Bessel Distributions and Automorphic Spectral Reciprocity for General Linear Groups

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Xinchen Miao

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Dihua Jiang

June, 2023

Acknowledgements

Firstly, I want to thank my advisor Dihua Jiang for his consistent support and encouragement in the past six years. He introduced me to the topics of automorphic forms, representation theory, and Langlands program, suggested me thinking about problems in relative trace formulae, provided practical and thought-provoking viewpoints that lead to solutions of the problems, and carefully reviewed the first draft of this thesis. Besides all these, he also spent a lot time on training me to work as a professional mathematician, and helping me to go through the period of cultural adjustment. Without his help, it will be very difficult for me to understand this subject and finish my thesis. I certainly cannot feel more grateful to him.

Besides my advisor, there are many other mathematicians who have helped me in various ways, including (in alphabetical order) Valentin Blomer, Benjamin Brubaker, Jack Buttcane, James Cogdell, Adrian Diaconu, Solomon Friedberg, Paul Garrett, Jayce Getz, Heekyoung Hahn, Peter Humphries, Herve Jacquet, Kai-Wen Lan, Spencer Leslie, Baiying Liu, Zhilin Luo, Zhengyu Mao, William Messing, Chufeng Nien, Omer Offen, Yiannis Sakellaridis, Freydoon Shahidi, Fangyang Tian, Chen Wan, Jiaping Wang, Han Wu, Yangbo Ye, Lei Zhang, and Wei Zhang. Some of them kindly answered my questions; some of them patiently listened to me explain my work and provided useful feedbacks; some of them suggested me to consider other possible aspects of this problem; and all of them have given me enormous encouragements. I want to thank all of them.

I thank Professor Benjamin Brubaker, Adrian Diaconu, and Kai-Wen Lan for serving on my committee.

I would like to thank fellow graduate students in automorphic forms, representation theory and number theory group at University of Minnesota these years, including (in

alphabetical order) Shengmei An, Cheng Chen, Zhaolin Li, Zhilin Luo, Fangyang Tian, and Guodong Xi. Holding seminars and having discussions with them have always been illuminating and enjoyable.

I would like to thank my friends these years at University of Minnesota, including but not limited to (in alphabetical order) Shengmei An, Shan Chen, Lu Dai, Zanbing Dai, Juerui Feng, Dev Hegde, Yichang Liu, Yifan Liu, Wenjie Lu, Liya Ouyang, Tong Shi, Wei Wang, Wuzhe Xu and Jiangchen Zhu for all their help and encouragement, both in academics and daily life. It would have been much more difficult to complete this thesis without the support of all the above people. Moreover, I would like to thank Lionel Messi and the Argentina national soccer team, for their incredible and marvelous performance during the 2022 FIFA World Cup.

During the different stages in the preparation of this work, I have been giving lectures in various seminars and conferences at University of Minnesota, Rutgers University-Newark, John Hopkins University, Duke University, and Zhejiang University. I would like to thank all the organizers and participants of these events.

Finally, I would like to thank my parents, all family members and friends for their consistent support in various stages of my life.

Dedication

I would like to dedicate this paper to my parents and grandparents.

Abstract

In this thesis, we focus on the local and global theory of automorphic forms and relative trace formulae. In the local aspect, I work on the $\mathrm{GL}(n)$ Bessel distributions (functions) over non-archimedean local fields and the local Kloosterman-type orbital integrals. We prove the local integrability of Bessel functions for $\mathrm{GL}(n)$ (p-adic case) by using the relations between Bessel functions and local Kloosterman (orbital) integrals proved in several papers of E. M. Baruch [Ba03] [Ba04] [Ba05], the theory of the (relative) Shalika germs established by H. Jacquet and Y. Ye in [JY96] [JY99] and G. Stevens' approach [Ste87] on estimating certain $\mathrm{GL}(n)$ generalized Kloosterman sums. For the global theory, I study the automorphic spectral reciprocity formulae for certain L-functions. We prove a spectral reciprocity formula for the product of $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ and $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ Rankin-Selberg L-functions ($n \geq 3$). This generalizes the work of V. Blomer and R. Khan in [BK17] where the case of $n = 2$ was established. We extend the method develooped by R. Nunes in [Nun20] from $n = 2$ to general $n \geq 3$, by using the global zeta integrals for the $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ Rankin-Selberg L-functions, the spectral theory of square-integrable automorphic forms, the language of automorphic representations and representation theoretic view.

Contents

Chapter 1

Introduction

1.1 Relative Trace Formulae

The study of relative trace formulae has a long history. The mother of all trace formulae is the Poisson summation formula. If $f : \mathbb{R} \to \mathbb{C}$ is a sufficiently nice function, say Schwartz function (with rapid decay) for simplicity, then we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) \tag{1.1}$$

where

$$\widehat{f}(y) = \int_{-\infty}^{\infty} f(x)e(-xy)\,dx, \quad e(x) = e^{2\pi ix}$$

is the Fourier transform. Applying (1.1) formally with $f(x) = x^{-s}$, one obtains the functional equation of the Riemann zeta function. More generally, the Poisson summation formula implies the functional equations for all Dirichlet L-functions, and is in fact essentially a re-statement of those.

Why is the Poisson summation formula a trace formula? Given a function f as above, we consider the convolution operator

$$L_f : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$$

given by

$$L_f(g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy = \int_{\mathbb{R}/\mathbb{Z}} k(x,y)g(y)\,dy$$

where

$$k(x,y)=\sum_{n\in\mathbb{Z}} f(x-y+n). \tag{1.2}$$

The exponential functions $e_n(x)=e(nx)$ are an orthonormal basis of eigenfunctions: $L_f(e_n)=\widehat{f}(n)e_n$. Computing the trace of L_f in two ways (geometric and spectral expansion) we obtain

$$\operatorname{tr} L_f=\sum_{n\in\mathbb{Z}}\widehat{f}(n)=\int_{\mathbb{R}/\mathbb{Z}} k(x,x)\,dx=\sum_{n\in\mathbb{Z}} f(n),$$

since $\operatorname{vol}(\mathbb{R}/\mathbb{Z})=1$.

Now we start with an important and special class of relative trace formulae, which is called the relative Kuznetsov trace formula.

Let G and G' be two quasi-split algebraic reductive groups defined over a number field F and the corresponding adele ring $\mathbb{A}_F$. Let N be a maximal unipotent subgroup of G and θ be a non-degenerate additive character of $N(\mathbb{A}_F)$ which is trivial on $N(F)$. Let N' and θ' be similarly defined for G'. In addition, we let H be a subgroup of G' defined over F and χ a multiplicative character of $H(\mathbb{A}_F)$ which is trivial on $H(F)$. We consider the kernel functions. We let f and f' be smooth functions of compact support on $G(\mathbb{A}_F)$ and $G'(\mathbb{A}_F)$ respectively. We set

$$K_f(x,y):=\sum_{\gamma\in G(F)} f(x^{-1}\gamma y),$$

and

$$K_{f'}(x,y)=\sum_{\gamma'\in G'(F)} f'(x^{-1}\gamma' y).$$

Then the geometric side of the relative Kuznetsov trace formula reads as follows:

$$\int\int K_f(u_1,u_2)\overline{\theta}(u_1)\theta(u_2)du_1du_2=\int\int K_{f'}(h,u')\overline{\chi}(h)\theta'(u')dhdu'. \tag{1.3}$$

This identity is satisfied when f and f' are related by a certain geometric correspondence. The left hand side of (1.3) can be computed as a finite sum of orbital integrals

$$\sum_{\xi}\int\int f(u_1\xi u_2)\theta(u_1u_2)du_1du_2$$

by the Bruhat decomposition. The sum is over all orbits (or double cosets) $N(F)\xi N(F)$ and the integral is over the quotient of the product $N(\mathbb{A}_F)\times N(\mathbb{A}_F)$ by the stabilizer

of ξ. In order for the orbital integral to make sense (non-vanishing), we have to assume that the additive character $\theta(u_1u_2)$ is trivial on the stabilizer of ξ. Equivalently, we have to assume that $\theta(u_1u_2)$ depends only on the product $u_1\xi u_2$. We shall say that ξ (or its double coset) is relevant if this condition is satisfied. Therefore, the sum is over the set of all relevant double cosets $N(F)\xi N(F)$.

Similarly, the right hand side of (1.3) can be computed as a sum of orbital integrals of the form

$$\int\int f'(h\xi' u')\chi(h)\theta'(u')dhdu'.$$

The sum is over all relevant double cosets $H(F)\xi' N'(F)$. One expects to have a bijection $N(F)\xi N(F) \leftrightarrow H(F)\xi' N'(F)$ between the set of relevant double cosets. The geometric linear relation is that

$$\int\int f(u_1\xi u_2)\theta(u_1u_2)du_1du_2 = \int\int f'(h\xi' u')\chi(h)\theta'(u')dhdu'$$

for every $N(F)\xi N(F) \leftrightarrow H(F)\xi' N'(F)$. More precisely, the functions f and f' are product of local functions f_v and f'_v. For each place v, we have a set of equalities which read as follows:

$$\begin{aligned}&\int\int f_v(u_1\xi_v u_2)\theta_v(u_1u_2)du_1du_2\\ =&\gamma_v(\xi_v,\xi'_v)\cdot\int\int f'_v(h\xi' u')\chi_v(h)\theta'_v(u')dhdu'.\end{aligned} \tag{1.4}$$

The factor $\gamma_v(\xi_v,\xi'_v)$ is a transfer factor, to be defined appropriately. We shall say that the functions f_v and f'_v match if this relation is satisfied for every $N(F_v)\xi_v N(F_v) \leftrightarrow H(F_v)\xi'_v N'(F_v)$. The following assertions have to be proved.

- Every f_v matches a f'_v and conversely (**smooth matching**).
- At almost all finite places v, the characteristic function of the suitable maximal compact subgroup K_v and K'_v match (**fundamental lemma**).

The product of the local transfer factors $\prod_v \gamma_v(\xi_v,\xi'_v)$ is one when ξ and ξ' are rational. One expects to have at almost all places an homomorphism between the Hecke algebras $\mathcal{H}(G_v,K_v)$ and $\mathcal{H}(G'_v,K'_v)$. Moreover, Hecke functions which correspond to one another by this homomorphism match.

This setup is quite general. For instance, one expects such a relative trace formula identity with a geometric correspondence to hold true each time H is the group of fixed

points of an automorphism of order 2 of a group G', with the group G and G' being appropriately defined.

Now let ϕ be an automorphic form on G and ϕ' be an automorphic form on G'. We will set

$$\mathcal{W}(\phi) := \int_{N(F)\backslash N(\mathbb{A}_F)} \phi(u)\overline{\theta(u)}du; \tag{1.5}$$

$$\mathcal{W}'(\phi') := \int_{N'(F)\backslash N'(\mathbb{A}_F)} \phi'(u)\overline{\theta'(u)}du; \tag{1.6}$$

and

$$\mathcal{P}(\phi') := \int_{H(F)\backslash H(\mathbb{A}_F)} \phi'(h)\overline{\chi}(h)dh. \tag{1.7}$$

We will say that an automorphic representation π' of G' is distinguished by (H, χ) (or $\mathcal{P}$) if $\mathcal{P}$ is not identically 0 on the space of π'. Similarly, we will say that π' is distinguished by θ' if the restriction of $\mathcal{W}'$ to the space of π' is non-zero. Similarly, we shall say that an automorphic representation π of G is distinguished by θ if the restriction of $\mathcal{W}$ to the space of π is non-zero.

By the spectral decomposition of the kernel function $K_f(x, y)$ and $K_{f'}(x, y)$, one expects to be able to associate to every automorphic cuspidal representation π' of G' distinguished by (H, χ) and θ' an automorphic representation π of G distinguished by θ and conversely, such that (the spectral side of the relatvie Kuznetsov trace formula)

$$\sum_i \mathcal{W}(\pi(f)\phi_i)\overline{\mathcal{W}(\phi_i)} = \sum_j \mathcal{P}(\pi'(f')\phi'_j)\overline{\mathcal{W}'(\phi'_j)}. \tag{1.8}$$

In the left hand side, ϕ_i is an orthonormal basis of the space of π and f is a smooth function of compact support on $G(\mathbb{A}_F)$. On the right hand side, ϕ'_j is an orthonormal basis of the space of π' and f' is a smooth function of compact support on $G'(\mathbb{A}_F)$. The functions f and f' are related by the geometric correspondence and the representations π and π' are related by the principle of functoriality.

Formally, the left hand side of (1.8) computes a trace. Indeed, let A be the operator such that

$$\langle A\phi_1, \phi_2\rangle = \mathcal{W}(\phi_1)\overline{W(\phi_2)}.$$

Then the left hand side is the trace of the operator $A\pi(f)$. In particular, the left and right hand side does not depend on the choice of the basis. Therefore, the related

identities (1.3) and (1.8) in the relative Kuznetsov trace formula appear as an extension of the duality between characters (spectral side) and conjugacy classes (geometric side) which is at the heart of the Arthur-Selberg trace formula.

1.2 Bessel Distributions

In order to study the relative Kuznetsov trace formula, we need to have a better understanding of the geometric and spectral side of the relative trace formula globally.

After the milestone work of John Tate (Tate's thesis) to reformulate the functional equation of Hecke L-function in 1950, the local-global principle and the adelic languages became more and more important in Number Theory. Therefore, it suffices to study the geometric and spectral side of the relative trace formula locally for every local place (non-archimedean and archimedean local places). In the first part of the thesis, we will mainly work over the non-archimedean field $F = \mathbb{Q}_p$, where p is a fixed prime number.

In particular, we consider $G = \mathrm{GL}(n)$. Moreover, we write $G_n := \mathrm{GL}_n(F) = \mathrm{GL}_n(\mathbb{Q}_p)$. Let B be the Borel subgroup of upper triangular matrices, A or T the subgroup of diagonal matrices and N the subgroup of upper unipotent matrices. Let $K := \mathrm{GL}_n(\mathbb{Z}_p)$ be the maximal compact open subgroup of G_n. Let ψ be a non-degenerate character of N, which is of the form

$$\psi(n) := \psi'\left(\sum_{i=1}^{n-1} \alpha_i \cdot n_{i,i+1}\right)$$

for $n = (n_{i,j}) \in N$, $\alpha_1, \cdots, \alpha_{n-1} \in F^\times$ and ψ' is a smooth complex valued nontrivial additive character of F. We further assume that $\alpha_i = \alpha_{n-i}$ for all $1 \leq i \leq n-1$. For G_n, we define the normalizer of T in G_n to be

$$N_{G_n}(T) := \{g \in G_n : \ ghg^{-1} \in T \ for \ all \ h \in T\}.$$

The Weyl group W_{G_n} is defined as $W_{G_n} := N_{G_n}(T)/T$. The Weyl group is a finite group and is isomorphic to symmetric group on n letters S_n. Let w_{G_n} be the longest Weyl element in the Weyl group W_{G_n}, which can be written as

$$w_{G_1} = 1, \ \ w_{G_n} = \begin{pmatrix} 0 & 1 \\ w_{G_{n-1}} & 0 \end{pmatrix},$$

i.e. the $n \times n$ permutation matrix whose anti-diagonal entries are 1. We fix the (normalized) Haar measure with the volumes of K and $N \cap K$ both equal to one.

We recall the definition of Whittaker functionals over a p-adic field. Let (π, V) be a smooth irreducible representation of G_n. A Whittaker functional L is a linear functional on V such that $L(\pi(n)v) = \psi(n)L(v)$ for all $n \in N$ and $v \in V$. The following well-known theorem of Whittaker functionals is proved by J. Shalika [Sha74].

Theorem 1.2.1 (*Uniqueness of Whittaker functionals*)**.** *Let (π, V) be an irreducible smooth representation of G_n. Then the space of Whittaker functionals L has dimension at most equal to one.*

In other words, the Whittaker functional L is unique up to scalar. If this Whittaker functional L is non-zero, the representation π is called generic. For a non-zero Whittaker functional L, we define that $W_v(g) := L(\pi(g)v),\ v \in V,\ g \in G_n$ and let G_n act on the space of these functions by right translations. That is, if $g_1 \in G_n$ and W is a function on G_n then we define $(\rho(g_1)W)(g) = W(gg_1)$ for $g \in G_n$. The map $v \to W_v$ gives a realization of π on a space of Whittaker functions which satisfy $W(ng) = \psi(n)W(g)$ for all $n \in N$ and $g \in G_n$. We denote this space by $\mathcal{W}(\pi, \psi)$ and call it the Whittaker model of π.

To study the spectral side of the relative Kuznetsov trace formula locally, it is equivalent to study the local Bessel distributions, which is defined as follows:

Let π be a smooth irreducible generic representation of $G_n = \mathrm{GL}_n(\mathbb{Q}_p)$ with contragredient $\widetilde{\pi}$. We use π^* and $\widetilde{\pi}^*$ to denote the linear dual of π and $\widetilde{\pi}$ respectively. Let $f \in C_c^\infty(G_n)$ be a locally constant function with compact support on G_n. We take $l \in \pi^*$ and $l' \in \widetilde{\pi}^*$ to be fixed nonzero Whittaker functionals with repect to the non-degenerate additive character ψ and ψ^{-1}, respectively. We define $\widetilde{\pi}(f)l'$ as

$$\widetilde{\pi}(f)l' := \int_{G_n} f(g)\widetilde{\pi}(g)l'dg$$

or equivalently, for any $\widetilde{v} \in \widetilde{\pi}$, we have

$$\begin{aligned} \langle \widetilde{\pi}(f)l', \widetilde{v}\rangle &= \int_{G_n} f(g)\langle \widetilde{\pi}(g)l', \widetilde{v}\rangle dg \\ &= \int_{G_n} f(g)\langle l', \widetilde{\pi}(g^{-1})(\widetilde{v})\rangle dg. \end{aligned} \tag{1.9}$$

Then $\widetilde{\pi}(f)l'$ is a smooth linear functional on $\widetilde{\pi}$, hence can be identified with a vector $v_f \in \pi$.

Definition 1.2.2. *We define Bessel distribution $B(f)$ as*

$$B(f) := l(v_f).$$

E. M. Baruch obtained the first regularity result about the Bessel distribution $B(f)$. By Theorem 2.3 in [Ba01], when restricted to the open Bruhat cell $\Omega = Bw_{G_n}B$, this Bessel distribution $B(f)$ is given by integration against a locally constant kernel function $j_{0,\pi}(g)$ on Ω, which is called the relative Bessel function. More precisely,

Theorem 1.2.3. *[Ba01] Let $\Omega = Bw_{G_n}B$ be the open Bruhat cell in G_n and dx be a Haar measure on G_n restricted to Ω. Then there exists a locally constant function $j_{0,\pi} : \Omega \to \mathbb{C}$ such that*

$$B(f) = \int_{\Omega} j_{0,\pi}(x)f(x)dx,$$

for every $f \in C_c^{\infty}(\Omega)$.

We can extend this function $j_{0,\pi}(g)$ to the whole group G_n by letting zero when $g \in G_n - \Omega = G_n - Bw_{G_n}B$. Note that the function $j_{0,\pi}(g)$ is no longer a locally constant function on the whole group G_n.

There is another way to define the Bessel function $j_{\pi}(g)$ on the group G_n. The key ingredients in the definition are the stable integral and the uniqueness of Whittaker functionals and models.

We introduce the definition of the stable integral. Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be a filtration of N with compact open subgroups $N_i, i = 1, 2, \cdots$ such that $N = \cup_{i=1}^{\infty} N_i$. We denote this filtration by $\mathcal{N}$. Let $f : N \to \mathbb{C}$ be a locally constant function.

Definition 1.2.4. *We define the stable integral*

$$\int_N^{\mathcal{N}} f(n)dn := \lim_{m \to +\infty} \int_{N_m} f(n)dn$$

if this limit exists. If the limit exists, we say that the stable integral is convergent.

The following theorem is proved in [Ba05].

Theorem 1.2.5. *[Ba05] Let $\mathcal{N} := \{N_i, i \geq 1\}$ be a filtration of N for G_n as above. Let $g \in Bw_{G_n}B$ and $W \in \mathcal{W}(\pi, \psi)$. Then the (stable) integral*

$$\int_N^{\mathcal{N}} W(gn)\psi^{-1}(n)dn$$

is convergent. Moreover, the value for this integral is independent on the choice of filtration $\mathcal{N}$.

Let $g \in Bw_{G_n}B$ and we define the linear functional $L_g : V \to \mathbb{C}$ by

$$L_g(v) := \int_N^{\mathcal{N}} W_v(gn)\psi^{-1}(n)dn.$$

Since $L_g(\pi(n)v) = \psi(n)L_g(v)$ for all $n \in N$, we see that L_g is also a Whittaker functional, hence it follows from Theorem 1.2.1 that there exists a scalar $j_{\pi,\psi}(g)$ such that

$$L_g(v) = j_{\pi,\psi}(g)L(v)$$

for all $v \in V$. We call $j_\pi = j_{\pi,\psi}$ the Bessel function of π. The Bessel function $j_\pi(g)$ is independent on the choice of the Whittaker functional L which is unique up to scalar multiplication (See Theorem 1.2.1). We see that the Bessel function j_π is defined on the open Bruhat cell $Bw_{G_n}B$ of G_n. Moreover we know that $j_\pi(g)$ satisfies

$$j_\pi(n_1gn_2) = \psi(n_1n_2)j_\pi(g)$$

for all $n_1, n_2 \in N$ and $g \in Bw_{G_n}B$. The value of j_π is determined by its values on the set $w_{G_n}A = w_{G_n}T$ and the Bessel function j_π is locally constant on the set $Bw_{G_n}B$ (See [Ba05]). If $g \in G_n - Bw_{G_n}B$, it is defined that $j_\pi(g) \equiv 0$.

By Theorem 1.1 in [Chai19a] and Theorem 7.2 in [Chai17], we know that $j_{0,\pi}(g) = j_\pi(g)$ for all $g \in \Omega$ (See Theorem 1.1, 3.2 in [Chai19a] and Lemma 3.2 in [Chai17]).

As in the theory of distribution characters of smooth irreducible admissible representations of G_n (See [HC70] and [HC99]), the Bessel function j_π is expected to be locally integrable on G_n. We prove this result for all $n \geq 2$, which states as follows:

Theorem 1.2.6. *The Bessel function $j_\pi(g)$ is locally integrable on $\mathrm{GL}_n(\mathbb{Q}_p)$.*

Remark 1.2.7. *The study of (classical) Bessel functions has a very long history and can be traced back to the 19th century. Bessel functions are first defined by D. Bernoulli*

and then generalized by F. Bessel. The classical Bessel functions are canonical solutions $y = y(x)$ *of Bessel's differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

for an arbitrary complex number α*, which is defined as the order of the Bessel function [Wat95].*

Among all complex numbers, the most important cases are when α *is an integer or half-integer. Bessel functions for integers* α *are also known as cylinder functions because they show up in the solution to Laplace's equation in cylindrical coordinates. Bessel functions for half-integers* α *appear in the solution to Helmholtz equation in spherial coordinates [Wat95]. This shows the natural relations between Bessel functions and the solutions to PDEs.*

In the 20th century, more connections and applications of the Bessel functions and their generalizations were found in many other fields of mathematics, in particular, analytic number theory, automorphic forms and the Langlands Program.

For example, classical Bessel functions appears natrually in Voronoi's summation formula as well as Petersson's and Kuznetsov's trace formula for $\mathrm{GL}_2(\mathbb{R})$*. These formulae have become fundamental analytic tools in attacking some deep problems in analytic number theory, most notably the subconvexity problem for automorphic L-functions. A version of the Voronoi's summation formula, which is not in its most general form, roughly reads as follows:*

$$\sum_{n=1}^{\infty}\lambda_F(n)e\left(\frac{an}{c}\right)v(n) = \frac{1}{c}\cdot\sum_{n=1}^{\infty}\lambda_F(n)e\left(-\frac{\bar{a}n}{c}\right)\Gamma\left(\frac{n}{c^2}\right).$$

In the above formula, $a, \bar{a}$ *and* c *are integers such that* $(a, c) = 1$ *and* $a\bar{a} \equiv 1(\mod c)$*,* $\lambda_F(n)$ *are certain normalized Fourier coefficients of a Hecke-Maass cusp form* F *for* $\mathrm{SL}_2(\mathbb{Z})$ *(Here Hecke-Maass cusp forms can be defined as the eigenfunctions of a collection of differential and arithmetic operators, for example the Laplacian operator),* v *is a smooth weight function compactly supported on* $(0, \infty)$ *and* Γ *is the Hankel transform of* v*,*

$$\Gamma(x) = \int_0^{\infty} v(y)J_F(xy)dy,\ x \neq 0.$$

Here the intgral kernel J_F *of Hankel transform* Γ *has an expression in terms of classical Bessel functions. Moreover, under the representation theoretical viewpoint, the Hankel*

transform associated to classical Bessel functions is closely related to the local functional equation of the $\mathrm{GL}(2) \times \mathrm{GL}(1)$ *Rankin-Selberg L-function in the real place.*

The local integrability property is important to the understanding of general Bessel functions over local fields. The local integrability of $\mathrm{GL}(2)$ *Bessel functions over p-adic fields is well-known by the analytic properties of* $\mathrm{GL}(2)$ *Whittaker functions (See [So84] and [Ba97]). In [Ba04], E. M. Baruch proved that the Bessel functions for* $\mathrm{GL}(3)$ *over p-adic fields are locally integrable. However, the local integrability for Bessel functions on* $\mathrm{GL}(n)$ *(*$n \geq 4$*) is still an open problem in the subject. We prove the well-expected local integrability property for* $\mathrm{GL}_n(\mathbb{Q}_p)$ *(*$n \geq 2$*).*

An important corollary of the local integrability of $\mathrm{GL}_n(\mathbb{Q}_p)$ Bessel functions is the so called regularity theorem.

Theorem 1.2.8. *The Bessel distribution* $B(f)$ *is given by integration against the Bessel function* $j_\pi(g)$ *on* G_n*, that is, for any* $f \in C_c^\infty(G_n)$*, we have*

$$B(f) = \int_{G_n} j_\pi(g) f(g) dg.$$

Remark 1.2.9. *The regularity theorem (Theorem 1.2.8) may have applications in the study of relative trace formulae (especially the test vector problem in the Kuznetsov trace formula). Moreover, Theorem 1.2.8 tells us that all the information of local Bessel distribution* $B(f)$ *which is really hard to handle is contained in the Bessel function* $j_\pi(g)$*, which is still very difficult to study but more accessible. This would make it possible to generalize the results of Baruch and Mao [BM03] [BM05] [BM07] to higher rank groups.*

Remark 1.2.10. *The key ingredients in the proof of Theorem 1.2.6 and 1.2.8 are reduction steps to the careful study of the local orbital integrals (the geometric side of the relative Kuznetsov trace formula), relative Shalika germ expansions of the local orbital integrals, and a non-trivial upper bound for the local Kloosterman integrals and sums. Readers may see Chapter 3 for more details. The proof of Theorem 1.2.6 and 1.2.8 shows the power of the relative trace formulae. A better understanding of the geometric side of the relative trace formulae can tell us a lot of further information on the spectral side of the relative trace formula.*

1.3 Automorphic Spectral Reciprocity Formulae

The spectral decomposition of the kernel function plays an important role in the study of relative trace formulae. A careful study of the geometric side of the relative trace formula can always lead to a better understanding of the spectral side of the relative trace formula. Moreover, by finding the intrinsic symmetries in the geometric side of the relative trace formula, we may avoid performing a dedicate study of the geometric side and deduce some interesting spectral identites between moments of automorphic L-functions from the spectral side, which is called automorphic spectral reciprocity formulae.

The most famous reciprocity law in the number theory is the quadratic reciprocity law first proved by F. Gauss. Let q and ℓ be two distinct odd prime numbers. The celebrated law of quadratic reciprocity expresses the quadratic Legendre symbol $\left(\frac{\ell}{q}\right)$ in terms of $\left(\frac{q}{\ell}\right)$. This result is always viewed as the first result of algebraic number theory and has more than one hundred different proofs nowadays. The quadratic reciprocity law is very important since it connects the arithmetic properties of two unrelated finite fields $\mathbb{F}_q$ and $\mathbb{F}_\ell$.

Similar phenomenon will also appear in the theory of automorphic forms, which is known as the (automorphic) spectral reciprocity formula. Motivated by the work of Motohashi [Mot93], Michel-Venkatesh [MV10] and many other related works in lower ranks (degree 2 or 4 L-functions), V. Blomer suggested to investigate spectral identities of the following shape from the point of view of automorphic representations:

$$\sum_{\pi\in\mathcal{F}} \mathcal{L}(\pi)\mathcal{H}(\pi) = \sum_{\pi\in\tilde{\mathcal{F}}} \widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi),$$

where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are different families of automorphic representations, $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are certain (may be different) (products of) L-functions corresponding to automorphic representations. Moreover, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some (global) weight functions related to automorphic representation π. The map from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$ is given by an explicit integral transform. Such kind of automorphic spectral reciprocity formulae have recently attracted a lot of attentions in analytic number theory ([AK18] [BK17] [BK18] [BML19] [Nel19] [Nun20] [Zac19] [Zac20] etc.).

Beyond the simplicity and beauty of spectral reciprocity formulae, these formulae

are expected to have powerful applications to non-vanishing and subconvexity problems for the associated L-functions, which are very important in analytic number theory. This motivates an intension to understand the intrinsical symmetry behind such kind of identities. In other word, it is highly desirable to ask whether there exists a master formula of representation-theoretic nature to control all these reciprocity formulae related to L-functions. Moreover, there are several works on automorphic L-functions on which one can find such spectral reciprocity formulae hidden as the key ingredients inside the proof (For example, see [KY21] [Kha21]).

These kind of spectral reciprocity formulae in higher rank (degree 8 L-function) first appeared in a paper by Blomer, Miller and Li [BML19]. They established a spectral reciprocity formula for $\mathrm{GL}(4) \times \mathrm{GL}(2)$ Rankin-Selberg L-functions in the $\mathrm{GL}(2)$ archimedean (spectral) aspect. Later, Blomer and Khan's work [BK17] shows another kind of interesting and deep spectral reciprocity formula which contains the product of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin-Selberg L-functions and the standard $\mathrm{GL}(2)$ L-functions. This is a degree 8 L-function case again. In [BK17], Blomer and Khan actually used powerful analytic number theory tools to go further. They showed that if the input of $\mathrm{GL}(3)$ automorphic form is an Eisenstein series (which is not of rapid decay) other than a cusp form, the spectral reciprocity formula is still true. Therefore, they may successfully apply this kind of spectral reciprocity formula to study the $\mathrm{GL}(2)$ subconvexity problem in level aspect. It is worthwhile to mention that such kind of spectral reciprocity identities appeared at least since Motohashi's formula [Mot93] which connects the fourth moment of the Riemann zeta-function ($\mathrm{GL}(1)$ L-functions) to the cubic moment of standard L-functions of cusp forms on $\mathrm{GL}(2)$.

To be more precise, we briefly recall the main result of Blomer and Khan's paper ([BK17]).

Let f_1 be an automorphic form (cusp form or Eisenstein series) for the group $\mathrm{SL}_3(\mathbb{Z})$. In the following summations, we let f be a cusp form for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (f is ramifed at some finite places). They proved a spectral reciprocity formula of the following shape (For simplicity, I do not write down the Eisenstein series contribution from the spectral decomposition)

$$\sum_{f \text{ of level } q} L(s, f_1 \times f) L(w, f) \lambda_f(\ell) \leftrightsquigarrow \sum_{f \text{ of level } \ell} L(s', f_1 \times f) L(w', f) \lambda_f(q),$$

where q and ℓ are coprime integers and λ_f is the corresponding Hecke eigenvalue for the cusp form f and

$$s' = \tfrac{1}{2}(1+w-s), \quad w' = \tfrac{1}{2}(3s+w-1). \tag{1.10}$$

Furthermore, if f_1 is an Eisenstein series, we will also see that

$$\sum_{f \text{ of level } q} |L(1/2,f)|^4 \lambda_f(\ell) \leftrightsquigarrow \sum_{f \text{ of level } \ell} |L(1/2,f)|^4 \lambda_f(q).$$

Roughly speaking, this formula is a spectral reciprocity formula between the spectrum of the Laplacian operator on two different arithmetic hyperbolic surfaces $\Gamma_0(q)\backslash\,\mathbb{H}$ and $\Gamma_0(\ell)\backslash\,\mathbb{H}$ featuring a product of L-functions of total degree 8 (the product of degree 6 $\mathrm{GL}(3)\times\mathrm{GL}(2)$ Rankin-Selberg L-functions and degree 2 standard $\mathrm{GL}(2)$ L-functions).

In 2020, R. Nunes [Nun20] gave a new and nice proof of Blomer-Khan's result [BK17] and partially generalized their result to general number field instead of $\mathbb{Q}$. Instead of using tools in the analytic number theory (such as Kuznetsov trace formula, Voronoi Summation formula etc.), R. Nunes used mainly the spectral decomposition formula and the integral representation of Rankin-Selberg L-functions. The reciprocity of two coprime ideal $\mathfrak{q}$ and $\mathfrak{l}$ are given by the action of a Weyl element $\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. We extend the method in [Nun20] and the setup applied by R. Nunes can be generalized from $n=2$ to general $n \geq 3$. This is given in [Miao21], where we find the following abstract reciprocity formula.

Let $F/\mathbb{Q}$ be a fixed number field with ring of intergers $\mathcal{O}_F$ and discriminant Δ_F. The adele ring of F is denoted by $\mathbb{A}_F$ and its unit group is given by $\mathbb{A}_F^\times$ (idele group). We also let $\mathbb{A}_F^1 := \{x \in \mathbb{A}_F^\times \ : \ |x| = 1\}$, where $|\cdot| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$ is the adelic norm map. Note that $\mathbb{A}_F^1$ is exactly the kernel of the adelic norm map. We also call $\mathbb{A}_F^1$ norm one ideles.

Suppose that $\Phi \in C^\infty(Z_{n+1}(\mathbb{A}_F)\mathrm{GL}_{n+1}(F)\backslash \mathrm{GL}_{n+1}(\mathbb{A}_F))$ is a cuspidal automorphic form (automorphic form with zero constant term). Therefore, it is of rapid decay.

Then, we can define the projection by

$$\mathcal{A}_s\Phi(h_n) := |\det h_n|^{s-\frac{1}{2}} \cdot \int_{F^\times\backslash\mathbb{A}_F^\times} \Phi\left(\begin{pmatrix} z_n(u)h_n & \\ & 1 \end{pmatrix}\right) |u|^{n(s-\frac{1}{2})} d^\times u. \tag{1.11}$$

Since Φ is of rapid decay, the above average projection map over the center $\mathcal{A}_s\Phi$ is well-defined for every complex number s and is again of rapid decay in terms of

$h_n \in \mathrm{GL}(n)$. Moreover, we see that $\mathcal{A}_s\Phi$ is invariant under the action of the center $\mathrm{Z}_n(\mathbb{A}_F)$.

We also give the following definition of the period integral

$$I(w,\phi,\varphi) := \int_{\mathrm{GL}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \phi \begin{pmatrix} h_{n-1} & \\ & 1 \end{pmatrix} \varphi(h_{n-1}) |\det h_{n-1}|^{w-\frac{1}{2}} dh_{n-1},$$

whenever it is converges. Here ϕ is a smooth function defined on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$ and φ is a fixed everywhere unramified cuspidal automorphic form for $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Moreover, we assume that φ is invariant under the center $\mathrm{Z}_n(\mathbb{A}_F)$, therefore its central character is trivial. If $n = 2$, we see that $\varphi = 1$ which is the trivial character.

Theorem 1.3.1. *[Abstract Reciprocity Formula]*

Let $\Phi \in C^\infty(Z_{n+1}(\mathbb{A}_F)\mathrm{GL}_{n+1}(F)\backslash\mathrm{GL}_{n+1}(\mathbb{A}_F))$ be a cusp form. Then, for every s, $w \in \mathbb{C}$, we have the following abstract reciprocity equation

$$I(w,\mathcal{A}_s\Phi,\varphi) = I(w',\mathcal{A}_{s'}\check{\Phi},\varphi),$$

where $s' = \frac{1+(n-1)w-s}{n}$, $w' = \frac{(n+1)s+w-1}{n}$, and $\check{\Phi}$ is given by

$$\check{\Phi} = \Pi(w_{12})\cdot\Phi, \quad w_{12} := \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{1.12}$$

Actually, we have the following diagram:

$$\begin{array}{ccccc} & & \mathrm{GL}(n+1) & & \\ & \nearrow & & \nwarrow & \\ \mathrm{GL}(n) & & & & w_{12}\mathrm{GL}(n)w_{12}^{-1}. \\ & \nwarrow & & \nearrow & \\ & & \mathrm{GL}(n-1) & & \end{array} \tag{1.13}$$

Here the embedding of $\mathrm{GL}(n-1)$ in $\mathrm{GL}(n)$ is given by the embedding $g_{n-1} \to \begin{pmatrix} g_{n-1} & \\ & 1 \end{pmatrix}$.

Remark 1.3.2. *More generally, we may view spectral reciprocity formulae as a strong Gelfand formation, which is due to Reznikov (See [Rez08]). First, we say that a pair of*

reductive groups (G, H) over a number field is a strong Gelfand pair if for every place v and every pair of irreducible (admissible) representations Π and σ of $G(F_v)$ and $H(F_v)$, respectively, we have that the space of $H(F_v)$-invariant maps from Π to σ is at most one-dimensional. Now let G, H_1, H_2 and J be reductive groups with natural embeddings

$$\begin{array}{ccccc} & & G & & \\ & \nearrow & & \nwarrow & \\ H_1 & & & & H_2. \\ & \nwarrow & & \nearrow & \\ & & J & & \end{array} \tag{1.14}$$

Then we say that (G, H_1, H_2, J) is a strong Gelfand formation if the pairs (G, H_i) and (H_i, J) are strong Gelfand pairs, for $i = 1, 2$.

It is well-known that if (G, H) is a Gelfand pair, Π and σ are automorphic representations of G and H, respectively, and $\Phi \in \Pi$ and $\xi \in \sigma$ are automorphic forms than the period

$$\int_{[H]} \Phi(h)\overline{\xi}(h)dh,$$

where $[H] := H(F)\backslash H(\mathbb{A}_F)$ and $[J] := J(F)\backslash J(\mathbb{A}_F)$ can often be linked to an L-function (For simplicity, we assume that such quotients are compact). Therefore, in order to obtain a reciprocity formula one might consider $\Phi \in \Pi$ and $\phi \in \pi$ automorphic forms of G and J, respectively, and consider the period $\int_{[J]} \Phi(j)\overline{\phi}(j)dj$. Expanding the vector Φ in the spaces of automorphic forms in H_1 and H_2, one should get

$$\sum_{\xi_1} \left(\int_{[H_1]} \Phi\overline{\xi_1} \right) \left(\int_{[J]} \xi_1\overline{\phi} \right) = \sum_{\xi_2} \left(\int_{[H_2]} \Phi\overline{\xi_2} \right) \left(\int_{[J]} \xi_2\overline{\phi} \right),$$

where ξ_i runs through an orthonormal basis of automorphic forms for H_i $(i = 1, 2)$, and in the above discussion we ignore all convergence issues or non-discrete spectrum.

The reciprocity formula which we prove here can be seen as a special case of the above discussion where take $G = \mathrm{GL}(n+1)$, $H_1 = \mathrm{GL}(n)$, $J = \mathrm{GL}(n-1)$, and $H_2 = w_{12}\mathrm{GL}(n)w_{12}^{-1}$, with $w_{12} = \begin{pmatrix} I_{n-1} & \\ & \begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix} \end{pmatrix}$. For these groups, inclusion is given by $h \hookrightarrow \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ which is an embedding of $\mathrm{GL}(n)$ inside $\mathrm{GL}(n+1)$.

The period integral is closely related to the product of Rankin-Selberg L-functions. Applying the integral representations of Rankin-Selberg L-functions, the spectral decomposition of square-integrable automorphic representations and the local test-vectors theory of Rankin-Selberg L-functions [BKL19] to the period integral, we can generalize the spectral reciprocity formulae of Blomer-Khan and Nunes from $n = 2$ to general $n \geq 3$. More precisely, we establish a spectral reciprocity formula for the product of $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ and $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ $(n \geq 2)$ Rankin-Selberg L-functions, which is a degree $2n^2$ L-functions. The reciprocity of two coprime ideal $\mathfrak{q}$ and $\mathfrak{l}$ are given by the action of a Weyl element $w_{12} = \begin{pmatrix} I_{n-1} & & \\ & 1 & 1 \\ & 1 & \end{pmatrix}$. Note that if $n = 2$, the result has already been established in [Nun20, Theorem 1.1, 1.2]. The spectral reciprocity formula we have expresses the following reciprocity relation:

$$\sum_{\pi \text{ of level } \mathfrak{q}} L(s, \Pi \times \widetilde{\pi}) L(w, \pi \times \pi_1) H(\pi) \leftrightsquigarrow \sum_{\pi \text{ of level } \mathfrak{l}} L(s', \Pi \times \widetilde{\pi}) L(w', \pi \times \pi_1) \widetilde{H}(\pi)$$

for different unramified prime ideal $\mathfrak{q}$ and $\mathfrak{l}$, where $\widetilde{\pi}$ is the contragredient representation of π. Here both Π and π_1 are cuspidal automorphic forms on $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n-1)$ which are unramified everywhere and have trivial central character. Note that the automorphic representation π on $\mathrm{GL}(n)$ also have trivial central character. Moreover, we have

$$s' = \frac{1 + (n-1)w - s}{n}, \quad w' = \frac{(n+1)s + w - 1}{n}.$$

To be more precise, the automorphic spectral reciprocity formula we achieve is as follows:

Theorem 1.3.3. *Let Π and π_1 be everywhere unramified tempered cuspidal automorphic representation for $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n-1)$ over F with trivial central character. Let $\widetilde{\pi}$ be the contragredient representation of π. Suppose that $\mathfrak{q}$ and $\mathfrak{l}$ are unramified, coprime ideals. Futhermore, we assume that $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < \frac{n+1}{n+2}$. Then we will have the following identity*

$$\mathcal{M}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l}) = \mathcal{N}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{D}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{M}(\Pi, \pi_1, s', w', \mathfrak{l}, \mathfrak{q}),$$

where the complex parameters s', w' satisfy the relation

$$s' = \frac{1 + (n-1)w - s}{n}, \quad w' = \frac{(n+1)s + w - 1}{n}.$$

Here the definition of $\mathcal{M}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l})$, $\mathcal{N}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l})$ *and* $\mathcal{D}(\Pi, \pi_1, s, w, \mathfrak{q}, \mathfrak{l})$ *are given in* (11.20) (11.21), (10.11) *and* (11.22) *respectively.*

The spectral reciprocity formula as obtained in Theorem 1.3.3 for general $n \geq 3$ is expected to have interesting applications on the simultaneous non-vanishing problems of relevant Rankin-Selberg L-functions as indicated in [Nun20, Corollary 1.3]. In order to carry out such deep applications, we have to bound all the terms in continuous spectrum which contain the moment of higher-rank L-functions. We hope to discuss those applications in our future work.

1.4 Organization of the Paper

The first part of my thesis (Chapter 2-Chapter 7) will contribute to the proof of the local integrability of GL(n) Bessel functions (Theorem 1.2.6 and 1.2.8) and its applications. In the second part of my thesis (Chapter 8-Chapter 11), we will focus on the automorphic spectral reciprocity formulae for general linear groups (Theorem 1.3.1 and 1.3.3).

- Chapter 2: We consider the Bessel functions in lower rank (GL(2)). We give the definition and prove some basic properties of GL(2) Bessel functions, which is a special case in Section 1.2. Moreover, we give some applications of Bessel functions, including gamma factors, stability and local converse theorem. Readers may realize the importance and power of Bessel functions even in the small rank case.

- Chapter 3: We start the reduction steps for the proof of Theorem 1.2.6. Applying the relations between Bessel functions and local Kloosterman (orbital) integrals proved in several papers of E. M. Baruch [Ba03] [Ba04] [Ba05], the theory of the (relative) Shalika germs established by H. Jacquet and Y. Ye in [JY96] [JY99], we can reduce the proof to estimate certain local Kloosterman-type integral with a special characteristic function, which is locally constant and compact support.

- Chapter 4: We continue the reduction steps for the proof of Theorem 1.2.6. It is almost impossible to compute the GL(n) ($n \geq 4$) Kloosterman integral explicitly, which can be done in the case of GL(2) and GL(3) (See [JY96] and [JY99]).

To overcome this difficulty, after writing Kloosterman integrals as generalized Kloosterman sums, we apply and modify G. Stevens' approach [Ste87] on estimating certain $\mathrm{GL}(n)$ generalized Kloosterman sums. We need to estimate how many orbits are contributing. G. Stevens' approach gives an effective method to yield a non-trivial upper bound for the generalized Kloosterman sums.

- Chapter 5: We follow the method of G. Stevens (Chapter 4) to estimate Kloosterman sums for $\mathrm{GL}_n(\mathbb{Q}_p)$ and prove a non-triival upper bound (See Theorem 5.0.1).

- Chapter 6: We complete the proof of the local integrability of $\mathrm{GL}(n)$ Bessel functions (Theorem 1.2.6).

- Chapter 7: We give some applications of Theorem 1.2.6, including the regularity theorem (Theorem 1.2.8), the kernel formula (Corollary 7.1.5) and the non-trivial absolute convergence region for the Kloosterman zeta function (Section 7.2).

- Chapter 8: We give some preliminaries of automorphic spectral reciprocity formulae.

- Chapter 9: We recall some basic properties of global and local theory for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ Rankin-Selberg L-functions. Moreover, we consider the global zeta integral regularization when the automorphic representation is not cuspidal.

- Chapter 10: We prove the abstract reciprocity formula (Theorem 1.3.1) by comparing two periods with matrix computation. This is an identity between two periods. Moreover, by spectral decomposition, we are able to give the abstract pre-spectral reciprocity identity (Theorem 10.2.2).

- Chapter 11: By careful choosing the local test vectors for both non-archimedean and archimedean places, we are able to establish the explicit spectral reciprocity formula. Moreover, in order to get the information for global L-functions at the central value, we need a meromorphic continuation for certain moment of L-functions. Finally, we reach the explicit automorphic spectral reciprocity formula (Theorem 1.3.3).

- Finally, in Appendix A, after a slightly different and more careful approach in counting via G. Stevens' approach, we are able to give a better bound for the

Kloosterman sums estimation on $\mathrm{GL}_4(\mathbb{Q}_p)$ (Comparing Theorem A.0.1 with Theorem 5.0.1).

Chapter 2

Bessel Functions on $\mathrm{GL}(2)$

In the first part of my thesis, we will focus on the local integrability problem for $\mathrm{GL}(n)$ Bessel functions and its applications (Theorem 1.2.6 and 1.2.8). Readers may see Section 1.2 for introduction. Before we give the proof of the local integrability for Bessel functions in arbitrary rank, we will focus on the Bessel functions in lower rank (GL(2)) and consider some applications in this Chapter 2. Readers may see the importance and power of Bessel functions in the study of number theory and representation theory.

2.1 Definition and Properties of Bessel functions on $\mathrm{GL}_2(F)$

In this section, we will mainly give the definition and some properties of Bessel functions on $\mathrm{GL}_2(F)$, where F is a non-archimedean local field (See [Cog14] and [So84]).

Before we give the precise definition of Bessel functions, let's first recall the Whittaker model of an infinite dimensional irreducible admissible representation π on $\mathrm{GL}_2(F)$.

Let F be a non-Archimedean local field with characteristic 0, i.e. a finite extension of a p-adic field $\mathbb{Q}_p$. Let $\mathcal{O}_F$ and $\mathcal{O}_F^\times$ be the ring of p-adic integers and multiplicative group of units respectively. Let $\mathfrak{p}$ be the unique prime ideal in $\mathcal{O}_F$ and ω be the corresponding uniformizer. Let q be the order of the residue field of F, i.e. $q = |\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F|$. Moreover, we fix a nontrivial additive character ψ from F to S^1 whose conductor is exact $\mathcal{O}_F$.

Let $B_2 = T_2U_2$ be the standard Borel subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular matrices, with unipotent radical $U := U_2 = \begin{pmatrix} 1 & u_{12} \\ & 1 \end{pmatrix}$ and diagonal group

T_2. We can define a non-degenerate character ψ_U of U as follows:

$$\psi_U(u) := \psi\left(u_{12}\right),\ u \in U.$$

An infinite dimensional irreducible admissible representation (π, V) of $\mathrm{GL}_2(F)$ is generic if

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}\left(V, \mathrm{Ind}_U^{\mathrm{GL}_2(F)}\psi_U\right) \neq 0.$$

From the uniqueness of Whittaker model, it is known that if representation π is generic, then the above Hom-space is of dimension 1. Moreover, all infinite dimensional irreducible admissible representations (π, V) of $\mathrm{GL}_2(F)$ are generic. Now let π be an irreducible admissible generic representation of $\mathrm{GL}_2(F)$, fix a nonzero functional l in the above Hom-space, then the image of V under l is called the Whittaker model of π, denoted by $\mathcal{W}(\pi, \psi)$. It is known that $\mathcal{W}(\pi, \psi)$ is independent of the choice of l. For each vector $v \in V$, let $W_v := l(v)$. Then for $\forall u \in U,\ g \in \mathrm{GL}_2(F)$, we have

$$W_v(ug) = \psi_U(u)W_v(g),$$

$$W_v(g) = l(v)(g) = l(\pi(g)v)(I_2) = W_{\pi(g)v}(I_2).$$

In order to define Bessel functions for $\mathrm{GL}_2(F)$, we need the following lemma.

Lemma 2.1.1. *Let π be an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(F)$, then the integral*

$$I_x(W) := \int_F W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(z)dz, \quad W \in \mathcal{W}(\pi, \psi),\ x \in F^\times$$

converges (in the sense that the integral stabilizes for sufficiently large compact subgroups of F).

Proof. From classical theory, it is known that π will have a Whittaker model $\mathcal{W}(\pi, \psi)$ and Kirillov model $\mathcal{K}(\pi, \psi)$. They are related by

$$\varphi_v(x) = W_v\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathcal{K}(\pi, \psi) \text{ for } v \in V_\pi.$$

Assume first that

$$W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathcal{S}(F^\times),$$

where $\mathcal{S}(F^\times)$ is the space of Schwartz functions on $F^\times$ (Here we can make this assumption because $\mathcal{S}(F^\times) \subseteq \mathcal{K}(\pi, \psi)$). Then we claim that for a fixed $x \in F^\times$, the function

$$W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)$$

has a compact support in z.

Note that from the smoothness of Whittaker functions, let $|z|$ sufficiently large we know that

$$\pi\begin{pmatrix} 1 & 0 \\ -z^{-1} & 1 \end{pmatrix} W = W \quad W \in \mathcal{W}(\pi, \psi).$$

Then we have

$$\begin{aligned} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) &= W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z^{-1} & 1 \end{pmatrix}\right) \\ &= W\left(\begin{pmatrix} -xz^{-1} & x \\ 0 & -z \end{pmatrix}\right) \\ &= \omega_\pi(-z)\psi\left(-\frac{x}{z}\right) W\left(\begin{pmatrix} \frac{x}{z^2} & 0 \\ 0 & 1 \end{pmatrix}\right), \end{aligned}$$

where ω_π is the central character of representation π. Now from our previous assumption, if $|z|$ is large enough (depending on x), we have

$$W\left(\begin{pmatrix} \frac{x}{z^2} & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

This means that the function

$$W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)$$

has a compact support in z for a fixed x. So the integral converges under our assumption.

Now let W be any Whittaker function in the space $\mathcal{W}(\pi, \psi)$. We fix an integer $m_0 > 0$. Let $n \geq m_0$, then

$$\begin{aligned} &\int_{|z| \leq q^n} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \psi^{-1}(z) dz \\ =&\int_{|z| \leq q^n} W_{m_0}\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \psi^{-1}(z) dz, \end{aligned} \tag{2.1}$$

where

$$W_{m_0}(g) := q^{-m_0} \cdot \int_{|u| \leq q^{m_0}} W\left(g \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \psi^{-1}(u) du.$$

From the identity

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & yu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix},$$

it is easy to check that the function

$$W_{m_0}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

belongs to the space $\mathcal{S}(F^\times)$ from the orthogonality of characters. Moreover, from the properties of Whittaker functions, it is known that for $\forall u \in U,\ g \in \mathrm{GL}_2(F)$,

$$W_{m_0}(ug) = \psi_U(u) W_{m_0}(g),$$

and W_{m_0} is a smooth function from the smoothness of Whittaker functions. Therefore, the left hand side of the equation 3.4 stabilizes for n large enough. Lemma is proved.

□

By definition I_x is a Whittaker functional of representation π, that is

$$I_x\left(\pi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\right) W = \psi(z) I_x(W) \quad z \in F.$$

From the uniqueness of Whittaker model, it is known that such a Whittaker functional is unique up to scalars. Thus, we can define Bessel functions $j_\pi(x)$ for $\mathrm{GL}_2(F)$ as follows:

Definition 2.1.2. *The Bessel function $j_\pi(x)$ is the constant of proportionality (a function of x as $I_x(W)$ varies), that is,*

$$\int_F W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \psi^{-1}(z) dz = j_\pi(x) W(I_2) \quad W \in \mathcal{W}(\pi, \psi).$$

Remark 2.1.3. *From above lemma and the uniqueness of Whittaker model, we know that our definition of Bessel functions is well-defined. We also note that Bessel function $j_\pi(x)$ is independent on our choice of Whittaker function W in the space $\mathcal{W}(\pi, \psi)$.*

After the definition of Bessel function $j_\pi(x)$, we will give some properties of $j_\pi(x)$ in the following propositions.

Proposition 2.1.4 (**Weak kernel formula for** $\mathrm{GL}_2(F)$)**.** *Let $W \in \mathcal{W}(\pi, \psi)$ be such that the function*

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathcal{S}(F^\times),$$

then we have

$$W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) = \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y.$$

Proof. We let

$$\varphi_{W,x}(z) := W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

Then from the definition of Fourier transform and Bessel functions, we have

$$\widehat{\varphi}_{W,x}(1) = \int_F \varphi_{W,x}(z)\psi^{-1}(z)dz = j_\pi(x)W(I_2).$$

Since the function $\varphi_{W,x}(z)$ is compactly supported, our Fourier transform is well-defined. Now changing the varible, we have

$$\widehat{\varphi}_{W,x}(y) = |y|^{-1}\widehat{\varphi}_{\pi\begin{pmatrix} 1 & 0 \\ 0 & y^{-1} \end{pmatrix}W,xy}(1) = |y|^{-1}\omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right),$$

and hence

$$\begin{aligned} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) = \varphi_{W,x}(0) = \widehat{\widehat{\varphi}}_{W,x}(0) &= \int_F \widehat{\varphi}_{W,x}(y)dy \\ &= \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y. \end{aligned}$$

Here the above second equation is correct because of Fourier inversion formula. □

Remark 2.1.5. *The above Proposition 2.1.4 is called the weak kernel formula for* $\mathrm{GL}(2)$ *Bessel functions. It is a very important proposition of Bessel functions. We can view Bessel function as a "kernel" of integral representations. In 2004, E. M. Baruch proved the weak kernel formula for* $\mathrm{GL}(3)$ *Bessel functions [Ba04]. In 2017, J. Chai proved the weak kernel formula for* $\mathrm{GL}(n)$ *Bessel functions [Chai17].*

We give more propositions of Bessel functions.

Proposition 2.1.6. *1. For any nonnegative integer n_0, there exists a positive integer N, depending on π and n_0 only such that if $|x| > q^{2N}$, then*

$$j_\pi(x) = \int_{xz^{-2}-1 \in \mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_\pi(-z)\psi^{-1}\left(\frac{x}{z}+z\right)dz.$$

2. Using the same notations as part one of the proposition, we have $j_\pi(xu) = j_\pi(x)$ for $|x| \leq q^{2N}$ and $|u-1| \leq q^{-N}$.

Proof. 1. Take a Whittaker function $W = W_0$ in the space $\mathcal{W}(\pi, \psi)$ such that the function

$$W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is the characteristic function of $1 + \mathfrak{p}^{n_0}\mathcal{O}_F$.

Let A be such that if $|t| \leq A$, then

$$\pi\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} W_0 = W_0.$$

From our proof in Lemma 2.1.1, we have

$$\begin{aligned} j_\pi(x) = &\int_{|z|\leq A^{-1}} W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(z)dz \\ &+ \int_{|z|>A^{-1}} \omega_\pi(-z)\psi^{-1}\left(\frac{x}{z}+z\right) W_0\left(\begin{pmatrix} \frac{x}{z^2} & 0 \\ 0 & 1 \end{pmatrix}\right)dz. \end{aligned} \tag{2.2}$$

From the identity

$$W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = W_0\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right),$$

and using the local kernel formula in Proposition 2.1.4, we have

$$\begin{aligned} W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) &= \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(y) W_0\left(\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) d^\times y \\ &= \int_{F^\times} \omega_\pi^{-1}(x^{-1}y) j_\pi(y)\psi(yx^{-1}z) W_0\left(\begin{pmatrix} yx^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y. \end{aligned}$$

So for all $|z| \leq A^{-1}$, there exists a positive constant C_π such that for all $|x| > C_\pi$, we have

$$W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Now the part one of proposition is proved.

2. In part two, we use the same notations as the above part one. Since the function

$$W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is the characteristic function of $1 + \mathfrak{p}^{n_0}\mathcal{O}_F$, so if $|x| \leq q^{2N}$ and $|z| > q^N$, we have

$$W_0\left(\begin{pmatrix} \frac{x}{z^2} & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

So for $|x| \leq q^{2N}$, we have

$$j_\pi(x) = \int_{|z|\leq q^N} W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(z)dz.$$

Now using the local kernel formula in Proposition 2.1.4, we have

$$\begin{aligned} j_\pi(x) &= \int_{|z|\leq q^N} W_0\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(z)dz \\ &= \int_{F^\times}\int_{|z|\leq q^N} \omega_\pi^{-1}(y) j_\pi(xy)\psi(z(y-1))W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)dz\, d^\times y \qquad (2.3) \\ &= q^N \int_{|y-1|\leq q^{-N}} j_\pi(xy)dy. \end{aligned}$$

Changing the varible, we know that for $|x| \leq q^{2N}$ and $|u-1| \leq q^{-N}$, we have $j_\pi(xu) = j_\pi(x)$. This means that Bessel function $j_\pi(x)$ is a locally constant function on $F^\times$.

□

2.2 Mellin transform of Bessel functions and Gamma factors

In this section, we will show that the Mellin transform of $j_\pi(x)$ is the (Rankin-Selberg) gamma factor, which is defined by the local functional equation of local L-function for GL_2 or $\mathrm{GL}_2 \times \mathrm{GL}_1$.

Gamma factors of local GL_2 L-function

We will prove that the Mellin transform of $j_\pi(x)$ is the gamma factor $\gamma(s, \pi, \psi)$, which is given by the local functional equation of local L-function for GL_2.

Before we prove the theorem, let's firstly recall the local functional equation for local GL_2 L-function in Jacquet-Langlands' famous book Chapter 1 Section 3 [JL70].

Theorem 2.2.1 (*Local Functional Equation for* $\mathrm{GL}(2)$)**.** *Let π be an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(F)$. For every Whittaker function $W \in \mathcal{W}(\pi, \psi)$, we can define the local zeta integral as follows:*

$$Z_W(g, \pi, s) := \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) |x|^{s-\frac{1}{2}} d^\times x.$$

The above local zeta integral converges for $\mathrm{Re}(s) >> 0$, and can be analytically continued to a meromorphic function on the whole complex plane $\mathbb{C}$ with at most two poles. Furthermore, there exists a meromorphic function $\gamma(s, \pi, \psi)$, which depends neither on our choice of Whittaker function $W \in \mathcal{W}(\pi, \psi)$ nor on element $g \in \mathrm{GL}_2(F)$, and this meromorphic function satisfies the following functional equation:

$$Z_W(wg, \omega_\pi, 1-s) = \gamma(s, \pi, \psi) Z_W(g, \pi, s)$$

for all $W \in \mathcal{W}(\pi, \psi)$ and $g \in \mathrm{GL}_2(F)$, where

$$Z_W(wg, \omega_\pi, s) := \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right) \omega_\pi^{-1}(x) |x|^{s-\frac{1}{2}} d^\times x$$

and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From above local functional equation for GL(2), we can do the calculation of local gamma factor $\gamma(s, \pi, \psi)$, using a special family of Whittaker functions since local gamma

factor is independent on our choice of Whittaker functions $W \in \mathcal{W}(\pi, \psi)$. Now we have the following theorem which indicates that the gamma factor is exact the Mellin transform of Bessel function.

Theorem 2.2.2. *Assume that* $\mathrm{Re}(s) << 0$*, then we have*

$$\gamma(s, \pi, \psi) = \int_{F^\times} \omega_\pi^{-1}(x) j_\pi(x) |x|^{\frac{1}{2}-s} d^\times x.$$

Proof. Since the local gamma factor $\gamma(s, \pi, \psi)$ is independent on our choice of element $g \in \mathrm{GL}_2(F)$, we just fix $g = I_2$.

Now since $\mathcal{S}(F^\times) \subseteq \mathcal{K}(\pi, \psi)$, we can take a Whittaker function $W = W_0$ in the space $\mathcal{W}(\pi, \psi)$ such that the function

$$W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is the characteristic function of $\mathcal{O}_F^\times$. From our choice of Whittaker function W_0, we know that

$$Z_{W_0}(I_2, \pi, s) = \int_{\mathcal{O}_F^\times} 1 \; d^\times x = 1.$$

Since the local gamma factor $\gamma(s, \pi, \psi)$ is independent on our choice of Whittaker function $W_0 \in \mathcal{W}(\pi, \psi)$, it is known that

$$\begin{aligned} \gamma(s, \pi, \psi) &= Z_{W_0}(w, \omega_\pi, 1-s) \\ &= \int_{F^\times} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) \omega_\pi^{-1}(x) |x|^{\frac{1}{2}-s} d^\times x. \end{aligned}$$

Now recall the weak kernel formula for GL_2 Bessel functions (Proposition 2.1.4), we have

$$\begin{aligned} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) &= \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y \\ &= \int_{\mathcal{O}_F^\times} \omega_\pi^{-1}(y) j_\pi(xy) d^\times y. \end{aligned} \tag{2.4}$$

Since we assume that $\mathrm{Re}(s) << 0$, so the local zeta integral $Z_{W_0}(w, \omega_\pi, 1-s)$ is absolutely convergent. From Fubini's Theorem, we have

$$\begin{aligned} Z_{W_0}(w, \omega_\pi, 1-s) &= \int_{F^\times} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) \omega_\pi^{-1}(x)|x|^{\frac{1}{2}-s} d^\times x \\ &= \int_{F^\times} \left(\int_{\mathcal{O}_F^\times} \omega_\pi^{-1}(y) j_\pi(xy) d^\times y \right) \omega_\pi^{-1}(x)|x|^{\frac{1}{2}-s} d^\times x \\ &= \int_{\mathcal{O}_F^\times} \left(\int_{F^\times} \omega_\pi^{-1}(xy) j_\pi(xy)|x|^{\frac{1}{2}-s} d^\times x \right) d^\times y \\ &= \int_{\mathcal{O}_F^\times} |y|^{s-\frac{1}{2}} d^\times y \cdot \int_{F^\times} \omega_\pi^{-1}(z) j_\pi(z)|z|^{\frac{1}{2}-s} d^\times z \ \text{ (Changing Varible)} \\ &= \int_{F^\times} \omega_\pi^{-1}(x) j_\pi(x)|x|^{\frac{1}{2}-s} d^\times x. \end{aligned} \tag{2.5}$$

In conclusion, we have

$$\gamma(s, \pi, \psi) = Z_{W_0}(w, \omega_\pi, 1-s) = \int_{F^\times} \omega_\pi^{-1}(x) j_\pi(x)|x|^{\frac{1}{2}-s} d^\times x$$

for $\mathrm{Re}(s) << 0$. □

Gamma factors of local $\mathrm{GL}_2 \times \mathrm{GL}_1$ L-function

We will relate Bessel functions $j_\pi(x)$ to the Rankin-Selberg convolution gamma factor $\gamma(s, \pi \times \chi, \psi)$, which is defined by the local functional equation of local L-function for $\mathrm{GL}_2 \times \mathrm{GL}_1$.

Before we prove the theorem, let's firstly recall the local functional equation for local $\mathrm{GL}_2 \times \mathrm{GL}_1$ L-function in Roger Godement's notes on Jacquet-Langlands' theory Chapter 1 Section 12 [God18].

Theorem 2.2.3 (*Local Functional Equation for* $\mathrm{GL}(2) \times \mathrm{GL}(1)$)**.** *Let π be an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(F)$. For every Whittaker function $W \in \mathcal{W}(\pi, \psi)$ and every multiplicative quasicharacter χ of $F^\times$, we can define the local zeta integral as follows:*

$$Z_W(g, \pi, \chi, s) := \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi^{-1}(x)|x|^{s-\frac{1}{2}} d^\times x.$$

The above local zeta integral converges for $\mathrm{Re}(s) >> 0$*, and can be analytically continued to a meromorphic function on the whole complex plane* $\mathbb{C}$ *with at most two poles. Furthermore, there exists a meromorphic function* $\gamma(s, \pi \times \chi, \psi)$*, which depends neither on our choice of Whittaker function* $W \in \mathcal{W}(\pi, \psi)$ *nor on element* $g \in \mathrm{GL}_2(F)$*, and this meromorphic function satisfies the following functional equation:*

$$Z_W(wg, \omega_\pi \chi^{-1}, 1-s) = \gamma(s, \pi \times \chi, \psi) Z_W(g, \pi, \chi, s)$$

for all $W \in \mathcal{W}(\pi, \psi)$ *and* $g \in \mathrm{GL}_2(F)$*, where*

$$Z_W(wg, \omega_\pi \chi^{-1}, s) := \int_{F^\times} W\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} wg\right) \omega_\pi^{-1}(x)\chi(x)|x|^{s-\frac{1}{2}} d^\times x$$

and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

From above local functional equation for GL(2), we can do the calculation of local gamma factor $\gamma(s, \pi \times \chi, \psi)$, using a special family of Whittaker functions since local gamma factor is independent on our choice of Whittaker functions $W \in \mathcal{W}(\pi, \psi)$. Now we have the following theorem which indicates that the gamma factor is exact the Mellin transform of Bessel function.

Theorem 2.2.4. *Assume that* $\mathrm{Re}(s) << 0$*, then we have*

$$\gamma(s, \pi \times \chi, \psi) = \int_{F^\times} \omega_\pi^{-1}(x)\chi(x) j_\pi(x) |x|^{\frac{1}{2}-s} d^\times x.$$

Proof. Since the local gamma factor $\gamma(s, \pi \times \chi, \psi)$ is independent on our choice of element $g \in \mathrm{GL}_2(F)$, we just fix $g = I_2$. Let the conductor of quasicharacter χ be $1 + \mathfrak{p}^n \mathcal{O}_F$, i.e. $\chi_{1+\mathfrak{p}^n\mathcal{O}_F} = 1$.

Now since $\mathcal{S}(F^\times) \subseteq \mathcal{K}(\pi, \psi)$, we can take a Whittaker function $W = W_0$ in the space $\mathcal{W}(\pi, \psi)$ such that the function

$$W_0\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = q^n \cdot 1_{1+\mathfrak{p}^n\mathcal{O}_F}.$$

From our choice of Whittaker function W_0, we know that

$$Z_{W_0}(I_2, \pi, s) = q^n \cdot \int_{1+\mathfrak{p}^n\mathcal{O}_F} 1 \; d^\times x = q^n \cdot q^{-n} = 1.$$

Since the local gamma factor $\gamma(s, \pi \times \chi, \psi)$ is independent on our choice of Whittaker function $W_0 \in \mathcal{W}(\pi, \psi)$, it is known that

$$\begin{aligned}\gamma(s, \pi \times \chi, \psi) &= Z_{W_0}(w, \omega_\pi \chi^{-1}, 1-s) \\ &= \int_{F^\times} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) \omega_\pi^{-1}(x)\chi(x)|x|^{\frac{1}{2}-s} d^\times x.\end{aligned}$$

Now recall the weak kernel formula for GL_2 Bessel functions (Proposition 2.1.4), we have

$$\begin{aligned}W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) &= \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y \\ &= q^n \cdot \int_{1+\mathfrak{p}^n \mathcal{O}_F} \omega_\pi^{-1}(y) j_\pi(xy) d^\times y.\end{aligned} \tag{2.6}$$

Since we assume that $\mathrm{Re}(s) << 0$, so the local zeta integral $Z_{W_0}(w, \omega_\pi \chi^{-1}, 1-s)$ is absolutely convergent. From Fubini's Theorem, we have

$$\begin{aligned}Z_{W_0}(w, \omega_\pi \chi^{-1}, 1-s) &= \int_{F^\times} W\left(\begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}\right) \omega_\pi^{-1}(x)\chi(x)|x|^{\frac{1}{2}-s} d^\times x \\ &= q^n \cdot \int_{F^\times} \left(\int_{1+\mathfrak{p}^n \mathcal{O}_F} \omega_\pi^{-1}(y) j_\pi(xy) d^\times y \right) \omega_\pi^{-1}(x)\chi(x)|x|^{\frac{1}{2}-s} d^\times x \\ &= q^n \cdot \int_{1+\mathfrak{p}^n \mathcal{O}_F} \left(\int_{F^\times} \omega_\pi^{-1}(xy) j_\pi(xy) \chi(x)|x|^{\frac{1}{2}-s} d^\times x \right) d^\times y \\ &= q^n \cdot \int_{1+\mathfrak{p}^n \mathcal{O}_F} \chi^{-1}(y)|y|^{s-\frac{1}{2}} d^\times y \cdot \int_{F^\times} \omega_\pi^{-1}(z) j_\pi(z)\chi(z)|z|^{\frac{1}{2}-s} d^\times z \\ &= q^n \cdot q^{-n} \cdot \int_{F^\times} \omega_\pi^{-1}(x)\chi(x) j_\pi(x)|x|^{\frac{1}{2}-s} d^\times x \\ &= \int_{F^\times} \omega_\pi^{-1}(x)\chi(x) j_\pi(x)|x|^{\frac{1}{2}-s} d^\times x.\end{aligned} \tag{2.7}$$

In conclusion, we have

$$\gamma(s, \pi \times \chi, \psi) = Z_{W_0}(w, \omega_\pi \chi^{-1}, 1-s) = \int_{F^\times} \omega_\pi^{-1}(x)\chi(x) j_\pi(x)|x|^{\frac{1}{2}-s} d^\times x$$

for $\mathrm{Re}(s) << 0$. □

2.3 Applications of GL_2 Bessel functions

Using the results in the previous Section 2.2 for the Mellin transform, we will discuss some applications of Bessel functions for $\mathrm{GL}_2(F)$ in this section.

Stability of Rankin-Selberg gamma factors for $\mathrm{GL}_2 \times \mathrm{GL}_1$

We will prove the stability of Rankin-Selberg gamma factors $\gamma(s, \pi \times \chi, \psi)$ for local $\mathrm{GL}_2 \times \mathrm{GL}_1$ L-functions. The statement of the theorem is as follows:

Theorem 2.3.1 (*Stability of Rankin-Selberg gamma factors for* $\mathrm{GL}_2 \times \mathrm{GL}_1$*).* *Let π_1, π_2 be two infinite dimensional irreducible admissible representations of $\mathrm{GL}_2(F)$. Let χ be a multiplicative quasicharacter of $F^\times$. We further assume that $\omega_{\pi_1} = \omega_{\pi_2}$. Now if character χ is highly ramified (i.e. Let the conductor of χ be $1+\mathfrak{p}^n\mathcal{O}_F$ and let $n >> 0$), then we have*

$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi).$$

Proof. Recall that from Theorem 2.2.2 and Theorem 2.2.4, it is known that

$$\gamma(s, \pi_1 \times \chi, \psi) = \int_{F^\times} \omega_{\pi_1}^{-1}(x)\chi(x)j_{\pi_1}(x)|x|^{\frac{1}{2}-s}d^\times x,$$

$$\gamma(s, \pi_2 \times \chi, \psi) = \int_{F^\times} \omega_{\pi_2}^{-1}(x)\chi(x)j_{\pi_2}(x)|x|^{\frac{1}{2}-s}d^\times x,$$

for all $\mathrm{Re}(s) << 0$.

In order to prove the stability of Rankin-Selberg gamma factors, we also need some properties of Bessel functions $j_\pi(x)$ we have already proved in Section 2.1. Applying Proposition 2.1.6, we know that

- For any nonnegative integer n_0, there exists a positive integer N, depending on π_1, π_2 and n_0 only such that if $|x| > q^{2N}$, then

  $$j_{\pi_1}(x) = \int_{xz^{-2}-1 \in \mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_{\pi_1}(-z)\psi^{-1}\left(\frac{x}{z} + z\right) dz,$$

  and

  $$j_{\pi_2}(x) = \int_{xz^{-2}-1 \in \mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_{\pi_2}(-z)\psi^{-1}\left(\frac{x}{z} + z\right) dz,$$

- Using the same notations as above, we have $j_{\pi_i}(xu) = j_{\pi_i}(x)$ $(i = 1, 2)$ for $|x| \leq q^{2N}$ and $|u - 1| \leq q^{-N}$.

Now for $i = 1, 2$, let's write

$$\begin{aligned}\gamma(s, \pi_i \times \chi, \psi) = &\int_{0<|x|\leq q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x \\ &+ \int_{|x|>q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x.\end{aligned} \tag{2.8}$$

By Proposition 2.1.6 and above discussion, for $i = 1, 2$, we know that

$$\begin{aligned}&\int_{|x|>q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x \\ =&\int_{|x|>q^{2N}} \int_{xz^{-2}-1\in\mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_{\pi_i}(-zx^{-1})\psi^{-1}\left(\frac{x}{z}+z\right)\chi(x)|x|^{\frac{1}{2}-s}d^{\times}x.\end{aligned} \tag{2.9}$$

Since the central character is the same, i.e. $\omega_{\pi_1} = \omega_{\pi_2}$, this means that

$$\int_{|x|>q^{2N}} \omega_{\pi_1}^{-1}(x)\chi(x)j_{\pi_1}(x)|x|^{\frac{1}{2}-s}d^{\times}x = \int_{|x|>q^{2N}} \omega_{\pi_2}^{-1}(x)\chi(x)j_{\pi_2}(x)|x|^{\frac{1}{2}-s}d^{\times}x.$$

Now for $i = 1, 2$, let's consider the integral:

$$\int_{0<|x|\leq q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x.$$

From absolute convergence of the above integral, we can write

$$\begin{aligned}\int_{0<|x|\leq q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x &= \sum_{k=-\infty}^{2N} \int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^{\times}x \\ &= \sum_{k=-\infty}^{2N} q^{k(\frac{1}{2}-s)} \cdot \int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)d^{\times}x.\end{aligned} \tag{2.10}$$

Suppose that the conductor of central character ω_{π_i} $(i = 1, 2)$ is $1 + \mathfrak{p}^t\mathcal{O}_F$ $(t \geq 0)$, and the conductor of the multiplicative quasicharacter χ is $1 + \mathfrak{p}^l\mathcal{O}_F$. Since χ is highly ramified, we can choose $l > \max\{2N, 2t\}$.

Fix an element $u \in 1 + \mathfrak{p}^{l-1}\mathcal{O}_F$ and $u \notin 1 + \mathfrak{p}^l\mathcal{O}_F$. Since the conductor of quasicharacter χ is $1 + \mathfrak{p}^l\mathcal{O}_F$, we know that $\chi(u) \neq 1$. From our assumption our l, it is known that $u \in 1 + \mathfrak{p}^N\mathcal{O}_F$ and $u \in 1 + \mathfrak{p}^t\mathcal{O}_F$. Now changing the variable, we have

$$\begin{aligned}\int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)d^{\times}x &= \int_{|t|=q^k} \omega_{\pi_i}^{-1}(tu^{-1})\chi(tu^{-1})j_{\pi_i}(tu^{-1})d^{\times}t \\ &= \chi^{-1}(u) \cdot \int_{|t|=q^k} \omega_{\pi_i}^{-1}(t)\chi(t)j_{\pi_i}(t)d^{\times}t.\end{aligned} \tag{2.11}$$

Since $\chi^{-1}(u) \neq 1$, we know that for $i = 1, 2$,

$$\int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)d^\times x = 0.$$

Therefore,

$$\int_{0<|x|\leq q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^\times x = 0.$$

Now in conclusion, if quasicharacter χ is highly ramified, we have

$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi).$$

The stability of Rankin-Selberg gamma factors for $\mathrm{GL}_2 \times \mathrm{GL}_1$ is proved. □

Jacquet's local converse conjecture for $\mathrm{GL}_2(F)$

We will mainly use properties of Bessel functions $j_\pi(x)$ (Section 2.1) to prove Jacquet's local converse conjecture for $\mathrm{GL}_2(F)$.

Before the proof, let's firstly recall Jacquet's general local converse conjecture for $\mathrm{GL}_n(F)$.

Conjecture 2.3.2 (*Jacquet's local converse conjecture for* $\mathrm{GL}_n(F)$*).* *Let π_1, π_2 be two infinite dimensional irreducible admissible generic representations of* $\mathrm{GL}_n(F)$*, and suppose that they have the same central character, i.e. $\omega_{\pi_1} = \omega_{\pi_2}$. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, and for all infinite dimensional irreducible admissible generic representations τ of $GL_r(F)$ with $1 \leq r \leq \left[\frac{n}{2}\right]$, then $\pi_1 \cong \pi_2$.

Remark 2.3.3. *Jacquet's local converse conjecture for* $\mathrm{GL}_n(F)$ *was proved by Jacquet-Liu [JL16] and Jingsong Chai [Chai19b] almost at the same time, using different methods. Jacquet-Liu's paper mainly used Rankin-Selberg's convolution, while Jingsong Chai's paper mainly used the properties of Bessel functions and Howe vectors.*

For $n = 2$, we have the following version of Jacquet's local converse theorem for $\mathrm{GL}_2(F)$.

Theorem 2.3.4 (*Jacquet's local converse conjecture for* $\mathrm{GL}_2(F)$)**.** *Let* π_1, π_2 *be two infinite dimensional irreducible admissible (generic) representations of* $\mathrm{GL}_2(F)$*, and suppose that they have the same central character, i.e.* $\omega_{\pi_1} = \omega_{\pi_2}$*. If*

$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi),$$

as functions of the complex variable s*, and for all multiplicative quasicharacter* χ *of* $F^\times$*, then* $\pi_1 \cong \pi_2$*.*

Proof. First note that we have the following important observation.

Claim: $\pi_1 \cong \pi_2$ if and only if for $x \in F^\times$, we have $j_{\pi_1}(x) = j_{\pi_2}(x)$.

Proof of Claim: If $\pi_1 \cong \pi_2$, then from the definition of Bessel functions it is known that $j_{\pi_1}(x) = j_{\pi_2}(x)$. Now we need to prove the "only if" part. From the uniqueness of Whittaker model, we know that $\pi_1 \cong \pi_2$ is equivalent to $\mathcal{W}(\pi_1, \psi) \bigcap \mathcal{W}(\pi_2, \psi) \neq \varnothing$, i.e. there exists $W(g) \in \mathcal{W}(\pi_1, \psi) \bigcap \mathcal{W}(\pi_2, \psi)$.

We choose $v_1 \in V_{\pi_1}, v_2 \in V_{\pi_2}$, such that

$$W_{v_1}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = W_{v_2}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = 1_{\mathcal{O}_F^\times}.$$

Since $\omega_{\pi_1}(x) = \omega_{\pi_2}(x)$, we have

$$W_{v_1}\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right) = W_{v_2}\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right).$$

Now recall the weak kernel formula for $\mathrm{GL}_2(F)$ Bessel functions (Proposition 1.2.8), since $j_{\pi_1}(x) = j_{\pi_2}(x)$, we have

$$\begin{aligned} W_{v_1}\left(\begin{pmatrix} 0 & -x \\ 1 & 0 \end{pmatrix}\right) &= W_{v_2}\left(\begin{pmatrix} 0 & -x \\ 1 & 0 \end{pmatrix}\right) \\ &= \int_{F^\times} \omega_\pi^{-1}(y) j_\pi(xy) W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) d^\times y. \end{aligned} \tag{2.12}$$

Now from Bruhat decomposition, for $g \in \mathrm{GL}_2(F)$, we have two cases.

1. If $g \in B$, where B is the Borel subgroup of $\mathrm{GL}_2(F)$. We can write

$$g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

for some suitable $x, y \in F^\times, z \in F$. Now from above discussion and properties of Whittaker functions, we know that $W_{v_1}(g) = W_{v_2}(g)$.

2. If $g \in BwB$, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the nontrivial element of GL(2). Now we can write
$$g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -y \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}$$
for some suitable $x, y, v \in F^\times, z, u \in F$. From above discussion, using the weak kernel formula for $\mathrm{GL}_2(F)$ and properties of Whittaker functions, we know that $W_{v_1}(g) = W_{v_2}(g)$.

This means that $W_{v_1} = W_{v_2}$. Therefore, it is known that $\mathcal{W}(\pi_1, \psi) \bigcap \mathcal{W}(\pi_2, \psi) \neq \varnothing$. So $\pi_1 \cong \pi_2$. We proved the claim.

Recall some properties of Bessel functions $j_\pi(x)$ we have already proved in Section 2. Applying Proposition 2.1.6, we know that

- For any nonnegative integer n_0, there exists a positive integer N, depending on π_1, π_2 and n_0 only such that if $|x| > q^{2N}$, then
$$j_{\pi_1}(x) = \int_{xz^{-2}-1 \in \mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_{\pi_1}(-z)\psi^{-1}\left(\frac{x}{z} + z\right) dz,$$
and
$$j_{\pi_2}(x) = \int_{xz^{-2}-1 \in \mathfrak{p}^{-n_0}\mathcal{O}_F} \omega_{\pi_2}(-z)\psi^{-1}\left(\frac{x}{z} + z\right) dz,$$
- Using the same notations as above, we have $j_{\pi_i}(xu) = j_{\pi_i}(x)$ $(i = 1, 2)$ for $|x| \leq q^{2N}$ and $|u - 1| \leq q^{-N}$.

Since $\omega_{\pi_1} = \omega_{\pi_2}$, it is known that if $|x| > q^{2N}$, $j_{\pi_1}(x) = j_{\pi_2}(x)$ from above proposition of Bessel functions. Now it remains to prove that $j_{\pi_1}(x) = j_{\pi_2}(x)$ for all $|x| \leq q^{2N}$.

For $|x| \leq q^{2N}$, it is known that for every quasicharacter χ of $F^\times$, we have
$$\gamma(s, \pi_1 \times \chi, \psi) = \gamma(s, \pi_2 \times \chi, \psi). \tag{2.13}$$
From previous Theorem 2.2.4 and the proof of Theorem 2.3.1, we know that above equation 2.13 of Rankin-Selberg gamma factors is equivalent to
$$\int_{0<|x|\leq q^{2N}} \omega_{\pi_1}^{-1}(x)\chi(x)j_{\pi_1}(x)|x|^{\frac{1}{2}-s} d^\times x = \int_{0<|x|\leq q^{2N}} \omega_{\pi_2}^{-1}(x)\chi(x)j_{\pi_2}(x)|x|^{\frac{1}{2}-s} d^\times x \tag{2.14}$$

for all quasicharacter χ of $F^\times$.

Note that from absolute convergence of the above integral, for $i = 1, 2$, we can write

$$\begin{aligned}\int_{0<|x|\leq q^{2N}} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^\times x &= \sum_{k=-\infty}^{2N} \int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)|x|^{\frac{1}{2}-s}d^\times x \\ &= \sum_{k=-\infty}^{2N} q^{k(\frac{1}{2}-s)} \cdot \int_{|x|=q^k} \omega_{\pi_i}^{-1}(x)\chi(x)j_{\pi_i}(x)d^\times x.\end{aligned} \tag{2.15}$$

This means that above equation 2.14 is equivalent to

$$\int_{|x|=q^k} \omega_{\pi_1}^{-1}(x)\chi(x)j_{\pi_1}(x)d^\times x = \int_{|x|\leq q^k} \omega_{\pi_2}^{-1}(x)\chi(x)j_{\pi_2}(x)d^\times x \tag{2.16}$$

for all quasicharacter χ and all integers k such that $k \leq 2N$.

Since the central character is same, we write $\widetilde{\chi} := \chi\omega_{\pi_i}^{-1}$ for $i = 1, 2$. Changing the variable and let $x = \omega^{-k}u$, where ω is the corresponding uniformizer of the unique prime ideal $\mathfrak{p}$ and $|u| = 1$, i.e. $u \in \mathcal{O}_F^\times$.

Therefore, the above equation 2.16 is equivalent to

$$\int_{|u|=1} \left(j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)\right)\widetilde{\chi}(u)d^\times u = 0 \tag{2.17}$$

for all quasicharacter $\widetilde{\chi}$ and all integers k such that $k \leq 2N$.

We write $j_{\pi_1,\pi_2,k}(u) := j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)$. From properties of Bessel functions (Proposition 2.1.6), it is known that $j_{\pi_1,\pi_2,k}$ is a locally constant function defined on $\mathcal{O}_F^\times$. Moreover, for $v \in \mathcal{O}_F^\times$ such that $|v - 1| \leq q^{-N}$, we have $j_{\pi_1,\pi_2,k}(u) = j_{\pi_1,\pi_2,k}(uv)$.

Since each characteristic function $1_{a+\mathfrak{p}^n\mathcal{O}_F}$ ($n \geq 0$ and $a \in \mathcal{O}_F^\times$) can be written as a finite sum of multiplicative character μ of $\mathcal{O}_F^\times$ (For each multiplicative character μ of $F^\times$, we can restrict it on $\mathcal{O}_F^\times$), it is known that $\{\mu \cdot 1_{\mathcal{O}_F^\times} | \mu \text{ multiplicative character}\}$ forms a basis of Schwartz functions on $\mathcal{O}_F^\times$, i.e. locally constant functions on $\mathcal{O}_F^\times$ ($\mathcal{S}(F^\times) \cdot 1_{\mathcal{O}_F^\times}$).

Now from above discussion, we know there exist $t \in \mathbb{N}^*$, multiplicative character $\widetilde{\mu}_i$ and $C_i \in \mathbb{C}$, where $1 \leq i \leq t$ such that

$$\overline{j_{\pi_1,\pi_2,k}(u)} = \overline{j_{\pi_1}(\omega^{-k}u)} - \overline{j_{\pi_2}(\omega^{-k}u)} = \sum_{i=1}^{t} C_i\widetilde{\mu}_i(u) \tag{2.18}$$

for all $u \in \mathcal{O}_F^\times$ (i.e. $|u| = 1$).

From our condition 2.17, we know that

$$\int_{|u|=1} \left(j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)\right) \sum_{i=1}^{t} C_i \widetilde{\mu_i}(u) d^{\times}u = 0 \tag{2.19}$$

for all integers k such that $k \leq 2N$. From equation 2.18, the above equation 2.19 is equivalent to

$$\int_{|u|=1} \left(j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)\right) \left(\overline{j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)}\right) d^{\times}u = 0 \tag{2.20}$$

for all integers k such that $k \leq 2N$.

The above equation 2.20 is equivalent to

$$\int_{|u|=1} |j_{\pi_1}(\omega^{-k}u) - j_{\pi_2}(\omega^{-k}u)|^2 d^{\times}u = 0 \tag{2.21}$$

for all integers k such that $k \leq 2N$.

From above equation 2.21, it is known that $j_{\pi_1}(\omega^{-k}u) = j_{\pi_2}(\omega^{-k}u)$ for all $u \in \mathcal{O}_F^{\times}$ and integer $k \leq 2N$. Therefore, we know that $j_{\pi_1}(x) = j_{\pi_2}(x)$ for all $x \in F^{\times}$ such that $0 < |x| \leq q^{2N}$.

In conclusion, we proved that $j_{\pi_1}(x) = j_{\pi_2}(x)$ for all $x \in F^{\times}$. From our proof of previous claim, we get $\pi_1 \cong \pi_2$.

□

Chapter 3

Reduction of Proof for Theorem 1.2.6

Now we go back from GL(2) Bessel functions to general GL(n) Bessel functions. In the following Chapter 3 and 4, we will explain the idea of the proof for Theorem 1.2.6. Our setup is several reduction steps. By applying the results in [Ba05], [JY96] and [JY99], we will reduce the proof of local integrability of Bessel functions to finding a non-trivial upper bound for the generalized Kloosterman sums under certain test (characteristic) functions in the case of $\mathrm{GL}_n(\mathbb{Q}_p)$.

3.1 Reduction Step 1: Asymptotic behaviour of the local orbital integrals

We recall our main theorem on the local integrability of GL(n) Bessel functions firstly.

Theorem 3.1.1 (Theorem 1.2.6)**.** *The Bessel function $j_\pi(g)$ is locally integrable on* $\mathrm{GL}_n(\mathbb{Q}_p)$.

Harish-Chandra's proof of the local integrability of the distribution characters (See [HC70] and [HC99]) depends on certain relations between the asymptotics of the character and certain orbital integrals. In the Bessel function case, we have a similar result for such kind of relations.

For $\phi \in C_c^\infty(G_n)$ and $g \in Bw_{G_n}B$, we can define the local orbital integral as follows:

$$J_{\phi,\psi}(g) := \int_{N\times N} \phi(n_1 g n_2)\psi^{-1}(n_1 n_2)dn_1 dn_2.$$

It follows from the results in [JY96] and [JY99] that the above orbital integral converges absolutely and defines a locally constant function when $g \in Bw_{G_n}B$. If $g \in G_n - Bw_{G_n}B$, it is defined that $J_{\phi,\psi}(g) = 0$.

Note that $w_{G_n}Nw_{G_n} = \bar{N}$, where $\bar{N}$ means the opposite of the unipotent radical N. We can consider a similar orbital integral.

Let $f \in C_c^\infty(G_n)$ and $g \in \bar{N}B$, we define the orbital integral similarly:

$$I_{f,\psi}(g) := \int_{N\times N} f(n_1^t g n_2)\psi^{-1}(n_1 n_2)dn_1 dn_2.$$

Here n_1^t means the transpose of the unipotent radical n_1. From our assumptions on the additive character ψ, it is clear that $\psi(n) = \psi(w_{G_n}n^t w_{G_n})$ for all $n \in N$. The above integral converges absolutely and defines a locally constant function on $\bar{N}B$. Moreover, if $g \in G_n - \bar{N}B$, it is defind that $I_{f,\psi}(g) = 0$ which is the same way as before. We note that $I_{f,\psi}(g) = J_{\phi,\psi}(w_{G_n}g)$, where $f(w_{G_n}g) = \phi(g) \in C_c^\infty(G_n)$ and $f(g) \in C_c^\infty(G_n)$.

We may also consider the following local orbital integral.

Let $f \in C_c^\infty(G_n)$ and $g \in \bar{N}B$. Let Z_n be the center of G_n and ω be a quasicharacter of Z_n. We define:

$$I_{f,\omega,\psi}(g) := \int_{N\times Z_n\times N} f(n_1^t z g n_2)\omega^{-1}(z)\psi^{-1}(n_1 n_2)dn_1 dz dn_2.$$

The above integral converges absolutely and defines a locally constant function on $\bar{N}B$. Similarly, we extend the orbital integral $I_{f,\omega,\psi}(g)$ to a function on G_n by setting $I_{f,\omega,\psi}(g) = 0$ when $g \in G_n - \bar{N}B$.

Moreover, we define that

$$J_{f,\omega,\psi}(g) := \int_{N\times Z_n\times N} f(n_1 z g n_2)\omega^{-1}(z)\psi^{-1}(n_1 n_2)dn_1 dz dn_2.$$

This orbital integral converges absolutely and defines a locally constant function when $g \in Bw_{G_n}B$. If $g \in G_n - Bw_{G_n}B$, it is defined that $J_{f,\omega,\psi}(g) = 0$. We note that $I_{f,\omega,\psi}(g) = J_{\phi,\omega,\psi}(w_{G_n}g)$, where $f(g) = \phi(w_{G_n}g) \in C_c^\infty(G_n)$ and $\phi(g) \in C_c^\infty(G_n)$.

The following theorem is proved in [Ba05].

Theorem 3.1.2. *Let π be an smooth irreducible representation of G_n. Let $x \in G_n$, then there exists a neighbourhood U_x of x in G_n and a function $\phi \in C_c^\infty(G_n)$ such that*

$$j_{\pi,\psi}(g) = J_{\phi,\omega_\pi,\psi}(g)$$

for all $g \in U_x$. Here ω_π is the central character of π.

From this theorem, we know that the Bessel function $j_{\pi,\psi}(g)$ is a locally integrable function on G_n if the orbital integral $J_{\phi,\omega_\pi,\psi}(g)$ is locally integrable as a function on G_n for every $\phi \in C_c^\infty(G_n)$.

Similarly, we see that $j_{\pi,\psi}(g)$ is a locally integrable function on G_n if the local orbital integral $I_{f,\omega_\pi,\psi}(g)$ is locally integrable as a function on G_n for every $f \in C_c^\infty(G_n)$.

If for every $f \in C_c^\infty(G_n)$, $I_{f,\omega_\pi,\psi}(g)$ is locally integrable as a function on G, then the Bessel function $j_{\pi,\psi}$ is locally integrable in G_n. This is given by Theorem 3.1.2. In order to prove that $I_{f,\omega_\pi,\psi}(g)$ is locally integrable as a function on G_n, it is important to study the asymptotic behaviour of the local orbital integral when g approaches to the boundary of the domain. From Bruhat decomposition, we can further assume that $g \in A = T$.

3.2 Reduction Step 2: An estimation and comparison

For $g \in G_n := \mathrm{GL}_n(F)$, let $\Delta_r(g)$, $1 \leq r \leq n$, be the determinant of the sub-matrix $g_{r,r}$ of $g \in G_n$ formed with the first r rows and the first r columns of g. Hence $\Delta_n(g) = \det g$. We define $\Delta : G_n \to \mathbb{R}_{\geq 0}$ by

$$\Delta(g) := \left| \frac{(\Delta_1(g))^2 \cdot (\Delta_2(g))^2 \cdots (\Delta_{n-1}(g))^2}{(\Delta_n(g))^2} \right|.$$

It is known that $g \notin \bar{N}B$ (the open Bruhat cell) if and only if we have $\Delta(g) = 0$. Moreover

$$\Delta(\mathrm{diag}(a_1, a_2, \cdots, a_n)) = \left| a_1^{2(n-2)} \cdot a_2^{2(n-3)} \cdots a_{n-2}^2 \cdot a_n^{-2} \right|.$$

Let δ be the modulus character of B, we have

$$\delta(\mathrm{diag}(a_1, a_2, \cdots, a_n)) = \left| a_1^{n-1} \cdot a_2^{n-3} \cdots a_{n-1}^{3-n} \cdot a_n^{1-n} \right|.$$

Hence, we have

$$\Delta(a) = \delta(a) \cdot |a_1 a_2 \cdots a_n|^{n-3} = \delta(a) \cdot |\Delta_n(a)|^{n-3}.$$

We first recall the main Theorem 0.3 proved by Dabrowski and Reeder [DR98].

For $i, j \in \{1, 2, \cdots, n\}, i \neq j$, we let $\alpha_{i,j} : A \to F^\times$ be the functions defined by

$$\alpha_{i,j}(\mathrm{diag}(a_1, a_2, \cdots, a_n)) := \frac{a_i}{a_j}.$$

Let $\Phi = \{\alpha_{i,j}\}, 1 \leq i, j \leq n$ be the root system of G_n and let $\check{\Phi} = \{\check{\alpha} : \alpha \in \Phi\}$. We have $\Phi = \Phi^+ \bigsqcup \Phi^-$, where $\Phi^+ = \{\alpha_{i,j} : 1 \leq i < j \leq n\}$ is the set of positive roots and $\Phi^- = \{\alpha_{i,j} : 1 \leq j < i \leq n\}$ is the set of negative roots. Let $\Delta = \{\alpha_{i,i+1} : 1 \leq i \leq n-1\}$ be the set of simple roots. Similarly, we write $\check{\Phi} = \check{\Phi}^+ \bigsqcup \check{\Phi}^-$.

Let $f \in C_c^\infty(G_n)$ and let $g \in \bar{N}B$. We define the following orbital integral:

$$O_f(g) := \int_{N \times N} f(n_1^t g n_2) dn_1 dn_2.$$

The convergence of this integral follows from [JY96] and [JY99]. In 1998, Dabrowski and Reeder [DR98] studied this orbital integral when $f = f_0$ is the characteristic function of the maximal compact subgroup K. We recall that the Haar measure is normalized with the volume of K and $N \cap K$ equals to one. For each $a \in A$, there exists a unique $\lambda_a \in \check{\Phi}$ such that $a = a_K \lambda_a(\omega)$, where $a_K \in A_k := A \cap K$ and ω is the uniformizer. The following result is the main theorem which is proved by Dabrowski and Reeder [DR98].

Theorem 3.2.1. *[DR98, Theorem 0.3] Let f_0 be the characteristic function of K. Then $O_{f_0}(a) = 0$ if $\lambda = \lambda_a \notin \mathbb{Z}_{\geq 0}\check{\Phi}^+$, i.e., λ_a is not a nonnegative integral linear combination of positive coroots. If λ_a is such a linear combination, we can write*

$$\lambda_a = \sum_{\beta \in \check{\Phi}^+} m_\beta \beta = \sum_{1 \leq i < j \leq n} m_{i,j} \check{\alpha}_{i,j}, \tag{3.1}$$

for $m_{i,j} \geq 0$ and we also write $\overline{m} = (m_\beta)_{\beta \in \check{\Phi}^+} = (m_{i,j})_{1 \leq i < j \leq n}$. Then we have

$$O_{f_0}(a) = \Delta^{-\frac{1}{2}}(a) \times \sum_{\overline{m}} \left(1 - \frac{1}{q}\right)^{\kappa(\overline{m})}, \tag{3.2}$$

where $\kappa(\overline{m})$ is the number of strictly positive coordinates of $\overline{m}$ and $\overline{m}$ runs over all possible decompositions for λ_a.

Using an idea similar to that in [Ba04, Section 3], we can prove the following theorem

Theorem 3.2.2. *$\Delta^{-\frac{1}{2}+\epsilon}$ is locally integrable as a function on $G_n = \mathrm{GL}_n(F)$ for every $\epsilon > 0$.*

Proof. If $\epsilon \geq \frac{1}{2}$, then $\Delta^{-\frac{1}{2}+\epsilon}$ is a locally constant and smooth function on G_n. Therefore, there is nothing to prove.

If $0 < \epsilon < \frac{1}{2}$, then $\Delta^{-\frac{1}{2}+\epsilon}$ is not well-defined if $g \notin \bar{N}B = \bar{N}AN$. However, since $\bar{N}AN$ is Zariski open dense in G_n, the complement subset of $\bar{N}AN$ in G_n is closed and of measure zero. So we can define $\Delta^{-\frac{1}{2}+\epsilon}(g) := 0$ if $g \notin \bar{N}AN$. This will not affect the local integrability of the function $\Delta^{-\frac{1}{2}+\epsilon}$. Note that if $0 < \epsilon < \frac{1}{2}$, then $\Delta^{-\frac{1}{2}+\epsilon}$ is not a locally constant and smooth function on G_n.

We note that it is sufficient to prove that

$$\int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f(g) dg < +\infty$$

for every characteristic function f of Kg_0, where $K = \mathrm{GL}_n(\mathcal{O}_F)$ is the maximal compact open subgroup of G_n and some fixed point $g_0 \in G_n$. From the Iwasawa decomposition, we can write $g_0 = k_0 b_0$ for some $b_0 \in B$ and $k_0 \in K$. Hence it is enough to show that $\int_G \Delta^{-\frac{1}{2}+\epsilon}(g) f(g) dg < +\infty$ holds for $f = \rho_r(b_0^{-1}) f_0$ where ρ_r is the right translation and f_0 is the characteristic function of K.

By writing $b_0 = a_0 n_0$ for $a_0 \in A$ and $n_0 \in N$ we have

$$\begin{aligned}
\int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g)(\rho_r(b_0^{-1}) f_0)(g) dg &= \int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f_0(g b_0^{-1}) dg \\
&= \int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g b_0) f_0(g) dg \\
&= \Delta^{-\frac{1}{2}+\epsilon}(a_0 n_0) \cdot \int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f_0(g) dg \\
&= \Delta^{-\frac{1}{2}+\epsilon}(a_0) \cdot \int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f_0(g) dg.
\end{aligned}$$

Hence it is enough to prove that $\int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f(g) dg < +\infty$ for $f = f_0$. Now applying the invariance properties of Δ and writing $dg = \delta(a) dn_1 da dn_2 = \Delta(a) dn_1 da dn_2$ on the Zariski open dense subset that consists of elements of the form $g = n_1^t a n_2$ where $n_1, n_2 \in N$ and $a \in A$ (Note that since $g = n_1^t a n_2 \in K$, we have $|\det(a)| = |\Delta_n(a)| = 1$,

which gives that $\delta(a) = \Delta(a)$), we can get the following

$$\begin{aligned}\int_{G_n} \Delta^{-\frac{1}{2}+\epsilon}(g) f_0(g) dg &= \int_{\bar{N}\times A\times N} \Delta^{-\frac{1}{2}+\epsilon}(g) f_0(g) dg \\ &= \int_{N\times A\times N} \Delta^{-\frac{1}{2}+\epsilon}(n_1^t a n_2) f_0(n_1^t a n_2) \Delta(a) dn_1 da dn_2 \\ &= \int_A \Delta^{\frac{1}{2}+\epsilon}(a) \cdot O_{f_0}(a) da,\end{aligned}$$

where

$$O_{f_0}(a) := \int_{N\times N} f_0(n_1^t a n_2) dn_1 dn_2.$$

Here we can change the order of integrations because $\Delta^{-\frac{1}{2}+\epsilon} \geq 0$ and $f_0 \geq 0$. The Haar measures on G_n, N and A are all normalized such that the volumes of K and $N \cap K$ equal to one.

Now, we let $X = X(A) = \mathrm{Hom}(A, F^\times)$ be the group of F-rational characters. Let $\check{X} = \check{X}(A) = \mathrm{Hom}(F^\times, A)$ be the set of co-characters. For each $a \in A$, there exists a unique $\lambda_a \in \check{X}$ such that $a = a_K \lambda_a(\omega)$ where $a_K \in T_K := T \cap K$ and ω is the uniformizer of F. We can see that

$$\begin{aligned}\int_T \Delta^{\frac{1}{2}+\epsilon}(a) \cdot O_{f_0}(a) da &= \sum_{\lambda\in\check{X}} \int_{(T\cap K)\cdot\lambda(\omega)} \Delta^{\frac{1}{2}+\epsilon}(a) \cdot O_{f_0}(a) da \\ &= \left(\int_{T\cap K} da\right) \cdot \left(\sum_{\lambda\in\check{X}} \Delta^{\frac{1}{2}+\epsilon}(\lambda(\omega)) \cdot O_{f_0}(\lambda(\omega))\right).\end{aligned}$$

It is easy to see that

$$\int_{T\cap K} da = Vol(T \cap K) = \left(1 - \frac{1}{q}\right)^n < 1.$$

From Theorem 3.2.1 ([DR98]), we know that the sum over $\lambda \in \check{X}$ takes place for λ of the form

$$\lambda = \sum_{i=1}^{n-1} m_i \cdot \check{\alpha}_{i,i+1},$$

where $\check{\alpha}_{i,i+1}$ are positive simple coroots and m_i are nonnegative integers. Moreover, we have the following estimation:

$$O_{f_0}(\lambda(\omega)) = \Delta^{-\frac{1}{2}}(\lambda(\omega)) \times \sum_{\overline{m}} \left(1 - \frac{1}{q}\right)^{\kappa(\overline{m})} < q^{\sum_{i=1}^{n-1} m_i} \times R(a).$$

Here $\Delta(a) = \Delta(\lambda_a(\omega)) = \Delta(\lambda(\omega)) = q^{-2\cdot\sum_{i=1}^{n-1} m_i}$ and $R(a) := R(\lambda_a)$ is the number of possibilities of writing λ_a as in Equation (3.1) (See Theorem 3.2.1). We let $1 \leq i \leq n-1$, the multiplicity of the simple coroot $\check{\alpha}_{i,i+1}$ in the coroot $\check{\alpha}_{s,t}$ $(1 \leq s < t \leq n)$ equals to $i(n-i)$.

Now by direct estimation, we have

$$R(a) \leq \prod_{i=1}^{n-1} (m_i + 1)^{i(n-i)}.$$

We compute $\sum_{i=1}^{n-1} i(n-i) < n \times \sum_{i=1}^{n-1} i = \frac{(n-1)n^2}{2} < \frac{n^3}{2}$, which gives that

$$R(a) \leq \prod_{i=1}^{n-1} (m_i + 1)^{i(n-i)} < \left(\sum_{i=1}^{n-1} m_i + n \right)^{\frac{n^3}{2}}.$$

From above discussion, we see that

$$O_{f_0}(\lambda(\omega)) < q^{\sum_{i=1}^{n-1} m_i} \times \left(\sum_{i=1}^{n-1} m_i + n \right)^{\frac{n^3}{2}}.$$

Note that

$$\Delta(a) = \Delta(\lambda_a(\omega)) = \Delta(\lambda(\omega)) = q^{-2\cdot\sum_{i=1}^{n-1} m_i},$$

therefore we have

$$\begin{aligned} &\Delta^{\frac{1}{2}+\epsilon}(\lambda(\omega)) \cdot O_{f_0}(\lambda(\omega)) \\ <&q^{-2\epsilon\cdot\sum_{i=1}^{n-1} m_i} \times \left(\sum_{i=1}^{n-1} m_i + n \right)^{\frac{n^3}{2}}. \end{aligned}$$

Hence our integral is controlled by

$$\begin{aligned} 1 + \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} \cdots \sum_{m_{n-1} \geq 0} q^{-2\epsilon\cdot\sum_{i=1}^{n-1} m_i} \times \left(\sum_{i=1}^{n-1} m_i + n \right)^{\frac{n^3}{2}} &\leq 1 + \sum_{k=1}^{+\infty} \frac{(k+n)^n \times (k+n)^{\frac{n^3}{2}}}{q^{2\epsilon k}} \\ &\leq 1 + \sum_{k=1}^{+\infty} \frac{(k+n)^{n^3}}{q^{2\epsilon k}} \end{aligned}$$

which is finite when $\epsilon > 0$ (Note that here n is a fixed positive integer). This proves that the function $\Delta^{-\frac{1}{2}+\epsilon}$ is locally integrable for every $\epsilon > 0$.

□

According to the above Theorem 3.2.2 and Theorem 3.1.2, the local integrability of Bessel function (See Theorem 1.2.6) can be reduced to prove the following conjecture.

Conjecture 3.2.3. *Fix $f \in C_c^\infty(G_n)$. Then $|I_{f,\omega_\pi,\psi}(g)\Delta^{\frac{1}{2}-\delta}(g)|$ is bounded on compact sets in G_n for some given $\delta > 0$.*

Remark 3.2.4. *In* GL(3) *case, the Conjecture 3.2.3 was proved in [Ba04] for $\delta = \frac{1}{8} > 0$. Hence the local integrability for* GL(3) *Bessel functions is proved.*

Similarly, we can establish the following conjecture:

Conjecture 3.2.5. *Fix $f \in C_c^\infty(G_n)$. Then $|I_{f,\psi}(g)\Delta^{\frac{1}{2}-\delta}(g)|$ is bounded on compact sets in G_n for some given $\delta > 0$.*

We have the following quick proposition.

Proposition 3.2.6. *Assume that Conjecture 3.2.5 holds for general n, then Conjecture 3.2.3 also holds in general.*

Proof. Since $f \in C_c^\infty(G_n)$, we let Q_1 be the support of f. Since Q_1 is compact, it follows that $|\det(g)|$ have both lower and upper bounds for $g \in Q_1$. Hence the support of the orbital integral $I_{f,\psi}(g)$ is also on a set on which the determinant is bounded.

Now let Q_2 be a compact set in G_n. We will have to show that $|I_{f,\omega_\pi,\psi}(g)\Delta^{\frac{1}{2}-\delta}(g)|$ is bounded on Q_2 in G_n for the fixed $\delta > 0$ in Conjecture 3.2.5. If $g \in Q_2$, $z \in Z_n$ and gz is in the support of the orbital integral $I_{f,\psi}(g)$, we have $\det(gz) = \det(g) \times \det(z)$ is in some fixed compact set in $F^\times$. Hence we know that z is in a fixed compact set P in Z_n which is independent on the choice of $g \in Q_2$.

We let $C_1 := \max_{z\in P}\left(|\omega_\pi(z)^{-1}|\right) > 0$ and $g \in Q_2$. By Conjecture 3.2.5, there exists a constant $C_2 > 0$ such that

$$|I_{f,\psi}(gz)| \leq C_2 \cdot \Delta^{-\frac{1}{2}+\delta}(gz)$$

for all $g \in Q_2$ and $z \in P$. Hence, if $g \in Q_2$, then we have

$$\begin{aligned} |I_{f,\omega_\pi,\psi}(g)| &= \left|\int_{Z_n} I_{f,\psi}(gz)\omega_\pi^{-1}(z)dz\right| \\ &= \left|\int_{P} I_{f,\psi}(gz)\omega_\pi^{-1}(z)dz\right| \\ &\leq C_1C_2 \times \int_P \left|\Delta^{-\frac{1}{2}+\delta}(gz)\right| dz. \end{aligned} \tag{3.3}$$

Note that $I_{f,\omega_\pi,\psi}(g) = 0$ if $g \notin \bar{N}B$. Now we may write $g = n_1^t a n_2$ for $n_1, n_2 \in N$ and $a \in A$. Hence we have $\Delta^{-\frac{1}{2}+\delta}(gz) = \Delta^{-\frac{1}{2}+\delta}(n_1^t a n_2 z) = \Delta^{-\frac{1}{2}+\delta}(n_1^t a z n_2) = \Delta^{-\frac{1}{2}+\delta}(az) = \Delta^{-\frac{1}{2}+\delta}(a) \times \Delta^{-\frac{1}{2}+\delta}(z) = \Delta^{-\frac{1}{2}+\delta}(g) \times \Delta^{-\frac{1}{2}+\delta}(z)$. Since P is compact, there exists $C_3 > 0$ such that

$$\int_P \left|\Delta^{-\frac{1}{2}+\delta}(z)\right| dz < C_3.$$

Therefore we have

$$|I_{f,\omega_\pi,\psi}(g)| < C_1 C_2 C_3 \left|\Delta^{-\frac{1}{2}+\delta}(g)\right|.$$

This is equivalent to

$$\left|I_{f,\omega_\pi,\psi}(g)\Delta^{\frac{1}{2}-\delta}(g)\right| < C_1 C_2 C_3,$$

which is exactly Conjecture 3.2.3.

□

3.3 Reduction Step 3: (Relative) Shalika germs

In order to study the asymptotic behaviour of the orbital integrals, we need to introduce the (relative) Shalika germs of Jacquet-Ye [JY96] [JY99].

We define the generalized orbital integrals of a function $f \in C_c^\infty(G_n)$, that is, the functionals

$$I(g, f) := \int f(n_1^t g n_2)\psi^{-1}(n_1 n_2) dn_1 dn_2$$

for $g \in G_n$. This is an extension of the previous orbital integral $I_{f,\psi}(g)$. The integration domain is explicitly given as follows. By the Bruhat decomposition, we can write $g = wa$ where $w \in W$ and $a \in A$. In order for the integral to make sense we must assume that $\psi(n_1 n_2) = 1$ if $n_1^t w a n_2 = wa$. However, this assumption is not always true for all the Weyl elements. A Weyl element w is said to be relevant if this condition is satisfied. The integral is then taken over the quotient of $N \times N$ by the subgroup of elements (n_1, n_2) of $N \times N$ satisfying $n_1^t w a n_2 = wa$. If $w = I_n \in G_n$, we have $I(g, f) = I(wa, f) = I(a, f) = I_{f,\psi}(a) = I_{f,\psi}(g)$, where the orbital integral $I_{f,\psi}$ is defined in Section 2. We let an element $a \in A$ operate on G_n by $g \to ga$.

Let W_{G_n} be the Weyl group of G_n and R_{G_n} be the subset of relevant elements of W_{G_n}. For each positive integer r we denote by w_{G_r} the $r \times r$ permutation matrix whose

anti-diagonal entries are 1. If $w \in R_{G_n}$ then w has the form

$$w = \begin{pmatrix} w_{G_{n_1}} & 0 & 0 & \cdots & 0 \\ 0 & w_{G_{n_2}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & w_{G_{n_r}} \end{pmatrix}$$

for a suitable r-tuple of positive integers $(n_1, n_2, \cdots, n_r)$ with $\sum_{i=1}^{r} n_i = n$. This is given in [JY96]. Hence, there are 2^{n-1} relevant Weyl elements in $\mathrm{GL}(n)$ case (We may label them as $e = I_n, w_1, w_2, \cdots \cdots, w_{2^n-2}, w_{G_n}$).

A parabolic subgroup P_n of G_n is called standard if it contains AN. Then $P_n = M_n U_n$ where M_n is the unique Levi subgroup of P_n containing A and U_n is the unipotent radical of P_n. A Levi subgroup of this type is said to be standard. It is said to be of type $(n_1, n_2, \cdots, n_r)$ where $\sum_{i=1}^{r} n_i = n$ if it consists of matrices of the following form

$$m = \begin{pmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & m_r \end{pmatrix}$$

with $m_i \in \mathrm{GL}(n_i)$. If M_n is the standard Levi subgroup of type $(n_1, n_2, \cdots, n_r)$, we can write $M_n = M_w$ and $w = w_{M_n}$.

Example 3.3.1. *In the special case $n = 4$, all the $2^{4-1} = 8$ relevant Weyl elements can be represented as follows:*

$$e = I_4, \quad w_{G_4},$$

$$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & w_{G_3} \end{pmatrix}, \quad w_2 = \begin{pmatrix} w_{G_2} & 0 \\ 0 & w_{G_2} \end{pmatrix}, \quad w_3 = \begin{pmatrix} w_{G_3} & 0 \\ 0 & 1 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 1 & & \\ & w_{G_2} & \\ & & 1 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & w_{G_2} \end{pmatrix}, \quad w_6 = \begin{pmatrix} w_{G_2} & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Now we continue for the general case. Let M_n be a standard Levi subgroup of G_n. We set $W_{M_n} := W_{G_n} \cap M_n$ and $R_{M_n} = R_{G_n} \cap M_n$. We let A_{M_n} be the center of M_n.

If w is in R_{G_n} then we set $A_w = A_{M_w}$. Any relevant orbit of $N \times N$ contains a unique representative of the form wa with $w \in R_{M_n}$ and $a \in A_w$.

If w_1 and w_2 are in R_{G_n}, we write $w_1 \to w_2$ if $M_{w_1} \subseteq M_{w_2}$. This is equivalent to $A_{w_1} \supseteq A_{w_2}$ and also to $w_1 \in R_{M_{w_2}}$.

We recall that $\Delta_r(g)$ is the determinant of principal $r \times r$ minor of g for $1 \leq r \leq n$. Let $\tilde{\Delta}(g)$ be the set of the functions Δ_r on G_n. These functions are invariant under the action of $N \times N$.

Suppose $w_1 \to w_2$, that is, $w_1 \in R_{M_{w_2}}$. This implies that if $\tilde{\Delta} \in \tilde{\Delta}(G_n)$ and $\tilde{\Delta}(w_2) \neq 0$ then $\tilde{\Delta}(w_1 a) \neq 0$ as well for any $a \in A_{w_1}$. We can therefore define the subset $A_{w_1}^{w_2}$ of $a \in A_{w_1}$ such that

$$\tilde{\Delta}(w_1 a) = \tilde{\Delta}(w_2)$$

for each $\tilde{\Delta} \in \tilde{\Delta}(G_n)$ such that $\tilde{\Delta}(w_2) \neq 0$.

Using above notations, we are able to state the follow lemma, which was proved in [JY99].

Lemma 3.3.2. *Suppose $w_1 \to w_2$ and $a \in A_{w_1}$. Then there are only finitely many pairs (b, c), $b \in A_{w_1}^{w_2}$, $c \in A_{w_2}$ such that $a = bc$.*

From the above lemma, we can state the following definition. If f_1 and f_2 are any complex valued and locally constant functions on $A_{w_1}^{w_2}$ and A_{w_2} respectively, we can define a new function $f_1 \star f_2$ on A_{w_1} by the formula

$$f_1 \star f_2(a) := \sum f_1(b) f_2(c),$$

where the sum is over all pairs (b, c), $b \in A_{w_1}^{w_2}$, $c \in A_{w_2}$ such that $a = bc$. Moreover, $f_1 \star f_2(a) = 0$ if there is no such pair. We can now state the main result in [JY96] and [JY99].

Theorem 3.3.3. *For each pair $(w, w') \in R_{G_n} \times R_{G_n}$ with $w \to w'$, there exist a family of locally constant functions $K_w^{w'}$ defined on $A_w^{w'}$, satisfying the following properties.*

- *If $w' = w$ then $K_w^{w'} = \delta_e$ the Dirac delta function on the finite set A_w^w.*
- *For each function $f \in C_c^\infty(G_n)$, there are functions $\omega_w \in C_c^\infty(A_w)$ which depend on the choice of function f, $w \in R_{G_n}$, such that, for any $w \in R_{G_n}$, the*

corresponding generalized orbital integral satisfies

$$I(w\cdot, f) = \sum_{w' \in R_{G_n},\, w \to w'} K_w^{w'} \star \omega_{w'}.$$

In this expression, locally constant functions $K_w^{w'}$ are independent on the choice of the test function f.

We recall that $I(w\cdot, f)$ is the generalized orbital integral defined from A_w to $\mathbb{C}$ as follows: For any $a \in A_w$, we define

$$I(wa, f) := \int f(n_1^t wan_2)\psi^{-1}(n_1 n_2)dn_1 dn_2,$$

where the integral is taken over the quotient of $N(F) \times N(F)$ by the subgroup of pairs (n_1, n_2) such that $n_1^t wan_2 = wa$.

In above Theorem 3.3.3, $K_w^{w'}$ are called the (relative) Shalika germs according to Jacquet-Ye's paper [JY96] and [JY99].

For the special case $G_4 = \mathrm{GL}_4(F)$, let $K_e^{w_i}$ $(i = 1, 2, 3, 4, 5, 6)$ and $K_e^{w_{G_4}}$ be the relative Shalika germs. We can state a special case of above theorem as follows:

Example 3.3.4. *In the* GL(4) *case, the generalized orbital integral $I(w\cdot, f)$ has the following asymptotic behaviour if we fix $w = e = I_4$:*

$$I(e\cdot, f) = I_{f,\psi} = \omega_e + \sum_{i=1}^{6} K_e^{w_i} \star \omega_{w_i} + K_e^{w_{G_4}} \star \omega_{w_{G_4}}, \tag{3.4}$$

where ω_e, $\omega_{w_{G_4}}$ and ω_{w_i} $(i = 1, 2, 3, 4, 5, 6)$ are all locally constant and compact supported functions which are dependent on the choice of f.

Remark 3.3.5. *The above theorem actually gives the asymptotic behaviour of orbital integrals at the boundary. Assume that C_1, C_2, C_3, C_4 are four fixed positive real number and ϵ is a small enough positive real number, we have the following seven different cases.*

(a) If $|\Delta_1(a)| = C_1$, $|\Delta_2(a)| < \epsilon$, $|\Delta_3(a)| < \epsilon$ and $|\Delta_4(a)| = C_4$, then we can write

$$I_{f,\psi}(a) = K_e^{w_1} \star \omega_{w_1} = \sum_{a=bc} K_e^{w_1}(b)\omega_{w_1}(c).$$

(b) If $|\Delta_1(a)| < \epsilon$, $|\Delta_2(a)| = C_2$, $|\Delta_3(a)| < \epsilon$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_2} \star \omega_{w_2} = \sum_{a=bc} K_e^{w_2}(b)\omega_{w_2}(c).$$

(c) If $|\Delta_1(a)| < \epsilon$, $|\Delta_2(a)| < \epsilon$, $|\Delta_3(a)| = C_3$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_3} \star \omega_{w_3} = \sum_{a=bc} K_e^{w_3}(b)\omega_{w_3}(c).$$

(d) If $|\Delta_1(a)| = C_1$, $|\Delta_2(a)| < \epsilon$, $|\Delta_3(a)| = C_3$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_4} \star \omega_{w_4} = \sum_{a=bc} K_e^{w_4}(b)\omega_{w_4}(c).$$

(e) If $|\Delta_1(a)| = C_1$, $|\Delta_2(a)| = C_2$, $|\Delta_3(a)| < \epsilon$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_5} \star \omega_{w_5} = \sum_{a=bc} K_e^{w_5}(b)\omega_{w_5}(c).$$

(f) If $|\Delta_1(a)| < \epsilon$, $|\Delta_2(a)| = C_2$, $|\Delta_3(a)| = C_3$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_6} \star \omega_{w_6} = \sum_{a=bc} K_e^{w_6}(b)\omega_{w_6}(c).$$

(g) If $|\Delta_1(a)| < \epsilon$, $|\Delta_2(a)| < \epsilon$, $|\Delta_3(a)| < \epsilon$ *and* $|\Delta_4(a)| = C_4$, *then we can write*

$$I_{f,\psi}(a) = K_e^{w_{G_4}} \star \omega_{w_{G_4}} = \sum_{a=bc} K_e^{w_{G_4}}(b)\omega_{w_{G_4}}(c).$$

Remark 3.3.6. *An analogy of above theorem in* GL(3) *case can be found in [Ba04, Theorem 4.2].*

Now we come back to the general case. In order to prove Conjecture 3.2.3, from above Theorem 3.3.3 of Jacquet-Ye, it is sufficient to study the (relative) Shalika germs $K_e^{w_i}$ $(i = 1, 2, \cdots, 2^n - 3, 2^n - 2)$ and $K_e^{w_{G_n}}$.

In other words, Conjecture 3.2.3 can be reduced to the proof of following Conjecture on the boundness of relative Shalika germs:

Conjecture 3.3.7. *The absolute value* $\left|K_e^{w_i}\Delta^{\frac{1}{2}-\delta}\right|$ *(*$i = 1, 2, \cdots, 2^n - 3, 2^n - 2$*) and* $\left|K_e^{w_{G_n}}\Delta^{\frac{1}{2}-\delta}\right|$ *are bounded on* $A_e^{w_i}$ *(respectively,* $A_e^{w_{G_n}}$*) for some given* $\delta > 0$.

Conjecture 3.3.8. *The absolute value* $\left|K_e^{w_{G_n}}\Delta^{\frac{1}{2}-\delta_n}\right|$ *is bounded on* $A_e^{w_{G_n}}$ *for some given* $\delta_n > 0$.

Although Conjecture 3.3.8 is a part of Conjecture 3.3.7, we will see that Conjecture 3.3.8 is exactly equivalent to Conjecture 3.3.7.

Theorem 3.3.9. *Suppose that Conjecture 3.3.8 holds for general n, then Conjecture 3.3.7 is true in general.*

Proof. It is sufficient to prove the following result:

If $\left|K_e^{w_{G_n}}\Delta^{\frac{1}{2}-\delta_n}\right|$ is bounded on $A_e^{w_{G_n}}$ for some given $\delta_n > 0$, then $\left|K_e^{w_i}\Delta^{\frac{1}{2}-\delta}\right|$ $(i = 1, 2, \cdots, 2^n-3, 2^n-2)$ is bounded on $A_e^{w_i}$ for some given $\delta > 0$.

From the discussion of Section 2 in [JY99], the system of (relative) Shalika germs is inductive. More precisely, if $w' \neq w_{G_n}$, then for every a in $A_e^{w'}$, we can write $a = \mathrm{diag}(a_1, a_2, \cdots, a_s)$ with $a_i \in A_e^{w_i'}$. Therefore, we have $A_e^{w'} = \prod A_{e_i}^{w_i'}$ and the equation of the (relative) Shalika germs is as follows: $K_e^{w'}(a) = \prod K_{e_i}^{w_i'}(a_i)$. Now we pick $\delta := \frac{1}{2}\cdot\min\{\delta_2, \delta_3, \cdots, \delta_n\}$. Hence by the equation $K_e^{w'}(a) = \prod K_{e_i}^{w_i'}(a_i)$ and $\Delta^{\frac{1}{2}-\delta}(a) = \prod \Delta^{\frac{1}{2}-\delta}(a_i)$, we prove the reduction. □

Remark 3.3.10. *We give a remark for the above theorem in* GL(4) *case. Since* $w_i \neq w_{G_4}$ *(*$i = 1, 2, 3, 4, 5, 6$*), we can reduce the calculation of the (relative) Shalika germs* $K_e^{w_i}$ *to the* GL(3) *or* GL(2) *case. Explicitly, we can reduce the (relative) Shalika germs* $K_e^{w_1}$ *and* $K_e^{w_3}$ *to* GL(3) *(relative) Shalika germs. The (relative) Shalika germs* $K_e^{w_i}$ *(*$i = 4, 5, 6$*) can be reduced to* GL(2) *(relative) Shalika germs. Finally, the (relative) Shalika germ* $K_e^{w_2}$ *can be reduced to the product of two* GL(2) *(relative) Shalika germs. In conclusion, we can reduce these* GL(4) *(relative) Shalika germs to lower rank cases. Therefore by the results in [Ba04] [JY96] [JY99], we can prove the first part of above Conjecture 3.3.7 by setting* $\delta = \frac{1}{8}$.

The proof of above Conjecture 3.3.8 is the key ingredient of the proof of local integrability of Bessel functions. We will prove above Conjecture 3.3.8 in the following Chapter 5 and 6 by using G. Stevens' idea to estimate the corresponding Kloosterman integral (sum). In order to estimate the Kloosterman integral, we need to give a fine stratification of Kloosterman sets and use the Weil bound to prove a nontrivial bound for the Kloosterman sum attached to the longest Weyl element w_{G_n}.

3.4 Reduction Step 4: An explicit formula for $K_e^{w_{G_n}}$

By [JY99, Section 2], we give the explicit formula for $K_e^{w_{G_n}}$. The relation between $K_e^{w_{G_n}}$ and local Kloosterman sums for G_n is given in the next section.

For the p-adic local field F, let $\mathcal{O}_F$ be the ring of integers and $\mathfrak{p}\mathcal{O}_F$ be the (unique) maximal ideal. Let q be the cardinality of the residue field $\mathcal{O}_F/\mathfrak{p}\mathcal{O}_F$. Let $K_{m'}$ be the principal congruence subgroup of maximal compact open subgroup of K, i.e. we have

$$K_{m'} = I_n + M_{n\times n}(\mathfrak{p}^{m'}\mathcal{O}_F),$$

where $M_{n\times n}$ is the n-by-n matrix. We fix the positive integer m'. Here m' should be chosen large enough such that $a = 1$ when $a^n = 1$ and $a \in 1 + \mathfrak{p}^{m'}\mathcal{O}_F$. Moreover, without loss of generality, we can further assume that $m' >> n$.

For $\alpha = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ where $\prod_{i=1}^n a_i = (-1)^{\frac{(n+1)(n+2)}{2}+1}$ and $|a_1| \leq q^{-k}, |a_1a_2| \leq q^{-k}, \cdots, \cdots, |a_1a_2\cdots a_i| \leq q^{-k}, \cdots, \cdots, |a_1a_2\cdots a_{n-1}| \leq q^{-k}$ for some large enough positive integer k ($k >> m'$), we have

$$K_e^{w_{G_n}}(\alpha) = I(\alpha, \Phi) = \int_{N\times N} \Phi(n_1^t \alpha n_2)\psi^{-1}(n_1n_2)dn_1dn_2.$$

The measure here is the standard Haar measure such that the volume of the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ and the subgroup $N \cap \mathrm{GL}_n(\mathcal{O}_F) = N(\mathcal{O}_F)$ both equal to one. Moreover, we have $\Phi(x) = q^{\frac{n(n-1)}{2}m'} \cdot 1_{w_{G_n}K_{m'}}$, where $1_{w_{G_n}K_{m'}}$ is the characteristic function of the compact subset $w_{G_n}K_{m'}$. It is easy to see that $\Phi \in C_c^\infty(G_n)$.

Chapter 4

Summary of Stevens' Results

We continue our reduction steps in Chapter 3 for the proof of Theorem 1.2.6.

4.1 Definition of $\mathrm{GL}(n)$ Kloosterman sums

We will mainly use the method in [Ste87] to estimate the Kloosterman integrals for $\mathrm{GL}(n)$. In this section, we will briefly review the paper [Ste87].

The classical (GL(2)) Kloosterman sum is given by

$$S(m,n;c)=\sum_{d,\bar{d} \mod c} e\left(\frac{md+n\bar{d}}{c}\right)$$

where $d\bar{d}\equiv 1 \pmod{c}$ and $e(x)=e^{2\pi ix}$, which arises when one computes the Fourier expansion of the GL(2) Poincaré series.

Now we define the $\mathrm{GL}(n)$ Kloosterman sums. We first recall some notations which are given in above Section 3. Let F be a non-archimedean local field. From this section to the end of our paper, we will further assume that $F=\mathbb{Q}_p$ where p is a prime number. Let $\mathbb{Z}_p$ be the ring of integers and $\mathfrak{p}\mathbb{Z}_p$ be the (unique) maximal ideal. Let p be the cardinality of the residue field $\mathbb{Z}_p/\mathfrak{p}\mathbb{Z}_p$. Let $K_{m'}$ be the principal congruence subgroup of maximal compact open subgroup of $K=\mathrm{GL}_n(\mathbb{Z}_p)$.

We let ψ_p be a nontrivial additive character from $N(\mathbb{Q}_p)$ to $\mathbb{C}^\times$ which is trivial on $N(p^m\mathbb{Z}_p)$. Here m is a fixed large enough positive integer with $m>2m'>>n$. The

character ψ_p has the following form:

$$\psi_p\left(\begin{pmatrix} 1 & x_1 & \cdots & * & * \\ 0 & 1 & x_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right) = \xi\left(\sum_{i=1}^{n-1} \nu_i x_i\right),$$

where ξ is the standard nontrivial additive character of $\mathbb{Q}_p$ as in [Ste87, Section 1] and $\nu_1, \nu_2, \cdots, \nu_{n-1} \in p^{-m}\mathbb{Z}_p - \{0\}$. We also assume that $\nu_i = \nu_{n-i}$ for all $1 \leq i \leq n-1$. Since the positive integer m is large enough, without loss of generality, we can further assume that $p^{-m} \leq |\nu_i| \leq p^m$ for all $1 \leq i \leq n-1$. Similarly, we can define an additive character from from $N(\mathbb{Q}_p)$ to $\mathbb{C}^\times$ which is trivial on $N(p^m\mathbb{Z}_p)$ if we are given the parameters $\nu_1', \nu_2', \nu_3', \cdots, \nu_{n-1}' \in p^{-m}\mathbb{Z}_p - \{0\}$ with $p^{-m} \leq |\nu_i'| \leq p^m$ for all $1 \leq i' \leq n-1$. The definition is as follows:

$$\psi_p'\left(\begin{pmatrix} 1 & x_1 & \cdots & * & * \\ 0 & 1 & x_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right) = \xi\left(\sum_{i=1}^{n-1} \nu_i' x_i\right).$$

We set $c = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in T$. Now following [Ste87, Section 2], we define

$$C(w_{G_n} c) := N(\mathbb{Q}_p) w_{G_n} c N(\mathbb{Q}_p) \cap K_m,$$

and

$$X(w_{G_n} c) := N(p^m\mathbb{Z}_p) \backslash C(w_{G_n} c) / N(p^m\mathbb{Z}_p).$$

By the Bruhat decomposition we have natural maps

$$u : X(w_{G_n} c) \to N(p^m\mathbb{Z}_p) \backslash N(\mathbb{Q}_p), \quad u' : X(w_{G_n} c) \to N(\mathbb{Q}_p) / N(p^m\mathbb{Z}_p).$$

defined by the relation $x = u(x) w_{G_n} c u'(x)$ for $x \in X(w_{G_n} c)$.

Now the *local Kloosterman sum for* $\mathrm{GL}_n(\mathbb{Q}_p)$ is defined as follows:

$$Kl_p(\psi_p, \psi_p'; c, w_{G_n}) := \sum_{x \in X(w_{G_n} c)} \psi_p(u(x)) \cdot \psi_p'(u'(x)).$$

Remark 4.1.1. *Using the notation in Chapter 3. If we further assume that $\psi_p = \psi_p'$, in this special case we have the following equation:*

$$Kl_p(\psi_p^{-1}, \psi_p'^{-1}; c, w_{G_n}) = p^{n(n-1)m} \cdot J_{1_{K_m}, \psi_p}(w_{G_n} c) = p^{n(n-1)m} \cdot I_{1_{w_{G_n} K_m}, \psi_p}(c).$$

By Section 3.4, we have the explicit formula as follows:

$$K_{e,\psi_p}^{w_{G_n}}(c) = p^{\frac{n(n-1)}{2} m} \cdot I_{1_{w_{G_n} K_m}, \psi_p}(c).$$

Hence we see that

$$K_{e,\psi_p}^{w_{G_n}}(c) = p^{-\frac{n(n-1)}{2} m} \cdot Kl_p(\psi_p^{-1}, \psi_p'^{-1}; c, w_{G_n}).$$

So the local Kloosterman sum is exact a special case of the orbital (Kloosterman) integral when the test function is the multiplication of the characteristic function for the congruence subgroup K_m and a positive constant $p^{n(n-1)m}$. Now in order to prove the non-trivial bound for the relative Shalika germs and orbital integrals, it suffices to give a non-trivial bound for the local Kloosterman sum.

4.2 Decomposition of $\mathrm{GL}(n)$ Kloosterman sums

We need an orbit decomposition of $X(\tau)$ where $\tau = w_{G_n} c$. Let $t' \in T(1 + p^m \mathbb{Z}_p)$ and set $s := \tau^{-1} t' \tau \in T(1 + p^m \mathbb{Z}_p)$. If $\gamma \in C(\tau)$, we can write $\gamma = u \cdot \tau \cdot u'$ with $u \in N(\mathbb{Q}_p)$ and $u' \in N(\mathbb{Q}_p)$. We also have

$$t' \gamma s^{-1} = (t' u t'^{-1}) \cdot \tau \cdot (s u' s^{-1}) \in N(\mathbb{Q}_p) \tau N(\mathbb{Q}_p) \cap K_m = C(\tau).$$

Since conjugation by t and s preserves $N(p^m \mathbb{Z}_p)$, we know that the map

$$T(1 + p^m \mathbb{Z}_p) \times C(\tau) \to C(\tau), \quad (t, \gamma) \to t \gamma s^{-1}$$

descends to an action of $T(1 + p^m \mathbb{Z}_p)$ on $X(\tau)$ as follows

$$T(1 + p^m \mathbb{Z}_p) \times X(\tau) \to X(\tau), \quad t,\ x \to t * x.$$

For characters ψ_p, the decomposition of $X(\tau)$ into $T(1 + p^m \mathbb{Z}_p)$-orbits leads to the following decomposition of the Kloosterman sums:

$$Kl_p(\psi_p, \psi_p'; c, w_{G_n}) := \sum_{x \in T(1+p^m \mathbb{Z}_p) \backslash X(\tau)} \sum_{T(1+p^m \mathbb{Z}_p) * x} \psi_p(u(x)) \cdot \psi_p'(u'(x)).$$

Here $T(1+p^m\mathbb{Z}_p)\backslash X(\tau)$ is a set of representations $x \in X(\tau)$ for the $T(1+p^m\mathbb{Z}_p)$-orbits and $T(1+p^m\mathbb{Z}_p) * x$ is the orbit through x.

Now we want to write down the above formula explicitly. We need the following notations.

The roots of the standard torus T in $\mathrm{GL}(n)$ are the characters $\underline{\lambda}_{ij} : T \to \mathrm{GL}(1)$ given by

$$\underline{\lambda}_{ij}\left(\begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & \cdots & & \\ & & & t_{n-1} & \\ & & & & t_n \end{pmatrix}\right) = t_i t_j^{-1}, \quad 1 \le i,j \le n,\ i \ne j. \tag{4.1}$$

For $w \in W_{G_n}$, we recall that $w(j)$, $j \in \{1,2,3,\cdots,n\}$ is given by the formula

$$w \cdot e_j = e_{w(j)},$$

where $e_1, e_2, e_3, \cdots, e_n$ is the standard basis of column vectors. Let $\Delta = \{\underline{\lambda}_{i,i+1} : 1 \le i \le n-1\}$ be the root basis associated to the standard unipotent subgroup U. Let

$$\Delta_w := \{\underline{\lambda}_{i,i+1} : w(i+1) < w(i)\}. \tag{4.2}$$

We write

$$u(x) = \begin{pmatrix} 1 & u_1 & * & \cdots & * \\ & 1 & u_2 & \cdots & * \\ & & \cdots & \cdots & \cdots \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{pmatrix}, \quad u'(x) = \begin{pmatrix} 1 & u'_1 & * & \cdots & * \\ & 1 & u'_2 & \cdots & * \\ & & \cdots & \cdots & \cdots \\ & & & 1 & u'_{n-1} \\ & & & & 1 \end{pmatrix}.$$

For $1 \le i \le n-1$ we define $\kappa_i : X(w_{G_n}c) \to \mathbb{Q}_p/p^m\mathbb{Z}_p$, and if also $w(i+1) < w(i)$ then we define $\kappa'_i : X(w_{G_n}c) \to \mathbb{Q}_p/p^m\mathbb{Z}_p$ by

$$\kappa_i(x) = u_i, \quad \kappa'_i(x) = u'_i. \tag{4.3}$$

For $t \in T(1+p^m\mathbb{Z}_p)$ we then have

$$\kappa_i(t*x) = \underline{\lambda}_{i,i+1}\kappa_i(x), \quad \kappa'_i(t*x) = \underline{\lambda}_{w(i),w(i+1)}\kappa'_i(x). \tag{4.4}$$

Since

$$u'(t*x) = s \cdot u'(x) \cdot s^{-1}$$

for $t \in T(1+p^m\mathbb{Z}_p)$ and $s = \tau^{-1}t\tau \in T(1+p^m\mathbb{Z}_p)$, we see that the orbits in $X(w_{G_n}c) = X(\tau)$ correspond to $T(1+p^m\mathbb{Z}_p)$-conjugacy classes in $N(\mathbb{Q}_p)/N(p^m\mathbb{Z}_p)$.

The following definition is an analogy of Definition 4.9 in [Ste87].

Definition 4.2.1. *[Ste87, Definition 4.9]*

(a) For $\ell > 0$ and $w \in W$, we let

$$\begin{aligned} A_w(\ell) &:= (\mathbb{Z}/p^{\ell+m}\mathbb{Z})^{\Delta} \times (\mathbb{Z}/p^{\ell+m}\mathbb{Z})^{\Delta_w} \\ &= \prod_{i=1}^{n-1}(\mathbb{Z}/p^{\ell+m}\mathbb{Z}) \times \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} (\mathbb{Z}/p^{\ell+m}\mathbb{Z}). \end{aligned}$$

A typical element of $A_{w_{G_n}}(\ell)$ will be denoted by

$$\underline{\lambda} \times \underline{\lambda}' = (\lambda_i)_{i=1,2,3,\cdots,n-1} \times (\lambda_i')_{\substack{i=1,2,3,\cdots,n-1 \\ w(i+1)<w(i)}}$$

.

If $w = w_{G_n}$ which is the longest Weyl element, we will simply have $A_{w_{G_n}}(\ell) = (\mathbb{Z}/p^{\ell+m}\mathbb{Z})^{2(n-1)}$.

(b) Let

$$\begin{aligned} V_w(\ell) :=& \Big\{ \underline{\lambda} \times \underline{\lambda}' \in A_w(\ell) : \lambda_i \in (\mathbb{Z}/p^{\ell+m}\mathbb{Z})^{\times}, \\ & \lambda_i \equiv 1 \mod p^m \text{ and } \lambda_i' \cdot \prod_{w(i+1)\le j<w(i)} \lambda_j = 1 \Big\}. \end{aligned}$$

If $w = w_{G_n}$ which is the longest Weyl element, we will simply have

$$\begin{aligned} V_{w_{G_n}}(\ell) :=& \Big\{ \underline{\lambda} \times \underline{\lambda}' \in A_{w_{G_n}}(\ell) : \lambda_i \in (\mathbb{Z}/p^{\ell+m}\mathbb{Z})^{\times}, \\ & \lambda_i \equiv 1 \mod p^m \text{ and } \lambda_i' \times \lambda_{n-i} = 1 \Big\}. \end{aligned}$$

(c) For a character $\theta : A_w(\ell) \to \mathbb{C}^{\times}$, we define

$$S_w(\theta;\ell) := \sum_{v \in V_w(\ell)} \theta(v).$$

We fix

$$\tilde{c} = \begin{pmatrix} p^{a_1}u_1 & & & & & \\ & p^{a_2-a_1}u_2 & & & & \\ & & \cdots & & & \\ & & & p^{a_{n-1}-a_{n-2}}u_{n-1} & & \\ & & & & p^{-a_{n-1}}u_n \end{pmatrix}, \tag{4.5}$$

where $u_i \in \mathbb{Z}_p^\times$ for $i = 1, 2, 3, \cdots, n$ and $u_1u_2u_3\cdots u_n = (-1)^{\frac{(n+1)(n+2)}{2}+1}$. Moreover, a_i $(i = 1, 2, \cdots, n-1)$ are positive integers which are larger than m.

The following lemma is an analogy of Theorem 4.10 in [Ste87].

Lemma 4.2.2. *[Ste87, Theorem 4.10]*

Let ℓ be large enough so that the matrix entries of $u(x)$, $u'(x)$ lie in $p^{-\ell}\mathbb{Z}_p/p^m\mathbb{Z}_p$ for every $x \in X(w_{G_n}c)$ (For example, we can simply pick $\ell := \max\{a_1, a_2, \cdots, a_{n-1}\}$.). Let $\kappa_i(x), \kappa_i'(x)$ be as in (4.3) and define the character $\theta_x : A_{w_{G_n}}(\ell) \to \mathbb{C}^\times$ by

$$\theta_x(\underline{\lambda} \times \underline{\lambda}') = \prod_{i=1}^{n-1} \xi(\lambda_i\nu_i\kappa_i(x)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\lambda_i'\nu_i'\kappa_i'(x)).$$

Let $N(x)$ denote the number of elements in the orbit through an element $x \in X(w_{G_n}\tilde{c})$. Then

$$Kl_p(\psi_p, \psi_p'; \tilde{c}, w_{G_n}) = |V_{w_{G_n}}(\ell)|^{-1} \cdot \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(w_{G_n}c)} N(x) S_{w_{G_n}}(\theta_x; \ell).$$

Here $|V_{w_{G_n}}(\ell)|$ is the cardinality of the set $V_{w_{G_n}}(\ell)$.

Proof. The proof is the same as Theorem 4.10 in [Ste87]. We can rewrite the Kloosterman sum as follows:

$$\begin{aligned} Kl_p(\psi_p, \psi_p'; c, w) &= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} \sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \psi_p(u(y)) \cdot \psi_p'(u'(y)) \\ &= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} \sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \prod_{i=1}^{n-1} \xi(\nu_i\kappa_i(y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\nu_i'\kappa_i'(y)). \end{aligned}$$

We note that for every pair $(\underline{\lambda}, \underline{\lambda}') \in V_w(\ell)$, we can find $t \in T(1 + p^m\mathbb{Z}_p)$ such that $\kappa_i(t * x) = \lambda_i\kappa_i(x)$ and $\kappa_j'(t * x) = \lambda_i'\kappa_j'(x)$ for $x \in X(\tau)$, where $1 \le i, j \le n$ and

$w(j+1) < w(j)$. Hence we have the following equation:

$$\begin{aligned}
&\sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \prod_{i=1}^{n-1} \xi(\nu_i\kappa_i(y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\nu_i'\kappa_i'(y)) \\
&= \sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \prod_{i=1}^{n-1} \xi(\nu_i\kappa_i(t*y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\nu_i'\kappa_i'(t*y)) \\
&= \sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \prod_{i=1}^{n-1} \xi(\lambda_i\nu_i\kappa_i(y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\lambda_i'\nu_i'\kappa_i'(y)).
\end{aligned}$$

Now we take the summation over the finite set $V_w(\ell)$. We have

$$\begin{aligned}
&|V_w(\ell)| \times Kl_p(\psi_p, \psi_p'; c, w) \\
&= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} \sum_{y \in T(1+p^m\mathbb{Z}_p)*x} \sum_{\lambda \times \lambda' \in V_w(\ell)} \prod_{i=1}^{n-1} \xi(\lambda_i\nu_i\kappa_i(y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\lambda_i'\nu_i'\kappa_i'(y)) \\
&= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} N(x) \cdot \sum_{\lambda \times \lambda' \in V_w(\ell)} \prod_{i=1}^{n-1} \xi(\lambda_i\nu_i\kappa_i(y)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\lambda_i'\nu_i'\kappa_i'(y)) \\
&= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} N(x) \cdot \sum_{\lambda \times \lambda' \in V_w(\ell)} \prod_{i=1}^{n-1} \xi(\lambda_i\nu_i\kappa_i(x)) \cdot \prod_{\substack{i=1 \\ w(i+1)<w(i)}}^{n-1} \xi(\lambda_i'\nu_i'\kappa_i'(x)) \\
&= \sum_{x \in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)} N(x) \cdot S_w(\theta_x; \ell).
\end{aligned}$$

We divide both sides by the cardinality $|V_w(\ell)|$ to prove the statement.

In summary, we have the following equation:

$$Kl_p(\psi_p,\psi_p';c,w)=\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)}\sum_{y\in T(1+p^m\mathbb{Z}_p)*x}\psi_p(u(y))\cdot\psi_p'(u'(y))$$

$$=\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)}\sum_{y\in T(1+p^m\mathbb{Z}_p)*x}\prod_{i=1}^{n-1}\xi(\nu_i\kappa_i(y))\prod_{\substack{i=1\\ w(i+1)<w(i)}}^{n-1}\xi(\nu_i'\kappa_i'(y))$$

$$=|V_w(\ell)|^{-1}\cdot\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)}\sum_{y\in T(1+p^m\mathbb{Z}_p)*x}\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{n-1}\xi(\lambda_i\nu_i\kappa_i(y))\prod_{\substack{i=1\\ w(i+1)<w(i)}}^{n-1}\xi(\lambda_i'\nu_i'\kappa_i'(y))$$

$$=|V_w(\ell)|^{-1}\cdot\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)}\sum_{y\in T(1+p^m\mathbb{Z}_p)*x}\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{n-1}\xi(\lambda_i\nu_i\kappa_i(x))\prod_{\substack{i=1\\ w(i+1)<w(i)}}^{n-1}\xi(\lambda_i'\nu_i'\kappa_i'(x))$$

$$=|V_w(\ell)|^{-1}\cdot\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(\tau)}N(x)\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{n-1}\xi(\lambda_i\nu_i\kappa_i(x))\prod_{\substack{i=1\\ w(i+1)<w(i)}}^{n-1}\xi(\lambda_i'\nu_i'\kappa_i'(x))$$

$$=|V_w(\ell)|^{-1}\cdot\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(wc)}N(x)\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{n-1}\xi(\lambda_i\nu_i\kappa_i(x))\prod_{\substack{i=1\\ w(i+1)<w(i)}}^{n-1}\xi(\lambda_i'\nu_i'\kappa_i'(x))$$

$$=|V_w(\ell)|^{-1}\times\sum_{x\in T(1+p^m\mathbb{Z}_p)\backslash X(wc)}N(x)S_w(\theta_x;\ell).$$

□

Remark 4.2.3. *In the above Lemma 4.2.2, we have*

$$S_{w_{G_n}}(\theta_x;\ell)=\prod_{i=1}^{n-1}S_2(p^\ell\nu_i\kappa_i(x),p^\ell\nu_{n-i}'\kappa_{n-i}'(x);p^\ell),$$

where

$$S_2(\nu,\nu';p^\ell):=\sum_{\substack{\lambda,\lambda'\in(\mathbb{Z}/p^{\ell+m}\mathbb{Z}),\\ \lambda\equiv 1 \mod p^m,\ \lambda\cdot\lambda'=1}}\xi\left(\frac{\nu\lambda+\nu'\lambda'}{p^\ell}\right)$$

is the (restricted) GL(2) *Kloosterman sum. Here we can write* $\nu=p^{-m}\tilde{\nu}$ *and* $\nu'=p^{-m}\tilde{\nu}'$*, where* $\tilde{\nu},\tilde{\nu}'\in\mathbb{Z}_p-\{0\}$*.*

Remark 4.2.4. *One may find the* GL(2) *Kloosterman sums we get above is a little bit different from the classical* GL(2) *Kloosterman sums which we defined at the beginning of the section. The* GL(2) *Kloosterman sums here have more restrictions. Note that by*

the orthogonality of multiplicative characters, such kind of restricted GL(2) *Kloosterman sums are closely related to so called twisted* GL(2) *Kloosterman sums. We have the following identity:*

$$\begin{aligned} S_2(\nu,\nu';p^\ell) :=& \sum_{\substack{\lambda,\lambda'\in(\mathbb{Z}/p^{\ell+m}\mathbb{Z}),\\ \lambda\equiv 1 \mod p^m,\ \lambda\cdot\lambda'=1}} \xi\left(\frac{\nu\lambda+\nu'\lambda'}{p^\ell}\right) \\ =& \sum_{\lambda,\lambda'\in(\mathbb{Z}/p^{\ell+m}\mathbb{Z}),\lambda\cdot\lambda'=1} \frac{1}{\phi(p^m)} \times \sum_{\chi \mod p^m} \xi\left(\frac{\nu\lambda+\nu'\lambda'}{p^\ell}\right)\chi(\lambda) \\ =& \frac{1}{\phi(p^m)} \times \sum_{\chi \mod p^m} \sum_{\lambda,\lambda'\in(\mathbb{Z}/p^{\ell+m}\mathbb{Z}),\lambda\cdot\lambda'=1} \xi\left(\frac{\nu\lambda+\nu'\lambda'}{p^\ell}\right)\chi(\lambda) \\ =& \frac{1}{p^{m-1}(p-1)} \times \sum_{\chi \mod p^m} \sum_{\lambda,\lambda'\in(\mathbb{Z}/p^{\ell+m}\mathbb{Z}),\lambda\cdot\lambda'=1} \xi\left(\frac{\nu\lambda+\nu'\lambda'}{p^\ell}\right)\chi(\lambda). \end{aligned} \tag{4.6}$$

Therefore, by [KL13, Section 9] (Weil bounds for twisted GL(2) *Kloosterman sums), we can still have an analogy of Weil bounds for this kind of restricted Kloosterman sum, which is enough for us to get the non-trivial bound of higher rank Kloosterman sums. More precisely, we have the refined Weil bound as follows:*

$$|S_2(\nu,\nu';p^\ell)| \leq (\ell+m+1)\cdot C_m \cdot (\gcd(|p^m\nu|_p^{-1},|p^m\nu'|_p^{-1},p^{\ell+m}))^{1/2}p^{(\ell+m)/2},$$

where $p^m\nu, p^m\nu' \in \mathbb{Z}_p - \{0\}$. *Here* C_m *is a positive explicit constant only depends on* p, m *and is independent on the value of* ℓ. *For example, we can pick* $C_m = p^{m/2}$ *in our case. In the calculation of Section 5 and Appendix, we will see that the constant* C_m *is not sensitive.*

Chapter 5

Estimation of Kloosterman sums on $\mathrm{GL}(n)$

In this chapter, we follow Stevens' approach [Ste87] to bound the $\mathrm{GL}(n)$ Kloosterman sums introduced in the previous Chapter 4.

For $w \in W_{G_n}$, we define $w(j)$, $j \in \{1, 2, 3, \cdots, n\}$ by the formula

$$w \cdot e_j = \pm e_{w(j)},$$

where $e_1, e_2, e_3, \cdots, e_n$ is the standard basis of column vectors.

Let $\nu_1, \nu_2, \nu_3, \cdots, \nu_{n-1}, \nu_1', \nu_2', \mu_3', \cdots, \nu_{n-1}' \in p^{-m}\mathbb{Z}_p - \{0\}$. Moreover, we further assume that $p^{-m} \leq |\nu_i| \leq p^m$ and $p^{-m} \leq |\nu_i'| \leq p^m$ for all $1 \leq i, i' \leq n-1$. We recall that the non-degenerated character ψ_p of $N(\mathbb{Q}_p)$ which is trivial on $N(p^m\mathbb{Z}_p)$ is given by

$$\psi_p\left(\begin{pmatrix} 1 & u_1 & * & \cdots & * \\ & 1 & u_2 & \cdots & * \\ & & \cdots & \cdots & \cdots \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{pmatrix}\right) = \xi(\nu_1 u_1 + \nu_2 u_2 + \nu_3 u_3 + \cdots + \nu_{n-1} u_{n-1}). \quad (5.1)$$

The definition of additive character ψ_p' is given in a similar way.

We fix

$$c = \mathrm{diag}(p^{a_1} v_1, p^{a_2 - a_1} v_2, \cdots, p^{a_{n-1} - a_{n-2}} v_{n-1}, p^{-a_{n-1}} v_n). \quad (5.2)$$

In other word, we can write

$$c = \begin{pmatrix} p^{a_1}v_1 & & & & & \\ & p^{a_2-a_1}v_2 & & & & \\ & & \cdots & & & \\ & & & p^{a_{n-1}-a_{n-2}}v_{n-1} & & \\ & & & & p^{-a_{n-1}}v_n \end{pmatrix}, \tag{5.3}$$

where $v_i \in \mathbb{Z}_p^\times$ for $i = 1, 2, 3, \cdots, n$ and $v_1v_2v_3\cdots v_n = (-1)^{\frac{(n+1)(n+2)}{2}+1}$. Here $a_1, a_2, a_3, \cdots, a_n$ are all nonnegative integers. Moreover, we have $a_1, a_2, a_3, \cdots, a_n \geq m$, where m is a fixed positive integer.

We will use the same notation as in Stevens [Ste87, Section 4]. And we need Definition 4.2.1 and Lemma 4.2.2 in above Chapter 4.

Theorem 5.0.1. *Let $Kl_p(\psi_p, \psi_p'; c, w_{G_n})$ be the local Kloosterman sum attached to the long element w_{G_n}. Let ψ be as in* (5.1)*, $\ell = \max(a_1, a_2, \cdots, a_{n-1})$, $\varrho = \max(a_{n-1}, a_1)$, $\sigma = \min(a_{n-1}, a_1)$, and*

$$C_n := 2^{n^2-1} \cdot p^{2(n+3)(n-1)m} \cdot \left(\prod_{j=1}^{n-1} (|\nu_j \nu_{n-j}' p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \right)$$
$$\times (\ell + (n-1)m + 1)^{(n^2-1)} \cdot ((n-1)\ell + n)^{\frac{n^3}{2}}.$$

Then

$$|Kl_p(\psi_p, \psi_p'; c, w_{G_n})| \leq C_n \min(p^{\sigma + a_2 + \cdots + \varrho/2 + \frac{n(n-1)}{2}m}, p^{\ell/2 + 2\sigma + (n-3)\varrho + a_2 + \cdots + a_{n-2} - \ell + \frac{n(n-1)}{2}m}). \tag{5.4}$$

In particular, we have $|Kl_p(\psi_p, \psi_p'; c, w_{G_n})| \leq C_n p^{(1 - \frac{1}{4n^2-18n+22}) \cdot (a_1+a_2+a_3+\cdots+a_{n-1}) + \frac{n(n-1)}{2}m}$.

We first recall Lemma 5.2 of [Ste87].

Lemma 5.0.2. *[Ste87, Lemma 5.2] Let $r \geq 1$ and for each $k = 1, 2, \cdots, r$ let $I_k = \{r-k+1, \cdots, r\}$ be the final k-element subset of $\{1, \cdots, r\}$. Let $g, g' \in \mathrm{GL}_r(\mathbb{Q}_p)$. Then $g' \in N(\mathbb{Q}_p)g$ if and only if for each $k = 1, \cdots, r$ and every k-element subset $I \subseteq \{1, \cdots, r\}$ we have $g_{I_k, I} = g'_{I_k, I}$. In other word, this last equality asserts that the bottom row of $k \times k$ subdeterminants of g agrees with that for g'.*

Note that from the definition of the set $X(w_{G_n}c)$, we can assume that the element

$$\begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ & 1 & x_{23} & \cdots & x_{2n} \\ & & \cdots & \cdots & \cdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \in N(\mathbb{Q}_p)/N(p^m\mathbb{Q}_p).$$

So we can let $x_{ij} = p^{-b_{ij}}c_{ij}$, for $1 \leq i < j \leq n$, where b_{ij} are all integers and they satisfy $b_{ij} \geq -m$. We also have $c_{ij} \in \mathbb{Z}_p^\times$ for all $1 \leq i < j \leq n$.

Since $X(w_{G_n}c) \in K_m$, we have to put some conditions on b_{ij} and c_{ij}. Thanks to the Lemma 5.2 in [Ste87], we can have the conditions of b_{ij} and c_{ij} explicitly. The conditions are restrictions of the subdeterminants of the unipotent radical subgroups. For example, we have $-m \leq b_{ij} \leq a_i$ for all $1 \leq i \leq n-1$. In particular, we have $b_{1,n} = a_1$.

Conversely, if we are given integers b_{ij} $(1 \leq i < j \leq n)$ with $b_{ij} \geq -m$ and $c_{ij} \in \mathbb{Z}_p^\times$ satisfying all the conditions in Lemma 5.2 of [Ste87],then there exists an elememt $x_{b_{ij}}^{c_{ij}} \in X(w_{G_n}c)$ for which

$$u'(x_{b_{ij}}^{c_{ij}}) = \begin{pmatrix} 1 & p^{-b_{1,2}}c_{1,2} & p^{-b_{1,3}}c_{1,3} & \cdots & p^{-a_1}c_{1,n} \\ & 1 & p^{-b_{2,3}}c_{2,3} & \cdots & p^{-b_{2,n}}c_{2,n} \\ & & \cdots & \cdots & \cdots \\ & & & 1 & p^{-b_{n-1,n}}c_{n-1,n} \\ & & & & 1 \end{pmatrix} \pmod{N(p^m\mathbb{Z}_p)}. \quad (5.5)$$

Now let ψ_p be the nontrivial character of $N(\mathbb{Q}_p)$ which is trivial on $N(p^m\mathbb{Z}_p)$, i.e. the nontrivial character of $N(\mathbb{Q}_p)/N(p^m\mathbb{Z}_p)$ which is defined in the previous section. For b_{ij}, and c_{ij} satisfying the Propeties given in Lemma 5.2 in [Ste87], we define

$$X_{b_{ij}}^{c_{ij}}(w_{G_n}c) := T(1 + p^m\mathbb{Z}_p) * x_{b_{ij}}^{c_{ij}}$$

be the orbit through $x_{b_{ij}}^{c_{ij}}$, and we also define

$$S_{b_{ij}}^{c_{ij}}(\psi_p, \psi_p'; c, w_{G_n}) := \sum_{x \in X_{b_{ij}}^{c_{ij}}(w_{G_n}c)} \psi_p(u(x))\psi_p'(u'(x))$$

be the Kloosterman sum restricted to the given orbit. Now for fixed b_{ij}, c_{ij} which satisfy the previous Properties given in Lemma 5.2 of [Ste87], we let

$$X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c) := \bigcup_{b_{ij},c_{ij},j-i\geq 2} X_{b_{ij}}^{c_{ij}}(w_{G_n}c),$$

where b_{ij} run over all integers bigger than $-m$, and c_{ij} run over all the elements of $\mathbb{Z}_p^\times$ satisfying previous properties for $j-i\geq 2$. Let

$$S_{b_{i,i+1},c_{i,i+1}}(\psi_p,\psi_p';c,w_{G_n}) := \sum_{x\in X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c)} \psi_p(u(x))\psi_p'(u'(x)).$$

Lemma 5.0.3. *We have $X(w_{G_n}c)=\coprod_{b_{i,i+1},c_{i,i+1}} X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c)$, where $b_{i,i+1}$ run over all integers bigger than $-m$ and smaller than ℓ, and $c_{i,i+1}$ run over all the elements in the finite coset of $\mathbb{Z}_p^\times/(1+p^m\mathbb{Z}_p)$ satisfying previous properties in Lemma 5.2 of [Ste87].*

Proof. The proof is the same as Lemma 5.2 and 5.7 in [Ste87]. Note that $K_m = N(p^m\mathbb{Z}_p)\cdot(B^-\cap K_m)\cdot N(p^m\mathbb{Z}_p)$ is true for $G_n=\mathrm{GL}_n(\mathbb{Q}_p)$, where B^- is the opposite Borel subgroup. Therefore, the matrix $u(x)$ is uniquely determined by the matrix $u'(x)$. □

Lemma 5.0.4. *Let $\ell=\max(a_1,a_2,\cdots,a_{n-1})$, and $b_{i,i+1}+m\leq a_i$ (for all $1\leq i\leq n-1$) be integers which are bigger than $-m$. Then*

$$\begin{aligned}
&|S_{b_{i,i+1},c_{i,i+1}}(\psi_p,\psi_p';c,w_{G_n})|\\
\leq\ & 2^{n-1}\cdot p^{2(n-1)m}\cdot(|\nu_1\nu_{n-1}'p^{2m}|_p^{-1},p^{\ell+m})^{1/2}(|\nu_2\nu_{n-2}'p^{2m}|_p^{-1},p^{\ell+m})^{1/2}\cdots\\
&\times(|\nu_{n-1}\nu_1'p^{2m}|_p^{-1},p^{\ell+m})^{1/2}\cdot(\ell+m+1)^{n-1}\cdot p^{-\frac{b_{12}+b_{23}+\cdots+b_{n-1,n}}{2}}\cdot\#(X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c)).
\end{aligned}$$

Proof. The involution map $\iota: g\to g^\iota := w_{G_n}\cdot(g^t)^{-1}\cdot w_{G_n}$ sends $X_{b_{ij},c_{ij}}(w_{G_n}c)$ to $X_{b_{ij},c_{ij}}((w_{G_n}c)^\iota)$. Composing ψ_p with ι has the effect of replacing $(\nu_1,\nu_2,\nu_3,\cdots,\nu_{n-1})$ by $(-\nu_{n-1},-\nu_{n-2},\cdots,-\nu_1)$ and $(\nu_1',\nu_2',\nu_3',\cdots,\nu_{n-1}')$ by $(-\nu_{n-1}',-\nu_{n-2}',\cdots,-\nu_1')$. For $g_0=u_1w_{G_n}cu_2\in X(w_{G_n}c)$, we have $g_0^\iota\in X((w_{G_n}c)^\iota)=X(w_{G_n}(w_{G_n}c^{-1}w_{G_n}))\subseteq K_m$ by definition. Hence, applying ι to the element $w_{G_n}c$ reverses the roles of a_{n-1} and a_1. Thus we may assume that $a_{n-1}\geq a_1$ without loss of generality.

Recall that $\ell=\max(a_1,a_2,\cdots,a_{n-1})=\max(a_2,\cdots,a_{n-1})$. Then Properties in Lemma 5.2 of [Ste87] imply that the matrix entries of $u(x)$ and $u'(x)$ lie in $p^{-\ell}\mathbb{Z}_p/p^m\mathbb{Z}_p$

for every $x \in X(w_{G_n}c)$. Indeed, by Lemma 5.0.3, it is enough to verify this for $x = x_{b_{ij}}^{c_{ij}}(w_{G_n}c)$. By definition, it is easy to verify that the matrix entries of $u'(x)$ lies in $p^{-\ell}\mathbb{Z}_p/p^m\mathbb{Z}_p$. Moreover, for $G_n = \mathrm{GL}_n(\mathbb{Q}_p)$, we have the following decomposition

$$K_m = N(p^m\mathbb{Z}_p) \cdot (B^- \cap K_m) \cdot N(p^m\mathbb{Z}_p),$$

where B^- is the opposite Borel subgroup. Hnece the matrix $u(x)$ is uniquely determined by the matrix $u'(x)$. By the uniqueness of the Bruhat decomposition, Proposition 3.1 and Lemma 3.4 in [KN22], we see that the matrix entrices of $u(x)$ also lies in $p^{-\ell}\mathbb{Z}_p/p^m\mathbb{Z}_p$.

Now let $\mathcal{S}$ be a finite subset of $\mathbb{Z}_{\geq -m}^{\frac{(n-1)(n-2)}{2}-1} \times (\mathbb{Z}_p^\times)^{\frac{(n-1)(n-2)}{2}}$ such that $X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c)$ is the disjoint union of the $X_{b_{ij}}^{c_{ij}}(w_{G_n}c)$ with $(b_{ij}, c_{ij}, j-i \geq 2) \in \mathcal{S}$. Then as in Lemma 4.2.2 we have

$$\begin{aligned} S_{b_{i,i+1},c_{i,i+1}}(\psi_p, \psi_p'; c, w_{G_n}) < p^{-(n-1)\ell}(1-p^{-1})^{-(n-1)} \cdot \\ \cdot \sum_{(b_{ij},c_{ij},j-i\geq 2)\in\mathcal{S}} \#(X_{b_{ij}}^{c_{ij}}(cw_{G_n}))S_{w_{G_n}}(\theta_{b_{ij}}^{c_{ij}}; \ell), \end{aligned} \tag{5.6}$$

where $S_{w_{G_n}}$ is defined in Definition 4.2.1, and $\theta_{b_{ij}}^{c_{ij}} : A_{w_G}(\ell) \to \mathbb{C}^\times$ is the character defined in the previous section by

$$\begin{aligned} \theta_{b_{ij}}^{c_{ij}}(\underline{\lambda} \times \underline{\lambda}') = e(\nu_1 u_1 \lambda_1 + \nu_2 u_2 \lambda_2 + \cdots + \nu_{n-1}u_{n-1}\lambda_{n-1} \\ + \nu_1' p^{-b_{12}} c_{12}\lambda_1' + \nu_2' p^{-b_{23}} c_{23}\lambda_2' + \cdots + \nu_{n-1}' p^{-b_{n-1,n}} c_{n-1,n}\lambda_{n-1}'). \end{aligned}$$

By Remark 4.2.3 (See also Example 4.12 in Stevens [Ste87]), we have

$$\begin{aligned} S_{w_{G_n}}(\theta_{b_{ij}}^{c_{ij}}; \ell) = S_2(\nu_1 u_1 p^\ell, \nu_{n-1}' c_{n-1,n} p^{\ell - b_{n-1,n}}; p^\ell) \\ \cdot S_2(\nu_2 u_2 p^\ell, \nu_{n-2}' c_{n-2,n-1} p^{\ell - b_{n-2,n-1}}; p^\ell) \cdots \\ \cdot S_2(\nu_{n-1} u_{n-1} p^\ell, \nu_1' c_{12} p^{\ell - b_{12}}; p^\ell), \end{aligned} \tag{5.7}$$

where S_2 is the restricted GL(2)-Kloosterman sum defined in Remark 4.2.3. By refined Weil's bound established in Remark 4.2.4, we have the inequality

$$|S_2(\nu, \nu'; p^\ell)| \leq (\ell + m + 1)p^{m/2}(\gcd(|p^m\nu|_p^{-1}, |p^m\nu'|_p^{-1}, p^{\ell+m}))^{1/2} p^{(\ell+m)/2}, \tag{5.8}$$

for $\nu, \nu' \in p^{-m}\mathbb{Z}_p - \{0\}$ (See also Section 9 in [KL13]).

In order to apply the refined Weil's bound, we note that

$$\begin{aligned}
&\gcd(|\nu_{n-1}u_{n-1}p^{\ell+m}|_p^{-1}, |\nu_1' p^{\ell+m-b_{12}}|_p^{-1}, p^{\ell+m}) \le \gcd(|\nu_{n-1}\nu_1' p^{2m}|_p^{-1}, p^{\ell+m})p^{\ell+m-b_{12}},\\
&\gcd(|\nu_{n-2}u_{n-2}p^{\ell+m}|_p^{-1}, |\nu_2' p^{\ell+m-b_{23}}|_p^{-1}, p^{\ell+m}) \le \gcd(|\nu_{n-2}\nu_2' p^{2m}|_p^{-1}, p^{\ell+m})p^{\ell+m-b_{23}},\\
&\qquad\qquad\qquad\qquad\cdots\cdots\\
&\gcd(|\nu_1 u_1 p^{\ell+m}|_p^{-1}, |\nu_{n-1}' p^{\ell+m-b_{n-1,n}}|_p^{-1}, p^{\ell+m}) \le \gcd(|\nu_1\nu_{n-1}' p^{2m}|_p^{-1}, p^{\ell+m})p^{\ell+m-b_{n-1,n}}.
\end{aligned}$$

since $\gcd(a,b) \le \min(a,b)$. Hence we have

$$\begin{aligned}
|S_{w_{G_n}}(\theta_{b_{ij}}^{c_{ij}};\ell)| \le (\ell+m+1)^{n-1}p^{2(n-1)m}\cdot\left(\prod_{j=1}^{n-1}(|\nu_j\nu_{n-j}'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}\right)\\
\cdot\, p^{(n-1)\ell-\frac{b_{12}+b_{23}+\cdots+b_{n-1,n}}{2}}.
\end{aligned}$$

This inequality, together with (5.6), gives

$$\begin{aligned}
&|S_{b_{i,i+1},c_{i,i+1}}(\psi_p,\psi_p';c,w_{G_n})|\\
&\le (\ell+m+1)^{n-1}\cdot\left(\prod_{j=1}^{n-1}(|\nu_j\nu_{n-j}'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}\right)\cdot p^{2(n-1)m-\frac{1}{2}\sum_{j=1}^{n-1}b_{j,j+1}}\\
&\quad\cdot(1-p^{-1})^{-(n-1)}\cdot\sum_{(b_{ij},c_{ij},j-i\ge 2)\in\mathcal{S}}\#(X_{b_{ij}}^{c_{ij}}(w_{G_n}c)).
\end{aligned}\tag{5.9}$$

The sum appearing on the right hand side is equal to $\#(X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c))$. Since $p\ge 2$ we have $(1-p^{-1})^{-(n-1)} \le 2^{n-1}$, by (5.9). This completes the proof of the lemma. □

Proof of Theorem 5.0.1. By the involution map ι, we can assume that $a_{n-1} \ge a_1$ without loss of generality. Let

$$\begin{aligned}
C = 2^{n^2-1}\cdot p^{2(n+3)(n-1)m}\cdot\left(\prod_{j=1}^{n-1}(|\nu_j\nu_{n-j}'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}\right)\\
\times(\ell+(n-1)m+1)^{(n^2-n)}\cdot((n-1)\ell+n)^{\frac{n^3}{2}}.
\end{aligned}$$

We define

$$X_{b_{i,i+1}}(w_{G_n}c) := \coprod_{c_{i,i+1}} X_{b_{i,i+1},c_{i,i+1}}(w_{G_n}c)$$

and

$$S_{b_{i,i+1}}(\psi_p,\psi_p';c,w_{G_n}) := \coprod_{c_{i,i+1}} S_{b_{i,i+1},c_{i,i+1}}(\psi_p,\psi_p';c,w_{G_n}).$$

At first, we deal with the case $a_{n-1} = \max(a_1, a_2, \cdots, a_{n-1})$.

If $b_{12} + b_{23} + \cdots + b_{n-1,n} > a_{n-1}$, we have $p^{-\frac{b_{12}+b_{23}+\cdots+b_{n-1,n}}{2}} < p^{-\frac{a_{n-1}}{2}}$. Recall Theorem 3.2.1 and the proof of Theorem 3.2.2, we know that

$$\begin{aligned} X(w_{G_n}c) &\leq p^{n(n-1)m} \times O_{f_0}(c) \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times |\kappa(\overline{m})| \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times (a_1 + a_2 + \cdots + a_{n-1} + n)^{\frac{n^3}{2}} \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times ((n-1)\ell + n)^{\frac{n^3}{2}}. \end{aligned}$$

Hence by Lemma 5.0.4, we have

$$|S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})| \leq C \times p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Applying the above Lemma 5.0.3, we have

$$|Kl_p(\psi_p, \psi'_p; c, w_{G_n})| \leq C_n \times p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Hence we prove the Theorem 5.0.1 for the case $b_{12} + b_{23} + \cdots + b_{n-1,n} > a_{n-1}$.

Now we assume that $b_{12} + b_{23} + \cdots + b_{n-1,n} \leq a_{n-1}$.

If

$$\begin{aligned} &b_{13} + b_{24} + \cdots + b_{n-2,n} \leq a_{n-2}; \\ &b_{14} + b_{25} + \cdots + b_{n-3,n} \leq a_{n-3}; \\ &\cdots\cdots\cdots \\ &b_{1,k} + b_{2,k+1} + \cdots + b_{n-k+1,n} \leq a_{n-k+1}; \\ &\cdots\cdots\cdots \\ &b_{1,n-1} + b_{2,n} \leq a_2; \end{aligned} \tag{5.10}$$

then we have $\#(b_{ij}) \leq (\ell + m + 1)^{\frac{(n-2)(n-1)}{2}}$ and $\#(c_{ij}) \leq p^{\sum_{1\leq i<j\leq n} b_{ij} + \frac{(n-2)(n-1)}{2}m}$ for $j - i \geq 2$. Therefore, we obtain that

$$\begin{aligned} \#(X_{b_{i,i+1}}(w_{G_n}c) &\leq (\ell + m + 1)^{\frac{(n-2)(n-1)}{2}} \cdot p^{\sum_{1\leq i<j\leq n} b_{ij} + \frac{n(n-1)}{2}m} \\ &\leq (\ell + m + 1)^{\frac{(n-2)(n-1)}{2}} \cdot p^{a_1+a_2+\cdots+a_{n-2}+(b_{12}+b_{23}+\cdots+b_{n-1,n})+\frac{n(n-1)}{2}m}. \end{aligned}$$

Hence by Lemma 5.0.4, we have

$$|S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})| \leq C \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Applying the above Lemma 5.0.3, we have

$$|Kl_p(\psi_p, \psi_p'; c, w_{G_n})| \leq C_n \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

If there exists k $(2 \leq k \leq n-1)$ such that $b_{1,k} + b_{2,k+1} + \cdots + b_{n-k+1,n} > a_{n-k+1}$, we will show that

$$\prod_{i=1}^{n-k+1} \#(c_{i,i+k-1}) \leq 2^{k-1} \cdot (\ell + (n-1)m + 1)^k \cdot p^{a_{n-k+1}+(n-k+1)m}.$$

We will consider the following $n-1$ submatrices:

$$M_1 := (p^{-b_{1,n}} c_{1,n});$$

$$M_2 := \begin{pmatrix} p^{-b_{1,n-1}} c_{1,n-1} & p^{-b_{1,n}} c_{1,n} \\ p^{-b_{2,n-1}} c_{2,n-1} & p^{-b_{2,n}} c_{2,n} \end{pmatrix};$$

$$M_3 := \begin{pmatrix} p^{-b_{1,n-2}} c_{1,n-2} & p^{-b_{1,n-1}} c_{1,n-1} & p^{-b_{1,n}} c_{1,n} \\ p^{-b_{2,n-2}} c_{2,n-2} & p^{-b_{2,n-1}} c_{2,n-1} & p^{-b_{2,n}} c_{2,n} \\ p^{-b_{3,n-2}} c_{3,n-2} & p^{-b_{3,n-1}} c_{3,n-1} & p^{-b_{3,n}} c_{3,n} \end{pmatrix};$$

$$\cdots\cdots;$$

$$M_{n-1} := \begin{pmatrix} p^{-b_{12}} c_{12} & p^{-b_{13}} c_{13} & \cdots & \cdots & p^{-a_1} c_{1,n} \\ 1 & p^{-b_{23}} c_{23} & \cdots & \cdots & p^{-b_{2,n}} c_{2,n} \\ 0 & 1 & p^{-b_{34}} c_{34} & \cdots & p^{-b_{3,n}} c_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & p^{-b_{n-1,n}} c_{n-1,n} \end{pmatrix},$$

where $b_{i,i} = 0, c_{i,i} = 1$ and $c_{i,j} = 0$ if $i > j$.

Applying Lemma 5.2 in [Ste87] to the submatrix $M_1, M_2, \cdots, M_{n-1}$, we see that

$$p^{a_1}|M_1| \in 1 + p^m\mathbb{Z}_p, p^{a_2}|M_2| \in 1 + p^m\mathbb{Z}_p, \cdots, p^{a_{n-1}}|M_{n-1}| \in 1 + p^m\mathbb{Z}_p.$$

For certain k $(2 \leq k \leq n-1)$, we further consider the submatrix:

$$M_k := \begin{pmatrix} p^{-b_{1,n-k+1}} c_{1,n-k+1} & \cdots & p^{-b_{1,n-1}} c_{1,n-1} & p^{-b_{1,n}} c_{1,n} \\ p^{-b_{2,n-k+1}} c_{2,n-k+1} & \cdots & p^{-b_{2,n-1}} c_{2,n-1} & p^{-b_{2,n}} c_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ p^{-b_{k,n-k+1}} c_{k,n-k+1} & \cdots & p^{-b_{k,n-1}} c_{k,n-1} & p^{-b_{k,n}} c_{k,n} \end{pmatrix},$$

where $b_{i,i} = 0, c_{i,i} = 1$ and $c_{i,j} = 0$ if $i > j$. For the matrix M_k, we consider its $k-1$ submatrices:

$$M_{1,k} := (p^{-b_{k,n}} c_{k,n});$$

$$M_{2,k} := \begin{pmatrix} p^{-b_{k-1,n-1}} c_{k-1,n-1} & p^{-b_{k-1,n}} c_{k-1,n} \\ p^{-b_{k,n-1}} c_{k,n-1} & p^{-b_{k,n}} c_{k,n} \end{pmatrix};$$

$$M_{3,k} := \begin{pmatrix} p^{-b_{k-2,n-2}} c_{k-2,n-2} & p^{-b_{k-2,n-1}} c_{k-2,n-1} & p^{-b_{k-2,n}} c_{k-2,n} \\ p^{-b_{k-1,n-2}} c_{k-1,n-2} & p^{-b_{k-1,n-1}} c_{k-1,n-1} & p^{-b_{k-1,n}} c_{k-1,n} \\ p^{-b_{k,n-2}} c_{k,n-2} & p^{-b_{k,n-1}} c_{k,n-1} & p^{-b_{k,n}} c_{k,n} \end{pmatrix};$$

$$\cdots\cdots;$$

$$M_{k-1,k} := \begin{pmatrix} p^{-b_{2,n-k+2}} c_{2,n-k+2} & \cdots & p^{-b_{2,n-1}} c_{2,n-1} & p^{-b_{2,n}} c_{2,n} \\ p^{-b_{3,n-k+2}} c_{3,n-k+2} & \cdots & p^{-b_{3,n-1}} c_{3,n-1} & p^{-b_{3,n}} c_{3,n} \\ \cdots & \cdots & \cdots & \cdots \\ p^{-b_{k,n-k+2}} c_{k,n-k+2} & \cdots & p^{-b_{k,n-1}} c_{k,n-1} & p^{-b_{k,n}} c_{k,n} \end{pmatrix};$$

$$M_{k,k} := M_k,$$

where $b_{i,i} = 0, c_{i,i} = 1$ and $c_{i,j} = 0$ if $i > j$.

Applying Lemma 5.2 in [Ste87] to the submatrices $M_{1,k}, M_{2,k}, \cdots, M_{k-1,k}, M_k$, we see that

$$p^{a_k}|M_k| \in 1 + p^m\mathbb{Z}_p,\ p^{a_k}|M_{1,k}| \in p^m\mathbb{Z}_p,\ p^{a_k}|M_{2,k}| \in p^m\mathbb{Z}_p,\ \cdots,\ p^{a_k}|M_{k-1,k}| \in p^m\mathbb{Z}_p.$$

Hence we know that there exist $t_1, t_2, \cdots, t_{k-1} \in \mathbb{Z}_{\geq -(n-1)m}$, where $t_i \leq a_k \leq \ell$ for all $1 \leq i \leq k-1$ such that

$$p^{t_1}|M_{1,k}| \in \mathbb{Z}_p^\times,\ p^{t_2}|M_{2,k}| \in \mathbb{Z}_p^\times,\ \cdots,\ p^{t_{k-1}}|M_{1,k-1}| \in \mathbb{Z}_p^\times.$$

Furthermore, without loss of generality, we can assume that $t_1 = b_{k,n} \geq -m$ and $t_i \leq t_{i+1} + m$ for all $1 \leq i \leq k-1$, where $t_k := a_k$. Therefore, we have $t_i \geq -i \times m$ for all $1 \leq i \leq k-1 \leq n-1$. Otherwise, we may substitute the element $x_{b_{ij}}^{c_{ij}} \in C(w_{G_n}c)$ by $x_{b_{ij}}^{c_{ij}}u' \in C(w_{G_n}c)$, where u' is an element in $N(p^m\mathbb{Z}_p) \subseteq N(\mathbb{Z}_p)$. For example, if $t_2 < t_1 - m$, we may focus on the matrix $M_{2,k}$ and substitute the element $p^{-b_{k-1,n-1}}c_{k-1,n-1}$ by $p^{-b_{k-1,n-1}}c_{k-1,n-1} + p^m$. This is given by the right multiplication of $u' = (u_{i,j})$ with

$u_{i,i} = 1$ $(1 \le i \le n)$, $u_{k-1,n-1} = p^m$ and all the other entrices equal to zero. Note that $x_{b_{ij}}^{c_{ij}}$ and $x_{b_{ij}}^{c_{ij}} u'$ represent the same element in $X(w_{G_n} c)$. These right multiplications of u' will give at most 2^{k-1} $(\{0, p^m\}^{k-1})$ different cases. Therefore by induction, we see that

$$\begin{aligned}
&\prod_{i=k}^{k} \#(c_{i,i+n-k}) \le p^{t_1+m}, \\
&\prod_{i=k-1}^{k} \#(c_{i,i+n-k}) \le 2 \cdot p^{t_2+2m}, \\
&\qquad\qquad \cdots, \\
&\prod_{i=2}^{k} \#(c_{i,i+n-k}) \le 2^{k-2} \cdot p^{t_{k-1}+(k-1)m}, \\
&\prod_{i=1}^{k} \#(c_{i,i+n-k}) \le 2^{k-1} \cdot p^{t_k+km} = 2^{k-1} \cdot p^{a_k+km}.
\end{aligned}$$

We also note that

$$\prod_{i=1}^{k} \#(t_i) \le (\ell + (n-1)m + 1)^k,$$

which will give that

$$\prod_{i=1}^{n-k+1} \#(c_{i,i+k-1}) \le 2^{k-1} \cdot (\ell + (n-1)m + 1)^k \cdot p^{a_{n-k+1}+(n-k+1)m}$$

for $2 \le k \le n-1$.

For the general case, using the same argument as above discussions, we write

$$\begin{aligned}
&b_{13} + b_{24} + \cdots + b_{n-2,n} = a_{n-2} + d_{n-2}, \\
&b_{14} + b_{25} + \cdots + b_{n-3,n} = a_{n-3} + d_{n-3}, \\
&\cdots\cdots, \\
&b_{1,k} + b_{2,k+1} + \cdots + b_{n-k+1,n} = a_{n-k+1} + d_{n-k+1}, \\
&\cdots\cdots, \\
&b_{1,n-2} + b_{2,n-1} + b_{3,n} = a_3 + d_3, \\
&b_{1,n-1} + b_{2,n} = a_2 + d_2.
\end{aligned} \tag{5.11}$$

We can obtain an upper bound as follows:

$$\#(c_{ij}) \leq 2^{\frac{(n-2)(n-1)}{2}} \cdot (\ell+(n-1)m+1)^{\frac{(n-2)(n-1)}{2}} \cdot p^{\sum_{1\leq i<j\leq n} b_{ij}+\frac{(n-2)(n-1)}{2}m-\sum_{j=2}^{n-1}\max(0,d_j)},$$

where $j-i \geq 2$ and by Lemma 5.0.4, we have

$$|S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})| \leq C \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Applying the above Lemma 5.0.3, we have

$$|Kl_p(\psi_p, \psi'_p; c, w_{G_n})| \leq C_n \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Note that in this case, since $a_{n-1} = \max(a_1, a_2, \cdots, a_{n-1})$, we always have

$$a_1 + a_2 + \cdots + a_{n-2} + \frac{a_{n-1}}{2} \leq \ell/2 + 2a_1 + (n-3)a_{n-1} + a_2 + a_3 + \cdots + a_{n-2} - \ell.$$

Thus Theorem 5.0.1 follows from the equality $Kl_p(\psi_p, \psi'_p; c, w_{G_n}) = \sum_{b_{i,i+1}} S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})$.

If we have $a_{n-1} \neq \max(a_1, a_2, \cdots, a_{n-1})$, by a similar argument as above, we obtain

$$|S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})| \leq C \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Applying the above Lemma 5.0.3, we will have

$$|Kl_p(\psi_p, \psi'_p; c, w_{G_n})| \leq C_n \cdot p^{a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2}+\frac{n(n-1)}{2}m}.$$

Note that if $a_{n-1} << \max(a_1, a_2, \cdots, a_{n-1})$, in other word a_{n-1} is small, this bound is not good enough. So we have to bound this Kloosterman sum in other way.

Now we assume that $a_k := \max(a_1, a_2, \cdots, a_{n-1})$. Since we assume that $a_{n-1} \geq a_1$, we have $2 \leq k \leq n-2$. Moreover, we will consider extra $n-1$ submatrices as follows:

$$U_0 := (1);$$

$$U_1 := (p^{-b_{n-1,n}} c_{n-1,n});$$

$$U_2 := \begin{pmatrix} p^{-b_{n-2,n-1}} c_{n-2,n-1} & p^{-b_{n-2,n}} c_{n-2,n} \\ 1 & p^{-b_{n-1,n}} c_{n-1,n} \end{pmatrix};$$

$$\cdots\cdots;$$

$$U_{n-2} := \begin{pmatrix} p^{-b_{23}}c_{23} & p^{-b_{24}}c_{24} & \cdots & \cdots & p^{-b_{2,n}}c_{2,n} \\ 1 & p^{-b_{34}}c_{34} & \cdots & \cdots & p^{-b_{3,n}}c_{3,n} \\ 0 & 1 & p^{-b_{45}}c_{45} & \cdots & p^{-b_{4,n}}c_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & p^{-b_{n-1,n}}c_{n-1,n} \end{pmatrix},$$

where $b_{i,i} = 0, c_{i,i} = 1$ and $c_{i,j} = 0$ if $i > j$. Applying Lemma 5.2 in [Ste87] to the submatrices $U_1, U_2, \cdots, U_{n-2}$, we see that

$$p^{a_{n-1}}|U_1| \in p^m\mathbb{Z}_p,\ p^{a_{n-1}}|U_2| \in p^m\mathbb{Z}_p,\ \cdots,\ p^{a_{n-1}}|U_{n-2}| \in p^m\mathbb{Z}_p.$$

By our assumption, we have $a_k = \max(a_1, a_2, \cdots, a_{n-1})$. We will focus on the following $k-1$ elements:

$$p^{-b_{k,n}}c_{k,n},\ p^{-b_{k-1,n-1}}c_{k-1,n-1},\ \cdots,\ p^{-b_{2,n-k+2}}c_{2,n-k+2}.$$

We have the following cases:

If $b_{k,n} \leq a_{n-1}$, then we have $\#(c_{k,n}) \leq p^{a_{n-1}+m}$.

If $b_{k,n} > a_{n-1}$, since $p^{a_{n-1}}|U_{n-k}| \in p^m\mathbb{Z}_p$ and the coefficient of $p^{-b_{k,n}}c_{k,n}$ is given by $(-1)^{n-k+1} \cdot U_0 = (-1)^{n-k+1} \cdot (1) = (-1)^{n-k+1}$, then we have

$$\#(c_{k,n}) \leq p^{b_{k,n}-(b_{k,n}-a_{n-1})+m} \leq p^{a_{n-1}+m}.$$

Following above idea, we continue our induction steps. If $b_{k-j,n-j} \leq a_{n-1}$, then we have $\#(c_{k-j,n-j}) \leq p^{a_{n-1}+m}$ for $0 \leq j \leq k-2$.

If $b_{k-j,n-j} > a_{n-1}$, since $p^{a_{n-1}}|U_{n-k+j}| \in p^m\mathbb{Z}_p$ and the coefficient of $p^{-b_{k-j,n-j}}c_{k-j,n-j}$ is given by above matrix $(-1)^{n-k+1} \cdot |U_j|$. Note that without loss of generality, we can assume that the norm of the determinant of U_j satisfies $|U_j| \geq p^{-(n-1)m}$. Otherwise, we may again substitute the element $x_{b_{ij}}^{c_{ij}} \in C(w_{G_n}c)$ by $x_{b_{ij}}^{c_{ij}}u' \in C(w_{G_n}c)$, where u' is an element in $N(p^m\mathbb{Z}_p) \subseteq N(\mathbb{Z}_p)$. For example, if the norm of U_2 satisfies $|U_2| < p^{-(n-1)m}$, we may focus on the matrix U_2 and substitute the element $p^{-b_{n-2,n-1}}c_{n-2,n-1}$ by $p^{-b_{n-2,n-1}}c_{n-2,n-1} + p^m$. This is given by the right multiplication of $u' = (u_{i,j})$ with $u_{i,i} = 1$ $(1 \leq i \leq n)$, $u_{n-2,n-1} = p^m$ and all the other entrices equal to zero. Note that $x_{b_{ij}}^{c_{ij}}$ and $x_{b_{ij}}^{c_{ij}}u'$ represent the same element in $X(w_{G_n}c)$. These right multiplications of u' will give at most 2^{n-2} different cases. Therefore we have

$$\#(c_{k-j,n-j}) \leq p^{b_{k-j,n-j}-(b_{k-j,n-j}-a_{n-1})+(n-1)m} \leq p^{a_{n-1}+(n-1)m}.$$

Now we vary j from $0 \leq j \leq k-2$, we will have

$$\#(c_{2,n-k+2}, c_{3,n-k+3}, \cdots, c_{k-1,n-1}, c_{k,n}) \leq 2^{n-2} \times p^{(k-1)a_{n-1}+(n-1)(k-1)m}.$$

It remains to bound the remaining $\#(c_{ij})$ for $j-i \geq 2$.

If $b_{12}+b_{23}+\cdots+b_{n-1,n} > a_k > a_{n-1}$, then using the similar argument as above (when $a_{n-1} = \max(a_1, a_2, \cdots, a_{n-1})$), we have $p^{-\frac{b_{12}+b_{23}+\cdots+b_{n-1,n}}{2}} < p^{-\frac{a_k}{2}}$. Recall Theorem 3.2.1 and the proof of Theorem 3.2.2, we know that

$$\begin{aligned} X(w_{G_n}c) &\leq p^{n(n-1)m} \times O_{f_0}(c) \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times |\kappa(\overline{m})| \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times (a_1+a_2+\cdots+a_{n-1}+n)^{\frac{n^3}{2}} \\ &\leq p^{n(n-1)m} \times p^{a_1+a_2+\cdots+a_{n-1}} \times ((n-1)\ell+n)^{\frac{n^3}{2}}. \end{aligned}$$

Hence by Lemma 5.0.4, we have

$$\begin{aligned} |S_{b_{i,i+1}}(\psi_p, \psi'_p; c, w_{G_n})| &\leq C \times p^{a_1+a_2+\cdots+a_{n-2}+a_{n-1}-\frac{a_k}{2}+\frac{n(n-1)}{2}m} \\ &= C \times p^{a_1+a_2+\cdots+a_{k-1}+\frac{a_k}{2}+a_{k+1}+\cdots+a_{n-2}+a_{n-1}+\frac{n(n-1)}{2}m}. \end{aligned} \tag{5.12}$$

Applying the above Lemma 5.0.3, we have

$$\begin{aligned} |Kl_p(\psi_p, \psi'_p; c, w_{G_n})| &\leq C_n \times p^{a_1+a_2+\cdots+a_{n-2}+a_{n-1}-\frac{a_k}{2}+\frac{n(n-1)}{2}m} \\ &= C_n \times p^{a_1+a_2+\cdots+a_{k-1}+\frac{a_k}{2}+a_{k+1}+\cdots+a_{n-2}+a_{n-1}+\frac{n(n-1)}{2}m}. \end{aligned} \tag{5.13}$$

Hence we prove the Theorem 5.0.1 for the case $b_{12}+b_{23}+\cdots+b_{n-1,n} > a_k$.

If $b_{12}+b_{23}+\cdots+b_{n-1,n} \leq a_k$, then using the similar argument as above, we will obtain the bound as follows (note that $\#(c_{2,n-k+2}, c_{3,n-k+3}, \cdots, c_{k-1,n-1}, c_{k,n}) \leq 2^{n-2} \cdot p^{(k-1)a_{n-1}+(n-1)(k-1)m}$):

$$\begin{aligned} \#(c_{ij}) \leq & 2^{\frac{n(n-1)}{2}} \times (\ell+(n-1)m+1)^{\frac{(n-2)(n-1)}{2}} \\ & \times p^{2a_1+a_2+\cdots+a_{k-1}+a_{k+1}+\cdots+a_{n-2}+(k-1)a_{n-1}+(k-1)(n-1)m+\frac{(n-2)(n-1)}{2}m} \end{aligned} \tag{5.14}$$

for $j-i \geq 2$. Since $2 \leq k \leq n-2$, we will have

$$\begin{aligned} \#(c_{ij}) \leq & 2^{\frac{n(n-1)}{2}} \times (\ell+(n-1)m+1)^{\frac{(n-2)(n-1)}{2}} \\ & \times p^{2a_1+a_2+\cdots+a_{k-1}+a_{k+1}+\cdots+a_{n-2}+(n-3)a_{n-1}+(n-3)(n-1)m+\frac{(n-2)(n-1)}{2}m} \end{aligned} \tag{5.15}$$

where $j-i \geq 2$. Hence by Lemma 5.0.4, we have

$$|S_{b_{i,i+1}}(\psi_p, \psi_p'; c, w_{G_n})| \leq C \times p^{a_k/2+2a_1+a_2+\cdots+a_{k-1}+a_{k+1}+\cdots+a_{n-2}+(n-3)a_{n-1}+\frac{n(n-1)}{2}m}.$$

Applying the above Lemma 5.0.3, we have

$$|Kl_p(\psi_p, \psi_p'; c, w_{G_n})| \leq C_n \times p^{a_k/2+2a_1+a_2+\cdots+a_{k-1}+a_{k+1}+\cdots+a_{n-2}+(n-3)a_{n-1}+\frac{n(n-1)}{2}m}.$$

This proves the main inequality in our Theorem 5.0.1.

For the second claim, we recall that $a_k = \max(a_1, a_2, \cdots, a_{n-1}) \geq a_{n-1} \geq a_1$. We note that if $a_k \leq (2n-5)a_{n-1}$, the following inequality holds:

$$a_1+a_2+\cdots+a_{n-2}+\frac{a_{n-1}}{2} \leq a_k/2+2a_1+(n-3)a_{n-1}+a_2+a_3+\cdots+a_{k-1}+a_{k+1}+\cdots+a_{n-2}.$$

Therefore we will have

$$\begin{aligned} a_1 + a_2 + \cdots + a_{n-2} + \frac{a_{n-1}}{2} &\leq \left(1 - \frac{0.5}{(2n-5)(n-2)+1}\right) \cdot (a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}) \\ &\leq \left(1 - \frac{1}{4n^2-18n+22}\right) \cdot (a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}). \end{aligned}$$

If $a_k \geq (2n-5)a_{n-1}$, we will also have (Note that $n \geq 3$)

$$\begin{aligned} \ell/2 + 2a_1 + (n-3)a_{n-1} + \sum_{j=2, j\neq k}^{n-2} a_j &\leq \left(1 - \frac{0.5}{(2n-5)(n-3)+2}\right) \sum_{j=1}^{n-1} a_j \\ &\leq \left(1 - \frac{0.5}{(2n-5)(n-2)+1}\right) \sum_{j=1}^{n-1} a_j \\ &\leq \left(1 - \frac{1}{4n^2-18n+22}\right) \sum_{j=1}^{n-1} a_j. \end{aligned}$$

This proves the second claim and therefore Theorem 5.0.1.

□

Remark 5.0.5. *Note that the trivial bound for the local Kloosterman sum (integral) (see [DR98]) is the following:*

$$|Kl_p(\psi_p, \psi_p'; c, w_{G_n})| \leq A_\epsilon \cdot p^{(1+\epsilon)(a_1+a_2+a_3+\cdots+a_{n-1})},\ \epsilon > 0$$

where A_ϵ is a positive constant independent on the choice of c. Since $1-\frac{1}{4n^2-18n+22} < 1$, so applying Steven's method, we get a nontrivial bound for the local Kloosterman sum (integral).

Remark 5.0.6. *The result is not optimal when* $n \geq 4$. *When* $n = 2$ *and* $n = 3$, *the constant* $\frac{1}{2}$ *and* $\frac{1}{4}$ *are sharp (See [Ba97] and [DF97]). To improve the bound in other cases, one may use the stationary phase formulas as Dabrowski and Fisher did for* $\mathrm{GL}(3)$ *([DF97]). It is also a very interesting question to ask whether we can improve the exponent* $1 - \frac{1}{4n^2-18n+22}$ *to some* $1 - \delta$, *where* $0 < \delta < 1$ *and* δ *is independent on the choice of the positive integer* n *(See [Ste87] for the conjecture:* $\delta = \frac{1}{4}$*).*

Chapter 6

Proof of Local Integrability for $\mathrm{GL}(n)$ Bessel functions

In this chapter, we use the results in the previous chapters to prove the local integrability of Bessel functions for GL(n) ($n \geq 4$). The local integrability of Bessel functions has only been proved for GL(2) [So84] [Ba97] and GL(3) [Ba04]. The main ingredients of their proofs are some basic analytic properties of the Bessel functions and orbital integrals. Moreover, an explicit expression for the GL(3) (or GL(2)) relative Shalika germ $K_e^{w_{G_3}}$ (or $K_e^{w_{G_2}}$) (See [JY96] [JY99]) and the p-adic stationary phase methods are also needed in [Ba04]. However, an explicit expression for the (relative) Shalika germs is invalid for GL(n) ($n \geq 4$) and the p-adic stationary phase calculations become very complicated. In order to overcome these obstacles, we apply Stevens' method in [Ste87] to bound the (relative) Shalika germs $K_e^{w_{G_n}}$. The key ingredient of our proof is a nontrivial bound for the local Kloosterman sum (integral) attached to the longest Weyl element which we have already established in Theorem 5.0.1.

First, let's recall the Conjecture 3.2.3 in Chapter 3.

Conjecture 6.0.1 (Conjecture 3.2.3)**.** *Fix $f \in C_c^\infty(G_n)$. Then $|I_{f,\psi}(g)\Delta^{\frac{1}{2}-\delta_n}(g)|$ is bounded on compact sets in G_n for some given $\delta_n > 0$.*

According to the discussion in Chapter 3, we know that Conjecture 3.2.3 is equivalent to the local integrability of Bessel functions for GL(n). Applying Jacquet-Ye's theory of the relative Shalkia germs (Theorem 3.3.3 and [JY96],[JY99]), we can reduce Conjecture

3.2.3 to Conjecture 3.3.8 from the discussion in Chapter 3. Now it suffices to prove Conjecture 3.3.8.

We will give the proof of Conjecture 3.3.8. Therefore, we prove the local integrability of Bessel functions for $\mathrm{GL}_n(\mathbb{Q}_p)$. We state the following theorem.

Theorem 6.0.2. *The absolute value* $\left|K_e^{w_{G_n}}\Delta^{\frac{1}{2}-\delta}\right|$ *is uniformly bounded on the set* $A_e^{w_{G_n}}$ *for any* δ *satisfying* $0<\delta<\frac{1}{8n^2-36n+44}$.

Proof. According to the definition of (relative) Shalika germ $K_e^{w_{G_n}}$ in [JY96] and [JY99], for any element g in the set $A_e^{w_{G_n}}$, without loss of generality, we can assume that $|\Delta_i(g)|<\epsilon$, where $1\leq i\leq n-1$ and ϵ is any arbitrary small positive real number. Here Δ_i is the determinant of principal $i\times i$ minor of g which we have already defined in Section 3. Moreover, as in Section 4 and 5, we further assume that $\Delta_n(g)=(-1)^{\frac{(n+1)(n+2)}{2}+1}$. Hence, we have $|\Delta_n(g)|=1$.

Note that $|\Delta_i(g)|<\epsilon$ for all $1\leq i\leq n-1$, we can write $g=c\in T$. From above assumption, we let

$$c:=\begin{pmatrix} p^{a_1}v_1 & & & & & \\ & p^{a_2-a_1}v_2 & & & & \\ & & \cdots & & & \\ & & & p^{a_{n-1}-a_{n-2}}v_{n-1} & & \\ & & & & p^{-a_{n-1}}v_n \end{pmatrix},$$

where $v_i\in\mathbb{Z}_p^\times$ for $i=1,2,3,\cdots,n$. Here $a_1,a_2,\cdots,a_{n-1}$ are all large enough positive integers. Moreover, we have $a_1,a_2,\cdots,a_{n-1}>>m$ if we let positive real number ϵ to be small enough, where m is a fixed positive integer in Section 4.

By direct calculation, we have

$$\Delta(c):=\left|\frac{(\Delta_1(c))^2\cdot(\Delta_2(c))^2\cdots(\Delta_{n-1}(c))^2}{(\Delta_n(c))^2}\right|=p^{-2(a_1+a_2+a_3+\cdots+a_{n-1})}.$$

So we have $\Delta^{\frac{1}{2}-\delta}(g)=\Delta^{\frac{1}{2}-\delta}(c)=p^{-(1-2\delta)(a_1+a_2+a_3+\cdots+a_{n-1})}$.

Now applying Theorem 5.0.1, we further assume that $a_1,a_2,\cdots,a_{n-1}>>m$ (For example, $a_j>100000(n-1)m$ for $j=1,2,\cdots,n-1$) and $\nu_1,\nu_1',\nu_2,\nu_2',\cdots,\nu_{n-1},\nu_{n-1}'$ satisfy $p^{-m}\leq|\nu_i|\leq p^m$ and $p^{-m}\leq|\nu_i'|\leq p^m$ for all $1\leq i,i'\leq n-1$. Therefore, we

have

$$\begin{aligned}
&|K_{e,\psi_p}^{w_{G_n}}(c)| < |Kl_p(\psi_p^{-1}, \psi_p'^{-1}; c, w_{G_n})| \\
&\leq 2^{n^2-1} \cdot (\ell + (n-1)m + 1)^{(n^2-1)} \cdot p^{(1-\frac{1}{4n^2-18n+22})\cdot(a_1+a_2+\cdots+a_{n-1})} \\
&\qquad \times p^{2(n-1)m+2(n+3)(n-1)m} \times ((n-1)\ell + n)^{\frac{n^3}{2}} \times p^{\frac{n(n-1)}{2}m} \\
&< 2^{n^2-1} \cdot (\ell + (n-1)m + 1)^{(n^2-1)} \cdot p^{(1-\frac{1}{4n^2-18n+22})\cdot(a_1+a_2+\cdots+a_{n-1})} \\
&\qquad \times p^{2(n-1)m+3(n+3)(n-1)m} \times ((n-1)\ell + n)^{\frac{n^3}{2}} \\
&< 2^{n^2-1} \cdot (\ell + (n-1)m + 1)^{(n^2-1)} \cdot p^{(1-\frac{1}{4n^2-18n+22})\cdot(a_1+a_2+\cdots+a_{n-1})} \\
&\qquad \times p^{3(n+4)(n-1)m} \times ((n-1)\ell + n)^{n^3},
\end{aligned} \tag{6.1}$$

where $\ell = \max(a_1, a_2, a_3, \cdots, a_{n-1})$.

In conclusion, for any $g \in A_e^{w_{G_n}}$, we have

$$\begin{aligned}
|K_e^{w_{G_n}}(g)\Delta^{\frac{1}{2}-\delta}(g)| &= |K_e^{w_{G_n}}(c)\Delta^{\frac{1}{2}-\delta}(c)| \\
&\leq 2^{n^2-1} \cdot p^{3(n+4)(n-1)m} \cdot (\ell + (n-1)m + 1)^{(n^2-1)} \cdot ((n-1)\ell + n)^{n^3} \\
&\qquad \times p^{-(\frac{1}{4n^2-18n+22}-2\delta)(a_1+a_2+a_3+\cdots+a_{n-1})}.
\end{aligned} \tag{6.2}$$

Now since $0 < \delta < \frac{1}{8n^2-36n+44}$, we have $0 < \frac{1}{4n^2-18n+22} - 2\delta < \frac{1}{4n^2-18n+22}$. This means that the positive real number

$$2^{n^2-1} \cdot p^{3(n+4)(n-1)m} \cdot (\ell + (n-1)m + 1)^{(n^2-1)} \cdot ((n-1)\ell + n)^{n^3} \cdot p^{-(\frac{1}{4n^2-18n+22}-2\delta)(a_1+a_2+a_3+\cdots+a_{n-1})}$$

approaches to 0 when $a_1, a_2, \cdots, a_{n-1}$ tends to infinity. Hence, it is bounded uniformly in terms of $a_1, a_2, \cdots, a_{n-1}$. This finishes the proof.

□

Chapter 7

Applications

7.1 Applications of Theorem 1.2.6

In this section, we will give some applications of our main Theorem 1.2.6. We first recall the definition of Bessel distribution. We will follow the notations in Chapter 3, 4 and 5.

We recall some notations in Section 1.2.

Let π be a smooth irreducible generic representation of $G_n = \mathrm{GL}_n(\mathbb{Q}_p)$ with contragredient $\widetilde{\pi}$. We use π^* and $\widetilde{\pi}^*$ to denote the linear dual of π and $\widetilde{\pi}$ respectively. Let $f \in C_c^\infty(G_n)$ be a locally constant function with compact support on G_n. We take $l \in \pi^*$ and $l' \in \widetilde{\pi}^*$ to be fixed nonzero Whittaker functionals with repect to the non-degenerate additive character ψ and ψ^{-1}, respectively. We define $\widetilde{\pi}(f)l'$ as

$$\widetilde{\pi}(f)l' := \int_{G_n} f(g)\widetilde{\pi}(g)l'dg$$

or equivalently, for any $\widetilde{v} \in \widetilde{\pi}$, we have

$$\begin{aligned} \langle \widetilde{\pi}(f)l', \widetilde{v}\rangle &= \int_{G_n} f(g)\langle \widetilde{\pi}(g)l', \widetilde{v}\rangle dg \\ &= \int_{G_n} f(g)\langle l', \widetilde{\pi}(g^{-1})(\widetilde{v})\rangle dg. \end{aligned} \tag{7.1}$$

Then $\widetilde{\pi}(f)l'$ is a smooth linear functional on $\widetilde{\pi}$, hence can be identified with a vector $v_f \in \pi$.

Definition 7.1.1. *We define Bessel distribution $B(f)$ as*

$$B(f) := l(v_f).$$

E. M. Baruch obtained the first regularity result about the Bessel distribution $B(f)$. By Theorem 2.3 in [Ba01], when restricted to the open Bruhat cell $\Omega = Bw_{G_n}B$, this Bessel distribution $B(f)$ is given by integration against a locally constant kernel function $j_{0,\pi}(g)$ on Ω, which is called the relative Bessel function. In other words, for any $f \in C_c^\infty(\Omega)$, we have

$$B(f) = \int_\Omega j_{0,\pi}(g)f(g)dg.$$

We can extend this function $j_{0,\pi}(g)$ to the whole group G_n by letting zero when $g \in G_n - \Omega = G_n - Bw_{G_n}B$. By Theorem 1.1 in [Chai19a] and Theorem 7.2 in [Chai17], we know that $j_{0,\pi}(g) = j_\pi(g)$ for all $g \in \Omega$ after certain normalizations on l or l' (See Theorem 1.1, 3.2 in [Chai19a] and Lemma 3.2 in [Chai17]).

We recall the following regularity theorem introduced in Section 1.2, which naturally connects the Bessel function $j_\pi(g)$ to the above Bessel distribution $B(f)$.

Theorem 7.1.2 (Theorem 1.2.8)**.** *The Bessel distribution $B(f)$ is given by integration against the Bessel function $j_\pi(g)$ on G_n, that is, for any $f \in C_c^\infty(G_n)$, we have*

$$B(f) = \int_{G_n} j_\pi(g)f(g)dg.$$

Proof. For $f \in C_c^\infty(G_n)$, we define the following distribution

$$B_1(f) := \int_{G_n} j_\pi(g)f(g)dg.$$

Note that the Bessel function $j_\pi(g)$ is local integrable on G_n (Theorem 1.2.6), this distribution is well defined. Now we consider the distribution $\widetilde{B}(f) := B(f) - B_1(f)$. By Theorem 2.3 in [Ba01], Theorem 1.1 in [Chai19a] and Theorem 7.2 in [Chai17], the restriction of this distribution $\widetilde{B}(f)$ to the open Bruhat cell $\Omega = Bw_{G_n}B$ is zero. Therefore, this distribution is supported on $G_n - \Omega = G_n - Bw_{G_n}B$. Now by Theorem A in [AGS15], the wave-front set of the distribution $\widetilde{B} = B - B_1$ is contained in $(G_n - \Omega) \times \mathcal{N}$, where $\mathcal{N}$ is the set of nilpotent elements (cone) in the dual of Lie algebra of G_n. Hence, by Corollary B and C in [AGK15], we have $\widetilde{B} \equiv B - B_1 \equiv 0$, which gives that $B \equiv B_1$. This finishes the proof. □

Remark 7.1.3. *The regularity theorem (Theorem 1.2.8) may have applications in the study of relative trace formulae (especially Kuznetsov trace formula).*

We will give an application of above Theorem 1.2.8.

Corollary 7.1.4. *Let π_1 and π_2 be two generic smooth irreducible representations of G_n, with the corresponding Bessel functions $j_{\pi_1}(g)$ and $j_{\pi_2}(g)$. If there exists a non-zero constant c, such that for any $g \in G_n$, we have $j_{\pi_1}(g) = cj_{\pi_2}(g)$. Then $\pi_1 \cong \pi_2$.*

Proof. By Lemma 2.2 part (2) in [FLO12], the Bessel distributions are linear independent for two inequivalent smooth irreducible generic representations. This Corollary now follows directly from above Theorem 1.2.8. □

We recall that $N = N_n$ is the standard unipotent radical subgroup of G_n and w_{G_n} is the longest Weyl element in G_n. We embed G_{n-1} into G_n on the upper and left corner by the map $g_{n-1} \to \begin{pmatrix} g_{n-1} & \\ & 1 \end{pmatrix}$. We have the following kernel formula.

Corollary 7.1.5. *Let π be a generic smooth irreducible representation of G_n with the corresponding Bessel function $j_\pi(g)$. If $W_v(g)$ is a Whittaker function in the Whittaker model $\mathcal{W}(\pi, \psi)$, then for any diagonal matrix $b = \mathrm{diag}(b_1, \cdots, b_n) \in A_n \subseteq G_n$ and $W_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} \in C_c^\infty(N_{n-1}\backslash G_{n-1}, \psi)$, we have*

$$W_v(bw_{G_n}) = \int_{N_{n-1}\backslash G_{n-1}} j_\pi \left(bw_{G_n} \begin{pmatrix} h^{-1} & \\ & 1 \end{pmatrix} \right) W_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} dh.$$

Proof. By applying the local integrability of the Bessel function $j_\pi(g)$ and using the same method in Lemma 5.3 of [Ba04], we can show that the right hand side of above integral is absolutely convergent. This corollary now follows directly from Theorem 4.2 (the weak kernel formula) in [Chai17]. □

Remark 7.1.6. *The kernel formula in the above Corollary 7.1.5 actually gives the action of the longest Weyl element w_{G_n} on the Kirillov model of π. A special case of the kernel formula in* GL(2) *case is proved in Proposition 2.1.4 (Chapter 2).*

7.2 Kloosterman Zeta Functions

Non-trivial bounds on certain Kloosterman sums will give the non-trivial absolute convergence region for the Kloosterman zeta functions.

Let $M, N \in \mathbb{Z}$ and $s \in \mathbb{C}$. The Kloosterman zeta function for GL(2) is defined to be

$$Z(M, N; s) := \sum_{c=1}^{\infty} S(M, N; c) c^{-2s}, \tag{7.2}$$

where

$$S(M, N; c) = \sum_{a=1,(a,c)=1}^{c} e^{2\pi i(aM+\overline{a}N)/c}$$

for $a\overline{a} \equiv 1 \mod c$, is the classical Kloosterman sum. The Weil bound $S(M, N; c) = O_\epsilon(c^{\frac{1}{2}+\epsilon})$ [Wei48] shows that the Kloosterman zeta function (7.2) converges absolutely for $\mathrm{Re}(s) > \frac{3}{4}$.

The function (7.2) was first introduced by A. Selberg [Sel65] who obtained its meromorphic continuation to the whole complex s-plane. In [GS83], Goldfeld and Sarnak have shown that

$$Z(M, N; s) = O_\epsilon(|s|^{\frac{1}{2}+\epsilon})$$

for $\mathrm{Re}(s) > \frac{1}{2} + \epsilon$ and $|\mathrm{Im}(s)| > \epsilon$. This bound leads to the following Kuznetsov's Theorem [Kuz80]

$$\sum_{c \le x} \frac{S(M, N; c)}{c} = O_\epsilon(x^{\frac{1}{6}+\epsilon}).$$

In order to define the Kloosterman zeta function for GL(n), we recall a slightly different and general definition of local Kloosterman sums on $\mathrm{GL}_n(\mathbb{Q}_p)$ in Chapter 4.

We let ψ_p be a nontrivial additive character from $N(\mathbb{Q}_p)$ to $\mathbb{C}^\times$ which is trivial on $N(\mathbb{Z}_p)$. The character ψ_p has the following form:

$$\psi_p\left(\begin{pmatrix} 1 & x_1 & \cdots & * & * \\ 0 & 1 & x_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right) = \xi\left(\sum_{i=1}^{n-1} \nu_i x_i\right),$$

where ξ is the standard nontrivial additive character of $\mathbb{Q}_p$ as in [Ste87, Section 1] and $\nu_1, \nu_2, \cdots, \nu_{n-1} \in \mathbb{Z}_p - \{0\}$. We also assume that $\nu_i = \nu_{n-i}$ for all $1 \le i \le n-1$. Similarly, we can define an additive character from from $N(\mathbb{Q}_p)$ to $\mathbb{C}^\times$ which is trivial

on $N(\mathbb{Z}_p)$ if we are given the parameters $\nu'_1, \nu'_2, \nu'_3, \cdots, \nu'_{n-1} \in \mathbb{Z}_p - \{0\}$. The definition is as follows:

$$\psi'_p\left(\begin{pmatrix} 1 & x_1 & \cdots & * & * \\ 0 & 1 & x_2 & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & x_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\right) = \xi\left(\sum_{i=1}^{n-1} \nu'_i x_i\right).$$

We set $c = \mathrm{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \cdots, c_2/c_1, c_1) \in T(\mathbb{Q}_p)$. Following [Ste87, Section 2], for every $w \in W_{G_n}$, we define

$$C(wc) := N(\mathbb{Q}_p)wcN_w(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Z}_p),$$

and

$$X(wc) := N(\mathbb{Z}_p)\backslash C(wc)/N_w(\mathbb{Z}_p),$$

where $N_w := N \cap (wc)^{-1}\overline{N}wc$ and $\overline{N}$ is the opposite unipotent radical subgroup.

By the Bruhat decomposition we have natural maps

$$u : X(wc) \to N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p), \quad u' : X(wc) \to N_w(\mathbb{Q}_p)/N_w(\mathbb{Z}_p).$$

defined by the relation $x = u(x) \cdot wc \cdot u'(x)$ for $x \in X(wc)$.

Now the local Kloosterman sum for $\mathrm{GL}_n(\mathbb{Q}_p)$ is defined as follows:

$$Kl_p(\psi_p, \psi'_p; c, w) := \sum_{x \in X(wc)} \psi_p(u(x)) \cdot \psi'_p(u'(x)).$$

Note that the local Kloosterman sum will vanish automatically unless w is a relevant Weyl element and $c_i \in \mathbb{Z}_p$ for all $1 \leq i \leq n-1$ (See Chapter 3 and 4 for more details).

For the global Kloosterman sum on $\mathrm{GL}(n)$, we let $c = \mathrm{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \cdots, c_2/c_1, c_1) \in T(\mathbb{Q})$. Let $\psi = \prod_{p<\infty} \psi_p$ and $\psi' = \prod_{p<\infty} \psi'_p$ be additive characters of $N(\mathbb{A}_\mathbb{Q})$ (respectively $N_w(\mathbb{A}_\mathbb{Q})$), which is trivial on $\prod_{p<\infty} N(\mathbb{Z}_p)$ (respectively $\prod_{p<\infty} N_w(\mathbb{Z}_p)$). The global Kloosterman sum associated to this data is

$$Kl(\psi, \psi'; c, w) := \prod_{p<\infty} Kl_p(\psi_p, \psi'_p; c_p, w).$$

Note that the global Kloosterman sum will vanish automatically unless $c_i \in \mathbb{Z}$ for all $1 \leq i \leq n-1$.

Now we are ready to define the Kloosterman zeta function for $\mathrm{GL}(n)$.

Let $c = \mathrm{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \cdots, c_2/c_1, c_1) \in T(\mathbb{Q})$ and $s = (s_1, s_2, \cdots, s_{n-1}) \in \mathbb{C}^{n-1}$. The Kloosterman zeta function associated to the relevant Weyl element w is defined to be

$$Z_w(\psi, \psi'; s) := \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} Kl(\psi, \psi'; c, w) c_1^{-ns_1} \cdots c_{n-1}^{-ns_{n-1}}. \tag{7.3}$$

The global Kloosterman zeta function for $\mathrm{GL}(n)$ is defined to be

$$Z(\psi, \psi'; s) := \sum_{w \in W_{G_n}} Z_w(\psi, \psi'; s). \tag{7.4}$$

Note that the trivial bound for the local Kloosterman sum (integral) (See [DR98]) is the following:

$$\left|Kl_p(\psi_p, \psi'_p; c, w)\right| \leq A_\epsilon \cdot \left|\prod_{i=1}^{n-1} c_i\right|^{1+\epsilon}, \ \epsilon > 0, \tag{7.5}$$

where A_ϵ is a positive constant independent on the choice of c. By this "trivial" estimation, we have the following absolute convergence region for the Kloosterman zeta function.

Theorem 7.2.1. *The global Kloosterman zeta function $Z(\psi, \psi'; s)$ converges absolutely for $\mathrm{Re}(s_i) > \frac{2}{n}$, where $1 \leq i \leq n-1$.*

The non-trivial upper bound for the local Kloosterman sum is

$$\left|Kl_p(\psi_p, \psi'_p; c, w)\right| \leq A_\epsilon \cdot \left|\prod_{i=1}^{n-1} c_i\right|^{1-\delta_n+\epsilon}, \ \epsilon > 0, \tag{7.6}$$

for some $\delta_n > 0$, and A_ϵ is a positive constant independent on the choice of c.

Non-trivial bounds on certain Kloosterman sums will give the non-trivial absolute convergence region for the Kloosterman zeta function $Z(\psi, \psi'; s)$.

Theorem 7.2.2. *Assume that the local Kloosterman sum has the above non-trivial upper bound* (7.6)*, then the global Kloosterman zeta function $Z(\psi, \psi'; s)$ converges absolutely for $\mathrm{Re}(s_i) > \frac{2-\delta_n}{n}$, where $1 \leq i \leq n-1$.*

Remark 7.2.3. *If $w = w_{G_n}$, which is the longest Weyl element, we have the non-trivial upper bound for the local Kloosterman sum. From Theorem 5.0.1 and A.0.1, we can pick $\delta_n = \frac{1}{4n^2-18n+22}$. Moreover, for $n = 4$, we have a stronger bound $\delta_4 = \frac{1}{8}$.*

For other relevant Weyl elements, we only have some partial results in the small rank $\mathrm{GL}(n)$ *($n \leq 4$) case. For $n = 2$, the celebrated Weil bound [Wei48] tells that $\delta_2 = \frac{1}{2}$. For $n = 3$, by the careful analysis in [DF97], we have $\delta_3 = \frac{1}{4}$, which is a sharp bound for all the relevant Weyl elements. For $n = 4$, we know the existence of $\delta_4 > 0$ in [BM22].*

An ambitious conjecture of G. Stevens reads as follows:

Conjecture 7.2.4. *[Ste87] The global Kloosterman zeta function $Z(\psi, \psi'; s)$ converges absolutely for* $\mathrm{Re}(s_i) > \frac{7}{4n}$, *for all $n \geq 3$ and $1 \leq i \leq n-1$.*

Remark 7.2.5. *The conjecture 7.2.4 is only known for $n = 3$ and is widely open for $n \geq 4$.*

We may also expect the following result:

Conjecture 7.2.6. *The global Kloosterman zeta function $Z(\psi, \psi'; s)$ has a meromorphic continuation to the whole complex s-plane.*

Chapter 8

Automorphic Spectral Reciprocity Formula: Preliminaries

In the second part of my thesis, we will focus on automorphic spectral reciprocity formulae. Readers may see Section 1.3 for introduction. We start with some preliminaries.

8.1 Number Fields and Local Fields

Let $F/\mathbb{Q}$ be a fixed number field with ring of intergers $\mathcal{O}_F$ and discriminant Δ_F.

For a place v of F, we let F_v be a local field which is the completion of F at the place v. If v is non-Archimedean, we write $\mathfrak{o}_v$ for the ring of integers in F_v with maximal ideal $\mathfrak{m}_v$ and uniformizer ϖ_v. The cardinality of the residue field is $p_v := |\mathfrak{o}_v/\mathfrak{m}_v|$. For $s \in \mathbb{C}$, we define the local zeta function $\zeta_{F_v}(s)$ to be $(1-p_v^{-s})^{-1}$ if v is a finite place; $\zeta_{F_v}(s) = \pi^{-s/2}\Gamma(s/2)$ if v is real and $\zeta_{F_v}(s) = 2(2\pi)^{-s}\Gamma(s)$ if v is complex.

The adele ring of F is denoted by $\mathbb{A}_F$ and its unit group is given by $\mathbb{A}_F^\times$ (idele group). We also let $\mathbb{A}_F^1 := \{x \in \mathbb{A}_F^\times \ : \ |x| = 1\}$, where $|\cdot| : \mathbb{A}_F^\times \to \mathbb{R}_{>0}$ is the adelic norm map. Note that $\mathbb{A}_F^1$ is exactly the kernel of the adelic norm map. We also call $\mathbb{A}_F^1$ norm one ideles.

We fix $\psi = \prod_v \psi_v$ be the additive character with the form as $\psi_\mathbb{Q} \circ \mathrm{Tr}_{F/\mathbb{Q}}$, where $\mathrm{Tr}_{F/\mathbb{Q}}$

is the trace map and $\psi_{\mathbb{Q}}$ is the standard additive character on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$. For $v<\infty$, we let d_v be the conductor of additive character ψ_v, which is the smallest non-negative integer such that ψ_v is trivial on $\mathfrak{m}_v^{d_v}$. In this case, we will have $\Delta_F=\prod_{v<\infty}q_v^{d_v}$. We may also set $d_v=0$ when v is Archimedean.

8.2 Subgroups of $\mathrm{GL}(n)$ and Measure

Now we consider some subgroups of $\mathrm{GL}(n)$.

Let $\mathrm{G}=\mathrm{GL}(n)$. If R is a field, by definition, $\mathrm{G}(R)$ is the group of $n\times n$ matrices with coefficients in R and determinant in $R-\{0\}$. We also define the following standard and also important subgroups

$$\mathrm{Z}_n(R):=\left\{\begin{pmatrix}u&&&\\&u&&\\&&\cdots&\\&&&u\end{pmatrix},\ u\in R^*\right\},$$

$$\mathrm{N}_n(R):=\left\{\begin{pmatrix}1&x_{1,2}&x_{1,3}&\cdots&x_{1,n}\\&1&x_{2,3}&\cdots&x_{2,n}\\&&\cdots&\cdots&\cdots\\&&&1&x_{n-1,n}\\&&&&1\end{pmatrix},\ x_{i,j}\in R,\ 1\le i<j\le n\right\},$$

$$\mathrm{A}_n(R):=\left\{\begin{pmatrix}y_{n-1}\cdots y_1&&&&\\&y_{n-2}\cdots y_1&&&\\&&\cdots&&\\&&&y_1&\\&&&&1\end{pmatrix},\ y_i\in R^*,\ 1\le i\le n-1\right\},$$

and the Borel subgroup

$$\mathrm{B}_n(R):=\mathrm{Z}_n(R)\mathrm{N}_n(R)\mathrm{A}_n(R).$$

We let W be the Weyl group of G. We write

$$w_0 := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

be the longest Weyl element.

Moreover, for any place v, we let K_v be the maximal compact subgroup of $\mathrm{G}(F_v)$ which is defined by

$$\mathrm{K}_v = \begin{cases} \mathrm{G}(\mathfrak{o}_v) & \text{if} \quad v \text{ is finite} \\ \mathrm{O}_n(\mathbb{R}) & \text{if} \quad v \text{ is real} \\ \mathrm{U}_n(\mathbb{C}) & \text{if} \quad v \text{ is complex.} \end{cases} \tag{8.1}$$

We also let $\mathrm{K} := \prod_v \mathrm{K}_v$. If $v < \infty$ and $n \geqslant 0$, we define the congruence subgroup [JPS81] by (This will be used in the computation of local vectors and is useful in the local new-vector theory)

$$\mathrm{K}_{v,0}(\varpi_v^n) = \left\{ \begin{pmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ k_{n,1} & k_{n,2} & \cdots & k_{n,n} \end{pmatrix} \in \mathrm{K}_v, \quad k_{n,i} \in \mathfrak{m}_v^n, \ 1 \leq i \leq n-1 \right\}.$$

If $\mathfrak{a}$ is an ideal of $\mathcal{O}_F$ with prime decomposition $\mathfrak{a} = \prod_{v<\infty} \mathfrak{p}_v^{f_v(\mathfrak{a})}$ where $\mathfrak{p}_v$ is the unique prime (also maximal) ideal corresponding to the finite place v, then we define

$$\mathrm{K}_0(\mathfrak{a}) := \prod_{v<\infty} \mathrm{K}_{v,0}(\varpi_v^{f_v(\mathfrak{a})}) \subseteq K \subseteq \mathrm{GL}_n(\mathbb{A}_F).$$

Now we have to normalize the measures we need.

At each place v, dx_v denotes a self-dual measure on F_v with respect to the standard additive character ψ_v. If $v < \infty$, dx_v gives a Haar measure on F_v which gives the volume $q_v^{-d_v/2}$ to the integer ring $\mathfrak{o}_v$. If v is real, the measure dx_v is the standard Lebesgue measure on $\mathbb{R}$. If v is complex, the measure dx_v is the multiplication of two and the

standard Lebesgue measure on $\mathbb{C}$. We define $dx := \prod_v dx_v$ on $\mathbb{A}_F$. Moreover, we will take $d^\times x_v := \zeta_{F_v}(1)\frac{dx_v}{|x_v|}$ as the Haar measure on the multiplicative group $F_v^\times$ and we let $d^\times x := \prod_v d^\times x_v$ be the Haar measure on the idele group $\mathbb{A}_F^\times$.

We equip K_v with the probability Haar measure dk_v. In other word, the volume of K_v equals to one.

Using the Iwasawa decomposition on $\mathrm{G}(F_v)$ which gives $\mathrm{G}(F_v) = \mathrm{Z}_n(F_v)\mathrm{N}_n(F_v)\mathrm{A}_n(F_v)\mathrm{K}_v$, a Haar measure on $\mathrm{G}(F_v)$ can be given by

$$dg_v = d^\times u \prod_{1\le i<j\le n} dx_{i,j} \frac{\prod_{i=1}^{n-1} d^\times y_i}{\delta(\mathrm{A}_n)} dk_v, \tag{8.2}$$

where $\delta(A_n) = \delta(y_1, y_2, \cdots, y_{n-1})$ is the modular character defined on $\mathrm{A}_n(F_v)$. The measure on the adelic points of the subgroups in $\mathrm{GL}(n)$ are just the product of the local measures defined above. We also denote by dg the quotient measure on the space

$$\mathrm{X} := \mathrm{Z}_n(\mathbb{A}_F)\mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}_F),$$

with total volume $\mathrm{vol}(\mathrm{X}) < \infty$.

8.3 Whittaker functions

We recall some basic background of Whittaker functions.

We start from the generic representations. Let π be a global genreic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and let $\phi \in \pi$ be a generic automorphic form. Let $W_\phi : \mathrm{GL}_n(\mathbb{A}_F) \to \mathbb{C}$ be the Whittaker function of ϕ which is given by

$$W_\phi(g) := \int_{\mathrm{N}_n(F)\backslash\mathrm{N}_n(\mathbb{A}_F)} \phi(ng)\,\psi_1^{-1}(n)dn. \tag{8.3}$$

We will give the definition of the additive character ψ_1 later.

Since π is generic, the Whittaker function does not vanish. The integral defined above (8.3) is absolute convergent since the integral domain $\mathrm{N}_n(F)\backslash\mathrm{N}_n(\mathbb{A}_F)$ is compact and the integral is moreover uniformly comvergent on any compact subsets in $\mathrm{GL}_n(\mathbb{A}_F)$.

By changing variables, we note that $W_\phi(ng) = \psi_1(n)W_\phi(g)$ for all $n \in \mathrm{N}_n(\mathbb{A}_F)$, where ψ_1 is a nontrivial standard additive character from $\mathrm{N}_n(F)\backslash\mathrm{N}_n(\mathbb{A}_F)$ to $\mathbb{C}$. In our paper, we will choose that $\psi_1(n) := \psi\left(\sum_{i=1}^{n-1} x_{i,i+1}\right)$, where $n = (x_{i,j}) \in \mathrm{N}_n(\mathbb{A}_F)$. Here ψ is a standard additive character from $\mathbb{A}_F/F$ to $\mathbb{C}$ which is defined in Section 2.1.

We also have the following Fourier series expansion (see [Cog07, Theorem 1.1]) when we further assume that ϕ is a cusp form. Therefore, a cusp form is automatically generic by the following equation.

$$\phi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\phi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right). \tag{8.4}$$

We can also define local Whittaker functions for smooth generic admissible representations of GL_n over local fields F_v. We have the following well known decomposition theorem. If π is a generic smooth irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then we know that π factors as a restricted tensor product by $\pi \simeq \otimes'_v \pi_v$. For each v, π_v is a local generic smmoth irreducible admissible representation of $\mathrm{GL}_n(F_v)$. For each local place v, we can define local Whittaker functions. For every $\phi \in \pi$, if we write $\phi = \otimes'_v \phi_v$, then we have the decomposition

$$W_\phi(g) = \prod_v W_{\phi_v}(g_v), \text{ where } g = (g_v)_v \in \mathrm{GL}_n(\mathbb{A}_F). \tag{8.5}$$

In fact, we know that the map $\phi \mapsto W_\phi$ intertwines the space of π and the space

$$\mathcal{W}(\pi, \psi_1) := \{W_\phi;\ \phi \in \pi\},$$

which is called the Whittaker model of π. We can similarly define the local Whittaker models $\mathcal{W}(\pi_v, \psi_{1,v})$.

It is also important to consider Whittaker functions with respect to the character $\psi'_1 = \psi_1^{-1}$ since it will appear in the integral representations for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ Rankin-Selberg L-functions (see Section 3.2). It also has a close relation with the Whittaker model of the contragredient representation $\widetilde{\pi}$ when the representation π is unitarizable (see [FLO12, Appendix A]). Locally, we can define W'_{ϕ_v} by simply replacing $\psi_{1,v}$ by $\psi'_{1,v} = \psi_{1,v}^{-1}$. Globally, we can define W'_ϕ by replacing ψ_1 by ψ'_1. Moreover, we have the following equation

$$W'_{\overline{\phi_v}} = \overline{W_{\phi_v}} \text{ for all places } v \text{ of } F. \tag{8.6}$$

For $W \in \mathcal{W}(\pi_v, \psi_{1,v})$, the corresponding Whittaker model $\mathcal{W}(\widetilde{\pi}_v, \psi_{1,v}^{-1})$ for contragredient representation is given by $\widetilde{W}(g) := W\left(w_0 (g^t)^{-1}\right)$, where w_0 is the longest Weyl

element in $\mathrm{GL}(n)$ and $(g^t)^{-1}$ means the transpose inverse, therefore it is an involution in $\mathrm{GL}(n)$. The global Whittaker model $\mathcal{W}(\widetilde{\pi}, \psi_1^{-1})$ is defined in the same way. If a local generic smooth irreducible admissible representation π_v is unramified, this is equivalent to say that there exists a vector which is right invariant by the action of the maximal compact subgroup of $\mathrm{G}(F_v)$ in the space of π. We call such a vector spherical vector and spherical vectors are unique up to multiplication by scalars. We say that a vector $\phi_v \in \pi_v$ is normalized spherical if it is spherical. Moreover, its related Whittaker function W_{ϕ_v} satisfies $W_{\phi_v}(I_{n\times n}) = 1$.

8.4 Spectral Decomposition

Now we consider the L^2 space $L^2(\mathrm{X})$ which is the Hilbert space of complex valued square integrable functions with the domain X. The L^2-norm is defined by

$$||\varphi||^2_{L^2(\mathrm{X})} := \int_{\mathrm{X}} |\varphi(g)|^2 dg. \tag{8.7}$$

For any $\varphi_1, \varphi_2 \in L^2(\mathrm{X})$, we have

$$(\varphi_1, \varphi_2)_{L^2(\mathrm{X})} := \int_{\mathrm{X}} \varphi_1(g)\overline{\varphi_2(g)} dg.$$

An important closed subspace of $L^2(\mathrm{X})$ is the closed subspace of cusp forms. We let $L^2_{\mathrm{cusp}}(\mathrm{X})$ be the closed subspace of cusp forms. A cusp form is the function $\varphi \in L^2(\mathrm{X})$ with the following additional equation

$$\int_{\mathrm{N}_n(F)\backslash \mathrm{N}_n(\mathbb{A}_F)} \varphi(ng) dn = 0,$$

for almost all $g \in \mathrm{GL}_n(\mathbb{A}_F)$.

The group $\mathrm{G}(\mathbb{A}_F)$ acts by right translations on both spaces $L^2(\mathrm{X})$ and $L^2_{\mathrm{cusp}}(\mathrm{X})$ and the corresponding representation is unitary with respect to the norm in (8.7). The decomposition theorem of automorphic representations is well known, which states that each irreducible component π factors as a restricted tensor product $\pi = \otimes'_v \pi_v$ for all places k, where π_v are irreducible and unitary representations of the local groups $\mathrm{G}(F_v)$. The Langlands theorem on the spectral decomposition as in [MW95] gives the following:

$$L^2(\mathrm{X}) = L^2_{\mathrm{cusp}}(\mathrm{X}) \oplus L^2_{\mathrm{res}}(\mathrm{X}) \oplus L^2_{\mathrm{cont}}(\mathrm{X}). \tag{8.8}$$

Here the closed subspace $L^2_{\mathrm{cusp}}(\mathrm{X})$ decomposes as a direct sum of irreducible $\mathrm{G}(\mathbb{A}_F)$-representations which are called the cuspidal automorphic representations. $L^2_{\mathrm{res}}(\mathrm{X})$ is called the residue spectrum which is the direct sum of all residue automorphic representations of $L^2(\mathrm{X})$. Finally, the continuous part $L^2_{\mathrm{cont}}(\mathrm{X})$ is a direct integral of irreducible $\mathrm{G}(\mathbb{A}_F)$-representations and it is expressed by the meromorphic continuation of Eisenstein series.

For any ideal $\mathfrak{a}$ of $\mathcal{O}_F$, we let $L^2(\mathrm{X},\mathfrak{a}) := L^2(\mathrm{X})^{\mathrm{K}_0(\mathfrak{a})}$ be the closed subspace of level $\mathfrak{a}$ automorphic forms. This is the closed subspace of L^2 functions that are invariant under the subgroup $\mathrm{K}_0(\mathfrak{a})$.

We now have the following spectral orthogonal decomposition with the level restriction:

$$L^2(\mathrm{X},\mathfrak{a}) = L^2_{\mathrm{cusp}}(\mathrm{X},\mathfrak{a}) \oplus L^2_{\mathrm{res}}(\mathrm{X},\mathfrak{a}) \oplus L^2_{\mathrm{cont}}(\mathrm{X},\mathfrak{a}). \tag{8.9}$$

8.5 Automorphic Representations

Now we consider automorphic representations. Let $\hat{\mathrm{X}}$ be the isomorphism class of unitary irreducible automorphic representations which will appear in the spectral decomposition of $L^2(\mathrm{X})$. Since we will later use the integral representation of $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$ Rankin-Selberg L-functions, we will only focus on the unitary irreducible generic automorphic representations. We will consider $\hat{\mathrm{X}}_{\mathrm{gen}}$ be the subset of $\hat{\mathrm{X}}$ which is the isomorphism class of generic representations in $\hat{\mathrm{X}}$, which is the unitary irreducible automorphic representation class that have (unique) Whittaker models. We fix an automorphic Plancherel measure $d\mu_{\mathrm{aut}}$ on $\hat{\mathrm{X}}$ which is compatible with the Haar measure on X.

Fortunately, we have the Langlands classification for $\hat{\mathrm{X}}_{\mathrm{gen}}$ (see [CPS94] and [MW95]). We take a partition $n = r_1 + \cdots + r_k$. Let π_j be a unitary cuspidal automorphic representation for $\mathrm{GL}_{r_j}(\mathbb{A}_F)$ (If $r_j = 1$, we simply take π_j to be a unitary Hecke character). We now consider the unitary induced representation Π from the Levi subgroup $\mathrm{GL}(r_1)\times\cdots\times\mathrm{GL}(r_k) \subseteq \mathrm{GL}(n)$ to $\mathrm{GL}(n)$ with the tensor product representation $\pi_1\otimes\cdots\otimes\pi_k$. There exists a unique irreducible constituent of Π which we denote by the isobaric sum $\pi_1\boxplus\cdots\boxplus\pi_k$. Then, Langlands classification says that every element in $\hat{\mathrm{X}}_{\mathrm{gen}}$ is isomorphic to such an isobaric sum.

Moreover, we know that all residual automorphic representations in $L^2_{\mathrm{res}}(\mathrm{X})$ is not

generic (see [JL13] Proposition 2.1).

Now we note that all the unitary generic Eisenstein series will have the form of the isobaric sum $\pi_1|\cdot|^{it_1} \boxplus \cdots \boxplus \pi_k|\cdot|^{it_k}$, where $t_1, t_2, \cdots, t_k \in \mathbb{R}$. We will write

$$\pi = \pi(\pi_1, \cdots, \pi_k, t_1, \cdots, t_k) = \pi_1|\cdot|^{it_1} \boxplus \cdots \boxplus \pi_k|\cdot|^{it_k}$$

with parameters $t_1, \cdots, t_k$ $(k \geq 2)$.

8.6 Spectral Decomposition Formula

We have the following abstract spectral decomposition formula in $\mathrm{GL}(n)$ case (see [MV10, Section 2.2]). The discrete part is generated by cusp forms and residue representations. The continuous spectrum part which is expressed by Eisenstein series is complicated. However, we know that it depends on the partition of the positive integer n.

Proposition 8.6.1. *Suppose that* $\mathrm{X} := \mathrm{Z}_n(\mathbb{A}_F)\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$. *Let* $F \in C^\infty(\mathrm{X}/\mathrm{K}_0(\mathfrak{a}))$ *and of rapid decay, then we have the following equation:*

$$\begin{aligned} F(g) &= \int_{\substack{\pi \in \hat{\mathrm{X}} \\ \mathrm{cond}(\pi)|\mathfrak{a}}} \left(\sum_{\phi \in \mathcal{B}(\pi, \mathfrak{a})} \langle F, \phi \rangle \phi(g) \right) d\mu_{\mathrm{aut}}(\pi), \\ &= \int_{\substack{\pi \text{ automorphic} \\ \mathrm{cond}(\pi)|\mathfrak{a}}} \left(\sum_{\phi \in \mathcal{B}(\pi, \mathfrak{a})} \langle F, \phi \rangle \phi(g) \right) d\mu_{\mathrm{aut}}(\pi), \\ &= \sum_{\substack{\pi \text{ cuspidal} \\ \mathrm{cond}(\pi)|\mathfrak{a}}} \sum_{\phi \in \mathcal{B}(\pi, \mathfrak{a})} \langle F, \phi \rangle \phi(g) + \sum_{\substack{\pi \text{ residue} \\ \mathrm{cond}(\pi)|\mathfrak{a}}} \sum_{\phi \in \mathcal{B}(\pi, \mathfrak{a})} \langle F, \phi \rangle \phi(g) \\ &+ \int_{\substack{\pi \text{ continuous} \\ \mathrm{cond}(\pi)|\mathfrak{a}}} \left(\sum_{\phi \in \mathcal{B}(\pi, \mathfrak{a})} \langle F, \phi \rangle \phi(g) \right) d\mu_{\mathrm{aut}}(\pi). \end{aligned} \tag{8.10}$$

Remark 8.6.2. *Above Proposition 8.6.1 is a smooth version of the spectral decomposition 8.9.*

Chapter 9

Rankin-Selberg L-functions

9.1 Rankin-Selberg L-functions

We need to recall integral representations of $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ $(n \geq 3)$ L-functions.

The theory is quite similar to the adelic Hecke-Jacquet-Langlands' theory [God18] of twisted L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_1$. We let Π be irreducible automorphic representations of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$. Let $\Phi \in \Pi$ be an automorphic form. Let π be irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and let $\phi \in \pi$ be an automorphic form. We first assume that Φ is a cusp form and is of rapid decay. Therefore, for every $s \in \mathbb{C}$, we can consider the following period integral

$$I(s, \Phi, \phi) := \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)} \Phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) |\det h|^{s-\frac{1}{2}} dh,$$

which defines an entire function of s and is bounded on vertical strips.

From the Whittaker-Fourier expansion of cusp forms (8.4) [Cog07, Theorem 1.1], if Φ is a cusp form, we will have (for $\mathrm{Re}(s)$ large enough)

$$I(s, \Phi, \phi) = \Psi(s, W_\Phi, W'_\phi), \quad (\mathrm{Re}(s) \gg 1), \tag{9.1}$$

where the global zeta integral is given by

$$\Psi(s, W_\Phi, W'_\phi) := \int_{N_n(\mathbb{A}_F)\backslash \mathrm{GL}_n(\mathbb{A}_F)} W_\Phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} W'_\phi(h) |\det h|^{s-\frac{1}{2}} dh. \tag{9.2}$$

The following result can be found in [Cog07] [JPS79] [JPS83] [JS90] [Jac09].

Proposition 9.1.1. *Let $\Phi = \otimes_v' \Phi_v \in \Pi$ and $\phi = \otimes_v' \phi_v \in \pi$ be factorizable automorphic forms on $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_F)$. Let W_{Φ_v} and W'_{ϕ_v} be the corresponding Whittaker functions defined in Section 2.3. Then, for $\mathrm{Re}(s)$ large enough, the global zeta integral $\Phi(s, W_\Phi, W'_\phi)$ converges and we have the following factorization (Euler product)*

$$\Psi(s, W_\Phi, W'_\phi) = \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\phi_v}),$$

where the local zeta integral is given by

$$\Psi_v(s, W_{\Phi_v}, W'_{\phi_v}) := \int_{N_n(F_v)\backslash \mathrm{GL}_n(F_v)} W_{\Phi_v} \begin{pmatrix} h_v & \\ & 1 \end{pmatrix} W'_{\phi_v}(h_v) |\det h_v|^{s-\frac{1}{2}} dh_v. \tag{9.3}$$

Moreover, if both Π_v and π_v are unramified and Φ_v and ϕ_v are normalized spherical vectors, we will have

$$\Psi_v(s, W_{\Phi_v}, W'_{\phi_v}) = L(s, \Pi_v \times \pi_v).$$

One of the key ingredients in our paper is the following generalization of integral representation on Rankin-Selberg L-functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ given by Ichino and Yamana [IY15].

Proposition 9.1.2. *Let $\Phi \in \Pi$ and $\phi \in \pi$ be automorphic forms on $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_F)$. Assume that the following period integral*

$$I(s, \Phi, \phi) := \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} \Phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) |\det h|^{s-\frac{1}{2}} dh$$

is absolute convergent, we will still have the following equation for $\mathrm{Re}(s)$ large enough:

$$I(s, \Phi, \phi) = \Psi(s, W_\Phi, W'_\phi), \quad (\mathrm{Re}(s) \gg 1), \tag{9.4}$$

where

$$\Psi(s, W_\Phi, W'_\phi) := \int_{N_n(\mathbb{A}_F)\backslash \mathrm{GL}_n(\mathbb{A}_F)} W_\Phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} W'_\phi(h) |\det h|^{s-\frac{1}{2}} dh. \tag{9.5}$$

Proof. This is a combination of Corollary 3.10 and Main Theorem (Theorem 1.1) of Ichino and Yamana's paper [IY15]. □

9.2 Regularization

Given an automorphic form ϕ in a generic representation π of $\mathrm{GL}_n(\mathbb{A})$, we can write its Fourier expansion using W_ϕ. For example, if ϕ is cuspidal then we can write

$$\phi(g) = \sum_{\gamma \in N_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\phi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right)$$

by [Cog07, Theorem 1.1]. The above Fourier expansions converge absolutely and uniformly on compact subsets.

If ϕ is not cuspidal then the Fourier expansion of ϕ is more complicated. Readers may find [IY15, Proposition 4.2] as a good reference. We do not need full Fourier expansion for non-cuspidal automorphic forms. However, we only need a partial Fourier expansion with respect to the unipotent subgroup U_n, where U_n is the unipotent radical of the parabolic subgroup in $\mathrm{GL}(n)$ with respect to the partition $n = (n-1) + 1$.

From abelian Fourier theory we have

$$(\phi - \phi_{U_n})(g) = \sum_{\gamma \in P_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\phi^{U_n} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right). \tag{9.6}$$

Here P_n is the standard mirabolic subgroup of $\mathrm{GL}(n)$ defined by $\mathrm{GL}(n-1) \rtimes U_n$. In other words, P_n is the stabilizer of $(0, 0, \cdots, 0, 1)$ of the right action of $\mathrm{GL}(n)$ on the row vectors, thus consists of matrices in $\mathrm{GL}(n)$ with last row being $(0, 0, \cdots, 0, 1)$. ϕ_{U_n} is the constant term of ϕ along U_n defined by

$$\phi_{U_n}(g) := \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \phi(ug) du$$

and $W_\phi^{U_n}$ is a partial Whittaker function defined by

$$W_\phi^{U_n}(g) := \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \phi(ug)\overline{\psi(u)} du = \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} (\phi(ug) - \phi_{U_n}(ug))\overline{\psi(u)} du, \tag{9.7}$$

which follows as ϕ_{U_n} is left $U_n(\mathbb{A}_F)$-invariant.

Now we are ready to formulate the regularization of the global L-functions of $\sigma \times \pi$ on $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$ while σ is not a cuspidal representation. In this case, the usual zeta integral which is divergent. We have the following proposition.

Proposition 9.2.1. *[JN21, Proposition 4.1]*

Let φ be a generic automorphic form on $\mathrm{GL}_n(\mathbb{A}_F)$ and ϕ be a cusp form on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ with trivial central character. Then for $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ large enough

$$\int_{\mathrm{GL}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \left(\varphi\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) - \varphi_{U_n}\left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right) \right) \overline{\phi(g)} \cdot |\det(g)|^{s-\frac{1}{2}} dg$$

is absolutely convergent and equals to $\Psi(s, W_\varphi, \overline{W_\phi})$.

Chapter 10

Abstract Spectral Reciprocity Formula: Theorem 1.3.1 and 10.2.2

10.1 Abstract Spectral Identity

We can briefly summarize the proof of Theorem 10.2.2 now. The proof of Theorem 10.2.2 is a combination of above decomposition formula 8.6.1 and an identity between two periods (Abstract Reciprocity Formula, Theorem 1.3.1). Later, we will relate the period to moments of certain L-functions. The proof of an identity between two periods is a rather simple matrix computation and is really important to abstract pre-spectral reciprocity formula.

Suppose that $\Phi \in C^\infty(Z_{n+1}(\mathbb{A}_F)\mathrm{GL}_{n+1}(F)\backslash \mathrm{GL}_{n+1}(\mathbb{A}_F))$ is a cuspidal automorphic form. Therefore, it is of rapid decay.

Then, we can define the projection by

$$\mathcal{A}_s\Phi(h_n) := |\det h_n|^{s-\frac{1}{2}} \cdot \int_{F^\times\backslash\mathbb{A}_F^\times} \Phi\left(\begin{pmatrix} z_n(u)h_n & \\ & 1 \end{pmatrix}\right) |u|^{n(s-\frac{1}{2})} d^\times u. \tag{10.1}$$

Since Φ is of rapid decay, the above average projection map over the center $\mathcal{A}_s\Phi$ is well-defined for every complex number s and is again of rapid decay in terms of $h_n \in \mathrm{GL}(n)$. Moreover, we may easily check that $\mathcal{A}_s\Phi$ is invariant under the action of

the center $\mathrm{Z}_n(\mathbb{A}_F)$.

We also give the following definition of the period integral

$$I(w,\phi,\varphi) := \int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \phi\begin{pmatrix} h_{n-1} & \\ & 1 \end{pmatrix}\varphi(h_{n-1})|\det h_{n-1}|^{w-\frac{1}{2}}dh_{n-1},$$

whenever it is converges. Here ϕ is a smooth function defined on $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)$ and φ is a fixed everywhere unramified cuspidal automorphic form for $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Moreover, we assume that φ is invariant under the center $\mathrm{Z}_n(\mathbb{A}_F)$, therefore its central character is trivial. If $n = 2$, we see that $\varphi = 1$ which is the trivial character. We can now state the abstract spectral reciprocity formula.

Theorem 10.1.1 (Theorem 1.3.1)**.** *[Abstract Reciprocity Formula]*

Let $\Phi \in C^\infty(Z_{n+1}(\mathbb{A}_F)\mathrm{GL}_{n+1}(F)\backslash\mathrm{GL}_{n+1}(\mathbb{A}_F))$ be a cusp form. Then, for every s, $w \in \mathbb{C}$, we have the following abstract reciprocity equation

$$I(w,\mathcal{A}_s\Phi,\varphi) = I(w',\mathcal{A}_{s'}\check{\Phi},\varphi),$$

where $s' = \frac{1+(n-1)w-s}{n}$, $w' = \frac{(n+1)s+w-1}{n}$, and $\check{\Phi}$ is given by

$$\check{\Phi} = \Pi(w_{12})\cdot\Phi, \quad w_{12} := \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{10.2}$$

Proof. From the definition, we may write

$$\begin{aligned} I(w,\mathcal{A}_s\Phi,\varphi) &= \int_{F^\times\backslash\mathbb{A}^\times}\int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \Phi\left(\begin{pmatrix} z_{n-1}(u)h_{n-1} & & \\ & u & \\ & & 1 \end{pmatrix}\right)\varphi(h_{n-1}) \\ &\quad\cdot|u|^{n(s-\frac{1}{2})}|\det h_{n-1}|^{s+w-1}dh_{n-1}d^\times u \\ &= \int_{F^\times\backslash\mathbb{A}^\times}\int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \Phi\left(\begin{pmatrix} z_{n-1}(u)h_{n-1} & & \\ & u & \\ & & 1 \end{pmatrix}\right)\varphi(h_{n-1}) \\ &\quad\cdot|u|^{n(s-\frac{1}{2})}|\det h_{n-1}|^{s+w-1}dh_{n-1}d^\times u. \end{aligned} \tag{10.3}$$

Since Φ is a cusp form, our integral is well-defined for all complex parameters s and w. Now, since Φ is left invariant by $Z_{n+1}(\mathbb{A})\mathrm{GL}_{n+1}(F)$, we see that for every u, h_{n-1}, we have (Note that $w_{12} \cdot w_{12} = I_{n+1}$)

$$\begin{aligned}
&\Phi\left(\begin{pmatrix} z_{n-1}(u)h_{n-1} & & \\ & u & \\ & & 1 \end{pmatrix}\right) \\
=&\Phi\left(\begin{pmatrix} z_{n-1}(u) & & \\ & u & \\ & & u \end{pmatrix}\begin{pmatrix} h_{n-1} & & \\ & 1 & \\ & & u^{-1} \end{pmatrix}\right) \\
=&\Phi\left(\begin{pmatrix} z_{n-1}(u) & & \\ & u & \\ & & u \end{pmatrix} w_{12}\begin{pmatrix} h_{n-1} & & \\ & u^{-1} & \\ & & 1 \end{pmatrix} w_{12}\right) \\
=&\Phi\left(\begin{pmatrix} h_{n-1} & & \\ & u^{-1} & \\ & & 1 \end{pmatrix} w_{12}\right) \\
=&\check{\Phi}\left(\begin{pmatrix} h_{n-1} & & \\ & u^{-1} & \\ & & 1 \end{pmatrix}\right).
\end{aligned} \tag{10.4}$$

This gives that

$$\begin{aligned}
I(w, \mathcal{A}_s\Phi, \varphi) = &\int_{F^\times\backslash\mathbb{A}^\times}\int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \Phi\left(\begin{pmatrix} z_{n-1}(u)h_{n-1} & & \\ & u & \\ & & 1 \end{pmatrix}\right)\varphi(h_{n-1}) \\
&\cdot |u|^{n(s-\frac{1}{2})}|\det h_{n-1}|^{s+w-1}dh_{n-1}d^\times u \\
= &\int_{F^\times\backslash\mathbb{A}^\times}\int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \check{\Phi}\left(\begin{pmatrix} h_{n-1} & & \\ & u^{-1} & \\ & & 1 \end{pmatrix}\right)\varphi(h_{n-1}) \\
&\cdot |u|^{n(s-\frac{1}{2})}|\det h_{n-1}|^{s+w-1}dh_{n-1}d^\times u.
\end{aligned} \tag{10.5}$$

Applying this to (10.3) and using the change of variables which is given by $(u, h_{n-1}) = (u'^{-1}, z_{n-1}(u')h'_{n-1})$. We will see that the following equations hold:

$$n(s' - \frac{1}{2}) = n(\frac{1}{2} - s) + (n-1)(s + w - 1)$$

and

$$s' + w' - 1 = s + w - 1,$$

which gives the result by solving two linear equations.

□

10.2 Pre-spectral Reciprocity Formula

Now we will give a spectral decomposition of the period $I(w, \mathcal{A}_s\Phi, \varphi)$ which we consider in Theorem 1.3.1.

Let Π be an automorphic (everywhere unramified) cuspidal representation for $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ with trivial central character and let $\Phi = \otimes'_v \Phi_v \in \Pi$ be a cusp form. Let π_1 be an automorphic (everywhere unramified) cuspidal representation for $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ with trivial central character and let $\varphi = \otimes'_v \varphi_v \in \pi_1$ be a cusp form. We note that since Φ is a cusp form, therefore $\mathcal{A}_s\Phi$ is of rapid decay. Hence, we can apply the abstract spectral decomposition formula in Proposition 8.6.1 as follows:

$$\begin{aligned}\mathcal{A}_s\Phi(h) = &\sum_{\pi \in C(S)} \sum_{\phi \in \mathcal{B}_{\mathrm{cusp}}(\pi)} \langle \mathcal{A}_s\Phi, \phi\rangle \phi(h) + \sum_{\pi \in R(S)} \sum_{\phi \in \mathcal{B}_{\mathrm{res}}(\pi)} \langle \mathcal{A}_s\Phi, \phi\rangle \phi(h) \\ &+ \int_{\pi \in E(S)} \sum_{\phi \in \mathcal{B}_{\mathrm{cont}}(\pi)} \langle \mathcal{A}_s\Phi, \phi\rangle \phi(h) d\mu_{\mathrm{aut}}(\pi).\end{aligned} \tag{10.6}$$

Here S is any finite set of places that contain all the archimedean places and those finite places for which π_v is ramified. Moreover, we let $C(S)$ be the collection of cuspidal automorphic representations of $\mathrm{GL}(n)$ which are unramified everywhere outside S. We let $E(S)$ be the collection of Eisenstein series of $\mathrm{GL}(n)$ which are unramified everywhere outside S. And $R(S)$ is the collection of residual automorphic representations of $\mathrm{GL}(n)$ that are unramified everywhere outside S.

Now we take the integration on both sides of above equation (10.6) against a standard additive character ψ_1 which is defined in Section 2.3 for Whittaker models and over the compact set $N_n(F)\backslash N_n(\mathbb{A})$, we get the following equation for Whittaker functions:

$$W_{\mathcal{A}_s\Phi}(h) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{cusp}}(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle W_\phi(h) + \int_{\pi\in E(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{eisen}}(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle W_\phi(h) d\mu_{\mathrm{aut}}(\pi).$$

We note that since the residue spectrum for $\mathrm{GL}(n)$ are not generic ([JL13] Proposition 2.1), they do not contribute to the above expression and therefore vanish. We recall that not all Eisenstein series from continuous spectrum are generic. They are generic if and only if they are induced from the cuspidal data. For example, in the case of GL(3), it only contains two parts, which are minimal and maximal cuspidal Eisenstein series with the corresponding partition $3 = 1+1+1$ and $3 = 2+1$. Now, if we let $\mathrm{Re}(w)$ be large enough, we see that

$$\begin{aligned}\Psi(w, W_{\mathcal{A}_s\Phi}, W'_\varphi) = &\sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{cusp}}(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle \Psi(w, W_\phi, W'_\varphi) \\ &+ \int_{\pi\in E(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{eisen}}(\pi)} \langle \mathcal{A}_s\Phi,\phi\rangle \Psi(w, W_\phi, W'_\varphi) d\mu_{\mathrm{aut}}(\pi).\end{aligned} \tag{10.7}$$

We note that in fact the terms related to inner product and zeta integral $\langle \mathcal{A}_s\Phi,\phi\rangle$ and $\Psi(w, W_\phi, W'_\varphi)$ are a product of local (zeta) integrals when $\mathrm{Re}(w)$ is large enough since

$$\Psi\left(w, W_\phi, W'_\varphi\right) = \prod_v \Psi_v(w, W_{\phi_v}, W'_{\varphi_v}).$$

Moreover, by changing variables, we note that

$$\begin{aligned}\langle \mathcal{A}_s\Phi,\phi\rangle &= \int_{\mathrm{X}} |\det h|^{s-\frac{1}{2}} \left(\int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z_n(u)h_n & \\ & 1\end{pmatrix} |u|^{n(s-\frac{1}{2})} du\right) \overline{\phi(h_n)} dh_n \\ &= \int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)} \Phi\begin{pmatrix} h_n & \\ & 1\end{pmatrix} \overline{\phi(h_n)} |\det h_n|^{s-\frac{1}{2}} dh_n = I(s,\Phi,\overline{\phi}),\end{aligned} \tag{10.8}$$

since ϕ is invariant under the center $\mathrm{Z}_n(\mathbb{A})$.

By the Rankin-Selberg theory for $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$, we can also write $I(s,\Phi,\overline{\phi})$ as a product of local zeta integrals (see Section 3.2). For large enough $\mathrm{Re}(s)$, we have

$$I(s,\Phi,\overline{\phi})=\Psi(s,W_\Phi,\overline{W_\phi})=\Psi(s,W_\Phi,W'_{\overline{\phi}})=\prod_v\Psi_v(s,W_{\Psi_v},W'_{\overline{\phi_v}})=\prod_v\Psi_v(s,W_{\Psi_v},\overline{W_{\phi_v}}).$$

Note that if ϕ is not generic, then the inner product $\langle\mathcal{A}_s\Phi,\phi\rangle$ will vanish since $W_\phi\equiv 0$ by definition when we unfold the period integral.

By Proposition 9.2.1, we know that when $\mathrm{Re}(w)$ is large enough,

$$I(w,\phi-\phi_{U_n},\varphi)=\Psi(w,W_\phi,W'_\varphi).$$

Let $\widetilde{U}_n$ be the image of U_n under the embedding $\mathrm{GL}(n)\to\mathrm{GL}(n+1)$ by $g\to\begin{pmatrix}g&\\&1\end{pmatrix}$ and $\Phi_{\widetilde{U}_n}$ is the constant term along $\widetilde{U}_n$, that is

$$\Phi_{\widetilde{U}_n}(g):=\int_{U_n(F)\backslash U_n(\mathbb{A}_F)}\Phi\left(\begin{pmatrix}u&\\&1\end{pmatrix}g\right),$$

for every $g\in\mathrm{GL}_{n+1}(\mathbb{A}_F)$.

The above integral does not vanish identically. Since $U_n(F)\backslash U_n(\mathbb{A}_F)$ is compact, we have the spectral decomposition of $\mathcal{A}_s\Phi_{\widetilde{U}_n}$ similarly.

We note again that the residue spectrum for $\mathrm{GL}(n)$ are not generic ([JL13] Proposition 2.1). Hence the inner product $\langle\mathcal{A}_s\Phi,\phi\rangle$ in the residue spectrum will vanish (See (10.6)). By definition, we have

$$\begin{aligned}I(w,\mathcal{A}_s\Phi-\mathcal{A}_s\Phi_{\widetilde{U}_n},\varphi)=&\sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{cusp}}(\pi)}\langle\mathcal{A}_s\Phi,\phi\rangle I(w,\phi-\phi_{U_n},\varphi)\\&+\int_{\pi\in E(S)}\sum_{\phi\in\mathcal{B}_{\mathrm{eisen}}(\pi)}\langle\mathcal{A}_s\Phi,\phi\rangle I(w,\phi-\phi_{U_n},\varphi)d\mu_{\mathrm{aut}}(\pi),\end{aligned}\tag{10.9}$$

for every w with $\mathrm{Re}(w)$ large enough.

By Proposition 9.2.1, for $\mathrm{Re}(w)$ large enough, we have

$$I(w,\phi-\phi_{U_n},\varphi)=\Psi(w,W_\phi,W'_\varphi).$$

Therefore, we see that

$$\begin{aligned}\Psi(w, W_{\mathcal{A}_s\Phi}, W'_\varphi) &= \int_{N_{n-1}(\mathbb{A}_F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} W_{\mathcal{A}_s\Phi}\left(\begin{pmatrix} h_{n-1} & \\ & 1\end{pmatrix}\right) W'_\varphi(h_{n-1})|\det h_{n-1}|^{w-\frac{1}{2}} dh_{n-1} \\ &= I\left(w, \mathcal{A}_s\Phi - \mathcal{A}_s\Phi_{\widetilde{U}_n}, \varphi\right) \\ &= I\left(w, \mathcal{A}_s\Phi, \varphi\right) - I\left(w, \mathcal{A}_s\Phi_{\widetilde{U}_n}, \varphi\right)\end{aligned} \tag{10.10}$$

for $\mathrm{Re}(w)$ large enough.

Now, we define the degenerate term by

$$\begin{aligned}&\mathcal{D}(s, w, \Phi, \varphi) := I\left(w, \mathcal{A}_s\Phi_{\widetilde{U}_n}, \varphi\right) \\ &= \int_{F^\times\backslash\mathbb{A}_F^\times}\int_{\mathrm{GL}_{n-1}(F)\backslash\mathrm{GL}_{n-1}(\mathbb{A}_F)} \Phi_{\widetilde{U}_n}\left(\begin{pmatrix} zh & & \\ & z & \\ & & 1\end{pmatrix}\right)\overline{\varphi(h)}\cdot|\det(h)|^{s+w-1}\cdot|z|^{n(s-\frac{1}{2})}dhdz.\end{aligned} \tag{10.11}$$

Since Φ is a cusp form, the rapid decay of Φ ensures that the above integral of the degenerate term (10.11) converges absolutely for any $s, w \in \mathbb{C}$.

Combining above discussions together, we have the following Proposition:

Proposition 10.2.1. *Let Π be a cuspidal tempered automorphic (everywhere unramified) representation with trivial central character and let $\Phi = \otimes'_v\Phi_v \in \Pi$ be a cusp form for $\mathrm{GL}(n+1)$ over F. Let π_1 be a cuspidal tempered automorphic (everywhere unramified) representation for $\mathrm{GL}(n-1)$ over F with trivial central character and let $\varphi = \otimes'_v\varphi_v \in \pi_1$ be an automorphic form. Let $\tilde{\pi}$ be the contragredient representation of π. Then, we have*

$$\begin{aligned}\Delta_F^{1/2} I\left(w, \mathcal{A}_s\Phi - \mathcal{A}_s\Phi_{\widetilde{U}_n}, \varphi\right) &= \sum_{\pi\in C(S),\,\pi\,\mathrm{cusp}} \frac{\Lambda(s, \Pi\times\widetilde{\pi})\Lambda(w, \pi\times\pi_1)}{\Lambda(1, \mathrm{Ad}\,\pi)} H(\pi) \\ &\quad + \int_{\pi\in E(S),\,\pi\,\mathrm{eisen}} \frac{\Lambda(s, \Pi\times\widetilde{\pi})\Lambda(w, \pi\times\pi_1)}{\Lambda^*(1, \mathrm{Ad}\,\pi)} H(\pi) d\mu_{\mathrm{aut}}(\pi),\end{aligned} \tag{10.12}$$

where $H(\pi) = \prod_v H_v(\pi_v)$ is the global weight function. We note that H_v depends on

the choice of Φ_v *and* φ_v, s *and* w, *which is given by*

$$H_v(\pi_v) := \sum_{W \in \mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v}, \overline{W})\Psi_v(w, W, W'_{\varphi_v})}{L_v(s, \Pi_v \times \widetilde{\pi}_v)L_v(w, \pi_v \times \pi_{1,v})}. \tag{10.13}$$

Here for each (generic) automorphic representation π *of* $\mathrm{GL}(n)$, *we will consider the following completed L-functions*

$$\Lambda(s, \mathrm{Ad}, \pi),\ \Lambda(s, \Pi \times \pi) \text{ and } \Lambda(s, \pi \times \pi_1).$$

They are, the Adjoint L-function of π, *the Rankin-Selberg L-function of* $\Pi \times \pi$ *and* $\pi \times \pi_1$, *respectively.*

Moreover, $\Lambda^*(1, \mathrm{Ad}\,\pi)$ *in the denominator means the non-zero residue of the completed adjoint L-functions. If* π *is cuspdial, we know that* $\Lambda^*(1, \mathrm{Ad}\,\pi) = \Lambda(1, \mathrm{Ad}\,\pi) \neq 0$.

We actually have the following:

Let F be a number field, with ring of integers $\mathcal{O}_F$. Let Π be a cuspidal automorphic representation of $\mathrm{GL}(n+1)$ over F with trivial central character. Let π_1 be a cuspidal automorphic representation of $\mathrm{GL}(n-1)$ over F with trivial central character.

Let H be a global weight function which is defined above. We consider the following sums:

$$\mathcal{C}(H) := \mathcal{C}(s, w, \Phi, \varphi) = \sum_{\pi \in C(S)} \frac{\Lambda(s, \Pi \times \widetilde{\pi})\Lambda(w, \pi \times \pi_1)}{\Lambda(1, \mathrm{Ad}, \pi)} H(\pi),$$

which is the cuspidal contribution.

We should also consider the following continuous (Eisenstein) contribution

$$\mathcal{E}(H) := \mathcal{E}(s, w, \Phi, \varphi) = \int_{\pi \in E(S)} \frac{\Lambda(s, \Pi \times \widetilde{\pi})\Lambda(w, \pi \times \pi_1)}{\Lambda^*(1, \mathrm{Ad}, \pi)} H(\pi) d\mu_{\mathrm{aut}}(\pi), \tag{10.14}$$

where S is any finite set of places that contain all the archimedean places and those finite places for which π_v is ramified. Moreover, we let $C(S)$ be the collection of cuspidal automorphic representations of $\mathrm{GL}(n)$ which are unramified everywhere outside S. We let $E(S)$ be the collection of Eisenstein series of $\mathrm{GL}(n)$ which are unramified everywhere outside S.

By definition, we may write $H(\pi) = H(s, w, \pi_1, \Phi)$. We define $\check{H}(\pi) := H(s', w', \pi_1, \check{\Phi})$, where s, w' and $\check{\Phi}$ is defined in the proof of abstract reciprocity formula (Theorem 1.3.1).

We want to understand the following first moment of generic spectrum $\hat{\mathrm{X}}_{\mathrm{gen}}$ which is the summation of the cuspidal contribution, the Eisenstein contribution and the degenerate term:

$$\mathcal{M}(H) := \mathcal{C}(H) + \mathcal{E}(H) + \mathcal{D}(s, w, \Phi, \varphi).$$

We end this subsection with the following result which can be seen as a pre-spectral reciprocity formula.

Theorem 10.2.2. *Let $s, w \in \mathbb{C}$ and define*

$$(s', w') := \left(\frac{1 + (n-1)w - s}{n}, \frac{(n+1)s + w - 1}{n} \right). \tag{10.15}$$

Let S be a finite set of places which contain all the archimedean places and those finite places for which π_v is ramified. Suppose that the real parts of four parameters s, w, s', w' are all sufficiently large. Then we have the following identity

$$\mathcal{M}(H) = \mathcal{M}(\check{H}).$$

Proof. This is a direct corollary from Theorem 1.3.1 and Proposition 10.2.1. □

Chapter 11

Explicit Spectral Reciprocity Formula: Theorem 1.3.3

11.1 Local Vectors and Computations

Now in order to give the explicit spectral reciprocity formula (Theorem 1.3.3), we have to pick local vectors. We follow the method in [Nun20, Section 7]. We will show how to choose local vectors on some special non-archmediean places and archmediean places. We also give some very basic local estimations on these places for local weight functions H_v. Our local estmations are enough for us to establish the spectral reciprocity formula.

We recall that π_1 is an automorphic cuspidal (everywhere unramified) representation for $\mathrm{GL}(n-1)$ with trivial central character. Let $\varphi = \otimes_v' \varphi_v \in \pi_1$ be a cusp form. Moreover, for every place v, we simply fix $\varphi_v := \varphi_v^0$ be the normalized spherical vector in the Whittaker model.

Let Π be an unramified (everywhere) cuspidal automorphic representation of $\mathrm{GL}(n+1)$ with trivial central character. For all the place v, we let Φ_v^0 be the normalized spherical vector in the Whittaker model. Let $\mathfrak{q}$ and $\mathfrak{l}$ be two fixed (may not unramified) coprime integral ideals of F. We will write $\Phi^{\mathfrak{q},\mathfrak{l}} = \otimes_v' \Phi_v^{\mathfrak{q},\mathfrak{l}} \in \Pi$ be a cusp form. For all $v \nmid \mathfrak{q}\mathfrak{l}$, we

will simply pick $\Phi_v^{\mathfrak{q},\mathfrak{l}} = \Phi_v^0$. For $v \mid \mathfrak{l}$, we will pick

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)k}} \sum_{\beta_i \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} \right), \quad (11.1)$$

where $k = v(\mathfrak{l})$. Finally, for $v \mid \mathfrak{q}$, we will pick

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)m}} \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \right), \quad (11.2)$$

with $m = v(\mathfrak{q})$.

The choice for the local vector here for $v|\mathfrak{q}$ is compatible with the local new-vector computation made in [BKL19].

Remark 11.1.1. *In order to design a spectral reciprocity formula, we do not have much freedom when choosing the local vectors. For the reciprocal relation of two unramified coprime ideals $\mathfrak{q}$ and $\mathfrak{l}$, we only have the freedom for one finite place. For example, after we pick the local vector for the place $v|\mathfrak{q}$, the local vector for the place $v|\mathfrak{l}$ is automatically fixed. They are related by the following simple matrix identity:*

$$\begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

We let S be a finite set with the definition $S = \{v \mid \mathfrak{q}\} \cup \{v \mid \mathfrak{l}\} \cup \{v \mid \infty\}$. We have the following local properties for several different cases (The computation will be given later).

- If $v \notin S$, we know that π_v is unramified and $\Phi_v^{\mathfrak{q},\mathfrak{l}} = \Phi_v^0$ is spherical. And we also have $H_v(\pi_v) = 1$ in this case with only one term survives in the summation of H_v. This is a direct corolloary from the discussion in Section 3.2.

- If $v \mid \mathfrak{q}$, let $\Phi_v = \Phi_v^{\mathfrak{q},\mathfrak{l}}$ be as in (11.2) and let $t = v(\mathfrak{q})$. Then we have

  (1) $H_v(\pi_v)$ vanishes if $\mathrm{cond}(\pi_v) > t$.

  (2) $H_v(\pi_v) = p^{-(n-1)t}$ if $\mathrm{cond}(\pi_v) = t$.

- If $v \mid \infty$ which is the Archimedean place, we have the following Proposition

  Proposition 11.1.2. *Let v be an Archimedean place of F. Let Π_v be an irreducible admissible generic representation for $\mathrm{GL}_{n+1}(F_v)$, then there exists a Whittaker function $W_{\Pi_v} \in \mathcal{W}(\Pi_v, \psi_v)$ such that for every irreducible admissible generic representation for $\mathrm{GL}_n(F_v)$, we have*

$$H_v(\pi_v) = \begin{cases} 1, & \text{if } \pi_v \text{ is unramified,} \\ 0, & \text{otherwise.} \end{cases}$$

  Proof. This Proposition is proved in [Nun20] Section 7.4 by applying Stade's formula (Theorem 3.4 in [Sta01]) for Archimedean $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$ Rankin-Selberg L-functions.

□

11.2 Local computations for two fixed finite places

For $v \mid \mathfrak{l}$, we pick

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)k}} \sum_{\beta_i \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 \\ & & & & & 1 \end{pmatrix} \right), \quad (11.3)$$

where $k = v(\mathfrak{l})$.

For $v \mid \mathfrak{q}$, we will pick

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)m}} \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & \beta_1 \\ & 1 & & & \beta_2 \\ & & \cdots & & \\ & & & 1 & \beta_{n-1} \\ & & & & 1 \end{pmatrix} \right), \quad (11.4)$$

with $m = v(\mathfrak{q})$.

For $v \mid \mathfrak{l}$, we recall that

$$H_v(\pi_v) = \sum_{W \in \mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v^{\mathfrak{q},\mathfrak{l}}}, \overline{W}) \Psi_v(w, W, W'_{\varphi_v})}{L_v(s, \Pi_v \times \widetilde{\pi}_v) L_v(w, \pi_v \times \pi_{1,v})},$$

where

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)k}} \sum_{\beta_i \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & \beta_1 & \\ & 1 & & & \beta_2 & \\ & & \cdots & & & \\ & & & 1 & \beta_{n-1} & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \right),$$

with $k = v(\mathfrak{l})$.

Following the same method line by line in [Nun20, Section 7], we will see that H_v vanishes unless π_v is unramified. By right $\mathrm{GL}_n(F_v)$-invariance of the Haar measure, for any fixed element $h \in \mathrm{GL}_n(F_v)$, we see that

$$\Psi \left(s, \Pi_v \begin{pmatrix} h & \\ & 1 \end{pmatrix} W_{\Phi_v}, \overline{W} \right) = \Psi \left(s, W_{\Phi_v}, \overline{\pi_v(h) W} \right).$$

Now we note that given a basis $\mathcal{B}^W(\pi_v)$ of $\mathcal{W}(\pi_v, \psi_v)$, we may create a different one by considering the set

$$\left\{ \pi_v(h) \cdot W,\, W \in \mathcal{B}^W(\pi_v) \right\},$$

for some fixed element $h \in \mathrm{GL}_n(F_v)$.

Applying this idea to the element $h = \begin{pmatrix} 1 & & & \beta_1 \\ & 1 & & \beta_2 \\ & & \cdots & \\ & & 1 & \beta_{n-1} \\ & & & 1 \end{pmatrix}$ for $\beta_i \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v$, $i = 1, 2, \cdots, n-1$, we deduce that

$$H_v(\pi_v) = \sum_{W \in \mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v^0}, \overline{W})\Psi_v(w, W^{(m)}, W'_{\varphi_v})}{L_v(s, \Pi_v \times \widetilde{\pi}_v)L_v(w, \pi_v \times \pi_{1,v})},$$

where

$$\begin{aligned} W^{(m)}(h') :&= \sum_{\beta \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v} W\left(h' \begin{pmatrix} 1 & & & \beta_1 \\ & 1 & & \beta_2 \\ & & \cdots & \\ & & 1 & \beta_{n-1} \\ & & & 1 \end{pmatrix}^{-1} \right) \\ &= \sum_{\beta \in \mathfrak{m}_v^{-k}/\mathfrak{o}_v} W\left(h' \begin{pmatrix} 1 & & & -\beta_1 \\ & 1 & & -\beta_2 \\ & & \cdots & \\ & & 1 & -\beta_{n-1} \\ & & & 1 \end{pmatrix} \right). \end{aligned} \tag{11.5}$$

Since $W \mapsto \overline{\Psi_v(s, W_{\Phi_v^0}, \overline{W})}$ is a right K_v invariant linear functional, it vanishes if π_v is not unramified. This kind of idea will also be used in the calculation of another local vector when $v|\mathfrak{q}$. Moreover, this linear functional is invariant by orthogonal projection into the space $\mathcal{W}(\pi_v, \psi_v)^{K_v}$ of right K_v-invariant vectors of $\mathcal{W}(\pi_v, \psi_v)$. Since K_v is the maximal open compact subgroup, we know that the space $\mathcal{W}(\pi_v, \psi_v)^{K_v}$ is a one-dimensional space that is spanned by the normalized spherical vector. Therefore, we may restrict the sum defining H_v to a sum over a basis of $\mathcal{W}(\pi_v, \psi_v)^{K_v}$. Hence, only one term survives in the sum defining H_v over the basis. Now, by the unramified calculation, we have $\Psi(s, W_{\Phi_v^0}, \overline{W_{\pi_v}}) = L_v(s, \Pi_v \times \widetilde{\pi}_v)$

By above discussion, we see that

$$H_v(\pi_v) = \frac{\Psi_v(w, W^{(m)}, W'_{\varphi_v})}{L_v(w, \pi_v \times \pi_{1,v})}$$

if π_v is unramified.

We will write $H_v(\pi_v)$ explicitly. From the definition of $W^{(m)}(h)$, we see that

$$\begin{aligned} W^{(m)}\begin{pmatrix} h_{n-1} & \\ & 1 \end{pmatrix} &= \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v, i=1,2,\cdots,n-1} \psi\left(-\sum_{i=1}^{n-1}\beta_i h_{n-1,i}\right)\cdot W_{\pi_v}\begin{pmatrix} h_{n-1} & \\ & 1 \end{pmatrix} \\ &= p_v^{(n-2)m} \times \delta_{v(h_{n,i})\geq m,\, m=1,2,\cdots,n-1}\cdot W_{\pi_v}\begin{pmatrix} h_{n-1} & \\ & 1 \end{pmatrix}. \end{aligned} \tag{11.6}$$

We need to continuous our local computation by applying Iwasawa decomposition to h_{n-1}. We write $h_{n-1} = z(h)n(h)a(h)k(h)$. Since the valuation $v(h_{n-1,i}) \geq m$ for all $i = 1, 2, \cdots, n-1$, we see that $v(zk_{n-1,i}) \geq m$ for all $i = 1, 2, \cdots, n-1$.

From above discussion, we may have the decomposition:

$$\Psi_v(w, W^{(m)}, W'_{\varphi_v}) = \sum_{\nu=m}^{\infty} p_v^{-(n-1)\nu(s-\frac{1}{2})}\cdot p_v^{(n-2)m}\cdot \Psi_\nu(W_{\pi_v}), \tag{11.7}$$

where

$$\begin{aligned} \Psi_\nu(W_{\pi_v}) &= \int_{A_{n-1}(F_v^\times)}\int_{K_v} W_{\pi_v}\left(z(\varpi_v^\nu)a\right) W'_{\varphi_v}(ak)\cdot|a|^{s-\frac{n}{2}}\, dA_{n-1}(F_v^\times)\, dk \\ &= \int_{A_{n-1}(F_v^\times)} W_{\pi_v}\left(z(\varpi_v^\nu)a\right) W_{\varphi_v}(a)\cdot|a|^{s-\frac{n}{2}}\, dA_{n-1}(F_v^\times). \end{aligned} \tag{11.8}$$

Here we use the fact that both the Whittaker function W_{π_v} and W'_{φ_v} are unramified. Hence they are invariant under the maximal compact subgroup K_v and we also note that the total mass for K_v is one. Here the local weight function H_v should be related to the local Hecke eigenvalue λ_{π_v}.

For $v \mid \mathfrak{q}$, we recall that

$$H_v(\pi_v) = \sum_{W\in\mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v^{\mathfrak{q},\mathfrak{l}}}, \overline{W})\Psi_v(w, W, W'_{\varphi_v})}{L_v(s, \Pi_v\times\widetilde{\pi}_v)L_v(w, \pi_v\times\pi_{1,v})},$$

where

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^{(n-1)m}} \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} \Phi_v^0 \left(g \begin{pmatrix} 1 & & & & & \beta_1 \\ & 1 & & & & \beta_2 \\ & & \cdots & & & \\ & & & 1 & & \beta_{n-1} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \right),$$

with $m = v(\mathfrak{q})$. This means that

$$W_{\Phi_v^{\mathfrak{q},\mathfrak{l}}}(g) := \frac{1}{p_v^{(n-1)m}} \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v,\, i=1,2,\cdots,n-1} W_{\Phi_v^0} \left(g \begin{pmatrix} 1 & & & & & \beta_1 \\ & 1 & & & & \beta_2 \\ & & \cdots & & & \\ & & & 1 & & \beta_{n-1} \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \right),$$

with $m = v(\mathfrak{q})$. Here $W_{\Phi_v^0}$ is the normalized spherical vector in the Whittaker space. By the theory of newvectors [JPS81], we will see that $H_v(\pi_v)$ vanishes unless $\mathrm{cond}(\pi_v) \leq m$. If $\mathrm{cond}(\pi_v) = m$, $H_v(\pi_v) = p_v^{-(n-1)m}$. We expect that in general, $H_v(\pi_v) \ll_\epsilon p_v^{(n-1)m(\theta-1+\epsilon)}$, where θ is a positive constant satisfying $0 \leq \theta < \frac{1}{2}$ towards the Ramanujan-Petersson's Conjecture.

The computation in this part follows the method developed in [BKL19] and [Nun20, Section 7].

By definition and some matrix computations, we note that

$$\begin{aligned} W_{\Phi_v^{\mathfrak{q},\mathfrak{l}}} \begin{pmatrix} h_n & \\ & 1 \end{pmatrix} &= \frac{1}{p_v^{(n-1)m}} \times \sum_{\beta_i \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v, i=1,2,\cdots,n-1} \psi\left(\sum_{i=1}^{n-1} \beta_i h_{n,i} \right) \cdot W_{\Phi_v^0} \begin{pmatrix} h_n & \\ & 1 \end{pmatrix} \\ &= \delta_{v(h_{n,i}) \geq m,\, m=1,2,\cdots,n-1} \cdot W_{\Phi_v^0} \begin{pmatrix} h_n & \\ & 1 \end{pmatrix}. \end{aligned} \tag{11.9}$$

We need to continuous our local computation by applying Iwasawa decomposition to h_n. We write $h_n = z(h)n(h)a(h)k(h)$. Since the valuation $v(h_{n,i}) \geq m$ for all $i = 1, 2, \cdots, n-1$, we see that $v(zk_{n,i}) \geq m$ for all $i = 1, 2, \cdots, n-1$. We set $a_1 :=$

$\min(m, v(z))$ and $a_2 := m - a_1$, we can see that this is equivalent to say that k belonging to the congruence subgroup $K_{v,0}(\varpi_v^{a_2})$ (See the definition in Section 2.2). We will simply write this congruence subgroup as $K_v[a_2]$. Here we must have $v(z) \geq 0$. Otherwise, the spherical Whittaker function will vanish. Therefore we have $0 \leq a_1, a_2 \leq m$.

Now we may choose an orthonormal basis for $\mathcal{W}(\pi_v, \psi_v)$. From the local new-vector theory for GL(n) [JPS81], if we let $W_0 = W_{\pi_v}$ be the new vector and for each $j \geq 0$, we define

$$W_j := \pi_v \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & \varpi_v^j \end{pmatrix} W_0.$$

Therefore, we see that $\{W_0, W_1, W_2, \ldots\}$ is a basis for $\mathcal{W}(\pi_v, \psi_v)$. Moreover, we know that for each $j \geq 0$, $\{W_0, W_1, \ldots, W_j\}$ is a basis for the $K_v[n_0 + j]$-invariant vectors in $\mathcal{W}(\pi_v, \psi_v)$, where $n_0 = \mathrm{cond}(\pi_v)$. Applying Gram-Schmidt method to the basis $\{W_0, W_1, W_2, \ldots\}$, we obtain an orthonormal basis of $\mathcal{W}(\pi_v, \psi_v)$ with $\{\widetilde{W_0}, \widetilde{W_1}, \widetilde{W_2}, \ldots\}$. We choose the basis as $\mathcal{B}^W(\pi_v) = \{\widetilde{W_0}, \widetilde{W_1}, \widetilde{W_2}, \ldots\}$ and continue to do some computation on $H_v(\pi_v)$.

From above discussion, we have the following decomposition:

$$\Psi_v(s, W_{\Phi_v^{\mathfrak{q},\mathfrak{l}}}, \overline{W}) = \sum_{a_1+a_2=m} \sum_{\min(\nu,m)=a_1} p_v^{-n\nu(s-\frac{1}{2})} \Psi_{\nu,a_2}(W), \tag{11.10}$$

where

$$\begin{aligned} \Psi_{\nu,a_2}(W) &= \int_{A_n(F_v^\times)} \int_{K_v[a_2]} W_{\Phi_v^0}\left(z(\varpi_v^\nu)a\right) \overline{W}(ak) \cdot |a|^{s-\frac{n+1}{2}} dA_n(F_v^\times)\, dk \\ &= \int_{A_n(F_v^\times)} W_{\Phi_v^0}\left(z(\varpi_v^\nu)a\right) \int_{K_v[a_2]} \overline{W}(ak) \cdot |a|^{s-\frac{n+1}{2}} dA_n(F_v^\times)\, dk. \end{aligned} \tag{11.11}$$

Here we use the fact that Whiitaker function $W_{\Phi_v^0}$ is normalized spherical and W is also invariant by the center. Now, if $W = \widetilde{W_j}$ is an element of in our orthonormal basis $\{\widetilde{W_0}, \widetilde{W_1}, \widetilde{W_2}, \ldots\}$, then it follows that

$$\int_{K_v[f]} \widetilde{W_j}(hk)dk = \begin{cases} \mathrm{vol}(K_v[f])\widetilde{W_j}(h), \text{ if } j + n_0 \leq f, \\ 0, \text{ otherwise.} \end{cases} \tag{11.12}$$

from the orthogonality of the elements in the given orthonormal basis (See the discussion in [Nun20] for more details).

Now applying (11.10) and (11.12) to the definition of the local weight function $H_v(\pi_v)$. We will have the following equation:

$$H_v(\pi_v) = \frac{1}{L_v(s,\Pi_v \times \widetilde{\pi}_v)L_v(w,\pi_v \times \pi_{1,v})} \sum_{a_1+a_2=m} \sum_{j \leq a_2 - n_0} \mathrm{vol}(K_v[j]) \sum_{\min(\nu,m)=a_1} p^{-n\nu(s-1/2)}$$
$$\times \int_{A_n(F_v^\times)} W_{\Phi_v^0} \begin{pmatrix} z(\varpi_v^\nu)a & \\ & 1 \end{pmatrix} \overline{\widetilde{W}_j}(a))|a|^{s-(n+1)/2} dA_n(F_v^\times) \Psi_v(w, \widetilde{W}_j, W'_{\varphi_v}). \tag{11.13}$$

Note that the above equation is actually a finite sum. If $\mathrm{cond}(\pi_v) > m$, this gives that $n_0 > m$. We see that $0 \leq j \leq a_2 - n_0 \leq m - n_0 < 0$. Contradiction! Therefore, $H_v(\pi_v)$ vanishes in this case.

If $\mathrm{cond}(\pi_v) = m$, we see that only one term survive. Moreover, we have $a_2 = m$, and $j = a_1 = 0$. We see that $H_v(\pi_v) = \mathrm{vol}(K_v[a_2])^{-1} = \mathrm{vol}(K_v[m])^{-1} = p_v^{-(n-1)m}$ in this case.

Note that since $0 \leq j \leq a_2 - n_0 \leq a_2 \leq m$, the local weight function H_v is a finite sum in terms of the element $\widetilde{W}_j$ in the orthonormal basis.

For two places $\mathfrak{l}$ and $\mathfrak{q}$, we see that the local weight function H_v is a finite sum with the elements in the orthonormal basis $\mathcal{W}(\pi_v, \psi_v)$ for both two cases.

Remark 11.2.1. *In order to find applications for the spectral reciprocity formula, we have to find a good estimation for the local weight function $H_v(\pi_v)$.*

11.3 Meromorphic Continuation

Let v be a non-archimedean place of F. We consider the local weight function $H_v(\pi_v)$. It is known that for all the finite place $v \neq \mathfrak{q}, \mathfrak{l}$, we have $H_v(\pi_v) = 1$.

Assume that π_v is a local component of the unitary generic Eisenstein series $\pi = \pi(\pi_1, \pi_2, \cdots, \pi_k, t_1, \cdots, t_k) = \pi_1|\cdot|^{it_1} \boxplus \cdots \boxplus \pi_k|\cdot|^{it_k}$. We fix the cuspidal data $\pi_1, \cdots, \pi_k$ and vary the remaining parameters $(t_1, t_2, \cdots, t_k) \in \mathbb{C}^k$ $(k \geq 2)$.

The following Proposition is a direct Corollary of Proposition 4.1, Proposition 4.2 and Theorem 4.1 in [CPS17].

Proposition 11.3.1. *The following two ratios*

$$\frac{\Psi_v(s, W_{\Phi_v}, \overline{W})}{L_v(s, \Pi_v \times \widetilde{\pi}_v)}$$

and

$$\frac{\Psi_v(w, W, W'_{\varphi_v})}{L_v(w, \pi_v \times \pi_{1,v})}$$

for any $W \in \mathcal{W}(\pi_v, \psi_v)$ have no poles and hence define entire rational functions in terms of their complex parameters. In other word, we have

$$\frac{\Psi_v(s, W_{\Phi_v}, \overline{W})}{L_v(s, \Pi_v \times \widetilde{\pi}_v)} \in \mathbb{C}[p_v^s, p_v^{-s}, p_v^{t_1}, p_v^{-t_1}, \cdots, p_v^{t_k}, p_v^{-t_k}],$$

and

$$\frac{\Psi_v(w, W, W'_{\varphi_v})}{L_v(w, \pi_v \times \pi_{1,v})} \in \mathbb{C}[p_v^w, p_v^{-w}, p_v^{t_1}, p_v^{-t_1}, \cdots, p_v^{t_k}, p_v^{-t_k}].$$

Since the local weight function H_v is a finite sum with the elements in the orthonormal basis $\mathcal{W}(\pi_v, \psi_v)$, we have the following Proposition.

Proposition 11.3.2. *The local weight function $H_v(\pi_v)$ has no poles and hence define entire rational functions in terms of their complex parameters. In other word, we have*

$$H_v(\pi_v) \in \mathbb{C}[p_v^s, p_v^{-s}, p_v^w, p_v^{-w}, p_v^{t_1}, p_v^{-t_1}, \cdots, p_v^{t_k}, p_v^{-t_k}].$$

From our choice of local vectors in the previous subsection and the definition of global and local weight function, we know that

$$H(\pi) = \prod_v H_v(\pi_v) = H_{\mathfrak{q}}(\pi_{\mathfrak{q}}) \times H_{\mathfrak{l}}(\pi_{\mathfrak{l}}),$$

since $H_v(\pi_v) = 1$ if $v \nmid \mathfrak{q}\mathfrak{l}$.

Therefore, we have the following proposition:

Proposition 11.3.3. *The global weight function $H(\pi)$ has no poles and hence define entire rational functions in terms of their complex parameters.*

We are going to deduce the meromorphic continuation of the term $\mathcal{E}(H)$ which is the continuous contribution of Eisenstein series in the spectral decomposition.

Proposition 11.3.4. *Let Π and π_1 be everywhere unramified cuspidal automorphic representation of $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n-1)$ which have trivial central character. Let $\mathcal{E}(H)$ (10.14) be given previously, defined initially for the absolute convergence domain $\mathrm{Re}(s), \mathrm{Re}(w) > 1$. It admits a meromorphic continuation to $\mathrm{Re}(s), \mathrm{Re}(w) \geq \frac{1}{2}$. If $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < 1$, its analytic continuation is given by $\mathcal{E}(H) + \mathcal{R}(H)$, where*

$$\mathcal{R}(H) := \sum_{\omega \in F^\times \widetilde{U_\infty} \backslash \mathbb{A}_F^1} \operatorname*{Res}_{t_1=(1-w)/i} (2\pi i) \times \frac{\Lambda(s-it_1, \Pi \times \pi_1)\Lambda(s-it_2, \Pi \times \omega^{-1})\Lambda(w+it_1, \widetilde{\pi}_1 \times \pi_1)\Lambda(w+it_2, \pi_1 \times \omega)}{\Lambda^*(1, \mathrm{Ad}, \pi)} H(\pi). \tag{11.14}$$

Note that the summation over the unitary Hecke character ω which is only ramified at two finite places $\mathfrak{q}$ and $\mathfrak{l}$ is a finite sum. We write $U_\infty := \prod_{v|\infty} \{y \in F_v^*; |y_v| = 1\}$ and recall that $\mathbb{A}_F^1$ is the norm one ideles. Here we pick π to be the maximal cuspidal Eisenstein series given by $\pi = \pi(\widetilde{\pi}_1, \omega, t_1, t_2) = \widetilde{\pi}_1 |\cdot|^{it_1} \boxplus \omega |\cdot|^{it_2}$, where $\widetilde{\pi_1}$ is the contrigredient representation of π_1. Hence it is a cuspdial representation of $\mathrm{GL}(n-1)$ with trivial central character. We have the following decomposition of completed L-functions:

$$\Lambda(s, \Pi \times \widetilde{\pi}) = \Lambda(s-it_1, \Pi \times \pi_1)\Lambda(s-it_2, \Pi \times \omega^{-1}),$$

and

$$\Lambda(w, \pi \times \pi_1) = \Lambda(w+it_1, \widetilde{\pi}_1 \times \pi_1)\Lambda(w+it_2, \pi_1 \times \omega).$$

Moreover, for general $\pi = \pi(\sigma_1, \sigma_2, t_1, t_2)$, we see that $t_1 + t_2 = A$, where A is a complex constant only depends on the central characters of the cuspidal data σ_1 and σ_2 since π has trivial central character.

Proof. The meromorphic continuation part of $\mathcal{E}(H)$ is given by the meromorphic continuation of Rankin-Selberg L-functions and the entireness of the global weight function $H(\pi)$ (see Proposition 11.3.3). Since $H(\pi)$ has no poles and define entire rational functions in terms of their parameters (Proposition 11.3.3), by the contour and residue theorem in complex analysis, we see that the term $\mathcal{R}(H)$ will vanish unless the ratio of completed L-functions

$$\frac{\Lambda(s, \Pi \times \widetilde{\pi})\Lambda(w, \pi \times \pi_1)}{\Lambda^*(1, \mathrm{Ad}, \pi)}$$

have poles. Note that the denominator $\Lambda^*(1, \mathrm{Ad}, \pi)$ is always finite and non-zero, by the locations of possible poles of Rankin-Selberg L-functions, we must have

$$\pi = \pi(\widetilde{\pi}_1, \omega, t_1, t_2) = \widetilde{\pi}_1|\cdot|^{it_1} \boxplus \omega|\cdot|^{it_2},$$

where $\widetilde{\pi_1}$ is the contrigredient representation of π_1 and ω is a unitary Hecke character.

Hence we have the decomposition of completed L-functions:

$$\Lambda(s, \Pi \times \widetilde{\pi}) = \Lambda(s - it_1, \Pi \times \pi_1)\Lambda(s - it_2, \Pi \times \omega^{-1}),$$

and

$$\Lambda(w, \pi \times \pi_1) = \Lambda(w + it_1, \widetilde{\pi}_1 \times \pi_1)\Lambda(w + it_2, \pi_1 \times \omega).$$

We note that $\Lambda(s, \Pi \times \widetilde{\pi})$ is entire for all $s, t_1, t_2 \in \mathbb{C}$ since Π is a cuspidal automorphic representation for $\mathrm{GL}(n+1)$. The completed L-function $\Lambda(w, \pi \times \pi_1)$ will have a simple pole if and only if $w + it_1 = 1$. The correponding residue is

$$\Lambda^*(w, \pi \times \pi_1) = \Lambda(1, \mathrm{Ad}, \pi_1)\Lambda(w + it_2, \pi_1 \times \omega).$$

Now applying the coutour and residue theorem in complex analysis, the remaining part of the proof is the same as the proof in [Nun20, Proposition 8.1], [BK17, Lemma 16] and [BK18, Lemma 3]. □

11.4 The Degenerate Term

We consider some properites of the degenerate term.

Proposition 11.4.1. *Let Φ and φ be given as in the previous section. We have*

$$\mathcal{D}\left(\left(\frac{1}{2}, \frac{1}{2}\right), \Phi, \varphi\right) = \mathcal{D}_\infty \times \mathcal{D}_\mathfrak{l} \times \prod_{v \neq \mathfrak{l},\, v finite} \frac{L_v(1, \Pi \times \widetilde{\pi_1}) L_v(\frac{n}{2}, \widetilde{\Pi})}{L_v(1 + \frac{n}{2}, \widetilde{\pi_1})}, \tag{11.15}$$

where $\mathcal{D}_\infty \asymp 1$ and $\mathcal{D}_\mathfrak{l} \asymp 1$.

Proof. This is almost Proposition 10.1 of [JN21]. For any $\Phi \in \Pi$ and $\varphi \in \pi_1$, we have

$$\mathcal{D}(s, w, \Phi, \varphi) = \prod_v \Omega_v(s, w, W_{\Phi_v}, W_{\varphi_v})$$

for $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ large enough. Here we define $\Omega_v\left(s, w, W_{\Phi_v}, W_{\varphi_v}\right)$ by

$$\int_{N_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} \int_{F_v^\times} W_{\Phi_v}\left(\begin{pmatrix} zh & & \\ & z & \\ & & 1 \end{pmatrix}\right) \overline{W_{\varphi_v}(h)} \cdot |\det(h)|^{s+w-1} \cdot |z|^{n(s-\frac{1}{2})} d^\times z dh.$$

The integral defined above is absolutely convergent for sufficiently large $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$.

We also define

$$\mathcal{D}_\infty = \mathcal{D}_\infty(s, w) := \prod_{v|\infty} \Omega_v\left(s, w, W_{\Phi_v}, W_{\varphi_v}\right).$$

By Lemma 10.2 of [JN21], we have

$$\Omega_v\left(s, w, W_{\Phi_v}, W_{\varphi_v}\right) = \frac{L_v(s+w, \Pi \times \widetilde{\pi_1}) L_v(ns, \widetilde{\Pi})}{L_v((n+1)s+w, \widetilde{\pi_1})}$$

for all finite place $v \neq \mathfrak{l}$.

For the finite place $v = \mathfrak{l}$, by our choice of the local test vectors and Iwasawa decomposition, we have

$$\begin{aligned} &\Omega_{\mathfrak{l}}\left(s, w, W_{\Phi_v}, W_{\varphi_v}\right) = \Omega_{\mathfrak{l}}\left(s, w, W_{\Pi_v}, W_{\pi_{1,v}}\right) \\ &= \sum_{m_1 \geq \cdots \geq m_{n-1} \geq m,\, \ell \geq 0} \frac{\lambda_\Pi(m_1+\ell, \cdots, m_{n-1}+\ell, \ell, 0) \lambda_{\widetilde{\pi_1}}(m_1, m_2, \cdots, m_{n-1})}{p^{(s+w)\sum_{i=1}^{n-1} m_i + n\ell s}}. \end{aligned} \tag{11.16}$$

Since the cuspidal automorphic representations Π and π_1 are tempered, we have

$$\lambda_\Pi(m_1+\ell, \cdots, m_{n-1}+\ell, \ell, 0) \asymp 1$$

and

$$\lambda_{\widetilde{\pi_1}}(m_1, m_2, \cdots, m_{n-1}) \asymp 1.$$

Hence, for $\mathrm{Re}(s), \mathrm{Re}(w) > 0$, we have

$$\mathcal{D}_{\mathfrak{l}} := \Omega_{\mathfrak{l}}\left(s, w, W_{\Phi_v}, W_{\varphi_v}\right) \asymp 1.$$

□

Similarly, on the dual side, we have

Proposition 11.4.2. *Let Φ and φ be given as in the previous section. We have*

$$\mathcal{D}\left(\left(\frac{1}{2},\frac{1}{2}\right),\check{\Phi},\varphi\right)=\mathcal{D}_{\infty}\times\mathcal{D}_{\mathfrak{q}}\times\prod_{v\neq\mathfrak{q},\,v finite}\frac{L_v(1,\Pi\times\widetilde{\pi_1})L_v(\frac{n}{2},\widetilde{\Pi})}{L_v(1+\frac{n}{2},\widetilde{\pi_1})},\tag{11.17}$$

where $\mathcal{D}_{\infty}\asymp 1$ and $\mathcal{D}_{\mathfrak{q}}\asymp 1$.

11.5 Explicit Spectral Reciprocity Formula

Now we can give the statement of spectral reciprocity formula.

Let Π and π_1 be everywhere unramified cuspidal automorphic representation for $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n-1)$ over F with trivial central character. Let $s,w\in\mathbb{C}$, $\mathfrak{q}$ and $\mathfrak{l}$ be unramified coprime ideals.

Let H be the global weight function with kernel function $\Phi=\Phi^{\mathfrak{q},\mathfrak{l}}$ which we pick in the previous subsection by local new-vectors.

Note that we have

$$H(\pi)=\prod_{v}H_v(\pi_v),$$

where H_v is given by local Rankin-Selberg integral as follows:

$$H_v(\pi_v):=\sum_{W\in\mathcal{B}^W(\pi_v)}\frac{\Psi_v(s,W_{\Phi_v},\overline{W})\Psi_v(w,W,W'_{\varphi_v})}{L_v(s,\Pi_v\times\widetilde{\pi}_v)L_v(w,\pi_v\times\pi_{1,v})},\tag{11.18}$$

where $\Phi=\otimes\Phi_v\in\Pi$ is a cusp form and $s,w\in\mathbb{C}$.

We may write

$$H(\pi)=H(\Pi,\pi,\pi_1,s,w;\mathfrak{q},\mathfrak{l}).$$

Using the notations in Chapter 10, we have

$$\mathcal{M}(H)=\mathcal{M}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l}),\tag{11.19}$$

where

$$\mathcal{M}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l}):=\mathcal{C}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})+\mathcal{E}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l}),\tag{11.20}$$

with

$$\mathcal{C}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})=\mathcal{C}(H)=\sum_{\substack{\pi\ \mathrm{cusp}^0\\ \mathrm{cond}(\pi)|\mathfrak{q}}}\frac{\Lambda(s,\Pi\times\widetilde{\pi})\Lambda(w,\pi\times\pi_1)}{\Lambda(1,\mathrm{Ad},\pi)}H(\Pi,\pi,\pi_1,s,w;\mathfrak{q},\mathfrak{l}),$$

which is the cuspidal contribution.

And

$$\mathcal{E}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})=\mathcal{E}(H)=\int_{\substack{\pi\ \mathrm{eisen}^0\\ \mathrm{cond}(\pi)|\mathfrak{q}}}\frac{\Lambda(s,\Pi\times\widetilde{\pi})\Lambda(w,\pi\times\pi_1)}{\Lambda^*(1,\mathrm{Ad},\pi)}H(\Pi,\pi,\pi_1,s,w;\mathfrak{q},\mathfrak{l})d\mu_{\mathrm{aut}}(\pi)$$

which is the continuous (Eisenstein) contribution.

The notation cusp^0 and eisen^0 means that we are restricting to irreducible generic automorphic forms which are unramified at every archimedean place. We can define

$$\mathcal{N}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l}):=\mathcal{R}(\check{H})-\mathcal{R}(H) \tag{11.21}$$

and

$$\mathcal{D}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l}):=\mathcal{D}\left(s',w',\check{\Phi},\varphi\right)-\mathcal{D}\left(s,w,\Phi,\varphi\right). \tag{11.22}$$

From the above discussion in Section 3, we can give the statement of our main theorem finally.

Theorem 11.5.1 (Theorem 1.3.3)**.** *Let Π and π_1 be everywhere unramified tempered cuspidal automorphic representation for $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n-1)$ over F with trivial central character. Let $\widetilde{\pi}$ be the contragredient representation of π. Suppose that $\mathfrak{q}$ and $\mathfrak{l}$ are unramified, coprime ideals. Futhermore, we assume that $\frac{1}{2}\leq \mathrm{Re}(s),\mathrm{Re}(w)<\frac{n+1}{n+2}$. Then we will have the following identity*

$$\mathcal{M}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})=\mathcal{N}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})+\mathcal{D}(\Pi,\pi_1,s,w,\mathfrak{q},\mathfrak{l})+\mathcal{M}(\Pi,\pi_1,s',w',\mathfrak{l},\mathfrak{q}),$$

where the complex parameters s',w' satisfy the relation

$$s'=\frac{1+(n-1)w-s}{n},\quad w'=\frac{(n+1)s+w-1}{n}.$$

References

[AGK15] A. Aizenbud, D. Gourevitch and A. Komarsky, *Vanishing of certain equivarient distributions on p-adic spherical spaces, and nonvanishing of spherical Bessel functions.* International Mathematics Research Notices, (2015), no.18, 8471-8483.

[AGS15] A. Aizenbud, D. Gourevitch and E. Sayag, *z-finite distributions on p-adic groups.* Advances in Mathematics 285 (2012), 1376-1414.

[AK18] N. Anderson and E. M. Kiral, *Level reciprocity in the twisted second moment of Rankin-Selberg L-functions.* arXiv: 1801.06089. Mathematika, (64) 2018, no.3, 770-784.

[Ba97] E. M. Baruch, *On Bessel distributions of* GL(2) *over a p-adic field.* Journal of Number Theory 67 (1997), 190-202.

[Ba01] E. M. Baruch, *On Bessel distributions for quasi-split groups.* Transactions American Math. Society, 353, no.7, (2001), 2601-2614.

[Ba03] E. M. Baruch, *Bessel functions for* GL(3) *over a p-adic field.* Pacific Journal of Mathematics 211(1) (2003), 1-34.

[Ba04] E. M. Baruch, *Bessel Distribution for* GL(3) *over the p-adic field.* Pacific Journal of Mathematics 217(1) (2004), 11-27.

[Ba05] E. M. Baruch, *Bessel functions for* GL(n) *over a p-adic field.* Automorphic representations, L-functions and applications: Progress and prospects 1-40, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[BM03] E. M. Baruch, Z. Mao, *Bessel identities in the Waldspurger correspondence over a p-adic field.* Am. J. Math. 125, no.2 (2003), 225-288.

[BM05] E. M. Baruch, Z. Mao, *Bessel identities in the Waldspurger correspondence over the real numbers.* Israel J. Math. 145 (2005), 1-81.

[BM07] E. M. Baruch, Z. Mao, *Central value of automorphic L-functions.* Geom. Funct. Anal. 17, no.2 (2007), 333-384.

[BFG88] D. Bump, S. Friedberg and D. Goldfeld, *Poincare series and Kloosterman sums for* $\mathrm{SL}(3,\mathbb{Z})$. Acta. Arith. 50 (1988), 31-89.

[Blo19] V. Blomer, *Density theorems for* $\mathrm{GL}(n)$. arXiv: 1906.07459. preprint, 2019.

[BM22] V. Blomer, S. Man, *Bounds for Kloosterman sums on* $\mathrm{GL}(n)$. arXiv: 2208.01295. preprint, 2022.

[BK17] V. Blomer and R. Khan, *Twisted moments of L-functions and spectral reciprocity.* arXiv: 1706.01245. Duke Math Journal, (168) 2019, no.6, 1109-1177.

[BK18] V. Blomer and R. Khan, *Uniform subconvexity and symmetry breaking reciprocity.* arXiv: 1804.01602. Journal of Functional Analysis (276) 2019, no.7, 2315-2358.

[BKL19] A. Booker, M. Krishnamurthy and M. Lee, *Test vectors for Rankin-Selberg L-functions.* arXiv: 1903.03458. Journal of Number Theory, (209) 2020, 37-48.

[BML19] V. Blomer, S. Miller and X. Li, *A spectral reciprocity formula and non-vanishing for L-functions on* $\mathrm{GL}(4)\times\mathrm{GL}(2)$. arXiv: 1705.04344. Journal of Number Theory Prime (205) 2019, 1-43.

[Chai17] J. Chai, *A weak kernel formula for Bessel functions.* Transactions American Math. Society, 369, no.10, (2017), 7139-7167.

[Chai19a] J. Chai, *On Bessel functions over p-adic fields.* International Mathematics Research Notices, (2019), no.3, 673-699.

[Chai19b] J. Chai, *Bessel functions and local converse conjecture of Jacquet.* Journal of European Math. Society, 21 (2019), no.6, 1703-1728.

[CQ20] J. Chai, Z. Qi, *Bessel identities in the Waldspurger correspondence over the complex numbers.* Israel J. Math. 235 (2020), 439-463.

[Cog07] J. Cogdell, *L-functions and converse theorems for* GL_n, Automorphic forms and applications, volume 12 of IAS/Park City Math. Ser., page 97-177. Amer. Math. Soc., Providence, RI.

[Cog14] J. Cogdell, *Bessel functions for* GL_2. Indian J. Pure Appl. Math. 45 (2014), no.5, 557-582.

[CKPSS04] J. Cogdell, H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups.* Publ. Math. Inst. Hautes Etudes Sci., (99) 2004, 163-233.

[CPS94] J. W. Cogdell, I. I. Piatetski-Shapiro, *Converse Theorem for* GL_n. Publ. Math. Inst. Hautes Etudes Sci., (79) 1994, 157-214.

[CPS98] J. W. Cogdell, I. I. Piatetski-Shapiro, *Stability of gamma factors for* $\mathrm{SO}(2n+1)$. Manuscripta Math. 95, no.4 (1998), 437-461.

[CPS17] J. Cogdell, I. I. Piatetski-Shapiro, *Derivatives and L-functions for* GL_n. Representation Theory, Number Theory, and Invariant Theory pp 115-173.

[CPSS08] J. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi, *Stability of* γ*-factors for quasi-split groups.* J. Inst. Math. Jussieu 7. no.1 (2008), 27-66.

[CST17] J. Cogdell, F. Shahidi, and T. Tsai, *Local Langlands correspondence for* GL_n *and the exterior and symmetric square* ϵ*-factors.* Duke Math. J. 166 (2017), no.11, 2053-2132.

[DF97] R. Dabrowski, B. Fisher, *A stationary phase formula for exponential sums over $\mathbb{Z}/p^m\mathbb{Z}$ and applications to* GL(3)*-Kloosterman sums.* Acta Arith. 80 (1997), no.1, 1-48.

[DR98] R. Dabrowski, M. Reeder, *Kloosterman sets in reductive groups.* Journal of Number Theory 73 (1998), no.2, 228-255.

[FLO12] B. Feigon, E. Lapid and O. Offen, *On representations distinguished by unitary groups.* Publ. Math. Inst. Hautes Etudes Sci. 115 (2012), 185-323.

[Fr87] S. Friedberg, *Poincare series for* GL(n)*: Fourier expansion, Kloosterman sums and algebreo-geometric estimates.* Math. Z. 196 (1987), 165-188.

[God18] R. Godement, *Notes on Jacquet-Langlands' Theory*, CTM 8 Classical Topics in Mathematics, Higher Education Press, 2018.

[GS83] D. Goldfeld, P. Sarnak, *Sums of Kloosterman sums.* Invent. Math. 71 (1983), 243-250.

[GSW21] D. Goldfeld, E. Stade, and M. Woodbury, *An orthogonality relation for* GL($4, \mathbb{R}$) *(with an appendix by Bingrong Huang).* arXiv: 1910.13586. Forum of Mathematics, Sigma (2021), 1-83.

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups.* Lecture Notes in Mathematics, 162, Springer-Verlag, Berlin, 1970. Notes by G. van Dijk.

[HC99] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups.* University Lecture Series, vol. 16, American Mathematical Society, Providence, RI. 1999, Preface and notes by Stephen DaBacker and Paul J. Sally, Jr.

[IT13] A. Ichino, N. Templier, *On the Voronoi formula for* GL(n). Amer. J. Math. 135 (2013), no.1, 65-101.

[IY15] A. Ichino, S. Yamana, *Periods of automorphic forms: the case of* $(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \mathrm{GL}_n)$, Compositio Math. 151 (2015) 665-712.

[Jac70] H. Jacquet *Automorphic Forms on* GL(2)*, Part II.* Lecture Notes in Mathematics 278, Springer-Verlag.

[Jac03] H. Jacquet, *Smooth transfer of Kloosterman integrals.* Duke Math. Journal, 120(1) (2003), 121-152.

[Jac04] H. Jacquet, *Kloosterman identities over a quadratic extension.* Annals of Mathematics, 160(2) (2004), 755-779.

[Jac05] H. Jacquet, *A guide to the relative trace formula.* Automorphic representations, L-functions and applications: Progress and prospects, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[Jac09] H. Jacquet, *Archimedean Rankin-Selberg integrals.* Automorphic forms and L-functions II: local aspects, Contemporary Mathematics, vol. 489 (American Mathematical Society, Providence, RI. 2009), 57-172.

[JL70] H. Jacquet, and R. Langlands, *Automorphic Forms on* GL(2). Lecture Notes in Mathematics 114, Springer-Verlag.

[JL16] H. Jacquet, and B. Liu, *On the local converse theorem for p-adic* GL_n. To appear in American Journal of Mathematics.

[JPS79] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on* GL(3)*, I, II.* Ann. of Math. (2) 109 (1979), 169-258.

[JPS81] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des representations du groupe lineaire.* Math. Ann. 256 (2), 199-214, 1981.

[JPS83] H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Rankin-Selberg convolutions.* Amer. J. Math. 105 (1983), 367-464.

[JS90] H. Jacquet, J. Shalika, *Rankin-Selberg convolutions: Archimedean theory.* Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, part I, Israel Mathematical Conference Proceedings, vol. 2 (Weizmann, Jerusalem, 1990), 125-207.

[JY92] H. Jacquet, Y. Ye, *Relative Kloosterman integrals for* GL(3). Bull. Soc. Math. France 120 (1992), 263-295.

[JY96] H. Jacquet, Y. Ye, *Distinguished representation and quadratic base change for* GL(3). Transactions American Math. Society, 348, no.3, (1996), 913-939.

[JY99] H. Jacquet, Y. Ye, *Germs of Kloosterman integrals for* GL(3). Transactions American Math. Society, 351, no.3, (1999), 1227-1255.

[Ja20] S. Jana, *Second Moment of* $\mathrm{GL}(n) \times \mathrm{GL}(n)$ *Rankin-Selberg L-functions*. arXiv: 2012.07817. Forum Math. Sigma, vol.10, e47, 2022.

[JN21] S. Jana, R. Nunes, *Spectral Reciprocity for* $\mathrm{GL}(n)$ *and simultaneous non-vanishing of central L-values*. arXiv: 2111.02297. preprint, 2021.

[JL13] D. Jiang, B. Liu, *On Fourier Coefficients of Automorphic Forms of* $\mathrm{GL}(n)$. International Mathematics Research Notices Vol. 2013, No. 17, 4029-4071.

[Kha12] R. Khan, *Simultaneous non-vanishing of* $\mathrm{GL}(3) \times \mathrm{GL}(2)$ *and* $\mathrm{GL}(2)$ *L-functions*. Math. Proc. Cambridge Philos. Soc., (152) 2012, no.3, 535-553.

[Kha21] R. Khan, *Subconvexity bounds for twisted L-functions II*. arXiv: 2103.12330. To appear in Transactions American Math. Society.

[KN22] E. Mehmet Kiral, M. Nakasuji, *Parametrization of Kloosterman Sets and* SL_3*-Kloosterman Sums*. arXiv: 2001.01936. Advances in Mathematics 403 (2022).

[KY21] E. M. Kiral, M. Young, *The fifth moment of modular L-functions*. arXiv: 1701.07507. J. Eur. Math. Soc. 23 (2021), no.1, 237-314.

[KL13] A. Knightly, C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*. Memoirs American Math. Society, Volume 224, Number 1055, 2013.

[Kuz80] N. V. Kuznetsov, *Petersson hypothesis for parabolic forms of weight zero and linnik hypothesis, Sums of Kloosterman Sums.* Mat. Sb. 111 (153) (1980), 334-383.

[LM09] E. Lapid, Z. Mao, *On the asymptotics of Whittaker functions.* Representation Theory, 13, (2009), 63-81.

[LM13] E. Lapid, Z. Mao, *Stability of certain oscillotory integrals.* International Mathematics Research Notices, (2013), no.3, 525-547.

[LM15] E. Lapid, Z. Mao, *A conjecture on Whittaker-Fourier coefficients of cusp forms.* Journal of Number Theory 146 (2015), 448-505.

[Man22] S. Man, *Symplectic Kloosterman sums and Poincare series.* Ramanujan J. 57 (2022), 707-753.

[Miao21] X. Miao, *Spectral Reciprocity for the product of Rankin-Selberg L-functions.* arXiv: 2110.11529. preprint, 2021.

[Miao22] X. Miao, *Bessel Functions and Kloosterman Integrals on* GL(n). arXiv: 2208.01016. preprint, 2022.

[Mot93] Y. Motohashi, *An explicit formula for the fourth power mean of the Riemann zeta-function.* Acta Math., (170) 1993, no.2, 181-220.

[MV10] P. Michel, A. Venkatesh, *The subconvexity problem for* GL(2). arXiv: 0903.3591. Publ. Math. Inst. Hautes Etudes Sci., (111) 2010, 171-271.

[MW95] C. Moeglin, J. L. Waldspurger, *Spectral Decomposition and Eisenstein Series.* Cambridge Tracts in Mathematics, 113, Cambridge University Press, Cambridge, 1995.

[Nel19] P. Nelson, *Eisenstein series and the cubic moment for* PGL(2). arXiv: 1911.06310.

[Nun20] R. Nunes, *Spectral reciprocity via integral representations.* arXiv: 2002.01993.

[Qi20] Z. Qi, *Theory of Fundamental Bessel Functions of High Rank*. Memoirs American Math. Society, Volume 267, Number 1303, 2020.

[Rez08] A. Reznikov, *Rankin-Selberg without unfolding and bounds for spherical Fourier coefficients of Maass forms*. J. Amer. Math. Soc., 21 (2), 2008, 439-477.

[SV17] Y. Sakellaridis, A. Venkatesh, *Periods and harmonic analysis on spherical varieties*. arXiv: 1203.0039.

[Sar90] P. Sarnak, *Diophantine Problems and Linear Groups*. Proceedings of the ICM Kyoto (1990), 459-471.

[Sel65] A. Selberg, *On the estimation of Fourier coefficients of modular forms*. Proc. Sympos. Pure Math., no.8, Amer. Math. Soc., Providence, R.I., 1965, pp. 1-15.

[Sha74] J. Shalika, *The multiplicity one theorem for* $\mathrm{GL}(n)$. Annals of Mathematics, 100(2) (1974), 171-193.

[So84] D. Soudry, *The L and* γ *factors for generic representations of* $\mathrm{GSp}_4 \times \mathrm{GL}_2$ *over a non-Archimedean p-adic field*. Duke Math. Journal, 51 (1984), 355-394.

[Sta01] E. Stade, *Mellin transform of* $\mathrm{GL}(n, \mathbb{R})$ *Whittaker functions*. Amer. J. Math. 123(1), 2001, 121-161.

[Ste87] G. Stevens, *Poincare series on* $\mathrm{GL}(r)$ *and Kloostermann sums*. Math. Annalen 277 (1987), 25-51.

[Tsu21] M. Tsuzuki, *Hecke-Maass cusp forms on* $\mathrm{PGL_n}$ *of large levels with non-vanishing L-values.* arXiv: 2103.07835. Journal of Functional Analysis 281 (2021).

[Wat95] G. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1995.

[Wei48] A. Weil, *On some exponential sums.* Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204-207.

[Wu14] H. Wu, *Burgess-like subconvex bounds for* GL(2) $\times$ GL(1). arXiv: 1209.5950. Geometric and Functional Analysis (24) 2014, no.3, 968-1036.

[Ye93] Y. Ye, *The fundamental lemma of a relative trace formula for* GL(3). Compositio Math. 89 (1993), 121-162.

[Zac19] R. Zacharias, *Periods and Reciprocity I.* arXiv: 1809.10593, International Mathematics Research Notices Vol. 2021, No. 3, 2191-2209.

[Zac20] R. Zacharias, *Periods and Reciprocity II.* arXiv: 1912.01512.

Appendix A

GL(4) Kloosterman Sums

In this appendix, we follow [Ste87] to bound the GL(4) Kloosterman sums introduced in the previous section. The bound we get in Theorem A.0.1 for GL(4) is slightly stronger than that in Theorem 5.0.1 (Chapter 5). We recall and apply the results in Chapter 4 for $n = 4$.

For $w \in W_{G_4}$, we recall that $w(j)$, $j \in \{1, 2, 3, 4\}$ is given by the formula

$$w \cdot e_j = e_{w(j)},$$

where e_1, e_2, e_3, e_4 is the standard basis of column vectors. Fix six numbers $\nu_1, \nu_1', \nu_2, \nu_2', \nu_3, \nu_3' \in p^{-m}\mathbb{Z}_p - \{0\}$. Moreover, without loss of generality, we further assume that $p^{-m} \leq |\nu_i| \leq p^m$ and $p^{-m} \leq |\nu_i'| \leq p^m$ for all $1 \leq i, i' \leq 3$. Recall the definition of additive character ψ_p on $N(\mathbb{Q}_p)$ which is trivial on $N(p^m\mathbb{Z}_p)$

$$\psi_p\left(\begin{pmatrix} 1 & u_1 & * & * \\ & 1 & u_2 & * \\ & & 1 & u_3 \\ & & & 1 \end{pmatrix}\right) = \xi(\nu_1 u_1 + \nu_2 u_2 + \nu_3 u_3). \tag{A.1}$$

The definition of additive character ψ_p' is given in a similar way.

Fix

$$\tilde{c} = \mathrm{diag}(p^s v_4, p^{r-s} v_3, p^{t-r} v_2, p^{-t} v_1), \tag{A.2}$$

where $v_i \in \mathbb{Z}_p^\times$ for $i = 1, 2, 3, 4$ and $v_1 v_2 v_3 v_4 = 1$. Moreover, we further assume that $t, r, s \geq m$, where m is a fixed positive integer.

Theorem A.0.1. *Let $Kl_p(\psi_p, \psi_p'; c, w_{G_4})$ be the local Kloosterman sum attached to the longest Weyl element w_{G_4}. Let ψ_p be as in* (A.1), $\ell = \max(r,s,t)$, $\varrho = \max(t,s)$, $\sigma = \min(t,s)$, *and*

$$C_8 = 8p^{9m}(|\nu_1\nu_3'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_2\nu_2'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_3\nu_1'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}$$
$$\cdot (\ell+m+1)^3(\varrho+m+1)(r+m+1)^2(\sigma+m+1)^2.$$

Then

$$|Kl_p(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| \leq C_8 \min(p^{r+\sigma+\varrho/2+3m}, p^{\varrho+3\sigma/2+r/2+3m}). \tag{A.3}$$

In particular, we have $|Kl_p(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| \leq C_8 p^{7(t+r+s)/8+3m}$.

We have the following observation of the matrix identity

$$g_0 := \begin{pmatrix} \frac{-a_1}{xyz-xv-uz+w} & & & \\ \frac{a_2(yz-v)}{uv-wy} & \frac{a_2(xyz-xv-uz+w)}{uv-wy} & & \\ \frac{-a_3v}{w} & \frac{a_3(w-xv)}{w} & \frac{a_3(wy-uv)}{w} & \\ a_4 & a_4x & a_4u & a_4w \end{pmatrix}$$
$$= \begin{pmatrix} 1 & u_1 & u_4 & u_6 \\ & 1 & u_2 & u_5 \\ & & 1 & u_3 \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \cdot \begin{pmatrix} a_4 & & & \\ & a_3 & & \\ & & a_2 & \\ & & & a_1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x & u & w \\ & 1 & y & v \\ & & 1 & z \\ & & & 1 \end{pmatrix}, \tag{A.4}$$

where

$$u_1 = \frac{a_1(u-xy)}{a_2(xyz-xv-uz+w)}, \quad u_2 = \frac{a_2(w-uz)}{a_3(uv-wy)},$$
$$u_3 = -\frac{a_3v}{a_4w}, \quad u_4 = \frac{a_1x}{a_3(xyz-xv-uz+w)},$$

and

$$u_5 = \frac{a_2(yz-v)}{a_4(uv-wy)}, \quad u_6 = -\frac{a_1}{a_4(xyz-xv-uz+w)}.$$

Now, using the notation above, we let

$$\begin{pmatrix} a_4 & & & \\ & a_3 & & \\ & & a_2 & \\ & & & a_1 \end{pmatrix} = \tilde{c} = \begin{pmatrix} p^s v_4 & & & \\ & p^{r-s}v_3 & & \\ & & p^{t-r}v_2 & \\ & & & p^{-t}v_1 \end{pmatrix},$$

i.e. $a_1 = p^{-t}v_1$, $a_2 = p^{t-r}v_2$, $a_3 = p^{r-s}v_3$, $a_4 v = p^s v_4$. Note that $v_1v_2v_3v_4 = 1$ and $v_i \in \mathbb{Z}_p^\times$ for $1 \le i \le 4$.

From the definition of the set $X(w_{G_4}\tilde{c})$, we can assume that the element

$$\begin{pmatrix} 1 & x & u & w \\ & 1 & y & v \\ & & 1 & z \\ & & & 1 \end{pmatrix} \in N(\mathbb{Q}_p)/N(p^m\mathbb{Z}_p).$$

So we can put $x = p^{-a}x'$, $y = p^{-b}y'$, $z = p^{-c}z'$, $u = p^{-d}u'$, $v = p^{-f}v'$ and $w = p^{-s}w'$, where $x', y', z', u', v', w' \in \mathbb{Z}_p^\times$. Moreover, a, b, c, d, f are all integers and they satisfy $a, b, c, d, f \ge -m$.

From the above matrix identities in (A.4), assuming that the element $g_0 \in X(w_{G_4}\tilde{c}) \subseteq K_m$, we can deduce that

(1) $-\frac{a_1}{xyz-xv-uz+w} \in 1 + p^m\mathbb{Z}_p$, i.e.

$$\mu := p^t v_1^{-1}(p^{-a-f}x'v' + p^{-c-d}u'z' - p^{-a-b-c}x'y'z' - p^{-s}w') \in 1 + p^m\mathbb{Z}_p.$$

(2) $\frac{a_2(xyz-xv-uz+w)}{uv-wy} \in 1 + p^m\mathbb{Z}_p$, i.e.

$$\lambda := p^r(v_1v_2)^{-1}(p^{-b-s}w'y' - p^{-d-f}u'v') \in 1 + p^m\mathbb{Z}_p.$$

(3) $a_4 w \in 1 + p^m\mathbb{Z}_p$, i.e. $v_4 w' \in 1 + p^m\mathbb{Z}_p$.

(4) $a_4x, a_4u \in p^m\mathbb{Z}_p$, i.e. $|a_4x| \le p^{-m}$ and $|a_4u| \le p^{-m}$. This means that $a + m \le s$ and $d + m \le s$.

(5) $-\frac{a_3 v}{w} \in p^m\mathbb{Z}_p$, i.e.$r - f \ge m$, so $f + m \le r$.

(6) $\frac{a_3(w-xv)}{w} \in p^m\mathbb{Z}_p$, i.e.

$$\tilde{m} := p^r(p^{-a-f}x'v' - p^{-s}w') \in p^m\mathbb{Z}_p.$$

Therefore, we have $a + f \le \max(r, s)$.

(7) $\frac{a_2(yz-v)}{uv-wy} \in p^m\mathbb{Z}_p$, i.e.

$$\tilde{n} := p^t(p^{-b-c}y'z' - p^{-f}v') \in p^m\mathbb{Z}_p.$$

Hence we have $b + c \le \max(t, f)$.

Applying Lemma 5.2 in [Ste87], we have more properties on the relevant data:

(8) $a_3a_4y \in p^m\mathbb{Z}_p$, i.e. $k := p^r v_3 v_4 p^{-b} y' \in p^m\mathbb{Z}_p$. Therefore, we have $b + m \leq r$.

(9) $a_3a_4(xy - u) \in p^m\mathbb{Z}_p$, i.e.

$$p^r v_3 v_4 (p^{-a-b}x'y' - p^{-d}u') \in p^m\mathbb{Z}_p, \quad \tilde{t} := p^{r-a-b}x'y' - p^{r-d}u' \in p^m\mathbb{Z}_p.$$

Therefore, we have $a + b \leq \max(r, d)$.

(10) $a_2a_3a_4z \in p^m\mathbb{Z}_p$, i.e. $p^t v_2 v_3 v_4 p^{-c} z' \in p^m\mathbb{Z}_p$. Hence we have $c + m \leq t$.

Remark A.0.2. *The above properties (8)—(10) on $a, b, c, d, f, x', y', z', u', v', w'$ can also be deduced from the following:*

By the matrix identities in (A.4)*, we have $g_0 = uw_{G_4}\tilde{c}u'$. We set $g_0^\iota := w_{G_4} \cdot (g^t)^{-1} \cdot w_{G_4}$. Since $g_0 \in X(w_{G_4}\tilde{c})$, we have $g_0^\iota \in X((w_{G_4}\tilde{c})^\iota) = X(w_{G_4}(w_{G_4}\tilde{c}^{-1}w_{G_4})) \subseteq K_m$. We can deduce above properties (8)—(10) from the definition of K_m. Actually, all the properties (1)—(10) can be achieved from Lemma 5.0.2 (See also Lemma 5.2 in [Ste87]).*

Conversely, if we are given integers a, b, c, d, f with $a, b, c, d, f \geq -m$ and x', y', z', $u', v', w' \in \mathbb{Z}_p^\times$ satisfying the above properties (1)—(10), there exists an elememt $x_{a,b,c,d,f}^{x',y',z',u',v',w'}$ $\in X(w_{G_4}\tilde{c})$ (Lemma 5.0.2) for which

$$u'(x_{a,b,c,d,f}^{x',y',z',u',v',w'}) = \begin{pmatrix} 1 & p^{-a}x' & p^{-d}u' & p^{-s}w' \\ & 1 & p^{-b}y' & p^{-f}v' \\ & & 1 & p^{-c}z' \\ & & & 1 \end{pmatrix} \pmod{N(p^m\mathbb{Z}_p)}. \tag{A.5}$$

Applying the matrix identities in (A.4). Since all the above properties (1)—(10) are satisfied, we see that the element $g_0 \in X(w_{G_4}\tilde{c})$ from Lemma 5.0.2. Hence we can pick $x_{a,b,c,d,f}^{x',y',z',u',v',w'} = g_0$.

Using above notations, we can rewrite $u_i, 1 \leq i \leq 6$ as follows:

$$\begin{aligned}
u_1 &= \mu^{-1} p^{r-t} v_2^{-1} (p^{-a-b} x'y' - p^{-d} u'); \\
u_2 &= \lambda^{-1} p^{t-r} (v_1 v_3)^{-1} (p^{s-c-d} u'z' - w'); \\
u_3 &= -v_3 v' (v_4 w')^{-1} p^{r-s-f}; \\
u_4 &= -\mu^{-1} v_3^{-1} x' p^{s-r-a}; \\
u_5 &= \lambda^{-1} (v_1 v_4)^{-1} p^{t-s} (p^{-f} v' - p^{-b-c} y'z'); \\
u_6 &= \mu^{-1} v_4^{-1} p^{-s}.
\end{aligned}$$

Recall that ψ_p is the nontrivial additive character of $N(\mathbb{Q}_p)$ which is trivial on $N(p^m\mathbb{Z}_p)$. For a, b, c, d, f, and x', y', z', u', v', w' satisfying the above Property (1)—(10), let

$$X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c}) := T(1 + p^m\mathbb{Z}_p) * x^{d,f,u',v',w'}_{a,b,c,x',y',z'}$$

be the orbit through $x^{d,f,u',v',w'}_{a,b,c,x',y',z'}$, and let

$$S^{d,f,u',v',w'}_{a,b,c,x',y',z'}(\psi_p, \psi'_p; \tilde{c}, w_{G_4}) := \sum_{x \in X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})} \psi_p(u(x))\psi'_p(u'(x))$$

be the Kloosterman sum restricted to the given orbit. Now for a, b, c, x', y', z' satisfying the previous Property (1)—(10), we let

$$X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c}) := \bigcup_{d,f,u',v',w'} X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c}),$$

where d, f run over all integers bigger than $-m$, and u', v', w' run over all the elements of $\mathbb{Z}_p^\times$ satisfying Property (1)—(10). Let

$$S_{a,b,c,x',y',z'}(\psi_p, \psi'_p; \tilde{c}, w_{G_4}) := \sum_{x \in X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})} \psi_p(u(x))\psi'_p(u'(x)).$$

Lemma A.0.3. *We have $X(w_{G_4}\tilde{c}) = \coprod_{a,b,c,x',y',z'} X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})$, where a, b, c run over all integers bigger than $-m$, and x', y', z' run over all the elements in the finite coset of $\mathbb{Z}_p^\times/(1 + p^m\mathbb{Z}_p)$ satisfying Property (1)—(10).*

Proof. The proof is the sams as Lemma 5.2 and 5.7 in [Ste87]. Note that for $G_n = \mathrm{GL}_n(\mathbb{Q}_p)$, we have the decomposition

$$K_m = N(p^m\mathbb{Z}_p) \cdot (B^- \cap K_m) \cdot N(p^m\mathbb{Z}_p),$$

where B^- is the opposite Borel subgroup. Hence the matrix $u(x)$ is uniquely determined by the matrix $u'(x)$.

□

Lemma A.0.4. *Let $\ell = \max(s,r,t)$, and $a \leq s-m,\ b \leq r-m,\ c \leq t-m$ be integers which are larger than $-m$. Then we have the inequality*

$$|S_{a,b,c,x',y',z'}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| \leq 8 \cdot p^{6m} \cdot (|\nu_1\nu'_3 p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} (|\nu_2\nu'_2 p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}$$
$$\times (|\nu_3\nu'_1 p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \cdot (\ell+m+1)^3 \cdot p^{-\frac{a+b+c}{2}} \cdot \#(X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})).$$

Proof. The order two involution map $\iota : g \to g^\iota := w_{G_4} \cdot (g^t)^{-1} \cdot w_{G_4}$ sends $X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})$ to $X_{c,b,a,z',y',x'}((w_{G_4}\tilde{c})^\iota)$. Composing ψ_p with ι has the effect of replacing (ν_1,ν_2,ν_3) by $(-\nu_3,-\nu_2,-\nu_1)$ and (ν'_1,ν'_2,ν'_3) by $(-\nu'_3,-\nu'_2,-\nu'_1)$. For $g_0 = u_1 w_{G_4}\tilde{c}u_2 \in X(w_{G_4}\tilde{c})$, we have $g_0^\iota \in X((w_{G_4}\tilde{c})^\iota) = X(w_{G_4}(w_{G_4}\tilde{c}^{-1}w_{G_4})) \subseteq K_m$ by definition. Hence, applying ι to the element $w_{G_4}\tilde{c}$ reverses the roles of t and s. Therefore we may assume that $t \geq s$ without loss of generality.

Note that $\ell = \max(r,t)$. Property (1)—(10) imply that the matrix entries of $u(x)$ and $u'(x)$ lie in $p^{-\ell}\mathbb{Z}_p/p^m\mathbb{Z}_p$ for every $x \in X(w_{G_4}\tilde{c})$. Indeed, by Lemma A.0.3, it is enough to verify this for $x = x^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})$. Appiying the Property (1)—(10). Since

$$\begin{aligned}\mu &= p^{-s}p^t v_1^{-1}(p^{s-d-c}u'z' - w') + p^{-a}p^t v_1^{-1}(p^{-f}v' - p^{-b-c}y'z') \\ &= p^{-s}\lambda v_3 u_2 p^r + p^{-a}v_1^{-1}\tilde{n} \in 1 + p^m\mathbb{Z}_p,\end{aligned}$$

and

$$u_1 = \mu^{-1}p^{-t}v_2^{-1}\tilde{t},\ \ u_5 = -\lambda^{-1}(v_1v_4)^{-1}p^{-s}\tilde{n}$$

for $\tilde{n},\tilde{t} \in p^m\mathbb{Z}_p$, we have

$$u_1 \in p^{-t+m}\mathbb{Z}_p \subseteq p^{-\ell+m}\mathbb{Z}_p,\ \ u_2 \in p^{-r+m}\mathbb{Z}_p \subseteq p^{-\ell+m}\mathbb{Z}_p$$

and

$$u_5 \in p^{-s+m}\mathbb{Z}_p \subseteq p^{-\ell+m}\mathbb{Z}_p.$$

Moreover, $u_3 \in p^{-s+m}\mathbb{Z}_p \subseteq p^{-\ell+m}\mathbb{Z}_p$, $u_4 \in p^{-r+m}\mathbb{Z}_p \subseteq p^{-\ell+m}\mathbb{Z}_p$ and $u_6 \in p^{-\ell}\mathbb{Z}_p^\times$ are directly from the properties. The claim is now easily verified.

Now let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2_{\geq -m} \times (\mathbb{Z}_p^\times)^3$ such that $X_{a,b,c,x',y',z'}(w_{G_n}c)$ is the disjoint union of the $X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_n}c)$ with $(d,f,u',v',w') \in \mathcal{S}$. Then as in Lemma 4.2.2 [Ste87, Th. 4.10], we have

$$S_{a,b,c,x',y',z'}(\psi_p, \psi_p'; \tilde{c}, w_{G_4}) < p^{-3\ell}(1-p^{-1})^{-3} \times \sum_{(d,f,u',v',w') \in \mathcal{S}} \#(X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})) S_{w_{G_4}}(\theta^{d,f,u',v',w'}_{a,b,c,x',y',z'}; \ell), \tag{A.6}$$

where $S_{w_{G_4}}$ is defined in Definition 4.2.1 [Ste87, Def. 4.9] and $\theta^{d,f,u',v',w'}_{a,b,c,x',y',z'} : A_{w_{G_4}}(\ell) \to \mathbb{C}^\times$ is also the character defined in Definition 4.2.1 [Ste87, Def. 4.9] by

$$\begin{aligned}
\theta^{d,f,u',v',w'}_{a,b,c,x',y',z'}(\underline{\lambda} \times \underline{\lambda}') :&= \xi\left(\nu_1 u_1 \lambda_1 + \nu_2 u_2 \lambda_2 + \nu_3 u_3 \lambda_3 + \nu_1' p^{-a} x' \lambda_1' + \nu_2' p^{-b} y' \lambda_2' + \nu_3' p^{-c} z' \lambda_3'\right) \\
&= \xi\Bigg(\frac{(\nu_1 \mu^{-1} v_2^{-1} p^{\ell+r-t}(p^{-a-b}x'y' - p^{-d}u'))\lambda_1}{p^\ell} \\
&\quad + \frac{\nu_2 \lambda^{-1}(v_1 v_3)^{-1} p^{\ell+t-r}(p^{s-c-d}u'z' - w'))\lambda_2}{p^\ell} \\
&\quad + \frac{(\nu_3 v_3 v'(v_4 w')^{-1} p^{\ell+r-s-f})\lambda_3}{p^\ell} \\
&\quad + \frac{\nu_1' p^{\ell-a} x' \lambda_1' + \nu_2' p^{\ell-b} y' \lambda_2' + \nu_3' p^{\ell-c} z' \lambda_3'}{p^\ell}\Bigg).
\end{aligned}$$

By Remark 4.2.3 and Example 4.12 in [Ste87], we have

$$\begin{aligned}
S_{w_{G_4}}(\theta^{d,f,u',v',w'}_{a,b,c,x',y',z'}; \ell) = S_2(&\nu_1 \mu^{-1} v_2^{-1} p^{\ell+r-t}(p^{-a-b}x'y' - p^{-d}u'), \nu_3' z' p^{\ell-c}; p^\ell) \\
&\cdot S_2(\nu_2 \lambda^{-1}(v_1 v_3)^{-1} p^{\ell+t-r}(p^{s-c-d}u'z' - w'), \nu_2' y' p^{\ell-b}; p^\ell) \\
&\cdot S_2(\nu_3 v_3 v'(v_4 w')^{-1} p^{\ell+r-s-f}, \nu_1' x' p^{\ell-a}; p^\ell),
\end{aligned} \tag{A.7}$$

where S_2 is the restricted GL(2)-Kloosterman sum defined in Remark 4.2.3.

By the refined Weil's bound in Remark 4.2.4, we have the inequality

$$|S_2(\nu, \nu'; p^\ell)| \leq (\ell + m + 1) \cdot B_m \cdot (\gcd(|\nu p^m|_p^{-1}, |\nu' p^m|_p^{-1}, p^{\ell+m}))^{1/2} p^{(\ell+m)/2}, \tag{A.8}$$

for $\nu, \nu' \in p^{-m}\mathbb{Z}_p - \{0\}$ (Here we may let $B_m = p^{m/2}$. See Section 9 in [KL13]).

In order to apply the refined Weil's bound, we note that

$$\begin{aligned}
\gcd(|\nu_3 p^{\ell+m+r-s-f}|_p^{-1}, |\nu_1' p^{\ell+m-a}|_p^{-1}, p^{\ell+m}) &\leq \gcd(|\nu_3 \nu_1' p^{2m}|_p^{-1}, p^{\ell+m}) p^{\ell+m-a}, \\
\gcd(|\nu_2 p^{\ell+m+t-r}(p^{s-c-d}u'z' - w')|_p^{-1}, |\nu_2' p^{\ell+m-b}|_p^{-1}, p^{\ell+m}) &\leq \gcd(|\nu_2 \nu_2' p^{2m}|_p^{-1}, p^{\ell+m}) p^{\ell+m-b}, \\
\gcd(|\nu_1 p^{\ell+m+r-t}(p^{-a-b}x'y' - p^{-d}u')|_p^{-1}, |\nu_3' p^{\ell+m-c}|_p^{-1}, p^{\ell+m}) &\leq \gcd(|\nu_1 \nu_3' p^{2m}|_p^{-1}, p^{\ell+m}) p^{\ell+m-c},
\end{aligned}$$

since $\gcd(a,b) \leq \min(a,b)$. Hence we have

$$|S_{w_{G_4}}(\theta^{d,f,u',v',w'}_{a,b,c,x',y',z'};\ell)| \leq (\ell+m+1)^3 \times (|\nu_1\nu_3'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_2\nu_2'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \cdot (|\nu_3\nu_1'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \cdot p^{3\ell+6m-\frac{a+b+c}{2}}.$$

This inequality, together with (A.6), gives

$$\begin{aligned} &|S_{a,b,c,x',y',z'}(\psi_p,\psi_p';\tilde{c},w_{G_4})| \leq (|\nu_1\nu_3'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_2\nu_2'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \\ &\times (|\nu_3\nu_1'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \cdot (\ell+m+1)^3 \cdot (1-p^{-1})^{-3} \cdot p^{6m-\frac{a+b+c}{2}} \\ &\times \sum_{(d,f,u',v',w')\in\mathcal{S}} \#(X^{d,f,u',v',w'}_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})). \end{aligned} \tag{A.9}$$

The sum appearing on the right hand side is equal to $\#(X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c}))$. Since $p \geq 2$, we have $(1-p^{-1})^{-3} \leq 8$, by (A.9). This completes the proof of the lemma. □

Proof of Theorem A.0.1. By the involution map ι, we can assume that $t \geq s$ without loss of generality. Let

$$\begin{aligned} C = {} & 8p^{9m}(|\nu_1\nu_3'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_2\nu_2'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2}(|\nu_3\nu_1'p^{2m}|_p^{-1}, p^{\ell+m})^{1/2} \\ & \times (\ell+m+1)^3(r+m+1)(s+m+1). \end{aligned} \tag{A.10}$$

We define

$$X_{a,b,c}(w_{G_4}\tilde{c}) := \coprod_{x',y',z'} X_{a,b,c,x',y',z'}(w_{G_4}\tilde{c})$$

and

$$S_{a,b,c}(\psi_p,\psi_p';\tilde{c},w_{G_4}) := \coprod_{x',y',z'} S_{a,b,c,x',y',z'}(\psi_p,\psi_p';\tilde{c},w_{G_4}).$$

At first, we deal with the case $t \geq r$.

- If $a+b+c \leq t$ and $d+f \leq r$, then we have

$$\begin{aligned} \#(d,f) &\leq (s+m+1)(r+m+1), \\ \#(u',v',w') &\leq p^{d+s+f+3m}. \end{aligned}$$

So we have

$$\begin{aligned} \#(X_{a,b,c}(w_{G_4}\tilde{c})) &\leq (r+m+1)(s+m+1)p^{a+b+c+3m+d+f+s+3m} \\ &\leq (r+m+1)(s+m+1)p^{r+s+a+b+c+6m}. \end{aligned}$$

Hence by Lemma A.0.4, we have

$$|S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| \leq Cp^{r+s+t/2+3m}.$$

Applying the above Lemma A.0.3, we have

$$\begin{aligned} |Kl_p(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| &\leq C(r+m+1)(s+m+1)(t+m+1)p^{r+s+t/2+3m} \\ &= C_8 p^{r+s+t/2+3m}. \end{aligned}$$

- If $a+b+c \leq t$ and $d+f > r$, then we assume that $d+f = r+k$, where $k \geq 1$. Note that $d+m \leq s, f+m \leq r$, which implies that $k \leq s-2m$. By Property (1)—(10), we have $b+s = d+f = r+k$. Since $\lambda \in 1+p^m\mathbb{Z}_p$, we have $\#\{(u', v', w')\} \leq p^{d+f+(s-k)+3m} = p^{r+s+3m}$. Hence

$$|S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| \leq Cp^{r+s+\frac{a+b+c}{2}+3m} \leq Cp^{r+s+t/2+3m}.$$

- If $a+b+c > t$ and $d+f \leq r$, then by Property (1)—(10) and a similar argument as above, we can assume that $a+b+c = t+h$, where $h \geq 1$. We also see that $h \leq d$ or $h \leq f$. Now since $\mu \in 1+p^m\mathbb{Z}_p$, we have $\#\{(u', v', w')\} \leq p^{(d-h)+f+s+3m}$. Hence we get that

$$\begin{aligned} |S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| &\leq Cp^{d-h+f+s+\frac{a+b+c}{2}+3m} \\ &= Cp^{d+f+s+(t-h)/2+3m} \leq Cp^{r+s+t/2+3m}. \end{aligned}$$

- If $a+b+c > t$ and $d+f > r$, then using the similar argument as above we have $\#\{(u', v', w')\} \leq p^{(d-h)+f+(s-k)+3m}$. Hence

$$\begin{aligned} |S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| &\leq Cp^{d-h+f+s-k+\frac{a+b+c}{2}+3m} \\ &= Cp^{d+f+s+(t-h)/2+3m} \leq Cp^{r+s+t/2+3m}. \end{aligned}$$

Note that in this case, we always have $r+s+t/2 \leq t+3s/2+r/2$. Theorem A.0.1 now follows from the equality

$$Kl_p(\psi_p, \psi'_p; \tilde{c}, w_{G_4}) = \sum_{-m \leq a \leq s, -m \leq b \leq r, -m \leq c \leq t} S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4}).$$

Now we handle the case $r > t$. By a similar argument as above, we obtain

$$|S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| \leq Cp^{r+s+t/2+3m}.$$

Note that if t is small, this bound is not good enough to get a nontrivial upper bound for Kloosterman sums. So we have to bound this in another way.

- Assume that $f > t$, then by previous Property (1)—(10), we have $b + c = f$, and $a + f \leq r$. By Property (1)—(10), we have $\#(u', v') \leq p^{d+f-(a+f-t)+2m}$. If $d + f \leq r$, we see that

$$\begin{aligned}|S_{a,b,c}(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| &\leq Cp^{d+f-(a+f-t)+s+\frac{a+b+c}{2}+3m} \leq Cp^{t+s+d+\frac{b+c}{2}+3m} \\ &\leq Cp^{t+s+\frac{d}{2}+\frac{d+f}{2}+3m} \leq Cp^{t+3s/2+r/2+3m}.\end{aligned}$$

- Assume that $f > t$ and $d + f > r$, by writing $d + f = r + k$, $1 \leq k \leq s + 2m$, we have

$$\begin{aligned}|S_{a,b,c}(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| &\leq Cp^{d+f-(a+f-t)+s-k+\frac{a+b+c}{2}+3m} \\ &\leq Cp^{t+s+\frac{d}{2}+\frac{d+f}{2}-k+3m} \leq Cp^{t+3s/2+r/2+3m}.\end{aligned}$$

  since $\mu, \lambda \in 1 + p^m\mathbb{Z}_p$.

- Assume that $f \leq t$ and $a + b + c > r > t$. Since $\mu \in 1 + p^m\mathbb{Z}_p$, we have $\#(u', v') \leq p^{d+f-(a+b+c-t)+2m}$. Hence by the same argument on the size of $d + f$, we have

$$|S_{a,b,c}(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| \leq Cp^{r+t+s-\frac{a+b+c}{2}+3m} \leq Cp^{t+s+r/2+3m}.$$

- Assume that $f \leq t$, and $a + b + c \leq r$. If $a + b + c \leq t$, then we have

$$|S_{a,b,c}(\psi_p; \tilde{c}, w_{G_4})| \leq Cp^{d+f+s+\frac{a+b+c}{2}+3m} \leq Cp^{t+2s+t/2+3m}.$$

  If $t < a+b+c \leq r$, we write $a+b+c = t+h$, where $1 \leq h \leq r-t$. Since $\mu \in 1+p^m\mathbb{Z}_p$ and $h \leq d$ or $h \leq f$, we have $\#(u', v', w') \leq p^{(d+f-h)+s+3m}$. Therefore we sill have

$$\begin{aligned}|S_{a,b,c}(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| &\leq Cp^{d+f-h+s+\frac{a+b+c}{2}+3m} \leq Cp^{d+f+s-\frac{a+b+c}{2}+t+3m} \\ &\leq Cp^{d+f+s+t/2+3m} \leq Cp^{t+2s+t/2+3m}.\end{aligned}$$

  So in this case, we have

$$|S_{a,b,c}(\psi_p, \psi_p'; \tilde{c}, w_{G_4})| \leq Cp^{d+f+s+\frac{a+b+c}{2}+3m} \leq Cp^{t+2s+t/2+3m}.$$

Note that $t+2s+t/2 \leq t+3s/2+r/2$ if and only if $s+t \leq r$, $t+2s+t/2 \leq r+s+t/2$ if and only if $s+t \leq r$, and $t+3s/2+r/2 \leq r+s+t/2$ if and only if $s+t \leq r$. Hence we have the following inequality:

$$|S_{a,b,c}(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| \leq C \min(p^{r+s+t/2+3m}, p^{t+3s/2+r/2+3m}).$$

This proves (A.3).

We now give a proof of the second claim. If $r+\sigma+\varrho/2 \leq \varrho+3\sigma/2+r/2$, i.e., $r \leq \varrho+\sigma$, then $\sigma+r \leq 3\varrho$, so $r+\sigma+\varrho/2 \leq 7(\varrho+r+\sigma)/8$. If $r+\sigma+\varrho/2 > \varrho+3\sigma/2+r/2$, i.e., $r > \varrho+\sigma$, then $4\sigma < 2r$ and $\varrho+5\sigma < 3r$, so we still have $\varrho+3\sigma/2+r/2 < 7(\varrho+r+\sigma)/8$. This proves that $\min(p^{r+\sigma+\varrho/2}, p^{\varrho+3\sigma/2+r/2}) \leq p^{7(t+r+s)/8}$, as claimed, and hence Theorem A.0.1. □

Remark A.0.5. *Note that the trivial bound for the local Kloosterman sum in [DR98] is the following:*

$$|Kl_p(\psi_p, \psi'_p; \tilde{c}, w_{G_4})| = O_\epsilon(p^{(1+\epsilon)(t+r+s)}).$$

Since $\frac{7}{8} < 1$*, we get a nontrivial bound for the local Kloosterman sum (integral) by applying Stevens' method.*

Remark A.0.6. *The bound in the appendix is better than the bound in Section 5. The bound in Section 5 is* $1 - \frac{1}{4\times4^2-18\times4+22} = 1 - \frac{1}{14} = \frac{13}{14}$*, which is larger than* $1 - \frac{1}{8} = \frac{7}{8}$*. Applying a similar method (Stevens' approach [Ste87]), such kind of non-trivial bound for* GL(4) *Kloosterman sums attached to the longest Weyl element* w_{G_4} *is also achieved by Bingrong Huang with the exponent* $\frac{9}{10} < 1$ *in the appendix of [GSW21]. After a more careful estimation for the case* $r > t$*, we can slightly improve the bound in [GSW21] by moving the exponent from* $1 - \frac{1}{10} = \frac{9}{10}$ *to* $1 - \frac{1}{8} = \frac{7}{8}$*.*

Remark A.0.7. *The result is not optimal. To improve the bound in some cases, one may use the stationary phase formulas as Dabrowski and Fisher did for* GL(3) *(See [DF97]).*