

SOME BLOW UP RESULTS FOR A NONLINEAR PARABOLIC EQUATION WITH

A GRADIENT TERM

BY

M. CHIPOT

AND

F.B. WEISSLER

IMA Preprint Series # 298

February 1987

**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS**  
**UNIVERSITY OF MINNESOTA**  
514 Vincent Hall  
206 Church Street S.E.  
Minneapolis, Minnesota 55455

- 65 D. Arnold and R.S. Falk, Continuous Dependence on the Elastic Coefficients for a Class of Anisotropic Materials
- 66 I.J. Bakelman, The Boundary Value Problems for Non-linear Elliptic Equation and the Maximum Principle for Euler-Lagrange Equations
- 67 Ingo Müller, Gases and Rubbers
- 68 Ingo Müller, Pseudoelasticity in Shape Memory Alloys - an Extreme Case of Thermoelasticity
- 69 Luis Magalhães, Persistence and Smoothness of Hyperbolic Invariant Manifolds for Functional Differential Equations
- 70 A. Damilamin and M. Vogelius, Homogenization Limits of the Equations of Elasticity in Thin Domains
- 71 H.-C. Simpson and S.J. Spector, On Hadamard Stability in Finite Elasticity
- 72 J.-L. Vazquez and C. Yarur, Isolated Singularities of the Solutions of the Schrödinger Equation with a Radial Potential
- 73 G. Dal Maso and U. Mosco, Wiener's Criterion and  $L^2$ -Convergence
- 74 John H. Maddocks, Stability and Folds
- 75 R. Hardt and D. Kinderlehrer, Existence and Partial Regularity of Static Liquid Crystal Configurations
- 76 M. Nerukar, Construction of Smooth Ergodic Cocycles for Systems with Fast Periodic Approximations
- 77 J.-L. Ericksen, Stable Equilibrium Configurations of Elastic Crystals
- 78 Patricio Aviles, Local Behavior of Solutions of Some Elliptic Equations
- 79 S.-N. Chow and R. Lauterbach, A Bifurcation Theorem for Critical Points of Variational Problems
- 80 R. Pego, Phase Transitions: Stability and Admissibility in One Dimensional Nonlinear Viscoelasticity
- 81 Mariano Giaquinta, Quadratic Functionals and Partial Regularity
- 82 J. Bona, Fully Discrete Galerkin Methods for the Korteweg De Vries Equation
- 83 J. Maddocks and J. Keller, Mechanics of Robes
- 84 F. Bernis, Qualitative Properties for some nonlinear higher order Nonlinear Higher Order Parabolic Equations with Absorption
- 85 F. Bernis, Finite Speed of Propagation and Asymptotic Rates for some Nonlinear Higher Order Parabolic Equations with Minimal Strategy Spaces
- 86 S. Reichstein and S. Reiter, Game Forms with Minimal Strategy Spaces  
An Answer to Littlewood's Problem on Boundedness
- 88 J. Rubinstein and R. Mauri, Dispersion and Convection in Periodic Media
- 89 W.H. Fleming and P.E. Souganidis, Asymptotic Series and the Method of Vanishing Viscosity
- 90 H. Beirão da Veiga, Existence and Asymptotic Behavior for Strong Solutions of Navier-Stokes Equations in the Whole Space
- 91 L.A. Caffarelli, J.L. Vazquez, and N.I. Wolanski, Lipschitz Continuity of Solutions and Interfaces of the N-Dimensional Porous Medium Equation
- 92 R. Johnson,  $m$ -Functions and Floquet Exponents for Linear Differential Systems
- 93 F.V. Atkinson and L.A. Peletier, Ground States and Dirichlet Problems for  $-\Delta = F(U)$  in  $\mathbb{R}^n$
- 94 G. Dal Maso, U. Mosco, The Wiener Modulus of a Radial Measure
- 95 H. A. Levine and H.F. Weinberger, Inequalities between Dirichlet and Neumann Eigenvalues
- 96 J. Rubinstein, On the Macroscopic Description of Slow Viscous Flow Past a Particle
- 97 G. Dal Maso and U. Mosco, Wiener Criteria and Energy Decay for Relaxed Dirichlet Problems
- 98 V. Ofikar and P. Walfman, On the Monge-Ampere Equation Arising in the Reflect Mapping Problem
- 99 M. Chipot, D. Kinderlehrer and L. Caffarelli, Some Smoothness Properties of Linear Laminae
- 100 Y. Giga and R. Kohn, Characterizing Blow-up Using Similarity Variables
- 101 P. Cannarsa and H. M. Soner, On the Singularities of the Viscosity Solutions to Hamilton-Jacobi-Bellman Equations
- 102 Andrew Majda, Nonlinear Geometric Optics for Hyperbolic Systems of Conservation Laws

- 203 G. Buttazzo, G. Dal Maso and U. Mosco, A Derivation Theorem for Capacities with Respect to a Radon Measure
- 204 S. Cowin, M. Mehrabadi, On the Identification of Material Symmetry for Anisotropic Elastic Materials
- 205 R.W.R. Darling, Constructing Nonhomeomorphic Stochastic Flows.
- 206 M. Chipot, On the Reynolds Lubrication Equation
- 207 R.V. Kohn and G.W. Milton, On Bounding the Effective Conductivity of Anisotropic Composites
- 208 I.J. Bakelman, Notes Concerning the Torsion of Hardening Rods and its N-Dimensional Generalizations
- 209 I.J. Bakelman, The Boundary Value Problems for Non-Linear Elliptic Equations
- 210 Guanglu Gong and Mingping Qian, On the Large Deviation Functions of Markov Chains
- 211 Arle Leizarowitz, Control Problems with Random and Progressively Known Targets
- 212 R.W.R. Darling, Ergodicity of a Measure-Valued Markov Chain Induced by Random Transformations
- 213 G. Gong, M. Qian & Zhongxin Zhao, Killed Diffusions and its Conditioning
- 214 Arle Leizarowitz, Controlling Diffusion Processes on Infinite Horizon with the Overtaking Criterion
- 215 Milford Beatty, The Poisson Function of Finite Elasticity
- 216 David Terman, Traveling Wave Solutions Arising From a Combustion Model
- 217 Yuh-Jia Lee, Sharp Inequalities and Regularity of Heat Semi-Group on Infinite Dimensional Spaces
- 218 D. Stroock, Lecture Notes
- 219 Claudio Canuto, Spectral Methods and Maximum Principle
- 220 Thomas O'Brien, A Two Parameter Family of Pension Contribution Functions and Stochastic Optimization
- 221 Takeyuki Hida, Analysis of Brownian Functionals
- 222 Leonid Hurwicz, On Informational Decentralization and Efficiency of Resource Allocation Mechanisms
- 223 E.B. Fabes and D.W. Stroock, A New Proof of Moser's Parabolic Harnack Inequality via the Old Ideas of Nash
- 224 Minoru Murata, Structure of Positive Solution to  $(-\Delta + V)u = 0$  in  $\mathbb{R}^n$
- 225 Paul Dupuis, Large Deviations Analysis of Reflected Diffusion
- 226 F. Bernis, Constrained Stochastic Approximation Algorithms in Convex Sets
- 227 F. Bernis, Existence Results for Doubly Nonlinear Higher Order Parabolic Equations on Unbounded Domains.
- 227 S. Orey and S. Pelikan, Large Deviations Principles for Stationary Processes
- 228 R. Gulliver and S. Hildebrandt, Boundary Configurations Spanning Continua of Minimal Surfaces.
- 229 J. Baxter, G. Dal Maso & U. Mosco, Stopping Times and  $T$ -Convergence.
- 230 Juffo Bouillet, Self-Similar Solutions, Having Jumps and Intervals of Constancy of a Diffusion-heat Conduction Equation
- 231 R. Hardt, D. Kinderlehrer & F.-H. Lin, A Remark About the Stability of Smooth Equilibrium Configurations of Static Liquid Crystal
- 232 M. Chipot and M. Luskin, The Compressible Reynolds Lubrication Equation
- 233 J.H. Maddocks, A Model for Disclinations in Nematic Liquid Crystal
- 234 C. Fofas, G.R. Sell and R. Temam, Inertial Manifolds for Nonlinear Evolutionary Equations
- 235 P.-L. Chow, Expectation Functionals Associated with Some Stochastic Evolution
- 236 Giuseppe Buttazzo, Reinforcement by a Thin Layer with Oscillating Thickness
- 237 W.H. Fleming, S.J. Sheu and H.M. Soner, On Existence of the Dominant Eigenfunction and its Application to the Large Deviation Properties of an Ergodic Markov Process
- 238 R. Jensen and P.E. Souganidis, A Regularity Result for Viscosity Solutions of Hamilton-Jacobi Equations in one Space Dimension
- 239 B. Boczar-Karakli, J.L. Bona and D.L. Cohen, Interaction of Shallow-Water Waves and Bottom Topography
- 240 F. Colonus and W. Killemann, Infinite Time Optimal Control and Periodicity Dirichlet Problems

Author(s)	Title	Author(s)	Title
41	<b>Harry Keaton</b> , Scaling Relations for 2D-Percolation	277	<b>Cockburn, B.</b> The Quasi-Monotone Schemes for Scalar Conservation Laws Part I
42	<b>A. Leizarowitz</b> , Infinite Horizon Optimization for Markov Process with Finite States Spaces	278	<b>M. Chipot, T. Sideris</b> , On the Abelian Higgs Model
43	<b>Louis H.Y. Chen</b> , The Rate of Convergence in A Central Limit Theorem for Dependent Random Variables with Arbitrary Index Set	279	<b>C. Fofas, B. Nicolaenko, G.R.Self, R. Temam</b> , Inertial Manifolds for the Kuramoto-Sivashinsky Equation and an Estimate of their Lowest Dimension
44	<b>G. Kallianpur</b> , Stochastic Differential Equations in Duals of Nuclear Spaces with some Applications	280	<b>R. Duran</b> , On the Approximation of Miscible Displacement in Porous Media by a Method of Characteristics Combined with a Mixed Method
45	<b>Tzsu-Shuh Chiang, Yunshong Chow and Yuh-Jia Lee</b> , Evaluation of Certain Functional Integrals	281	<b>H. Aizhang, Zhang Bo</b> , The Convergence for Model Expansion Method
46	<b>L. Karp and M. Pinsky</b> , The First Eigenvalue of a Small Geodesic Ball in a Riemannian Manifold	282	<b>V. Tversky</b> , Dispersive Bulk Parameters for Coherent Propagation in Correlated Random Distributions
47	<b>Chi-Sing Man</b> , Towards An Acoustoelastic Theory for Measurement of Residual Stress	283	<b>H.Th.F. den Hollander</b> , Mixing Properties for Random Walk in Random Scenery
48	<b>Andreas Stoll</b> , Invariance Principles for Brownian Intersection Local Time and Polymer Measures	284	<b>H.R. Jauslin</b> , Nondifferentiable Potentials for Nonequilibrium Steady States
49	<b>R.M.R. Darling</b> , Rate of Growth of the Coalescent Set in a Coalescing Stochastic Flow	285	<b>K. Meyer, G.R.Self</b> , Homoclinic Orbits and Bernoulli Bundles in Almost Periodic Systems
50	<b>R. Cohen, R. Hardt, D. Kinderlehrer, S.Y.Lin, M. Luskin</b> , Minimum Energy for Liquid Crystals: Computational Results	286	<b>J. Douglas, Jr., Y. Yuan</b> , Finite Difference Methods for the Transient Behavior of a Semiconductor Device
51	<b>Suzanne M. Lenhart</b> , Viscosity Solutions for Weakly Coupled Systems of First Order PDEs	287	<b>LJ Kaitai, Yan Ningning</b> , The Extrapolation for Boundary Finite Elements
52	<b>M. Cranston, E. Fabes, Z. Zhao</b> , Condition Gauge and Potential Theory for the Schrödinger Operator	288	<b>R. Durrett, R.H. Schonmann</b> , Stochastic Growth Models
53	<b>H. Brezis, J.M.Caron, E.H. Lieb</b> , Harmonic Maps with Defects	289	<b>David Kinderlehrer</b> , Remarks about Equilibrium Configurations of Crystals
54	<b>A. Carverhill</b> , Flows of Stochastic Dynamical Systems: Nontriviality of the Lyapunov Spectrum	290	<b>D.G. Aronson, J.L. Vazquez</b> , Eventual $C^\infty$ -Regularity and Concavity for Flows in One-Dimensional Porous Media
55	<b>A. Carverhill</b> , Conditioning A 'Lifted' Stochastic System in a Product Case	291	<b>L.R. Scott, J.M. Boyle, B. Bagheri</b> , Distributed Data Structures for Scientific Computation
56	<b>R.J. Williams</b> , Local Time and Excursions of Reflected Brownian Motion	292	<b>J. Douglas, Jr., P.J.Paes Leme, T. Arbogast, T. Schmitt</b> , Simulation of Flow in Naturally Fractured Petroleum Reservoirs
57	<b>H. Foilmer, S. Orey</b> , Large Deviations for the Empirical Field of a Gibbs Measure	293	<b>D.G. Aronson, L.A. Caffarelli</b> , Optimal Regularity for One-Dimensional Porous Medium Flow
58	<b>A. Leizarowitz</b> , Characterization of Optimal Trajectories On an Infinite Horizon	294	<b>Haim Brezis</b> , Liquid Crystals and Energy Estimates for $S^2$ -Valued Maps
59	<b>Y.Giga, T. Miyakawa, H. Osada</b> , Two Dimensional Navier Stokes Flow with Measures As Initial Vorticity	295	<b>Todd Arbogast</b> , Analysis of the Simulation of Single Phase Flow Through a Naturally Fractured Reservoir
60	<b>M. Chipot, V. Ollker</b> , Sur Une Propriete Des Fonctions Propres De L'Operateur De Laplace Beltrami	296	<b>He Yinnian, LJ Kaitai</b> , The Coupling Method of Finite Elements and Boundary Elements for Radiation Problems
61	<b>V. Perez-Abreu</b> , Decompositions of Semimartingales On Duals of Countably Nuclear Spaces	297	<b>T. Cazenave, A. Haroux, L. Vazquez, F.B. Weissler</b> , Nonlinear Effects in the Wave Equation with a Cubic Restoring Force
62	<b>J.M. Ball</b> , Does Rank-One Convexity Imply Quasiconvexity	298	<b>M. Chipot, F.B. Weissler</b> , Some Blow Up Results for a Nonlinear Parabolic Parabolic Equation with a Gradient Term
63	<b>B. Cockburn</b> , The Quasi-Monotone Schemes for Scalar Conservation Laws. Part I	299	<b>LJ Kaitai</b> , Perturbation Solutions of Simple and Double Bifurcation Problems for Navier-Stokes Equations
64	<b>K.A. Pericak-Spector</b> , On Radially Symmetric Simple Waves in Elasticity	300	<b>Chen Zhangxin, LJ Kaitai</b> , The Convergence on the Multigrid Algorithm for Navier-Stokes Equations
65	<b>P.N. Shivakumar, Chi-Sing Man, Simon W. Rabkin</b> , Modelling of the Heart and Pericardium at End-Diastole		
66	<b>Jose-Luis Menaldi</b> , Probabilistic View of Estimates for Finite Difference Methods		
67	<b>Robert Hardt, Harold Rosenberg</b> , Open Book Structures and Unicity of Minimal Submanifolds		
68	<b>Bernardo Cockburn</b> , The Quasi-Monotone Schemes for Scalar Conservation Laws		
69	<b>H.R. Jauslin, W. Zimmermann, Jr.</b> , Dynamics of a Model for an Ac Josephson Effect in Superfluid $^4He$		
70	<b>A.K. Kapila</b> , Introductory Lecture on Reacting Flows		
71	<b>J.C. Taylor</b> , Do Minimal Solutions of Heat Equations Characterize Diffusions?		
72	<b>J.C. Taylor</b> , The Minimal Eigenfunctions Characterize the Ornstein-Uhlenbeck Process		
73	<b>Chi-Sing Man, Quan-Xin Sun</b> , On the Significance of Normal Stress Effects in the Flow of Glaciers		
74	<b>Omer Hijab</b> , On Partially Observed Control of Markov Processes		
75	<b>Lawrence Gray</b> , The Behavior of Processes with Statistical Mechanical Properties		
76	<b>R. Hardt, D. Kinderlehrer, M. Luskin</b> , Remarks About the Mathematical Theory of Liquid Crystals		

SOME BLOW UP RESULTS FOR A NONLINEAR PARABOLIC EQUATION

WITH A GRADIENT TERM

M. CHIPOT

Université de Metz  
Département de mathématiques  
et informatique  
Ile du Saulcy  
57045 Metz-Cedex (FRANCE)

F.B. WEISSLER\*

Department of Mathematics  
Texas A & M University  
College Station, Texas 77843

\* Partially supported by NSF Grant DMS 8201639.

## 1. INTRODUCTION

In this paper we study solution of the semilinear parabolic problem

$$\left. \begin{aligned} u_t &= \Delta u - |\nabla u|^q + |u|^{p-1}u, & t > 0, x \in \Omega \\ u(t,y) &= 0, & t > 0, y \in \Gamma \\ u(0,x) &= \phi(x), & x \in \Omega. \end{aligned} \right\} \quad (1.1)$$

Here  $\Omega \subset \mathbb{R}^N$  is a bounded domain with smooth boundary  $\Gamma$ ,  $u = u(t,x)$ ,  $\Delta$  and  $\nabla$  apply only to the spatial variables, and  $p > 1$  and  $q > 1$  are fixed (finite) parameters. Our main goal is to show that under appropriate conditions on  $q$ ,  $p$ , and  $n$ , there exists a suitable initial value  $\phi$  so that the corresponding solution of (1.1) blows up in a finite time.

In the case where there is no gradient term, i.e.

$$\left. \begin{aligned} u_t &= \Delta u + |u|^{p-1}u, & t > 0, x \in \Omega \\ u(t,y) &= 0, & t > 0, y \in \Gamma \\ u(0,x) &= \phi(x), & x \in \Omega, \end{aligned} \right\} \quad (1.2)$$

the following result has been known for some time, (see for example Ball [2]).

Theorem 1.1. Let  $p > 1$  and let  $\phi: \bar{\Omega} \rightarrow \mathbb{R}$  be sufficiently smooth (e.g.  $C^2$ ) with  $\phi|_{\Gamma} \equiv 0$ . If  $\phi$  is large enough in the sense that its "energy"

$$E(\phi) = \frac{1}{2} \|\nabla \phi\|_2^2 - \frac{1}{p+1} \|\phi\|_{p+1}^{p+1} \quad (1.3)$$

is negative, then the corresponding solution of (1.2) blows up in finite time.

We remark that local existence of solutions for (1.2) follows by standard iteration methods (see for example Segal [28]) on the Banach space  $C_0(\Omega)$ . Thus, if the existence time  $T$  of the maximal solution to (1.2) is finite, i.e. if the solution blows up in finite time  $T$ , then  $\lim_{t \rightarrow T} \|u(t, \cdot)\|_{\infty} = \infty$ .

In the past few years, a great deal of work has been done to study the precise behavior of solutions to (1.2) as  $t$  approaches the finite blow up time.

(See [3,12,14,15,16,24,25,29,32,33].) A corresponding theory is also being developed in the case where  $|u|^{p-1}u$  in (1.2) is replaced by  $\lambda e^u$ . (See [4,5,6,7,8,12,22,23,30].)

One is naturally led to consider more general parabolic problems of the form

$$u_t = \Delta u + f(u, \nabla u). \quad (1.4)$$

To our knowledge, there has not been much study of solutions to equations of the form (1.4) which blow up in finite time. (For an example, see [11].) Moreover, we are not aware of any finite time blow up results which would apply to equation (1.1). Furthermore, the gradient term in (1.1) has a damping effect, working against blow up; and so it is not clear if problem (1.1) has solutions which blow up in finite time. Our goal is therefore somewhat modest: to find an analogue of Theorem 1.1 for problem (1.1).

Theorem 1.2. Let  $1 < q < 2p/(p + 1)$  and let  $\phi \in W^{3,S}(\Omega)$  for  $s$  sufficiently large,  $\phi$  not identically zero. Suppose in addition that

- i)  $\phi = 0$  on  $\Gamma$ ,
- ii)  $\Delta \phi - |\nabla \phi|^q + |\phi|^{p-1} \phi = 0$  on  $\Gamma$ ,
- iii)  $\phi > 0$  in  $\Omega$ ,
- iv)  $\Delta \phi - |\nabla \phi|^q + \phi^p > 0$  in  $\Omega$ ,
- v)  $E(\phi) < 0$ ,
- vi) if  $q < 2p/(p + 1)$ , then  $\|\phi\|_{p+1}$  is sufficiently large,
- vii) if  $q = 2p/(p + 1)$ , then  $p$  is sufficiently large.

Then the corresponding solution of (1.1) blows up in finite time, in the  $L^\infty$  norm.

The obvious difficulty with this result is that it is not at all clear if such a  $\phi$  exists. A natural candidate for  $\phi$  is a regular solution of the following elliptic problem,

$$\left. \begin{aligned} \Delta\phi - |\nabla\phi|^q + \lambda\phi^p &= 0 && \text{in } \Omega \\ \phi &> 0 && \text{in } \Omega \\ \phi &= 0 && \text{on } \Gamma \end{aligned} \right\} \quad (1.5)$$

where  $\lambda > 0$  is sufficiently small.

Theorem 1.3 Let  $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$ . Suppose  $1 < q < 2p/(p + 1)$  and (if  $n > 3$ )  $p < (n + 2)/(n - 2)$ . Then for all  $\lambda > 0$  there exists a regular solution  $\phi$  of (1.5). If  $\lambda$  is sufficiently small, then  $\phi$  satisfies conditions i)-vi) in Theorem 1.2.

Suppose  $n = 1$  and  $q = 2p/(p + 1)$ . Then for all  $\lambda > \lambda_p$ , where

$$\lambda_p = \frac{(2p)^p}{(p+1)^{2p+1}}, \quad (1.6)$$

there exists a regular solution  $\phi$  of (1.5). If in addition,  $\lambda < 2/(p + 1)$ , then  $\phi$  satisfies conditions i)-v) in Theorem 1.2. ("Regular" above means regular enough to apply Theorem 1.2.)

The paper is organized as follows. In Section 2 we prove local existence and uniqueness, and regularity for problem (1.1) with initial values in an appropriate Sobolev space. Moreover, we indicate precisely the conditions on  $s$  required for Theorem 1.2 and prove that conditions i)-iv) on  $\phi$  imply  $u(t, \cdot) > 0$  and  $u_t(t, \cdot) > 0$  throughout the trajectory. In Section 3 we prove Theorem 1.2, using energy arguments based on the methods found in Ball [2]. We have attempted to write Section 3 so that it is, at least formally, independent of the technicalities of Section 2. In Section 4 we begin the study of (1.5) and prove Theorem 1.3. Finally, in Section 5 we present some additional results concerning (1.5). In particular, we show that the value of  $\lambda_p$  claimed in Theorem 1.3 is in fact sharp.

We remark that the value  $q = 2p/(p+1)$  is "critical" in many respects. The condition  $q \leq 2p/(p + 1)$  arises naturally in the energy arguments. When  $q = 2p/(p + 1)$ ; both equations (1.1) and (1.5) have the same scaling properties as the same equations without the gradient term; and the character of solutions to (1.5) changes considerably as  $q$  is smaller than, equal to, or bigger than  $2p/(p + 1)$ .

### Acknowledgements

Both authors were research fellows at the Institute for Mathematics and its Applications, the University of Minnesota, during the entire academic year 1984-85. Most of the research for this article was done during this time, and we are grateful to the IMA for its support. Also, while at the IMA, we were fortunate to be able to discuss this work with S. Hastings, R. Pego, and P. Souganidis. We thank them for their ideas and suggestions.



## 2. LOCAL EXISTENCE AND REGULARITY FOR THE EVOLUTION EQUATION

In this section  $\Omega \subset \mathbb{R}^n$  is a bounded domain with smooth boundary  $\Gamma = \partial\Omega$ . Also,  $p$  and  $q$  are fixed real numbers, strictly larger than 1. Our goal is to construct a local theory for the parabolic problem (1.1). The first step is to write the corresponding variation of parameters integral equation

$$u(t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta} J(u(s)) ds \quad (2.1)$$

where  $u(t) = u(t, \cdot)$  and  $J = J_1 + J_2$  with

$$\begin{aligned} J_1(u) &= -|\nabla u|^q, \\ J_2(u) &= |u|^{p-1}u. \end{aligned}$$

Also,  $e^{t\Delta}$  denotes the heat semigroup on  $\Omega$  with Dirichlet boundary conditions. Recall that for  $1 < s < \infty$ ,  $e^{t\Delta}$  is an analytic, contraction,  $C_0$  semigroup on  $L^s = L^s(\Omega)$ . Furthermore, the domain of its generator in  $L^s$  is

$$D_s(\Delta) = W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega).$$

Moreover, it is known, [10], that  $e^{t\Delta}$  restricts to a  $C_0$  semigroup on  $W_0^{1,2}(\Omega)$ ,  $1 < s < \infty$ .

We will construct a local theory for the integral equation (2.1) in the Banach space  $W_0^{1,s} = W_0^{1,s}(\Omega)$ ,  $s$  sufficiently large, using the framework developed in [31]. Note that for suitable  $r_1, r_2 \geq 1$ ,

$$\begin{aligned} J_1: W_0^{1,s} &\rightarrow L^{r_1} \\ J_2: W_0^{1,s} &\rightarrow L^{r_2} \end{aligned}$$

are continuously Frechet differentiable maps, Lipschitz on bounded sets in  $W_0^{1,s}$ . Indeed,  $r_1$  can clearly be chosen  $s/q$ , provided  $s > q$ ; and allowable



values for  $r_2$  can easily be computed by first determining when  $W_0^{1,s}$  is embedded in  $L^{r_2}$ . Moreover, if  $1 \leq r \leq s < \infty$ , then for  $t > 0$

$$e^{t\Delta}: L^r \rightarrow W_0^{1,s}$$

is bounded with norm bounded by  $Ct^{-\alpha}$ , where

$$\alpha = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{s} \right) + \frac{1}{2}$$

and  $C$  can be chosen uniformly up to any finite time. (See Lemma 4.1 in [31] and Theorem 4.17 in [1].) Therefore, for each  $t > 0$ , the map  $K_t \equiv e^{t\Delta}J$  is a continuously Frechet differentiable mapping of  $W_0^{1,s}$  into itself, Lipschitz on bounded sets. In order to apply the results in [31], it suffices to choose  $s$  so that  $\alpha < 1$  with both  $r = r_1 > 1$  and  $r = r_2 > 1$ . Routine calculations (albeit somewhat tedious) show that this can be done if

$$\begin{aligned} s > q, & \quad s > n(q - 1), \\ s > np/(n + p), & \quad s > n(p - 1)/(p + 1). \end{aligned} \tag{2.2}$$

(The conditions on the left side of (2.2) come from the requirement that  $r_1, r_2 > 1$ ; and the conditions on the right side, from the requirement that  $\alpha < 1$  for both  $r_1$  and  $r_2$ .) From now on we assume that  $s$  satisfies (2.2). (We will later need an additional assumption on  $s$ .) Thus, by Theorem 1 in [31], for every  $\phi \in W_0^{1,s}$ , there is a unique maximal solution  $u \in C([0, T_\phi); W_0^{1,s})$  to the integral equation (2.1).  $T_\phi$  is the existence time of the solution starting at  $\phi$ ; and if  $T_\phi < \infty$ , then  $\|u(t)\|_{W_0^{1,s}} \rightarrow \infty$  as  $t \rightarrow T_\phi$ .

We remark that if  $q < 2$ , then a local theory for the integral equation (2.1) can be constructed in  $L^r(\Omega)$ , using the framework developed in [34]. In fact, Theorem 2 of [34] needs to be modified slightly to handle a nonlinearity of the form  $J = J_1 + J_2$ . (Two spaces  $E_{J_1}$  and  $E_{J_2}$  are needed instead of just  $E_J$ .) We omit the details and simply indicate that if

$$r > n(p-1)/2 \tag{2.2a}$$

$$r > n(q-1)/(2-q), \quad q < 2,$$

then we have local existence and uniqueness for (2.1) in  $L^r$ . In particular, if the existence time  $T_\phi$  is finite, then  $\|u(t, \cdot)\|_r \rightarrow \infty$  as  $t \rightarrow T_\phi$ . It follows, of course, that  $\|u(t, \cdot)\|_\infty \rightarrow \infty$  as  $t \rightarrow T_\phi$ . Since  $q \leq 2p/(p+1)$  implies  $q < 2$ , this is the case under the hypotheses of Theorem 1.2.

We would like to show that if  $\phi$  is sufficiently regular, then the resulting solution of (2.1) is also a solution of the original problem (1.1) and has some additional regularity properties. Recall that the integral equation (2.1) gives rise to a "semi-flow"  $W_t$  on  $W_0^{1,s}$ , i.e.  $W_t$  takes  $\phi \in W_0^{1,s}$  to its value at time  $t$  under the action of (2.1). In other words,  $W_t \phi = u(t)$ , where  $u(t)$  is the maximal solution in  $W_0^{1,s}$  of (2.1) with initial value  $\phi$ . In particular,  $W_t \phi$  is defined precisely for  $t \in [0, T_\phi)$ . The generator of the semi-flow  $W_t$  is

$$B\phi = \lim_{t \rightarrow 0^+} \frac{W_t \phi - \phi}{t}, \tag{2.3}$$

where the limit is taken in  $W_0^{1,s}$ . The domain of  $B$ ,  $D(B)$ , is simply the set of  $\phi \in W_0^{1,s}$  for which the limit (2.3) exists. Formally,  $B = \Delta + J$ , i.e.

$$B\phi = \Delta\phi - |\nabla\phi|^q + |\phi|^{p-1}\phi. \tag{2.4}$$

However, the characterization of  $B$  and  $D(B)$  in [31] (Theorem 3.1) is somewhat abstract, and some care is needed to describe  $B$  and  $D(B)$  in the present context. For technical convenience, we make the following additional restrictions on  $s$ :

$$s > 2q, \quad s > nq. \tag{2.5}$$

Proposition 2.1. Suppose  $s \in \mathbb{R}$  satisfies (2.2) and (2.5). Let  $W_t$  be the semi-flow on  $W_0^{1,s}$  resulting from the integral equation (2.1), with generator  $B$  and domain  $D(B)$ . Then  $D(B)$  is the set of all  $\phi \in W^{3,s} \cap W_0^{1,s}$  such that

$$\Delta\phi - |\nabla\phi|^q + |\phi|^{p-1}\phi \in W_0^{1,s}. \quad (2.6)$$

For  $\phi \in D(B)$ ,  $B\phi$  is given by (2.4).

Proof. Suppose first  $\phi \in D(B)$ . By Theorem 3.1 part (iv) in [31], it follows that

$$\Delta e^{t\Delta}\phi + e^{t\Delta}(-|\nabla\phi|^q + |\phi|^{p-1}\phi) \quad (2.7)$$

converges strongly in  $W_0^{1,s}$  to  $B\phi$  as  $t \rightarrow 0^+$ . Now certainly  $|\nabla\phi|^q \in L^{s/q}$  and, thanks to (2.5),  $|\phi|^{p-1}\phi \in W_0^{1,s} \subset L^{s/q}$ . Thus, the second term in (2.7) converges in  $L^{s/q}$  as  $t \rightarrow 0^+$  to  $(-|\nabla\phi|^q + |\phi|^{p-1}\phi)$ . Furthermore, again by (2.5),  $\phi \in W_0^{1,s} \subset H_0^1(\Omega)$ ; and so  $\Delta e^{t\Delta}\phi \rightarrow \Delta\phi$  in  $H^{-1}(\Omega)$  as  $t \rightarrow 0^+$ . Thus, the distributional limit of (2.7) as  $t \rightarrow 0^+$  is the desired expression

$$\Delta\phi - |\nabla\phi|^q + |\phi|^{p-1}\phi.$$

This must be the same as the  $W_0^{1,s}$  limit, which proves (2.6) and (2.4).

To prove that  $\phi \in W^{3,s}$ , note first that  $B\phi \in W_0^{1,s} \subset L^{s/q}$  and  $-|\nabla\phi|^q + |\phi|^{p-1}\phi \in L^{s/q}$ . Hence  $\Delta\phi \in L^{s/q}$ . Since  $\phi \in W_0^{1,s/q}$ , elliptic regularity gives that  $\phi \in W^{2,s/q}$ . Therefore  $\nabla\phi \in W^{1,s/q}$ . Again by (2.5),  $W^{1,s} \subset W^{1,s/q} \subset L^\infty$ . Thus  $|\nabla\phi|^q$  and  $|\phi|^{p-1}\phi$  are both in  $L^\infty$ . Since  $B\phi \in W_0^{1,s} \subset L^\infty$ , it follows that  $\Delta\phi \in L^\infty$ . Thus  $\phi \in W^{2,r}(\Omega)$  for every finite  $r$ . Therefore  $|\nabla\phi|^q$  and  $|\phi|^{p-1}\phi$  are both in  $W^{1,s}$ ; and since  $B\phi \in W^{1,s}$ , we get that  $\Delta\phi \in W^{1,s}$ . By higher order elliptic regularity (see for example Theorem IX.32 in [9]), it follows that  $\phi \in W^{3,s}$ .

On the other hand, suppose  $\phi \in W^{3,s} \cap W_0^{1,s}$  satisfies (2.6). To show that  $\phi \in D(B)$ , we must show, by Theorem 3.1 part (iv) in [31], that for all  $t > 0$ ,

$e^{t\Delta}\phi$  is in the domain of  $\Delta$  as a semigroup generator in  $W_0^{1,S}$  and that (2.7) has a limit in  $W_0^{1,S}$  as  $t \rightarrow 0^+$ . Now  $e^{t\Delta}$  is an analytic semigroup on  $L^S$ . Thus,  $e^{t\Delta}$  restricts to an analytic semigroup on  $D_S(\Delta)$ , considered as a Banach space with its graph norm. Since  $\phi \in D_S(\Delta) = W^{2,S} \cap W_0^{1,S}$ , it follows that for all  $t > 0$   $e^{t\Delta}\phi$  is in the domain of  $\Delta$  as a semigroup generator in  $D_S(\Delta)$ . Since  $D_S(\Delta)$  is continuously embedded in  $W_0^{1,S}$ ,  $e^{t\Delta}\phi$  is in the domain of  $\Delta$  as a semigroup generator in  $W_0^{1,S}$ . Finally, again since  $\phi \in D_S(\Delta)$ ,  $\Delta e^{t\Delta}\phi = e^{t\Delta}\Delta\phi$ , where both expressions make sense in  $L^S$ . Consequently, (2.7) equals

$$e^{t\Delta}(\Delta\phi - |\nabla\phi|^q + |\phi|^{p-1}\phi),$$

which clearly has a limit in  $W_0^{1,S}$  as  $t \rightarrow 0^+$  because of (2.6).

Remark. The above proof shows that Proposition 2.1 remains correct if  $W^{3,S}$  is replaced by  $W^{2,S}$ . It was higher order elliptic regularity which allowed us to conclude  $\phi \in W^{3,S}$ .

Proposition 2.2. Under the same conditions as in Proposition 2.1, let  $\phi \in D(B)$  and  $u(t) = W_t\phi$ ; i.e.  $u(t)$  is the maximal solution of equation (2.1). Then:

i)  $u \in C^1([0, T_\phi); W_0^{1,S})$  and

$$u'(t) = \Delta u(t) - |\nabla u(t)|^q + |u(t)|^{p-1}u(t), \quad (2.8)$$

where each term on the right side of (2.8) is in  $C([0, T_\phi); L^{S/q})$ ;

ii)  $u \in C([0, T_\phi); W^{2,S/q})$ ;

iii)  $\|u(t)\|_\infty$  and  $\|\nabla u(t)\|_\infty$  are bounded on any interval  $[0, T]$  with  $T < T_\phi$ .

Proof. By Theorem 2.2 in [31],  $u \in C^1([0, T_\phi); W_0^{1,S})$ ,  $u(t) \in D(B)$  for all  $t \in [0, T_\phi)$  and  $u'(t) = Bu(t)$ . The previous proposition now implies (2.8).

Furthermore  $|\nabla u(t)|^q$  and  $|u(t)|^{p-1}u(t)$  are clearly continuous into  $L^{s/q}$  (again using (2.5)) and since  $Bu(t)$  is continuous into  $W_0^{1,s}$ , it follows that  $\Delta u(t)$  is continuous into  $L^{s/q}$ . This proves i).

Since  $u(t) \in D(B)$ , the previous proposition implies  $u(t) \in W^{2,s/q} \cap W_0^{1,s/q}$ . Also  $u(t)$  and  $\Delta u(t)$  are both continuous in  $L^{s/q}$ . Since the graph norm for  $\Delta$  on  $W^{2,s/q} \cap W_0^{1,s/q}$  is equivalent to the  $W^{2,s/q}$  norm, it follows that  $u(t)$  is continuous into  $W^{2,s/q}$ , which proves ii).

Finally, iii) follows easily since  $W^{1,s/q}$  is continuously embedded in  $L^\infty$ , thanks to assumption (2.5).

Proposition 2.3. Under the same conditions as in Proposition 2.1, let  $\phi \in W_0^{1,s}$  with  $\phi \geq 0$  a.e. in  $\Omega$ . Then  $u(t) = W_t \phi \geq 0$  for all  $t \in [0, T_\phi)$ .

Proof. Any  $\phi \geq 0$  in  $W_0^{1,s}$  can be approximated in  $W_0^{1,s}$  by non-negative  $C^\infty$  functions with compact support in  $\Omega$ , in particular by non-negative functions in  $D(B)$ . By the continuity properties of the semi-flow  $W_t$  (see Theorem 1 in [31]), we may therefore assume  $\phi \geq 0$  is in  $D(B)$ . (In fact, we only need the result for  $\phi \in D(B)$ .)

Multiplying (2.8) by

$$u(t)^- \equiv \frac{|u(t)| - u(t)}{2}$$

and integrating over  $\Omega$  yields

$$\int u' u^- = \int (\Delta u) u^- - \int |\nabla u|^q u^- + \int |u|^{p-1} u u^-,$$

where we have suppressed the dependence on  $t$ . By the previous proposition, we clearly have  $u \in C^1((0, T_\phi); L^2)$  and  $u \in C((0, T_\phi); H_0^1)$ . Thus

$$\int u' u^- = -\frac{1}{2} \frac{d}{dt} \int (u^-)^2$$

and

$$\int (\Delta u) u^- = \int |\nabla u^-|^2.$$

The first formula above follows from the definition of  $u^-$  (multiply out  $(u^-)^2$ ) and the second formula from well-known facts about  $\nabla u^-$  ([17], Section 7.4). We now restrict ourselves to  $t \in [0, T]$  for a fixed  $T < T_\phi$ . By part iii) of the previous proposition, there is a constant  $C = C(T)$  such that

$$\int |u|^p |u^-| \leq C \int |u^-|^2$$

and

$$\begin{aligned} \int |\nabla u|^q |u^-| &\leq C \int |\nabla u^-| |u^-| \\ &\leq \varepsilon C \int |\nabla u^-|^2 + C_\varepsilon C \int |u^-|^2, \end{aligned}$$

where  $\varepsilon > 0$  is arbitrary, but  $C_\varepsilon$  depends on the choice of  $\varepsilon$ . Putting all this together, and choosing  $\varepsilon > 0$  so that  $\varepsilon C < 1$ , we get that for  $t \in (0, T]$ ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |u^-|^2 &\leq - \int |\nabla u^-|^2 + \varepsilon C \int |\nabla u^-|^2 + C_\varepsilon C \int |u^-|^2 \\ &\leq C_\varepsilon C \int |u^-|^2. \end{aligned}$$

Since  $u^- \in C([0, T]; L^2)$  and  $u(0)^- = \phi^- = 0$ , Gronwall's lemma now implies  $\int |u^-|^2 = 0$  for all  $t \in [0, T]$ . Since  $T < T_\phi$  is arbitrary, we see that  $u(t)^- = 0$  for all  $t \in [0, T_\phi)$ .

For the energy arguments in the next section we need not only  $u(t) \geq 0$ , but also  $u'(t) \geq 0$ , throughout the trajectory. In order to prove this with the weak maximum principle methods of previous proof, we need some higher order regularity in  $t$ . We begin with the following lemma. Its proof is modeled on the proof of Theorem 3 in [20]. (See also Proposition 1.2 in [31].)

Lemma 2.4. Under the same conditions as in Proposition 2.1, let  $\phi \in D(B)$  and  $u(t) = W_t \phi \in C^1([0, T_\phi]; W_0^{1,S})$ . Denote  $v(t) = u'(t)$ , so  $v \in C([0, T_\phi]; W_0^{1,S})$ . Then for any compact subinterval  $[\varepsilon, T] \subset (0, T_\phi)$ ,  $v: [\varepsilon, T] \rightarrow W_0^{1,S}$  is Holder continuous.



Proof. By Theorem 2.2 in [31],  $v(t)$  satisfies the following integral equation:

$$v(t) = e^{t\Delta}(B\phi) - q \int_0^t e^{(t-s)\Delta} (|\nabla u(s)|^{q-2} \nabla u(s) \circ \nabla(s)) ds \\ + p \int_0^t e^{(t-s)\Delta} |u(s)|^{p-1} v(s) ds. \quad (2.9)$$

Since  $B\phi \in W_0^{1,S} \subset L^S$  and  $e^{t\Delta}$  is an analytic semigroup on  $L^S$ , it follows that  $e^{t\Delta}(B\phi)$  is in  $C^1((0,\infty); D_S(\Delta))$  and thus is certainly Holder continuous in  $W_0^{1,S}$  on  $[\varepsilon, T]$ . Thus, it suffices to show that the two integral terms are Holder continuous on  $[\varepsilon, T]$ . We consider only the first one, the second one being easier to handle.

By Proposition 2.2 part ii) and the fact that  $W^{1,S/q} \subset L^\infty$  by (2.5), we have that  $\nabla u$  and hence  $|\nabla u|^{q-2} \nabla u$  must be in  $C([0, T]; L^\infty)$ . (Obviously, we mean that  $|\nabla u|^{q-2} \nabla u = 0$  in case  $|\nabla u| = 0$ . This presents no problem since  $q > 1$ .) Moreover,  $\nabla w$  is clearly in  $C([0, T]; L^S)$ . Therefore,

$$w(t) \equiv |\nabla u(t)|^{q-2} \nabla u(t) \circ \nabla w(t) \quad (2.10)$$

is in  $C([0, T]; L^S)$ . Let

$$z(t) = \int_0^t e^{(t-s)\Delta} \Delta w(s) ds.$$

Since  $W_0^{1,S} = D_S(\sqrt{-\Delta})$ , the domain of  $\sqrt{-\Delta}$  in  $L^S$  (see [26,27]), to show  $z(t)$  is Holder continuous in  $W_0^{1,S}$ , it suffices to show  $\sqrt{-\Delta} z(t)$  Holder continuous in  $L^S$ . For  $0 \leq t < t + \tau \leq T$ , we have

$$\sqrt{-\Delta}(z(t + \tau) - z(t)) = \sqrt{-\Delta}(e^{\tau\Delta} - I) \int_0^t e^{(t-s)\Delta} \Delta w(s) ds \\ + \sqrt{-\Delta} \int_0^\tau e^{s\Delta} w(t + \tau - s) ds$$

$$= (e^{\tau\Delta} - I)(-\Delta)^{-\alpha} \int (-\Delta)^{\alpha+1/2} e^{(t-s)\Delta} w(s) ds \\ + \int_0^{\tau} (-\Delta)^{1/2} e^{s\Delta} w(t + \tau - s) ds,$$

where  $0 < \alpha < 1/2$ . Using the facts (Theorems 11.3 and 12.1 in [21]) that for  $t \in (0, T]$  and  $0 < \nu < 1$

$$\|(-\Delta)^{\nu} e^{t\Delta}\|_S \leq Ct^{-\nu}$$

$$\|(e^{t\Delta} - I)(-\Delta)^{-\nu}\|_S \leq Ct^{\nu},$$

we deduce that

$$\|\sqrt{-\Delta} (z(t + \tau) - z(t))\|_S \leq C\tau^{\alpha} \int_0^t (t - s)^{-\alpha - 1/2} ds \sup_{[0, T]} \|w(t)\|_S \\ + C \int_0^{\tau} s^{-1/2} ds \sup_{[0, T]} \|w(t)\|_S.$$

This proves the Holder continuity of  $\sqrt{-\Delta} z(t)$  in  $L^S$  and thereby completes the proof of the lemma.

Remark. For the above result we do not need the rather strong result that  $W_0^{1, S} = D_S(\sqrt{-\Delta})$ . The easier result that  $D_S((-\Delta)^{\nu})$  is continuously embedded in  $W_0^{1, S}$  (Theorem 9.2 in [21]) can be used with only a slight modification of the proof.

Proposition 2.5. Under the same conditions and with the same notation of Lemma 2.4, we have that  $v \in C^1((0, T_{\phi}); L^{S/q})$  and

$$v'(t) = \mathfrak{w}(t) - q|\mathfrak{u}(t)|^{q-2}\mathfrak{u}(t) \circ \mathfrak{w}(t) + p|u(t)|^{p-1}v(t). \quad (2.11)$$

Proof. Since  $\phi \in D(B)$ ,  $u \in C^1([0, T_\phi); W_0^{1,s})$ , and so  $\nabla u \in C^1([0, T_\phi); L^s)$ . Furthermore, by the previous lemma  $\nabla v$  is Holder continuous in  $L^s$  on  $[\epsilon, T]$ . It follows that  $w(t)$  given by (2.10) is Holder continuous in  $L^{s/q}$  on  $[\epsilon, T]$ . Similarly,  $|u|^{p-1}v$  is Holder continuous in  $L^{s/q}$  on  $[\epsilon, T]$ .

We now consider the integral equation (2.9) as an equation in  $L^{s/q}$ . The semigroup is analytic in  $L^{s/q}$  and the two integrands (not including  $e^{(t-s)\Delta}$ ) are both Holder continuous functions into  $L^{s/q}$  on any compact subinterval  $[\epsilon, T] \subset (0, T_\phi)$ . The result now follows from well known properties of analytic semigroups. (See, for example, Theorem 1.27 in Chapter IX of [19].)

We are now able to prove the desired positivity of  $u'(t)$ .

Proposition 2.6. Suppose  $s \in \mathbb{R}$  satisfies (2.2) and (2.5) and that  $\phi$  is in  $D(B)$ . Let  $u(t) = W_t \phi$  and  $v(t) = u'(t)$ , and suppose further that  $v(0) = u'(0) = B\phi \geq 0$ . Then  $u'(t) \geq 0$  for all  $t \in [0, T_\phi)$ .

Proof. We know  $v \in C([0, T_\phi); W_0^{1,s}) \subset C([0, T_\phi); H_0^1)$  and  $v^\circ \in C^1((0, T_\phi); L^{s/q}) \subset C^1((0, T_\phi); L^2)$ . Thus, multiplying (2.11) by  $v(t)^-$  and integrating over  $\Omega$  yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int |v^-|^2 &= -\int |\nabla v^-|^2 + q \int |\nabla u|^{q-2} \nabla u \circ \nabla(v^-) - p \int |u|^{p-1} v v^- \\ &\leq -\int |\nabla v^-|^2 + C \int |\nabla v^-| |v^-| + C \int |v^-|^2, \end{aligned}$$

where we use Proposition 2.2, iii) to estimate  $\nabla u$  and  $u$ , and  $C$  can be chosen uniformly for  $t \in (0, T]$ ,  $T < T_\phi$ . The proof is completed exactly as in the proof of Proposition 2.3.

### 3. ENERGY ARGUMENTS

In this section we prove Theorem 1.2. Throughout this section,  $\Omega$ ,  $\Gamma$ ,  $p$ , and  $q$  are as in the introduction and we assume  $s \in \mathbb{R}$  satisfies (2.2) and (2.5). Also, we take  $\phi \in W^{3,s}$ , not identically zero satisfying hypotheses i)-iv) of Theorem 1.2. In the language of Section 2, that means  $\phi \in D(B) \subset W_0^{1,s}$  with  $\phi > 0$  and  $B\phi > 0$ .  $T_\phi$  is the existence time of the maximal solution  $u(t)$  of the integral equation (2.1). By Propositions 2.2, 2.3 and 2.6,  $u \in C^1([0, T_\phi); W_0^{1,s})$ , satisfying equation (2.8), with  $u(t) > 0$  and  $u'(t) > 0$  for all  $t \in [0, T_\phi)$ .

Lemma 3.1. The energy of the solution  $u(t)$ ,

$$E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{p+1} \|u(t)\|_{p+1}^{p+1}$$

is a non-increasing function of  $t \in [0, T_\phi)$ .

Proof. Since  $u \in C^1([0, T_\phi); W_0^{1,s})$  and, by (2.5),  $W_0^{1,s} \subset H_0^1$  and  $W_0^{1,s} \subset L^\infty$ , it follows that  $E(u(t))$  is a  $C^1$  function of  $t \in [0, T_\phi)$ . We easily calculate from (2.8) that

$$\begin{aligned} \frac{d}{dt} E(u(t)) &= \langle -\Delta u(t), u'(t) \rangle - \langle u(t)^p, u'(t) \rangle \\ &= \langle -u'(t) + |\nabla u(t)|^q, u'(t) \rangle \\ &< 0. \end{aligned}$$

Lemma 3.2. Suppose  $E(\phi) = E(u(0)) < 0$  and that  $q < 2p/(p+1)$ . Then for all  $t \in [0, T_\phi)$

$$|\langle u(t), |\nabla u(t)|^q \rangle| < \left( \frac{2}{p+1} \right)^{q/2} C(p, q) \|u(t)\|_{p+1}^{p+1-\alpha}$$

where

$$\alpha = p - \frac{q(p+1)}{2} > 0$$

and  $C(p,q) = 1$  in case  $q = 2p/(p + 1)$ .

Proof. From Holder's inequality, it follows that

$$\begin{aligned} |\langle u, |\nabla u|^q \rangle| &\leq \|u\|_{p+1} \|\nabla u\|_{q(p+1)/p}^q \\ &= \|u\|_{p+1} \|\nabla u\|_{q(p+1)/p}^q. \end{aligned}$$

Since  $q \leq 2p/(p + 1)$ , we have  $q(p + 1)/p \leq 2$ ; and so

$$|\langle u, |\nabla u|^q \rangle| \leq C(p,q) \|u\|_{p+1} \|\nabla u\|_2^q,$$

where  $C(p,q) = 1$  if  $q = 2p/(p + 1)$ .

By the previous lemma  $E(u(t)) \leq 0$  for all  $t \in [0, T_\phi)$ , or

$$\|\nabla u\|_2^q \leq \left(\frac{2}{p+1}\right)^{q/2} \|u\|_{p+1}^{q(p+1)/2}.$$

The result follows by combining the last two inequalities.

Proof of Theorem 1.2.

Suppose to the contrary that  $T_\phi = \infty$ . Let  $F(t) = \|u(t)\|_2^2$ . Then  $F \in C^1([0, \infty))$ ,  $F(0) = \|\phi\|_2^2 > 0$ , and

$$\begin{aligned} F'(t) &= 2\langle u(t), u'(t) \rangle \\ &= -2\|\nabla u(t)\|_2^2 - 2\langle u(t), |\nabla u(t)|^q \rangle + 2\|u(t)\|_{p+1}^{p+1} \\ &= -4E(t) + 2\left(\frac{p-1}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - 2\langle u, |\nabla u|^q \rangle \\ &> 2\left(\frac{p-1}{p+1}\right) \|u(t)\|_{p+1}^{p+1} - 2\left(\frac{2}{p+1}\right)^{q/2} C(p,q) \|u(t)\|_{p+1}^{p+1-\alpha}, \end{aligned}$$

where we have used Lemma 3.1 ( $E(u(t)) \leq E(\phi) \leq 0$ ) and Lemma 3.2. Continuing the calculation, we have

$$\begin{aligned} F'(t) &> 2\|u(t)\|_{p+1}^{p+1} \left[ \left( \frac{p-1}{p+1} \right) - \left( \frac{2}{p+1} \right)^{q/2} C(p,q) \|u(t)\|_{p+1}^{-\alpha} \right] \\ &> CF(t)^{(p+1)/2} \left[ \left( \frac{p-1}{p+1} \right) - \left( \frac{2}{p+1} \right)^{q/2} C(p,q) \|u(t)\|_{p+1}^{-\alpha} \right]. \end{aligned}$$

Suppose first  $q < 2p/(p+1)$  and that  $\|\phi\|_{p+1}$  is sufficiently large so that

$$\left( \frac{p-1}{p+1} \right) - \left( \frac{2}{p+1} \right)^{q/2} C(p,q) \|\phi\|_{p+1}^{-\alpha} \equiv k > 0.$$

Then since  $u'(t) > 0$ , it follows that

$$F'(t) > kCF(t)^{(p+1)/2}$$

for all  $t \in [0, \infty)$ . Since  $(p+1)/2 > 1$ , this is impossible for a function  $F \in C^1([0, \infty))$  with  $F(0) > 0$ . This contradiction shows  $T_\phi < \infty$ .

Now suppose  $q = 2p/(p+1)$ . Then  $C(p,q) = 1$  and  $\alpha = 0$ ; so

$$F'(t) > CF(t)^{(p+1)/2} \left[ \left( \frac{p-1}{p+1} \right) - \left( \frac{2}{p+1} \right)^{p/(p+1)} \right].$$

This again yields a contradiction if  $p$  is large enough that the coefficient above is positive.

Remark. The expression

$$\left( \frac{p-1}{p+1} \right) - \left( \frac{2}{p+1} \right)^{p/(p+1)}$$

is increasing in  $p$  for  $1 < p < \infty$  with limit 1 as  $p \rightarrow \infty$ . Thus, if we let  $p_*$  be its unique zero in this range, then the above argument works for all  $p > p_*$ . An easy computation shows  $3.3 < p_* < 3.4$ .

#### 4. THE ELLIPTIC PROBLEM

For the moment,  $\Omega$ ,  $\Gamma$ ,  $p$ , and  $q$  are as in the introduction. Also,  $\lambda$  is always positive. Our goal is first to show the connection between the elliptic problem (1.5) and the hypotheses of Theorem 1.2. Then for  $\Omega = B_R$  we will study the existence of solutions to (1.5). Together, this will prove Theorem 1.3.

Proposition 4.1. Let  $2 < s < \infty$  and suppose  $\phi \in W^{2,s}(\Omega)$  is a solution of (1.5) with  $\lambda < 2/(p+1)$ . Then  $\phi$  satisfies hypotheses i)-v) of Theorem 1.2.  
If in addition,  $s$  satisfies (2.5) then  $\phi \in W^{3,s}$ .

Proof. Hypotheses i) and iii) are stated in (1.5), and so there is nothing to prove. Now

$$\Delta\phi - |\nabla\phi|^q + \phi^p = (1 - \lambda)\phi^p,$$

from which immediately follow ii) and iv). Finally, since  $\phi = 0$  on  $\Gamma$  implies  $\phi \in W_0^{1,s}$ ,

$$\begin{aligned} E(\phi) &= \frac{1}{2} \|\Delta\phi\|_2^2 - \frac{1}{p+1} \|\phi\|_{p+1}^{p+1} \\ &= -\frac{1}{2} \int_{\Omega} \phi \Delta\phi - \frac{1}{p+1} \int_{\Omega} \phi^{p+1} \\ &= -\frac{1}{2} \int_{\Omega} \phi |\nabla\phi|^q - \left( \frac{1}{p+1} - \frac{\lambda}{2} \right) \int_{\Omega} \phi^{p+1}. \end{aligned}$$

Thus  $E(\phi) < 0$  because  $\lambda < 2/(p+1)$ . The regularity of  $\phi$  follows exactly as in the proof of Proposition 2.1. Simply note at the start that, thanks to (2.5),  $\phi^p \in W_0^{1,s}$  and so

$$B\phi = (1 - \lambda)\phi^p \in W_0^{1,s}.$$

Proposition 4.2. Assume that  $p < (n + 2)/(n - 2)$ . (If  $n = 1$  or  $2$ , this condition is vacuous.) Suppose that  $\phi_k \in H_0^1$ ,  $k = 1, 2, 3, \dots$ , satisfy

$$\left. \begin{aligned} \Delta \phi_k + \lambda_k \phi_k^p &> 0 \\ \phi_k &> 0 \\ \phi_k &\not\equiv 0 \end{aligned} \right\} \quad (4.1)$$

where  $\lambda_k > 0$  and  $\lambda_k \rightarrow 0$  as  $k \rightarrow \infty$ . Then  $\|\phi_k\|_{p+1} \rightarrow \infty$  as  $k \rightarrow \infty$ .

Proof. Suppose not. Then, by passing to a subsequence, we may assume

$\|\phi_k\|_{p+1} \leq M$  independent of  $k$ . Let  $\|\phi_k\|_{p+1} = N_k$  and

$$\psi_k = \phi_k / N_k.$$

Obviously  $\|\psi_k\|_{p+1} = 1$ . ( $N_k \neq 0$  since  $\phi_k \not\equiv 0$ .) Moreover, multiplying the inequality in (4.1) by  $\phi_k$  and integrating over  $\Omega$ , we have that

$$\|\nabla \phi_k\|_2^2 \leq \lambda_k \|\phi_k\|_{p+1}^{p+1},$$

or

$$\|\Delta \psi_k\|_2^2 \leq \lambda_k N_k^{p-1} \|\psi_k\|_{p+1}^{p+1}.$$

Since  $\|\psi_k\|_{p+1} = 1$ ,  $N_k \leq M$ , and  $\lambda_k \rightarrow 0$ , it follows that  $\|\nabla \psi_k\|_2 \rightarrow 0$  as  $k \rightarrow \infty$ , i.e.  $\psi_k \rightarrow 0$  in  $H_0^1(\Omega)$  as  $k \rightarrow \infty$ . However, the condition on  $p$  implies that  $H_0^1(\Omega)$  is embedded in  $L^{p+1}$ , and so  $\psi_k \rightarrow 0$  in  $L^{p+1}$ . This contradicts the fact that  $\|\psi_k\|_{p+1} = 1$ , thereby proving the proposition.



The following corollary to the above two proposition states explilly how solutions of (1.5) yield solutions of (1.1) which blow up in finite time.

Corollary 4.3. Assume  $s \in \mathbb{R}$  satisfies (2.2) and (2.5). Suppose first that  $1 < q < 2p/(p+1)$  and that  $1 < p < (n+2)/(n-2)$ , ( $1 < p < \infty$  if  $n = 1$  or  $2$ .) If  $\phi \in W^{2,s}$  is a solution of (1.5) with  $\lambda$  sufficiently small, then  $\phi$  satisfies all the hypotheses of Theorem 1.2; and so the solution of (1.1) with initial value  $\phi$  blows up in finite time.

Suppose next that  $q = 2p/(p+1)$  with  $p > p_*$ . (See the remark at the end of Section 2.) If  $\phi \in W^{2,s}$  is a solution of (1.5) with  $\lambda \leq 2/(p+1)$ , then  $\phi$  satisfies all the hypotheses of Theorem 1.2; and so the solution of (1.1) with initial value  $\phi$  blows up in finite time.

Now we let  $\Omega = B_R = \{x \in \mathbb{R}^n : |x| < R\}$ , and we look for solutions of (1.5) on  $B_R$ . In fact we are going to look for radially symmetric solutions of (1.5). This is not a genuine restriction, because the techniques of [13] can be used to show that any solution of (1.5) in  $B_R$  must be radially symmetric. We are therefore led to consider the following initial value problem:

$$\left. \begin{aligned} u''(r) + \frac{n-1}{r} u'(r) - |u'(r)|^q + \lambda |u(r)|^p &= 0, & r > 0 \\ u(0) &= a > 0 \\ u'(0) &= 0. \end{aligned} \right\} \quad (4.2)$$

If  $u \in C^2([0,R])$  is a solution of (4.2) with  $u(r) > 0$  for  $0 \leq r < R$  and  $u(R) = 0$ , then  $\phi(x) = u(|x|)$  is the desired solution of (1.5). (Note, for the rest of the paper, we will no longer be directly concerned with problem (1.1). Thus, the letter "u" will henceforth be used to denote solutions of (4.2).)

Proposition 4.4. Fix  $\lambda > 0$ . For every  $a > 0$  there exists a (unique) maximal solution  $u \in C^2([0,R_a])$  of (4.2). Furthermore:

i)  $u'(r) < 0$  for all  $r$ ,  $0 < r < R_a$ ;

ii) the function

$$H(r) = \frac{1}{2} u'(r)^2 + \frac{\lambda}{p+1} |u(r)|^p u(r)$$

is decreasing on  $[0, R_a)$ ;

iii) if  $u(r) > 0$  for all  $0 < r < R_a$ , then  $R_a = \infty$  and

$$\lim_{r \rightarrow \infty} u(r) = 0,$$

$$\lim_{r \rightarrow \infty} u'(r) = 0,$$

$$\lim_{r \rightarrow \infty} u''(r) = 0.$$

Proof. We first prove the existence of a unique solution to (4.2) on some interval  $[0, \epsilon]$ . Consider the system

$$\left. \begin{aligned} u(r) &= a + \int_0^r v(s) ds \\ v(r) &= r^{-(n-1)} \int_0^r s^{n-1} (|v(s)|^q - \lambda |u(s)|^p) ds. \end{aligned} \right\} \quad (4.3)$$

It is easy to see that a solution of (4.2) is also a solution of (4.3) with  $v = u'$ . Indeed, simply multiply the equation in (4.2) by  $r^{n-1}$  and integrate. On the other hand, by standard iteration techniques, there is certainly a unique solution  $u, v \in C([0, \epsilon])$  to (4.3) for some  $\epsilon > 0$ . Clearly  $u \in C^1([0, \epsilon])$  with  $u'(r) = v(r)$ . In particular,  $u'(0) = 0$ . Moreover,  $v$  is immediately seen to be in  $C^1((0, \epsilon])$  and so  $u \in C^2((0, \epsilon])$  and satisfies (4.2). It remains to

show that  $u$  is  $C^2$  at  $r = 0$ , i.e. that  $v$  is  $C^1$  at  $r = 0$ . From (4.3), l'Hopital's rule easily gives

$$v'(0) = \lim_{r \rightarrow 0} \frac{v(r)}{r} = -\frac{\lambda a^p}{n}.$$

On the other hand, from (4.2)

$$\begin{aligned} \lim_{r \rightarrow 0} v'(r) &= \lim_{r \rightarrow 0} u''(r) = \frac{(n-1)\lambda a^p}{n} - \lambda a^p \\ &= -\frac{\lambda a^p}{n}. \end{aligned}$$

Thus,  $u \in C^2([0, \epsilon])$ .

Since for  $r > 0$ , there are no singularities in (4.2), the solution on  $[0, \epsilon]$  can be locally continued to a maximal solution  $u \in C^2([0, R_a))$ . Since the continuation procedure treats (4.2) as a system  $(u(r), v(r))$  with  $u'(r) = v(r)$ , it follows that if  $R_a < \infty$  then either  $|u(r)| \rightarrow \infty$  or  $|u'(r)| \rightarrow \infty$  as  $r \rightarrow R_a$ .

To prove i), note first that  $u'(0) = 0$  and  $u''(0) < 0$ . Hence  $u'(r) < 0$  on some interval  $(0, \delta)$ . Let  $r_0$  be the first positive zero of  $u'$ . Then  $u(r_0) \neq 0$  since if  $u(r_0) = u'(r_0) = 0$ , it follows by uniqueness that  $u(r) \equiv 0$ . Consequently, from the equation in (4.2)

$$u''(r_0) = -\lambda |u(r_0)|^p < 0,$$

which implies that  $u'(r) > 0$  for  $r$  in some interval  $(r_0 - \epsilon, r_0)$ . This contradicts the choice of  $r_0$ , and thereby proves i).

Next, we compute easily that for  $r > 0$

$$\begin{aligned} H'(r) &= u'(r)u''(r) + \lambda |u(r)|^p u'(r) \\ &= u'(r) \left[ -\left(\frac{n-1}{r}\right)u'(r) + |u'(r)|^q \right] \\ &< 0. \end{aligned}$$

This proves ii).

Finally, if  $u(r) > 0$  for all  $r \in [0, R_a)$ , then  $0 < H(r) < H(0)$  for all  $r \in [0, R_a)$ . Thus,  $u(r)$  and  $u'(r)$  are a priori bounded and so  $R_a = \infty$ . Now  $u'(r) < 0$  and  $u(r) > 0$ . Hence

$$\lim_{r \rightarrow \infty} u(r) = u_{\infty} \text{ (finite)}$$

exists. Likewise  $H(r)$  has a finite limit as  $r \rightarrow \infty$ . It follows therefore that

$$\lim_{r \rightarrow \infty} u'(r) = v_{\infty}$$

exists. In fact we must have  $v_{\infty} = 0$  in order for  $\lim_{r \rightarrow \infty} u(r)$  to exist.

Finally, from (4.2) we now deduce that

$$\lim_{r \rightarrow \infty} u''(r) = -\lambda |u_{\infty}|^p.$$

Thus, the only way we can have  $\lim_{r \rightarrow \infty} u'(r) = 0$  is if  $u_{\infty} = 0$ . This completes the proof of iii).

For a fixed  $\lambda > 0$ , we denote the first zero of the solution to (4.2) by  $z(a)$ . We make the convention that  $z(a) = \infty$  in case  $u(r) > 0$  for all  $r \geq 0$ . Thus, the solution  $u(r)$  of (4.2) yields the desired solution of (1.5) precisely if  $z(a) = R$ . This certainly motivates studying the function  $z(a)$ .

Proposition 4.5.  $\{a > 0: z(a) < \infty\}$  is open and  $z(\cdot)$  is continuous on this set. Moreover

$$\lim_{a \rightarrow 0} z(a) = \infty. \tag{4.4}$$

Also, if  $z(a_0) = \infty$  for some  $a_0 \in \mathbf{R}$ , then  $\lim_{a \rightarrow a_0} z(a) = \infty$ .

Proof. If  $z(a) < \infty$ , then  $u(r) < 0$  for  $r$  slightly larger than  $z(a)$ . By continuous dependence on the data, if we change  $a$  by only a little bit,  $u(r)$  must still be negative somewhere, and therefore have a zero. Continuity of  $z(\cdot)$  follows from continuous dependence of  $u(r)$  on  $a$  and the fact that  $u$  can have at most one zero since  $u'(r) < 0$  for  $r > 0$ .

To prove (4.4), we note that since  $H(r)$  is decreasing, it follows that

$$\frac{1}{2} u'(r)^2 \leq H(r) \leq H(0) = \frac{\lambda a^{p+1}}{p+1}$$

or

$$|u'(r)| \leq \sqrt{2\lambda/(p+1)} a^{(p+1)/2}. \quad (4.5)$$

Consequently

$$\begin{aligned} a &= u(0) - u(z(a)) \\ &= - \int_0^{z(a)} u'(s) ds \\ &\leq z(a) a^{(p+1)/2} \sqrt{2\lambda/(p+1)}, \end{aligned}$$

or

$$z(a) \geq \frac{a^{-(p-1)/2}}{\sqrt{2\lambda/(p+1)}}. \quad (4.6)$$

This proves (4.4).

For the last statement, we show that given  $M > 0$ , then  $z(a) > M$  if  $a$  is sufficiently close to  $a_0$ . Now for  $a = a_0$ ,  $u(r) > 0$  for all  $r > 0$ ; hence  $u(r) \geq \delta > 0$  on  $[0, M]$ . By continuous dependence on the data, if  $a$  is sufficiently close to  $a_0$  then  $u(r) \geq \delta/2$  on  $[0, M]$ . Hence  $z(a) > M$  for such  $a$ .

Next we would like to study the behavior of  $z(a)$  as  $a \rightarrow \infty$ . We first consider the case  $q < 2p/(p+1)$ .

Proposition 4.6. Assume that  $q < 2p/(p+1)$  and (in case  $n \geq 3$ )  $p < (n+2)/(n-2)$ . Then, for all  $\lambda > 0$ , we have

$$\limsup_{a \rightarrow \infty} a^{(p-1)/2} z(a) < \infty, \quad (4.7)$$

$$\lim_{a \rightarrow \infty} z(a) = 0. \quad (4.8)$$

Proof. Fix  $\lambda > 0$ . Denote by  $u(\cdot; a)$  the solution of (4.2) with initial value  $a$ ; and for all  $a > 0$ , set

$$v_a(r) = a^{-1} u(ra^{-(p-1)/2}; a).$$

Then  $v_a$  is easily seen to satisfy

$$\left. \begin{aligned} v_a'' + \frac{n-1}{r} v_a' - a^{q(p+1)/2} - p|v_a'| + \lambda|v_a|^p &= 0 \\ v_a(0) &= 1 \\ v_a'(0) &= 0 \end{aligned} \right\} \quad (4.9)$$

Also,  $v_a'(r) < 0$  for  $r > 0$  and  $v_a(r) > 0$  for  $0 \leq r < a^{(p-1)/2} z(a)$ . Hence

$$0 \leq v_a(r) \leq 1, \quad 0 \leq r \leq a^{(p-1)/2} z(a). \quad (4.10)$$

Moreover, (4.5) translates into

$$|v_a'(r)| \leq \sqrt{2\lambda/(p+1)}, \quad r \geq 0. \quad (4.11)$$

Suppose now there exists a sequence  $a_m \rightarrow \infty$  such that  $a_m^{(p-1)/2} z(a_m) \rightarrow \infty$  as  $m \rightarrow \infty$ . By the Arzela-Ascoli theorem and a standard diagonal argument,

there is a subsequence, which we still denote  $a_m$ , and a continuous function  $v: [0, \infty) \rightarrow [0, 1]$  such that  $v_{a_m} \rightarrow v$  uniformly on all compact subsets  $[0, M] \subset [0, \infty)$ . In particular,  $v(0) = 1$ ,  $v$  is nonincreasing on  $[0, \infty)$ , and  $v$  is Lipschitz continuous with a Lipschitz constant no greater than  $\sqrt{2\lambda/(p+1)}$ . (Each  $v_a$  has these properties.) Finally, since  $q < 2p/(p+1)$  and  $|v'_a|$  is bounded independent of  $r$  and  $a$ , it follows from (4.9) that

$$v'' + \frac{n-1}{r} v' + \lambda v^p = 0 \quad (4.12)$$

in the sense of distributions on  $(0, \infty)$ .

It is well known that since  $p > (n+2)/(n-2)$ , such a  $v$  cannot exist. This is proved on pp. 293-294 in [33] in the case  $\lambda = 1$ . (See also Proposition 3.9 in [18].) The same arguments work for any  $\lambda > 0$ , or else  $\lambda$  can be scaled away by multiplying  $v$  by a suitable factor. This proves (4.7), and hence (4.8).

We now turn to the case  $q = 2p/(p+1)$ . This will be quite different since the scaled solution  $v_a$  satisfies the same equation as  $u$ .

Lemma 4.7. If  $q = 2p/(p+1)$ , then for all  $a > 0$ ,

$$z(a) = a^{-(p-1)/2} z(1). \quad (4.13)$$

Proof. By (4.9), we see that  $v_a = u(\cdot; 1)$  for all  $a > 0$ . Hence the first zero of  $v_a$  is  $z(1)$ . However, by the definition of  $v_a$ , its first zero is  $a^{(p-1)/2} z(a)$ . This proves (4.13).

In other words, whether or not  $z(a)$  is finite depends entirely on whether or not  $z(1)$  is finite. This in turn depends on  $\lambda$ .

Lemma 4.8 Let  $r_0 > 0$ ,  $\lambda > 0$ ,  $q = 2p/(p+1)$  and suppose  $u: (r_0, \infty) \rightarrow \mathbb{R}$  is  $C^2$  and satisfies

- i)  $u(r) > 0, r > r_0, \text{ and } \lim_{r \rightarrow \infty} u(r) = 0;$
- ii)  $u'(r) < 0, r > r_0, \text{ and } \lim_{r \rightarrow \infty} u'(r) = 0;$
- iii)  $u''(r) - |u'(r)|^q + \lambda u(r)^p = 0, r > r_0.$

If in addition  $u(r)$  satisfies

$$u''(r) \leq k|u'(r)|^q, r > r_0 \tag{4.14}$$

where  $k > 0$  is a fixed constant, then  $u$  must also satisfy

$$u''(r) \leq [1 - \lambda(\frac{p+1}{2k})^p] |u'(r)|^q, r > r_0. \tag{4.15}$$

Proof. Since  $u'(r) < 0$ , inequality (4.14) can be rewritten

$$(-u'(r))^{-(q-1)} u''(r) \leq -ku'(r).$$

Integrating this from  $r$  to  $\infty$ , we get

$$\frac{(-u'(r))^{2-q}}{2-q} \leq ku(r),$$

or

$$u(r) \geq \frac{p+1}{2k} (-u'(r))^{2/(p+1)}.$$

(Recall  $q = 2p/(p + 1) < 2$ .)

Hence

$$\begin{aligned} u''(r) &= (-u'(r))^q - \lambda u(r)^p \\ &\leq [1 - \lambda(\frac{p+1}{2k})^p] (-u'(r))^q. \end{aligned}$$



Proposition 4.9. Suppose  $n = 1$  and  $q = 2p/(p + 1)$ . If  $\lambda > (2/(p + 1))^p$ , then  $z(a) < \infty$  for all solutions  $u(r)$  of (4.2).

Proof. If  $z(a) = \infty$ , then  $u(r) > 0$  for all  $r > 0$ . By Proposition 4.4, and the fact that  $n = 1$ ,  $u(r)$  satisfies conditions i)-iii) of Lemma 4.8 with  $r_0 = 0$ . Furthermore, by (4.2) with  $n = 1$ , we have that (4.14) holds with  $k = 1$ . Thus (4.15) holds with  $k = 1$ . Since  $\lambda > [2/(p+1)]^p$ , it follows that  $u''(r) < 0$  for  $r > 0$ . This is impossible because  $u'(r) < 0$  and  $u(r) > 0$  for  $r > 0$ .

The result in Proposition 4.9 is already enough to give us a solution of (1.5) with  $\lambda < 2/(p + 1)$ . Indeed,  $[2/(p + 1)]^p < 2/(p + 1)$ . However, this result can be improved.

Lemma 4.10. Suppose  $\lambda > \lambda_p$ , given by (1.6). Define the following sequence inductively:

$$k_0 = 1,$$

$$k_m = 1 - \lambda \left( \frac{p+1}{2k_{m-1}} \right)^p,$$

as long as  $k_{m-1} > 0$ . Then either  $k_m$  is eventually non-positive or

$$\lim_{m \rightarrow \infty} k_m = 0.$$

Proof. It is easy to verify by induction that the sequence  $k_m$  is decreasing as long as it is defined. Consequently, if the conclusion is false, then

$$\lim_{m \rightarrow \infty} k_m = k > 0,$$

and  $k$  must satisfy

$$k = 1 - \lambda \left( \frac{p+1}{2k} \right)^p.$$

In other words,  $k$  is a positive solution to

$$f(x) = x^{p+1} - x^p = -\lambda \left( \frac{p+1}{2} \right)^p$$

However, the minimum value of  $f(x)$  for  $x > 0$  is easily computed to be

$$-\frac{1}{p+1} \cdot \left( \frac{p}{p+1} \right)^p.$$

Hence we must have

$$\lambda \leq \left( \frac{2}{p+1} \right)^p \cdot \frac{1}{p+1} \left( \frac{p}{p+1} \right)^p = \lambda_p.$$

Therefore, if  $\lambda > \lambda_p$ , the conclusion holds.

Proposition 4.11. Suppose  $n = 1$  and  $q = 2p/(p+1)$ . If  $\lambda > \lambda_p$ , then  $z(a) < \infty$  for all solutions of (4.2).

Proof. If  $z(a) = \infty$ , then  $u(r) > 0$  for all  $r > 0$ . By Proposition 4.4 and the fact that  $n = 1$ ,  $u(r)$  satisfies conditions i)-iii) of Lemma 4.8 with  $r_0 = 0$ . Furthermore, by (4.2) with  $n = 1$ , we have that (4.14) holds with  $k = 1$ . Hence by Lemma 4.8, (4.14) and therefore (4.15) hold with all values  $k = k_m$  defined in Lemma 4.10. Thus, by Lemma 4.10,  $u''(r) \leq 0$  for all  $r > 0$ . This is impossible since  $u'(r) < 0$  and  $u(r) > 0$  for  $r > 0$ .

Proof of Theorem 1.3. Suppose first that  $1 < q < 2p/(p+1)$  and (if  $n > 3$ )  $p < (n+2)/(n-2)$ . By Propositions 4.5 and 4.6, for all  $\lambda > 0$  and  $R > 0$ , there exists  $a > 0$  such that  $z(a) = R$ . In other words, if  $u(r)$  is the solution to (4.2) with this initial value  $a$ , then  $u(r) > 0$  for  $0 \leq r < R$  and  $u(R) = 0$ . Then  $\phi(x) = u(|x|)$ ,  $|x| \leq R$ , is the desired solution of (1.5). The other properties of  $\phi$  follow from Corollary 4.3.

Now suppose that  $n = 1$ ,  $q = 2p/(p+1)$ , and  $\lambda > \lambda_p$ . Then by Proposition 4.11 and Lemma 4.7, there exists (a unique)  $a > 0$  such that  $z(a) = R$ . The rest of the proof is as in the previous case.

## 5. FURTHER RESULTS ON THE ELLIPTIC PROBLEM

In this section we continue in the same context and with the same notation established in the previous section. In particular, for a fixed  $\lambda > 0$ ,  $z(a)$  is the first zero of the solution to the initial value problem (4.2), with the convention that  $z(a) = \infty$  in case the solution always remains positive. Our goal is to study more completely the problem (4.2), i.e. the elliptic problem (1.5) in  $B_R$ . We first gather as much information as we can in the general case, and then specialize to dimension  $n = 1$ . The following result is a variation on the estimate (4.6).

Lemma 5.1. Let  $\lambda > 0$  and  $u(r)$  be the solution of (4.2), with  $z(a)$  the first zero of  $u(r)$ . Then

$$z(a) > \lambda^{-1/q} a^{1-(p/q)}. \quad (5.1)$$

Proof. We may certainly assume  $z(a) < \infty$ . We claim first that the maximum value of  $-u'(r)$  on  $[0, z(a)]$  is achieved in the interior. Indeed,

$$-u''(0) = \frac{\lambda a^p}{n} > 0,$$

$$-u''(z(a)) = \left( \frac{n-1}{z(a)} \right) u'(z(a)) - |u'(z(a))|^q < 0.$$

So if  $r_0$  is such that  $-u'(r_0)$  is a maximum on  $[0, z(a)]$ , then  $u''(r_0) = 0$ , i.e.

$$-(-u'(r_0))^q + \lambda u(r_0)^p = -\frac{n-1}{r_0} u'(r_0) > 0$$

or

$$(-u'(r_0))^q < \lambda u(r_0)^p < \lambda a^p.$$

This implies that for  $0 < r \leq z(a)$ ,

$$-u'(r) \leq \lambda^{1/q} a^{p/q}.$$

Hence,

$$\begin{aligned} a &= u(0) - u(z(a)) \\ &= -\int_0^{z(a)} u'(s) ds \\ &\leq z(a) \lambda^{1/q} a^{p/q}, \end{aligned}$$

which proves (5.1).

Lemma 5.2. Let  $q > 2p/(p + 1)$  and (in case  $n \geq 3)$   $p < (n+2)/(n-2)$ . Then,  
for all  $\lambda > 0$ , we have

$$\limsup_{a \rightarrow 0} a^{(p-1)/2} z(a) < \infty.$$

In particular, for all sufficiently small  $a$ ,  $z(a) < \infty$  and

$$z(a) \leq Ca^{-(p-1)/2}. \tag{5.2}$$

Proof. This is an obvious modification to the proof of Proposition 4.6.

Let us consider for a moment whether or not a regular (i.e.  $C^2$ ) solution of the elliptic problem (1.5) exists on  $\Omega = B_R$ . As noted in Section 4, the methods of [13] can be used to prove that such a solution,  $\phi(x)$ , must be radially symmetric. In other words,  $\phi(x) = u(|x|)$ , where  $u(r)$  is a solution of (4.2) with  $z(a) = R$ . Therefore, for a fixed  $\lambda > 0$ , the number of solutions to (1.5) on  $B_R$  is precisely the cardinality of the set  $z^{-1}(R)$ . For each  $\lambda > 0$ , we define

$$R(\lambda) = \inf_{a>0} z(a). \quad (5.3)$$

Note that if  $R(\lambda) < \infty$ , then by Proposition 4.5,  $z(a) = R$  has at least one solution whenever  $R(\lambda) < R < \infty$ .

Proposition 5.3. i) If  $q \geq p$  then  $R(\lambda) > 0$ .

ii) If  $q > 2p/(p+1)$  and (in case  $n \geq 3$ )  $p < (n+2)/(n-2)$ , then  $R(\lambda) < \infty$ .

iii) If  $q > p$  and (in case  $n \geq 3$ )  $p < (n+2)/(n-2)$ , then  $z(a) = R(\lambda) < \infty$  for some  $a > 0$ , and  $z(a) = R$  has at least two solutions  $a > 0$  for each  $R > R(\lambda)$ .

Proof. Statement i) follows from the lower estimates for  $z(a)$  given by (4.6) and (5.1). Statement ii) follows from Lemma 5.2. For statement iii) note that (4.6) and (5.1) imply  $\lim_{a \rightarrow 0} z(a) = \infty$  and  $\lim_{a \rightarrow \infty} z(a) = \infty$ . Moreover, by Lemma 5.2,  $z(a)$  is finite for some values of  $a$  and hence, by Proposition 4.5, assumes its (positive) minimum  $R(\lambda)$  at some  $a_m$ ,  $0 < a_m < \infty$ . Clearly, then for each  $R > R(\lambda)$ , there exist  $a_1$  and  $a_2$  with  $0 < a_1 < a_m < a_2 < \infty$  such that  $z(a_1) = z(a_2) = R$ .

Corollary 5.4: i) If  $q \geq p$  and  $0 < R < R(\lambda)$ , then there is no regular solution of (1.5) on  $B_R$ .

ii) If  $q > 2p/(p+1)$  and (in case  $n \geq 3$ )  $p < (n+2)/(n-2)$ , then for  $R > R(\lambda)$ , there is at least one regular solution of (1.5) on  $B_R$ .

iii) If  $q > p$  and (in case  $n \geq 3$ )  $p < (n+2)/(n-2)$ , then for  $R = R(\lambda)$ , there is at least one regular solution of (1.5) on  $B_R$ ; and for  $R > R(\lambda)$ , there are at least two regular solutions of (1.5) on  $B_R$ .

We next focus our attention on the case  $q = 2p/(p+1)$ . This is particularly interesting since it is the critical value for both the energy arguments in Section 3 and the scaling argument in (the proof of) Proposition 4.6.

Proposition 5.5. Let  $q = 2p/(p + 1)$  and  $\lambda > 0$ . Suppose first  $n=1,2$  or  $n \geq 3$  and  $p < n/(n-2)$ . Then there exists a positive constant  $k$  such that

$$U(r) = kr^{-2/(p-1)} \tag{5.4}$$

satisfies

$$u''(r) + \frac{n-1}{r} u'(r) - |u'(r)|^q + \lambda u(r)^p = 0 \tag{5.5}$$

if and only if  $\lambda \leq \lambda_{p,n}$ , where

$$\lambda_{p,n} = \frac{1}{p+1} \left[ \frac{2p}{(p+1)(2p-np+n)} \right]^p.$$

On the other hand, if  $n \geq 3$  and  $p \geq n/(n-2)$ , then such a solution exists for all  $\lambda > 0$ .)

Proof. By direct calculation, we see that  $U(r)$ , given by (5.4), satisfies (5.5) precisely when

$$\lambda k^{p-1} - \left( \frac{2}{p-1} \right)^q k^{q-1} = - \frac{2}{p-1} \left( \frac{2p}{p-1} - n \right). \tag{5.6}$$

If  $n \geq 3$  and  $p \geq n/(n-2)$ , then the right side of (5.6) is non-negative. In this case, since  $q = 2p/(p + 1) < p$ , a positive solution  $k$  to (5.6) can always be found.

Suppose instead either  $n = 1,2$  or in case  $n \geq 3$ ,  $p < n/(n-2)$ , so the right side of (5.6) is negative. Then a positive solution  $k$  of (5.6) can be found precisely when

$$\inf_{x>0} f(x) < - \frac{2}{p-1} \left( \frac{2p}{p-1} - n \right), \tag{5.7}$$

where  $f(x) = \lambda x^{p-1} - [2/(p-1)]^q x^{q-1}$ . Using elementary calculus, and not forgetting that  $q = 2p/(p+1)$ , one can verify that (5.7) holds if and only if  $\lambda < \lambda_{p,n}$ .

Remark. Note that  $\lambda_{p,1} = \lambda_p$ , defined by (1.6). Also,  $\lambda_{p,n}$  is increasing as a function of  $n$ .

Proposition 5.6. Assume  $q = 2p/(p+1)$  and  $\lambda < \lambda_p = \lambda_{p,1}$ . Then  $z(a) = \infty$  for all  $a > 0$ . In other words, there is no regular solution of (1.5) on  $B_R$ , for any  $R > 0$ .

Proof. In dimension  $n = 1$  there is a particularly easy and elegant proof, which we present first. Suppose a  $C^2$  solution  $\phi$  of (1.5) exists in dimension  $n = 1$ . Let  $U(r)$  be the solution of (5.5) given by (5.4) with  $n = 1$ . (Recall  $\lambda < \lambda_{p,1}$ ). Set

$$\bar{\rho} = \sup\{\rho \in \mathbb{R}: \text{the graph of } \phi(r-\rho) \text{ does not touch the graph of } U(r)\}.$$

Clearly  $\bar{\rho} \in \mathbb{R}$ . Also, the graphs of  $\phi(r-\bar{\rho})$  and  $U(r)$  touch at some point  $r_0$ , i.e.  $\phi(r_0 - \bar{\rho}) = U(r_0) > 0$ ; and by the definition of  $\bar{\rho}$ , we must also have  $\phi'(r_0 - \bar{\rho}) = U'(r_0)$ . However, both  $\phi(r - \bar{\rho})$  and  $U(r)$  satisfy (5.5) with the same Cauchy data at  $r_0$ . Hence  $\phi(r - \bar{\rho}) = U(r)$  wherever both functions are defined. In particular,  $\phi$  can not be  $C^2([-R,R])$  with  $\phi(\pm R) = 0$ .

For the case  $n > 2$ , we assume that  $z(a) < \infty$  for some  $a > 0$ . For any fixed  $\gamma > 0$ , let

$$G(r) = \frac{u'(r)^2}{2} - \gamma |u(r)|^p u(r).$$

Then  $G(0) < 0$  and  $G(z(a)) > 0$ . ( $u'(z(a)) = 0$  by local existence and uniqueness.) Thus, the first zero of  $G(r)$  is between 0 and  $z(a)$ ; call it  $r_0$ .

Clearly  $G(r_0) = 0$  and  $G'(r_0) \geq 0$ . However, since  $G(r_0) = 0$ , we have

$$|u'(r_0)| = \sqrt{2\gamma} u(r_0)^{(p+1)/2}.$$

Therefore, at  $r = r_0$

$$\begin{aligned} G' &= u'u'' - \gamma(p+1)u^p u' \\ &= u' \left( -\frac{n-1}{r_0} u' + |u'|^q - \lambda u^p - \gamma(p+1)u^p \right) \\ &< u' (|u'|^q - (\lambda + \gamma(p+1))u^p) \\ &= u' ((2\gamma)^{q/2} u^{(p+1)q/2} - (\lambda + \gamma(p+1))u^p) \\ &= u' u^p f(\gamma), \end{aligned}$$

where

$$f(\gamma) = (2\gamma)^{p/(p+1)} - \gamma(p+1) - \lambda.$$

(We have used the fact that  $(p+1)q/2 = p$ .) Now  $G'(r_0) \geq 0$ ,  $u'(r_0) < 0$ , and  $u(r_0) > 0$ . Consequently, we must have  $f(\gamma) < 0$ . Since  $\gamma > 0$  was arbitrary, this must be true for all  $\gamma > 0$ . A straightforward calculation of the extreme points for  $f(\gamma)$  shows that we must have  $\lambda > \lambda_{p,1}$ . This proves the proposition.

Remarks. The estimate for  $G'(r_0)$  depends on the fact that  $n \geq 2$  in order to get strict inequality. Thus, it seems that for  $n = 1$ , the second argument misses the case  $\lambda = \lambda_{p,1}$ . However, since for a fixed value of  $a$ , the set of  $\lambda > 0$  for which  $z(a) < \infty$  is clearly open, we recover the case  $\lambda = \lambda_{p,1}$  when  $n = 1$ .

Also, in the case  $n = 1$ , Proposition 5.6 and Theorem 1.3 give a complete description of when there are solutions of (1.5) on  $B_R$  with  $q = 2p/(p+1)$ . There exists a solution if and only if  $\lambda > \lambda_p$ . It is natural to conjecture an



analogous result for higher dimensions. One wonders what the sharp cut-off value would be, in particular if it is  $\lambda_{p,n}$ .

A variation on the proof of Proposition 5.6 yields the following result.

Proposition 5.7. Assume  $q < 2p/(p + 1)$  and fix  $\lambda > 0$ . Then there exists  $a_* > 0$  such that  $z(a) = \infty$  for  $a < a_*$ .

Proof. Suppose  $z(a) < \infty$ . Let  $u(r)$  be the corresponding solution of (4.2) and set

$$G(r) = \frac{u'(r)^2}{2} - \lambda |u(r)|^p u(r).$$

Then  $G(0) < 0$  and  $G(z(a)) > 0$ . Let  $r_0$  be the first zero of  $G$ ; so  $G(r_0) = 0$  and  $G'(r_0) > 0$ . Reasoning as in the proof of Proposition 5.6 (with  $\gamma = \lambda$ ), we see that at  $r = r_0$

$$\begin{aligned} G' &\leq u'((2\lambda)^{q/2} u^{(p+1)q/2} - \lambda(p+2)u^p) \\ &\leq u' u^{(p+1)q/2} ((2\lambda)^{q/2} - \lambda(p+2)a^{p-[(p+1)q/2]}). \end{aligned}$$

Since  $p > (p+1)q/2$ , it follows that for a sufficiently small,  $G'(r_0) < 0$ . This contradicts the earlier observation that  $G'(r_0) > 0$ .

Hence  $z(a) = \infty$  for  $a > 0$  sufficiently small.

We now restrict ourselves to the special case  $n = 1$ . The problem (4.2) becomes the autonomous problem

$$\left. \begin{aligned} u''(r) - |u'(r)|^q + \lambda |u(r)|^p &= 0 \\ u(0) &= a > 0. \\ u'(0) &= 0. \end{aligned} \right\} \quad (5.8)$$

Problem (1.5) becomes

$$\left. \begin{aligned} \phi'' - |\phi'|^q + \lambda\phi &= 0 & \text{in } (-R, R) \\ \phi &> 0 & \text{in } (-R, R) \\ \phi(\pm R) &= 0, \end{aligned} \right\} \quad (5.9)$$

where  $\phi \in C^2([-R, R])$ .

Consider the case  $q < 2p/(p + 1)$ . By Propositions 4.5 and 4.6, i.e. the first part of Theorem 1.3, for all  $\lambda > 0$  and  $R > 0$ , there is a solution to problem (5.9). We will show it to be unique. (Note that in the case  $q = 2p/(p+1)$  the solution of (5.9) is unique when it exists because of formula (4.13). If  $q > p$ , we know it not to be unique for  $R$  large enough.)

Lemma 5.8. Let  $v(r)$  be the maximal solution (as in Proposition 4.4) to the problem

$$\left. \begin{aligned} v''(r) - b|v'(r)|^q + \lambda|v(r)|^p &= 0 \\ v(0) &= v_0 > 0 \\ v'(0) &= 0 \end{aligned} \right\} \quad (5.10)$$

where  $b > 0$  and  $\lambda > 0$  are parameters. Then  $v(r)$  is an increasing function of  $b$ .

Proof. The existence and uniqueness of  $v(r)$  follow exactly as in the proof of Proposition (4.1). In particular  $v'(r) < 0$  for  $r < 0$ . It is clear from the integral equation corresponding to (5.10) that  $v$  is a  $C^1$  function jointly in  $r$  and  $b$ . We denote  $v_r = \partial v / \partial r$  and  $v_b = \partial v / \partial b$ . Then considering  $v_r$  and  $v_b$  as functions of  $r$ , and using ' to denote  $d/dr$ , we have

$$v_r'' + bq|v'|^{q-1}v_r' + \lambda p|v|^{p-1}v_r = 0$$

$$v_b'' + bq|v'|^{q-1}v_b' + \lambda p|v|^{p-1}v_r = (-v')^q.$$

Hence, setting  $w = v_b'v_r - v_r'v_b$ , it follows that

$$w' + qb(-v')^{q-1}w = (-v')^qv' < 0;$$

or, if  $f$  denotes a primitive of  $qb(-v')^{q-1}$ ,

$$(e^f w)' < 0. \tag{5.11}$$

Since  $v_b(0) = v_r(0) = 0$ , we have  $w(0) = 0$ . Hence by (5.11),  $w(r) < 0$  for  $r > 0$ . This implies  $(v_b/v_r)' < 0$  for  $r > 0$ . Moreover,

$$\lim_{r \rightarrow 0} \frac{v_b(r)}{v_r(r)} = \lim_{r \rightarrow 0} \frac{v_b'(r)}{v_r'(r)} = \frac{0}{v''(0)} = 0,$$

and so  $v_b/v_r < 0$  for all  $r > 0$ , i.e.  $v_b(r) > 0$  for  $r > 0$ .

Proposition 5.9. Let  $q < 2p/(p + 1)$  and  $n = 1$ . Then, for all  $\lambda > 0$ ,  $a^{(p-1)/2}z(a)$  is a decreasing function of  $a$ . In particular  $z(a)$  is decreasing.

If  $q > 2p/(p+1)$  and  $n = 1$ , then for all  $\lambda > 0$ ,  $a^{(p-1)/2}z(a)$  is an increasing function of  $a$ .

Proof. By the previous lemma, if  $n = 1$  and  $q < 2p/(p+1)$ , then  $v_a(r)$  defined by (4.9) is a decreasing function of  $a$ . Consequently, the first zero of  $v_a$ , i.e.  $a^{(p-1)/2}z(a)$ , is a decreasing function of  $a$ . More precisely,  $a^{(p-1)/2}z(a)$  is nonincreasing for all  $a > 0$  and strictly decreasing where it is finite.

An analogous argument works for  $q > 2p/(p+1)$ .

Corollary 5.10. Let  $q < 2p/(p+1)$  and  $n = 1$ . Then for every  $\lambda > 0$  and  $R > 0$ , the  $C^2$  solution of (5.9) is unique.

Finally, we have a result which further contrasts the cases  $q < 2p/(p+1)$  and  $q > 2p/(p+1)$ .

Propositio 5.11. Suppose  $q > 2p/(p+1)$  and  $n = 1$ . Let  $\lambda > 0$  be arbitrary. Then  $z(a) < \infty$  for all  $a > 0$  and  $\lim_{a \rightarrow 0} z(a) = \infty$ .

Proof. By Lemma 5.2 and Proposition 4.5 we already know that  $z(a) < \infty$  for small  $a > 0$  and  $\lim_{a \rightarrow 0} z(a) = \infty$ . Suppose  $z(a) = \infty$  for some  $a > 0$ , and let  $u(r)$  be the corresponding solution of (5.8). Let  $v(r)$  be a solution to (5.8) with a smaller initial value  $a$  such that  $z(a) < \infty$ . Let

$$\bar{\rho} = \sup\{\rho \in \mathbf{R}: \text{the graph of } v(r-\rho) \text{ does not touch the graph of } u(r)\}.$$

As in the proof of Proposition 5.6, it is clear that  $\bar{\rho} \in \mathbf{R}$  and that  $u(r)$  and  $v(r - \bar{\rho})$  would have to coincide, which is impossible.

Remarks. We can make a few more observations in the case  $n = 1$ . First, if  $q < 2p/(p+1)$ , then by Proposition 4.6, 5.7, and 5.9 there exists  $a_* > 0$  such that  $z(a) = \infty$  for  $a \leq a_*$  and  $z(a) < \infty$  for  $a > a_*$ . Next, in the case  $q > 2p/(p+1)$ , if  $\phi_1$  and  $\phi_2$  are two different solutions of (5.9), then  $\phi_1(x) \neq \phi_2(x)$  for all  $x$  in  $(-R, R)$ . This follows from a translation argument similar to the proofs of Propositions 5.6 and 5.11. We mention without proof that if  $q > 2p/(p+1)$ , there can exist singular solutions of (5.9), i.e. solutions in  $C^2([-R, R] \setminus \{0\})$  with  $\lim_{x \rightarrow 0} \phi(x) = \infty$ .

Clearly, solutions of (1.5) exhibit radically different behavior depending on the relationship between  $p$  and  $q$ . However, the picture is certainly not complete.

REFERENCES

1. R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
2. J.M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math., Oxford Series, 28 (1977), 473-486.
3. P. Baras and L. Cohen, Complete blow-up after  $T_{\max}$  for the solution of a semilinear heat equation, preprint 1985.
4. J. Bebernes, A. Bressan, and D. Eberly, A description of blow-up for the fuel ignition model, preprint.
5. J. Bebernes and W. Fulks, The small heat loss problem, J. Diff. Eqns., to appear.
6. J. Bebernes and D.R. Kassoy, A mathematical analysis of blowup for thermal reactions - the spatially nonhomogeneous case, SIAM J. Appl. Math., 40 (1981), 476-484.
7. J. Bebernes and W. Troy, Non existence for the Kassoy problem, SIAM J. Math. Anal., to appear.
8. H. Bellout, A criterion for blow-up of solutions to semi-linear heat equations, SIAM J. Math. Anal., to appear.
9. H. Brezis, Analyse fonctionnelle, Masson, Paris, 1983.
10. H. Engler, Contractive properties for the heat equation in Sobolev spaces, preprint 1984.
11. A. Friedman and A.A. Lacey, to appear.
12. A. Friedman and J.B. McLeod, Blow-up of solutions of semilinear heat equations, Ind. Univ. Math. J. 34 (1985), 425-448.
13. B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Commun. Math. Phys. 68 (1979), 209-243.
14. Y. Giga, A remark on a priori bounds for global solutions of semilinear equations, preprint, 1985.
15. Y. Giga and R.V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Commun. Pure Appl. Math 38 (1985), 297-319.
16. Y. Giga and R.V. Kohn, Characterizing blow-up using similarity variables, Ind. Univ. Math. J., to appear.
17. D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1977.
18. A. Haraux and F.B. Weissler, Non-uniqueness for a semilinear initial value problem, Ind. Univ. Math. J. 31 (1982), 167-189.

19. T. Kato, Perturbation Theory for Linear Operators, Springer, New York, 1966.
20. T. Kato and H. Fujita, On the nonstationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova 32 (1962), 243-260.
21. H. Komatsu, Fractional powers of operations, Pacific J. Math 19 (1966), 285-346.
22. A.A. Lacey, Mathematical analysis of thermal runaway for spatially inhomogeneous reactions, SIAM J. Appl. Math. 43 (1983), 1350-1366.
23. A.A. Lacey, The form of blow-up for nonlinear parabolic equations, Proc. Royal Soc. of Edinburgh, 98A (1984), 183-202.
24. A.A. Lacey, Global blow-up of a nonlinear heat equation, preprint 1986.
25. C.E. Mueller and F.B. Weissler, Single point blow-up for a general semilinear heat equation, Ind. Univ. Math. Jnl. 34 (1985), 881-913.
26. R. Seeley, Norms and domains of complex powers  $A_B^Z$ , Amer. Jnl. Math. 93 (1971), 299-309.
27. R. Seeley, Interpolation in  $L^p$  with boundary conditions, Studia Math. 44 (1972), 47-60.
28. I. Segal, Non-linear semi-groups, Ann. of Math. 78 (1963), 339-364.
29. W.C. Troy, The existence of bounded solutions of a semilinear heat equation, preprint 1986.
30. W.C. Troy and D. Eberly, preprint 1985.
31. F.B. Weissler, Semilinear evolution equations in Banach spaces, Jnl. Func. Anal. 32 (1979), 277-296.
32. F.B. Weissler, Single point blow-up for a semilinear initial value problem, Jnl. Diff. Eqns., 55 (1984), 204-224.
33. F.B. Weissler, An  $L^\infty$  blow-up estimate for a nonlinear heat equation, Commun. Pure and Appl. Math. 38 (1985), 291-295.
34. F.B. Weissler, Local existence and nonexistence for semilinear parabolic equations in  $L^p$ , Ind. Univ. Math. Jnl. 29 (1980), 79-102.