Logarithmic lower bounds for Néel walls

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Abstract

Most mathematical models for interfaces and transition layers in materials science exhibit sharply localized and rapidly decaying transition profiles. We show that this behavior can largely change when non-local interactions dominate and internal length scales fail to be determined by dimensional analysis: we consider a reduced model for micromagnetic Néel walls which are observed in suitable regime of thin films. The typical phenomenon associated with this wall type is the very long logarithmic tail of transition profiles. Logarithmic upper bounds were recently derived by the author. In the present article we prove that the latter result is indeed optimal. In particular, we show that Néel wall profiles are supported by explicitly known comparison profiles that minimize relaxed variational principles and exhibit logarithmic decay behavior. This lower bound is established by a comparison argument based on a global maximum principle for the non-local field operator and the qualitative decay behavior of comparison profiles.

1 Introduction and statement of results

We are concerned with the following non-convex and non-local variational principle that emerges in the mathematical treatment of Néel walls, an important class of micromagnetic domain walls in the regime of thin films:

$$\begin{aligned} \mathcal{I}_\varepsilon(m) = \varepsilon \, \big\|m\big\|_{\dot{H}^1}^2 + \big\|u\big\|_{\dot{H}^{1/2}}^2 + \big\|u\big\|_{L^2}^2 \to \min \\ m = (u,v) : \mathbb{R} \to \mathbb{S}^1 \quad \text{with} \quad u(0) = 1 \quad \text{and} \quad v(\pm\infty) = \pm 1 \end{aligned} \tag{1}$$

Here, ε is supposed to be a small parameter related to the relative strength of anisotropy that favors the end-states to be attained at $\pm\infty$. A minimizer m_ε describes an energetically optimal in-plane rotation of magnetization vectors. These, so-called Néel walls, received remarkable attention in physics literature, see e.g. [6], [12], [13]. As one of the most important features, logarithmic tails of transition profiles were predicted by formal arguments in order to explain long-range

interactions between different Néel walls, when neighboring tails overlap.

The main analytical feature of (1) is that the energy only gives uniform control in $H^{1/2}(\mathbb{R})$ that just fails to control pointwise bounds and continuity. Thus the combination of fading H^1-control with pointwise constraints is delicate and suggests a logarithmic singularity at least in a renormalized setting.

Recently, logarithmic scaling for minimal energies of (1) in terms of the small anisotropy parameter ε was established using a perturbation argument, see [3]. A refined version is announced in [4]:

$$\inf \mathcal{I}_\varepsilon = \pi \, \log(1/\varepsilon)^{-1} \left(1 + o(1)\right) \quad \text{as} \quad \varepsilon \to 0,$$

where the infimum taken over all fields which are admissible according to the constraints in (1). Similar scaling laws where derived in case of periodic arrays (where the tails are confined by those of neighboring walls) that in particular serve to quantify the repulsive force between Néel wall tails, see [5].

At the same time, a finer analysis for minimizers of (1) was conceived in order to capture the logarithmic tails of profiles u_ε in a pointwise sense. In [8] we studied the blow up of L^p-norms in terms of the minimal energy when $p \to \infty$. These estimates imply in particular a pointwise logarithmic upper bound and allow to derive a singular limit and to identify a somehow universal profile in a renormalized setting.

The intention of this note is twofold: firstly we intend to show the remarkable fact that relaxation of (1) provides, in addition to the trivial energetic lower bound, a pointwise lower bound for the profile. Secondly, since the relaxation can easily be identified with a linear problem having an explicit solution that exhibits the typical logarithmic behavior, we prove the optimality of the pointwise upper bound in [8].

To demonstrate the effect of the non-local contribution, we consider the local counterpart, omitting the $H^{1/2}$-interaction, that formally agrees with the micromagnetic Bloch wall, see e.g. [6]. The corresponding variational principle can be expressed in terms of the second component function v.

$$\mathcal{G}_\varepsilon(v) = \varepsilon \int \frac{|v'|^2}{1-v^2} \, dx + \int \left(1 - v^2\right) \, dx \to \min \tag{2}$$
$$v : \mathbb{R} \to [-1, 1] \quad \text{with} \quad v(0) = 0 \quad \text{and} \quad v(\pm\infty) = \pm 1.$$

The latter variational principle is closely related to the (classical) transition problem that arises from the Modica-Mortola example, see [11], [1]. In fact, after scaling, by a standard equipartition argument, it is easy to see that the optimal profile

v for $\varepsilon = 1$ is associated with the first order problem

$$v' = 1 - v^2 \quad \text{with} \quad v(0) = 0.$$

This Cauchy problem is solved by $v(x) = \tanh(x)$ which approaches the end-states $v = \pm 1$ exponentially fast as x goes to $\pm\infty$, respectively. We recover, after rescaling, all solutions $v_\varepsilon(x) = \tanh\left(x/\sqrt{\varepsilon}\right)$ for $\varepsilon > 0$ and corresponding profiles $u_\varepsilon(x) = \operatorname{sech}\left(x/\sqrt{\varepsilon}\right)$.

In contrast, due to the presence of three energy components with different scaling behavior, the internal length scales of the Néel wall problem are more complex and cannot be determined by dimensional analysis. Our argument shows for optimal profiles the existence of a *uniform* logarithmic tail beyond a core region of order ε close to the origin:

Theorem 1. *Let u_ε be a minimizing profile for the variational principle (1). Then u_ε exhibits a logarithmic tail in the sense that*

$$u_\varepsilon(x) \approx \frac{\log(1/x)}{\log(1/\varepsilon)} \quad \textit{for all} \;\; \varepsilon < x < 1/e \;\; \textit{and} \;\; 0 < \varepsilon < 1/4.$$

The upper bound can be derived by arguments as conceived in [8] in the context of finite films, i.e. the intermediate regime of finite thickness, but the arguments carry over in a simplified manner. We will give a brief review in section 5. The main result of the present article is the lower bound. It is based on comparison with the solution of the following variational principle which turn out to be the relaxation for (1):

$$\mathcal{I}_\varepsilon(u) = \varepsilon \left\|u\right\|^2_{\dot{H}^1} + \left\|u\right\|^2_{\dot{H}^{1/2}} + \left\|u\right\|^2_{L^2} \to \min \quad \text{for} \quad u(0) = 1. \tag{3}$$

The idea is to derive suitable pseudo-differential inequalities for the profiles to be compared. The essential ingredient is the convexity of comparison profiles away from zero. Then as a consequence of a global maximum principle for the non-local field operator $(-\Delta)^{1/2}$ we obtain our main result.

Theorem 2. *Let u_ε be a minimizing profile for the variational principle (1). Then the unique solution $\bar{u}_\varepsilon$ of the relaxed variational principle (3) is a pointwise lower bound.*

Then the lower bound in Theorem 1 is a direct consequence of the well-known logarithmic decay behavior of the solutions and the energetics of the linear problem (3), see [3], [4], and [13], that can be stated the following way:

Proposition 1. *The unique solutions $\bar{u}_\varepsilon$ of the relaxed variational principles (3) exhibit logarithmic tails in the sense that for $0 < \varepsilon < 1/4$*

$$\bar{u}_\varepsilon(x) = \frac{\Lambda(\varepsilon)}{\pi}\left[\log(1/x) + r_\varepsilon(x)\right] \quad \textit{for} \quad \varepsilon < x < 1,$$

where the functions $r_\varepsilon(x)$ are uniformly bounded in the above regime. The factor $\Lambda(\varepsilon)$ agrees with the minimal energy and has the asymptotic behavior

$$\Lambda(\varepsilon) = \pi\left(1 + o(1)\right)\log(1/\varepsilon)^{-1} \quad \textit{as} \quad \varepsilon \to 0.$$

Moreover the profiles $\bar{u}_\varepsilon$ have quadratic decay for large x

$$\bar{u}_\varepsilon(x) = \frac{\Lambda(\varepsilon)}{\pi}\left[x^{-2} + s_\varepsilon(x)\right] \quad \textit{as} \quad |x| \to \infty,$$

where $|s_\varepsilon(x)| \le c\,|x|^{-4}$ for some universal constant c independent of ε.

This behavior can be shown by elaborate expansions, see [3]; for convenience we will give a brief motivation from the Fourier picture in Section 4.

2 Relaxation and Euler-Lagrange equations

The variational principles (1) is non-convex through the pointwise constraint $|m| = 1$ and non-local through the presence of the energy contribution

$$\|u\|^2_{\dot{H}^{1/2}} = \frac{1}{4\pi}\int\int \frac{|u(x) - u(y)|^2}{|x-y|^2}\,dy\,dx$$

or in Fourier representation $\|u\|^2_{\dot{H}^{1/2}} = \int |\xi|\,|\hat{u}(\xi)|^2\,d\xi$. This energy contribution induces the non-local field operator $(-\Delta)^{1/2}u = \mathcal{F}^*\big(\xi \mapsto |\xi|\,\hat{u}(\xi)\big)$, where $\mathcal{F}^*$ denotes the inverse Fourier transform.

Relaxation amounts to convexifying the constraint $|m| = 1$. But when substituting the saturation constraint $|m| = 1$ in (1) by the convex constraint $|m| \le 1$, the second component of a minimizer necessarily vanishes. Thus the relaxed variational principle is essentially scalar.

Definition. *For $u \in H^1(\mathbb{R})$, let $\mathcal{I}_\varepsilon(u) := \mathcal{I}_\varepsilon(m) < \infty$ where $m = (u, 0)$.*

Next we show that relaxation of the variational principle leads to linear Euler-Lagrange equations. This is accomplished by a symmetrization argument. Recall that, for a non-negative H^1-function u, symmetric rearrangement $u \mapsto u^*$ (see [7]

Chapter 3) preserves the L^2-norm and decreases the Dirichlet energy. The non-local energy component has a strict rearrangement property, see e.g. [7] Lemma 7.17,

$$\|u^*\|_{\dot{H}^{1/2}} \leq \|u\|_{\dot{H}^{1/2}}$$

with equality if and only if u^* is a translation of u. Thus we have the following strict rearrangement property of the entire energy:

Lemma 2.1. *Let $u \in H^1(\mathbb{R})$ be non-negative and let u^* be its symmetrically decreasing rearrangement, then*

$$\mathcal{I}_\varepsilon(u^*) \leq \mathcal{I}_\varepsilon(u),$$

and the inequality is strict unless u is a translation of u^*.

Since $u \mapsto |u|$ decreases energy, one easily infers (arguing by contradiction) that minimizing profiles for the relaxed variational principle are necessarily non-negative and symmetrically decreasing. Hence the residual side condition $|u| \leq 1$ is automatically fulfilled, and relaxation of the variational principle (1) can be written as

$$\mathcal{I}_\varepsilon(u) = \varepsilon \left\|u\right\|^2_{\dot{H}^1} + \left\|u\right\|^2_{\dot{H}^{1/2}} + \left\|u\right\|^2_{L^2} \to \min \quad \text{in} \quad \{u(0) = 1\}. \tag{4}$$

By uniform convexity of the functional restricted to the affine space $\{u(0) = 1\} \cap H^1(\mathbb{R})$ there is a unique minimizer $\bar{u}_\varepsilon$ for each $\varepsilon > 0$. Moreover we have the following linear Euler-Lagrange equation

$$\varepsilon\,(-\Delta)\,\bar{u} + (-\Delta)^{1/2}\bar{u} + \bar{u} = \Lambda(\varepsilon)\,\delta_0 \quad \text{in} \quad \mathcal{D}'\,(\mathbb{R}) \tag{5}$$

with Lagrange multiplier $\Lambda(\varepsilon) = \inf\left(\mathcal{I}_\varepsilon(u), u(0) = 1\right)$.

Let us return to the original situation. The variational principle (1) gives rise to the following Euler-Lagrange system

$$d\mathcal{I}_\varepsilon(m) = d\mathcal{I}_\varepsilon(m)\, m \otimes m. \tag{6}$$

Regarding our comparison argument it is crucial to consider a scalar version that is more comparable with (5), instead of the Euler-Lagrange system (6) that we extensively used in [8]. Such equations can be derived using admissible variations of a minimizing profile u, i.e. variations of the form $u_s = u + s\varphi$, where φ is compactly supported in the open set $\{|u| < 1\}$ and s is sufficiently small. For a variation field $m_s : \mathbb{R} \to \mathbb{S}^1$ with $m_s \cdot \mathbf{e}_1 = u_s$ we have

$$\int |m_s'|^2 \, dx = \int \frac{|u_s'|^2}{1 - u_s^2} \, dx$$

Carrying out the differentiation with respect to s we obtain the equation

$$\mathcal{R}_\varepsilon(u) + (-\Delta)^{1/2} u + u = 0 \quad \text{in} \quad \big\{|u| < 1\big\} \tag{7}$$

with the nonlinear singular perturbation

$$\mathcal{R}_\varepsilon(u) = \varepsilon \left\{ -\frac{d}{dx}\left(\frac{u'}{1-u^2}\right) + \left(\frac{u'}{1-u^2}\right)^2 u \right\}. \tag{8}$$

For a sufficiently smooth profile u, we can carry out the differentiation in (8) and get

$$\mathcal{R}_\varepsilon(u) = \varepsilon \left\{ -\frac{u''}{1-u^2} - \left(\frac{u'}{1-u^2}\right)^2 u \right\}. \tag{9}$$

Surprisingly, the only knowledge about the full profile that we need to establish a lower bound is the following monotonicity property which can be deduced from a slight generalization of the strict rearrangement inequality in Lemma 2.1, see [8].

Proposition 2. *Minimizing profiles for the variational principle (1) are smooth and symmetrically decreasing.*

3 Maximum principle for $(-\Delta)^{1/2}$

From the micromagnetic model we have a representation of $(-\Delta)^{1/2}$ as a limit of zero order operators $\mathcal{S}_\delta$, called reduced stray-field operator. The parameter δ corresponds to the relative thickness of the magnetic layer. The operator $\mathcal{S}_\delta$ is defined by the following tangential average of the (physical) magnetostatic field

$$\mathcal{S}_\delta u = \frac{1}{\delta} \fint_{-\delta}^{\delta} \mathcal{H}(\mathbf{m}) \cdot \hat{e}_1 \, dz \tag{10}$$

where $\mathcal{H}(\mathbf{m})$ denotes the 3D-Helmholtz projection of the physical magnetization field

$$\mathbf{m}(x,y,z) = (u(x), v(x), 0)\ \chi_{[-\delta,\delta]}(z),$$

the re-embedding of $m = (u,v) : \mathbb{R} \to \mathbb{S}^1$ into the 3D context. One finds by a standard calculation, see e.g. [3], [8], that in frequency space the operator $\mathcal{S}_\delta$ is given by the self-similarly rescaled multiplier $\sigma_\delta(\xi) = \frac{1}{\delta}\,\sigma(\delta\,\xi)$, where the function $\sigma : \mathbb{R} \to \mathbb{R}$ reads like

$$\sigma(\xi) = 1 - \frac{1-\exp(-2|\xi|)}{2|\xi|} \sim \begin{cases} |\xi| & \text{for small } \ \xi \\ 1 & \text{for large } \ \xi. \end{cases}$$

This symbol is regular in the sense that it is the composition of the modulus with a real analytic function, and it exhibits the following growth property

$$|\partial_\xi^\alpha \sigma(\xi)| \leq c(\alpha)\ (1+|\xi|)^{-\alpha} \quad \text{for all} \quad \xi \in \mathbb{R}\setminus\{0\} \quad \text{and} \quad \alpha \geq 0. \tag{11}$$

It is easy to see that $\sigma_\delta(\xi) \leq |\xi|$ and $\sigma_\delta(\xi) \to |\xi|$ pointwise as $\delta \to 0$, see [8]. Thus, by dominated convergence and Plancherel we observe that $\mathcal{S}_\delta$ converges towards $(-\Delta)^{1/2}$ in the sense that for every $u \in H^1(\mathbb{R})$

$$\mathcal{S}_\delta u \to (-\Delta)^{1/2} u \quad \text{strongly in} \quad L^2(\mathbb{R}). \tag{12}$$

Computation of the inverse Fourier transform of σ_δ yields the convolution kernel representation of the operator $\mathcal{S}_\delta$

$$\mathcal{S}_\delta u = \frac{1}{\delta}\left(u - \mathcal{K}_\delta * u\right). \tag{13}$$

Explicitly we have the following (strictly) symmetrically decreasing kernel function of unit integral, which can be found e.g. in [3] and [13]

$$\mathcal{K}(x) = \frac{1}{4\pi}\,\log\left(1+\frac{4}{x^2}\right) \quad , \quad \mathcal{K}_\delta(x) = \frac{1}{\delta}\,\mathcal{K}\left(\frac{x}{\delta}\right).$$

We observe that for fixed $\delta > 0$ the reduced stray field operator $\mathcal{S}_\delta$ is bounded on the space of bounded continuous functions. Since trivially a global maximum always dominates the average, we deduce from (13) the following maximum principle.

Lemma 3.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be bounded and continuous, and suppose that u attains a global maximum at $x_0 \in \mathbb{R}$. Then $\mathcal{S}_\delta u$ is continuous and $(\mathcal{S}_\delta u)(x_0) \geq 0$. Moreover the inequality is strict unless u is constant.*

A maximum principle for the operator $(-\Delta)^{1/2}$ can be derived from the maximum principle of the convolution operator $\mathcal{S}_\delta$ by means of a regularity argument.

Lemma 3.2. *Suppose that the function $u \in H^1(\mathbb{R})$ is locally H^2-regular in $\mathbb{R}\setminus\{0\}$. Then as δ tends to zero*

$$\mathcal{S}_\delta u(x) \to (-\Delta)^{1/2} u(x) \quad \textit{locally uniformly in} \quad \mathbb{R}\setminus\{0\},$$

in particular, the function $(-\Delta)^{1/2} u$ is continuous in $\mathbb{R}\setminus\{0\}$.

Proof. Let φ be any smooth cutoff function with compact support outside the origin. Then by (12) the functions $\varphi\,\mathcal{S}_\delta u$ converges to $\varphi\,(-\Delta)^{1/2}u$ in $L^2(\mathbb{R})$. We show that $\varphi\,\mathcal{S}_\delta u$ is uniformly bounded in $H^1(\mathbb{R})$, which proves that $\varphi\,\mathcal{S}_\delta u$ converges uniformly. Using the bounds on the operator $\mathcal{S}_\delta$ and the commutator estimate below we get

$$\begin{aligned}\big\|\varphi\,\mathcal{S}_\delta u\big\|_{\dot H^1} &\leq \big\|\mathcal{S}_\delta(\varphi\,u)\big\|_{\dot H^1} + \big\|[\varphi',\mathcal{S}_\delta]u\big\|_{L^2} + \big\|[\varphi,\mathcal{S}_\delta]u'\big\|_{L^2}\\ &\leq \|\varphi\,u\|_{\dot H^2} + c\,\|\varphi''\|_{L^\infty}\,\|u\|_{L^2} + c\,\|\varphi'\|_{L^\infty}\,\|u\|_{\dot H^1} \leq c,\end{aligned}$$

which proves the desired H^1-bound providing uniform convergence. □

Proposition 3. *Suppose that the function $u \in H^1(\mathbb{R})$ is locally H^2-regular in $\mathbb{R}\setminus\{0\}$ and that u attains a global maximum at $x_0 \neq 0$. Then $(-\Delta)^{1/2}u$ is continuous in a neighborhood of x_0 and $(-\Delta)^{1/2}u(x_0) \geq 0$.*

The main ingredient in our regularity argument is the following commutator lemma (proved in [9]) which, in analogy to to the classical commutator lemma for $(-\Delta)^{1/2}$, is a consequence of the uniform bounds of $\mathcal{S}_\delta$ in the class of first order differential operators, see [10] pp. 89-93.

Lemma 3.3. *There is a universal constant $c > 0$ (independent of δ) such that*

$$\big\|[\phi,\mathcal{S}_\delta]u\big\|_{L^2} \leq c\,\|\phi'\|_{L^\infty}\|u\|_{L^2}.$$

4 Properties of comparison profiles

Recall that the solution of the relaxed variational principle (3) is given by the unique solution of the pseudo-differential equation

$$\varepsilon\,(-\Delta)\,\bar u + (-\Delta)^{1/2}\bar u + \bar u = \Lambda(\varepsilon)\,\delta_0,$$

where the Lagrange multiplier $\Lambda(\varepsilon) > 0$ agrees with the minimal energy. Hence we have the relation $\bar u_\varepsilon = \Lambda(\varepsilon)\,G_\varepsilon$, where G_ε is the fundamental solution for the operator $\varepsilon(-\Delta) + (-\Delta)^{1/2} + 1$, which can be computed explicitly via Fourier transform: an expansion of the inverse multiplier into partial fractions shows that the fundamental solution is given by the function

$$G_\varepsilon(x) = c(\varepsilon)\,\big\{G\big(a(\varepsilon)\,x\big) - G\big(b(\varepsilon)\,x\big)\big\} \tag{14}$$

where $G \in L^2(\mathbb{R})$ denotes the fundamental solution of the operator $(-\Delta)^{1/2} + 1$, i.e. in frequency space $\hat G(\xi) = (|\xi| + 1)^{-1}$. The explicit real space representation of G is given by the symmetric extension of

$$G(x) = \frac{\cos x}{\pi}\int_x^\infty \frac{\cos t}{t}\,dt + \frac{\sin x}{\pi}\int_x^\infty \frac{\sin t}{t}\,dt, \quad \text{for} \quad x > 0. \tag{15}$$

For $0 < \varepsilon < 1/4$ the positive scaling numbers $a(\varepsilon)$ and $b(\varepsilon)$ are given by

$$a(\varepsilon) = \frac{1}{\varepsilon}\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \varepsilon}\right) \sim 1 \quad \text{and} \quad b(\varepsilon) = \frac{1}{\varepsilon}\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \varepsilon}\right) \sim \frac{1}{\varepsilon},$$

and the factor $c(\varepsilon)$ is given by

$$c(\varepsilon) = \frac{\varepsilon}{\pi}\big(b(\varepsilon) - a(\varepsilon)\big)^{-1} = \frac{1}{\pi}\sqrt{1 - 4\varepsilon} \sim \frac{1}{\pi}$$

as $0 < \varepsilon < 1/4$ tends to zero. The function G is well known as a Fourier integral, see e.g. [2]. It is smooth away from the origin, symmetrically decreasing and has the following expansion:

$$G(x) = \frac{1}{\pi}\begin{cases} \log(1/|x|) - \gamma + \mathcal{O}(x) & \text{as } |x| \to 0 \\ x^{-2} + \mathcal{O}(x^{-4}) & \text{as } |x| \to \infty, \end{cases}$$

where γ denotes Euler's constant. On the basis of this expansion and in combination with the following energy asymptotic

$$\Lambda(\varepsilon) = \inf\big(\mathcal{I}_\varepsilon(u), u(0) = 1\big) = \pi\ \log(1/\varepsilon)^{-1}(1 + o(1)) \quad \text{as } \varepsilon \to 0 \qquad (16)$$

the rather precise asymptotic for the $\bar{u}_\varepsilon$ that we stated in Proposition 1 can be derived from the representation $\bar{u}_\varepsilon = \Lambda(\varepsilon)\, G_\varepsilon(x)$ through (14), see [3] for details. This in particular shows the existence of a logarithmic tail in the context of the relaxed problem. We need the following property of G.

Lemma 4.1. *The function G satisfies $2\,G(y) + y\,G'(y) > 0$ for all $y \neq 0$.*

Proof. We have the trivial identity that $2\,G(y) + y\,G'(y) = G(y) + d/dy\big[y\,G(y)\big]$, and an easy computation in frequency space shows that the Fourier transform is given by $\big[\hat{G}(\xi)\big]^2$. Thus, since $G(x) > 0$, the expression, as a convolution of two positive functions, is in turn positive. □

The lemma will guarantee that $G_\varepsilon''(x) \geq 0$ for $x \neq 0$ and for all $0 < \varepsilon < 1/4$, as we will show below. The next proposition collects the properties of the profiles $\bar{u}_\varepsilon$ that enter the comparison argument.

Proposition 4. *For every $0 < \varepsilon < 1/4$, the solution $\bar{u}_\varepsilon$ of the relaxed variational principle is symmetrically decreasing and smooth and convex on the half-lines $\{x > 0\}$ and $\{x < 0\}$, respectively.*

Proof. In view of Lemma 2.1, we only need to show the statement about convexity. Since $\bar{u}_{eps} = \Lambda(\varepsilon)\, G_\varepsilon$, it is enough to show that $G_\varepsilon''(x) > 0$ for all $x \neq 0$. G_ε'' can be expressed in terms of the function G and the numbers $a(\varepsilon), b(\varepsilon)$, and $c(\varepsilon)$: For the equation $(-\Delta)^{1/2} G + G = \delta_0$ we deduce

$$(-\Delta) G(x) = (-\Delta)^{1/2} \big(\delta_0 - G\big)(x) = (-\Delta)^{1/2} (\delta_0)(x) + G(x) \quad \text{in} \quad \mathbb{R} \setminus \{0\}.$$

But away from zero we have $(-\Delta)^{1/2}(\delta_0)(x) = -x^{-2}/\pi$. In view of (14) this contribution cancels for G_ε'', and we obtain

$$\frac{d^2 G_\varepsilon}{dx^2}\,(x) = c(\varepsilon)\, \big\{ b(\varepsilon)^2\, G\big(b(\varepsilon)\, x\big) - a(\varepsilon)^2\, G\big(a(\varepsilon)\, x\big) \big\} \quad \text{in} \quad \mathbb{R} \setminus \{0\}.$$

Recall that $b(\varepsilon) > a(\varepsilon) > 0$. Hence the argument is complete when we show that for each $x \neq 0$ the function $a \mapsto f_x(a) = a^2\, G(a\, x)$ is non-decreasing for $a > 0$. But we observe that, according to Lemma 4.1, $f_x'(a) = a\, \big(2\, G(a\, x) + (a\, x)\, G'(a\, x)\big)$ is positive. □

Motivation of Proposition 1. It is enough to consider functions G_ε. The fact that G_ε decays only quadratically can easily be motivated: From the definition of the Fourier transform we have

$$\sup |x|^k G_\varepsilon(x) \leq c\, \left\| D_\xi^k \hat{G}_\varepsilon \right\|_M, \tag{17}$$

where $M = M(\mathbb{R})$ denotes the space of finite Radon measures on $\mathbb{R}$. Now the limited regularity of the Fourier representation $\hat{G}_\varepsilon(\xi) = c\, \big(\varepsilon\, |\xi|^2 + |\xi| + 1\big)^{-1}$ implies a (uniform) decay estimate through (17) for k not bigger than 2. Note that there is no ε-uniform estimate for $k = 0$, since $\int \hat{G}_\varepsilon(\xi) d\xi$ blows up logarithmically as ε tends to zero.

5 The comparison argument

Proof of Theorem 2. Fix some $0 < \varepsilon < 1/4$. Let $u = u_\varepsilon$ be a minimizing profile of the variational principle (1) and $\bar{u} = \bar{u}_\varepsilon$ the minimizing profile of its relaxation (3). With the positive coefficient $a(x) = \varepsilon\, (1 - u^2(x))^{-1}$ and the fact that by Proposition 4 we have that $(-\Delta)\bar{u} \leq 0$ in $\mathbb{R} \setminus \{0\}$, these profiles solve by (7) and (5) the following pseudo-differential inequalities

$$a(x)\, (-\Delta) u + (-\Delta)^{1/2} u + u = a(x)\, |u'|^2\, (1 - u^2)^{-1} \geq 0 \quad \text{in} \quad \{u < 1\},$$

$$a(x)\, (-\Delta) \bar{u} + (-\Delta)^{1/2} \bar{u} + \bar{u} = a(x)\, u^2\, (-\Delta) \bar{u} \leq 0 \quad \text{in} \quad \{u < 1\},$$

respectively. We consider $w = \bar{u} - u \in H^1(\mathbb{R})$. From Proposition 2 and Proposition 4 we deduce that $w \in H^2_{loc}(\mathbb{R} \setminus \{0\})$. From the above differential inequalities we infer that

$$a(x)(-\Delta)w + (-\Delta)^{1/2}w + w \leq 0 \quad \text{in} \quad \{u < 1\} \quad \text{and} \quad w \leq 0 \quad \text{in} \quad \{u = 1\}.$$

Thus the maximum principle for $(-\Delta)^{1/2}$, see Proposition 3, implies that w cannot attain a global positive maximum. But by Proposition 2 and Proposition 4 and since $|w(x)| \leq |u(x)| + |\bar{u}(x)|$, w decays to zero as $|x| \to \infty$. This proves that necessarily $w(x) \leq 0$ for all x in $\mathbb{R}$, i.e. u is bounded from below by $\bar{u}$.

Proof of Theorem 1. From Theorem 2 that we just proved and Proposition 1 we immediately get the logarithmic lower bound in the statement of the theorem. Let us briefly review the argument for the upper bound as given in [8]. Projection of (6) onto the first component equation gives by a decomposition of the non-linearity

$$\varepsilon\,(-\Delta)u + (-\Delta)^{1/2}u + u = e_\varepsilon[m]\,u + r[u] \tag{18}$$

where $e_\varepsilon[m]$ is the energy density and $r[u]$ is a defect distribution arising from the non-local contribution. The operator on the left hand side is uniformly first-order elliptic and the right hand side is essentially L^1-bounded by the energy. The uniform bounds for $r[u]$ are slightly weaker than L^1, but in view of asymptotic versions of critical Sobolev embedding still sufficient to imply the existence of a universal constant $c > 0$ that for a minimizer $m_\varepsilon = (u_\varepsilon, v_\varepsilon)$ we have

$$\|u_\varepsilon\|_{L^p} \leq c\,p\,\mathcal{I}_\varepsilon(m_\varepsilon) \quad \text{for all} \quad p \geq 1. \tag{19}$$

This linear blow up implies by monotonicity, see Proposition 2, that for $x > 0$

$$u_\varepsilon(x) \leq \fint_0^x u_\varepsilon(y)\,dy \leq \left(\fint_0^x |u_\varepsilon(y)|^p\,dy\right)^{1/p} \leq \frac{p}{x^{1/p}}\,\mathcal{I}_\varepsilon(m_\varepsilon) \tag{20}$$

and thus with the optimal choice of $p = \log(1/x)$ a logarithmic upper bound. For the right outer scaling one might use the energy asymptotic, see [3], [4]

$$\mathcal{I}_\varepsilon(m_\varepsilon) = \pi\,\big(1 + o(1)\big)\,\log(1/\varepsilon)^{-1}. \tag{21}$$

References

[1] G. Alberti, Variational models for phase transitions, An approach via Γ-convergence, published in L. Ambrosio and N. Dancer, *Calculus of variations and partial differential equations. Topics on geometrical evolution equations and degree theory*, Springer-Verlag, Berlin-Heidelberg (2000), pp 95-114

[2] I.N. Bronstein and K.A. Semendjajew, *Teubner-Taschenbuch der Mathematik. Teil 1*, Eds. G. Grosche, V. Ziegler, D. Ziegler, and E. Zeidler, Teubner, Leipzig (1996).

[3] C. J. Gracía Cervera, *Magnetic Domains and Magnetic Domain Walls*, Ph.D. thesis, New York University (1999).

[4] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Magnetic microstructure - a paradigm of multiscale problems, *Proc. ICIAM 99 (Edinburgh)*, Eds. J.M. Ball and J.C.R. Hunt, (1999), pp. 175-199.

[5] A. DeSimone, R. V. Kohn, S. Müller and F. Otto, Repulsive interaction of Néel walls, and the internal length scale of the Cross-tie wall, *SIAM Multiscale Model. Simul.* **1** (2003), pp. 57-104.

[6] A. Hubert and R. Schäfer, *Magnetic Domains, The Analysis of Magnetic Microstructures*, Springer-Verlag, Berlin-Heidelberg-New York (1998).

[7] E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics Vol.14 American Mathematical Society (1996).

[8] C. Melcher, The logarithmic tail of Néel walls in thin films, to appear in *Arch. Rational Mech. Anal.*

[9] C. Melcher, *Néel walls and regularity in thin film micromagnetics*, Dissertation, Universität Leipzig (2002)

[10] Y. Meyer and R. Coifman, *Wavelets - Calderon-Zygmund and multilinear operators* Cambridge studies in advanced mathematics 48, Cambridge University Press, Cambridge (1997).

[11] L. Modica and S. Mortola, Un esempio di $Gamma$-convergenza, Boll. Un. Mat. Ital. B (5), **14** (1977), pp 285-299.

[12] L. Néel, Energie des parois de Bloch dans les couches minces, *C .R.Acad. Sci. Paris* **241** (1955), pp. 533-536.

[13] H. Riedel and A. Seeger, Micromagnetic Treatment of Néel Walls, *Phys. Stat. Sol. (B)* **46** (1971), pp. 377-384.