ESSAYS ON REPEATED GAMES AND REPUTATIONS

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Abstract

This dissertation is comprised of two papers, which use and aim to extend the tools of repeated games and reputations to analyze strategic interactions between two parties that can explain various economic phenomena.

The first paper, "Reputation Effects in Two-Sided Incomplete-Information Games," studies reputation effects in a class of games with imperfect public monitoring and two long-lived players, both of whom have private information about their own type and uncertainty over the types of the other player. In particular, players may be either a *strategic* type who maximizes expected utility or a (simple) *commitment* type who finds it optimal to play a prespecified action every period. The strategic players may gain from opponent player's uncertainty about their types, by trying to convince the opponent that they are non-strategic. As in standard models, the (false) reputation of a strategic type of player for being the commitment type is established by mimicking the behavior of the commitment type. The distinct feature of my model is that *both* strategic players aim to establish a (false) reputation for being the commitment type. The class of games I consider, namely *one-sided binding moral hazard at the commitment profile*, encompasses a wide range of economic interactions between two parties that involve hidden-information (e.g. between a regulator and a regulatee) or hidden-action (e.g between an employer and an employee), where the reputation concerns of both parties are apparent. In both games, one party (principal) prefers that the other party (agent) play in a specific way and use costly auditing to enforce this behavior. The principal aims to establish a reputation for being *diligent*; whereas the agent want to build a reputation

for being *virtuous*. Long-run equilibrium analysis requires to examine the evolution of reputations, i.e. what happens to false reputations in the long-run. Extending the techniques of Cripps, Mailath, and Samuelson (2004), I find that neither strategic player can sustain a reputation for playing a noncredible behavior, i.e. a behavior which is not optimal given that the opponent is best responding in the stage game. Hence, in this class, the true types of both players will be revealed eventually in all Nash equilibria uniformly and the asymmetric information does not affect equilibrium analysis in the long-run.

The second paper, "Strategic Communication vs. Strategic Auditing with Reputation Concerns on Both Sides," studies misrepresentation of information, by a privately-informed agent, to an authority figure. Misrepresenting private information by agents, even under the existence of a regulator or monitor, is a common feature of many economic interactions. For instance, a bank who has private information about its financial health may misreport this information to a regulator; or a tax payer may fill out false income statements; or an investor may be engaged in fraudulent behavior by misrepresenting its books or have false fillings to a regulatory agency. Moreover, in most of such situations, the regulator or the monitor, who is supposed to detect deviations from the desirable behavior, may himself have an incentive to be engaged in moral hazard because of costly or timely monitoring. Moreover, both parties may have some prior or established beliefs about the other party and concerns about their reputation (for good behavior) or lack of it. The goal of this paper is to understand how private information can be manipulated by a strategic Sender (through false messages), in the *presence* of a strategic Receiver, who aims to deter the manipulation of information using costly auditing, when their interactions are not contractable and both parties have reputation concerns.

More specifically, the Sender ('she') has *noisy* private information about an underlying state of nature and is able to misrepresent this information by sending false messages. There is a strategic Receiver ('he') who aims to deter this manipulation using costly auditing. The magnitude of the Senders cost of lying is governed by the auditing strategy of the Receiver, which determines the probability of an audit and detecting an undesirable behavior of the Sender. The Receiver, on the other hand, may have an inventive not to audit intensively if he thinks that the Sender is going to give the accurate information (since auditing is costly). Receiver believes that the Sender could be an honest type with some strictly positive probability. An honest type always sends the true message, whereas a strategic Sender maximizes expected payoffs. Similarly, the Sender believes that the Receiver could be a tough type with some strictly positive probability. A tough Receiver always chooses high auditing whereas a strategic Receiver maximizes expected payoffs. The fact that the private information of the Sender is imperfect and the auditing by the Receiver is random prevent players from learning each others true types (strategic). To model this environment, I use a simultaneous-move version of an *inspection game*, with incomplete-information about the types of players, when their actions are not observable. This paper aims to analyze how uncertainty about each other's types and the concerns for (false) reputation pay of for both parties; and characterize the equilibria in the (1) one-shot game; (2) two-period game. The equilibrium strategies are determined by the parameters of the model, as well as the discount factors (in the two-period game). The infinitely-repeated game of strategic communication vs. strategic auditing fits into the class of games analyzed in the first paper; and thus, none of the parties can fool the other party indefinitely, i.e. their true types will be revealed eventually.

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Chapter 1

Reputation Effects in Two-Sided Incomplete-Information Games

1.1 Introduction

This paper studies long-run sustainability of (false) reputations in a class of games with imperfect public monitoring and two long-lived players, both of whom have private information about their own type and uncertainty over the types of the other player. In particular, players may be either a *strategic* type who maximizes expected utility or a (simple) *commitment* type who finds it optimal to play a prespecified action every period. The strategic players may gain from opponent player's uncertainty about their types, by trying to convince the opponent that they are non-strategic. As in standard models, the reputation of a strategic type of player for being commitment type is established by mimicking the behavior of the commitment type.[1] The distinct feature of my model is that, since there is uncertainty over the types of both players, *both* players aim to establish a false reputation for being the commitment type in order to induce the opponent player behave in a specific way. I believe that wanting to establish a reputation is a key concern for *all* parties involved in several economic interactions. Specifically, in the economic applications that can be described by the class of games I study (that is to be discussed in the subsequent sections), the reputation concerns of both parties are apparent. Long-run equilibrium analysis requires to examine the evolution of reputations, i.e. what happens to false reputations in the long-run. This paper examines the sustainability of reputations in these games when both players have concerns about their own reputation and the reputation of the opponent player. I find that neither (strategic type of) player can sustain a false reputation for playing a noncredible behavior,

[1] In most studies, a player aims to establish reputation of being the Stackelberg type, by mimicking this type's commitment action, i.e. the Stackelberg action (an action one would like to commit given that such a commitment induces a best response from the opponent player).

i.e. a behavior which is not optimal given that the opponent player is best responding to this in the stage game where the true types (strategic) were to be known. Hence, in this class, the true types of both players will be revealed eventually and the asymmetric information does not affect equilibrium analysis in the long-run. I believe this is the only class of two-sided incomplete-information games with simple commitment types and imperfect public monitoring, where reputations for noncredible behavior disappears in the long-run, in *all* Nash equilibria. To do so, I provide an example where reputations for noncredible behavior may be sustained in a Nash equilibrium.

This class of games encompass a wide range of economic applications between two parties that involve hidden-information (e.g. between a regulator and regulatee) or hidden-action (e.g. between an employer and employee). The common feature of these economic interactions is that one party (the principal) prefers that the other party (the agent) play in a certain way (e.g. to be truthful in a hidden-information setting, and to exert high effort in a hidden-action setting) and use costly auditing to enforce this behavior. The principal can choose either to be lazy or diligent in auditing the agent, which results in different probabilities of detecting an undesirable behavior of the agent. The agent prefers the principal to be lazy. Suppose that the payoffs of the stage game is defined such that if the agent thinks that the principal is diligent, she is better off by behaving properly, i.e. the way the principal wants her to play. Otherwise, if she thinks that the principal is lazy, she has an incentive to play some other action, which is not preferred by the principal. On the other hand, when the principal thinks that the agent chooses the proper action, he would like to be lazy in auditing. Otherwise, he is better off by being diligent. Suppose that the agent believes that with some probability the principal is a *tough* type who always chooses to be diligent in auditing. Then, the principal aims to establish a reputation for being the tough type, by choosing diligence every period, in order to enforce the agent to choose the proper behavior. Suppose also that the principal believes that the agent could be a *virtuous* type who acts properly every period. Then, the agent aims to establish a reputation for being the virtuous type to induce the principal to be lazy. The fact that the actions of the two parties are not observable to each other prevents them learning each other's true type.

These games belong to a class, which I call *one-sided binding moral hazard at the commitment profile*, as only one of the players has an incentive to deviate from the commitment action given that her opponent is of the commitment type or plays like the commitment type.[2] The properties of these games will be discussed in detail in subsequent sections. However, I would like to point out that this class captures games with the following

[2]The commitment action profile in the principal-agent game is (proper behavior, diligence). The only player who has an incentive to deviate from this profile is the principal, i.e. the principal wants to deviate to be lazy given that the agent chooses to behave properly. For the agent, the best reply against a diligent principal is to behave properly.

property: The interest of one of the players is aligned with the behavior of the commitment type of the opponent player; whereas the interest of the other player conflicts with the behavior of the commitment type of opponent. For instance, the interest of the strategic principal is aligned with the behavior of the virtuous type of the agent. However, the strategic agent's interest conflicts with the behavior of the tough principal since she prefers the principal to be lazy. I study long-run behavior of reputations in games with one-sided binding moral hazard at commitment profile when the actions of the players are not perfectly observable (imperfect public monitoring).

Long-run equilibrium analysis requires to examine the evolution of reputations. The theory of reputations in standard infinitely-repeated games mostly focuses on only one of the players having uncertainty over her types (call this player the *informed* player) and wanting to build a false reputation. The main results about how reputations evolve are: (1) There is a reputation-building phase in which the uninformed player is eventually convinced to see the commitment action if the informed player has been playing it. (2) There is a reputation-destruction phase, when the actions of the informed player is not observable, if the reputation is for a noncredible behavior, i.e. the commitment action that the informed player is mimicking is not optimal given that the uninformed player is best responding. The reputation is destructed optimally by the informed player as a trade off between the loss of reputation and the payoff gain by deviating to an action other that the commitment action, knowing that such a deviation cannot be detected unambiguously when her actions are not observable. The main contribution of this paper to the literature is to show that these central results about the evolution of reputations are robust introducing uncertainty over the types of the second player in games with one-sided binding moral hazard at the commitment profile. Extending the techniques of Cripps, Mailath, and Samuelson (2004), I find that neither player can sustain reputation for playing a strategy that is not equilibrium of the complete-information stage game - the game without uncertainty over the types.

In section 1.1.1, I provide an example of the class of games considered in this paper. The reader who wants to continue with the discussion of the general set up and results may jump to section 1.1.2.

1.1.1 Example

Consider the repeated interaction between a regulator and a regulatee where the possible actions for the regulatee are to be truthful or untruthful about some noisy information she gets regarding the state of nature that is realized at the end of the period; and those for the regulator are to be lazy or diligent in auditing the regulatee.[3] For instance, the regulatee could be a bank which gets a noisy information about its own financial health and the

[3]This game can be considered as a variant of *inspection games* extensively studied in the literature. I refer the reader Rudolf, Bernhard, and Zamir (2002) for a discussion on inspection games.

regulator could be a government official.[4,5]

The actions of the regulatee, i.e. to be truthful or untruthful, are not observable to the regulator. However, the regulator observes if the message sent by the regulatee is correct or not (i.e. if it matches the realized state of nature or not). Since the regulatee's information about the state of nature is noisy; an incorrect message can come from a truthful behavior, as well as a correct message can come from an untruthful behavior. Similarly, the actions of the regulator, being lazy or diligent that induce different probability of audit, are not observable to the regulatee. [6] However, she observes if there is an audit or not at the end of the period, which may result after a lazy or a diligent behavior. The probability of correct message is higher if the regulatee is truthful; similarly, the probability of an audit is higher if the regulator is diligent. The regulator prefers the regulatee to be truthful and the regulatee prefers the regulator to be lazy.

The expected (ex-ante) payoffs of the players are given by Table 1.1.1. Row player is the regulatee who chooses to be truthful (T) or untruthful

[4]The other possible applications for which these games can be used to model include analyzing the interaction between an employer and employee; tax evasion through the interaction between a tax payer and tax collecting agency; or the asset market manipulation via strategic announcements of an insider in the presence of a regulator.

[5]Benabou and Laroque (1992) provide a model of repeated strategic communication that analyzes manipulation in asset markets, where they extend Sobel (1985)'s model to the case in which the sender (insider) has noisy private information about the value of an asset. The sender can deceive public and distort the asset price through strategic announcements. However, their model is missing a strategic receiver who can audit the sender. The second chapter of this thesis provides a model with a strategic receiver.

[6]One can interpret this as the regulator chooses between two mixed strategies; or allocates some resources or time to auditing among its other tasks.

(U) and the column player is the regulator who chooses to be diligent (D) or lazy (L).

Table 1.1: Expected Payoff Matrix

	L	D
T	x, y	$x - l_1,\ y - c$
U	$x + g,\ z$	$x - l_2,\ z - c + d$

where $y, z, g, c, d > 0$, $l_2 > l_1 \geq 0$ and $y > y - c > z - c + d > z$.

The regulatee's best response against the choice of being lazy is to be untruthful, since she has an expected gain of g. However, the regulatee's best response when the regulator is diligent in auditing is to be truthful, since the expected loss from untruthfulness when the regulator is diligent l_2 is higher than l_1.[7] For the regulator, the best response when the regulatee is truthful is to be lazy, since the diligence in auditing has a cost of c. On the other hand, the regulator's best response is to be diligent when the regulatee is untruthful, since there is a expected gain d from possible detection of the untruthful behavior. The regulator gets his highest payoff when he is lazy and the regulatee is truthful; whereas the regulatee gets her highest payoff when the regulator is lazy and she is untruthful. Thus, the regulator wants to convince the regulatee that he is diligent to enforce truthfulness. However, the best response of the regulator if the regulatee is truthful is to be lazy.

[7] I allow l_1, the expected loss for the regulatee when the regulator is diligent and the regulatee is truthful, to be positive as well as zero because of the noise in signals regarding her behavior.

On the other hand, the regulatee prefers the regulator to be lazy, and thus the regulatee wants to convince the regulator that she is truthful to enforce laziness in auditing. However, the regulatee's best response once she thinks that the regulator is lazy is to be untruthful.

Let α_1 be the strategy of the regulatee and α_2 be the strategy of the regulator (in the stage game). There is unique mixed strategy Nash equilibrium of the stage game, which is $\alpha_1(T) = \frac{d-c}{d}$ and $\alpha_2(D) = \frac{g}{g+l_2-l_1}$ that provides a payoff of $x - \frac{g.l_1}{g+l_2-l_1}$ to the regulatee and $y - \frac{c(y-z)}{d}$ to the regulator. Note that the unique equilibrium profile is Pareto dominated by the profile $(Truthful, Lazy)$ which gives x and y to the regulatee and regulator, respectively. Also, as the expected gain d from detecting untruthfulness for the regulator approaches to the cost of monitoring c, then the regulatee, in equilibrium, chooses to be truthful with a smaller probability. On the other hand, as the expected gain g from untruthfulness increases, the regulator would choose to be diligent with a higher probability.

Suppose that there are private types for both of the players and players have uncertainty over the types of the other player. More specifically, the regulator believes that the regulatee is a *virtuous* type, who is truthful in every period, with some probability. The regulatee wants to use regulator's uncertainty over her types and pretend to be the virtuous type (by acting like the virtuous type) to enforce the regulator to be lazy in the continuation play. On the other hand, the regulatee believes that the regulator is a *tough* type, who is diligent in every period, with some probability. Then the regulator may find it worthwhile to exploit regulatee's uncertainty over his type by pretending to be the tough type to induce truthfulness. Since the actions of the players are not observable to each other, they can't learn each other's true types for sure. I would like to point out that the incentives of the regulator is aligned with the virtuous type of the regulatee in the sense that the regulator prefers the regulatee to be truthful and a virtuous regulatee is always truthful. On the other hand, the incentives of the regulatee conflicts with that of a tough regulator as the regulatee prefers the regulator to be lazy, but a tough regulator is always diligent. The set of equilibrium strategy profiles and payoffs can be found in Chapter 2.

I analyze what happens to the reputations (for being tough and virtuous) of the regulator and the regulatee in the limit in order to understand long-run equilibrium behavior and payoffs. For instance, if there were to be a Nash equilibrium where both of the reputations are sustainable, i.e. the types of the players are not revealed, this means that each player should be seeing similar public signals on average from both types of the opponent player, since they cannot distinguish between the types (given that the signals are statistically informative about both players' actions). But, this is only possible if both types of the players act the same on average in the limit. Then the regulator should be diligent and the regulatee should be truthful on average indefinitely, which wouldn't be efficient since this profile is Pareto inferior to always playing the profile $(Truthful, Lazy)$. I show this is not the case: neither of the players' reputation is sustainable, i.e. reputations of being tough and virtuous disappear in the limit, when players are not indeed those types. Hence, the asymmetric information over the types of players does not affect the equilibrium analysis, in the sense that any Nash equilibrium of the game converges to the Nash equilibrium of the game without uncertainty over the types.

1.1.2 Approach and Result

I present a class of games with imperfect public monitoring and incomplete-information over the types of both players. Each player can be either a (simple) *commitment* type who plays an exogenously specified stage game action every period independently of the history of the play[8] or a *strategic* type who maximizes expected payoffs. The stage games and the stage game commitment action profile I allow are restricted. A (strategic) player is *subject to binding moral hazard at the commitment action profile* if he/she finds it suboptimal to play the commitment action given that the opponent player is playing his/her commitment action or is the commitment type. The key condition is that the stage games considered in my model have "one-sided binding moral hazard at the commitment profile," which requires exactly

[8]A repeated game behavior strategy is called a *simple* strategy if it is a constant function of histories, i.e. for every period independent of history of the play, it assigns the same (possibly mixed) stage game action.

one player being subject to binding moral at the commitment action profile. Hence, for only one of the players (say player 2 - the principal), the best response to the commitment action of the opponent is not his commitment action, while for the other player (player 1 - the agent), her best response to the commitment action of the opponent player is her commitment action. So, there is only one player who has incentives to deviate to another action at the commitment action profile. Moreover, I require a monitoring structure that ensures that the public signals are statistically informative about each player's actions; and also, that allows players to infer the opponent's beliefs about their own type (thus makes the beliefs of each player *public*). The assumptions I make on monitoring structure enable players to identify any fixed stage game action of the other player from frequencies of signals after sufficiently many observations and compute the other player's posterior belief about their own types.

In this setting, I show that reputations of both of the players disappear eventually if the commitment actions are not part of an equilibrium for the strategic types in the stage game. More precisely, if for both of the players, their best response to the best response of the opponent to their commitment strategy is not their commitment strategy, then in any Nash equilibrium of the incomplete-information game, the true types will be revealed (almost surely).

The techniques of the proofs are borrowed from Cripps, Mailath, and

Samuelson (2004), who study games with imperfect public monitoring in which only one of the players has uncertainty about the types of the other player. The key condition I require on the stage games, i.e. the *one-sided binding moral hazard at the commitment action profile*, makes the long-run behavior of the reputation of the player who is subject to binding moral hazard (the principal) independent of the reputation of the player who is not (the agent); whereas, this condition necessitates the behavior of the reputation of the player who is *not* subject to binding moral hazard (the agent) depend on the behavior of the reputation of the player who is subject to binding moral hazard (the principal). Let player 2 (the principal) be the one who is subject to binding moral hazard at the commitment action profile. For instance, if player 2 were to play the commitment action all the time, there is no incentive for player 1 (the agent) to play something other than her commitment strategy since player 1's best response to the commitment action of player 2 is her commitment action. Thus, player 1's type won't be revealed unless player 2's type is revealed. In other words, there won't be any set of histories with positive probability on which player 1's type is revealed but not player 2's. After establishing that player 2's type is (almost) revealed in the long-run, I can show that player 1's true type should be revealed as well. Hence, the one-sided binding moral hazard condition allows us to break the analysis of the long-run behavior of the reputations of the two players into two stages:

1. I first show that the reputation of player 2 is not sustainable in the long-run: Suppose on the contrary that there is a positive probability set of events in which the type of player 2 is not revealed, i.e. if both types of player 2 are given strictly positive probability in the long-run. It means that the play of the strategic type of player 2 is not distinguishable from the play of the commitment type in the limit (given that the public signals are statistically informative about a player's actions). Hence, in the event in which the type is not revealed, the strategic and commitment type must be playing the same way on average. The strategic type of player 1 is going to best reply to the commitment action of player 2, which is the same as her commitment action. Thus, player 1 must be playing the same strategy on average, too. Hence, the posterior beliefs about player 1's type will not change in these histories. For any (fixed) belief about player 1's type, since the commitment strategy of player 2 is not his best response to player 1's commitment action, he can gain by playing the best response to player 1's commitment action (which is different than his commitment action), knowing that the deviations from equilibrium play cannot be unambiguously detected by player 1 (because of imperfect monitoring). Then, the strategic type of player 2 has a trade off between the loss of reputation and the current payoff gain. The convergence of beliefs in the limit ensures that any signal has an arbitrarily small effect

on players' beliefs. And thus, when reputations are public, player 2 eventually knows that player 1's beliefs have nearly converged and be sure that deviations from the commitment strategy have arbitrarily small effect on the payoffs from the continuation play (due to discounting). But then the strategic and the commitment type of player 2 are expected to play differently, which contradicts to the belief of player 1 about both types of player 1 playing the same strategy in the limit (on a positive set of histories). This establishes the fact that there cannot be positive set of events in which player 1's type is not revealed eventually. Moreover, the disappearance of player 2's reputation is uniform across all Nash equilibria, i.e. there is some period T after which reputation of Player 2 of being the commitment type converges to zero (almost surely) in all Nash equilibria (according to the probability measure induced by the strategic type of player 2's play).

2. Given that player 2's type is revealed (almost surely) after some T in any Nash equilibria, the game "approaches" to the one where there is uncertainty over the types of player 1 only. I show that player 1's reputation for being the commitment type converges to zero, in the same fashion. The only problem could be that even though player 2's type is revealed, he could keep playing the commitment strategy on average. In this case, the strategic player 1 would give a best response

which coincide with her commitment type and the type wouldn't be revealed. But this can't happen since in the events that type of player 1 is not revealed and the strategic and commitment type of player 1 play approximately the same way, so the strategic type of player 2 finds it optimal to play the best response against player 1's commitment action, which will decrease the reputation of player 2 even more after T.[9] So, player 1 expects to see a best reply to her commitment strategy from the strategic type of player 2 with a high probability. Thus, player 1 best responds to the strategic type of player 2 who gives a best reply to the commitment strategy of player 1 with high probability. However, as the strategic type player 1's best response to player 2's strategy is different than her commitment strategy, the strategic and the commitment type of player 1 are expected to play differently, which reveals her true type eventually.

These results imply that the asymmetric information about the types of players does not interfere in the long-run equilibrium behavior. I show that continuation play in every Nash equilibrium of the incomplete-information game is converges to an equilibrium of the complete-information game.

The implication of the results for the game between a regulator and a regulatee, presented in Section 1.1.1, is that the reputation of being tough for

[9]More precisely, the histories for which player 2's reputation can be rebuilt would have measure 0.

the regulator disappears in the long-run (regardless of the long-run behavior of the regulatee's reputation of being virtuous) since the regulator is the player who is subject to binding moral hazard at the commitment profile. It is shown that after his true type is almost known, the regulatee starts to take advantage of regulator's uncertainty over her type and regulatee's reputation of being virtuous disappears eventually as well. Intuitively, the regulatee waits for the revelation of the regulator's type, before revealing her type. These results suggest that if there is a possibility that the regulator is replaced every period so that the uncertainty about the regulator's type renewed every period, then the regulatee is never fully convinced that the regulator is tough. In this situation, the regulator keeps playing *diligent* and will not have incentive to deviate since the regulatee is not convinced.[10] So, one way to sustain the reputations is to introduce the possibility that keeps the uncertainty over the type of regulator every period. However, this will create an equilibrium which is not efficient. That is why the replacements should be strategically scheduled.

The requirements of my model satisfy all the assumptions of Cripps, Mailath, and Samuelson (2004), and thus if there is uncertainty about the types of one player only (either of the players), their result hold and the true type of the informed player is revealed eventually. In this paper, I am able to show that Cripps, Mailath, and Samuelson (2004)'s result is robust

[10] Mailath and Samuelson (2001) and Phelan (2006) provide models where the long-lived informed player's type is governed by a stochastic process that has long-run implications.

to introducing uncertainty over the type of the second player in this class of games.

1.1.3 Related Literature

Most of the early literature on games with reputation concerns focus on settings in which a long-lived player faces a sequence of short-lived players, each of whom plays only once but observes the previous play and believes that the long-lived player might be committed to some exogenously specified strategy. In this environment, Fudenberg and Levine (1989) and Fudenberg and Levine (1992) provide a lower bound on the long-lived player's average payoff, namely the stage game Stackelberg payoff,[11] given that she is sufficiently patient. Following the tradition of long-lived player gains from short-lived players' uncertainty over her types, Schmidt (1993) and Celentani, Fudenberg, Levine, and Pesendorfer (1996) show reputation effects arise in settings that involve two long-lived players and can be stronger than when the uninformed player is short-lived, i.e. the informed player may achieve a payoff that exceeds her stage game Stackelberg payoff. But, it is also well established that reputation results may fail when both players

[11]Stackelberg payoff is the payoff players receive in the stage game when they play their Stackelberg action (i.e. the action players would like to commit given that such a commitment induces a best response from the opponent player) and the opponent best responds to it. Stackelberg action is the action players would like to choose in an extensive-form game when they move the first.

are long-lived.[12] More specifically, reputation results hold in two-person repeated games, in which simultaneous-move stage game is played by equally patient agents and the actions are observable, if the stage game is a game of strictly conflicting interest[13] (Cripps, Dekel, and Pesendorfer (2005)) or the Stackelberg action is a dominant action in the stage game (Chan (2000)). For stage games other than those, Cripps and Thomas (1997) show that any individually rational and feasible payoff can be sustained in perfect equilibria of the infinitely repeated game, if the players are sufficiently patient. Atakan and Ekmekçi (2009b) provide that, if the uninformed player's actions are not observable, for a restricted class of stage games, the informed player receives her highest payoff in any Bayesian Nash equilibrium as players become patient. They restrict attention to locally non-conflicting interest game,[14] which also encompasses the class of games Cripps and Thomas (1997) and Chan (2000) consider to establish a folk theorem result under perfect observability of players' actions. Atakan and Ekmekçi (2009a) also show that a sufficiently patient informed player, whose type is unknown, can achieve his highest payoff in any perfect Bayesian equilibrium of the

[12]See Schmidt (1993), Celentani, Fudenberg, Levine, and Pesendorfer (1996), Cripps and Thomas (1997).

[13]A game has strictly conflicting interests if a best reply to the Stackelberg action of player 1 yields the best feasible and individually rational payoff for the informed player, say player 1, and the minmax for the uninformed player 2. See Mailath and Samuelson (2006) for a formal definition.

[14]A stage game is a locally non-conflicting interest game if player 2 (uninformed) receives a payoff that strictly exceeds her minmax value in the profile where player 1 (informed) receives his highest payoff. See Atakan and Ekmekçi (2009b) for a formal definition.

infinitely repeated game, where in each period equally patient players play an extensive-form stage game of perfect information. Moreover, Atakan and Ekmekçi (2009c) provide a two-sided reputation result with two equally patient players (for a restricted class of stage games) if there is a possibility that the players are Stackelberg types. The equilibrium behavior and payoff will look like a war of attrition.

All these papers aim to answer the effects of incomplete-information on the equilibrium payoff set and how an informed player can use the uncertainty over her types to achieve higher payoffs; rather than the asymptotic behavior of reputations. The main concern of this paper is to understand what happens to false reputations in the long-run. Cripps, Mailath, and Samuelson (2004) show that a long-lived player can maintain a permanent reputation for playing a commitment strategy in a game with imperfect monitoring only if that strategy plays an equilibrium of the corresponding complete-information stage game. They also prove that the continuation play in every Nash equilibrium of the incomplete-information game is Nash equilibrium of the complete-information game. Thus, the powerful results about the lower bounds on the long-lived informed player's average payoff are short-run reputation effects, where the long-lived informed player's payoff is calculated at the beginning of the game. Cripps, Mailath, and Samuelson (2007) extend their earlier result to games with two long-lived players where the uninformed long-lived player has private beliefs over the types of informed long-lived player, so that the reputation of the informed player is *private*. Also, an earlier result on the long-run properties of reputations is by Benabou and Laroque (1992), who study a game with a long-lived player who can be one of two types, honest or opportunist, and a continuum of myopic players in asset markets. They focus on the Markov perfect equilibrium of this game where the actions of the long-lived player is not observable. They show that the long-lived player reveals her type in any Markov perfect equilibrium. All these results are for games with imperfect public monitoring and uncertainty over the types of only one of the players. I show that this result extends to games of two-sided incomplete information with one-sided binding moral hazard at the commitment action profile. Best to my knowledge, this is the first paper that attempts to answer what happens to reputations in the long-run in two-sided incomplete-information games.

The paper is organized as follows: Section 1.2 describes the model, Section 1.3 states the main results of the paper and Section 1.4 proves Proposition 1.3.2 and 1.3.5, which lead to the proof of the main result Theorem 1.3.1.

1.2 Model

This section first defines the complete-information game, the game without uncertainty over the types of players (i.e. the game when both players are *strategic* types). Then I present the incomplete-information game by adding commitment types of players to the model.

1.2.1 Complete-information game

This is an infinitely repeated game with imperfect public monitoring. The stage game is a two-player finite simultaneous-move game. Player 1 ("she") chooses an action $i \in I \equiv \{1, ..., \bar{I}\}$ and player 2 ("he") chooses an action $j \in J \equiv \{1, ..., \bar{J}\}$. The public signal y is drawn from the finite set Y. The probability that the public signal is realized under the action profile (i, j) is given by ρ_{ij}^{y}. The ex post (realized) stage game payoff to player 1 (resp., 2) from action i (resp., j) and signal y is given by $u_1(i, y)$ (resp. $u_2(j, y)$). The ex ante (expected) stage game payoffs are $\pi_1(i, j) = \sum_y u_1(i, y)\rho_{ij}^{y}$ and $\pi_2(i, j) = \sum_y u_2(j, y)\rho_{ij}^{y}$.

Both players are long-lived with discount factor $\delta_1 < 1$ for player 1 and $\delta_2 < 1$ for player 2. The set of histories is $h_t^f \equiv ((i_0, j_0, y_0), ..., (i_{t-1}, j_{t-1}, y_{t-1})) \in H_t^f \equiv (I \times J \times Y)^t$. Each player observes the realization of the public signal and his or her own past actions. Player 1's private history is denoted by $h_{1t} \equiv ((i_0, y_0), ..., (i_{t-1}, y_{t-1})) \in H_{1t} \equiv (I \times Y)^t$. Similarly, player 2's private history is denoted by $h_{2t} \equiv ((j_0, y_0), ..., (j_{t-1}, y_{t-1})) \in H_{2t} \equiv (J \times Y)^t$. And, the public history observed by both players is $h_t \equiv (y_0, ..., y_{t-1}) \in H_t \equiv Y^t$. The filtration on $(I \times J \times Y)^\infty$ induced by the private histories of player $m = 1, 2$ is denoted by $\{\mathcal{H}_{mt}\}_{t=0}^\infty$, while the filtration induced by the public histories is denoted by $\{\mathcal{H}_t\}_{t=0}^\infty$. Player 1's strategy, $\sigma \equiv \{\sigma_t\}_{t=0}^\infty$, is a sequence of maps $\sigma_t : H_{1t} \to \Delta(I)$. Similarly, Player 2's strategy, $\tau \equiv \{\tau_t\}_{t=0}^\infty$, is a sequence of maps $\tau_t : H_{2t} \to \Delta(J)$.

The payoffs in the infinitely repeated game are normalized discounted sum of stage game payoffs, $(1 - \delta_m) \sum_{s=0}^\infty \delta_m^s \pi_m(i_s, j_s)$ for player $m = 1, 2$. The average discounted payoffs in period t is denoted by $\pi_{mt} \equiv (1 - \delta_m) \sum_{s=t}^\infty \delta_m^{s-t} \pi_m(i_s, j_s)$.

It is assumed that the public signals have full support (Assumption 1). So, every public signal is possible after any action profile and players can not infer the actions chosen by the other player perfectly after a signal. Full support assumption prevents perfect inference of actions after any signal. We also assume "individual full rank" conditions, so that after sufficiently many observations, any fixed stage game action of either player can be identified from the frequencies of the signals (Assumptions 2 and 3).

Assumption 1 *(Full support) For all $(i, j) \in I \times J$ and $y \in Y$, $\rho_{ij}^y > 0$.*

Assumption 2 *(Individual 1 full rank) For all $j \in J$, the I columns in the matrix $(\rho_{ij}^y)_{y \in Y, i \in I}$ are linearly independent.*

Assumption 3 *(Individual 2 full rank) For all $i \in I$, the J columns in the matrix $(\rho_{ij}^y)_{y \in Y, j \in J}$ are linearly independent.*

Assumption 2 and 3 ensure that, for each player, the distribution of signals generated by any (possibly mixed) action is statistically distinguishable from any other for any given action of the other player. Note that these conditions require that $|Y| \geq \max\{|I|, |J|\}$.

A strategy profile (σ, τ) induces a probability distribution $P_{(\sigma,\tau)}$ over $H_\infty^f \equiv (I \times J \times Y)^\infty$. We denote the expectation with respect to this distribution by $E_{(\sigma,\tau)}$.

Definition 1.2.1 *A Nash equilibrium of the complete information game is a strategy profile (σ^*, τ^*) such that $E_{(\sigma^*,\tau^*)}[\pi_{10}] \geq E_{(\sigma',\tau^*)}[\pi_{10}]$ for all σ' and $E_{(\sigma^*,\tau^*)}[\pi_{20}] \geq E_{(\sigma^*,\tau')}[\pi_{20}]$ for all τ'.*

This definition implies that under the equilibrium strategy profile, player m's strategy maximizes continuation expected utility after any history that occurs with positive probability. Note that, with the full support assumption, all public histories occur with positive probability. Hence, any Nash equilibrium outcome is also the outcome of a perfect Bayesian equilibrium.

1.2.2 Incomplete-information game

The uncertainty about players' preferences is modeled with Harsanyi (1967)'s notion of games with incomplete information by introducing a *com-*

mitment type for each player. At time $t = -1$, before the game starts, nature selects a type for both players and tells each player her or his own type privately. With probability $1 - \mu_0 > 0$, player 1 is a "strategic" type, denoted by n, with the preferences described above and with probability $\mu_0 > 0$, she is a "commitment" type, denoted by c, who plays the action $s_1 \in \Delta(I)$ in each period regardless of history. Similarly, with probability $1 - \gamma_0 > 0$, player 2 is a "strategic" type, denoted by n, whose preferences are described above and with probability $\gamma_0 > 0$, he is a "commitment" type, denoted by c, who plays the action $s_2 \in \Delta(J)$ in each period independent of history. [15]

The stage games I consider are restricted. The stage game commitment actions are denoted by (s_1, s_2) and defined to be the *commitment profile.* I assume that each player has unique best reply to the commitment action of the opponent player. Let $r_1 \equiv BR_1(s_2)$ and $r_2 \equiv BR_2(s_1)$ be the best responses of strategic type of player 1 and 2 against the commitment action of their opponent, respectively.

Definition 1.2.2 *Player $m = 1, 2$ is subject to binding moral hazard at the commitment profile (s_m, s_{-m}) if $r_m \neq s_m$.*

Since Assumption 1 ensures that deviations by players are not unambiguously detectable, if player m is subject to binding moral hazard at strategy

[15]Instead of modeling the incomplete-information by behavioral types, I could have modeled the commitment types as agents whose payoffs are different from those of the strategic ones, as in Koren (1992). Then players would know their own payoffs and have uncertainty over the payoffs of the other player.

profile (s_m, s_{-m}), then he has strict incentive to deviate from the profile (s_m, s_{-m}).

Definition 1.2.3 *A game has one-sided binding moral hazard at the commitment profile if there is exactly one player who is subject to binding moral hazard at the commitment strategy profile* (s_1, s_2)*.*

From hereafter, the term one-sided binding moral hazard refers to the one at the commitment profile. The stage games considered in this paper have one-sided binding moral hazard at the commitment profile. The importance of this condition will be discussed in the following sections.

Assumption 4 *The stage game satisfies the following:*

1. *The stage game has one-sided binding moral hazard at the commitment profile* (s_1, s_2)*.*

2. *Each player has a unique best reply* r_m *to* s_{-m} *and* (s_1, r_2) *and* (r_1, s_2) *are not stage game Nash equilibria.*

An example of a stage game with one-sided binding moral hazard is given in Table 1.2.2. It is a numerical example of the games discussed in section 1.1.1 between a regulator and regulatee. The commitment profile is (T, D) (which corresponds to $Truthful$ and *Diligent*). This game has one-sided binding moral hazard, since only player 2 is subject to binding moral hazard at (T, D). However, for a prisoners' dilemma game, if the commitment profile is given by (C, C), both players are subject to binding moral hazard at the commitment profile.

Table 1.2: One-sided binding moral hazard

	L	D
T	2, 3	0, 2
U	3, 0	-1, 1

Table 1.3: Two-sided binding moral hazard

	C	D
C	2, 2	-1, 3
D	3, -1	0, 0

Note that the first requirement already implies that either (s_1, r_2) or (r_1, s_2) is not Nash equilibrium, depending on the player who is subject to binding moral hazard at (s_1, s_2). Let player 2 be the one who is subject to binding moral hazard, then $r_1 = s_1$ and (r_1, s_2) is not a Nash equilibrium. Moreover, the second condition necessitates s_1 to be a pure action, whereas the commitment action of player 2 could be a mixed-action.

The first requirement of Assumption 4 implies that the commitment profile (s_1, s_2) is not a Nash equilibrium of the stage game, thus repetition of this commitment profile in every period independent of history can not be a Nash equilibrium of the complete-information infinitely repeated game (when the types are known and indeed strategic). So, at least one of the players has a profitable deviation from the repeated commitment profile. Since only one player is subject to binding moral hazard, for exactly one player, there is a profitable deviation in the complete-information repeated game. This condition requires that one of the players best response against the commitment action of the opponent player should be the same as her commitment action. Hence, for one player the best response in the complete-information infinitely repeated game is the same as her commitment strategy in the repeated game. For notational clarity, I let player 2 be the player who is subject to binding moral hazard at (s_1, s_2) and player 1 be the one whose best response to $\hat{\tau}$ in repeated game is the same as her commitment strategy $\hat{\sigma}$.

The stage game 1.2.2 has a unique Nash equilibrium in mixed strategies. Let $\alpha_1 \in \Delta(I)$ be the strategy of player 1 (regulatee) and $\alpha_2 \in \Delta(J)$ be the strategy of player 2 (regulator). The unique Nash equilibrium of the stage game is $\alpha_1(T) = \frac{1}{2}$ and $\alpha_2(D) = \frac{1}{2}$, providing a payoff vector of (1, 1.5) to player 1 and player 2, respectively. The minmax value for player 1 is 0 (the action that minmaxes player 1 is D) and the minmax value for player 2 is 1 (the action that minmaxes player 2 is U). I introduce incomplete information about the types of both players by assuming there is a probability $\mu_0 > 0$ that player 1 is the Stackelberg type who plays her pure Stackelberg action T in every period and there is a probability $\gamma_0 > 0$ that player 2 is the

Stackelberg type who plays his pure Stackelberg action D in every period.[16] This game satisfies the conditions of Assumption 4. The first requirement of Assumption 4 is met since the unique best response for player 1 against D, the commitment action of player 2, is T and (T, D) is not a Nash equilibrium of the stage game; and for player 2, the unique best response against the commitment action of player 1 is L and (T, L) is not a Nash equilibrium of the stage game. It is easy to check that the commitment profile (T, D) is not Nash equilibrium of the stage game and only player 2 is subject to binding moral hazard at (T, D). Thus, player 1's best response to D is the same as her commitment action (Stackelberg action) T. Hence, only player 2 has an incentive to deviate from the commitment profile in the infinitely repeated complete-information game.

Let $\hat{\sigma}$ denote the repeated game strategy of playing s_1 in each period independent of history and $\hat{\tau}$ denote the repeated game strategy of playing s_2 in each period independent of history. Since r_1 is unique best response to s_2, by Assumption 4, the best response of player 1 in the repeated game, i.e. $BR_1(\hat{\tau})$, is a singleton and prescribes playing r_1 in each period for every history. Similarly, the best response of player 2 against the commitment strategy of player 1, $BR_2(\hat{\sigma})$, is a singleton and prescribes playing r_2 in each

[16] This is an action to which a player would like to commit given that such a commitment induces a best response from the opponent player, i.e. this is the action a player would like to choose in an extensive-form game when she/he moves the first. Note that the commitment types in the game between a regulatee and regulator, i.e. *virtuous* and *tough*, are indeed their Stackelberg types and the commitment profile $(Truthful, Diligent)$ are the Stackelberg actions of regulatee and regulator, respectively.

period for every history. Since (s_1, r_2) and (r_1, s_2) are not stage game Nash equilibrium, $(\hat{\sigma}, BR_2(\hat{\sigma}))$ and $(BR_1(\hat{\tau}), \hat{\tau})$ are not Nash equilibrium of the complete-information infinitely repeated game. The unique stage game best responses guarantee that there are no multiple best responses to the commitment strategies in the infinitely repeated game. Since $(\hat{\sigma}, BR_2(\hat{\sigma}))$ and $(BR_1(\hat{\tau}), \hat{\tau})$ are not Nash equilibrium in the complete-information infinitely repeated game, each player has an incentive to deviate to a strategy other than the repeated game commitment strategy, given that opponent is best responding to the commitment strategy.

Let $K = \{c, n\}$ and $L = \{c, n\}$ be the type spaces for player 1 and player 2, respectively. The repeated game strategy for player 1, σ, is a sequence of maps $\sigma_t : H_{1t} \times K \rightarrow \Delta(I)$. For player 2, the repeated game strategy is denoted by τ is a sequence of maps $\tau_t : H_{2t} \times L \rightarrow \Delta(J)$. Let σ be denoted as $\sigma \equiv (\hat{\sigma}, \tilde{\sigma})$ where $\hat{\sigma}$ is the strategy of the c type of player 1 that prescribes playing s_1 in each period independent of history and $\tilde{\sigma}$ is the repeated game strategy of n type of player 1. Similarly, I denote the repeated game strategy of c type of player 2 by $\hat{\tau}$ which prescribes playing s_2 every period regardless of history and $\tilde{\tau}$ is the strategy of n type of player 2, and $\tau \equiv (\hat{\tau}, \tilde{\tau})$. A state of the world in the incomplete information game, ω, is a type for player 1, a type for player 2, and a sequence of actions and public signals. The set of states is $\Omega \equiv K \times L \times H^f_\infty$, where $H^f_\infty = (I \times J \times Y)^\infty$. The priors (μ_0, γ_0), the strategies $\sigma \equiv (\hat{\sigma}, \tilde{\sigma})$ of player 1, and the strategies $\tau \equiv (\hat{\tau}, \tilde{\tau})$ of player 2 jointly induce a probability measure $Q_{(\sigma,\tau;\mu,\gamma)}$ on $(\Omega, \mathcal{F}) \equiv (K \times L \times H^f_\infty, 2^K \otimes 2^L \otimes \mathcal{H}^f_\infty)$. The probability measure $Q_{(\sigma,\tau;\mu,\gamma)}$ describes how an uninformed observer of the game expects the play to evolve. I denote the expectation with respect to $Q_{(\sigma,\tau;\mu,\gamma)}$ by $E_{(\sigma,\tau)}$. Let $E_{(\sigma,\tau)}[.|\mathcal{H}_{1t}]$ and $E_{(\sigma,\tau)}[.|\mathcal{H}_{2t}]$ denote players expectations with respect to $Q_{(\sigma,\tau)}$ conditional on the filtration induced by the private histories, $\mathcal{H}_{1t}$ and $\mathcal{H}_{2t}$, respectively.

The strategy profile $(\hat{\sigma}, \tau)$ and $(\tilde{\sigma}, \tau)$, where $\tau = (\hat{\tau}, \tilde{\tau})$, induce probability measure $Q^{c\cdot}$ and $Q^{n\cdot}$, which describes how the play evolves when player 1 is the commitment and strategic type, respectively. The probability measure $Q^{k\cdot} \equiv Q_{(\sigma_k,\tau)}$, where σ_k is the strategy of the k type of player 1, describes how the game evolves if player 1 is of type k. The associated expectation is denoted by $E^{k\cdot} \equiv E_{(\sigma_k,\tau)}$. Similarly, the strategy profile $(\sigma, \hat{\tau})$ and $(\sigma, \tilde{\tau})$, where $\sigma = (\hat{\sigma}, \tilde{\sigma})$, induce probability measure $Q^{\cdot c}$ and $Q^{\cdot n}$, which describe how the play evolves when player 2 is the commitment and strategic type, respectively. So, the probability measure $Q^{\cdot l} \equiv Q_{(\sigma,\tau_l)}$, where τ_l is the strategy of the l type of player 2, describes how the game evolves if player 2 is of type l, and the associated expectation is $E^{\cdot l} \equiv E_{(\sigma,\tau_l)}$.

I will denote $Q_{(\sigma,\tau)}$, $E_{(\sigma,\tau)}$, $Q_{(\sigma_k,\tau)}$, $E_{(\sigma_k,\tau)}$, $Q_{(\sigma,\tau_l)}$ and $E_{(\sigma,\tau_l)}$ by Q, E, $Q^{k\cdot}$, $E^{k\cdot}$, $Q^{\cdot l}$ and $E^{\cdot l}$, respectively. Players' payoffs in the repeated game is then

$$
\begin{aligned}
E^{k\cdot}[\pi_{10}] &= E^{k\cdot}\big[(1-\delta_1)\sum_{t=0}^{\infty}\delta_1^t\pi_1(i_t,j_t)\big] \\
E^{\cdot l}[\pi_{20}] &= E^{\cdot l}\big[(1-\delta_2)\sum_{t=0}^{\infty}\delta_2^s\pi_2(i_t,j_t)\big]
\end{aligned}
$$

Players are indeed "strategic." Then,

Definition 1.2.4 *A Nash equilibrium of the incomplete information game is a strategy profile* $(\tilde{\sigma}, \tilde{\tau})$ *such that*

$$E^{n\cdot} \equiv E_{\tilde{\sigma},\tau}[\pi_{10}] \geq E_{\sigma',\tau}[\pi_{10}], \quad \forall \sigma'$$

$$E^{\cdot n} \equiv E_{\sigma,\tilde{\tau}}[\pi_{20}] \geq E_{\sigma,\tau'}[\pi_{20}], \quad \forall \tau'$$

Player 1's posterior belief in period t about player 2's type is given by $\mathcal{H}_{1t}$ - measurable random variable

$$\gamma_t \equiv Q^{n\cdot}(c \mid \mathcal{H}_{1t}) : \Omega \to [0,1],$$

and player 2's posterior belief in period t about player 1's type is given by $\mathcal{H}_{2t}$ - measurable random variable

$$\mu_t \equiv Q^{\cdot n}(c \mid \mathcal{H}_{2t}) : \Omega \to [0,1].$$

The main theorem will establish that the reputations for being the commitment types cannot be sustainable indefinitely; in other words the true types will be learned eventually in almost all histories, i.e. $\mu_t \to 0$ and $\gamma_t \to 0$ almost surely (with respect to the probability distribution induced by the strategies of the strategic type of the players). I should point out that with this specification, players' beliefs about each other's type is private. This means players do not know the beliefs of the other player about their own types perfectly. I impose a condition on the public monitoring structure that rules out the dependence of beliefs about the opponent on player's own past actions, and thus enables players to infer opponent's beliefs about their own types. I assume that the monitoring structure is such that the informativeness of the public signal about a player's action is independent of the other player's action (Assumption 5), and as a consequence, the reputations become *public*. Let $\text{Prob}(i \mid y, j, \alpha_1)$ be the posterior probability of "player 1 having chosen pure action i", given mixed α_1 and given that player 2 observed signal y after playing action j, and $\text{Prob}(j \mid y, i, \alpha_2)$ is the corresponding posterior probability of player 2's action.

Assumption 5 *(Independence) For any $\alpha_1 \in \Delta(I)$ and $\alpha_2 \in \Delta(J)$, and* *any signal* $y \in Y$,

$$
\begin{aligned}
Prob(i \mid y, j, \alpha_1) &= Prob(i \mid y, j', \alpha_1), \quad \textit{for all} \quad j, j' \\
Prob(j \mid y, i, \alpha_2) &= Prob(j \mid y, i', \alpha_2), \quad \textit{for all} \quad i, i'.
\end{aligned}
$$

Assumption 5 implies that for all $\alpha_1 \in \Delta(I)$ and $j, j' \in J$,

$$
\frac{\text{Prob}(y \mid i, j)\alpha_1(i)}{\sum_{i \in I} \alpha_1(i)\text{Prob}(y \mid i, j)} = \frac{\text{Prob}(y \mid i, j')\alpha_1(i)}{\sum_{i \in I} \alpha_1(i)\text{Prob}(y \mid i, j')} \tag{1.2.1}
$$

Similarly, for all $\alpha_2 \in \Delta(J)$ and $i, i' \in I$,

$$
\frac{\text{Prob}(y \mid i, j)\alpha_2(j)}{\sum_{j \in J} \alpha_2(j)\text{Prob}(y \mid i, j)} = \frac{\text{Prob}(y \mid i', j)\alpha_2(j)}{\sum_{j \in J} \alpha_2(j)\text{Prob}(y \mid i', j)} \tag{1.2.2}
$$

Assumption 5 ensures that public signals allow players to infer the other player's beliefs about their type since the information that public signal provides about the player's action is independent of the opponent's behavior. In other words, this monitoring structure allows players to calculate opponent's inference about their reputation without knowing the opponent's action, thus reputations of both players becomes *public* and beliefs are common knowledge.

Assumption 5 holds if each player has individual specific signals, i.e. the public signal y is such that $y = (y_1, y_2) \in Y = Y_1 \times Y_2$ where y_1 is a signal of player 1's action and y_2 is a signal of player 2's, with

$$\rho_{ij}^{y} = \rho_i^{y_1} \rho_j^{y_2}, \quad \forall i, j, y.$$

Such monitoring structure is called *product structure.* In the motivating example game presented in section 1.1.1, each player had separate public signal, distribution of each depended only on player's own action, independent of the other players action. [17,18]

For games with product structure, every sequential equilibrium payoff in the complete-information infinitely repeated game (equilibrium when private histories are used as beliefs) is also a public perfect equilibrium payoff. [19] I can show that Assumption 5 implies every sequential equilibrium payoff in the complete-information game is also a public perfect equilibrium payoff, in the same spirit of Fudenberg and Levine (1994). Thus, there is no loss of generality if I restrict attention to public strategies. A public strategy $\sigma \equiv \{\sigma_t\}_{t=0}^{\infty}$ for player 1 is a sequence of maps $\sigma_t : H_t \to \Delta(I)$ and that for player 2, $\tau \equiv \{\tau_t\}_{t=0}^{\infty}$, is a sequence of maps $\tau_t : H_t \to \Delta(J)$. The strategy profile

[17]For regulatee, the public signals are correct and incorrect message; whereas, for regulator, the public signals are there has been an audit or no audit; probability of which depends on their own actions only.

[18]For games with product structure, pure-action profiles satisfy pairwise identifiability condition. Thus Fudenberg, Levine, and Maskin (1994) Folk theorem for games with imperfect public monitoring holds. Hence, any feasible and Pareto efficient payoff dominating a Nash equilibrium payoff of the stage game can be attained as an equilibrium payoff of the repeated (complete-information) game if players are sufficiently patient.

[19]See Mailath and Samuelson (2006) (p.330) and Fudenberg and Levine (1994) (Theorem 5.2) for further discussion.

(σ, τ) induces a probability distribution $Q_{(\sigma,\tau)}$ over $H^f_\infty \equiv (I \times J \times Y)^\infty$. Let $E_{(\sigma,\tau)}[. \mid \mathcal{H}_t]$ denote players expectations with respect to $Q_{(\sigma,\tau)}$ conditional on the filtration induced by the public history, $\mathcal{H}_t$.

Note that due to Assumption 5 (i.e. under a monitoring structure such as the product structure), γ_t and μ_t can be viewed as $\mathcal{H}_t$ - measurable random variable $Q(c \mid \mathcal{H}_t)$ on Ω. This property enables both players to compute the opponent player's beliefs about themselves. So, in period t, strategic type of player 1 is maximizing $E_{\tilde{\sigma},\tau}[\pi_{1t} \mid \mathcal{H}_t]$, and a strategic player 2 is maximizing $E_{\sigma,\tilde{\tau}}[\pi_{2t} \mid \mathcal{H}_t]$, that depend on the information sets generated by public histories.

At any Nash equilibrium of the incomplete information game, γ_t is a bounded martingale with respect to the measure Q and filtration $\{\mathcal{H}_{1t}\}_t$ (and also with respect to filtration $\{\mathcal{H}_t\}_t$ by Assumption 5). Therefore, γ_t converges Q-almost surely (and also $Q^{\cdot n}$ - almost surely and Q^{nn} - almost surely, since $Q^{\cdot n}$ and Q^{nn} are absolutely continuous with respect to Q) to a random variable γ_∞ on Ω. Similarly, at any Nash equilibrium of the incomplete information game, μ_t is a bounded martingale with respect to the measure Q and filtration $\{\mathcal{H}_{2t}\}_t$ (and also with respect to filtration $\{\mathcal{H}_t\}_t$), and thus converges Q-almost surely (and hence $Q^{n\cdot}$ - almost surely and Q^{nn} - almost surely, since $Q^{n\cdot}$ and Q^{nn} are absolutely continuous with respect to Q) to a random variable μ_∞ on Ω. [20]

[20] The proof is presented in the Appendix, Lemma 2.6.1.

1.3 Main Results

1.3.1 Reputations in the long-run

The main result of this paper is that neither player can sustain a reputation for playing a strategy that is not part of a Nash equilibrium of the complete-information stage game for games with one-sided binding moral hazard at the commitment profile under imperfect public monitoring.

Theorem 1.3.1 *Suppose Assumptions 1-5 hold. In any Nash equilibrium of the incomplete-information game, reputations of players cannot be sustained indefinitely:*

$$\mu_t \to 0, \quad Q^{n\cdot} - \textit{almost surely},$$

$$\gamma_t \to 0, \quad Q^{\cdot n} - \textit{almost surely}.$$

Moreover, the convergence is uniform. [21]

Theorem 1.3.1 will be proved with the help of two propositions. The first proposition argues that reputation of player 2, who is subject to binding moral hazard at the commitment profile, disappears uniformly in any Nash equilibrium of the incomplete-information game, i.e. $\gamma_t \to 0, \quad Q^{\cdot n} -$

[21]Note that $\mu_t \to 0 \quad Q^{nn} -$almost surely, and $\gamma_t \to 0 \quad Q^{nn} -$almost surely, since Q^{nn} is absolutely continuous with respect to $Q^{n\cdot}$ and $Q^{\cdot n}$.

almost surely and convergence is uniform across all Nash equilibria. The second proposition argues that if player 2's reputation disappears uniformly, then player 1's reputation disappears uniformly in any Nash equilibrium as well.

One-sided binding moral hazard condition implies that the player whose reputation disappears in the long-run, independent of the asymptotic behavior of the other player's reputation, is the one who is subject to binding moral hazard at the commitment profile. After establishing that player's type is (almost) revealed in the long-run, one can show that other player's true type should be revealed as well. Hence, the one-sided binding moral hazard condition allows us to break the analysis of the long-run behavior of the reputations of the two players into two stages.

The proof of Theorem 1.3.1 is immediate by Proposition 1.3.2 and 1.3.5. Section 1.4 is devoted to the proofs of Proposition 1.3.2 and 1.3.5.

Proposition 1.3.2 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game satisfies one-sided binding moral hazard at the commitment profile. Then, in any Nash equilibrium of the incomplete-information game, reputation of player 2, who is subject to binding moral hazard at the commitment profile, cannot be sustained indefinitely:*

$$\gamma_t \to 0, \quad Q^{\cdot n} - \textit{almost surely.}$$

Moreover, the disappearance of player 2's reputation is uniform. That is for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma,\tilde{\tau}) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{\cdot n}_{\sigma=(\hat{\sigma},\tilde{\sigma}),\tilde{\tau}}$ is the probability measure induced on Ω by $(\sigma, \tilde{\tau})$ and the strategic type of player 2 and $\gamma_t(\sigma, \tilde{\tau})$ is the associated reputation of player 2.

The disappearance of player 2's reputation is independent of the asymptotic behavior of player 1's reputation. So, player 2's reputation of being the commitment type converges to zero $Q^{\cdot n}$-almost surely, whether the uncertainty over the types of player 1 is resolved or not. This leads to the following corollaries.

Corollary 1.3.3 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game satisfies one-sided moral hazard at the commitment profile. Suppose there exists a Nash equilibrium $(\tilde{\sigma}, \tilde{\tau})$ that induces a set of histories on which the reputation of player 2 (who is subject to binding moral hazard) does not disappear, i.e. there exists $A \in \Omega$ such that $\gamma_t(\omega) \to \gamma_\infty > \eta$ for some $\eta > 0$, given that the reputation of player 1 is* *sustained on these histories, $\mu_t(\omega) \to \mu_\infty > \epsilon$ for some $\epsilon > 0$ for all $\omega \in A$. Then $Q^{\cdot n}(A) = 0$, where $Q^{\cdot n}$ denotes $Q_{(\sigma,\tilde{\tau})}$.*

Suppose that there exists a Nash equilibrium profile that induces histories (with positive measure) on which the reputation of player 1 does not disappear. This means that player 2 believes that player 1 plays the commitment action s_1 on average in the long-run, which induces him to deviate from (s_1, s_2) eventually. Also, the other immediate corollary of Proposition 1.3.2 is that there is no histories with positive measure where player 1's reputation disappears in the long-run, but player 2's not.

Corollary 1.3.4 *Suppose the monitoring technology satisfies Assumptions 1, 3 and 5, and the stage game satisfies one-sided moral hazard at the commitment profile. Suppose there exists a Nash equilibrium $(\tilde{\sigma}, \tilde{\tau})$ that induces a set of histories on which player 1's reputation disappears but not player 2's (who is subject to binding moral hazard), i.e. there exists $A \in \Omega$ such that $\mu_t(\omega) \to 0$, but $\gamma_t(\omega) \to \gamma_\infty > \eta$ for some $\eta > 0$ and for all $\omega \in A$. Then $Q^{\cdot n}(A) = 0$, where $Q^{\cdot n}$ denotes $Q_{(\sigma,\tilde{\tau})}$.*

Having established that player 2 reveals his true type eventually regardless of the asymptotic behavior of player 1's reputation, the game can be considered to be the one with "almost" one-sided incomplete-information where the uncertainty is about the types of player 1 only. The next proposition gives the sufficient conditions for the disappearance of player 1's reputation.

Proposition 1.3.5 *Suppose the monitoring technology satisfies Assumptions 1, 2 and 5, and player 2's reputation γ_t converges uniformly to zero $Q^{\cdot n}$-almost surely in any equilibrium. Suppose also (s_1, r_2) is not a Nash equilibrium of the stage game. Then, in any Nash equilibrium of the incomplete-information game, player 1's reputation disappears eventually:*

$$\mu_t \to 0, \quad Q^{n\cdot} - \text{almost surely}$$

where the convergence is uniform, i.e. for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$,

$$Q^{n\cdot}_{\tilde{\sigma},\tau}(\mu_t(\tilde{\sigma}, \tau) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{n\cdot}_{\tilde{\sigma},\tau=(\hat{\tau},\tilde{\tau})}$ is the probability measure induced on Ω by $(\tilde{\sigma}, \tau)$ and the strategic type of player 1 and $\mu_t(\tilde{\sigma}, \tau)$ is the associated reputation of player 1.

Proposition 1.3.5 implies that the sustainability of player 1's reputation depends on that of player 2's reputation. If player 2's reputation disappears, player 1's reputation disappears eventually, given that (s_1, r_2) is not a Nash equilibrium of the stage game (and thus $(\hat{\sigma}, BR_2(\hat{\sigma}))$ is not a Nash equilibrium of the repeated complete-information game).

Corollary 1.3.6 *Suppose the monitoring technology satisfies Assumptions 1, 2 and 5. Suppose there exists a Nash equilibrium that induces a set of histories $A \in \Omega$ with $Q(A) > 0$ on which the reputation of player 2 does not disappear, i.e. $\gamma_t(\omega) \to \gamma_\infty > \eta$ for some $\eta > 0$ and for all $\omega \in A$. Suppose also the stage game best reply of player 1 against s_2 is the same as her commitment action, i.e. $r_1 = s_1$. Then, $\mu_t(\omega) \to \mu_\infty > \epsilon$ for some $\epsilon > 0$ and $Q^{n.}$-almost surely in A.*

Corollary 1.3.6 says that if the uncertainty over player 2's type persists, the uncertainty over player 1's type persists as well, since then player 1 expects to see s_2 on average in the long-run and gives a best response to it ($r_1 = s_1$). However, by Proposition 1.3.2, the uncertainty over player 1's type can persist only if either (s_1, s_2) a Nash equilibrium of the stage game or there is a mechanism that replenish the uncertainty over player 2's type. One such mechanism can be introducing a possibility for replacing the type of player 2 every period. With such a mechanism, player 2 need to mimic the commitment type always to convince player 1, since player 1 is never fully convinced because of the replacement possibility. Hence, player 2's type will not be revealed. As player 2's type is not revealed, player 1's type will not be revealed as well.

The implication of these results for the regulatee-regulator game presented in Section 1.1.1 is that the reputation of being tough for the regulator disappears in the long-run since regulator is the player who is subject to binding moral hazard at the commitment profile (by Proposition 1.3.2). After his true type is almost known, the regulatee starts to take advantage of regulator's uncertainty over her type and regulatee's reputation of being virtuous disappears eventually as well (by Proposition 1.3.5). Furthermore, the set of histories where the regulatee's true type is almost known, but regulator's true type is not revealed has measure zero. One way to make both reputations sustainable is to introduce the possibility that the type of the regulator changes every period (with some probability). Then, the regulator can never convince the regulatee perfectly that he is tough, so he needs to be diligent every period. Both reputations can be made permanent this way. However, from the welfare point of view, this is inefficient. In order to get the efficient stage game outcome played (frequently), one needs a mechanism that allows for some deterioration for the reputation of the regulator up to a lower bound, so that the regulatee will not start exploiting this deterioration. By strategically scheduled replacement periods, there are periods of $(Truthful, Lazy)$ which Pareto dominates $(Truhful, Diligent)$. There is an optimal schedule for replacement periods that depends on the parameter values of the payoffs, as well as the values for the prior beliefs and discount factors.

1.3.2 Equilibrium behavior

After establishing that the true types will be (almost) known in the long-run and the information structure of the game approaches to that of the complete-information game, one expects to see such a convergence result holds for the equilibrium behavior. I show that any Nash equilibrium of the incomplete-information game converges to a public perfect equilibrium of the complete-information game, following the definitions and the methods provided by Cripps, Mailath, and Samuelson (2004) for one long-lived and a sequence of short-lived players.

Let $t' = 0, 1, ...$ denote the time periods of the continuation play of the game that starts at some period t. A pure public (continuation game) strategy ς_1 for player 1 is a sequence of maps $\varsigma_{1t'} : H_{t'} \to I$ for $t' = 0, 1, ...$ Similarly, a pure public strategy for player 2 in the continuation game t' is ς_2, a sequence of maps $\varsigma_{2t'} : H_{t'} \to J$ for $t' = 0, 1, ...$ Let $S_1 = I^{\bigcup_{t'=0}^{\infty} Y^{t'}}$ and $S_2 = J^{\bigcup_{t'=0}^{\infty} Y^{t'}}$ be the set of pure strategies for player 1 and player 2, respectively.[22] Note that S_1 and S_2 include the pure strategies in the

[22]The sets S_1 and S_2 are countable products of finite sets I and J. Define σ-algebras for each set that are generated by cylinder sets and denote by $\mathcal{S}_m$, $m = 1, 2$. $(S_m, \mathcal{S}_m)$ is equipped with the product topology.

original game as well. Player m's payoff is given by,[23]

$$U_m(\varsigma_1, \varsigma_2) = E_{(\varsigma_1,\varsigma_2)}\left[(1-\delta_m)\sum_{t'=0}^{\infty}\delta_m^{t'}\pi_m(i_{t'}, j_{t'})\right]$$

The mixed strategies (of the repeated continuation game) are the probability distributions over the set of pure strategies, i.e. ϑ_m be probability measures on $(S_m, \mathcal{S}_m)$.[24] Let Θ_m denote the set of all probability measures ϑ_m, $m = 1, 2$. Note that Θ_m is sequentially compact with respect to the product topology. Since players' payoffs are discounted, the utility function $U_m : \Theta_1 \times \Theta_2 \to \Re$ is continuous for each $m = 1, 2$ with this topology.[25]

A sequence of measures ϑ_1^n converges to $\hat{\vartheta}_1$ if the following holds: For every $T \geq 0$,

$$\vartheta_1^n|_{I^{Y^T}} \to \hat{\vartheta}_1|_{I^{Y^T}}$$

and, similarly, ϑ_2^n converges to $\hat{\vartheta}_2$ if for every $T \geq 0$,

$$\vartheta_2^n|_{J^{Y^T}} \to \hat{\vartheta}_2|_{J^{Y^T}}$$

Pick an equilibrium $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game and a public history h_t. These strategies specifies behavior strategies in the con-

[23]Even though the strategies are pure, the payoffs are random because of imperfect public monitoring.

[24]Note that by Kuhn's theorem, one can replace mixed strategies by behavior strategies for games with perfect recall.

[25]Reader is referred to Fudenberg and Tirole (1991) for a detailed discussion and proof of the continuity of the utility function (due to discounting $\delta_1, \delta_2 < 1$).

tinuation game, $\tilde{\sigma}_{h_t}$ and $\tilde{\tau}_{h_t}$, which are realization equivalent to the mixed strategies $\tilde{\vartheta}_1^{h_t}$ and $\tilde{\vartheta}_2^{h_t}$ (for the continuation game), by Kuhn's Theorem. The following theorem states that the limit of every convergent subsequence of $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ is a Nash equilibrium of the complete-information game. Similar convergence result about the asymptotic equilibrium behavior can be found Cripps, Mailath, and Samuelson (2004) for one long-lived and a sequence of short-lived players. The appropriate modifications of their proofs for our model is given below.

Theorem 1.3.7 *Suppose Assumptions 1-5 are satisfied. For any Nash equilibrium of the incomplete-information game and for almost all sequences of public histories* $\{h_t\}_t$ *(with respect to measure* Q^{nn}*), the limit of every convergent subsequence of continuation equilibrium profiles* $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ *is a public perfect equilibrium of the complete-information game (game with strategic types of players).*

Proof. We modify the proof of Cripps, Mailath, and Samuelson (2004) for two long-lived player with uncertainty over the types of both players. Since $(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})$ are continuation equilibrium profile, for each public history h_t and pure strategies $\varsigma_1' \in S_1$ and $\varsigma_2' \in S_2$, the continuation expected payoffs should satisfy:

$$E_{(\tilde{\vartheta}_1^{h_t},\gamma_t\hat{\vartheta}_2^{h_t}+(1-\gamma_t)\tilde{\vartheta}_2^{h_t})}[U_1(\varsigma_1,\varsigma_2)] \geq E_{\gamma_t\hat{\vartheta}_2^{h_t}+(1-\gamma_t)\tilde{\vartheta}_2^{h_t}}[U_1(\varsigma_1',\varsigma_2)] \qquad (1.3.1)$$

$$E_{(\mu_t\hat{\vartheta}_1^{h_t}+(1-\mu_t)\tilde{\vartheta}_1^{h_t},\tilde{\vartheta}_2^{h_t})}[U_2(\varsigma_1,\varsigma_2)] \geq E_{\mu_t\hat{\vartheta}_1^{h_t}+(1-\mu_t)\tilde{\vartheta}_1^{h_t}}[U_1(\varsigma_1,\varsigma_2')] \qquad (1.3.2)$$

where $\hat{\vartheta}_1^{h_t}$ and $\hat{\vartheta}_2^{h_t}$ are the commitment mixed strategies corresponding to commitment behavior strategies $\hat{\sigma}_{h_t}$ and $\hat{\tau}_{h_t}$ in the continuation game. By Theorem 1.3.1, $\mu_t \to 0 \quad Q^{n\cdot}$-almost surely and $\gamma_t \to 0 \quad Q^{\cdot n}$-almost surely which imply $\gamma_t \to 0$ and $\mu_t \to 0 \quad Q^{nn}$- almost surely, by absolute continuity of Q^{nn} with respect to $Q^{n\cdot}$ and $Q^{\cdot n}$. Suppose $\{h_t\}_t$ is a sequence of public histories on which $\gamma_t, \mu_t \to 0$ and $\{(\tilde{\vartheta}_1^{h_t}, \tilde{\vartheta}_2^{h_t})\}_{t=1}^{\infty} \to (\tilde{\vartheta}_1^*, \tilde{\vartheta}_2^*)$ on this sequence. We need to show $(\tilde{\vartheta}_1^*, \tilde{\vartheta}_2^*)$ satisfies (1.3.1) and (1.3.2), which suffices to show expectation $E_{(\vartheta_1,\vartheta_2)}$ is continuous in $(\vartheta_1, \vartheta_2)$. The continuity of this expectation is given by Theorem 4.4 of Fudenberg and Tirole (1991) and it is due to discounting (since $\delta_1, \delta_2 < 1$). ■

Theorem 1.3.7 implies that the equilibrium of the incomplete-information regulatee-regulator game converges to a public perfect equilibrium of the complete-information regulatee-regulator game, and thus the the equilibrium payoff in the complete-information game converges to an equilibrium payoff of the complete-information game. I would like to point out that by our assumptions on the monitoring technology (Assumptions 2, 3 and 5), the imperfect public monitoring Folk theorem holds.[26] Hence, any feasible and individually rational payoff vector of stage game in the complete-information game can be attained as an equilibrium payoff of the complete-information repeated game, if players are sufficiently patient. However, Theorem 1.3.7 neither constrain the possible set of equilibrium payoffs, nor answers if any particular Nash equilibrium strategy profile (or payoff) of the complete-information game can be achieved as a limit of a Nash equilibrium of the incomplete-information game.

However, one can show that there is a lower bound for the equilibrium payoff vector if the discount factor of player 2 (who is subject to one-sided binding moral hazard) is allowed to vary, i.e. for any fixed prior beliefs (μ_0, γ_0), if player 2's discount factor is above some $\bar{\delta}_2(\mu_0, \gamma_0) \to 1$, the lower bound on his equilibrium payoff converges to $\pi_2(s_1, s_2) = \pi_2(r_1, s_2)$, whereas player 1's payoff converges to $\pi_1(r_1, s_2)$.

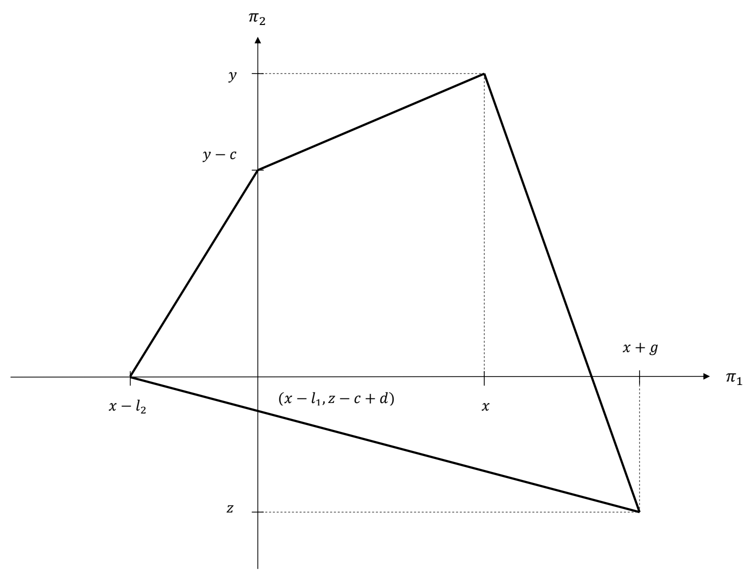
If player 2 is sufficiently patient, he can get a payoff of at least $\min_{\alpha_1 \in BR_1(s_2)} \pi_2(\alpha_1, s_2)$, i.e. for any $\epsilon > 0$, there exists a $\delta_2^*(\mu_0, \gamma_0) \in (0, 1)$ such that for all $\delta_2 \in (\delta_2^*, 1)$, strategic type of player 2's payoff is at least $\min_{\alpha_1 \in BR_1(s_2)} \pi_2(\alpha_1, s_2) - \epsilon$. However, the lower bound on equilibrium payoffs is a temporary lower bound on player 2's equilibrium payoff, in the sense that for any fixed discount factor δ_2 of player 2, γ_t will eventually be small enough so that $\delta_2 < \delta_2^*(\mu_t, \gamma_t)$. Hence, at the beginning of the game, there

[26]See Fudenberg, Levine, and Maskin (1994).

will be a period of (s_1, s_2) where players collect their payoffs $\pi_1(s_1, s_2)$ and $\pi_2(s_1, s_2)$), but eventually player 2 cannot resist to deviate which will reveal his true type.

I would like to make a couple observations. First, note that player 2's commitment action is allowed to be a mixed-action. In fact, if player 2 can commit to a mixed stage game action, he can get a higher lower bound for his equilibrium payoffs. Consider the regulator-regulatee game given in Table 1.2.2. Suppose the commitment action of the regulator is $s_2 = D$. Then $\min_{\alpha_1 \in BR_1(s_2)} \pi_2(\alpha_1, s_2) = \pi_2(r_1, s_2) = 2$. However, instead, if the regulator's commitment action randomizes between L and D, giving a weight on D a little more than $\frac{1}{2}$, then the best reply of the regulatee is still T and the regulator can achieve $\frac{5}{2}$. The regulatee's payoff, on the other hand, is bound by her minmax payoff (if $s_2 = D$).

The following graph is the set of feasible payoff vectors of the regulatee-regulator game, depending on the parameters of the stage game, where the origin $(x - l_1, z - c + d)$ is the minmax payoff of each player. Suppose the commitment action of the regulator is D. Then, the regulator collects a payoff of $y - c$ and the regulatee collects her minmax payoff $x - l_1$ at the beginning of the game. If the discount factor of the regulator is very high, the equilibrium payoffs they get in the beginning of the game is inefficient.



1.3.3 Discussion on one-sided binding moral hazard

The condition of one-sided binding moral hazard at the commitment profile (s_1, s_2) is crucial for the results and proofs.[27] If none of the players has an incentive to deviate at (s_1, s_2) in the complete-information stage game, it means (s_1, s_2) is a Nash equilibrium of the stage game and thus repetition of (s_1, s_2) every period independent of history is a Nash equilibrium of the repeated complete-information game. Hence, the reputations can be sustained in that situation as the commitment strategies $(\hat{\sigma}, \hat{\tau})$ is a Nash equilibrium for strategic types. If, on the other hand, the stage game

[27]Note that this is not only a restriction on the stage game payoff set, but also a restriction on the simple commitment types allowed.

has two-sided binding moral hazard at the commitment profile, i.e. both players have an incentive to deviate at (s_1, s_2) in the stage game, then the results are not clear. I believe one can construct a Nash equilibrium where the reputations do not necessarily disappear.

Consider the following stage game and suppose that there are *cooperative* types for both players, and thus both strategic players have an incentive to deviate from the commitment action profile (C, C). Let $\mu_1 = \mu_2 = \mu_0$ be the prior belief that the players are *cooperative* type.

Table 1.4: Two-sided binding moral hazard at (C, C)

	C	D_1	D_2
C	4, 4	0, 0	0, 5
D_1	5, 0	1, 3	0, 0
D_2	0, 0	0, 0	1, 3

Let $Y_1 = \{H, L\}$ and $Y_2 = \{h, l\}$ be the public signal spaces for player 1 and 2, $y \in Y_1 \times Y_2$, and the probability distributions over signals are given as below:

$$prob(H|C) = p, \quad prob(H|D_1) = r, \quad prob(H|D_2) = q$$

$$prob(h|C) = p, \quad prob(h|D_1) = q, \quad prob(h|D_2) = r$$

where $p > 1/2 > q > r$.

The strategy profile represented by the following automaton is a public perfect equilibrium for the complete-information infinitely repeated game. The states are $W = \{w_{CC}, w_1, w_2\}$ and the initial state is w_{CC}, and players choose the following action profiles corresponding to each state:

$$\begin{aligned} f(w_{CC}) &= CC, \\ f(w_1) &= D_1D_1, \\ f(w_2) &= D_2D_2. \end{aligned}$$

and the transition is

$$t(y) = \begin{cases} w_{CC} & \text{if} \quad y = Hh \quad \text{or} \quad Ll, \\ w_1 & \text{if} \quad y = Lh, \\ w_2 & \text{if} \quad y = Hl, \end{cases}$$

Note that w_1 is the punishment state for player 1 and w_2 is the punishment state for player 2. The equilibrium path can be described by an ergodic Markov chain on the state space with stationary distribution putting more weight on w_{CC}. This strategy profile is also an equilibrium profile for low enough μ_0, and the types will not be revealed because of frequent play of CC.

However, one can also construct Nash equilibria on which the uncertainty over the types of both players is going to be revealed eventually. In fact, if one of the player's type is revealed, the other's type is going to be revealed as well.

Lastly, I'd like to point out that Cripps, Mailath, and Samuelson (2004) show that if there is uncertainty over the types of only one of the players, the true type (which is strategic) of this player will be revealed eventually in any Nash equilibria in games with imperfect monitoring. Adding uncertainty over the types of the second player may change this result for some Nash equilibria in games other than one-sided moral hazard.

1.4 False reputations disappear uniformly

1.4.1 Player 2's reputation disappears uniformly

I first show that either the true type of player 2 is revealed or player 1's expectation of the strategy played by the strategic type of player 2 is in the limit the same as the strategy played by the commitment type, given that the public signals are statistically informative about player 2's behavior (Lemma 1.4.1). In other words, if player 1 is not eventually convinced that player 2 is strategic, she must be convinced that player 2 is mimicking the commitment type on average in the long-run. This is the standard merging of beliefs argument modified for imperfect public monitoring games.[28] Then, it is shown that if there is a set of histories with positive measure in which player 2's reputation does not disappear, in those histories, player

[28]See Sorin (1999) and Cripps, Mailath, and Samuelson (2004).

1 must be convinced that player 2 will play the commitment strategy in the continuation play; and moreover, reputations being public implies that player 2 knows about player 1's beliefs about his behavior (Lemma 1.4.3). Hence, player 2 believes that the strategic type of player 1 should be best responding to the commitment strategy of player 2 (Lemma 1.4.4), which also coincide with the strategy of commitment type of player 1 (since player 1 is *not* subject to binding moral hazard at the commitment type, and thus $r_1 = s_1$). Then player 2 has an incentive to deviate from his commitment strategy (as player 2 is subject to binding moral hazard at the commitment profile), knowing that these deviations will not be detected due to imperfect monitoring [29] and player 1's beliefs have nearly converged, and thus the effect of deviations on player 1's beliefs will be arbitrarily small. However, the long-run effect of many such deviations, which generate different distributions over the public signals (Assumption 3), reveals that player 2 plays a strategy different than the commitment strategy. This provides the ground for the desired contradiction to the hypothesis of having a positive measure set of histories in which player 1 is convinced that player 2 is playing the commitment strategy on average in the long run.

[29] Player 2's incentive to deviate from the commitment strategy is stronger in two-sided incomplete-information game compared to one-sided incomplete information game where there is uncertainty only over the types of player 2.

Player 1's posterior beliefs about player 2

The following Lemma and Corollary establish that either player 1's expectation of the strategy played by the strategic type of player 2 is in the limit the same as the strategy played by the commitment type of player 2, or player 1's posterior probability that player 2 is the commitment type converges to zero (given that player 2 is indeed strategic). The proof is as the one provided by Cripps, Mailath, and Samuelson (2004).

Lemma 1.4.1 *Suppose Assumptions 1, 3 and 5 are satisfied. In any Nash equilibrium of the incomplete-information game,*

$$\lim_{t\to\infty} \gamma_t(1-\gamma_t)\|\hat{\tau}_t - E^{\cdot n}[\tilde{\tau}_t|\mathcal{H}_t]\| = 0, \quad Q - a.s. \tag{1.4.1}$$

Note that since $\hat{\tau}_t$ is a simple strategy, it can be replaced by s_2.

Proof. Let $\gamma_{t+1}(h_t; i_t, y_t)$ denote player 1's belief in period $t+1$ after playing i_t and observing public signal y_t in period t, given public history h_t. By Bayes' rule,

$$\gamma_{t+1}(h_t; i_t, y_t) = \frac{\gamma_t \text{prob}[y_t \mid h_t, i_t, c]}{\gamma_t \text{prob}[y_t \mid h_t, i_t, c] + (1-\gamma_t)\text{prob}[y_t \mid h_t, i_t, n]}$$

Since the probability of observing the signal y_t from the commitment type of player 2 is $\sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}$, and from the strategic type is $E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid$

$h_t]$, one can rewrite the above expression as,

$$\begin{aligned}\gamma_{t+1}(h_t; i_t, y_t) &= \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t} + (1-\gamma_t) E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid h_t]} \\ &= \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\sum_{j\in J} \rho_{i_t j}^{y_t} \Big(\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t] \Big)}\end{aligned}$$

The difference between $\gamma_{t+1}(h_t; i_t, y_t)$ and $\gamma_t(h_t)$ gives,

$$\begin{aligned}|\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| &= \left| \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{\sum_{j\in J} \rho_{i_t j}^{y_t} (\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])} - \gamma_t \right| \\ &= \left| \frac{\gamma_t(1-\gamma_t) \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t} - \gamma_t(1-\gamma_t) \sum_{j\in J} \rho_{i_t j}^{y_t} E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])}{\sum_{j\in J} \rho_{i_t j}^{y_t} (\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])} \right| \\ &= \frac{\gamma_t(1-\gamma_t) \Big| \sum_{j\in J} \rho_{i_t j}^{y_t} (s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \Big|}{\sum_{j\in J} \rho_{i_t j}^{y_t} (\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t])}\end{aligned}$$

Note that the denominator $\sum_{j\in J} \rho_{i_t j}^{y_t} (\gamma_t s_2^j + (1-\gamma_t) E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) < \max_{j\in J} \rho_{i_t j}^{y_t} < 1$. Thus,

$$|\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| \geq \gamma_t(1-\gamma_t) \Big| \sum_{j\in J} \rho_{i_t j}^{y_t} (s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \Big|$$

which implies

$$\max_{y\in Y} |\gamma_{t+1}(h_t; i_t, y_t) - \gamma_t(h_t)| \geq \gamma_t(1-\gamma_t) \Big| \sum_{j\in J} \rho_{i_t j}^{y_t} (s_2^j - E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]) \Big|$$

Since γ_t is a martingale on $[0,1]$ (with respect to Q and filtration $\{\mathcal{H}_t\}_t$ by Lemma 2.6.1) and bounded martingales converge almost surely, $|\gamma_{t+1} - \gamma_t| \to 0$ Q-almost surely. This implies, for any $y \in Y$,

$$\gamma_t(1-\gamma_t)\left|\sum_{j \in J} \rho^{y_t}_{i_t j}(s^j_2 - E^{\cdot n}[\tilde{\tau}^j_t \mid h_t])\right| \to 0, \qquad Q-\text{a.s.} \tag{1.4.2}$$

Since (1.4.2) holds for all y, it can be restated as

$$\gamma_t(1-\gamma_t)\left\|\Pi_{i_t}(s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t])\right\| \to 0, \qquad Q-\text{a.s.}$$

where Π_{i_t} is a $|Y| \times |J|$ matrix that contains the values for $\rho^{y_t}_{i_t j}$. Since for all i_t and $y \in Y$, $\rho^{y_t}_{i_t j} > 0$ by Assumption 1 and J columns are linearly independent by Assumption 3, the unique solution to $\Pi_{i_t} x = 0$ is $x = 0$ and there exists a strictly positive constant $k = \inf_{i \in I, x \neq 0} \frac{\|\Pi_i x\|}{\|x\|}$. Thus, $\|\Pi_i x\| \geq k\|x\|$, which implies

$$\gamma_t(1-\gamma_t)\left\|\Pi_{i_t}(s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t])\right\| \geq \gamma_t(1-\gamma_t)k\left\|(s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t])\right\| \to 0, \quad Q-\text{a.s.}$$

This implies (1.4.1). ■

Note that Lemma 1.4.1 holds also $Q^{n\cdot}$ almost surely, since γ_t is also a bounded martingale with respect to $Q^{n\cdot}$, which is the probability measure that describes how the game evolves from the perspective of the strategic type of player 1. Note that any statement that holds Q almost surely, also holds $Q^{n\cdot}$ almost surely.

The immediate implication of Lemma 1.4.1 is Corollary 1.4.2:

Corollary 1.4.2 *At any Nash equilibrium of the incomplete-information game satisfying Assumptions 1, 3 and 5,*

$$\lim_{t\to\infty} \gamma_t \left\| s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t] \right\| \to 0, \qquad Q^{\cdot n} - a.s.$$

Corollary 1.4.2 says that if player 2 is indeed strategic and the game evolves according to the play of strategic type of player 2, either his reputation for being the commitment type disappears, i.e. $\gamma_t \to 0$, $Q^{\cdot n}$ - almost surely, or he is expected to play the commitment action in the limit. **Proof.** I first show that $\frac{\gamma_t}{(1-\gamma_t)}$ is a $Q^{\cdot n}$-martingale (with respect to filtration $\{\mathcal{H}_{1t}\}_t$ and $\{\mathcal{H}_t\}$ due to Assumption 5). For all h_{t+1}, i_t and for all i,

$$\begin{aligned}
E^{\cdot n}[\frac{\gamma_{t+1}}{(1-\gamma_{t+1})}|\mathcal{H}_{t+1}] &= \sum_{y\in Y} \text{prob}[y_t|h_t, n].\frac{\gamma_{t+1}(h_t, i_t, y_t)}{1-\gamma_{t+1}(h_t, i_t, y_t)} \\
&= \sum_{y\in Y} E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{ij}^{y_t} \mid h_t] \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{i_t j}^{y_t}}{(1-\gamma_t)E^{\cdot n}[\sum_{j\in J} \tilde{\tau}_t^j \rho_{i_t j}^{y_t} \mid h_t]} \\
&= \sum_{y\in Y} \frac{\gamma_t \sum_{j\in J} s_2^j \rho_{ij}^{y_t}}{(1-\gamma_t)} \\
&= \frac{\gamma_t \sum_{j\in J} s_2^j \sum_{y\in Y} \rho_{ij}^{y_t}}{(1-\gamma_t)} \\
&= \frac{\gamma_t}{(1-\gamma_t)}
\end{aligned}$$

The third equation is due to Assumption 5 and the last step is by Assumption 1. Thus, for all t,

$$E^{\cdot n}[\frac{\gamma_t}{(1-\gamma_t)}] = \frac{\gamma_0}{(1-\gamma_0)} \tag{1.4.3}$$

Since γ_t converging to some random variable Q - a.s. implies that γ_t converges $Q^{\cdot n}$ - a.s. (since $Q^{\cdot n}$ is absolutely continuous with respect to Q). Since $\frac{\gamma_0}{(1-\gamma_0)}$ is finite, $\lim_{t\to\infty} \gamma_t < 1 \quad Q^{\cdot n}$ - a.s. (Suppose on the contrary, there is a set $D \in \Omega$ with $Q^{\cdot n}(D) > 0$ such that $\gamma_t(\omega) \to 1$ for all $\omega \in D$. Then, $\frac{\gamma_t}{(1-\gamma_t)} \to \infty$ on D, which contradicts to (1.4.3)). ■

Note also that $\lim_{t\to\infty} \gamma_t \| s_2 - E^{\cdot n}[\tilde{\tau}_t \mid h_t] \| \to 0$, also $Q^{nn} -$ a.s.

Player 2's beliefs about player 1's beliefs

After showing that if player 1 does not eventually learn that player 2 is strategic (when player 2 is strategic and the histories are induced by the play of strategic player 2), then player 1 must think that strategic type of player 2's strategy should be close to that of commitment type of player 2's; now, I want to show that player 2 will know that player 1 eventually expects to see the commitment action of player 2 in the continuation game on these histories (since reputations are public by Assumption 5).

Lemma 1.4.3 *Suppose Assumptions 1, 3 and 5 hold and suppose there exists $A \in \Omega$ such that $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$, i.e. there exists a set of events with strictly positive measure in which reputation of* *player 2 does not necessarily disappear. Then, there exists $\eta > 0$ and $F \subset A$ with $Q^{\cdot n}(F) > 0$ (and $Q(F) > 0$) such that, for any $\xi > 0$, there exist T for which,*

$$\gamma_t > \eta, \quad \forall t \geq T,$$

$$E\left[\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi, \quad \forall t \geq T \tag{1.4.4}$$

for all $\omega \in F$; and for all $\psi > 0$

$$Q\left(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid \mathcal{H}_t\right) \to 1 \tag{1.4.5}$$

where the convergence is uniform on F.[30]

Proof. First observe that on set A, $0 < \lim_{t\to\infty} \gamma_t(\omega) < 1$ by Corollary 1.4.2. Since $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$, there exist sufficiently small $\nu > 0$ and $\eta > 0$ such that $Q^{\cdot n}(D) > 2\nu$, where $D := \{\omega \in A : 2\eta < \lim_{t\to\infty} \gamma_t(\omega) < 1\}$. Note that D has positive measure under Q, i.e. there exists ν such that $Q(D) > 2\nu$ (since $Q^{\cdot n}$ is absolutely continuous with respect to Q.) Then, by Lemma 1.4.1, $\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\|$ converge Q - almost surely

[30]The same statements of the Lemma hold for $E^{\cdot n}$ and $Q^{\cdot n}$. In subsequent sections, both versions are going to be used.

to zero on D. [31] So, the random variables $Z_t := \sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\|$ also converge Q - almost surely (also $Q^{\cdot n}$ - almost surely) to zero on D. Thus, on D, by an extension of Hart (1985) Lemma 4.24, given in Mailath and Samuelson (2006), [32]

$$E[Z_t|\mathcal{H}_t] \to 0, \quad Q - \text{a. s.} \quad \text{and} \quad Q^{\cdot n} - \text{a. s.}$$

and also $E^{\cdot n}[Z_t|\mathcal{H}_t] \to 0, \quad Q^{\cdot n} - \text{a. s.}$

Egorov's Theorem (Chung (1974)) [33] then implies that there exists an $F \subset D$ such that $Q^{\cdot n}(F) > \nu$ (note that $Q(F) > 0$) on which the convergence of γ_t and $E[Z_t|\mathcal{H}_t]$ (and $E^{\cdot n}[Z_t|\mathcal{H}_t]$) is uniform. The uniform convergence of $E[Z_t|\mathcal{H}_t]$ on F implies that, for any $\xi > 0$, there exist a T such that on F, for all $t > T$, $\gamma_t > \eta$ and

$$E[Z_t|\mathcal{H}_t] = E\left[\sup_{s \geq t} \|s_2 - E[\tilde{\tau}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi \tag{1.4.6}$$

[31] Note that $\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\|$ also converge $Q^{\cdot n}$ and $Q^{\cdot c}$ - almost surely to zero on D.

[32] This lemma states that if $\{X_n\}_{n=1}^{\infty}$ is a bounded sequence of real random variables on some $(\Omega, \mathcal{F}, P)$, converging 0 as $n \to \infty$ and $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of σ - fields, then $E[X_n \mid \mathcal{F}_n] \to 0$ P-a.s.

[33] Egorov's Theorem states that if $\{X_n\}$ converges on the set C, then for any $\epsilon > 0$, there exists $C_0 \subset C$ with measure $\mathcal{P}(C \backslash C_0) < \epsilon$ such that X_n converges uniformly in C_0.

In order to show (1.4.5), fix $\psi > 0$. Then, for all $\xi' > 0$ such that $\xi = \xi'\psi$, (1.4.6) holds. Hence,

$$\begin{aligned} E[Z_t|\mathcal{H}_t] &= E[Z_t|Z_t < \psi, \mathcal{H}_t].Q(Z_t < \psi \mid \mathcal{H}_t) \\ &+ E[Z_t|Z_t \geq \psi, \mathcal{H}_t].Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi'\psi. \end{aligned}$$

Since the first expression is greater and equal to 0 and $E[Z_t|Z_t \geq \psi, \mathcal{H}_t] \geq \psi$,

$$Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi',$$

or $Q(Z_t < \psi \mid \mathcal{H}_t) > 1 - \xi'$ for all $t > T$ on F. This implies (1.4.5) and completes the proof.[34] ■

Player 1's best response to player 2

If player 1 were to be short-lived, as long as she thinks that she is facing a commitment strategy, she gives the myopic best reply to the commitment strategy of the opponent, which is s_1. This may not be true if player 1 is long-lived. She may have an incentive to play something other than the best response to the commitment action of player 2. In this case, since player 1 discounts, any losses from not playing a current best response should be

[34] The other way to show this: $Q(Z_t \geq \psi \mid \mathcal{H}_t) \leq \frac{E[Z_t|\mathcal{H}_t]}{\psi} < \frac{\xi}{\psi}$ by Chebyshev-Markov inequality since Z_t has a finite mean and $Z_t \geq 0$. Since $\psi > 0$ and $\xi = \xi'\psi$, I get $Q(Z_t \geq \psi \mid \mathcal{H}_t) < \xi'$ for all $\xi' > 0$.

recovered within a finite period of time. However, if player 1 is convinced that the commitment action will be played not only now, but also in the future, there will be no opportunity to accumulate subsequent gains, and hence she might as well play the stage-game best response. The next lemma, which follows from Lemma 4 of Cripps, Mailath, and Samuelson (2004), uses this intuition. It shows that if the commitment type and strategic type of player 2 play sufficiently similar from some time on, strategic type of player 1 will be best responding to the commitment type's strategy for arbitrarily many periods.

Lemma 1.4.4 *Suppose $\hat{\tau}$ be a simple pure public strategy and $BR_1(\hat{\tau})$ is the set of best replies of strategic type of player 1 to $\hat{\tau}$.*[35] *Let $(\tilde{\sigma}, \tilde{\tau})$ be Nash equilibrium strategies in the incomplete-information game. If $\tilde{\sigma}$ is a pure strategy,*[36] *then for all $T > 0$, there exists $\psi > 0$ such that if the strategic player 1 observes a (public) history h_t so that*

$$Q\left(\sup_{s \geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1 - \psi \qquad (1.4.7)$$

then for $\hat{\sigma} \in BR_1(\hat{\tau})$, the continuation strategy of $\tilde{\sigma}$ after the history h_t

[35] Remember that $\hat{\tau}$ assigns s_2, which is a pure action, in each period independent of history and the repeated strategy best response $\hat{\sigma} \equiv BR_1(\hat{\tau})$ is a singleton, which assigns r_1 in every period after any history.

[36] If $\tilde{\sigma}$ is not pure, one could assume that there exists $k > 0$ such that for all h_t, if $\tilde{\sigma}^i(h_t) > 0$, then $\tilde{\sigma}^i(h_t) > k$.

agrees with $\hat{\sigma}$ for the next T periods.

Proof. Fix $T > 0$ and a (public) history h'_t. Let $\hat{\tau}(h_s) = s_2$ denote the continuation play of committed player 2 after the public history h_s, where h'_t is the initial segment of h_s.

Since player 1 is discounting, there exist $T' \geq T$ and $\epsilon > 0$ such that if for $s = t, ..., t + T'$ and for all h_s with initial segment h'_t

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid h_s]\| < \epsilon, \tag{1.4.8}$$

is satisfied, then the continuation strategy of $\tilde{\sigma}$ after the history h'_t agrees with $\hat{\sigma} \in BR_1(\hat{\tau})$, for the next T periods.

Now, one needs to show (1.4.8) holds for $s = t, ..., t+T'$ and for all h_s with initial segment h'_t. Suppose not, i.e. there exist h_s, for some $s = t, ..., t + T'$ such that

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid h_s]\| \geq \epsilon,$$

For a contradiction, define $\bar{\rho} \equiv \min_{y,i,j} \rho^y_{ij}$ and $\psi = \frac{1}{2}\min\{\epsilon, \bar{\rho}^{T'}\}$. Since player 1 is playing a pure strategy, the probability of the continuation history h_s, conditional on the history h'_t, is at least $\bar{\rho}^{T'}$. Thus,

$$Q\Big(\|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s])\| \geq \epsilon \mid h'_t\Big) \geq \bar{\rho}^{T'},$$

Since $\psi < \epsilon$, $Q\Big(\sup_{s \geq t} \|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s])\| \geq \psi \mid h'_t \Big) \geq \bar{\rho}^{T'}$, contradicting (1.4.7), since $\bar{\rho}^{T'} > \psi$. ■

Proof of disappearance of player 2's reputation

The complications created by the two-sided incomplete information is waved by the fact that only player 2 is subject to binding moral hazard at the commitment profile. On a subset of states F in Lemma 1.4.3, the strategic type of player 1 believes that she should be playing a best response to the commitment strategy of player 2, which also coincides with the strategy of commitment type of player 1. So, the strategic type of player 2, knowing what player 1 thinks about his future behavior, will best respond to both the strategic type and the commitment type of player 1's strategy with high probability. Since, player 2's best response to player 1's strategy is different than his commitment strategy, the strategic and the commitment type of player 2 are expected to play differently, which will provide the desired contradiction to $\gamma_t \nrightarrow 0$ on F.

So, what is needed to be shown is that player 1 eventually assigns high probability to player 2 believing with high probability that player 1 believes player 2's strategy is very close to commitment strategy (when the game evolves according to the play of the strategic type of player 2). And thus player 1 believes that the strategic and the commitment type of player 2 act differently, since he believes that the strategic player 1 is giving a best response, which coincides with the play of the commitment type of player 1. Thus player 1's second order beliefs about the future behavior of player 2 contradicts with her first order beliefs, leading to a contradicting to $\gamma_\infty > 0$ on F.

More specifically, one has to show that player 1 assigns a probability $1-\zeta$ to player 2 believing with probability at least $1-\eta$ that player 1 thinks player 2's strategy is within ξ of the commitment strategy when the probability measure over the histories are induced by the play of the strategic type of player 2,[37] i.e.

$$Q^{\cdot n}\left(Q^{\cdot n}\left(\sup_{s\geq t}\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1-\eta \mid \mathcal{H}_t\right) > 1-\zeta$$

By picking ξ, η and ζ accordingly, I am able to get the required contradiction. More specifically, choose ξ and ζ such that $\zeta < 1$ and $\xi < \min\{\psi, 1-\zeta\}$. First, I make a couple of observations that provide the basis for the contradiction.

Since the commitment strategy of player 2, $\hat{\tau}$ (i.e. playing always s_2), is a simple pure strategy, by Lemma 1.4.4, for any $T' > 0$, there exists $\psi > 0$

[37]In other words, player 1 assigns at most probability ζ to the commitment strategy of player 2. This is because player 2 believes with high probability player 1 thinks he is going to act like a commitment type and best respond to it. Since player 2 is subject to binding moral hazard at the commitment profile, player 1 eventually expects player 2 playing the commitment strategy with no more than ζ probability. But, this contradicts her initial belief of player 2's strategy being ξ close to the commitment strategy if I choose ξ and ζ properly.

such that if player 1 observes a public history h_t so that

$$Q\left(\sup_{s\geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1-\psi, \tag{1.4.9}$$

then the continuation strategy of $\tilde{\sigma}$ after the history h_t agrees with $\hat{\sigma} \in BR_1(\hat{\tau})$ for the next T' periods, where $\hat{\sigma} = \{r_1\}_{t=0}^{\infty}$.

Now, suppose for a contradiction, that there is a set of states A with $Q^{\cdot n}(A) > 0$ and $\gamma_\infty(\omega) > 0$ for all $\omega \in A$ (note that $\gamma_\infty(\omega) < 1$ on A by Corollary 1.4.2). Then, by Lemma 1.4.3, there is a set $F \subset A$ with $Q^{\cdot n}(F) > 0$ such that for any $\xi > 0$, there exists a T such that for any $t > T$ and $\omega \in F$,

$$Q\left(\sup_{s\geq t} \|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t\right) \to 1 \tag{1.4.10}$$

Hence, there exists a subset $G \in F$ with $Q^{\cdot n}(G) > 0$ such that on G,

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\| < \xi \quad Q - \text{a.s.}$$

Note that (1.4.11) implies, for any $\xi > 0$ and any $t > T$, on G,

$$\|s_2 - E^{\cdot n}[\tilde{\tau}_t \mid \mathcal{H}_t]\| < \xi \quad Q^{\cdot n} - \text{a.s.} \tag{1.4.11}$$

Also, by following the same argument in Lemma 1.4.3 (through extension of Hart's lemma given in footnote 32), I conclude that for some η and ζ, on G,

$$Q^{\cdot n}\left(\sup_{s\geq t}\|s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi|\mathcal{H}_t\right) > 1-\eta\zeta. \tag{1.4.12}$$

This shows that with a high probability $(1-\eta\zeta)$, player 2 believes that player 1 assigns player 2's strategy to be ξ close to the commitment strategy for any $t > T$.

Define

$$g_t := Q^{\cdot n}\left(\sup_{s\geq t}\|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi|\mathcal{H}_t\right)$$

$$\kappa_t := Q^{\cdot n}(g_t > 1-\eta|\mathcal{H}_t)$$

I want to show $\kappa_t > 1-\zeta$.[38] Since, $E^{\cdot n}[g_t \mid \mathcal{H}_t] > 1-\eta\zeta$ by condition (1.4.12), and

$$\begin{aligned} E^{\cdot n}[g_t \mid \mathcal{H}_t] &= E^{\cdot n}[g_t \mid g_t \leq 1-\eta, \mathcal{H}_t](1-\kappa_t) + E^{\cdot n}[g_t \mid g_t > 1-\eta, \mathcal{H}_t]\kappa_t \\ &\leq (1-\eta)(1-\kappa_t) + \kappa_t \end{aligned}$$

Thus,

$$1-\eta\zeta < (1-\eta)(1-\kappa_t) + \kappa_t$$

[38] Note that reputations are public and both players use $\mathcal{H}_t$ as their information sets.

which implies $\kappa_t > 1 - \zeta$ on F. So,

$$Q^{\cdot n}\left(Q^{\cdot n}\left(\sup_{s \geq t} \|(s_2 - E^{\cdot n}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1 - \eta \mid \mathcal{H}_t\right) > 1 - \zeta$$

This says that player 1 assigns a probability of at least $1 - \zeta$ (after observing histories generated by the play of strategic type of player 2) to strategic type player 2 believing with probability at least $1 - \eta$ that player 1 believes player 2's strategy is within ξ of the commitment strategy and thus is going to give a best response to the commitment strategy of player 2 for at least next T' periods if player 1 is strategic type and play the commitment strategy if player 1 is the commitment type, both of which are the same strategy.

At time $t > T$, player 2 believes that with probability μ_t, player 1 is the commitment type who plays s_1 every period, and with probability $1 - \mu_t$, he is the strategic type who believes that player 2's strategy is within ξ of the commitment strategy from then on and hence plays the best reply to it (since I have picked $\xi < \psi$) thereafter. That is why, after $t > T$, both types of player 1 is expected to play r_1 thereafter, there won't be any revision of posterior about player 1's type, and hence $\mu_{t>T} = \mu_T$.

Since s_2 is not a best response to r_1 (the myopic best reply of strategic player 1 to s_2 and also the strategy of the commitment type of player 1) for strategic type of player 2, there exists $\eta_\mu > 0$ such that for any repeated game strategy of the strategic type of player 1 that attaches probability at least $1-\eta_\mu$ to $\hat{\sigma}$ (to always playing r_1), s_2 is suboptimal for the strategic type of player 2 (by the upper-hemicontinuity of the best response correspondence) in period 0. Note that since player 2 believes that player 1 is commitment type who plays $s_1 = r_1$ with probability μ, η depends on μ. Let $\bar{\eta} \equiv \sup_{\mu\in(0,1)} \eta_\mu$ such that s_2 is suboptimal for the strategic player 2 in period 0 if strategic type of player 1 attaches $1-\bar{\eta}$ to $\hat{\sigma}$, regardless of player 2's belief about player 1's type. Define $\bar{\rho} := \min_{y,i,j} \rho_{ij}^y$ (> 0 by Assumption 1). So, if player 2 assigns probability at least $1 - \bar{\rho}\bar{\eta} \equiv 1 - \eta$ to $\hat{\sigma}$, then he assigns at least probability $1 - \bar{\eta}$ to $\hat{\sigma}$ after any deviation that leaves the probability of $\hat{\sigma}$ (conditional on any signal) unchanged. Note that s_2 is suboptimal for any belief μ, in particular $\mu_{t>T} = \mu_T$.

Since ξ was picked such that $\xi < \psi$, and condition (1.4.9) holds for all $t > T$, strategic type of player 1 chooses to play r_1, the unique best response to the commitment action thereafter, whenever he believes that player 2's strategy is within ξ of the commitment strategy. Hence, in any period $t > T$, player 1 assigns a probability of at least $1-\zeta$ to player 2 believing that player 1's subsequent play is r_1 thereafter with at least probability $1 - \eta$. Thus, player 1 assigns probability at least $1 - \zeta$ to player 2's play in period t being a best response to $\hat{\sigma}$. Since s_2 is pure, it specifies an action $\hat{j}$ with probability 1. However, player 1 must believe that that action is played with no more than ζ probability in period t. But since, $1-\zeta > \xi$, this contradicts

(1.4.11). Player 1's second order beliefs about strategic player 2's behavior (after observing the relevant game has been evolving and histories have been generated by the play of strategic player 2) contradicts with her first order beliefs. This completes the proof of the first part of Proposition 1.3.2, i.e. $\gamma_t \to 0 \quad Q^{\cdot n}$ - almost surely, which implies $\gamma_t \to 0 \quad Q^{nn}$ - almost surely.

Uniform disappearance of player 2's reputation

Uniform convergence of $\gamma_t \to 0$, $Q^{\cdot n}$-almost surely means that there exists some period T after which reputation converges to zero across all Nash equilibria. Suppose, on the contrary, there is a Nash equilibrium for each T after which reputation of player 2 survives. Then the sequence of these Nash equilibria where the reputation lasts beyond T converges to a limiting Nash equilibrium with a sustainable reputation, which contradicts to disappearance of reputation result for any Nash equilibria.

The uniform disappearance of player 2's reputation, can be proved as the proof of Theorem 3 of Cripps, Mailath, and Samuelson (2007). I present the proof here for the sake of completeness. One needs to show that for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{\cdot n}_{\sigma=(\hat{\sigma},\tilde{\sigma}),\tilde{\tau}}$ is the probability measure induced on Ω by $(\sigma, \tilde{\tau})$ and the strategic type of player 2 and $\gamma_t(\sigma, \tilde{\tau})$ is the associated reputation of player 2.

Proof. Suppose for a contradiction, there exists $\varepsilon > 0$ such that for all T, there is a Nash equilibrium profile α_T such that

$$Q^{\cdot n}_{\alpha_T}(\gamma_t(\alpha_T) < \varepsilon, \quad \forall t > T) \leq 1 - \varepsilon,$$

where $Q^{\cdot n}_{\alpha_T}$ is the measure induced by the strategic type of player 2 and Nash equilibrium profile α_T, and $\gamma_t(\alpha_T)$ is the posterior about player 2's type under α_T.

Since the space of strategy profiles is compact in the product topology, there is a convergent subsequence $\{\alpha_{T_k}\}$ with limit α^*. Relabel this sequence so that $\alpha_k \to \alpha^*$ and

$$Q^{\cdot n}_k(\gamma^k_t < \varepsilon, \quad \forall t > k) \leq 1 - \varepsilon, \quad \text{or}$$

$$Q^{\cdot n}_k(\gamma^k_t \geq \varepsilon, \quad \text{for some} \quad t > k) \geq \varepsilon.$$

Since each α_k is a Nash equilibrium, $\gamma^k_t \to 0$, $Q^{\cdot n}_k$ - almost surely. So, there exists $K_k > k$ such that

$$Q_k^{\cdot n}(\gamma_t^k < \varepsilon, \quad \forall t \geq K_k) \geq 1 - \varepsilon/2.$$

So, for all k,

$$Q_k^{\cdot n}(\gamma_t^k \geq \varepsilon, \quad \text{for some } t, k < t < K_k) \geq \frac{\varepsilon}{2}.$$

Let χ_k denote the stopping time

$$\chi_k = \min\{t > k : \gamma_t^k \geq \varepsilon\}$$

and Z_t^k be the associated stopped process,

$$p_t^k = \begin{cases} \gamma_t^k & \text{if} \quad t < \chi_k, \\ \varepsilon & \text{if} \quad t \geq \chi_k. \end{cases}$$

Note that p_t^k is a supermartingale under $Q_k^{\cdot n}$ and for $t < k$, $p_t^k = \gamma_t^k$. So, for all k and $t \geq K_k$,

$$\begin{aligned} E_k^{\cdot n} p_t^k &\geq \varepsilon Q_k^{\cdot n}(\chi_k \leq t) \\ &\geq \frac{\varepsilon^2}{2}. \end{aligned}$$

On the other hand, since α^* is a Nash equilibrium, $\gamma_t^* \to 0$, $Q_{\alpha^*}^{\cdot n}$- almost surely. So, there exists s such that

$$Q^{\cdot n}_{\alpha^*}(\gamma^*_s < \varepsilon^2/12) > 1 - \varepsilon^2/12.$$

Then,

$$E^{\cdot n}_{\alpha^*}\gamma^*_s \leq \frac{\varepsilon^2}{12}\left(1 - \frac{\varepsilon^2}{12}\right) + \frac{\varepsilon^2}{12} < \frac{\varepsilon^2}{6}$$

Since $\alpha_k \to \alpha^*$ in the product topology, there exists a $k' > s$ such that for all $k \geq k'$,

$$E^{\cdot n}_k \gamma^k_s < \frac{\varepsilon^2}{3}.$$

But since $k' > s$, $p^k_s = \gamma^k_s$ for $k \geq k'$ and so for any $t \geq K_k$,

$$\frac{\varepsilon^2}{3} > E^{\cdot n}_k \gamma^k_s = E^{\cdot n}_k p^k_s \geq E^{\cdot n}_k p^k_t \geq \frac{\varepsilon^2}{2},$$

which is a contradiction. ■

1.4.2 Player 1's reputation disappears uniformly

Suppose that player 2's reputation disappears uniformly in any Nash equilibrium of the two-sided incomplete information game, i.e. for all $\varepsilon > 0$, there exists T_2 such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) < \varepsilon, \forall t > T_2) > 1 - \varepsilon,$$

where $Q^{\cdot n}_{\sigma,\tilde{\tau}}$ is the probability measure induced on Ω by $(\sigma, \tilde{\tau})$ and the strategic type of player 2 and $\gamma_t(\sigma, \tilde{\tau})$ is the associated reputation of player 2. So, after T_2 on, player 1 attaches a very high probability to be facing the strategic type of player 2, i.e. facing the commitment type no more than ε probability,

$$Q^{\cdot n}_{\sigma,\tilde{\tau}}(\gamma_t(\sigma, \tilde{\tau}) \geq \varepsilon, \quad \text{for some} \quad t > T_2) \leq \varepsilon,$$

Player 1 thinks she will be seeing a strategy by the strategic type of player 2 after T_2. I proceed by a similar argument as the one provided for the proof of Proposition 1.3.2 in Section 1.4. The counterparts of Lemma 1.4.1, 1.4.3 and 1.4.4 hold for player 1. With these results at hand and the above assumption, I derive that player 1's reputation disappears (uniformly) as well.

The following Lemma argues that either player 2's expectation of the strategy played by the strategic type of player 1 is in the limit the same as the strategy played by the commitment type of player 1, or player 2's posterior probability that player 1 is the commitment type converges to zero (given that player 1 is indeed strategic). The key idea is the same: Strictly positive beliefs about player 1's types can exist in the long-run only if both types of player 1 play identically in the limit provided that the public signals are statistically informative about player 1's actions.

Lemma 1.4.5 *Suppose Assumptions 1, 2 and 5 are satisfied. In any Nash equilibrium of the incomplete-information game,*

$$\lim_{t\to\infty} \mu_t(1-\mu_t)\|\hat{\sigma}_t - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0, \quad Q - a.s. \tag{1.4.13}$$

Note that since $\hat{\sigma}_t$ is a simple commitment strategy, it can be replaced by s_1. The proof is the same as the one given for Lemma 1.4.1.

Corollary 1.4.6 *At any Nash equilibrium of the incomplete-information game satisfying Assumptions 1, 2 and 5,*

$$\lim_{t\to\infty} \mu_t\|s_1 - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0, \quad Q^{n\cdot} - a.s.$$

Note that $\lim_{t\to\infty} \mu_t\|s_1 - E^{n\cdot}[\tilde{\sigma}_t|\mathcal{H}_t]\| = 0$, also Q^{nn} - a.s.

Corollary 1.4.6 says that if player 2 does not eventually learn that player 1 is strategic, then player 2 must think that strategic type of player 1's strategy should be close to that of commitment type since the distributions of public signals induced by the two types are not distinguishable. I now show that strategic type of player 1 will know that player 2 believes this, since reputations are public by Assumption 5.

Lemma 1.4.7 *Suppose Assumptions 1, 2 and 5 hold and suppose there exists $A \in \Omega$ such that $Q^{n\cdot}(A) > 0$ and $\mu_\infty(\omega) > 0$ for all $\omega \in A$, i.e. there* *exists a set of events with strictly positive measure in which reputation of player 1 does not necessarily disappear. Then, there exists* $\eta > 0$ *and* $F \subset A$, *with* $Q^{n\cdot}(F) > 0$, *such that, for any* $\xi > 0$, *there exist* T_1 *for which,*

$$\mu_t > \eta, \quad \forall t \geq T_1,$$

$$E\left[\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| \middle| \mathcal{H}_t\right] < \xi, \quad \forall t \geq T_1 \tag{1.4.14}$$

for all $\omega \in F$*; and for all* $\psi > 0$

$$Q\left(\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\tau}_s \mid \mathcal{H}_s]\| < \psi \mid \mathcal{H}_t\right) \to 1 \tag{1.4.15}$$

where the convergence is uniform on F*.*

The next lemma shows that if the commitment type and strategic type of player 1 play sufficiently similar, strategic type of player 2 will be best responding to the commitment type's strategy for arbitrarily many periods.

Lemma 1.4.8 *Suppose* $\hat{\sigma}$ *be a simple pure public strategy and* $BR_2(\hat{\sigma})$ *is the set of best replies of strategic type of player 1 to* $\hat{\sigma}$*.*[39] *Let* $(\tilde{\sigma}, \tilde{\tau})$ *be Nash*

[39] Note that $\hat{\sigma}$ assigns s_1 in every period independent of history. The repeated game best response of player 2 in the complete-information game is $BR_2(\hat{\sigma})$ is a singleton that assigns r_2 in each period.

equilibrium strategies in the incomplete-information game. If $\tilde{\tau}$ is a pure strategy, then for all $T > 0$, there exists $\psi > 0$ such that if player 2 observes a (public) history h_t so that

$$Q\left(\sup_{s\geq t}\|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1 - \psi \tag{1.4.16}$$

then for $\tau' \in BR_2(\hat{\sigma})$, the continuation strategy of $\tilde{\tau}$ after the history h_t agrees with τ' for the next T periods.

For a contradiction that if there is a set of states with positive measure that is induced by the play of the strategic type of player 1 on which $\mu_t \nrightarrow 0$, then, on a subset of states F in Lemma 1.4.7, the strategic type of player 2 believes that player 1's strategy is very close to her commitment strategy and thus he should be playing a best response to the commitment strategy of player 1 (after T_1), which is different than the strategy of the commitment type of player 2. Since the both players can compute what the other player believes about themselves and their future play, the strategic type of player 1 will know how player 2 thinks her future behavior is going to be and act accordingly. By Proposition 1.3.2, after some T_2, the reputation of player 2 will disappear in all Nash equilibria, and thus after $T = \max\{T_1, T_2\}$, player 1 expects to see a best reply to her commitment strategy from the strategic type of player 2 with a high probability. Since strategic type of player 2's strategy is different than the commitment type's strategy, his reputation will disappear even more after T. More precisely, the histories for which player 2's reputation can be rebuilt would have measure 0. Thus, the strategic player 1 best responds to the strategic type of player 2 (who gives a best reply to the commitment strategy of player 1 which is different than commitment type of player 2's strategy) with high probability. However, since the strategic type player 1's best response to player 2's strategy is different than her commitment strategy, the strategic and the commitment type of player 1 are expected to play differently, which will provide the contradiction to $\mu_t \nrightarrow 0$ on F.

More specifically, I am going to show that player 2 assigns a probability $1 - \zeta$ to player 1 believing with probability at least $1 - \eta$ that player 2 thinks player 1's strategy is within ξ of the commitment strategy when the probability measure over the histories are induced by the play of the strategic type of player 1,

$$Q^{n\cdot}\left(Q^{n\cdot}\left(\sup_{s \geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1 - \eta \mid \mathcal{H}_t\right) > 1 - \zeta.$$

Choose ξ and ζ such that $\xi < \min\{\psi, 1 - \zeta\}$ and $\zeta < 1$ to arrive the desired contradiction. Suppose that there is a set of states A with $Q^{n\cdot}(A) > 0$ and $\mu_\infty(\omega) > 0$ for all $\omega \in A$ (note that $\mu_\infty(\omega) < 1$ on A by Corollary 1.4.6.) Then, by Lemma 1.4.7, there is a set $F \in A$ with $Q^{n\cdot}(F) > 0$ (also

$Q(F) > 0$) such that for any $\xi > 0$, there exists a T_1 such that for any $t > T_1$ and $\omega \in F$,

$$Q\left(\sup_{s \geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t \right) \to 1 \tag{1.4.17}$$

Then, there exists a subset $G \in F$ with $Q^{n\cdot}(G) > 0$ such that for any $t > T_1$, on G,

$$\|s_1 - E^{n\cdot}[\tilde{\sigma}_t \mid \mathcal{H}_t]\| < \xi, \quad Q - \text{a.s.} \tag{1.4.18}$$

Note that (1.4.18) implies $\|s_1 - E^{n\cdot}[\tilde{\sigma}_t \mid \mathcal{H}_t]\| < \xi \quad Q^{n\cdot}$- a.s., Q^{nn}-a.s. and $Q^{\cdot n}$-a.s. Also, by the extension of Hart's lemma given in footnote 32), one can conclude that for some η and ζ,

$$Q^{n\cdot}\left(\sup_{s \geq t} \|s_1 - E^{\cdot n}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t \right) > 1 - \eta\zeta. \tag{1.4.19}$$

This shows that with a high probability $(1 - \eta\zeta)$, player 1 believes that player 2 assigns player 1's strategy to be ξ close to the commitment strategy. Define,

$$g_t := Q^{n\cdot}\left(\sup_{s \geq t} \|(s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi | \mathcal{H}_t \right)$$

$$\kappa_t := Q^{n\cdot}(g_t > 1 - \eta | \mathcal{H}_t)$$

I want to show $\kappa_t > 1 - \zeta$. Since, $E^{n\cdot}[g_t \mid \mathcal{H}_t] > 1 - \eta\zeta$ by condition

(1.4.19), and

$$\begin{aligned} E^{n\cdot}[g_t \mid \mathcal{H}_t] &= E^{n\cdot}[g_t \mid g_t \leq 1-\eta, \mathcal{H}_t](1-\kappa_t) + E^{n\cdot}[g_t \mid g_t > 1-\eta, \mathcal{H}_t]\kappa_t \\ &\leq (1-\eta)(1-\kappa_t) + \kappa_t \end{aligned}$$

I have,

$$1 - \eta\zeta < (1-\eta)(1-\kappa_t) + \kappa_t$$

which implies $\kappa_t > 1 - \zeta$ on F. So,

$$Q^{n\cdot}\left(Q^{n\cdot}\left(\sup_{s\geq t} \|(s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \xi \mid \mathcal{H}_t\right) > 1-\eta \mid \mathcal{H}_t\right) > 1-\zeta$$

This says that player 2 assigns a probability of at least $1-\zeta$ (after observing histories generated by the play of the strategic type of player 1) to player 1 believing with probability at least $1-\eta$ that player 2 believes player 1's strategy is within ξ of the commitment strategy. Note that, by Lemma 1.4.8, for all $T' > 0$, there exists $\psi > 0$ such that if player 2 observes a (public) history h_t so that

$$Q\left(\sup_{s\geq t} \|s_1 - E^{n\cdot}[\tilde{\sigma}_s \mid \mathcal{H}_s]\| < \psi \mid h_t\right) > 1-\psi$$

then for $\tau' \in BR_2(\hat{\sigma})$, the continuation strategy of $\tilde{\tau}$ after the history h_t

agrees with τ' for the next T' periods. Since $\xi < \psi$, strategic player 2 best responds to the commitment strategy of player 1 for the next T' periods.

Define $T := \max\{T_1, T_2\}$. Note that after time $t > T$, by uniform disappearance of player 2's reputation, player 1 believes that player 2 is the commitment type with $\gamma_t < \varepsilon$ who plays s_2 every period, and with probability $1 - \gamma_t > \varepsilon$ he is the strategic type who believes that player 2's strategy is within ξ of the commitment strategy from then on and hence plays the best reply to it (since $\xi < \psi$) thereafter. That is why, after $t > T$, the strategic player 2 will give a best response to s_1, which is different than the commitment strategy of player 2, thus it is unlikely for player 2 to rebuilt after T_2. I can stop the process γ_t at T.

Since s_1 is not a best response to $r_2 \neq s_2$ (i.e. the myopic best reply of strategic player 2 to s_1), there exists $\eta > 0$ such that for any repeated game strategy of the strategic type of player 1 that attaches probability at least $1 - \eta$ to $\hat{\tau}$ (to always playing r_2), s_1 is suboptimal for the strategic type of player 1 (by the upper-hemicontinuity of the best response correspondence) in period 0. Let $\bar{\eta}$ such that s_1 is suboptimal for the strategic player 1 in period 0 if strategic type of player 2 attaches $1 - \bar{\eta}$ to $\hat{\sigma}$, regardless of player 2's belief about player 1 being commitment type. Define $\bar{\rho} := \min_{y,i,j} \rho_{ij}^{y}$ (> 0 by Assumption 1). So, if player 2 assigns probability at least $1 - \bar{\rho}\bar{\eta} \equiv 1 - \eta$ to $\hat{\sigma}$, then he assigns at least probability $1 - \bar{\eta}$ to $\hat{\sigma}$ after any deviation that leaves the probability of $\hat{\sigma}$ (conditional on any signal) unchanged.

Since $\xi < \psi$, for all $t > T$, strategic type of player 2 chooses to play r_2, the unique best response to the commitment action thereafter, whenever he believes that player 1's strategy is within ξ of the commitment strategy. Hence, in any period $t > T$, player 2 assigns a probability of at least $1 - \zeta$ to player 1 believing that player 2's subsequent play is r_2 thereafter with at least probability $1 - \eta$. Thus, player 2 assigns probability at least $1 - \zeta$ to player 1's play in period t being a best response to $\hat{\tau}$. Since s_1 is pure, it specifies an action $\hat{i}$ with probability 1. However, player 2 must believe that that action is played with no more than ζ probability in period t. But since, $1 - \zeta > \xi$, this contradicts (1.4.18). Player 2's second order beliefs about strategic player 1's behavior (after observing the relevant game has been evolving and histories have been generated by the play of strategic player 1) contradicts with his first order beliefs.

The uniform disappearance of player 2's reputation follows the same argument as in Section 1.4.1. Hence, for all $\varepsilon > 0$, there exists T, such that for all Nash equilibria $(\tilde{\sigma}, \tilde{\tau})$ of the incomplete-information game,

$$Q^{nn}_{\tilde{\sigma},\tilde{\tau}}(\mu_t(\tilde{\sigma}, \tilde{\tau}) < \varepsilon, \forall t > T) > 1 - \varepsilon,$$

where $Q^{nn}_{\tilde{\sigma},\tilde{\tau}}$ is the probability measure induced on Ω by $(\tilde{\sigma}, \tilde{\tau})$ and the strategic types of players and $\mu_t(\tilde{\sigma}, \tilde{\tau})$ is the associated reputation of player 1.

1.5 Concluding Remarks

The main result of this paper is that the reputations of players for playing a strategy that is not part of an equilibrium of the stage game can not be sustainable in the long-run for games with one-sided binding moral hazard (at the commitment profile) under imperfect public monitoring. The way I prove our result is by first showing that the reputation of the player who is subject to binding moral hazard at the commitment profile disappears (uniformly) and then after the type of that player is almost known, the reputation of the other player should disappear as well. The implication of this result on the regulatee-regulator game is: First the reputation of being tough for the regulator disappears. After his true type is almost known, the regulatee starts to take advantage of regulator's uncertainty over her type and the regulatee's reputation of being virtuous disappears eventually, too. Moreover, the continuation equilibrium of the incomplete-information game converges to an equilibrium of the complete-information game in the limit.

There are some interesting related questions left for future research such as how the rate of disappearance (convergence) is affected by different priors. For instance, I believe that the existence of a tough regulator postpones the revelation of the true type of the regulatee; whereas the existence of a virtuous regulatee speeds up the revelation of the type of the regulator. So, a regulator whose goal is to understand the type of the regulatee should not pretend to be the tough type.

The other important observation one could make about the regulatee-regulator game is that the reputations are sustainable for more complicated commitment strategies that are equilibria of the repeated complete-information game. For instance, if there is a grim trigger type for the regulator; I believe that the reputations would be sustainable and the equilibrium would be almost efficient, in the sense that players achieve the highest total payoff (very close to the efficient frontier of the feasible and individually rational payoff set). Hence, if a regulator could choose to establish a reputation for a type in the presence of a grim trigger and a tough type, he should choose to mimic the grim trigger type.

Chapter 2

Strategic Communication vs. Strategic Auditing with Reputation Concerns on Both Sides

2.1 Introduction

This paper studies misrepresentation of information, by a privately-informed agent, to an authority figure. Misrepresention of private information by agents, even under the existence of a regulator or a monitor, is a common feature of many economic interactions. For instance, a bank who has private information about its financial health may misreport this information to a regulator; or a tax payer may fill out false income statements; or an investor may be engaged in fraudulent behavior by misrepresenting its books or have false fillings to a regulatory agency. Moreover, in most of such situations, the regulator or the monitor, who is supposed to detect deviations from the desirable behavior, may himself have an incentive to be engaged in moral hazard because of costly or timely auditing or monitoring. Moreover, both parties may have some prior or established beliefs about the other party and concerns about their reputation (for good behavior) or lack of it. The goal of this paper is to understand how private information can be manipulated by a strategic agent- Sender (through false messages), in the *presence* of a strategic Receiver, who aims to deter the manipulation of information using costly auditing, when their interactions are not contractable.[1]

More specifically, the Sender ('she') has *noisy* private information about an underlying state of nature and is able to misrepresent this information by sending false messages. There is a strategic Receiver ('he') who aims to deter this manipulation using costly auditing. The magnitude of Senders cost of lying is governed by the auditing strategy of the Receiver, which determines the probability of an audit and detecting an undesirable behavior

[1]Rahman (2009) proposes a contractual arrangement that makes the monitor's deviations irrelevant.

of the Sender.[2] The Receiver, on the other hand, may have an incentive not to audit intensively (since auditing is costly) if he thinks that the Sender is going to give the accurate information. Receiver believes that the Sender could be an honest type with some strictly positive probability. An honest Sender always sends the true message, whereas a strategic Sender maximizes expected payoffs. Similarly, the Sender believes that the Receiver could be a tough type with some strictly positive probability. A tough Receiver always chooses high auditing, whereas a strategic Receiver maximizes expected payoffs. The fact that the private information of the Sender is imperfect (noisy) and the auditing by the Receiver is random prevent players from learning each others true types, i.e. that they are strategic. To model this environment, I use a simultaneous-move version of an *inspection game*,[3] with incomplete-information about the types of players, where their actions are not observable. This paper aims to analyze how uncertainty about each other's types and the concerns for (false) reputation pay of for both parties; and characterize the equilibria in the (1) one-shot game; (2) two-period game. The equilibrium strategies are determined by the parameters of the model, as well as the discount factors (in the repeated game). The infinitely-repeated game of strategic communication vs. strategic auditing fits into the class of games analyzed in Chapter 1; and thus, none of the parties can fool

[2]The Receiver can be thought to be allocating time and/or resources to auditing among its other tasks.

[3]See Rudolf, Bernhard, and Zamir (2002) for a discussion on inspection games.

the other party indefinitely, i.e. their true types will be revealed eventually.

In the one-shot game, I found that if the prior belief that the Receiver is a tough type is above a threshold (which is the ratio of the expected gain of the Sender by being untruthful against a lazy Receiver to the sum of this expected gain against a lazy Receiver and the expected net loss being untruthful against a diligent Receiver), then the Sender is truthful with probability one (since she believes that she faces a tough Receiver with a high enough probability); and anticipating this the Receiver is lazy with probability one. On the other hand, if the prior belief that the Receiver is a tough type is below this threshold, then the prior belief that the Sender is honest determines the equilibrium strategy profiles. When the prior belief on Sender being honest is above a threshold (which is the ratio of the expected net gain of the Receiver by being diligent against an untruthful Sender to the sum of this expected gain against an untruthful Sender and the cost of being diligent against a truthful one), then the Sender chooses to be untruthful with probability one, since the Receiver believes he faces an honest Sender with a higher probability and chooses to be lazy with probability one. In other cases, the players use mixed strategies in equilibrium, that depend on the parameters defining the expected payoffs. Moreover, there is no circumstances where the Receiver chooses to be diligent with probability one in equilibrium.

The other important observation one can make in the one-shot game is that the Sender's equilibrium strategy for playing truthful is nonincreasing in the prior belief on her own type (Sender being honest) for any given prior belief about the Receiver's type. Thus, the Sender becomes less truthful as the Receiver believes she is honest with a higher probability. On the other hand, it is nondecreasing (so she chooses to be truthful with a weakly higher probability) in the prior belief on the Receiver being tough, for any given prior belief on her own type. The Receiver's strategy for playing diligent is nonincreasing both in the prior belief on his own type (tough) and the Sender being honest. Hence, the Receiver, in equilibrium, becomes lazy with a weakly higher probability as the prior belief on his toughness or on Sender being honest increases.

This weak monotonicity of equilibrium strategy profiles in the prior beliefs vanishes if the game is two-period. The incentives to maintain or build higher reputation for the second period, or best reply against such a motive, make the equilibrium strategy profile non-monotonic in the first period of the two-period game.[4] For instance, there are circumstances (when the Receiver's reputation of being tough is sufficiently high) where the Receiver chooses to be diligent with probability one to restore his reputation high for the second period to obtain his highest possible payoff, that cannot happen in any equilibrium of the one-shot game.

In the long-run, on the other hand, the true types of both the Sender and

[4]In the second period, players choose the equilibrium profiles of the one-shot game that are associated with the posterior beliefs in the second period.

the Receiver will eventually be revealed, by the results of Chapter 1. The Receiver reveals his true type first; after his type (almost) known the Sender also reveals her true type in the long-run. Moreover, any equilibria of the incomplete-information game converges to an equilibrium of the complete-information game in the long-run. However, the set of equilibria that they converge is an open question.

2.1.1 Motivating examples

The recent examples of such environment are the investment frauds in financial markets. On December 11, 2008, Bernard Madoff was arrested for operating one of the largest investment frauds and Ponzi schemes in the history. On March 12, 2009, he pleaded guilty to 11 federal offenses, including securities fraud, making false statements and making false fillings with the U.S. Securities and Exchange Commission (SEC). However, Bernard Madoff said he began the Ponzi scheme in the early 1990s, yet he was arrested on December 11, 2008; even though the SEC had previously conducted several investigations into Madoff's business practices since 1992.[5] He said that the only problem with the SEC's officials is that he had "too much credibility with them and they dismissed" the idea. David Kotz, who is the SEC inspector, conducted an investigation into how regulators failed to detect the fraud despite numerous red flags. David Kotz said in a report: "Despite

[5] An online New York Times article by DuBois, Roth, Davies, Couturier, and Brustein (2009) gives "A Timeline of the Madoff Fraud."

three examinations and two investigations being conducted, a through and competent investigation or examination was never performed."[6] Bernard Madoff told David Kotz that in a prison interview that "It never entered the SEC's mind that it was a Ponzi scheme," because of "the reputation I had."[7] SEC has been criticized for failure to act on Madoff fraud.

Another investment fraud charge was against Robert Allen Stanford on February 17, 2009, by the SEC. In a 159 page Report of Investigation (released on March 31, 2010) by the U.S. Securities and Exchange Commission Office of the Inspector General (OIG), the SEC has been following Stanford and his companies for much longer. The Report reveals lack of diligence in SEC enforcement. The IOG states:

"[T]he SEC's Fort Worth office was aware since 1997 that Robert Allen Stanford was likely operating a Ponzi scheme, having come to that conclusion a mere two years after Stanford Group Company ('SGC'), Stanford's investment adviser, registered with the SEC in 1995. We found that over the next 8 years, the SEC's Fort Worth Examination group conducted four examinations of Stanford's operations, finding in each examination that the CDs could not have been 'legitimate,' and that it was 'highly unlikely' that the returns Stanford claimed to generate could have been achieved with the purported conservative investment approach.

Fort Worth examiners dutifully conducted examinations of Stanford in 1997,

[6]See the online article "SEC criticised for failure to act on Madoff" at *http://business.timesonline.co.uk* by Christine Seib.

[7]See the online article "Madoff Explains How He Concealed the Fraud" at *www.cbsnews.com.*

1998, 2002 and 2004, concluding in each case that Stanford's CDs were likely a Ponzi scheme or a similar fraudulent scheme. The only significant difference in the Examination group's findings over the years was that the potential fraud grew exponentially, from 250 million to 1.5 billion."[8]

These instances amongst many other investment and accounting frauds suggest that the reputations (for being credible and diligent) of both parties are important in analyzing their interaction.

2.1.2 Related Literature

Strategic communication where a Sender, who has private information, willing to misrepresent this information through false announcements are extensively studied in various contexts. Sobel (1985) presents a model of credibility in which one agent must trust to another, whose intentions are unknown.[9] He shows that it pays to build a reputation for reliable behavior by repeatedly providing the accurate information; and also describes situations where it pays to the Sender to cash in on his (false) reputation. Benabou and Laroque (1992) provide a model of repeated strategic communication that extends Sobel (1985)'s model, to the case in which Sender (insider trader) has *noisy* private information about the value of an asset, to analyze manipulation of asset markets. They show that insiders have both

[8]See $http://www.sec.gov/news/studies/2010/oig-526.pdf$.

[9]Some examples include: a consumer deciding to believe advertisements, banks judging the reliability of loan applicants etc.

the ability and the incentives to manipulate public information and asset prices through strategically distorted announcements.[10] Since sender's information is noisy, he can engage in manipulation repeatedly without being fully detected. They also examine the extent to which the sender can influence the market in the long-run. Their model, however, is missing a strategic receiver who can audit the sender. This paper incorporates a strategic receiver who also tries to benefit from Sender's uncertainty about his types into Benabou and Laroque (1992)'s model.

Crawford (2003) models misrepresentation of intentions to competitors or enemies starting from an example of the Allies' decision to feint at Calais and attack Normandy on D-Day.[11] The situations he considers involve misrepresentations that have no direct costs between parties with conflicting interests. He points out that the first attempt to model such situations could be a *cheap talk* followed by a zero-sum two-person game. However, it is well known that the costless messages must be ignored, and thus equilibrium no information is conveyed by costless messages in equilibrium in that model.[12] Yet, in the examples Crawford (2003) provides, the misrepresentation succeeds. Crawford (2003) proposes a model that allows for the interaction between possibly rational and boundedly rational senders and

[10]Benabou and Laroque (1992) gives several real life examples of lying to manipulate financial markets.

[11]Crawford (2003) also provides examples from political and economic situations as well as military.

[12]See Crawford and Sobel (1982) and Farrell (1993).

receivers, by introducing types for players.[13] The players do know their own type only, but the structure of the game is common knowledge. The model is based on the class of zero-sum two-person perturbed Matching Pennies games. Before playing the stage game, the Sender sends the Receiver a costless message about her intended action. Players then choose their actions simultaneously. The equilibria of this game with one-sided pre-play communication depends on the parameters of the model.

My model is closest to Crawford (2003), in the sense that the Sender and Receiver have conflicting interests and uncertainty about each others' type. However, in my setting, the Sender tries to misrepresent her private information, rather than her intentions about the action she is going to take; meanwhile the Receiver decides on the auditing strategy. Thus, there is no need for pre-play communication. Also, in the economic interaction considered in this paper, zero-sum games are extreme version of conflicting interest game; thus I use a simultaneous-move version of an *inspection* game where players' actions are not observable to each other. Moreover, I also analyze the equilibria of two-period game to understand how reputation building or restoring could affect the behavior in equilibrium.

The equilibrium strategies are determined by the parameters of the model, as well as the discount factors (in the repeated game). The behavioral predictions of the model could hopefully be tested by field or lab

[13]The Sender's and Receiver's types are chosen randomly from separate distributions. They can be either Mortal (boundedly rational) or Sophisticated (rational).

experiments in future research.

2.2 Model

The model is a two-person simultaneous-move game with imperfect public monitoring and incomplete information on both sides. First, I define the complete-information two-person game.

2.2.1 Complete-Information Game

The Stage-game

There are two players: Player 1 is the Sender and Player 2 is the Receiver. Sender gets a noisy private information about the state of nature, $n \in N = \{H, L\}$, which can take two values, high (H) or low (L), each with probability $\frac{1}{2}$ and becomes publicly observable at the end of the period. At the beginning of each period, the Sender privately observes a $s \in S = \{h, l\}$ that predicts the true state of nature with probability α, i.e. $prob\{H \mid h\} = prob\{L \mid l\} = \alpha > \frac{1}{2}$. Depending on the signal, Sender reports a message $m \in M = \{h', l'\}$ to the Receiver, regarding the state of nature. Sender has the ability and incentive to deceive the uninformed Receiver by strategically manipulating the information through reporting false message, i.e. the Sender can either be *truthful* or *untruthful* about the signal he has received.

However, deceiving the Receiver is not costless. The cost of untruthful-

Table 2.1: Sender's information

	h	l
H	α	1 - α
L	1 - α	α

ness depends on the auditing strategy of the Receiver. The Receiver can choose to be *diligent* or *lazy* in auditing the Sender. Being diligent provides higher probability of audit β_D. If the Receiver chooses to be lazy, the probability of auditing is β_L. In the case of an audit, the Receiver gets an imperfect signal about the action chosen by the Sender. [14] If the Sender has been untruthful, the probability of detecting the undesirable behavior is $1/2 < p < 1$ and probability of making an error and accusing a truthteller is $1 - p$. For simplicity the likelihood of detecting an untruthful behavior is independent of Receiver's action (of being diligent or lazy). The Receiver's action determines the probability of being audited. The probability of detection in case of an audit can be summarized as:

Table 2.2: Detection probabilities

	h	l
h'	$1 - p$	p
l'	p	$1 - p$

The action set of the Sender is $A_S = \{T, U\}$; T is for being truthful and

[14] There can be many factors that prevents the Receiver detect the action chosen by the Sender even if he audits the Sender. The Receiver could be inexperienced, lack of expertise or there is lack of evidence.

U is for being untruthful; and that of the Receiver is $A_R = \{D, L\}$ where D is for being diligent and L is for being lazy.

The timeline in the stage game can be summarized as follows:

(0) Sender sees signal s about n.

(1) Sender takes an action in $\{T, U\}$, Receiver takes an action $\{D, L\}$.

(2) There is an audit $a = 1$ or no audit $a = 0$, probability of which depends on the Receiver's action.

(3) There is a detection $d = 1$ or no detection $d = 0$, conditional on being audited, probability of which depends on the Sender's action.

(4) True state of nature n is revealed.

At the end of the period, the publicly observed outcomes are the messages reported by the Sender, realized state of the nature, if there is an audit or not, and if the Sender is detected or not (in the case of an audit). I assume that the realized state of nature is payoff irrelevant, i.e. what matters for the players (in terms of payoffs) is whether the messages sent by the Sender matches the true state of the world or not, rather than the actual state.

Let I_f be the indicator function of whether player 1's message matches the realized state of the world or not; and similarly, denote whether the Sender is audited or not by the indicator function I_a. And let I_d is the indicator function of whether the Sender is detected or not. Note that

$I_d = 0$ if $I_a = 0$. At the end of the period, players observe three possible public signals $I_f \in \{0,1\}$, $I_a \in \{0,1\}$ and $I_d \in \{0,1\}$ (if there is a detection or not in the case of an audit). The distribution over I_a, $\rho(I_a|a_R)$, is determined by only the Receiver's actions and that of I_f, $\rho(I_f|a_S)$, is determined by the sender's actions. Note that the distributions over signals have *full support.* The set of publicly observed outcomes (I_f, I_a, I_d) is $Y = \{(1,1,1),(1,1,0),(1,0),(0,1,1),(0,1,0),(0,0)\}$. [15] Player i's (ex post) payoff after realization of (I_f, I_a, I_d) is given by $u_i(I_f, I_a, I_d; a_i)$. Ex ante stage game payoffs are then given by,

$$\pi_i(a_S, a_R) = \sum_{(I_f, I_a, I_d)} u_i(I_a, I_f, I_d; a_i)\rho_{a_i, a_{-i}}(I_f, I_a, I_d)$$

Expected (ex ante) payoffs from the stage game is summarized in the following payoff matrix:

Table 2.3: Expected Payoff Matrix

	L	D
T	x, y	$x - l_1,\ y - c$
U	$x + g,\ z$	$x - l_2,\ z - c + d$

where $y, z, g, c, d > 0$, $l_2 > l_1 \geq 0$ and $y > y - c > z - c + d > z$.

[15]In this setting, there is an individual specific public signal for each player. I_f and I_d are public signals about player 1's behavior (note that although the distribution of I_d depends on player 1's behavior only, it is observed only if there is audit, i.e. $I_a = 1$.) and I_a is the public signal about player 2's behavior.

2.2.2 Incomplete-Information Game

The game $G(\mu, \gamma)$ incorporates incomplete information on both sides. I use Harsanyi (1967)'s notion of a game with incomplete information to model the uncertainty about players' payoffs. Players' payoffs are identified with their types. Each player can be of two types. Let $K = \{honest, strategic\}$ be the type space for the Sender; and $L = \{tough, strategic\}$ be for the Receiver. An *honest* Sender is always truthful and a *tough* Receiver is always diligent. Note that the action played by honest and tough types of players are (pure) Stackelberg actions of strategic types. The prior beliefs about $k \in K$ is represented by a probability measure μ on K and those about $l \in L$ is represented by a probability measure γ. Each player knows his/her own type; has uncertainty over the types of the opponent. The structure of the game is common knowledge.

Repeated Game

The timeline in the repeated game is as follows:

1. Nature chooses $k \in K$ according to μ and $l \in L$ according to γ. The choices are made independently and once for all. Each player is told his/her type privately.

2. At every $t = 0, 1, ...,$:

    - The sender privately observes the signal about the state of nature;

- The players chooses their actions simultaneously;
- At the end of the period, the state of nature is publicly observed. The Sender is audited or not; and if audited the Sender is detected or not.

The set of public histories up to t is denoted by $H_t^p = (\mathcal{I}_f \times \mathcal{I}_a \times \mathcal{I}_d)^t$. An element of H_t^p would be $h_t^p = ((I_{f,0}, I_{a,0}, I_{d,0}), ..., (I_{f,t-1}, I_{a,t-1}, I_{d,t-1}))$. The set of private histories for player 1 is denoted by $H_{1t} = A_S^t \times H_t^p$ and that for player 2 is denoted by $H_{2t} = A_R^t \times H_t^p$. A behavior strategy for player 1, i.e. Sender's reporting strategy, is a sequence of maps $\sigma = \{\sigma_t\}_{t=1}^{\infty}$, where $\sigma_t : H_{1t} \times K \to \Delta(A_S)$. Similarly, a behavior strategy for player 2, i.e. Receiver's auditing strategy, is a sequence of maps $\tau = \{\tau_t\}_{t=1}^{\infty}$, where $\tau_t : H_{2t} \times L \to \Delta(A_R)$. Without loss of generality, I restrict attention to *public* strategies. A behavior strategy is *public* if in every period t, it depends only on the public history H_t^p, not on the private history.[16]

In the *infinitely*-repeated game, on the set H_∞^p, the σ-algebras $\mathcal{H}_t^p$ are generated by H_t^p and generate $\mathcal{H}_\infty^p$. And, the probability space is $(H_\infty^p, \mathcal{H}_\infty^p)$. Each strategy profile (σ, τ) together with the probability measures μ and γ induces a probability distribution Q on $(F, \mathcal{F}) = (K \times L \times H_\infty^p, 2^K \otimes 2^L \otimes \mathcal{H}_\infty^p)$. Every (σ, τ, k, l) defines a probability distribution P on $(H_\infty^p, \mathcal{H}_\infty^p)$. The

[16] The public monitoring structure described here satisfies Assumption 5 stated in Chapter 1. Hence, there is no loss of generality in restricting attention to public strategies.

relation between Q and P is the following:

$$Q_{\sigma,\tau,\mu,\gamma}(k,l,h_t^p) := \mu(k)\gamma(l)P_{\sigma,\tau,k,l}(h_t^p)$$

I denote the expectation with respect to $Q_{\sigma,\tau;\mu,\gamma}$ by $E_{\sigma,\tau;\mu,\gamma}$ and the expectation with respect to $P_{\sigma,\tau;\mu,\gamma}$ as $E_{\sigma,\tau}^{k,l}$. From the perspective of player 1 whose type is k,

$$Q_{\sigma,\tau;\gamma}^{k\cdot}(l,h_t^p) := \gamma(l)P_{\sigma,\tau,k,l}(h_t^p)$$

The associated expectation is denoted by $E_{\sigma,\tau;\gamma}^{k\cdot}$. Similarly, from the perspective of player 2 whose type is l,

$$Q_{\sigma,\tau;\mu}^{\cdot l}(k,h_t^p) := \mu(k)P_{\sigma,\tau,k,l}(h_t^p)$$

with expectation $E_{\sigma,\tau;\mu}^{\cdot l}$. Players' payoffs in the repeated game is then

$$\begin{aligned} v_1(\sigma,\tau,k) &= E_{\sigma,\tau;\gamma}^{k\cdot}\big[(1-\delta)\sum_{t=0}^{\infty}\delta^t\pi_1(\sigma_t,\tau_t)\big] \\ v_2(\sigma,\tau,l) &= E_{\sigma,\tau;\mu}^{\cdot l}\big[(1-\delta)\sum_{t=0}^{\infty}\delta^t\pi_2(\sigma_t,\tau_t)\big] \end{aligned}$$

Let $v_1(\sigma,\tau;k \mid h_t^p)$ and $v_2(\sigma,\tau;l \mid h_t^p)$ denote players' expected payoffs and $(\sigma|_{h_t^p},\tau|_{h_t^p})$ the continuation strategy induced by the strategy profile (σ,τ) in the continuation game that follows the history h_t^p.

Definition 2.2.1 *A strategy profile* (σ, τ) *is a sequential equilibrium in the infinitely-repeated game if*

1. *For any public history* $h_t^p \in H_t^p$, $v^1(\sigma, \tau, k \mid h_t^p) \geq v^1(\sigma', \tau, k \mid h_t^p)$ *for all* σ' *and* k*; similarly* $v^2(\sigma, \tau, l \mid h_t^p) \geq v^2(\sigma, \tau', l \mid h_t^p)$ *for all* τ'*; in other words,* $\sigma(k)|_{h_t^p}$ *is a best reply to* $E_\gamma^{k\cdot}[\tau|_{h_t^p} \mid h_t^p]$ *and* $\tau(l)|_{h_t^p}$ *is a best reply to* $E_\mu^{\cdot l}[\sigma|_{h_t^p} \mid h_t^p]$.

2. *Beliefs are updated via Bayes' Rule.*

Sequential equilibrium combines sequential rationality with consistency conditions on each player' beliefs over the types of the other player. After all public histories, it requires each player's action to be optimal, given some beliefs over the opponent's type, with players updating the beliefs by Bayes' rule. Note that by the full support of the distributions of signals, any signal is possible after any action profile. So, every finite history occurs with a strictly positive probability, and thus Bayes' rule determines the posteriors after all sequences of signals. Also, since any history on the equilibrium path must be followed by an optimal behavior in any Nash equilirium, and any history is possible, Nash equilibrium and sequential equilibrium payoffs coincide.

2.2.3 Inference

Suppose the reputations, i.e. the beliefs that the Sender is honest and the Receiver is tough, at time t are μ_t and γ_t, respectively.

Reputation of the Sender

If the reputation at date t is μ_t and the strategic Sender's report is truthful with probability σ, then

$$\mu_{t+1}(I_f, I_a, I_d) = \begin{cases} \frac{\mu_t \alpha(1-p)}{\mu_t \alpha(1-p)+(1-\mu)[\sigma\alpha(1-p)+(1-\sigma)(1-\alpha)p]} & \text{if} \quad (I_f, I_a, I_d) = (1,1,1) \\ \frac{\mu_t \alpha p}{\mu_t \alpha p+(1-\mu)[\sigma\alpha p+(1-\sigma)(1-\alpha)(1-p)]} & \text{if} \quad (I_f, I_a, I_d) = (1,1,0) \\ \frac{\mu_t (1-\alpha)p}{\mu_t (1-\alpha)p+(1-\mu)[\sigma(1-\alpha)p+(1-\sigma)\alpha(1-p)]} & \text{if} \quad (I_f, I_a, I_d) = (0,1,0) \\ \frac{\mu_t (1-\alpha)(1-p)}{\mu_t (1-\alpha)(1-p)+(1-\mu)[\sigma(1-\alpha)(1-p)+(1-\sigma)\alpha p]} & \text{if} \quad (I_f, I_a, I_d) = (0,1,1) \\ \frac{\mu_t \alpha}{\mu_t \alpha+(1-\mu)[\sigma\alpha+(1-\sigma)(1-\alpha)]} & \text{if} \quad (I_f, I_a) = (1,0) \\ \frac{\mu_t (1-\alpha)}{\mu_t (1-\alpha)+(1-\mu)[\sigma(1-\alpha)+(1-\sigma)\alpha]} & \text{if} \quad (I_f, I_a) = (0,0) \end{cases}$$

Suppose that the precision of the information Sender receives about the state of nature is greater than the precision of information Receiver gets about the behavior of the Sender in the case of an audit, i.e. $\alpha > p$. That means Sender's forecast is a better indicator about his behavior than if he is detected or not.[17] In this case, Sender's reputation increases if she has a correct forecast, even if she is detected (i.e. $(I_f, I_a, I_d) = (1, 1, 1)$); however, it decreases if there is an incorrect forecast but no detection (i.e. $(I_f, I_a, I_d) = (0, 1, 0)$). So, when $\alpha > p$,

$$\mu_{t+1}(0,1,1) < \mu_{t+1}(0,0) < \mu_{t+1}(0,1,0) < \mu_t < \mu_{t+1}(1,1,1) < \mu_{t+1}(1,0) < \mu_{t+1}(1,1,0)$$

Suppose that $\alpha = p$. In this situation, Sender's forecast and detection are equally informative about Sender's behavior. Sender's reputation does not change if there is a correct forecast and a detection (i.e. $(I_f, I_a, I_d) = (1, 1, 1)$); or if there is an incorrect forecast but no detection (i.e. $(I_f, I_a, I_d) = (0, 1, 0)$). So, when $\alpha = p$,

$$\mu_{t+1}(0,1,1) < \mu_{t+1}(0,0) < \mu_{t+1}(0,1,0) = \mu_t = \mu_{t+1}(1,1,1) < \mu_{t+1}(1,0) < \mu_{t+1}(1,1,0)$$

When $\alpha < p$, getting detected or not is more informative about Sender's behavior than Sender's forecast. In this case, Sender's reputation decreases if there is a correct forecast and a detection (i.e. $(I_f, I_a, I_d) = (1, 1, 1)$); however, it increases if there is an incorrect forecast but no detection (i.e.

[17] This could be the case when the Receiver is believed to be incompetent or inexperienced.

$(I_f, I_a, I_d) = (0, 1, 0)$). The reputation also increases if there is a correct forecast and no audit, (i.e. $(I_a, I_d) = (1, 0)$). The relation between $\mu(1, 0)$ (correct forecast, no audit) and $\mu(0, 1, 0)$ (incorrect forecast, no detection) depends on how much p is greater than α.

Note that $\mu_t : \Omega \to [0, 1]$ is a bounded martingale with respect to the filtration $\{\mathcal{H}_t\}$ and measure Q. Hence, it converges Q- almost surely to a random variable μ_∞ in the long-run.[18]

Reputation of the Receiver

Let the reputation of the Receiver at time t be γ_t. The probability of an audit from an observer's point of view when the strategic Receiver is diligent with probability τ is given by:

$$\varrho(\mu_t, \gamma_t) = \gamma_t \beta_D + (1 - \gamma_t)[\tau \beta_D + (1 - \tau)\beta_L]$$

Note that the probability of an audit $\varrho \in [\beta_L, \beta_D]$.

$$\gamma_{t+1} = \begin{cases} \gamma_{t+1}^{+} = \frac{\gamma_t \beta_D}{\varrho(\mu_t, \gamma_t)} & \text{if} \quad I_a = 1 \\ \\ \gamma_{t+1}^{-} = \frac{\gamma_t (1 - \beta_D)}{1 - \varrho(\mu_t, \gamma_t)} & \text{if} \quad I_a = 0 \end{cases}$$

Note that $\gamma_t : \Omega \to [0, 1]$ is a bounded martingale with respect to $\{\mathcal{H}_t\}$ and Q. Hence, it converges Q- almost surely to γ_∞ in the limit.

[18]See 2.6.1 in the Appendix.

2.3 One-shot game

In the complete-information one-shot game, the unique Nash equilibrium is in mixed strategies. Suppose $\alpha_1(T)$ denote the probability the Sender puts on $T \in A_S$ and $\alpha_2(D)$ denote the probability the Receiver puts on $D \in A_R$. The equilibrium is $\alpha_1(T) = \frac{d-c}{d}$ and $\alpha_2(D) = \frac{g}{g+l_2-l_1}$, providing a payoff of $x - \frac{g.l_1}{g+l_2-l_1}$ to the Sender and $y - \frac{c(y-z)}{d}$ to the Receiver. The minmax value for the Sender is $x - l_1$, that for the Receiver is $z - c + d$.

For the two-sided incomplete-information stage game, the characterization of equilibria is given by Proposition 2.3.1. The proof is provided in the Appendix.

Proposition 2.3.1 *The equilibrium strategies* $(\alpha_1, \alpha_2) \in \Delta(A_S) \times \Delta(A_R)$ *are as follows:*

1. *If* $\gamma > \frac{g}{g+l_2-l_1}$ *and* $\mu \in [0,1]$, $\quad \alpha_1(T) = 1$ *and* $\alpha_2(D) = 0$ *;*
2. *If* $\gamma = \frac{g}{g+l_2-l_1}$ *and*
   (a) *If* $\mu > \frac{d-c}{d}$, $\quad \alpha_1(T) \in [0,1]$ *and* $\alpha_2(D) = 0$*;*
   (b) *If* $\mu = \frac{d-c}{d}$, $\quad \alpha_1(T) \in [0,1]$ *and* $\alpha_2(D) = 0$*;*
   (c) *If* $\mu < \frac{d-c}{d}$, $\quad \alpha_1(T) \in [1 - \frac{c}{(1-\mu)d}, 1]$ *and* $\alpha_2(D) = 0$*.*
3. *If* $\gamma < \frac{g}{g+l_2-l_1}$ *and*
   (a) *If* $\mu > \frac{d-c}{d}$, $\quad \alpha_1(T) = 0$ *and* $\alpha_2(D) = 0$*;*

(b) *If* $\mu = \frac{d-c}{d}$, $\quad \alpha_1(T) = 0$ *and* $\alpha_2(D) \in [0, 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)}]$;

(c) *If* $\mu < \frac{d-c}{d}$, $\quad \alpha_1(T) = 1 - \frac{c}{(1-\mu)d}$ *and* $\alpha_2(D) = 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)}$;

Corresponding equilibrium expected payoff vectors are: [19]

$$(\pi_1, \pi_2) = \begin{cases} (x - \gamma l_1, y) & \textit{Case 1:} \quad \gamma > \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu \in [0,1] \\ (x - \frac{g.l_1}{g+l_2-l_1}, [\mu y + (1-\mu)z, y]) & \textit{Case 2.a :} \quad \gamma = \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu > \frac{d-c}{d} \\ (x - \frac{g.l_1}{g+l_2-l_1}, [y - \frac{c(y-z)}{d}, y]) & \textit{Case 2.b :} \quad \gamma = \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu = \frac{d-c}{d} \\ (x - \frac{g.l_1}{g+l_2-l_1}, [y - \frac{c(y-z)}{d}, y]) & \textit{Case 2.c :} \quad \gamma = \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu < \frac{d-c}{d} \\ (x + g - \gamma(g + l_2), \mu y + (1-\mu)z) & \textit{Case 3.a :} \quad \gamma < \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu > \frac{d-c}{d} \\ ([x - \frac{g.l_1}{g+l_2-l_1}, x + g - \gamma(g + l_2)], y - \frac{c(y-z)}{d}) & \textit{Case 3.b :} \quad \gamma < \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu = \frac{d-c}{d} \\ (x - \frac{g.l_1}{g+l_2-l_1}, y - \frac{c(y-z)}{d}) & \textit{Case 3.c :} \quad \gamma < \frac{g}{g+l_2-l_1} \quad \textit{and} \quad \mu < \frac{d-c}{d} \end{cases}$$

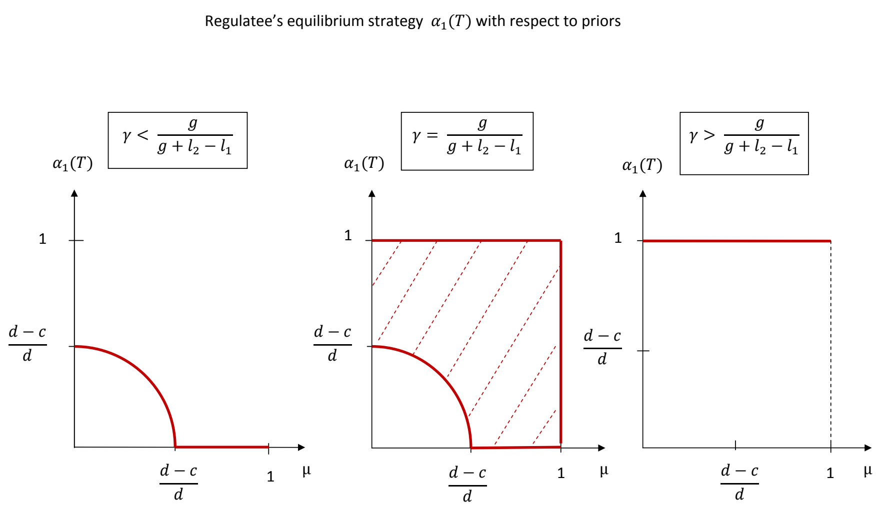
Note that if the prior belief that the Receiver is a tough type is above a threshold (which is the ratio of the expected gain of the Sender by being untruthful against a lazy Receiver to the sum of this expected gain against a lazy Receiver and the expected net loss being untruthful against a diligent Receiver), then the Sender is truthful with probability one (since she believes that she faces a tough Receiver with a high enough probability); and anticipating this the Receiver is lazy with probability one.

On the other hand, if the prior belief that the Receiver is a tough type is below this threshold, then the prior belief that the Sender is honest determines the equilibrium strategy profiles. When the prior belief on Sender being honest is above a threshold (which is the ratio of the expected net gain of the Receiver by being diligent against an untruthful Sender to the

[19]We denote the payoff correspondence for a player with a bracket [ ].

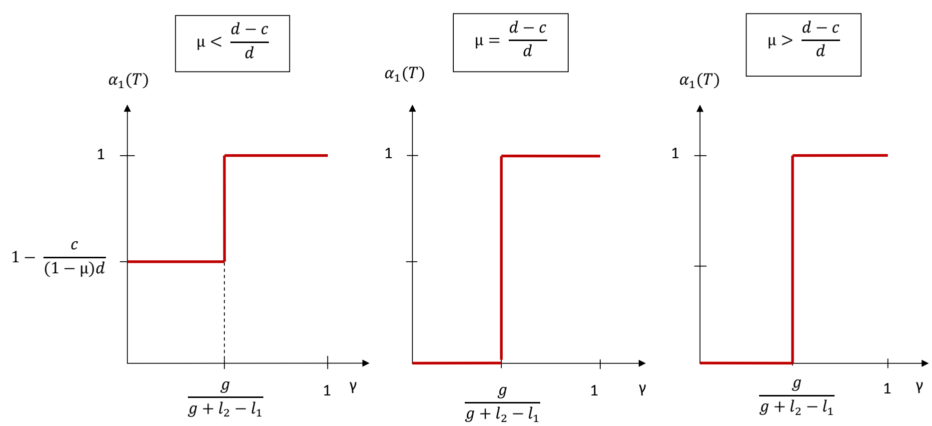
sum of this expected gain against an untruthful Sender and the cost of being diligent against a truthful one), then the Sender chooses to be untruthful with probability one, since the Receiver believes he faces an honest Sender with a higher probability and chooses to be lazy with probability one.

In other cases, the players use mixed strategies in equilibrium, that depend on the parameters defining the expected payoffs. Moreover, there is no circumstances where the Receiver chooses to be diligent with probability one in equilibrium. We can summarize the equilibrium strategies depending on the priors by the following graphs:



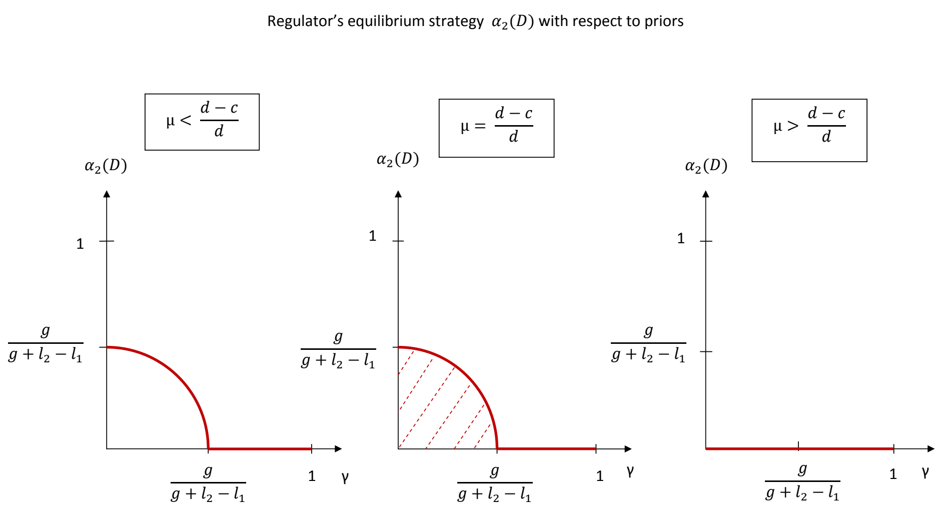
The equilibrium strategy of the Sender (for being truthful) is nonincreasing in the prior belief over her being honest (μ) increases, for any given prior on the Receiver's type; and it is nondecreasing in the prior belief that on the Receiver is tough (γ) for any fixed prior μ.

Regulatee's equilibrium strategy $\alpha_1(T)$ with respect to priors



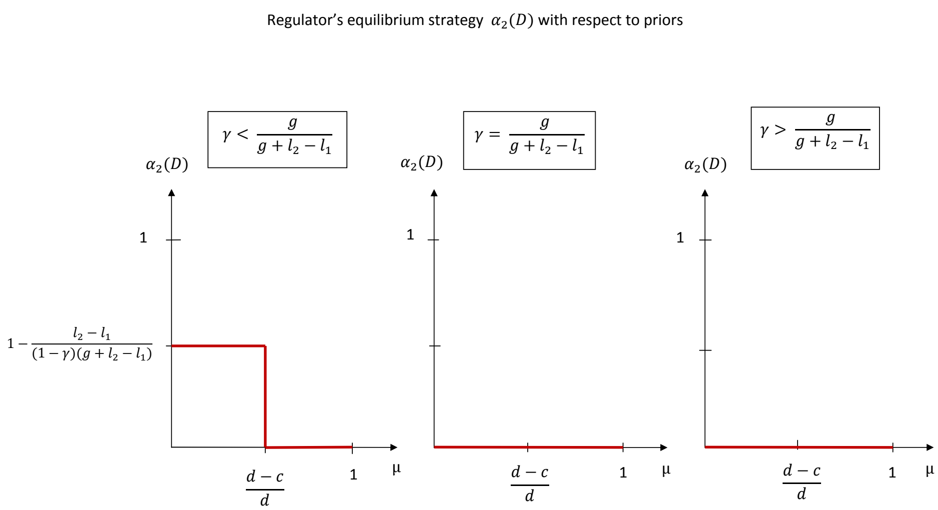
Moreover, if the prior on the toughness of the Receiver is sufficiently high (greater than $\frac{g}{g+l_2-l_1}$), the Sender always chooses to be truthful as she thinks she faces a tough regulator with a high probability. Note that threshold decreases as as the expected gain from being untruthful (g) decreases or the expected loss (l_2) increases.

The Sender can get her highest possible payoff $x + g$ only if she thinks that it is more likely she faces a *strategic* Receiver and the prior belief that she is honest is high enough ($\mu > \frac{d-c}{d}$) so that the strategic Receiver chooses to be lazy.



The Receiver's strategy (of being diligent) is nonincreasing both in the prior belief γ (for any fixed prior belief μ that the Sender is honest) and in the prior μ (for any given prior on the Receiver's type). The Receiver chooses to be diligent less and less often as he thinks that with a high probability he faces an honest Sender; or as he thinks that the Sender believes he is tough (so that she chooses to be truthful).

Also, note that the threshold (on the prior belief that the Sender is honest) above which the Receiver chooses to be lazy decreases as the expected cost of being diligent increases.



Note that the Receiver never (for no prior beliefs) chooses to be diligent for sure and when the prior on his toughness is above $\frac{g}{g+l_2-l_1}$, he always chooses to be lazy, knowing that the Sender is going to choose to be truthful. He randomizes between being lazy and diligent only when $\gamma < \frac{g}{g+l_2-l_1}$ and $\mu \leq \frac{d-c}{d}$, i.e. when he thinks that it is more likely he faces a strategic Sender and the Sender also believes that it is more likely that she faces a strategic Receiver.

2.4 Two-period game

In order to understand the concerns for building or maintaining (false) reputations, one needs to analyze the repeated interaction between these two parties. Focusing on two-period game captures the analysis and yet simplifies the calculations. In this section, I present some equilibria that provide the intuition for building or maintaining the reputations and kill the (weak) monotonicity of equilibrium behavior in prior beliefs.[20]

The prior belief that the Receiver is tough is denoted by γ_0, and γ_1 denotes the posterior belief at period 1, updated after the realization of I_a. Similarly, μ_0 denotes the prior belief that the Sender is honest, and μ_1 denotes the posterior at period 1, updated after the realization of (I_f, I_a, I_d). Let δ_1 and δ_2 be the discount factors of Sender and Receiver, respectively. And, $\sigma_0(T), \tau_0(D)$ denote the first period actions; $\sigma_1(T), \tau_1(D)$ are the second period actions. In the second period, the equilibrium strategies are determined according to Proposition 2.3.1 corresponding to the associated posterior beliefs. Note that the critical value for the equilibria in the second period are $\frac{g}{g+l_2-l_1}$ for γ_1 and $\frac{d-c}{d}$ for μ_1. One needs to find the critical values for prior beliefs μ_0, γ_0 that will determine the players' behavior, both in the first period and the second period.

[20]Characterizing the set of all equilibria requires to analyze many cases and gets very complicated to present.

The critical levels of prior beliefs about Receiver's type are as follows:

$$0 < \gamma_0^{'} < \frac{g}{g + l_2 - l_1} < \gamma_0^{''} < 1$$

where the prior $\gamma_0^{'}$ is determined by the requirement that the strategic Receiver is expected to play $\tau_0 = 0$ (lazy) in period 0 and $I_a = 1$ (audit) is observed, and the posterior belief that the Receiver is tough goes up to $\frac{g}{g+l_2-l_1}$ in the second period; and the prior $\gamma_0^{''}$ is determined by the requirement that if the strategic Receiver is expected to play $\tau(D) = 0$ in period 0 and the signal $I_a = 0$ (no audit) is observed, then the posterior belief goes below $\frac{g}{g+(l_2-l_1)}$.

So, if the prior is such that $\gamma_0 < \gamma_0^{'}$, then it is not possible to move the posterior above $\frac{g}{g+(l_2-l_1)}$, no matter what the strategic Receiver does in the first period.

$$\gamma_1^{+} = \frac{\gamma_0^{'}\beta_D}{\varrho} = \frac{\gamma_0^{'}\beta_D}{\gamma_0^{'}\beta_D + (1-\gamma_0^{'})\beta_L} = \frac{g}{g + l_2 - l_1}$$

and so

$$\gamma_0^{'} = \frac{g}{g + \frac{\beta_D}{\beta_L}(l_2 - l_1)} < \frac{g}{g + (l_2 - l_1)}$$

Similarly, the prior $\gamma_0^{''}$ is determined by the requirement that if the strategic Receiver is expected to play $\tau(D) = 0$ in period 0 and the signal $I_a = 0$

(no audit) is observed, then the posterior belief is $\frac{g}{g+(l_2-l_1)}$, that is,

$$\gamma_1^- = \frac{\gamma_0^{''}(1-\beta_D)}{1-\varrho} = \frac{\gamma_0^{''}(1-\beta_D)}{1-\gamma_0^{''}\beta_D-(1-\gamma_0^{''})\beta_L} = \frac{g}{g+(l_2-l_1)}$$

and so

$$\gamma_0^{''} = \frac{g}{g+\frac{(1-\beta_D)}{(1-\beta_L)}(l_2-l_1)} > \frac{g}{g+(l_2-l_1)}$$

If the prior $\gamma_0 > \gamma_0^{''}$, even after a negative signal, posterior probability exceeds $\frac{g}{g+(l_2-l_1)}$.

In order to find the critical levels of prior beliefs on Sender's type, I will assume $\alpha > p$ to simplify calculations. When $\alpha > p$ (Sender's forecast is more informative than if he is detected or not), there are seven prior beliefs μ_0 are critical for Sender's behavior. These priors can be ranked as follows:

$$0 < \mu_0^{'}(1,1,0) < \mu_0^{'}(1,0) < \mu_0^{'}(1,1,1) < \frac{d-c}{d} < \mu_0^{''}(0,1,0) < \mu_0^{''}(0,0) < \mu_0^{''}(0,1,1) < 1$$

The priors $\mu_0^{'}(I_f, I_a, I_d)$ is determined by the requirement that the strategic Sender is expected to choose $\sigma_0 = 0$ and (I_f, I_a, I_d) is observed and and the posterior increases to $\frac{d-c}{d}$, i.e.,

$$\mu_1(I_f, I_a, I_d) = \frac{\mu_0^{'}(I_f, I_a, I_d)\text{prob}[(I_f, I_d)|I_a, h]}{\text{prob}[(I_f, I_d)|I_a](\sigma_0 = 0)} = \frac{d-c}{d}$$

The values for $\mu_0^{'}(I_f, I_a, I_d)$ are the following:

$$\mu_0^{'}(1,1,0) = \frac{d-c}{d + \frac{(\alpha+p-1)c}{(1-\alpha)(1-p)}}, \qquad \mu_0^{'}(1,0) = \frac{d-c}{d + \frac{(2\alpha-1)c}{(1-\alpha)}}, \qquad \mu_0^{'}(1,1,1) = \frac{d-c}{d + \frac{(\alpha-p)c}{(1-\alpha)p}}$$

If the prior is such that $\mu_0 < \mu_0^{'}(I_f, I_a, I_d)$, even when the 'good' event (I_f, I_a, I_d) is realized, it is not possible to move the posterior above $\frac{d-c}{d}$.

The prior $\mu_0^{''}(I_f, I_a, I_d)$ is determined by the requirement that if the strategic Sender is expected to play $\sigma_0(T) = 0$ in period 0 and the 'bad' signal (I_f, I_a, I_d) is observed, then the posterior belief decreases $\frac{d-c}{d}$, that is,

$$\mu_1(I_f, I_a, I_d) = \frac{\mu_0^{''}(I_f, I_a, I_d)\text{prob}[(I_f, I_d)|I_a, h]}{\text{prob}[(I_f, I_d)|I_a](\sigma_0 = 0)} = \frac{d-c}{d}$$

The values for $\mu_0^{''}(I_f, I_a, I_d)$ are the following:

$$\mu_0^{''}(0,1,0) = \frac{d-c}{d - \frac{(\alpha-p)c}{\alpha(1-p)}}, \qquad \mu_0^{''}(0,0) = \frac{d-c}{d - \frac{(2\alpha-1)c}{\alpha}}, \qquad \mu_0^{''}(0,1,1) = \frac{d-c}{d - \frac{(\alpha+p-1)c}{\alpha p}}$$

If the prior $\mu_0 > \mu_0^{''}(I_f, I_a, I_d)$, then even when the 'bad' signal (I_f, I_a, I_d) is expected to be realized, it is not possible to have the posterior below $\frac{d-c}{d}$.

I assume $\alpha = p$ to simplify calculations. When $\alpha = p$, Sender's forecast is equally informative as his detection when an audit occurs. Hence, $(I_f, I_a, I_d) = (1,1,1)$ (correct forecast, but detected) and $(I_f, I_a, I_d) =$

$(0,1,0)$ (incorrect forecast, but no detection) does not change the posterior belief on Sender's type. In this case, there are five prior beliefs μ_0 are critical for Sender's behavior. These priors can be ranked as follows:

$$0 < \mu_0^{'}(1,1,0) < \mu_0^{'}(1,0) < \frac{d-c}{d} < \mu_0^{''}(0,0) < \mu_0^{''}(0,1,1) < 1$$

Characterizing the set of all equilibria even in the two-period game is very complicated. Below I present some equilibria to show how the intertemporal incentives influence equilibrium behavior of the Sender and Receiver. The proofs are omitted since they are straightforward calculations.

Proposition 2.4.1 *If the prior belief that the Receiver is tough is such that $\gamma_0 > \gamma_0^{''}$, then the first period equilibrium strategies are $\sigma_0(T) = 1$ and $\tau_0(D) = 0$; and the second period equilibrium strategies are $\tau_1(D) = 0, \sigma_1(T) = 1$.*

If the prior $\gamma_0 > \gamma_0^{''}$, the strategic Receiver will play according to the stage game equilibrium (no intertemporal incentive) and choose $\tau_0(D) = 0$ since even a negative signal leads to the most valuable posterior probability exceeding $\frac{g}{g+(l_2-l_1)}$. The best response of Sender to $\tau_0(D) = 0$ would be $\sigma_0(T) = 1$. Hence, the first period equilibrium strategy profile is $\tau_0(D) = 0, \sigma_0(T) = 1$. The second period equilibrium is the one-shot equilibrium corresponding to posterior beliefs (μ_1, γ_1), thus the second period equilibrium strategies are $\tau_1(D) = 0, \sigma_1(T) = 1$.

Proposition 2.4.2 *When the prior belief that Receiver is tough is such that* $\frac{g}{g+l_2-l_1} < \gamma_0 < \gamma_0^{''}$ *and the prior on Sender's type is* $\mu_0 < \mu_0^{'}(1,0)$*, then the equilibrium strategies in the first period are given as follows:*

- *If* $\delta_2 > \frac{d}{(1-\beta_L)(y-z)}$, $\sigma_0 = 1$ *and* $\tau_0 = 1$.
- *If* $\delta_2 = \frac{d}{(1-\beta_L)(y-z)}$, $\sigma_0 = 1$ *and* $\tau_0 \in [0,1]$.
- *If* $\delta_2 < \frac{d}{(1-\beta_L)(y-z)}$, $\sigma_0 = 1$ *and* $\tau_0 = 0$.

When the prior belief that Receiver is tough is between $\frac{g}{g+l_2-l_1}$ and $\gamma_0^{''}$, the posterior γ_1 goes down the critical level $\frac{g}{g+l_2-l_1}$ in the second period if the Receiver is expected to play $\tau_0 = 0$ and $I_a = 0$ (no audit) is realized. When the prior on Sender being honest is below $\mu_0^{'}(1,0)$, her reputation can only increase if there is an audit but no detection. However, if there is an audit, the Receiver reputation for being tough is going to either remain the same or increase even more, which induces the Sender to tell the truth in the first period. In this case, if the Receiver is sufficiently patient, he chooses to be diligent in the first period to sustain his high reputation for the second period, which cannot arise in the one-shot game. Remember in the one-shot game, there is no circumstances in which the Receiver chooses to be diligent with probability one.

Proposition 2.4.3 *Suppose that the prior on Receiver being tough is* $\gamma_0^{'} < \gamma_0 < \frac{g}{g+l_2-l_1}$ *and the prior on Sender being honest is* $\mu_0 < \mu_0^{'}(1,1,0)$. *The equilibrium strategies in the first period are as follows:*

- *If* $\delta_2 < \frac{(1-\mu_0)d^2-cd}{\beta_L c(y-z)}$, $\sigma_0 = 1 - \frac{cd+\delta_2\beta_L c(y-z)}{(1-\mu_0)d^2}$ *and* $\tau_0 = 1 - \frac{l_2-l_1}{(1-\gamma_0)(g+l_2-l_1)}$.
- *If* $\delta_2 = \frac{(1-\mu_0)d^2-cd}{\beta_L c(y-z)}$, $\sigma_0 = 0$ *and* $\tau_0 \in [0, 1 - \frac{l_2-l_1}{(1-\gamma_0)(g+l_2-l_1)}]$.
- *If* $\delta_2 > \frac{(1-\mu_0)d^2-cd}{\beta_L c(y-z)}$, $\sigma_0 = 0$ *and* $\tau_0 = 0$.

Receiver's reputation for being tough in the second period can increase above $\frac{g}{g+l_2-l_1}$, if the Receiver is expected to play $\tau_0 = 0$ and $I_a = 1$ (audit) is realized. When the prior on Sender's honesty is below this level, the posterior μ_1 on Sender's type cannot increase to $\frac{d-c}{d}$ or above in the second period. If the Receiver is sufficiently patient, then he plays $\tau_0(D) = 0$ (lazy) and hopes to see a good signal (audit) to increase his reputation above $\frac{g}{g+l_2-l_1}$ in the second period to get the highest payoff y in the second period. Anticipating this, the Sender chooses to be untruthful, i.e. $\sigma_0(T) = 0$.

Proposition 2.4.4 *Suppose that the prior on Receiver being tough is* $\gamma_0^{'} < \gamma_0 < \frac{g}{g+l_2-l_1}$ *and the prior on Sender being honest is* $\frac{d-c}{d} \leq \mu_0 \leq \mu_0^{''}(0,0)$. *Then the equilibrium strategies in the first period are as follows:*

- *If* $\delta_1 < K_1$, *then* $\sigma_0 = 0$ *and* $\tau_0 = 0$.
- *If* $\delta_1 = K_1$, *then* $\sigma_0 \in [0,1]$ *and* $\tau_0 = 0$.
- *If* $\delta_1 > K_1$, *then* $\sigma_0 = 1$ *and* $\tau_0 = 0$.

where $K_1 = \frac{g-\gamma_0(g+l_2-l_1)}{\alpha[(1-\gamma_0)(1-\beta_L)g-\gamma_0(1-\beta_D)(l_2-l_1)]}$.

Receiver's reputation of being tough in the second period increases above $\frac{g}{g+l_2-l_1}$ if there is an audit and $\tau_0 = 0$. The Sender's reputation, on the other hand, can go down to and below $\frac{d-c}{d}$ if she doesn't choose $\sigma_0(T) = 1$ and there is a bad signal such as 'incorrect forecast, no audit' and 'incorrect forecast, audit and detection.' Hence, if the Sender is sufficiently patient, to maintain her high reputation, she chooses $\sigma_0(T) = 1$ and hope for the Receiver's reputation going down in the second period after a bad signal 'no audit.'

Proposition 2.4.5 *Suppose that the prior on Receiver being tough is* $\gamma_0^{'} < \gamma_0 < \frac{g}{g+l_2-l_1}$ *and the prior on Sender being honest is* $\mu_0^{''}(0,0) \leq \mu_0$*. Then the equilibrium strategies in the first period are* $\sigma_0 = 0$ *and* $\tau_0 = 0$*.*

Receiver's reputation of being tough in the second period increases above $\frac{g}{g+l_2-l_1}$ if there is an audit and $\tau_0 = 0$. Sender's reputation cannot go below $\frac{d-c}{d}$ in the second period if $\mu_0^{''}(0,0) \leq \mu_0$. The Receiver has an incentive to increase his reputation above $\frac{g}{g+l_2-l_1}$ in the second period by choosing $\tau_0(D) = 0$ in the first period (also the Sender's reputation for being honest is high enough); whereas the Sender anticipating this and knowing that her reputation cannot go below $\frac{d-c}{d}$, chooses $\sigma_0(T) = 0$.

Proposition 2.4.6 *Suppose that the prior on Receiver's type is* $\gamma_0 < \gamma_0^{'}$

and the prior on Sender's type is $\mu_0 < \mu_0^{'}(1,1,0)$*. The first period strategies are* $\sigma_0 = 1 - \frac{c}{(1-\mu_0)d}$ *and* $\tau_0 = 1 - \frac{l_2 - l_1}{(1-\gamma_0)(g+l_2-l_1)}$*.*

In this case, even when there is a positive signal regarding the toughness of the Receiver when he is expected to play $\tau_0 = 0$, the posterior will still be below $\frac{g}{g+l_2-l_1}$. There is no way that the posterior about the Receiver being tough can be or above $\frac{g}{g+l_2-l_1}$, no matter what he does in the first period and which signal is realized. When the priors are below these critical levels, no matter what signals are observed and what the players choose, the posteriors μ_1 and γ_1 cannot be greater than and equal to $\frac{d-c}{d}$ and $\frac{g}{g+l_2-l_1}$, respectively. Hence, the second period equilibrium behavior are the mixed strategies $\sigma_1 = 1 - \frac{c}{(1-\mu_1)d}$ and $\tau_1 = 1 - \frac{l_2-l_1}{(1-\gamma_1)(g+l_2-l_1)}$, providing a payoff $x - \frac{gl_1}{g+l_2-l_1}$ to the Sender and $y - \frac{c(y-z)}{d}$ to the Receiver. The first period behavior would be independent of second period behavior and payoff, thus, the first period equilibrium is in mixed strategies as well, i.e. $\sigma_0 = 1 - \frac{c}{(1-\mu_0)d}$ and $\tau_0 = 1 - \frac{l_2-l_1}{(1-\gamma_0)(g+l_2-l_1)}$.

Proposition 2.4.7 *Suppose that the prior on Receiver's type is* $\gamma_0 < \gamma_0^{'}$ *and the prior on Sender's type is* $\mu_0 = \mu_0^{'}(1,1,0)$*. Equilibrium strategies in the first period are as follows:*

- *If* $\delta_1 < K_1$*,* $\sigma_0 = 1 - \frac{c}{(1-\mu_0)d}$ *and* $\tau_0 = \frac{g-\gamma_0(g+l)+(1-\alpha)(1-p)\delta_1[g(1-\gamma_0)\beta_L - l\gamma_0\beta_H]}{(1-\gamma_0)(g+l)-\delta_1(1-\alpha)(1-p)(1-\gamma_0)g(\beta_H-\beta_L)}$*.*
- *If* $\delta_1 = K_1$*,* $\sigma_0 \in [0, 1 - \frac{c}{(1-\mu_0)d}]$ *and* $\tau_0 = 1$*.*
- *If* $\delta_1 > K_1$*,* $\sigma_0 = 0$ *and* $\tau_0 = 1$*.*

where $K_1 = \frac{l_2 - l_1}{(1-\alpha)(1-p)\beta_D[g - \gamma_0(g + l_2 - l_1)]}$.

In this case, even when there is a positive signal regarding the toughness of the Receiver when he is expected to play $\tau_0 = 0$, the posterior will still be below $\frac{g}{g+l_2-l_1}$. On the other hand, the Sender's reputation can increase exactly to $\frac{d-c}{d}$ if σ_0 is expected to be 0 and $(1,1,0)$ (correct forecast, no detection) is realized. In this case, there exists an equilibrium action profile in the second period where the Receiver plays a mixed action and the Sender can get a higher payoff, if her reputation can increase to $\frac{d-c}{d}$ in the second period. Hence, if the Sender is sufficiently patient (depends on parameters of the model), then she lies for sure, hoping for a very good signal (correct forecast, audit, no detection) to get higher payoff in the second period. More importantly, the Receiver has to be diligent ($\tau_0 = 1$) in equilibrium (anticipating the Sender's behavior), which cannot occur in a one-shot game.

2.5 Infinitely Repeated Game

In order to understand the interaction between the Sender and Receiver in the long-run, or when the players do not know the terminal date of their relationship; one needs to analyze the infinitely repeated game and in particular what happens to false reputations in the long-run.

The set of public perfect equilibrium payoffs in the infinitely repeated complete-information game is the set of all feasible and individually rational payoffs; where the minmax payoff for the Sender is $x - l_1$ and that for the Receiver is $z - c + d$.[21]

For the incomplete-information game, one can show that the Receiver can guarantee himself at least his Stackelberg payoff $y - c$ by repeatedly playing his (pure) Stackelberg action D (diligent), if he is sufficiently patient. However, the main theorem (1.3.1) of Chapter 1 suggests that this lower bound is a temporary lower bound in the sense that the true types of the players are (almost) known eventually in the long-run.[22] Hence,

Proposition 2.5.1 *In any Nash equilibrium of the incomplete-information game, the false reputation of the Sender for being honest and the false reputation of the Receiver for being tough cannot be sustained indefinitely:*

$$\mu_t \rightarrow 0, \quad Q^{s\cdot} - \textit{almost surely},$$

$$\gamma_t \rightarrow 0, \quad Q^{\cdot s} - \textit{almost surely}.$$

Moreover, the convergence is uniform.

The similar convergence results hold in the behavioral level, i.e. the equilibria of the incomplete-information game converges to an equilibrium of the complete-information game in the long-run.

[21]Note that the rank conditions provided by Fudenberg, Levine, and Maskin (1994) for the imperfect public monitoring folk theorem are satisfied.

[22]The assumptions for 1.3.1 are satisfied in this setting.

2.6 Concluding Remarks

In this paper, I propose a model - a simultaneous-move inspection game with imperfect public monitoring and incomplete-information about the types of both players- to analyze situations where a privately-informed strategic Sender can misrepresent her information in the *presence* of a strategic Receiver who aims to deter misrepresentation of information via costly auditing.

I characterize the set of equilibria in the one-shot game and present some of the equilibria that show the incentives for reputation building or maintaining in the two-period game. I hope the predictions of the model in the one-shot game and two-period game can be tested by lab or field experiments in future research.

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Appendix

Chapter 1: Posteriors are bounded martingales

Lemma 2.6.1 *Suppose Assumption 1 holds. Then* γ_t *is a bounded martingale with respect to the measure* Q *and filtration* $\{\mathcal{H}_{1t}\}_t$*. Moreover, Assumption 5 implies that* γ_t *is also a bounded martingale with respect to the measure* Q *and filtration* $\{\mathcal{H}_t\}_t$*. Similarly, random variable* μ_t *is a bounded martingale with respect to the measure* Q *and filtration* $\{\mathcal{H}_{2t}\}_t$ *and* $\{\mathcal{H}_t\}_t$*.*

Proof. We first show γ_t is a martingale with respect to measure Q and filtration $\{\mathcal{H}_{1t}\}_t$. Let $\gamma_{t+1}(h_t; i_t, y_t)$ denote player 1's belief in period $t+1$ after playing i_t and observing public signal y_t in period t, given public history h_t. Note that $\gamma_{t+1} \equiv \text{prob}[c \mid h_t, i_t, y_t]$. So, for all $h_{1,t+1}$,

$$\begin{aligned} E[\gamma_{t+1}|\mathcal{H}_{1,t+1}] &= \sum_{y \in Y} \text{prob}[y_t|h_t, i_t].\gamma_{t+1}(h_t; i_t, y_t) \\ &= \sum_{y \in Y} \text{prob}[y_t|h_t, i_t].\frac{\gamma_t \text{prob}[y_t|h_t, i_t, c]}{\text{prob}[y_t|h_t, i_t]} \\ &= \sum_{y \in Y} \gamma_t \sum_{j \in J} \varsigma_2^j \rho_{i_t j}^{y_t} \\ &= \gamma_t. \end{aligned}$$

The random variable γ_t is also a martingale with respect to measure Q and filtration $\{\mathcal{H}_t\}_t$ under Assumption 5. In this case, for all h_{t+1} and for all i_t

and i,

$$
\begin{aligned}
E[\gamma_{t+1}|\mathcal{H}_{t+1}] &= \sum_{y\in Y} \text{prob}[y_t|h_t].\gamma_{t+1}(h_t;i_t,y_t) \\
&= \sum_{y\in Y} \text{prob}[y_t|h_t].\frac{\gamma_t \text{prob}[y_t|h_t,i_t,c]}{\text{prob}[y_t|h_t,i_t]} \\
&= \sum_{y\in Y}\sum_{j\in J} \rho_{ij}^{y_t}\left(\gamma_t \varsigma_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]\right)\frac{\gamma_t \sum_{j\in J}\varsigma_2^j \rho_{i_t j}^{y_t}}{\sum_{j\in J}\rho_{i_t j}^{y_t}\left(\gamma_t\varsigma_2^j + (1-\gamma_t)E^{\cdot n}[\tilde{\tau}_t^j \mid h_t]\right)} \\
&= \sum_{y\in Y}\sum_{j\in J} \frac{\text{prob}[y_t|i,j]\gamma_t\varsigma_2^j\rho_{i_t j}^{y_t}}{\text{prob}[y_t|i_t,j]} \\
&= \sum_{y\in Y}\sum_{j\in J} \frac{\rho_{ij}^{y_t}\gamma_t\varsigma_2^j\rho_{i_t j}^{y_t}}{\rho_{i_t j}^{y_t}} \\
&= \sum_{y\in Y}\gamma_t \sum_{j\in J}\varsigma_2^j\rho_{ij}^{y_t} \\
&= \gamma_t
\end{aligned}
$$

Note that the forth equality follows from Assumption 5 and the last line is due to Assumption 1.[23] By the same argument, we can show that μ_t is a bounded martingale with respect to the measure Q and filtration $\{\mathcal{H}_{2t}\}_t$ (and also with respect to filtration $\{\mathcal{H}_t\}_t$ by Assumption5), thus converges Q-almost surely (and hence $Q^{n\cdot}$ and Q^{nn} - almost surely) to a random variable μ_∞ on Ω. ■

[23]For the case of product structure,

$$
\begin{aligned}
E[\gamma_{t+1}|\mathcal{H}_{t+1}] &= \sum_{y\in Y} \text{prob}[y_t|h_t].\gamma_{t+1}(h_t;i_t,y_t) \\
&= \sum_{y\in Y} \text{prob}[y_t|h_t].\frac{\gamma_t \text{prob}[y_t|h_t,i_t,c]}{\text{prob}[y_t|h_t,i_t]} \\
&= \sum_{y\in Y} \rho_{ij}^{y}\frac{\gamma_t \text{prob}[y_t|h_t,i_t,c]}{\rho_{i_t j}^{y}} \\
&= \sum_{(y_1,y_2)\in Y} \rho_i^{y_1}\rho_j^{y_2}\frac{\gamma_t\sum_{j\in J}\varsigma_2^j\rho_{i_t}^{y_1}\rho_j^{y_2}}{\rho_{i_t}^{y_1}\rho_j^{y_2}} = \gamma_t.
\end{aligned}
$$

Chapter 2: Proof of Proposition 2.3.1 **Proof.** We find the best responses of players for each case.

The expected utility of the Sender from being truthful, i.e. choosing $\alpha_1 = 1$ is

$$\pi_1(T, \alpha_2) = \gamma(x - l_1) + (1 - \gamma)[\alpha_2(x - l_1) + (1 - \alpha_2)x]$$

Her expected utility from being untruthful, i.e. $\alpha_1 = 0$, is

$$\pi_1(U, \alpha_2) = \gamma(x - l_2) + (1 - \gamma)[\alpha_2(x - l_2) + (1 - \alpha_2)(x + g)]$$

Hence, the Sender's best response is given by the following:

$$BR_1(\alpha_2) = \begin{cases} 1 & \text{if} \quad \alpha_2 > 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)} \\ [0,1] & \text{if} \quad \alpha_2 = 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)} \\ 0 & \text{if} \quad \alpha_2 < 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)} \end{cases} \tag{2.6.1}$$

Note that the strategy of the Receiver that makes the Sender indifferent between being truthful and untruthful, $\alpha_2 = 1 - \frac{l_2 - l_1}{(1-\gamma)(g+l_2-l_1)}$, is greater than 0 if $\gamma < \frac{g}{g+l_2-l_1}$ and equals to 0 if $\gamma = \frac{g}{g+l_2-l_1}$. If $\gamma > \frac{g}{g+l_2-l_1}$, then $BR_1(\alpha_2) = 1$ for any α_2.

The expected utility of the Receiver by choosing to be diligent, i.e. choosing $\alpha_2 = 1$, is

$$\pi_2(\alpha_1, D) = \mu(y-c) + (1-\mu)[\alpha_1(y-c) + (1-\alpha_1)(z-c+d)]$$

The expected utility of the Receiver by choosing to be lazy, i.e. choosing $\alpha_2 = 0$, is

$$\pi_2(\alpha_1, L) = \mu y + (1-\mu)[\alpha_1 y + (1-\alpha_1)z]$$

Receiver's best response is given by the following:

$$BR_2(\alpha_1) = \begin{cases} 1 & \text{if} \quad \alpha_1 < 1 - \frac{c}{(1-\mu)d} \\ [0,1] & \text{if} \quad \alpha_1 = 1 - \frac{c}{(1-\mu)d} \\ 0 & \text{if} \quad \alpha_1 > 1 - \frac{c}{(1-\mu)d} \end{cases} \tag{2.6.2}$$

Note that the strategy of the Sender that makes the Receiver indifferent between choosing to be diligent and lazy, $\alpha_1 = 1 - \frac{c}{(1-\mu)d}$, is greater than 0 if $\mu < \frac{d-c}{d}$ and equals to 0 if $\mu = \frac{d-c}{d}$. If $\mu > \frac{d-c}{d}$, then $BR_2(\alpha_1) = 0$ for any α_1.

Case 1 $\gamma > \frac{g}{g+l_2-l_1}$ and $\mu \in [0,1]$.

In this case, $BR_1(\alpha_2) = 1$ for any τ. The unique fixed point of the best response correspondence $BR_1 \times BR_2$ is $\alpha_1 = 1$ and $\alpha_2 = 0$ (Receiver is truthful and Receiver is lazy). The equilibrium payoff vector (π_1, π_2)

corresponding to this strategy profile is $(x - \gamma l_1, y)$.

Case 2 $\gamma = \frac{g}{g+l_2-l_1}$. The strategy that makes the Sender indifferent between telling the truth and lying is $\tau = 0$. For $\alpha_2 > 0$, $BR_1(\alpha_2) = 1$.

2.a $\mu > \frac{d-c}{d}$: $BR_2(\alpha_1) = 0$ for any α_1. So, the equilibria in this case are $\alpha_1 \in [0, 1]$ and $\alpha_2 = 0$. Corresponding equilibrium payoffs for the Sender is $x - \frac{g.l_1}{g+l_2-l_1}$ and for the Receiver lies in $[\mu y + (1 - \mu)z, y]$.

2.b $\mu = \frac{d-c}{d}$: Receiver is indifferent between choosing to be lazy and diligent when $\alpha_1 = 0$. Otherwise, $BR_2(\alpha_1) = 0$. The equilibria in this case are $\alpha_1 \in [0, 1]$ and $\alpha_2 = 0$. Corresponding equilibrium payoffs for the Sender is $x - \frac{g.l_1}{g+l_2-l_1}$ and for the Receiver lies in $[y - \frac{c(y-z)}{d}, y]$.

2.c $\mu < \frac{d-c}{d}$: Best response correspondence of the Receiver is given by the expression 2.6.2. Sender is indifferent if $\alpha_2 = 0$ and $BR_1(\alpha_2) = 1$ if $\alpha_2 > 0$. The fixed point of $BR_1 \times BR_2$ are $\alpha_1 \in [1 - \frac{c}{(1-\mu)d}, 1]$ and $\alpha_2 = 0$, providing a payoff of $x - \frac{g.l_1}{g+l_2-l_1}$ to Sender and $[y - \frac{c(y-z)}{d}, y]$ to Receiver.

Case 3 $\gamma < \frac{g}{g+l_2-l_1}$. The best response correspondence of the Sender is given by expression 2.6.1.

3.a $\mu > \frac{d-c}{d}$: $BR_2(\alpha_1) = 0$ for any α_1. The unique equilibrium for these values of beliefs is $\alpha_1 = 0$ and $\alpha_2 = 0$, providing a payoff of $\gamma(x - l_2) + (1 - \gamma)(x + g) = x + g - \gamma(g + l_2)$ to Sender and $\mu y + (1 - \mu)z$ to Receiver.

3.b $\mu = \frac{d-c}{d}$: The Receiver is indifferent between choosing to be diligent and lazy when $\alpha_1 = 0$. Otherwise, $BR_2(\alpha_1) = 0$. The equilibria are $\alpha_1 = 0$ and $\alpha_2 \in [0, 1 - \frac{l_2 - l_1}{(1-\gamma)[g+l_2-l_1]}]$ with the associated equilibrium payoff in $[x - \frac{g.l_1}{g+l_2-l_1}, x + g - \gamma(g + l_2)]$ to the Sender and $y - \frac{c(y-z)}{d}$ to the Receiver.

3.c $\mu < \frac{d-c}{d}$: Best response correspondence of the Receiver is given by the expression 2.6.2. The unique equilibrium strategy profile for these values of beliefs is $\alpha_1 = 1 - \frac{c}{(1-\mu)d}$ and $\alpha_2 = 1 - \frac{l_2 - l_1}{(1-\gamma)[g+l_2-l_1]}$, providing equilibrium payoffs $x - \frac{g.l_1}{g+l_2-l_1}$ to the Sender and $y - \frac{c(y-z)}{d}$ to the Receiver.

■