ is diffeomorphic to the ball in $\mathbb{R}^N$, $N \geq 2$. The main result asserts that given any C^k-vector field V on $\mathbb{R}^{N+1}$ with $V(0) = 0$ one can adjust coefficients of L and the function g such that the corresponding problem (1),(2) has an $N+1$-dimensional invariant manifold through the equilibrium $u \equiv 0$ and the Taylor expansion at $u \equiv 0$ of the vector field representing the flow on this manifold coincides (in appropriate coordinates) with the Taylor expansion of V, up to k-th order terms. This result implies that a hyperbolic invariant N-torus can be found in (1),(2) (if L and g are appropriately chosen). This result also indicates that "chaotic dynamics" is likely to occur for some choices of L and g.

1. Introduction. In this paper we study the dynamics of the scalar semilinear parabolic equation

$$u_t - Lu = g(x, u, \nabla u), t > 0, x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N, N \geq 2$, is a domain such that $\overline{\Omega}$ is (as a manifold with boundary) C^∞ diffeomorphic so the unit ball in $\mathbb{R}^N$. Here L stands for a second order elliptic operator of the form

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a(x)u, \tag{1.2}$$

where the coefficients a_{ij}, a are smooth on $\overline{\Omega}$ and the matrix (a_{ij}) is symmetric and uniformly positive definite. The function $g(x, u, y) : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is assumed continuous together with all its partial derivatives

$$\frac{\partial g(x, u, y)}{\partial^{k_0} u \partial^{k_1} y_1 \dots \partial^{k_N} y_N}$$

with respect to u and y. (We separate $a(x)u$ from $g(x, u, \nabla u)$ for notational convenience.)

We subject (1.1) to Dirichlet boundary condition

$$u|_{\partial\Omega} = 0. \tag{1.3}$$

*Institute of Applied Mathematics, Comenius University, Mlynskà dolina, 842 15 Bratislava, *Czechoslovakia*.

This boundary value problem can be considered within the context of abstract semilinear parabolic equations, as examined in [He 1]. By [He 1], (1.1), (1.3) defines a local semiflow on an appropriate Sobolev-Slobodeckii space. (Details are given below.)

For various restricted classes of equations of the type (1.1), the dynamics has been well described. For instance, if $N = 1$, then the dynamics is simple: each bounded solution converges to an equilibrium [Ma 2,4,Ze]. A variety of other results giving more detailed description of the dynamics are available if $N = 1$ (see [An, B-F 1,2, F-R, He 1,2, L-P-S-S, C-L-S] and references given there).

If $N > 1$, we still observe a simple behaviour of solutions, provided $g = g(x, u)$ does not depend on ∇u. In this case (1.1), (1.3) admits a global Lyapunov functional [Ma 1]. Hence all bounded solutions approach a set of equilibria. If all equilibria are isolated which usually is, in a sense, a generic situation (see [B-C, B-P, He 2,4, Po, Ro] for various results concerning generic hyperbolicity of equilibria), then again each bounded solution is convergent. This convergence result remains valid if instead of the generic assumption on the equilibria one assumes that g is analytic in u [Si].

Now let us turn to the general case, when $N > 1$ and g is allowed to depend on ∇u. The semiflow defined by (1.1), (1.3) is no longer gradient-like and in fact an oscillatory behaviour of trajectories can occur (in [Hi 1], an equation of the type (1.1) which has a periodic orbit is given). Still there is a special structure which has important dynamical consequences. Namely, the problem (1.1), (1.3) falls into the class of so called strongly monotone dynamical systems [Hi 2] (cf. [Ma 3]). The concept of such systems is an abstraction of the strong comparison principle, which holds in (1.1), (1.3) as a consequence of the maximum principle for linearized equations [P-W]. One of the most important dynamical implications is that "almost all" bounded solutions of (1.1), (1.3) are convergent [Po 2]. So the "typical" behaviour of trajectories is again very simple (see [Hi 2, Po 3, S-T] for other typical properties of (1.1), (1.3)). However, no limitation emerges from the monotonicity structure on the at-all possible (atypical) dynamical behaviour. As a construction of Smale [Sm] shows, "any dynamics" can be found in strongly monotone systems. The aim of this paper is to show that trajectories of (1.1), (1.3) can indeed exhibit a complicated dynamical behaviour. As follows from our results, (1.1), (1.3) can have a trajectory dense in a torus of arbitrarily high dimension if N is sufficiently large, and the functions a_{ij}, a and g are appropriately chosen. Moreover, there is a strong indication that, even if $N = 2$, a chaotic shift dynamics can be detected in (1.1), (1.3). (Of course, by monotonicity, such dynamics must be unstable.)

So there is a big qualitative difference between the general equation (1.1) on one side and the gradient independent equation or the equation on an interval on the other side. The situation here is similar as when comparing equation (1.1) on an interval Ω with the same equation, where we add a simple nonlocal term

$$\int_\Omega \nu(x)u(x,t)dx.$$

As was shown in [F-P], in equations with such a term one can expect rather complicated dynamics (as opposed to local one-dimensional equations (1.1)).

Our approach toward complicated dynamics in (1.1), (1.3) is similar to that of [F-P]. We prove that any finite jet can be realized in (1.1), (1.3). There is a difference, however, in the way how this aim is achieved. Below we will outline the procedures used in both papers. First we give a precise meaning to the phrase "realize a jet in (1.1), (1.3)".

For this we write (1.1), (1.3) in the abstract form

$$u_t + Au = f(u). \tag{1.4}$$

Here A is the sectorial operator on $X := L_p(\Omega)$ (we choose $p > N$) defined by L and Dirichlet boundary condition and f is a Nemitskii operator defined by g. Specifically, we define A with the domain

$$D(A) := W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega) = \{u \in W^{2,p}(\Omega) | u|_{\partial\Omega} = 0\} \tag{1.5}$$

by

$$Au = -Lu. \tag{1.6}$$

For this sectorial operator, the fractional lower space $X^\alpha, 1/2 \leq \alpha < 1$, is the Sobolev-Slobodecsii space $W^{2\alpha,p}(\Omega) \cap W_0^{1,p}(\Omega)$ [Am, He 1]. Since $p > N$, we can choose $\alpha < 1$ sufficiently close to 1 such that we achieve the continuous imbedding

$$W^{2\alpha,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}). \text{ [Tr]} \tag{1.7}$$

Then defining $f : X^\alpha \to X$ by

$$f(u(\cdot))(x) = g(x, u(x), \nabla u(x)),$$

we clearly have $f \in C^\infty(X^\alpha, X)$. Thus, by [He 1], (1.4) defines a local semiflow on X^α.

Now assume that the differential operator L is chosen such that the corresponding operator A admits a decomposition

$$X = X_1 \oplus X_2$$

into close invariant subspaces, X_1 having finite dimension. Fix an integer $k > 0$ and consider the finite dimensional linear space $J_0^k(X_1)$ of the k-jets on X_1 for which 0 is the source and target (see [G-G]). Equivalently, any element in $J_0^k(X_1)$ can be understood as the Taylor expansion at 0 of a C^k-mapping $h : X_1 \to X_1$ such that $h(0) = 0$. (The Taylor expansion is taken up to the order k.)

We say that a jet $j^k \in J_0^k(X_1)$ can be realized in (1.1), (1.3), by adjusting the function g, if there exists a function g with the above regularity such that the equation (1.4) corresponding to (1.1), (1.3) has the following two properties:

P1) There exists a locally invariant manifold of (1.4) of the form

$$W = \{u_1 + \sigma(u_1) | u_1 \in U\}, \tag{1.8}$$

where U is a neighbourhood of 0 in X_1, and $\sigma : U \to X_2^\alpha := X_2 \cap X^\alpha$ is a C^k-mapping with $\sigma(0) = 0$.

P2) Consider the projected equation

$$u_1 = -Au_1 + Pf(u_1 + \sigma(u_1)), \tag{1.9}$$

representing the flow of (1.4) on W. Here $P : X \to X_1$ is the continuous projection with kernel X_2 (hence P commutes with A). The second property requires that the k-jet at 0 of the right hand side of (1.9) is equal to the given jet j^k.

Realization of finite jets of vector fields, in particular those with degenerate singularities, is an important prerequisite to finding an interesting dynamical behaviour in particular problems. Local bifurcation theory then supplies results, which can be applied to establish occurrence of interesting phenomena. As an example one can consider results of Langford and Iooss on interactions of Hopf and steady-state bifurcations [La, L-I]. In [I-L], they analyzed, via the normal form techniques, an unfolding of a vector field on $\mathbb{R}^3$ with $0, \pm iw$ degeneracy. Their analysis provides a bifurcation diagram, which is qualitatively unaffected by the terms of order greater than 5 in the Taylor expansion of the vector field at 0. The most interesting feature of this diagram is that it contains a parameter region, where the corresponding vector field has an invariant 2-torus.

In order to apply this result in a particular problem, it suffices to prove that any 5-jet on a three dimensional space can be realized. This is indeed the case with (1.1), (1.3) for $N = 2$. So, by the results of [L-I], if $N = 2$ an operator L and a function g can be found such that (1.1), (1.3) has an invariant 2-torus.

For a general $N \geq 2$ we prove that any finite jet on an $N+1$-dimensional space can be realized in (1.1), (1.3). Thus for $N > 2$ a mode interaction can be used leading to N-dimensional invariant tori [Bi, Ch-H].

It is likely that even more interesting invariant sets can be found in (1.1), (1.3). A system, where any finite jet on at least three-dimensional space can be realized, are expected to admit existence of a transverse homoclinic orbit of some return map. See [Gu 2] for a theoretical support of this expectation. Existence of such an orbit is known to be an evidence of presence of a Cantor invariant set with chaotic shift dynamics [G-H, Pa].

As was already mentioned, our jet realization result is similar to a result obtained in [F-P]. It was proved there, that any jet, the linear part of which has simple imaginary eigenvalues, can be realized in the nonlocal equation mentioned above. This result was proved in two steps. First coefficients in the linear part of the nonlocal equation were found such that the linear operators had a prescribed number of simple eigenvalues on the imaginary axis. Then higher order terms in the equations were adjusted such that the vector field on the corresponding center manifold of 0 had an arbitrarily prescribed jet (satisfying the restriction on the linear part). A similar procedure was used in [Gu 1,2] for realization of a 2-jet of a vector field on $\mathbf{R}^3$ in the Brusselator diffusive system.

The second step in the method of [F-P] is rather implicit, since transversality was used to prove that certain condition is generically satisfied.

The method we use here for realization of finite jets in (1.1), (1.3) is implicit already on the linear level. Unlike [Gu 1,2, F-P], we are not able to explicitly solve the complex inverse eigenvalue problem, i.e. to find a linear equation (1.1) with prescribed complex eigenvalues. (Note, however, that an existence results, based on the implicit function theorem, is presented in Section 2). Instead we start with an operator L which, subject to Dirichlet boundary conditions, has a kernel X_1 of dimension $N+1$. We then constitute a mapping which to each "small" g associates a k-jet on X_1. This jet is given by a vector field on an invariant manifold of (1.1), (1.3) close to X_1 (which exists for small g). The image of this mapping consists of jets in $J_0^k(X_1)$ which can be realized in (1.1), (1.3), by adjusting the function g (including linear terms). We use the implicit function theorem to prove that this mapping is locally surjective near $g=0$. In this way we prove that any jet in $J^k(X_1)(k>1-$ arbitrary), sufficiently close to 0 can be realized in (1.1), (1.3). This is quite sufficient, since any vector field can be obtained from an arbitrarily small one, via time rescaling. When completing the manuscript, the author was informed by J. Hale about his papers [Ha 1,2], where a similar result is proved for delay differential equations. Though a stronger result is claimed in these papers (realization of any vector field on a center manifold) the proof works only for the jet realization. We give more details about this in Section 2.

We now state our main theorem precisely. In its formulation and in the whole paper, if functions a_{ij}, a are mentioned refering to coefficients of a differential operator (1.2), it is always assumed that they are in $C^\infty(\overline{\Omega})$. Similarly a function $g(x,u,y)$ is always assumed continuous on $\overline{\Omega}\times\mathbf{R}^{N+1}$ together with all its partial derivatives with respect to (u,y).

THEOREM 1. *Let $\Omega\subset\mathbf{R}^N$ be any domain such that $\overline{\Omega}$ is C^∞-diffeomorphic to the unit ball in $\mathbf{R}^N$. Let $n=N$ or $n=N+1$. Then there exists coefficients a_{ij}, a such that the operator A defined by L and Dirichlet boundary condition (see (1.2), (1.5), (1.6)) has an n-dimensional kernel X_1 and the following property holds. For any integer $k>0$ there exists a neighbourhood B of 0 in $J_0^k(X_1)$ such that any jet in B can be realized in (1.1), (1.3), by adjusting the function g. Moreover, in the case $n=N, g$ can be chosen independent of u.*

Note that if A defined by (1.5), (1.6) has kernel X_1 then there exists a closed A-

invariant subspace X_2 complementary to X_1(i.e. $X = X_1 \oplus X_2$, as required for realization of jets on X_1).

The main reason why we have included the statement for $n = N$, though it gives us a weaker result (for $N = 2$ we can realize only jets of planar vector fields), is that in this case we can work with a more specific equation. If Ω is a ball in $\mathbb{R}^N$ then the conclusion of Theorem 1 for $n = N$ holds for $Lu = \Delta u + \mu u$, where Δ is the Laplacian and μ is a constant. Thus adjusting g independent of u, we stay in the class of equations

$$u_t = \Delta u + \mu u + g(x, \nabla u). \tag{1.10}$$

The paper is organized as follows.

Section 2 deals with the problem (1.1), (1.3) assuming that the operators L has an n-dimensional ($n = N$ or $n = N+1$) kernel X_1. An additional condition on L is derived, which assures that the mapping taking g onto a jet on X_1, as mentioned in the above outline, is locally surjective.

An operator with an n-dimensional kernel satisfying this additional condition is found in Section 3. In that section the proof of Theorem 1 is completed and a statement concerning equation (1.10) is presented.

In Section 4 we address some open problems related to our results.

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2. Local surjectivity. As was mentioned in the introduction, a starting ingradient of the proof of Theorem 1 is a differential operator L with an n-dimensional ($n = N, N+1$) kernel X_1. Such operator will be found in Section 3. In this section we derive an additional condition on L under which any small jet on X_1 can be realized in (1.1), (1.3), by adjusting function g.

Our method is based on two general theorems (center manifold theorem and a local surjectivity criterion). In order to allow the method to be applicable in other situations and to make the procedure more intelligible, we initially work in an abstract setting. We obtain an abstract surjectivity condition, which, interpreted in terms of (1.1), (1.3), gives us the sought additional condition on L.

For the abstract part of our investigation we assume that A is a sectorial operator on a Banach space X which has 0 as an eigenvalue of the same algebraic and geometric multiplicity n. We also assume that no other element of the spectrum of A lies on the imaginary axis. Let Λ be another Banach space and let V be an open neighbourhood of 0 in Λ. Consider the parametrized equation

$$u_t + Au = f(u, \lambda), \tag*{$(2.1)_\lambda$}$$

where $f : X^\alpha \times V \to X$ is a C^∞ function, and $X^\alpha, \alpha \in (0,1)$, is some fractional power space defined by A [He 1].

Introducing such a parametrized equation reflects our aim to investigate a problem where certain data may be adjusted. (Later the role of the parameter will be played by the function g.) We shall be concerned with jets on X_1 which can be realized in $(2.1)_\lambda$, by adjusting the parameter.

Let $X = X_1 \oplus X_2$ be the A-invariant spectral decomposition, where X_1 is the kernel of A. Let $P : X \to X_1$ be the spectral projection with kernel X_2 (see e.g. [Ka]). Assume that

$$f(0,\lambda) \equiv 0 \text{ and} \tag{2.2}$$

$$f(u,0) \equiv 0. \tag{2.3}$$

By (2.3), the subspace X_1 is an invariant manifold (consisting of equilibria) for the equation $(2.1)_0$. We now seek a locally invariant manifold for $(2.1)_\lambda$ of the form

$$W_\lambda = \{u_1 + \sigma(u_1,\lambda) | u_1 \in \mathcal{U}\}, \lambda \in \mathcal{W}, \tag{2.4}$$

where $\mathcal{U}$ is a neighbourhood of 0 in X_1, $\mathcal{W} \subset V$ is a neighbourhood of 0 in λ and $\sigma : \mathcal{U} \times \mathcal{W} \to X_2^\alpha = X^\alpha \cap X_2$ is a C^{k+1}-mapping. (Henceforth we fix an integer $k > 0$.) For this we assume that Λ admits a smooth cut off function, i.e. that the following property holds:

(CO) There exists a C^∞-function $\xi : \Lambda \to \mathbb{R}$ such that $\xi \equiv 1$ in a neighbourhood of 0 and $\xi \equiv 0$ outside the unit ball centered at 0 in Λ.

With this assumption one proves existence of the invariant manifold W_λ in a standard way using the extended equation

$$\begin{aligned} \dot{u} + Au &= f(u,\lambda), \\ \dot{\lambda} &= 0. \end{aligned} \tag{2.5}$$

(See [Ch - L 2] for another possible approach.) Indeed, by (2.2), (2.3), the right-hand side of (2.5) is of the second order for $(u,\lambda) \to (0,0)$. The linear operator in (2.5), i.e. the sectorial operator $(u,\lambda) \mapsto (Au, 0)$ on $X \times \Lambda$, has the same spectrum as A. The eigenspace of this operator corresponding to the eigenvalue 0 is the space $Y_1 := X_1 \times \Lambda$. Since X_1 is finite dimensional, (CO) implies that Y_1 admits a smooth cut off function. Therefore we can apply the center manifold theory [He 1, Ch-L, M-M] to conclude that (2.5) has a locally in variant C^{k+1}-manifold (a center manifold of the equilibrium (0,0)) of the form

$$W = \{(u_1,\lambda) + (\sigma(u_1,\lambda),\lambda) | (u_1,\lambda) \in \mathcal{U} \times \mathcal{W}\}.$$

Here $\mathcal{U} \times \mathcal{W}$ is a neighbourhood of $(0,0)$ in $X_1 \times \Lambda$ and $\sigma : \mathcal{U} \times \mathcal{W} \to X_2^\alpha$ is C^{k+1}. Clearly, each λ-section of this manifold is locally invariant manifold for $(2.1)_\lambda$, and it has the form (2.4).

From (2.2), (2.3) we further obtain

$$\sigma(u,0) \equiv 0 \text{ and} \tag{2.6}$$

$$\sigma(0,\lambda) \equiv 0 \tag{2.7}$$

(because the center manifold W must contain the equilibria $(u,0),(0,\lambda)$). In a usual way, we now represent the flow of (2.1) on W_λ by an ordinary differential equation. Namely, this flow is conjugate, via $P|_{W_\lambda}$, so the flow of the equation

$$\dot{u}_1 = Pf(u_1 + \sigma(u_1,\lambda),\lambda) \tag{2.8}$$

(Recall $A|_{X_1} = 0$).

Thus for each $\lambda \in \mathcal{W}$ we have defined a vector field

$$h(u_1,\lambda) := Pf(u_1 + \sigma(u_1,\lambda),\lambda) \tag{2.9}$$

on $\mathfrak{U} \subset X_1$. By (2.2), (2.6), (2.7), we have

$$h(0,\lambda) \equiv 0 \text{ and} \tag{2.10}$$

$$h(u,0) \equiv 0. \tag{2.11}$$

Now consider the k-jet

$$j_0^k h(\cdot,\lambda) = (D_{u_1}^r h(u_1,\lambda)|_{u_1=0})_{r=0}^k$$

at $u_1 = 0$ of the C^{k+1}-mapping $u_1 \mapsto h(u_1,\lambda)$. We are interested in the set of k-jets obtained in this way for all $\lambda \in \mathcal{W}$. Our aim is to find a condition guaranteeing that this set of jets contains a neighbourhood of 0 in $J_0^k(X_1)$. For this end we introduce a mapping ϕ, which maps each $\lambda \in \mathcal{W}$ onto the k-jet of $h(\cdot,\lambda)$:

$$\phi(\lambda) = j_0^k h(\cdot,\lambda).$$

since $h(u,\lambda)$ is $C^{k+1}, \phi : \mathcal{W} \to J_0^k(X_1)$ is C^1. By (2.11), we have $\phi(0) = 0$. Therefore the image of ϕ contains a neighbourhood of 0 in $J_0^k(X_1)$, provided $\phi'(0)$ is a surjective linear operator (see e.g. [Be]). We now calculate $\phi'(0)\nu$ for *a* $\nu \in \Lambda$. Using a change of the order of differentiation, we first obtain

$$\phi'(0)\nu = j_0^k\{h_\lambda(\cdot,0)\nu\}.$$

Now, by (2.9), (2.3), (2.6),

$$\begin{aligned} h_\lambda(u_1,0)\nu &= Pf_\lambda(u_1,0)\nu + Pf_u(u_1,0)\sigma_\lambda(u_1,0)\nu \\ &= Pf_\lambda(u_1,0)\nu. \end{aligned}$$

Thus $\phi'(0)\nu$ is the k-jet of the mapping $u_1 \mapsto Pf_\lambda(u_1,0)\nu$. It is now obvious that $\phi'(0)$ is surjective if the following property holds true

(SC) For any polynomial $H : X_1 \to X_1$ of degree k, satisfying $H(0) = 0$, there exists a $\nu \in \Lambda$ such that

$$Pf_\lambda(u_1,0)\nu = H(u_1) \tag{2.12}$$

for all $u_1 \in X_1$.

Let us mention that if $f(u,\lambda)$ is linear in λ (as will be the case in the forthcoming application) then (2.12) reads

$$Pf(u_1,\nu) = H(u_1). \tag{2.13}$$

With the surjectivity condition (SC) we have finished our abstract consideration. In applications of this abstract condition one has to choose a suitable parameter space Λ, admitting a cut off function, write a parametrized equation $(2.1)_\lambda$ corresponding to a particular problem, where the conditions (2.2), (2.3) are satisfied, and verify (SC). We now carry out this program for the problem (1.1), (1.3).

Assume that a differential operator L of the form (1.2) is given and let A be the operator on $L_p(\Omega)$ defined by L and Dirichlet boundary conditions (see (1.5), (1.6)). As in the introduction we choose $p > N$, but it is useful to note that the spectrum of A does not depend on the choice of $p > 1$. Since for $p = 2, L$ defines a self-adjoint operator with compact resolvent, the spectrum of A (on $L_p(\Omega), p > N$) consists of real eigenvalues with the same algebraic and geometric multiplicities.

Take $\alpha \in (0,1)$ as in (1.7) so that

$$X^\alpha \hookrightarrow (C^1(\overline{\Omega}).$$

Consider the equation (1.9) corresponding to (1.7), (1.3). We stress the dependence of the Nemitskii operator f on g,

$$f(u(\cdot))(x) = g(x, u(x), \nabla u(x)),$$

by writing $f = f(u,g)$. We thus consider the equation

$$u_t + Au = f(u,g),$$

where g is an element of some Banach space Λ. We postpone the specific choice of Λ for a while. As this point we only need to know that Λ consists of functions $g(x,u,y)$ continuous on $\overline{\Omega} \times \mathbb{R}^{N+1}$, together with all its partial derivatives with respect to (u,y) (this is the regularity required in Theorem 1), and the topology on Λ is at least as strong as the topology of locally uniform convergence (on $\overline{\Omega} \times \mathbf{R}^{N+1}$) of all partial derivatives with respect to (u, y). Then we clearly have $f(u, g) : X^\alpha \times \Lambda \to X$ of class C^∞. Moreover we shall assume that each function $g \in \Lambda$ satisfies $g(x, 0, 0) \equiv 0$, hence

$$f(0, g) = 0, \text{ for all } g \in \Lambda.$$

For $g \equiv 0$ we clearly have

$$f(u, 0) = 0.$$

So in our parametrized equation (g playing a role of the parameter) we have the conditions (2.2), (2.3), with λ replaced by g, satisfied.

We now reformulate the surjectivity condition (SC) for this particular equation parametrized by g.

Assume that A has 0 as an eigenvalue of multiplicity n. Let

$$X_1 = \text{ span } \{\varphi_1, \ldots, \varphi_n\}$$

be the corresponding eigenspace. we may assume the functions $\varphi_1, \ldots, \varphi_n$ to be orthogonal in $L_2(\Omega)$. Then the spectral projection $P : X \to X_1$ is given by

$$Pu = \sum_{j=1}^{n} \varphi_j \int_\Omega \varphi_j(x) u(x) dx \tag{2.14}$$

(P is just the restriction to $X = L_p(\Omega)$ of the spectral projection of the self adjoint operator defined by L and (1.3) on $L_2(\Omega)$). Using (2.14) and recalling linearity of $f(u, g)$ in g we can write the surjectivity condition (SC) for our particular case as follows.

(SCP) For any polynomial $H(u_1)$ on X_1 of degree k satisfying $H(0) = 0$, there exists a $g \in \Lambda$ such that

$$\sum_{j=1}^{n} \varphi_j \int_\Omega \varphi_j(x) g(x, u_1(x), \nabla u_1, (x)) dx = H(u_1) \tag{2.15}$$

for all $u_1 \in X_1$.

For the reader's convenience we now summarize all the conclusions from the above consideration which have to be remembered in the sequel.

Suppose we have a differential operator L with an n-dimensional kernel X_1 (assuming Dirichlet boundary condition). Then any jet in $J_0^k(X_1)$ sufficiently close to 0 can be realized in (1.1), (1.3), by adjusting g, provided a Banach space Λ of functions $g(x, u, y)$ can be found such that the following properties are satisfied:

Pa) Λ admits a smooth cut off function.

Pb) The topology on Λ is at least as strong as the topology of locally uniform convergence of all partial derivatives with respect to (u, j).

Pc) For each $g \in \Lambda$ one has $g(x, 0, 0) \equiv 0$.

Pd) (SCP) holds true.

With this preparation we can prove the main result of this section.

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^N$ be any bounded domain with smooth boundary. Let $n = N$ or $n = N + 1$. Assume that L is a differential operator of the form (1.2) which has the following two properties:*

i) L subject to Dirichlet boundary condition has an n-dimensional kernel

$$X_1 = \text{ span } \{\varphi_1, \ldots, \varphi_n\}.$$

ii) The $n \times n$ matrix $M(x)$ with the j-th row defined by

$$M_j(x) = \begin{cases} \nabla\varphi_j(x) \text{ if } n = N \\ (\varphi_j(x), \nabla\varphi_j(x) \text{ if } n = N + 1 \end{cases} \tag{2.16}$$

is regular (i.e. $\det M(x) \neq 0$) at some point $x \in \Omega$. (This of course does not depend on whether the basis $\varphi_1, \ldots, \varphi_n$ is orthogonal or not.)

Then for any integer $k > 0$ there exists a neighbourhood $\mathcal{B}$ of 0 in $J_0^k(X_1)$ much that any jet in $\mathcal{B}$ can be realized in (1.1), (1.3) by adjusting the function g. Moreover, in the case $n = N, g$ can be chosen independent of u.

Proof. Let the hypotheses be satisfied. Fix an integer $k > 0$. Lemma 2.1 will be proved if we find a Banach space Λ satisfying all the properties Pa) - Pd).

To define Λ, choose an integer $m > 1 + 2/N$, so that the following imbedding takes place:

$$H^m(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{2.17}$$

(see e.g. [Tr]). Let Λ be the set of all functions $g(x, u, y)$, which are polynomials of the degree k in variables (u, y) with (x-dependent) coefficients in $H^m(\Omega)$, and which satisfy

$$g(x, 0, 0) \equiv 0.$$

In case $n = N$, we further require each $g \in \Lambda$ to be independent of u. By (2.17), each $g \in \Lambda$ has the required regularity. We now define the norm on Λ.

Any $g \in \Lambda$ can be naturally identified with the vector of its coefficients. This defines a linear isomorphism between Λ and the space

$$(H^m(\Omega))^d, \tag{2.18}$$

where d is the dimensions of the space of all real polynomials in n variables, for which the origin is a zero point. We define the norm for Λ by requiring this isomorphism to be an isometry.

For such norm we clearly have the property Pb) satisfied. Further, since the space (2.18), as a product of Hilbert spaces is itself a Hilbert space, Pa) is also satisfied. The property Pc) was assumed in the definition of Λ. It only remains to verify Pd).

For this we first write (2.15) in real coordinates on $X_1, u_1 = r_1\varphi_1 + \cdots + r_n\varphi_n$, where we assume $\varphi_1, \ldots, \varphi_n$ to be orthonormal in $L_2(\Omega)$. Passing to this coordinates we see that (2.15) is equivalent so a system of equalities

$$\int_\Omega \varphi_j(x) g\left(x, \sum_{i=1}^n r_i\varphi_i(x), \sum_{i=1}^n r_i\nabla\varphi_i(x)\right) dx = H_j(r), \tag{2.19}$$

where $H_j, j = 1, \ldots, n$, are real polynomials in $r = (r_1, \ldots, r_n)$. In a vector notation, (2.19) can be written as

$$\int_\Omega \varphi_j(x) g(x, rM(x)) dx = H_j(r), \tag{2.20}$$

where $M(x)$ is defined by (2.16). Note that this notation is in agreement with our requirement that for $n = N, g$ does not depend on u.

Our task now stands as follows. Given any polynomial $H(r) = (H_1(r), \ldots, H_n(r))$ of degree k, satisfying $H(0) = 0$, we have to find a $g \in \Lambda$ such that (2.20) holds for any $r \in \mathbb{R}^n$.

Fix any such polynomial H. We define a function $g(x, u, y)$. By ii) the matrix $M(x)$ is regular at some point $x \in \Omega$. Since by elliptic regularity [Fr], the eigenfunctions $\varphi_1, \ldots, \varphi_n$ are smooth on $\overline{\Omega}$ (recall that L has smooth coefficients), $M(x)$ is regular for x in some subdomain $\Omega' \subset \Omega$. Let $M^{-1}(x)$ denote the inverse matrix. Put

$$g(x, z) = \begin{cases} \sum_{i=1}^n b_i(x) H_i(zM^{-1}(x)), \text{ for } x \in \Omega', z \in \mathbb{R}^n, \\ 0, \text{ for } x \in \Omega \backslash \Omega', z \in \mathbb{R}^n, \end{cases} \tag{2.21}$$

where

$$z = \begin{cases} (y_1, \ldots, y_N) \text{ if } u = N \\ (u, y_1, \ldots, y_N) \text{ if } u = N + 1, \end{cases}$$

and $b_i(x)$ are some C^∞-functions, with compact support in Ω', to be determined.

We claim that $g \in \Lambda$. Indeed, g is a polynomial in z of degree k and $g(x, 0) \equiv 0$. The coefficients of this polynomial are functions of x which vanish outside Ω'. On Ω' each of these coefficients is a product of a function $b_i(x)$ with several elements of $M^{-1}(x)$. Clearly, these products have compact support in Ω' and their regularly in X is determined by the regularity of $\varphi_1, \ldots, \varphi_n$. The latter being C^∞, we conclude that the coefficients of $g(x, z)$ are in $C^\infty(\overline{\Omega})$, hence they are certainly in $H^m(\Omega)$ (with m as in (2.17)). Finally we observe that g is independent of u if $n = N$. This shows that indeed $g \in \Lambda$.

Now we determine $b_1, \dots, b_n \in C_0^\infty(\Omega')$ such that (2.20) holds for all $r \in \mathbb{R}^n$ and $j = 1, \dots, n$. For g defined by (2.21), equality (2.20) reads as follows

$$\sum_{i=1}^{n} H_i(r) \int_\Omega \varphi_j(x) b_i(x) dx = H_j(r).$$

So in order to complete the proof it suffices to find $b_i \in C_0^\infty(\Omega')$ such that

$$\int_\Omega \varphi_j(x) b_i(x) dx = \delta_{ij}(\text{ the Kronecker symbol }). \tag{2.22}$$

To prove that such b_i exist, we first claim that the functions $\varphi_1, \dots, \varphi_n$ are linearly independent on Ω'. This follows easily from the unique continuation theorem [Mi]. Indeed, any notrivial linear combination of $\varphi_1, \dots, \varphi_n$ is again an eigenfunction of L and, as such, it cannot vanish on Ω' identically (for otherwise it must vanish on Ω contradicting linear independence of $\varphi_1, \dots, \varphi_n$ or Ω).

Using the linear independence of $\varphi_1, \dots, \varphi_n$, it is easy to see that the linear mapping

$$(b_1, \dots, b_n) \mapsto (\int_\Omega \varphi_j(x) b_i(x) dx)_{i,j=1}^n$$

is surjective from $\{L_2(\Omega')\}^n$ onto $\mathbb{R}^{n^2}$. Actually this mapping is surjective from the smaller space span $\{\varphi_1|_{\Omega'}, \dots, \varphi_n|_{\Omega'}\}$; this follows from the fact that the Gram matrix

$$\left(\int_{\Omega'} \varphi_i(x) \varphi_j(x) dx\right)_{i,j=1}^{n}$$

is regular. Now, since $C_0^\infty(\Omega')$ is dense in $L_2(\Omega')$ and the image of this continuous linear mapping is finite dimensional, the restriction of this mapping to the space $C_0^\infty(\Omega')$ must be surjective, as well. This shows that (2.22) holds for some functions $b_i \in C_0^\infty(\Omega')$. This shows that also the last property Pd) is satisfied. The proof is complete.

Some remarks are in order. First one concerns the property ii) of Lemma 2.1. It is not clear to us whether this property has to be assumed or it holds automatically. If for example $n = N = 2$, the latter is the case. Note that if $n = N$ then ii) is equivalent to functional independence of the eigenfunctions $\varphi_1, \dots, \varphi_n$. Important here is that these eigenfunctions correspond to one eigenvalue (otherwise they certainly need not be functionally independent). We will return to this problem in Section 4.

Now we discuss the possibility of proving a stronger result: realization of vector fields, rather than just finite jets. It would be nicer if we could prove that for g varying in some space of functions, the set of corresponding reduced vector fields $Pf(u_1 + \sigma(u_1, g), g)$ covers a neighbourhood of zero in a space of vector fields on X_1. An idea would be to apply the implicit function theorem to the mapping $g \mapsto Pf(u_1 + \sigma(u_1, g), g)$. One can indeed prove that the derivative of this mapping at $g = 0$ is surjective, provided the spaces are chosen suitably ($C^k - C^k$: C^k-norm for g, C^k-norm for fector fields). In [Ha 1,2], a similar observation led the author to the conclusion that an analogous nonlinear mapping in delay diffeential equations is locally surjective. There is a typical difficulty, however. This nonlinear mapping involves the composition of g with σ depending on g (f is the Nemitskii operator of g). Thus with the choice of spaces $C^k - C^k$, this mapping is not C^1. It is C^1 with the choice $C^k - C^{k-1}$, but this time the derivative of our mapping at $g = 0$ is not surjective. The range of this derivative consists of C^k vector fields, thus forms a proper dense subspace in C^{k-1}. Introducing a finite dimensional space (e.g. the space of k-jets as taken here) as the target space for our nonlinear mapping remedies the difficulty (a dense subspace in a finite dimensional space is the whole space).

Next remark is a preparation for the last result of this section. The proof of Lemma 2.1 shows that in order to realize any k-jet on X_1 (the kernel of L), it is sufficient to consider functions $g(x, u, y)$ which are polynomials in (u, y) of degree k. For $k = 1$ this has an interesting consequence, proving solvability of a linear inverse eigenvalue problem.

PROPOSITION 2.2. *Let $n = N$ or $n = N+1$ and let L satisfy the hypotheses of Lemma 2.1. Then there exists a neighbourhood $\mathcal{D}$ of 0 in the complex plane with the following property: For any n-tuple $\mu_1, \ldots, \mu_n \in \mathcal{D}$, satisfying the relations*

$$\mu_2 = \bar{\mu}_1, \ldots, \mu_{2l} = \bar{\mu}_{2l-1} \text{ and} \tag{2.23}$$

$$\mu_{2l+1}, \ldots, \mu_n \in \mathbf{R}, \tag{2.24}$$

there exist functions $a_0, a_1, \ldots, a_N \in C(\overline{\Omega})$ such that the differential operator

$$\tilde{L}u := Lu + a_0 u + \sum_{i=1}^{N} a_i u_{x_i}$$

subject to Dirichlet boundary conditions has $\mu_1, \ldots, \mu_n$ as eigenvalues.

Proof. The proof is a bend of some simple observations. First, as was remarked above, any jet in $J_0^1(X_1)$ can be realized in (1.1), (1.3), by adjusting a linear (in (u, y)) function $g(x, u, y)$.

Next we return to the abstract equation $(2.1)_\lambda$. We observe that if $f(u, \lambda)$ is linear in u, then the mapping $\sigma(u_1, \lambda)$, defining for λ near 0 an invariant manifold W_λ for $(2.1)_\lambda$ (see (2.4)), may be chosen linear in u_1. Indeed, the manifold W_λ can be chosen as an invariant subspace, close to X_1, of the linear operator

$$A(\lambda)w := Aw - f_u(\lambda, u)w.$$

(Recall that X_1 is an eigenspace of the unperturbed operator $A = A(0)$). It is a standard result that for λ near 0 such an invariant space exists [Ka, Theorems 2.14, 3.16 in Chapter

IV] and depends smoothly (if f is linear in λ, even analytically) on λ [Ka, Section VII.1]. Now, if $\sigma(u_1, \lambda)$ is linear in u_1, then so is the function $h(u_1, \lambda)$, representing the flow on W_λ (see (2.8), (2.9)). Therefore realization of the 1-jets of this function (by adjusting λ) actually means that the function $h(u_1, \lambda)$ itself can be realized (note that this property breaks down for k-jets, $k > 1$).

Finally, we observe that the restriction $A(\lambda)|W_\lambda$ to the invariant space W_λ has the same spectrum as the linear operator $u_1 \mapsto h(u_1, \lambda)$ (because these two operators define linearly conjugate flows).

Using these observations in conjunction with Lemma 2.1 we conclude that there is a neighbourhood $\mathcal{D}$ of 0 in $\mathbb{C}$ such that for any $\mu_1, \ldots, \mu_n \in \mathcal{D}$, satisfying (2.23), (2.24), we can adjust coefficients $a_0, \ldots, a_N \in C(\overline{\Omega})$ such that the operator $\tilde{L}$ subject to (1.3) has an invariant n-dimensional space $\tilde{X}_1$ and the spectrum of

$$\tilde{L}|_{\tilde{X}_1}$$

is given by the eigenvalues $\mu_1, \ldots, \mu_n$. Proposition 2.2 is proved. □

3. Multiple eigenvalues. In this section we find a linear operator which satisfies the hypotheses of Lemma 2.1.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^N$ be any smooth domain, such that $\overline{\Omega}$ is C^∞-diffeomorphic to the unit ball in $\mathbb{R}^N$. Let $u = N$ or $n = N + 1$. Then there exist smooth coefficients a_{ij}, a such that the differential operator L given by (1.2) has the properties i), ii) of Lemma 2.1.*

Lemma 3.1, in conjunction with Lemma 2.1, implies the assertion of Theorem 1. Later we will prove another more specific consequence for $n = N$.

In order to prove Lemma 3.1, we may without loss of generality assume that Ω is the unit ball

$$\Omega = B := \{x \in \mathbb{R}^n \,||x| = 1\}.$$

Indeed, if we find an operator L having the property i), ii) for $\Omega = B$ then we can pass to a general domain Ω using a change of coordinates $y = h(x)$: (h is a C^∞ diffeomorphism $\bar{B} \to \overline{\Omega}$). After such change of coordinates we obtain a differential operator $\tilde{L}_\Omega$ on Ω which clearly has an n-dimensional kernel (assuming Dirichlet boundary condition). The corresponding eigenspace is spanned by the functions

$$\psi_j(y) := \varphi_j(h^{-1}(y)), j = 1, \ldots, n,$$

where $\varphi_1, \ldots, \varphi_n$ span the kernel of L. It is easy to see that if $M(x)$ is the matrix defined by (2.23) and if $\tilde{M}(y)$ is defined in an analogous way for $\psi_1(y), \ldots, \psi_n(y)$ then

$$\det \tilde{M}(y) = c(y)M(h^{-1}(y)),$$

where $c^{-1}(y)$ is the Jacobian of $h^{-1}(y)$. (This is obvious for $n = N$, for $n = N+1$ expand $\det \tilde{M}(y)$ with respect to the first column). Thus if $M(x)$ is regular somewhere, then $\tilde{M}(y)$ is regular at some y. Finally we observe that the operator

$$L_\Omega := c^{-1}(y)\tilde{L}_\Omega$$

with Dirichlet boundary condition defines a self-adjoint operator on $L_2(\Omega)$ (just make a transformation in the scalar product integral). Hence L_Ω can be written in the divergence form (1.2). This follows from the explicit expression for the adjoint operator [Fr, He1] and from the fact that if two differential operators are equal (i.e., take the same values on their domain $H^2(\Omega) \cap H_0^1(\Omega)$) then the coefficients of their differential expressions are equal [Fr, Section 12]. Since L_Ω has the same kernel as $\tilde{L}_\Omega$, all requirements of Lemma 3.1 are met for L_Ω.

In the remaining part of this section we assume that $\Omega = B$ is the unit ball.

We prove Lemma 3.1 for $n = N$ and for $n = N+1$ separately.

Proof for $n = N$. We prove that for some eigenvalue $-\mu$ of the Laplace operator Δ, the operator $Lu := \Delta u + \mu u$ has the required properties. This is the content of the following lemma.

LEMMA 3.2. *Let μ be the least multiple eigenvalue of the eigenvalue problem*

$$\Delta u + \mu u = 0 \text{ on } \Omega, \tag{3.1}$$

$$u|_{\partial\Omega} = 0. \tag{3.2}$$

Then the differential operator

$$Lu := \Delta u + \mu u$$

has the properties i), ii) of Lemma 2.1.

Proof. In the proof we need some properties of eigenvalues and eigenfunctions of the Laplacian. For the reader convenience we start by recalling these properties. We introduce the spherical coordinates

$$x = rw, \quad r \in (0,1), \quad w \in S^{N-1},$$

(S^{N-1} is the unit sphere). In this coordinates, the problem (3.1), (3.2) takes the form

$$u_{rr} + \frac{N-1}{r}u_r + \frac{1}{r^2}\Delta_s u + \mu u = 0, \tag{3.3}$$

$$u|_{r=1} = 0 \tag{3.4}$$

where Δ_s is the spherical Laplacian [C-H, He 3]. It is a standard result [C-H] that the eigenvalues of (3.3), (3.4) form a sequence

$$\mu_{ml}, \quad m = 0, 1, \ldots, l = 1, 2, \ldots;$$

and for $\mu = \mu_{ml}$, the corresponding eigenspace is spanned by the functions

$$J_{ml}(r)v(w), \quad v(w) \in Y_m, \tag{3.5}$$

where J_{ml} is a nontrivial solution of

$$J_{rr} + \frac{N-1}{r} J_r + \left(\mu - \frac{m(m+N-2)}{r^2}\right) J = 0, \tag{3.6}$$

$$J(1) = 0, J - \text{regular at } r = 0, \tag{3.7}$$

and Y_m is the space of spherical harmonics of order m in N variables. By definition, Y_m consists of the restrictions to S^{N-1} of all harmonic polynomials on $\mathbb{R}^N$ of the degree m. Any elements of $Y_m \backslash \{0\}$ is an eigenfunction of Δ_s with the eigenvalue $-m(m + N - 2)$ (that's how the equation (3.6) comes out). [St].

Thus for a fixed m, the sequence

$$\mu_{m1} < \mu_{m2} < \ldots \tag{3.8}$$

is the sequence of the eigenvalues of the one dimensional problem (3.6), (3.7). Equivalently, $\sqrt{\mu_{m_1}}, \sqrt{\mu_{m_2}} \ldots$ is the sequence of all positive zeroes of a nontrivial solution of

$$J_{rr} + \frac{N-1}{r} J_r + \left(1 - \frac{m(m+N-2)}{r^2}\right) J = 0,$$

$$J \text{ regular at } 0.$$

Solutions of this equation (which is obtained from (3.6) via the transformation $r \to \sqrt{\mu}r$) can be handled similarly as the Bessel functions of integer order (i.e., the solutions in the case $N = 2$). Expanding the solutions in potential series and passing to functions J_{ml} via a corresponding inverse transformation $r \to \mu^{-\frac{1}{2}} r$, we find the values

$$J_{m1}(0) = 0, \text{ for } m = 1, 2, \ldots, \text{ and} \tag{3.9}$$

$$J'_{11}(0) \neq 0, J_{0l}(0) \neq 0, J'_{0l}(0) = 0, \text{ for } l = 1, 2, \ldots. \tag{3.10}$$

Since the first eigenfunction J_{m1} of (3.6), (3.7) has no zero in $(0, 1)$, using (3.9), and comparison arguments, we obtain

$$\mu_{01} < \mu_{11} < \mu_{21} < \ldots. \tag{3.11}$$

We also need the relation

$$\mu_{11} \neq \mu_{0l} \text{ for } l = 1, 2, \ldots. \tag{3.12}$$

To prove (3.10) one can use the following standard argument (for $N = 2$, (3.12) can be found in textbooks on Bessel functions) : If $\mu_{11} = \mu_{0l}$, then one proves, by differentiating (3.6), that $J_{11}(r)$ and $J'_{01}(r)$ solve the same equation. Since they are both regular at $0, J_{11}(r) = qJ'_{01}(r)$ for some constant q. From this and (3.7) we see that J_{01} has a double zero at $r = 1$, hence $J_{01} \equiv 0$, contradicting the fact that J_{01} is an eigenfunction.

Now we can complete the proof Lemma 3.2. From (3.8), (3.11), (3.12) it follows that μ_{11} has multiplicity N (the dimension of the space of spherical harmonics of the first order). The corresponding eigenfunctions (in Cartesian coordinates) are

$$\varphi_j(x) = J_n(r)\frac{x_j}{r}, r = |x|, j = 1, \ldots, N. \tag{3.13}$$

In fact μ_{1n} is the least multiple eigenvalue (the radial eigenvalues $\mu_{01}, \mu_{02}, \ldots$ are simple). So the operator $\Delta + \mu_{11}$ has the property i). It remains to be proved that the Jacobian $\mathfrak{J}(\varphi_1(x), \ldots, \varphi_N(x))$ does not vanish identically. In order to calculate this Jacobian we first change the coordinates:

$$\mathfrak{J}(\varphi_1(x), \ldots, \varphi_N(x)) = r^{1-N}\mathfrak{J}(\varphi_1(r, w), \ldots, \varphi_N(r, w)),$$

where $x = rw$ and the latter Jacobian is with respect to spherical coordinates. Taking into account the specific form (3.13) of $\varphi_1, \ldots, \varphi_N$, we find

$$\mathfrak{J}(\varphi_1(r, w), \ldots, \varphi_N(r, w)) = J'_{11}(r)\left(\frac{J_{11}(r)}{r}\right)^{N-1}\mathfrak{J}(x_1(r, w), \ldots, x_N(r, w))$$

where

$$\mathfrak{J}(x_1(r, w), \ldots, x_N(r, w)) = r^{N-1}$$

is the Jacobian of the transformation from spherical to Cartesian coordinates. Matching these equalities together, we obtain

$$\mathfrak{J}(\varphi_1(x), \ldots, \varphi_N(x)) = J'_{11}(r)\left(\frac{J_{11}(r)}{r}\right)^{N-1}, \tag{3.14}$$

which is certainly nonzero somewhere (J_{11} is not a constant). This completes the proof of Lemma 3.2 and the proof of Lemma 3.1 for $n = N$. $\square$

From Lemmas 2.1, 3.2 it follows that if Ω is the ball then any small finite jets on N-dimensional space can be realized in an equation of the following special form

$$u_t = \Delta u + \mu u + g(x, \nabla u), x \in \Omega. \tag{3.15}$$

We formulate this immediate consequence as

COROLLARY 3.3. *Let Ω be the unit ball in $\mathbf{R}^N$. Then there exists a constant μ such that the operator $\Delta+\mu$ has an N-dimensional kernel X_1 and the following property holds. For any integer $k>0$ there exists a neighbourhood $\mathcal{B}$ of 0 in $J_0^k(X_1)$ such that any jet in $\mathcal{B}$ can be realized in (3.15), (1.3) by adjusting the function $g=g(x,\nabla\mu)$.*

We now proceed in the proof of Lemma 3.1.

Proof for $n=N+1$. In seeking for an operator with $N+1$ dimensional kernel we use the following idea. Consider a differential operator on Ω which in spherical coordinates has the form

$$\mathsf{L}u = u_{rr} + \frac{N-1}{r}u_r + \frac{\gamma(r)}{r^2}\Delta_s u. \tag{3.16}$$

where $\gamma(r)$ is a smooth function on $(0,+\infty)$ such that $\gamma\equiv 1$ near 0. This operator subject to Dirichlet boundary condition has the same radial eigenvalues as the Laplacian. Nonradial eigenvalues, however, are affected by the choice of $\gamma(\cdot)$. Our goal is to find $\gamma(\cdot)$ such that the first nonradial eigenvalue coincides with a radial eigenvalue. This gives us an eigenvalue of multiplicity $N+1$. This goal is achieved in two steps. First we find $\gamma(\cdot)$ such that the first non radial eigenvalue is sufficiently large. In the next step we choose a path in space of admissible functions joining $\gamma(\cdot)$ to the constant function $\gamma=1$ (for which $\mathbb{L}$ is the Laplacian). Arguing by continuity we conclude that the first nonradial eigenvalue, which changes along the path, must meet some radial eigenvalue (which does not change with γ).

First observe that L defines a self-adjoint operator on $L_2(\Omega)$ (transform the coordinates in the scalar product integral). Thus in Cartesian coordinates, L takes the form (2.1). Also observe that L coincides with the Laplacian near the origin in $\mathbf{R}^N$ (because $\gamma\equiv 1$ near $r=0$).

In order to find the eigenvalues and the eigenfunctions of L one can use spherical harmonics in the same way as for the Laplacian. The eigenvalues form a sequence

$$\mu_{ml}, m=0,1,\dots,l=1,2,\dots$$

and for $\mu=\mu_{ml}$ the eigenfunctions have the form (3.5). The equation for $J_{ml}(r)$ now reads

$$J_{rr} + \frac{N-1}{r}J_r + \left(\mu - \frac{\gamma(r)m(m+N-2)}{r^2}\right)J = 0 \tag{3.17}$$

$$J(1)=0, J- \text{ regular at } 0. \tag{3.18}$$

Since $\gamma\equiv 1$ near $r=0$ (hence the transformation $r\mapsto\sqrt{\mu}r$ in (3.6) and (3.17) gives equations which are identical near 0), the evaluations (3.9), (3.10) remain valid. Hence, by comparison arguments, also (3.11) holds true. It follows that the eigenvalue μ_{11} has either multiplicity N (if $\mu_{11} \neq \mu_{0l}$ for $l = 1, 2, \dots$) or it has multiplicity $N+1$ (if $\mu_{11} = \mu_{0l}$ for some l). The latter is what we want to achieve.

Below we use the notation $\mu_{11}(\gamma)$ to emphasize the dependence of μ_{11} on γ.

We know that for $\gamma \equiv 1$ (L is the Laplacian) $\mu_{11}(1)$ has multiplicity N. Let l be the least integer such that

$$\mu_{11}(1) < \mu_{0l}. \tag{3.19}$$

such l certainly exists because $\mu_{0l} \to +\infty$ as $l \to \infty$.

We now find a γ_1 such that

$$\mu_{11}(\gamma_1) > \mu_{0l} \tag{3.20}$$

(Let us recall again that μ_{0l} is independent of γ). We claim that (3.20) holds if the inequality

$$\mu_{0l} - \frac{N-1}{r^2}\gamma_1(r) < 0 \tag{3.21}$$

is satisfied for all $r \in (0, 1)$. Indeed, if this is the case then the assumption $\mu_{0l} > \mu_{11}(\gamma_1)$ would imply

$$\mu_{11}(\gamma_1) - \frac{N-1}{r^2}\gamma_1(r) < 0.$$

For $J_{11}(r)$, as a solution of (3.17), (3.18) with $\mu = \mu_{11}(\gamma_1)$ and $m = 1$, this would clearly mean that it cannot achieve its positive maximum in $(0, 1)$. But since J_{11} can be assumed positive in $(0, 1)$ and since $J_{11}(0) = J_{11}(1) = 0$ (see (3.9), (3.18)), J_{11} has to achieve its positive maximum in $(0, 1)$. This shows that (3.21) indeed implies (3.20).

It is obvious that there exists a smooth function $\gamma_1(r)$ such that (3.21) holds and $\gamma_1(r) \equiv 1$ near 0. Fix such a $\gamma_1(r)$. Consider the homotopy

$$\gamma_t(r) := t + (1-t)\gamma_1(r).$$

For each $t \in (0, 1)$ we have $\gamma_t \equiv 1$ near 0 and $\gamma_0 \equiv 1$ on $(0, 1)$. Hence

$$\mu_{11}(\gamma_0) < \gamma_{0l} < \mu(\gamma_1).$$

Using the continuous dependence of $\mu_{11}(\gamma_t)$ on t we conclude that for some function $\gamma = \gamma_t, t \in (0, 1)$, we have

$$\mu_{11}(\gamma) = \mu_{0l}.$$

As we have already mentioned, for such γ, the eigenvalue $\mu_{11} := \mu_{11}(\gamma)$ has multiplicity $N+1$. The corresponding eigenfunctions in the Cartesian coordinates are

$$\varphi_1(x) := J_{0l}(r), \varphi_{1+j}(x) := J_{11}(r)\frac{x_j}{r}, r = |x|, j = 1, \dots, N. \tag{3.22}$$

We now calculate the determinant of the corresponding matrix $M(x)$. Again the first step in the calculation is the change of coordinates

$$
(3.23) \qquad M(x) = r^{1-N} \det \begin{pmatrix} \varphi_1(r,w) & \nabla_{r,w}\varphi_1(r,w) \\ \vdots & \vdots \\ \varphi_{N+1}(r,w) & \nabla_{rw}\varphi_{N+1}(r,w) \end{pmatrix},
$$

where $x = rw$ and $\nabla_{r,w}$ is the gradient with respect tot eh spherical coordinates. ((3.23) is obtained by expanding $M(x)$ with respect to the first column and by the change of coordinates in the resulting Jacobians). Now, since $\varphi_1(r,w) = J_{ol}(r)$ is independent of w, we have

$$
(3.24) \qquad M(x) = r^{1-N} \det \begin{pmatrix} J_{ol}(r) & J'_{ol}(r) \quad 0 \\ \vdots & \\ \varphi_{N+1}(r,w) & \tilde{M} \end{pmatrix}
$$

where $\tilde{M}$ is the $(N \times N)$ Jacobi matrix of $\varphi_2(r,w), \dots, \varphi_{N+1}(r,w)$. We can now use the calculations from the proof for $n = N$ to obtain

$$
\det \tilde{M} = \mathfrak{J}(\varphi_2(r,w), \dots, \varphi_{N+1}(r,w)) = r^{N-1} J'_{11}(r) \left(\frac{J_{11}(r)}{r} \right)^{N-1}
$$

Thus after expanding the determinant in (3.24) with respect to the first row we have

$$
(3.25) \qquad M(x) = J_{0l}(r) J'_{11}(r) \left(\frac{J_{11}(r)}{r} \right)^{N-1} - J'_{0l}(r) r^{1-N} D.
$$

where D stands for certain determinant. Using $J_{11}(0) = 0$ (see (3.9)) we see that D is of the class $0(r^{N-1})$ as $r \to 0$. Since we also have $J'_{0l}(0) = 0$ (see (3.10)), letting $r \to 0$ in (3.25) we find

$$
M(0) = J_{0l}(0)(J'_{11}(0))^N.
$$

Finally, referring to (3.10) again, we obtain

$$
M(0) \neq 0.
$$

We see that the operator L with a properly chosen γ has an $N+1$ dimensional eigenvalue μ and the matrix $M(x)$ for the corresponding eigenfunctions is not identically zero. Thus taking

$$
L = \mathsf{L} - \mu
$$

we obtain an operator with the required properties i), ii). The proof of Lemma 3.1 is complete. □

4. Problems. In this section we address some questions of interest which we cannot answer. First we return to the discussion of the hypothesis ii) of Lemma 2.1. We prove that if $u = N = 2$ then ii) is automatically satisfied. The arguments used breaks down for $N \geq 3$. We illustrate this on an example but leave unsolved the question whether ii) is always satisfied. Then we deal with some special cases of equation (1.1). We address a jet realization problem for such equations.

In Section 2 we have remarked that if $n = N = 2$ then any operator L, which has a two-dimensional kernel span $\{\varphi_1, \varphi_2\}$, has automatically the property ii) of Lemma 2.1, i.e., the Jacobian $\mathfrak{J}(\varphi_1(x), \varphi_2(x))$ does not vanish identically. This is a consequence of the following more general statement.

PROPOSITION 4.1. *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain and let $S \subset \overline{\Omega}$ be a C^2-curve. Let*

$$\mathsf{L}u = \sum_{i,j=1}^{2} \alpha_{ij} u_{x_i x_j} + \alpha_1 u_{x_1} + \alpha_2 u_{x_2} + \alpha_0 u$$

be any strongly elliptic operator with C^1(on $\overline{\Omega}$) coefficients. Assume that the functions $\varphi_1, \varphi_2 \in C^2(\overline{\Omega})$ are linearly independent on Ω and such that

$$\mathsf{L}\varphi_j = 0 \text{ on } \Omega, \text{ and} \tag{4.1}$$

$$\varphi_j|_s = 0, \text{ for } j = 1, 2. \tag{4.2}$$

Then $\mathfrak{J}(\varphi_1(x), \varphi_2(x))$ does not vanish identically on Ω.

Proof. Suppose that $\mathfrak{J}(\varphi_1(x), \varphi_2(x)) \equiv 0$ on Ω (hence on $\overline{\Omega}$). Then for each $x \in \overline{\Omega}$ we have

$$\rho(x) := \text{ rank } \begin{pmatrix} \nabla\varphi_1(x) \\ \nabla\varphi_2(x) \end{pmatrix} \leq 1.$$

We distinguish two possibilities:

a) $\rho(x) = 0$ for all $x \in S$.

b) $\rho(x_0) = 1$ for some $x_0 \in S$.

In both cases we find a contradiction. If a) holds then $\nabla(\varphi_1(x) \equiv 0$ on S. Hence

$$\frac{\partial \varphi_1}{\partial \eta} \equiv 0 \text{ on } S, \tag{4.3}$$

where η is any vector field on S. By (4.1) - (4.3) and uniqueness for the Cauchy problem [Mi], $\varphi_1 \equiv 0$ in a neighbourhood of S. Consequently $\varphi_1 \equiv 0$ on Ω, by the unique continuation theorem [Mi]. This contradicts linear independence of φ_1, φ_2.

Now assume that b) holds. Then $\rho(x) \equiv 1$ for x in some neighbourhood of x_0 in $\overline{\Omega}$. Let e.g. $\nabla\varphi_1(x_0) \neq 0$. A corollary to the implicit function theorem now tells us that there exists a C^1-function $F : \mathbf{R} \to \mathbf{R}$ such that

$$\varphi_2(x) \equiv F(\varphi_1(x))$$

for x in a neighbourhood U of x_0 in $\overline{\Omega}$. This in conjunction with (4.2) implies

$$\nabla\varphi_2(x) = F'(0)\nabla\varphi_1(x), \text{ for all } x \in U \cap S.$$

It follows that the function

$$\varphi := \varphi_2 - F'(0)\varphi_1$$

satisfies

$$\nabla\psi(x) = 0 \text{ on } U \cap S.$$

Since ψ also solves (4.1), (4.2), by the arguments from a), we obtain

$$\psi \equiv 0 \text{ on } \Omega$$

contradicting linear independence of φ_1, φ_2.

Thus both cases a) and b) lead to a contradiction, which shows that $\mathfrak{J}(\varphi_1(x), \varphi_2(x))$ cannot vanish identically. □

In the above proof, it was important that if rank of the Jacobi matrix of φ_1, φ_2 equal 1 at some point $x_0 \in S$ then this rank is constant in vicinity of x_0. This property does not hold if we have N functions on an N-dimensional domain $\Omega, N > 2$. For in this case the Dirichlet boundary condition forces the rank on S to be at most 1, while vanishing Jacobian on Ω only implies that the rank on Ω is at most $N - 1 \geq 2$. It is not difficult to find a counter example disproving Lemma 4.1 in dimension $N > 2$. Just take an operator L on the unit disk B which has a 3-dimensional kernel span $\{\varphi_1, \varphi_2, \varphi_3\}$ (such operator exists by Lemma 3.1) and define an operator $\tilde{L}$ on the 3-dimensional domain $\Omega := B \times (-1, 1)$ by

$$\tilde{L} := L + \frac{\partial^2}{\partial x_3^2}.$$

Clearly

$$\begin{aligned} \tilde{L}\varphi_j &= 0 \text{ on } \Omega \\ \varphi_j &= 0 \text{ on } \partial B \times (-1, 1), j = 1, 2, 3. \end{aligned}$$

However, the Jacobian of $\varphi_1, \varphi_2, \varphi_3$ vanish identically because these three functions depend only on two variables x_1, x_2.

This example shows that if the property of identically vanishing Jacobian is to be excluded, local arguments, as those in the proof of Proposition 4.7, are not sufficient, and one has to use the fact that the eigenfunctions are zero on the whole boundary of Ω.

The next comments concern the linear operator L. Our method requires that L has an n-dimensional kernel ($n = N$ or $n = N + 1$). For $n = N$, we have proved that on a general domain Ω such operator can be found by transforming the Laplacian on the ball. The transformation gives us an operator L of the divergence form with some smooth coefficients. It would be of more interest to prove that this inverse eigenvalue problem (finding an operator with an eigenvalue of multiplicity n) can be solved in a more special class of operators. For example, one could think of the class

$$Lu = \Delta u + a(x)u,$$

where only potential $a(\cdot)$ is allowed to vary. If such an operator L existed and if it also had the property ii) of Lemma 2.1, then in realization of finite jets on $\mathbb{R}^n$ one could stay within the class of equations

$$u_t = \Delta u + g(x, u, \nabla u), x \in \Omega \tag{4.4}$$

$$u|_{\partial\Omega} = 0. \tag{4.5}$$

Even more interesting is the question what kind of dynamics can be detected in (4.4), (4.5) if $g = g(u, \nabla u)$ does not depend explicitly on the spatial variable x. While we have shown that in spatially dependent problem (4.4), (4.5) complicated dynamics can occur, at least if Ω is the ball in $\mathbb{R}^N, N > 2$, we have nothing to support our expectation that (4.4), (4.5) with $g = g(u, \nabla u)$ can also provide dynamically interesting phenomena.

REFERENCES

[Am] H. AMANN, *On abstract parabolic fundamental solutions*, J. Math. Soc. Japan 39 (1987), pp. 93-116.

[An] S.B. ANGENENT, *The Morse-Smale property for a semilinear parabolic equation*, J. Diff. Eq. 62 (1986), pp. 427-442.

[B-V] A.V. BABIN AND M.I. VISHIK, *Regular attractors of semigroups and evolution equations*, J. Math. Pures et Appl. 62 (1983), pp. 441-491.

[Be] M.S. BERGER, *Nonlinearity and Functional Analysis*, Academic Press, New York 1977.

[Bi] Y.N. BIBIKOV, *Bifurcation of invariant tori for systems with degeneracy*(Russian), in "Questions of the Qualitative Theory of Differential Equations", V.M. Matrosov and L.Y. Anapolskii (eds.), Nauka, Novosibirsk (1988), pp. 4-8.

[B-C] P. BRUNOVSKÝ AND S.N. CHOW, *Generic properties of stationary state solutions of reaction-diffusion equations*, J. Diff. Eq., 53 (1984), pp. 1-23.

[B-F 1] P. BRUNOVSKÝ AND B. FIEDLER, *Connecting orbits in scalar diffusion equation*, in "Dynamics Reported", vol 1, V. Kirchgraber and H.O. Walther (eds.), Wiley & Sons (1988), pp. 57-89.

[B-F 2] P. BRUNOVSKÝ AND B. FIEDLER, *Connecting orbits in scalar reaction-diffusion equations II: The complete solution*, J. Diff. Eq., 81 (1989), pp. 106-136.

[B-P] P. BRUNOVSKÝ AND P. POLÁČIK, *Generic hyperbolicity for reaction diffusion equations on symmetric domains*, J. Applied Math. Physics (ZAMP), 38 (1987), pp. 172-182.

[C-L-S] T.F. CHEN, H.A. SEVINE AND P.E. SACKS, *Analysis of a convective reaction diffusion equation*, J. Nonlinear Analysis, 11 (1987), pp. 593-597.

[Ch-H] S.N. CHOW AND J.K. HALE, *Methods of Bifurcation Theory*, Springer-Verlag, New York 1982.

[Ch-L] S.N. CHOW AND K. LU, *Invariant manifolds for flows in Banach spaces*, J. Diff. Eq., 81 (1988), pp. 285-317.

[Ch-L 2] S.N. CHOW AND K.LU, *Smooth invariant foliations for flows in Banach spaces*, in preparation.

[Co-H] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, Interscience, New York 1953.

[F-P] P. FIEDLER AND P. POLÁČIK, *Complicated dynamics of scalar reaction diffusion equations with a nonlocal term*, preprint.

[Fr] A. FRIEDMAN, *Partial Differential Equations*, Holh, Rinehart & Winston, New York 1969.

[F-R] G. FUSCO AND C. ROCHA, *A permutation related to the dynamics of a scalar parabolic P.D.E.*, preprint.

[G-G] M. GOLUBITSKY AND V. GUILLEMIN, *Stable mappings and their Singularities*, Springer-Verlag, New York (1973).

[Gu 1] J. GUCKENHEIMER, *On a codimension two bifurcation*, in "Dynamical Systems and Turbulence", D.A. Rand and L.S. Young (eds.), Lecture Notes in Mathematics 898, Springer-Verlag, New York (1987), pp. 99-142.

[Gu 2] J. GUCKENHEIMER, *Multiple bifurcation problems of codimension two.*

[G-H] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York 1983.

[Ha 1] J.K. HALE, *Flows on centre manifolds for scalar functional differential equations*, Pro. Royal Soc. Edinburgh, 101A (1985), pp. 193-201.

[Ha 2] J.K. HALE, *Local flows for functional differential equations*, Contemporary Math. vol 56, Amer. Math. Soc, Providence (1986), pp. 185-192.

[He 1] D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer-Verlag, New York 1981.

[He 2] D.B. HENRY, *Some infinite-dimensional Morse-Smale systems defined by parabolic equations*, J. Diff. Eq., 59 (1985), pp. 165-205.

[He 3] D.B. HENRY, *Topics in Nonlinear Analysis*, Travalho de Matematica 192, Univ. Brasilia 1982.

[He 4] D.B. HENRY, *Generic properties of equilibrium solutions by perturbation of the boundary*, in "Dynamics of Infinite Dimensional Systems", S.N. Chow and J.K. Hale (eds.), Springer-Verlag, Berlin Heidelberg 1987.

[Hi 1] M.W. HIRSCH, *Differential equations and convergence almost everywhere in strongly monotone semi-flows*, in Contemporary Math. vol 17, Amer. Math. Soc., Providence (1983), pp. 267-295.

[Hi 2] M.W. HIRSCH, *Stability and convergence in strongly monotone dynamical systems*, J. Reine Angew. Math., 383 (1988), pp. 1-53.

[Ka] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1966.

[La] W.F. LANGFORD, *Periodic and steady state mode interaction lead to hori*, SIAM J. Appl. Math., 37 (1979), pp. 22-48.

[L-I] W.F. LANGFORD AND G. IOOSS, *Interactions of Hopf and pitchfork bifurcations*, in "Bifurcation Problems and their Numerical Solution", H.D. Mithelman and H. Weber (eds.), Birkhauser Verlag Basel 1980.

[L-P-S-S] H.A. LEVINE AND L.E. PAYNE AND P.E. SACKS AND B. STRANGHAM, *Analysis of a convective reaction-diffusion equation II*, SIAM Math. Analysis, 20 (1989), pp. 133-147.

[M-M] J.E. MARSDEN AND M. MCCRACKEN, *The Hopf Bifurcation and its Applications*, Springer-Verlag, New York 1976.

[Ma 1] H. MATANO, *Asymptotic behaviour and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci., 15 (1979), pp. 401-454.

[Ma 2] H. MATANO, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ, 18 (1978), pp. 221-227.

[Ma 3] H. MATANO, *Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems*, J. Fac. Sci. Univ. Tokyo, 30 (1984), pp. 645-673.

[Ma 4] H. MATANO, *Asymptotic behaviour of solutions of semilinear heat equations on* S^1, in "Nonlinear Diffusion Equations and their Equilibrium States, B Peletier and J. Serrin (eds.), Springer-Verlag, New York (1988), pp. 139-162.

[Mi] C. MIRANDA, *Partial Differential Equations of Elliptic type*, springer-Verlag, Berlin, Heidelberg 1970.

[Pa] K.J. PALMER, *Exponential dichotomies, the shadowing lemma and transversal homoclinic joints*, in "Dynamics Reported", vol 1, V. Kirchgraber and M.O. Walther (eds.), Willey & Sons (1988), pp. 265-306.

[Po 1] P. POLÁČIK, *Generic hyperbolicity in one-dimensional reaction diffusion equations with general boundary conditions*, J. Nonlinear Analysis, 11 (1987), pp. 593-597.

[Po 2] P. POLÁČIK, *Convergence in smooth strongly monotone flows defined by semilinear parabolic equations*, J. Diff. Eq., 79 (1989), pp. 89-110.

[Po 3] P. Poláčik, *Generic properties of strongly monotone semiflows defined by ordinary and parabolic differential equations*, in " Proceedings of the Third Colloquium on the Qualitative Theory of Differential Equations", L. Halvani (ed.), to appear.

[P-W] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Englewood Cliffs, NJ (1967).

[Ro] C. Rocha, *Generic properties of equilibria of reaction-diffusion equations with variable diffusion*, Proc. Royal Soc. Edinburgh, 101A (1985), pp. 45-55.

[Si] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Annals of Math., 118 (1983), pp. 525-571.

[Sm] S. Smale, *On the differential equations of species in competition*, J. Math. Biol, 3 (1976), pp. 5-7.

[St] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton 1970.

[Tr] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam 1978.

[Ze] T.J. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equations with one space variable*, Diff. Eq., 4 (1968), pp. 17-22.

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