A Uniform Approach towards the Local Gan-Gross-Prasad Conjecture

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Dedication

To my parents for their unconditional support to my career.

Abstract

The local Gan-Gross-Prasad conjecture is a generalization of the branching problem to classical groups over local fields of characteristic zero. The conjecture speculates on the multiplicity, that is, the dimension of the Bessel models and Fourier-Jacobi models in an irreducible admissible representation. Equivalent conditions for the multiplicity equaling to one is given in [GP92, GP94, GGP12].

J.-L. Waldspurger did the pioneer's work and proved the Bessel special orthogonal cases for tempered parameters over non-Archimedean local fields in [Wald10, Wald12a, Wald12b, Wald12c]. C. Mœglin and Waldspurger proved that case for generic parameters in [MW12]. The proof for the conjecture is almost completed but the proof for some cases used a different philosophy. This thesis aims to generalize Mœglin and Waldspurger's approach to formulate a relatively uniform proof for all cases.

Contents

Chapter 1

Introduction

The classical branching problem involves the rules of decomposing the restriction of irreducible representations of a compact group to its subgroup. We fix compact groups $H \subset G$ and specify a **parametrization** of their complex representations, that is, representations over $\mathbb{C}$. The branching problem also seeks an **equivalent condition** to determine when an irreducible representation π_H of H appears in the spectrum of the restriction $\pi_G|_H$ of an irreducible representation π_G of G. The branching problem also asks for a precise description of the **multiplicity** of π_H in $\pi_G|_H$.

Example 1.0.1 (Symmetric groups)**.** *Let* $G = \mathfrak{S}_n$ *and* $H = \mathfrak{S}_{n-1}$ *be permutation groups of order* n *and* $n-1$*, respectively. Their irreducible finite-dimensional complex representations* π_G, π_H *can be classified by the Young tableaux* $a = (a_1, \cdots, a_n)$*,* $b = (b_1, \cdots, b_{n-1})$ $(a_i, b_i \in \mathbb{N}$*,* $a_1 \geq \cdots \geq a_n$*,* $b_1 \geq \cdots \geq b_n)$*. The restriction* $\pi_G|_H$ *fully decomposes, and* π_H *appears in the decomposition of* $\pi_G|_H$ *if and only if*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq b_{n-1} \geq a_n.$$

Moreover, the multiplicity $\dim \mathrm{Hom}_H(\pi_G|_H, \pi_H)$ *is less than or equal to one. This result is known as* ***multiplicity-one theorem****. The spectrum of* $\pi_G|_H$ *can be fully described with this equivalent condition and multiplicity-one theorem.*

Example 1.0.2 (Compact unitary groups)**.** *Let* $G = \mathrm{U}(n)$ *and* $H = \mathrm{U}(n-1)$ *be compact unitary groups. Irreducible finite-dimensional complex representations* π_G, π_H *of* G, H *can* *be classified by the integral highest weights* $\mathbf{a} = (a_1, \cdots, a_n)$, $\mathbf{b} = (b_1, \cdots, b_{n-1})$ *(*$a_i, b_i \in \mathbb{N}$*,* $a_1 \geq \cdots \geq a_n$*,* $b_1 \geq \cdots \geq b_n$*) of the torus* $\mathrm{U}(1)^n$*,* $\mathrm{U}(1)^{n-1}$*. The restriction* $\pi_G|_H$ *fully decomposes, and* π_H *appears in the decomposition of* $\pi_G|_H$ *if and only if*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq b_{n-1} \geq a_n.$$

Moreover, the multiplicity $\dim \mathrm{Hom}_H(\pi_G|_H, \pi_H) \leqslant 1$*.*

When studying symmetry-breaking in Hamiltonian mechanics and quantum mechanics, similar problems for some non-compact real groups (e.g. the Lorentz group $\mathrm{SO}(1,3)$) need to be considered. However, when H is non-compact, the restriction $\pi_G|_H$ does not fully decompose in general. Researchers, such as those in the local theory of automorphic forms, address more interest in the **irreducible quotients** of $\pi_G|_H$. Besides, many computations, such as the branching problem for $(\mathrm{SO}(2,1), \mathrm{SO}(2))$ and $(\mathrm{SO}(3), \mathrm{SO}(2))$, suggested that the branching problem for **pure inner forms** are closed related to each other.

Targeting the irreducible quotients of the restriction and considering all pure inner forms, B. Gross and D. Prasad formulated a conjecture for admissible representations of special orthogonal groups in [GP92]. Their conjecture also applies to the special orthogonal groups over non-Archimedean local fields of characteristic zero. Later, they generalized the conjecture to a more general context, applicable for Bessel models of special orthogonal groups in [GP94]. In [GGP12], W. T. Gan, Gross, and Prasad extended the conjecture to other classical groups over local fields of characteristic zero, precisely describing the multiplicities. In this setting, the multiplicity is defined as the dimension of the Bessel or Fourier-Jacobi models the representation has and is known to be less than or equal to one ([AGRS10] [Wald12d] [GGP12] [Sun12a] [SZ12] [JSZ10] [Sun12b]). This conjecture is closely related to the study of automorphic forms and L-functions and is called the **local Gan-Gross-Prasad conjecture**. In this thesis, we will usually abbreviate it as the **local GGP conjecture**.

In [GGP12], Vogan L-packets are considered. Generally speaking, for an algebraic group $\mathbf{G}$ over a local field F of characteristic zero, and an L-parameter $\phi : \mathrm{WD}_F \to {}^L\mathbf{G}$, the Vogan L-packet $\Pi_\phi^{\mathrm{Vogan}}$ is the union of L-packets $\Pi_\phi(\mathbf{G}_\alpha(F))$ associated with a fixed L-parameter ϕ for all pure inner forms $\mathbf{G}_\alpha$ of the group $\mathbf{G}$. A parameter ϕ is called **generic** if Π_ϕ has a generic representation, that is, a representation with nontrivial Whittaker model. Based on

Vogan's conjecture ([Vog93]), for a generic parameter ϕ of $\mathbf{G}$, we have a parameterization of representations in Π_ϕ using characters of the component group $\mathcal{S}_\phi = \pi_0(Z_{\widehat{\mathbf{G}}}(\mathrm{Im}(\phi)))$, where $\widehat{\mathbf{G}}$ is the Langlands dual group of $\mathbf{G}$ and $Z_{\widehat{\mathbf{G}}}(\mathrm{Im}(\phi))$ denotes the centralizer of $\mathrm{Im}(\phi)$ in $\widehat{\mathbf{G}}$.

In [GGP12], Gan-Gross-Prasad conjectured for several pairs of classical groups that

- For every generic L-parameter, there is exactly one representation in the Vogan L-packet with multiplicity equal to one.
- Moreover, this unique representation is characterized by the local root numbers.

The first part of the conjecture is called the **Multiplicity One** part of the conjecture and the second part of the conjecture is called the **Epsilon Dichotomy** part of the conjecture.

There are four cases in the conjecture: Bessel cases for unitary groups, Bessel cases for special orthogonal groups, Fourier-Jacobi cases for (skew-)unitary groups, and Fourier-Jacobi cases for symplectic-metaplectic groups.

Recent years great progresses have been made on this conjecture. Over non-Archimedean local fields of characteristic zero, J.-L. Waldspurger was the pioneer who initiated such progress by completing the proof for tempered L-parameters of special orthogonal groups. His approach uses local trace formula and endoscopy in [Wald10] [Wald12a] [Wald12b] [Wald12c]. For this case, C. Mœglin and Waldspurger proved the conjecture for generic L-parameters in [MW12]. The ideas and the methods used in this approach can be outlined in the following diagram:

$$\begin{tikzcd}
\text{Uniqueness of generic repn in Vogan } L\text{-packet} \arrow[d, Rightarrow, swap, "\text{by Waldspurger}"] \arrow[d, Rightarrow, "\text{Local trace formula and endoscopy}"] & \\
\text{Tempered Bessel cases} \arrow[r, Rightarrow, "\text{by Atobe}", "\text{Theta corres.}"'] \arrow[d, Rightarrow, swap, "\text{by Mœglin,Waldspurger}"] \arrow[d, Rightarrow, "\text{Multiplicity formula}"] & \text{Tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, swap, "\text{by Atobe}"] \arrow[d, Rightarrow, "\text{Theta corres.}"] \\
\text{Bessel cases} & \text{Fourier-Jacobi cases}
\end{tikzcd}$$

R. Beuzart-Plessis proved the proof for tempered L-parameters of unitary groups over non-Archimedean local fields in [BP14] [BP16]. Based on this result, Gan and A. Ichino proved the conjecture for the Bessel and Fourier-Jacobi model of unitary groups in [GI16]. H. Atobe proved the conjecture for the Fourier-Jacobi models symplectic-metaplectic groups in [At18]. The proofs for this case can be outlined in the following diagram:

$$\begin{tikzcd}
\text{Uniqueness of generic repn in Vogan } L\text{-packet} \arrow[d, Rightarrow, "\text{by Beuzart-Plessis}"', "\text{Local trace formula and endoscopy}"] & \\
\text{tempered Bessel cases} \arrow[r, Rightarrow, "\text{by Gan-Ichino}", "\text{theta corres.}"'] \arrow[d, Rightarrow, "\text{by Gan-Ichino}"', "\text{Multiplicity formula}"] & \text{Tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, "\text{by Gan-Ichino}"', "\text{Theta corres.}"] \\
\text{Bessel cases} & \text{basic Fourier-Jacobi cases}
\end{tikzcd}$$

Over Archimedean local fields, i.e. $\mathbb{R}$ and $\mathbb{C}$, most results have also been proved. H. He proved the conjecture in unitary Bessel cases for discrete series L-parameters in [He17]. Beuzart-Plessis proved the uniqueness of the multiplicity-one representation for tempered L-parameters in unitary Bessel cases in [BP20]. H. Xue completed the proof for unitary Bessel cases in [X1] [X2] using a new approach based on theta correspondence and Schwartz homology. The proofs in the case can be outlined in the following diagram:

$$\begin{tikzcd}
\text{Small unitary cases} \arrow[d, Rightarrow, "\text{[X1]}"', "\text{Induction using theta corres.}"] & \\
\text{Tempered basic Bessel cases} \arrow[r, Leftrightarrow, "\text{Tempered intertwining form of theta corres.}", "\text{[X3]}"'] \arrow[d, Rightarrow, "\text{[X2]}"', "\text{Schwartz homology}"] & \text{Tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, "\text{[X3]}"', "\text{Schwartz homology}"] \\
\text{Bessel cases} & \text{Fourier-Jacobi cases}
\end{tikzcd}$$

Zhilin Luo proved the uniqueness of the multiplicity-one representation for tempered

L-parameters in special orthogonal cases in [Luo21]. Luo and the author of this thesis completed the proof for tempered special orthogonal cases, and based on these results, the author completed the proof for generic L-parameters in [Ch21]. Xue proved the Fourier-Jacobi cases for unitary groups in [X3], and the author reduced the conjecture for symplectic-metaplectic groups to basic tempered cases in [Ch23a]. The remaining open problem is the basic tempered symplectic-metaplectic cases, which will be treated in my ongoing work joint with R. Chen and J. Zou. The proofs for this case can be outlined in the following diagram:

$$\begin{tikzcd}
\text{Uniqueness of generic repn in Vogan } L\text{-packet} \arrow[d, Rightarrow, "\text{in [Luo21] [CL22]}"', "\text{Local trace formula and endoscopy}"] & \\
\text{Tempered basic Bessel cases} \arrow[r, Leftrightarrow, "\text{theta corres.}", "\text{[CCZ]}"'] \arrow[d, Rightarrow, "\text{in [Ch23a]}"', "\text{Multiplicity formula}"] & \text{Tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, "\text{in [Ch23a]}"', "\text{Multiplicity formula}"] \\
\text{Bessel cases} & \text{Fourier-Jacobi cases}
\end{tikzcd}$$

Most of these works treated the conjecture on a case-by-case basis. One of the motivations of this author is to provide a uniform treatment of the local Gan-Gross-Prasad conjecture for all classical groups and all local fields, at least in the following three aspects:

(1.0.1) Uniform Approach

1. over Archimedean and over non-Archimedean local fields;
2. for unitary cases (Bessel and Fourier-Jacobi cases for unitary groups) and for non-unitary cases (Bessel cases for special orthogonal groups and Fourier-Jacobi cases for symplectic-metaplectic groups);
3. for Bessel cases and for Fourier-Jacobi cases;

In [Ch23b], the author proposed a method to uniformly organize the proof, which can be explained in the following diagram:

$$\begin{tikzcd}
\text{Uniqueness of generic repn in Vogan } L\text{-packet} \arrow[d, Rightarrow, "\text{Local trace formula+endoscopy}"] & \\
\text{Tempered basic Bessel cases} \arrow[r, Leftrightarrow, "\text{theta correspondence}"] \arrow[d, Rightarrow, "\text{Multiplicity formula}"] & \text{Tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, "\text{Multiplicity formula}"] \\
\text{Bessel cases} & \text{Fourier-Jacobi cases}
\end{tikzcd}$$

More precisely, the proof is formulated in the following steps:

1. Use the **local converse theorem** to prove the uniqueness of generic representation in Vogan L-packets.
2. Use a **multiplicity formula** to reduce the conjecture to the tempered basic cases from the uniqueness of generic presentation in Vogan L-packet, which is a consequence of Vogan's conjecture ([Vog93]);
3. Use **theta correspondence** to build a bridge between tempered basic Bessel cases and tempered basic Fourier-Jacobi cases;
4. Use **local trace formula** to prove the tempered basic Bessel cases.

The multiplicity formula in the above diagram refers to Theorem 5.0.2. It has been proved in [MW12] [GI16] [CJLZ23] over non-Archimedean local fields and prove in [Ch23a] over Archimedean local fields. This thesis will focus on the multiplicity formula, and make attempt to uniformize the proof in the three aspects as outlined above.

The bridge between tempered basic cases have been built over non-Archimedean fields in [GI16] [At18]. One of the key ingredients in their approach is the result on "big theta=small theta", which has not been proved over Archimedean local fields in general. For unitary groups over Archimedean local fields in [X3] in an alternative method. The bridge for the remaining case will be treated in my ongoing work joint with R. Chen and J. Zou, only using "big theta=small theta" in the stable-range situations.

The tempered basic cases can be relatively uniformly proved by Waldspurger's trace formula approach, which have been established case-by-case in [Wald10, Wald12a, Wald12b, Wald12c, BP14, BP16, BP20, Luo21], respectively. Over Archimedean local fields, the twisted trace formula has not been established yet. In [CL22], Z. Luo and the author simplified Waldspurger's approach for the epsilon dichotomy over Archimedean local field such that the twisted local twisted trace is not necessary in the proof. This similification applies to the unitary Archimedean cases, but cannot be extend to the non-Archimedean situations.

The following is the structure of the thesis.

- Chapter 1 introduces the setting of the local Gan-Gross-Prasad conjecture following [GGP12].
- Chapter 2 introduces an analytic tool to prove one inequality of the multiplicity formula. This analytic tool can be traced back to the distributional analysis. In particular, over Archimedean local fields, we describe the contribution of derivatives of the distributions using the language of Schwartz homologies introduced in [CS20].
- Chapter 3 introduces an integral method to construct a nonzero element of the Hom-space defining the multiplicity.
- Chapter 4 aims proves the multiplicity by proving two inequalities. The proof for one of the inequalities uses distributional analysis and the proof for the other inequality uses results from the integral method.
- Chapter 5 describes a relatively uniform approach toward the local Gan-Gross-Prasad conjecture.

Conventions of notations:

- F is a local field of characteristic zero;
- The bold letters (e.g. $\mathbf{X}$, $\mathbf{U}$, $\mathbf{Z}$) denotes algebraic varieties over F;
- The Latin letters (e.g. X, U, Z) denote the set of F-points endowed with the natural topological structure.
- When F is Archimedean and $\mathbf{G}$ is an algebraic group, $G = \mathbf{G}(F)$ is a Lie group, we denote by $\mathfrak{g}$ its Lie algebra and by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}(F)$.
- The representations considered in this article are representations over the complex field $\mathbb{C}$.

- For a symplectic space V over F, the metaplectic group $\mathrm{Mp}(V)$ is the metaplectic double cover of $\mathrm{Sp}(V)$. We artificially introduce the notion $\mathbf{Mp}(V)$ and set $\mathbf{Mp}(V)(F) = \mathrm{Mp}(V)$. In this article, we do harmonic analysis in the algebraic setting and we can avoid this artificial notion in the proof using a reduction in Section 5.1.

Chapter 2

Local Gan-Gross-Prasad Conjecture

2.1 Bessel and Fourier-Jacobi models

Let F be a local field of characteristic zero and E is a field extension of F satisfying $[E:F]=1,2$.

The Bessel models and Fourier-Jacobi models are defined based on a pair of towers consisting of isometry groups $\mathcal{T}_1=\{\mathbf{G}_V\}_{V\in\mathcal{I}_1}$, $\mathcal{T}_2=\{\mathbf{H}_W\}_{W\in\mathcal{I}_2}$, where $\mathcal{I}_1,\mathcal{I}_2$ are families of ϵ-hermitian spaces, that is, quadratic/hermitian/symplectic/skew-hermitian forms.

For Bessel models, when $E\neq F$, the spaces V,W are hermitian spaces over E, and the towers are

$$\mathcal{T}_1=\{\mathbf{U}(V):\dim_E V\text{ is odd}\},\quad \mathcal{T}_2=\{\mathbf{U}(W):\dim_E W\text{ is even}\};$$

When $E=F$, the spaces V,W are quadratic spaces over E, and the towers are

$$\mathcal{T}_1=\{\mathbf{SO}(V):\dim_E V\text{ is odd}\},\quad \mathcal{T}_2=\{\mathbf{SO}(W):\dim_E W\text{ is even}\}.$$

For Fourier-Jacobi models, when $E\neq F$, the spaces V,W in $\mathcal{I}_1,\mathcal{I}_2$ are hermitian spaces over E,

$$\mathcal{T}_1=\{\mathbf{U}(V)\},\quad \mathcal{T}_2=\{\mathbf{U}(W)\};$$

When $E = F$, the spaces V, W in $\mathcal{I}_1, \mathcal{I}_2$ are symplectic spaces over E, and the towers are

$$\mathcal{T}_1 = \{\mathbf{Sp}(V)\}, \quad \mathcal{T}_2 = \{\mathbf{Mp}(W)\}.$$

Here we artificially introduce the notion $\mathbf{Mp}(V)$ and set $\mathbf{Mp}(V)(F) = \mathrm{Mp}(V)$, where $\mathrm{Mp}(V)$ is the metaplectic double cover of $\mathrm{Sp}(V)$. In Section 5.1, we change the tower into a tower of Jacobi groups to avoid using this artificial notion in the proof.

We fix a pair of towers $\mathcal{T}_1 = \{\mathbf{G}_V\}_{V \in \mathcal{I}_1}$ and $\mathcal{T}_2 = \{\mathbf{H}_W\}_{W \in \mathcal{I}_2}$ in any of the above cases. For $V \in \mathcal{I}_1$ and $W \in \mathcal{I}_2$, the groups $\mathbf{G}_V$ and $\mathbf{H}_W$ are **relevant** if and only if

$$V \subset W, \quad \text{and } V^{\perp} \text{ is split over } E, \text{ or}$$
$$W \subset V, \quad \text{and } W^{\perp} \text{ is split over } E.$$

When $\mathbf{G}_V$, $\mathbf{H}_W$ are relevant, we call (V, W) a **relevant pair**.

We denote by $\Pi_F(\mathbf{G}_V)$ the set of equivalence classes of irreducible admissible representations of $G_V = \mathbf{G}_V(F)$, and we require it to be Casselman-Wallach ([Cas89] [Wall94]) when F is Archimedean.

Given relevant $\mathbf{G}_V$ and $\mathbf{H}_W$, we define multiplicity $m(\pi_V, \pi_W)$ for $\pi_V \in \Pi_F(\mathbf{G}_V)$ and $\pi_W \in \Pi_F(\mathbf{H}_W)$. Since $\mathbf{G}_V$ and $\mathbf{H}_W$ are relevant, we may assume $W \subset V$, then we have a split S such that $V = W \perp S$, then $\dim_E S$ is odd in Bessel cases and $\dim S$ is even in Fourier-Jacobi cases. When $S \neq 0$, we set $r = \lfloor \frac{\dim_E S - 1}{2} \rfloor$. We choose a basis S over F, and denote it by $\{z_i\}_{i=-r}^{r}$ in Bessel cases and by $\{z_i\}_{1 \leqslant |i| \leqslant r}$ in Fourier-Jacobi cases such that

$$(z_i, z_j) = \delta_i^{-j}, i = 1, \cdots, r, j = -1, \cdots, -r, \text{ and}$$
$$(z_i, z_0) = 0, i = \pm 1, \cdots, \pm r, \text{ in Bessel cases,}$$

where $(\cdot, \cdot)$ is the ϵ-hermitian form on V.

Then we define $\mathbf{N}$ as the unipotent part of the parabolic group $\mathbf{P}_{V,r}$ stabilizing the full totally isotropic flag

$$X_1 \subset \cdots \subset X_r$$

of S, where $X_i = \mathrm{Span}_E(z_{r-i+1}, \cdots, z_r)$.

Fix an additive character ψ_F of F and let ξ be the generic character of $N = \mathbf{N}(F)$ defined by

$$\xi_N(n) = \begin{cases} \psi_E(\sum_{i=0}^{r}(z_i, z_{-(i-1)})) & \text{in Bessel cases,} \\ \psi_E(\sum_{i=1}^{r}(z_i, z_{-(i-1)})) & \text{in Fourier-Jacobi cases,} \end{cases}$$

where ψ_E is the additive character $\psi_E(z) = \psi_F(\mathrm{Re}(z))$, $z \in E$. Here

$$\mathrm{Re}(z) = \frac{1}{|\mathrm{Gal}(E/F)|} \sum_{\tau \in \mathrm{Gal}(E/F)} z^\tau, \text{ for } z \in E.$$

In Bessel cases, we define the multiplicity

$$m(\pi_V, \pi_W) = \dim \mathrm{Hom}_{G_W \ltimes N}(\pi_V|_{G_W \ltimes N} \otimes \pi_W, \xi),$$

where ξ is the character of $G_W \ltimes N$ induced from the character ξ_N of N.

In Fourier-Jacobi cases, for an additive character ψ_F of F, we denote by ω_{V,ψ_F} the Weil representation associated to ψ_F of

$$\begin{cases} \mathrm{Mp}(V) \ltimes \mathcal{H}(V)(F) & \text{when } V \text{ is a symplectic space,} \\ \mathrm{U}(V) \ltimes \mathcal{H}(V)(F) & \text{when } V \text{ is a skew-hermitian space,} \end{cases}$$

where $\mathcal{H}(W)$ is the Heisenberg group $\mathrm{Res}_{E/F} W \oplus \mathbb{G}_a$ over F with the multiplication defined by

$$(w, z) \cdot (w', z') = (z'w + zw', \frac{1}{2}\mathrm{Re}\langle w, w' \rangle_W + zz'), \quad w, w' \in \mathrm{Res}_{E/F}(W), \quad z, z' \in \mathbb{G}_a. \tag{2.1.1}$$

In Fourier-Jacobi cases, since $\pi_W \in \Pi_F(\mathbf{H}_W)$, $\pi_W \otimes \overline{\omega_{W,\psi_F}}$ is an irreducible representation of $\mathbf{G}_W^J = \mathbf{G}_W \ltimes \mathcal{H}(W)$. We define the multiplicity

$$m(\pi_V, \pi_W) = \begin{cases} \dim \mathrm{Hom}_{G_W^J \ltimes N}(\pi_V|_{G_W^J \ltimes N} \otimes (\pi_W \otimes \overline{\omega_{W,\psi_F}}), \xi) & \text{when } W \subsetneq V, \\ \dim \mathrm{Hom}_{G_V}(\pi_V \otimes (\pi_W \otimes \overline{\omega_{W,\psi_F}})|_{G_V}, 1) & \text{when } W = V, \end{cases}$$

where ξ is the character of $G_W^J \ltimes N$ induced from the character ξ_N of N.

To uniformize the notion with generalization in spherical varieties, we introduce an equivalent method to define the multiplicity in [BP20] [Ch23a]. The **Gan-Gross-Prasad triple** (G, H, ξ) associated to a relevant pair (V, W) is defined as $G = \mathbf{G}(F)$, $H = \mathbf{H}(F)$, where

$$\mathbf{G} = \begin{cases} \mathbf{G}_V \times \mathbf{H}_W & \text{in Bessel cases,} \\ \mathbf{G}_V \times (\mathbf{G}_W \ltimes \mathcal{H}(W)) & \text{in Fourier-Jacobi cases,} \end{cases}$$

$$\mathbf{H} = \begin{cases} \Delta\mathbf{H}_W \ltimes \mathbf{N} & \text{in Bessel cases,} \\ \Delta\mathbf{G}_W^J \ltimes \mathbf{N} & \text{in Fourier-Jacobi and } W \subsetneq V \text{ cases,} \\ \Delta\mathbf{G}_W & \text{in Fourier-Jacobi and } W = V \text{ cases,} \end{cases}$$

and ξ be the character of $H = \mathbf{H}(F)$ induced from ξ_N. In this setting, the multiplicity can be equivalently defined as

$$m(\pi) = \dim \mathrm{Hom}_H(\pi, \xi),$$

and it is compatible with the previous definition in the sense of

$$m(\pi_V, \pi_W) = \begin{cases} m(\pi_V \boxtimes \pi_W) & \text{in Bessel cases,} \\ m(\pi_V \boxtimes (\pi_W \otimes \omega_{W,\psi_F})) & \text{in Fourier-Jacobi case.} \end{cases}$$

This notation also helps treat Bessel and Fourier-Jacobi cases uniformly.

A GGP triple (G, H, ξ) associated to a relevant pair (V, W) is called **basic** if ξ is the trivial character. In Bessel cases, (G, H, ξ) is basic if $\dim_E V = \dim_E W + 1$. In Fourier-Jacobi cases, (G, H, ξ) is basic if $\dim_E V = \dim_E W$ (known as the **equal-rank cases**) or $\dim_F V = \dim_F W + 2$ (known as the **almost equal-rank cases**).

2.2 Vogan L-packets and its parameterization

As in the classical branching problem, we need a classification of representation. In this section, we recall the definition of Vogan L-packets for classical groups and metaplectic groups.

2.2.1 Vogan L-packet for classical groups.

When $\mathbf{G}_V$ is a connected reductive group, R. Langlands classifies the representations in $\Pi_F(\mathbf{G}_V)$ with L-parameters $\phi_V : \mathrm{WD}_F \to {}^L\mathbf{G}_V$, where WD_F is the Weil-Deligne group. Given an L-parameter ϕ_V, the local Langlands correspondence gives us an L-packet $\Pi_{F,\phi_V}(\mathbf{G}_V)$. The Vogan L-packet is defined as

$$\Pi_{F,\phi_V}^{\text{Vogan}} = \coprod_{\alpha \in H^1(F, \mathbf{G}_V)} \Pi_{F,\phi_V}(\mathbf{G}_{V_\alpha}).$$

An L-parameter ϕ_V is called **tempered** if $\mathrm{Im}(\phi_V)$ has bounded image. An L-parameter ϕ_V is called **generic** if $\Pi^{\mathrm{Vogan}}_{F,\phi_V}$ contains a generic representation, that is, a representation with a nontrivial Whittaker model. In particular, tempered parameters are generic. We call $\Pi_{F,\phi_V}(\mathbf{G}_V)$ tempered (respectively. generic) packet when ϕ_V is tempered (respectively. generic).

We denote by $\Pi_{F,\mathrm{temp}}(\mathbf{G}_V)$ the set of equivalent classes of tempered representations of $\mathbf{G}_V(F)$.

Conjecture 1 ([Vog93]). *For a generic parameter ϕ_V, fixing a Whittaker datum, there exists a bijection*

$$\Pi^{\mathrm{Vogan}}_{F,\phi_V} \longleftrightarrow \textit{characters of } \mathcal{S}_{\phi_V},$$

where $\mathcal{S}_\phi = \pi_0(\mathrm{Cent}_{\widehat{\mathbf{G}}_V}(\mathrm{Im}(\phi_V)))$ and $\widehat{\mathbf{G}}_V$ is the Langlands dual group of $\mathbf{G}_V$. Moreover, the trivial character corresponds to the generic representation.

Over Archimedean local fields, this conjecture has been confirmed in [Vog93]. Over non-Archimedean local fields, this conjecture has been proven for unitary groups in [MSTW], and for symplectic groups in [A13].

2.2.2 Vogan L-packet for metaplectic groups.

When $\mathbf{G}_V = \mathbf{Mp}(2n)$, the L-parameter $\widetilde{\phi}_V : \mathrm{WD}_F \to {}^L\mathbf{G}_V = \mathrm{Sp}(2n, \mathbb{C})$ correspond to an L-parameter $\phi_W : \mathrm{WD}_F \to {}^L\mathbf{SO}(W) = \mathbf{Sp}(2n, \mathbb{C})$ of $\mathbf{SO}(W)$ where $\dim W = 2n+1$ and $\mathrm{disc}(W) = 1$. The parameter $\widetilde{\phi}_V$ is called **generic** if ϕ_W is generic.

The Shimura-Waldspurger's correspondence gives a bijection

$$\theta_{V_{2n}} : \coprod_{\substack{\dim W = 2n+1 \\ \mathrm{disc}(W)=1}} \Pi_F(\mathbf{SO}(W)) \to \Pi^{\mathrm{gen}}_F(\mathbf{Mp}(2n)),$$

where $\Pi^{\mathrm{genu}}_F(\mathbf{Mp}(2n))$ is the set of equivalence classes of irreducible genuine representations. Using the Shimura-Waldspurger's correspondence, the Vogan L-packet is defined as

$$\Pi^{\mathrm{Vogan}}_{F,\widetilde{\phi}_V} = \theta_{V_{2n}}(\Pi^{\mathrm{Vogan}}_{F,\phi_V}),$$

and there is a bijection

$$\Pi^{\mathrm{Vogan}}_{F,\widetilde{\phi}_V} \longleftrightarrow \text{characters of } \mathcal{S}_{\widetilde{\phi}_V}$$

when $\widetilde{\phi}_V$ is generic. Here $\mathcal{S}_{\widetilde{\phi}_V} = \pi_0(\mathrm{Cent}_{\widehat{\mathbf{G}}_V}(\mathrm{Im}(\widetilde{\phi}_V)))$.

2.3 Local root numbers and the distinguished characters

Let $\mathbf{G}_V, \mathbf{H}_W$ be a relevant pair of reductive groups and $W \subset V$. For generic L-parameters ϕ_V, ϕ_W of $\mathbf{G}_V, \mathbf{H}_W$ respectively, Gan-Gross-Prasad defined the **relevant Vogan packet**

$$\Pi^{\text{Vogan,rel}}_{F,\phi_V\times\phi_W} = \coprod_{\alpha\in H^1(F,\mathbf{H}_W)} \Pi_{F,\phi_V}(\mathbf{G}_{V_\alpha}) \times \Pi_{F,\phi_W}(\mathbf{H}_{W_\alpha}).$$

Conjecture 2 ([GGP12]). *Given generic L-parameters ϕ_V of $\mathbf{G}_V$ and ϕ_W of $\mathbf{H}_W$, the following results hold.*

1. **Multiplicity-one:** *There is exactly one pair $(\pi_V, \pi_W) \in \Pi^{\text{Vogan,rel}}_{F,\phi_V\times\phi_W}$ such that*

   $$m(\pi_V, \pi_W) = 1.$$

2. **Epsilon-dichotomy:** *This unique pair (π_V, π_W) can be characterized by the characters*

   $$\eta^W_{\phi_V}, \eta^V_{\phi_W}$$

   defined by local root numbers as in [GGP12].

We denote by $\mathrm{std_V}$ the standard representation of ${}^L\mathbf{G}_V$. For an L-parameter $\phi_V : \mathrm{WD}_F \to {}^L\mathbf{G}_V$, we define $\phi_V^{\mathrm{ss}} = \mathrm{std_V} \circ \phi_{\mathrm{V}}$ the **semisimplification** of the parameter ϕ_V.

We set

$$\mathrm{M}_V = \begin{cases} \mathrm{std_V} \oplus \mathbb{C} & \text{when } \mathbf{G}_V = \mathbf{Sp}(V), \\ \mathrm{std_V} & \text{otherwise.} \end{cases}$$

For an L-parameter ϕ_W of $\mathbf{G}_W$ (here $\mathbf{G}_W$ does not necessary be of the same type as $\mathbf{G}_V$), we define ϕ_W^{ss} and M_W accordingly.

We follow [GGP12] define the following distinguished characters in the conjecture.

$$\begin{aligned} &\eta_{\phi_V,W}(s_V) = \mathrm{sgn}_{\phi_V,W}(s_V) \cdot \varepsilon\left(\frac{1}{2}, \mathrm{M}_V^{s_V=-1} \otimes \mathrm{M}_W, \psi_F\right), \quad s_V \in \mathcal{S}_V, \text{ and} \\ &\eta_{\phi_W,V}(s_W) = \mathrm{sgn}_{V,\phi_W}(s_W) \cdot \varepsilon\left(\frac{1}{2}, \mathrm{M}_W^{s_W=-1} \otimes \mathrm{M}_V, \psi_F\right), \quad s_W \in \mathcal{S}_W. \end{aligned} \tag{2.3.1}$$

where the space $\mathrm{M}_V^{s_V=-1}$, $\mathrm{M}_W^{s_W=-1}$ are respectively the $s_V = (-1)$-eigenspace of M_V, the $s_W = (-1)$-eigenspace of M_W, and $\varepsilon(\dots)$ is the local root number defined by Rankin-Selberg integral ([JPSS83]). The sign characters are defined as the following

$$\mathrm{sgn}_{\phi_V,W}(s_V) = \begin{cases} 1 & \text{when } \mathbf{G}_V = \mathbf{U}(V), \\ \det\left(-\mathrm{Id}_{\mathrm{M}_V^{s_V=-1}}\right)^{\frac{\dim \mathrm{M}_W}{2}} \cdot \det\left(-\mathrm{Id}_{\mathrm{M}_W}\right)^{\frac{\dim \mathrm{M}_V^{s_V=-1}}{2}} & \text{otherwise.} \end{cases}$$

$$\mathrm{sgn}_{V,\phi_W}(s_W) = \begin{cases} 1 & \text{when } \mathbf{G}_V = \mathbf{U}(V) \\ \det\left(-\mathrm{Id}_{\mathrm{M}_W^{s_W=-1}}\right)^{\frac{\dim \mathrm{M}_V}{2}} \cdot \det\left(-\mathrm{Id}_{\mathrm{M}_V}\right)^{\frac{\dim \mathrm{M}_W^{s_W=-1}}{2}} & \text{otherwise.} \end{cases}$$

Chapter 3

Distributional Analysis

This section reviews classical distributional analysis in the local theory of automorphic forms. Distributional analysis is a tool introduced by Shalika in [Sha74] to prove the uniqueness of Whittaker models. This method has been further developed in papers related to the local multiplicity-one theorem ([JSZ10], [GGP12], for instance). In this section, we follow these classical works for the statement over non-Archimedean local fields. Over Archimedean local fields, we rewrite the distributional analysis in terms of the Schwartz analysis developed in [CS20], [X2] so that the arguments are parallel to those for non-Archimedean cases.

3.1 Categories for distributional analysis

This section introduces the category of topological spaces, topological groups, and representations when we do distributional analysis over a local field F of characteristic zero. Over non-Archimedean local fields, we follow [Cas95]. Over Archimedean local fields, we use the notions of Schwartz analysis and refer readers to [dC91, AG08, Sun15] for details of the definitions of Nash manifolds and almost linear Nash groups.

The category *Top* of the topological spaces we consider is the category of

- locally compact totally disconnected topological spaces when F is non-Archimedean;
- Nash manifolds when F is Archimedean.

Example 3.1.1. *Let* $\mathbf{X}$ *be an algebraic variety over* F*, then its* F*-points* $X = \mathbf{X}(F)$ *has a canonical topological structure. With this structure,* X *is an element of* Top*.*

Definition 3.1.2. *For* $X \in Top$*, we denote by* $\mathcal{S}(X)$ *the space of*

- *locally constant compactly supported functions on* X *when* F *is non-Archimedean;*
- *Schwartz functions on* X *when* F *is Archimedean.*

The category Grp of topological groups we consider is the category of

- locally compact totally disconnected topological groups when F is non-Archimedean;
- almost linear Nash groups when F is Archimedean.

Example 3.1.3. *Let* $\mathbf{G}$ *be an algebraic group over* F*, then*

1. *its* F*-points* $\mathbf{X}(F)$ *has a canonical topological structure. With this structure,* $\mathbf{G}(F) \in Grp$*, and,*
2. *when* $\widetilde{G}$ *is a finite covering of* $\mathbf{G}(F)$*,* $\widetilde{G} \in Grp$*.*

In particular, for $\mathbf{G}_V$ *in Section 2.1, we have* $\mathbf{G}_V(F) \in Grp$*.*

For a group $G \in Grp$, the category of smooth complex representations $Rep(G)$ is the category whose objects are G-representations (π, V_π) on

- smooth spaces when F is non-Archimedean;
- Fréchet spaces of modernate growth when F is Archimedean.

When no confusion is possible, we do not distinguish a representation π with its underlying space V_π.

Example 3.1.4. *For* $G \in Grp$*,* $X \in Top$ *with a compatible left* G*-action, that is, the graph of* $G \times X \to X$ *lies in* Top*, we denote by* $\mathcal{S}(X)$ *the space of*

1. *compactly-supported functions on* X *when* F *is non-Archimedean;*
2. *Schwartz functions when* F *is Archimedean.*

Then $\mathcal{S}(X) \in Rep(G)$ via the action $(g.f)(x) = f(xg)$.

Given a center character ψ_G of G we denote by $Rep^{\psi_G}(G)$ the full subcategory consisting of objects with central character ψ_G.

Example 3.1.5. *Let $\mathbf{G}_V^J = \mathbf{G}_V \ltimes \mathcal{H}(V)$ be a Jacobi group in the setting of Section 2.1. Let $G_V^J = \mathbf{G}_V^J(F)$. For a fixed nontrivial character ψ_F of F, the Stone–von Neumann theorem ensures that $Rep^{\psi_F}(G_V^J)$ consists of representations in the form of*

$$\pi_V \otimes \omega_{V,\psi_F}, \quad \pi_V \in Rep(G_V).$$

Recall that, when studying the multiplicity, we work on the set $\Pi_F(\mathbf{G})$, which consists of the irreducible objects in a full category $Rep_{\mathrm{adm}}(G)$ of $Rep(G)$ for $G = \mathbf{G}(F)$. Here Rep_{adm} is defined as the category of

- smooth admissible representations;
- Casselman-Wallach representations, that is, smooth representations of moderate growth whose $(\mathfrak{g}_{\mathbb{C}}, K)$-module is admissible.

3.2 Some functors

This section reviews the properties of inductions and tensor products. These functors construct representations from representations of smaller groups.

Given groups $H, G \in Grp$ such that $H \subset G$, there is a functor $\mathrm{res}_H : Rep(G) \to Rep(H)$ defined by restriction to subgroup H. This restriction function res_H has a left adjoint, which is the compact induction when F is non-Archimedean, and is the Schwartz induction when F is Archimedean. These inductions are defined as follows:

Let $G \times_H \pi_H$ be the bundle over $H\backslash G$ defined by quotient $G \times \pi_H$ by H, where H acts on $G \times \pi_H$. We denote by Γ^{cpt} (respectively, Γ^{Sch}) the space of compactly supported sections (respectively, Schwartz section) of a smooth bundle (resp. Nash bundle). The **compact induction** is defined as $\Gamma^{\mathrm{cpt}}(H\backslash G, G \times \pi_H)$ and the **Schwartz induction** is defined as $\Gamma^{\mathrm{Sch}}(H\backslash G, G \times \pi_H)$. In the rest of the article, we use the notion Ind_H^G to be the compact induction when F is non-Archimedean and to be the Schwartz induction when F is Archimedean.

It is worth mentioning that there is a smooth induction $\mathrm{Ind}_H^{G,\mathrm{sm}}\pi_H$ defined as the space of smooth sections on the bundle $G \times_H \pi_H$. We have

$$\begin{aligned}\mathrm{Ind}_H^{G,\mathrm{cpt}}\pi_H \subset \mathrm{Ind}_H^{G,\mathrm{Sch}}\pi_H \subset \mathrm{Ind}_H^{G,\mathrm{sm}}\pi_H &\text{ when } F \text{ is Archimedean, and}\\ \mathrm{Ind}_H^{G,\mathrm{cpt}}\pi_H \subset \mathrm{Ind}_H^{G,\mathrm{sm}}\pi_H &\text{ when } F \text{ is non-Archimedean.}\end{aligned}$$

In particular, when $H\backslash G$ is compact, these sections being compactly supported, Schwartz, and smooth are equivalent. Therefore, in this case, these three inductions are the same. In particular, when H is a parabolic subgroup of G, these inductions are the same.

The following property on the inductions follows from the definition when F is non-Archimedean and has been confirmed in [CS20, Proposition 7.1, 7.2] when F is Archimedean.

Proposition 3.2.1. *1. Ind_H^G is an exact functor from Rep$(H) \to$ Rep(G).*

2. For algebraic subgroup H' of H, then

$$\mathrm{Ind}_H^G \circ \mathrm{Ind}_{H'}^H = \mathrm{Ind}_{H'}^G.$$

For $\pi_V, \pi_W \in Rep(G)$, we denote by $\pi_V \otimes \pi_W$ the tensor product of π_V, π_W when F is non-Archimedean and the projective tensor product of π_V, π_W when F is Archimedean. Here the projective tensor product is the completion of the tensor product with respect to the projective cross norm and is equal to the injective tensor project as the underlying space of π_V, π_W are Fréchet. Then the functor $\otimes$ has the following properties.

Let $\pi_G \in Rep_{\mathrm{adm}}(G)$. From [BK14], π_G is a nuclear nuclear Fréchet space, then the following proposition follows from [CHM00] and [CS20, Proposition 7.4] when F is Archimedean. Over non-Archimedean local fields, the result holds with weaker conditions, by assuming that the $\pi_G \in Rep(G)$.

Proposition 3.2.2. *The following hold.*

1. *$\cdot \otimes \pi_G$ is an exact functor in Rep(G).*

2. *For $\pi_H \in Rep(H)$, we have*

$$\mathrm{Ind}_H^G(\pi_H \otimes_H \pi_G|_H) = \mathrm{Ind}_H^G(\pi_H) \otimes \pi_G.$$

In the following, we recall Berstein's notion of parabolic induction of reductive groups and generalize the notion to Jacobi groups. For finite-dimensional vector spaces $V_1 \subset V$ and $\mathbf{G} = \mathbf{GL}(V)$, let $\mathbf{P} = \mathbf{LN}$ be the parabolic subgroup $\mathbf{G}$ stabilizing V_1. Then Levi component $\mathbf{L}$ is isomorphic to $\mathbf{GL}(V_1) \times \mathbf{GL}(V/V_1)$. For $\sigma_1 \in \mathit{Rep}(\mathrm{GL}(V_1))$ and $\sigma_2 \in \mathit{Rep}(\mathrm{GL}(V_2))$, we denote by $\sigma_1 \times \sigma_2$ the parabolic induction

$$\mathrm{Ind}_P^G(\sigma_1 \boxtimes \sigma_2).$$

Here $\sigma_1 \boxtimes \sigma_2$ denotes the inflation of the exterior tensor $\sigma_1 \boxtimes \sigma_2$ of $L = \mathbf{L}(F)$ to $P = \mathbf{P}(F)$. From Proposition 3.2.1(2), this binary operator is associative, that is,

$$(\sigma_1 \times \sigma_2) \times \sigma_3 = \sigma_1 \times (\sigma_2 \times \sigma_3). \tag{3.2.1}$$

Let $\mathcal{T} = \{\mathbf{G}_V\}_{V \in \mathcal{I}}$ be one of the towers $\mathcal{T}_1, \mathcal{T}_2$ any cases given in Section 2.1. For $V \in \mathcal{I}$ and a totally isotropic subspace X of V, then there exists nondegenerate V_0 such that $V = X \perp V_0 \perp X^*$. Let $\mathbf{P}$ be the parabolic subgroup of $\mathbf{G}$ stabilizing X, and the Levi component of $\mathbf{P}$ is isomorphic to

$$\begin{cases} \mathbf{GL}(X) \times \mathbf{G}_{V_0} & \text{when } \mathbf{G}_{V_0} \text{ is not metaplectic,} \\ \widetilde{\mathbf{GL}}(X) \times_{\pm 1} \mathbf{G}_{V_0} & \text{when } \mathbf{G}_{V_0} \text{ is metaplectic.} \end{cases}$$

Here $\widetilde{\mathbf{GL}}(V)$ denotes the algebraic group on $\mathbf{GL}(V) \times \{\pm 1\}$ with the multiplication rule given by

$$(g_1, \delta_1) \cdot (g_2, \delta_2) = (g_1 g_2, \delta_1 \delta_2 (\det(g_1), \det(g_2))_F), \quad g_1, g_2 \in \mathbf{GL}(V), \quad \delta_1, \delta_2 \in \{\pm 1\},$$

where $(\cdot, \cdot)_F$ is the Hilbert symbol. For $\sigma \in \mathit{Rep}(\mathrm{GL}(X))$ and $\pi_{V_0} \in \mathit{Rep}(G_{V_0})$, we denote by $\sigma \ltimes \pi_{V_0}$ the parabolic induction

$$\begin{cases} \mathrm{Ind}_P^G(\sigma \boxtimes \pi_{V_0}) & \text{when } \mathbf{G}_{V_0} \text{ is not metaplectic,} \\ \mathrm{Ind}_P^G(\sigma \boxtimes_{\pm 1} \pi_{V_0}) & \text{when } \mathbf{G}_{V_0} \text{ is metaplectic.} \end{cases}$$

Here $\sigma \boxtimes_{\pm} \pi_{V_0}$ denotes the inflation to $P = \mathbf{P}(F)$ of the $L = \mathbf{L}(F)$-representation that the exterior tensor of σ_1, π_{V_0} factors through. From Proposition 3.2.1(2), we have

$$(\sigma_1 \times \sigma_2) \ltimes \pi_{V_0} = \sigma_1 \times (\sigma_2 \ltimes \pi_{V_0}). \tag{3.2.2}$$

Therefore, we omit the bracelets when writing parabolic inductions in (3.2.1)(3.2.2).

In [Ch23a], when studying Fourier-Jacobi models, the author defined a Schwartz induction of the Jacobi group such that this induction has parallel properties as the parabolic induction of the reductive group on the Bessel side.

Let $\{\mathbf{G}_V\}_{V\in\mathcal{I}}, \{\mathbf{H}_W\}_{W\in\mathcal{I}'}$ be the towers $\mathcal{T}_1, \mathcal{T}_2$ in Fourier-Jacobi cases given in Section 2.1. Recall that the Jacobi group $\mathbf{G}_V^J = \mathbf{G}_V \ltimes \mathcal{H}(V)$ and we set up a parabolic type structure on $\mathbf{G}_V^J$. For a totally isotropic subspace X of V, there is a decomposition $V = X \oplus V_0 \oplus X^*$. We take $\mathbf{P} = \mathbf{L} \ltimes \mathbf{N}$ the parabolic subgroup of $\mathbf{G}_V$ stabilizing X and $X^\perp = V_0 \oplus X^*$. We define

$$\mathbf{P}^J = \mathbf{P} \ltimes \mathcal{H}(X^\perp).$$

When $\mathbf{G}_V$ is metaplectic, we denote by $\widetilde{\mathrm{GL}}(X^+)$ the algebraic group over $\mathbb{R}$ on the space $\mathbf{GL}(X^+) \times \{\pm 1\}$ with the multiplication rule given by

$$(g_1, \delta_1) \cdot (g_2, \delta_2) = (g_1 g_2, \delta_1 \delta_2 (\det(g_1), \det(g_2))_{\mathbb{R}}), \quad g_1, g_2 \in \mathrm{GL}(X^+), \quad \delta_1, \delta_2 \in \{\pm 1\},$$

where $(\cdot,\cdot)_{\mathbb{R}}$ is the Hilbert symbol. Then $\widetilde{\mathrm{GL}}(X^+)$ is a metaplectic cover of $\mathrm{GL}(X^+)$.

Let

$$\mathbf{L}^J = \mathbf{L} \ltimes \mathcal{H}(W) = \begin{cases} \mathrm{Res}_{E/F}\mathbf{GL}(X^+) \times \mathbf{G}_W^J, \text{ if } \mathbf{G}_W \text{ is not metaplectic,} \\ \widetilde{\mathbf{GL}}(X^+) \times_{\pm 1} \mathbf{G}_W^J, \text{ if } \mathbf{G}_W \text{ is metaplectic,} \end{cases}$$

and

$$\mathbf{N}^J = \mathbf{N} \ltimes \mathrm{Res}_{E/F}\mathbf{X},$$

where $\mathbf{X} = \mathbb{G}_a^{\dim_E X}$ and $\mathbf{X}(E) = X$ and $X = \mathbf{X}(E) = \mathrm{Res}_{E/F}\mathbf{X}(F)$. Then the group $\mathbf{P}^J$ has a decomposition

$$\mathbf{P}^J = \mathbf{L}^J \ltimes \mathbf{N}^J.$$

For $\sigma \in \mathit{Rep}(\mathbf{GL}(X))$ and $\pi_{V_0}^J \in \mathit{Rep}(\mathbf{G}_{V_0}^J(F))$, we denote by $\sigma \ltimes \pi_{V_0}$ the parabolic induction

$$\begin{cases} \mathrm{Ind}_P^G(\sigma \boxtimes \pi_{V_0}^J) & \text{when } \mathbf{G}_{V_0} \text{ is not metaplectic,} \\ \mathrm{Ind}_P^G(\sigma \boxtimes_{\pm 1} \pi_{V_0}^J) & \text{when } \mathbf{G}_{V_0} \text{ is metaplectic.} \end{cases}$$

This notion is compatible with the parabolic induction in the sense of the following lemma, which follows from the computation in [LS13, p10] using mixed models.

Lemma 3.2.3. *Let $\pi_{V_0} \in Rep(G_{V_0})$ and $\sigma \in Rep(\mathrm{GL}(X))$, we have*

$$(\sigma \ltimes \pi_{V_0}) \otimes \omega_{V,\psi} = \sigma \ltimes (\pi_{V_0} \otimes \omega_{V_0,\psi}).$$

3.3 Extended analysis to homology groups

This section reviews a Schwartz homology approach introduced in [CS20] and developed in [X2]. In this thesis, the Schwartz homology method is only applied to prove the reduction to basic cases (Section 5.4) over Archimedean local fields. The non-Archimedean proof for reduction to basic cases uses representation theory in terms of Bernstein components, which provides more precise information than the homology theory. Therefore, we assume F is Archimedean in this section.

Definition and basic properties

Recall that the multiplicity is defined as the dimension of a Hom-space $\mathrm{Hom}_H(\pi, \xi)$, which is equal to

$$\dim \mathrm{Hom}_H(\pi \otimes_H \xi^{-1}, 1_H) = \dim \mathrm{Hom}((\pi \otimes_H \xi^{-1})_H, \mathbb{C}),$$

where $(\cdot)_H$ denotes the H-coinvariant defined by the non-Hausdorff quotient

$$V_H = V / \sum_{h \in H} (h-1).V \tag{3.3.1}$$

The idea of the homological approach study the homological group of the coinvariant functor $(\cdot)_H$.

Definition 3.3.1. *Let $G \in Grp$. For $V \in Rep(G)$, the Schwartz homology $\mathrm{H}_i^{\mathcal{S}}(G, V)$ is defined to be the left derived functors of the G-coinvariant functor $V \mapsto V_G$. In particular, $\mathrm{H}_0^{\mathcal{S}}(G, V) = V_G$.*

By definition, we have the following equation that builds a bridge between Schwartz homologies and multiplicities.

$$\mathrm{Hom}_G(\mathrm{H}_0(G, V), 1_G) = \mathrm{Hom}_G(V, 1_G). \tag{3.3.2}$$

The following proposition is called Shapiro's lemma for Schwartz inductions and Schwartz homologies ([CS20, Theorem 7.5]).

Proposition 3.3.2. *Let H be a closed Nash subgroup of G and $\pi_H \in Rep(H)$*

$$\mathrm{H}_i^{\mathcal{S}}(G, \delta_G^{-1} \otimes \mathrm{Ind}_H^G(\delta_H \otimes \pi_H)) = \mathrm{H}_i^{\mathcal{S}}(H, \pi_H)$$

Vanishing results

Definition 3.3.3. *We say the Schwartz homologies of $\pi \in Rep(G)$ vanish, if*

$$\mathrm{H}_i^{\mathcal{S}}(G, \pi) = 0, \quad i = 0, 1, \cdots$$

The lemma below follows from the long exact sequence of the Schwartz homologies

Proposition 3.3.4. *Given a short exact sequence*

$$0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0, \quad \pi_1, \pi_2, \pi_3 \in Rep(G),$$

if the Schwartz homologies of two of $\pi_.$'s vanish, those of the remaining $\pi_.$ vanish.

Definition 3.3.5.

- *A **well-ordered** index set I is a total ordered set satisfying that every non-empty subset has a least element in this ordering. The **successor** α^+ of an element α is the least element of the subset*

$$\{\beta \in I, \beta > \alpha\}.$$

- *Let V be a Fréchet space, a **descending filtration** is a set of subspaces $\{V_\alpha\}_{\alpha \in I}$ satisfying*

$$V_\alpha \subset V_\beta \text{ for } \alpha > \beta \in I.$$

  *The descending filtration is **complete** if*

$$V \mapsto \varprojlim V/V_\alpha$$

  is an isomorphism of topological vector spaces.

- *The **graded pieces** are*

$$V_\alpha/V_{\alpha^+}, \alpha \in I.$$

Together with [X2, Lemma 2.12, Proposition 2.13], we obtain the following result.

Proposition 3.3.6. *Given representations $\pi \in Rep(G)$, $\pi' \in Rep_{\mathrm{adm}}(G)$ and a complete descending filtration of closed subspace π_α of π index by countably well-order set I, with graded pieces $\pi_{\alpha+}/\pi_\alpha$, suppose the Schwartz homologies of $(\pi_{\alpha+}/\pi_\alpha)\widehat{\otimes}\pi'$ vanish, then the Schwartz homologies of $\pi\widehat{\otimes}\pi'$ vanish.*

Chapter 4

Integral Method

This section reviews the classical integral method and applies it to Rankin-Selberg-type integrals and Igusa-type integrals. The idea of the integral method can be stated as follows. For $\pi_G \in Rep(G)$ and a linear function $f : \pi_G \to \mathbb{C}$. If the integral

$$f_G(g) = \int_G f(\pi(g)v)dg$$

is convergent and nonzero, it constructs a nonzero element in $\mathrm{Hom}_G(\pi_G, 1_G)$. However, the convergence and nonvanishing of the integral are not guaranteed in general. The classical approach to resolve the problem is to construct a family of integrals over G, prove meromorphic continuation, and then take the principal term of the meromorphic family. The principal term is nonzero and its G-invariance is guaranteed with the following result.

Lemma 4.0.1 (Lemma 2.2 of [GSS19]). *Let ψ be a character of G and F_s be a nonzero meromorphic family on π_G satisfying $\pi_G(g)F_s = |\psi(g)|^s F_s$. For every $s_0 \in \mathbb{C}$, there is a Laurant expansion*

$$F_s = \sum_{i=-n}^{\infty} a_{i,s_0}(s - s_0)^i \in E((s))$$

the term a_{-n,s_0} is G-invariant and is called the ***principal term*** *at $s = s_0$.*

There are two classical methods of constructing families of integrals satisfying the assumption of Lemma 4.0.1, construction by matrix coefficients and construction by equivariant functionals. When applied to studying zeta integrals related to standard L-functions of general linear groups, one gives Godement-Jacquet zeta integrals ([GJ06]) and the other gives Rankin-Selberg integrals ([JPSS83]).

In [MW12], Mœglin and Waldspurger proved "the second inequality" (Theorem 5.0.4(3)) using integrals constructed with matrix coefficients. In [JSZ10], Jiang-Sun-Zhu used an integral that can be regarded as a composition of a Rankin-Selberg-type integral and Igusa zeta integral([Ch23a]). This section provides a more general description and provides proof for meromorphic continuation for Rankin-Selberg-type integrals and Igusa zeta integrals.

4.1 Rankin-Selberg-type integrals

This section follows [JPSS83] [Sou93] [Jac09] to conceptualize and generalize the proof for the absolute convergence and meromorphic continuation of some Rankin-Selberg integrals.

4.1.1 Formulation

The first method to construct a family of integrals is the classical approach to constructing Rankin-Selberg integrals.

For a general linear group $\mathbf{GL}_r$, we denote by $\mathbf{N}_r$ is the maximal unipotent subgroup consisting of upper triangular unipotent matrices. Fix an additive character ψ_F of F, we denote by ψ_r the character of $N_r = \mathbf{N}_r(F)$ defined by

$$\psi_r(n) = \psi_F(\sum_{i=1}^{r-1} n_{i,i+1}), n \in N_r,$$

where $n_{i,i+1}$ denotes the $(i, i+1)$-entry of the matrix of n.

In [JPSS83], for generic representations π_m, π_n of $GL_m = \mathbf{GL}_m(F), GL_r = \mathbf{GL}_r(F)$ $(m = r+1)$, the Rankin-Selberg integral on $GL_m \times GL_r$ is defined as

$$\int_{N_r \backslash GL_r} \mathrm{W}_{v_m}(g)\overline{\mathrm{W}_{v_r}(g)}|\det(g)|^s dg,$$

where $v_m \in \pi_m$, $v_r \in \pi_r$ and the Whittaker functions $\mathrm{W}_{v_m} : GL_m \to \mathbb{C}, \mathrm{W}_{v_r} : GL_r \to \mathbb{C}$ are given by

$$\mathrm{W}_{v_m}(g) = \lambda_m(\pi_m(g)v_m), \mathrm{W}_{v_r}(g) = \lambda_r(\pi_n(g)v_r).$$

where λ_m, λ_r are Whittaker functionals of π_m, π_r with respect to (N_m, ψ_m), (N_r, ψ_r).

Let $\|\cdot\|$ be the norm on GL_r defined by

$$\|g\| = \max_{i,j=1,\cdots r}(|g_{i,j}|, \det(g)^{-1}).$$

Proposition 4.1.1. *Every family of Whittaker functions* $\{\mathrm{W}_v\}_{v\in V_\pi}$ *of a representation* $(\pi, V_\pi) \in$ *Rep*(GL_r) *has the following properties.*

1. *(a) When F is Archimedean, there are constants $C, N > 0$ and a continuous semi-norm ν on V_π such that*

   $$|\mathrm{W}_v(g)| \leqslant C\|g\|^N\nu(v), \quad g \in GL_r, v \in V_\pi.$$

   (b) When F is non-Archimedean, for every $v \in \pi$, there is a constant $C_v > 0$ such that

   $$|\mathrm{W}_v(g)| \leqslant C_v\|g\|^N.$$

2. *For $g \in GL_r$ and $n \in N_r$, we have*

   $$\mathrm{W}_v(ngg') = \psi_r(n)\mathrm{W}_{\pi(g')v}(g), \quad v \in V_\pi, n \in N_r, g, g' \in GL_r.$$

Based on these properties, we can formulate the following definition for Whittaker-type functions.

Definition 4.1.2. *We call a family of functions* $\{\mathrm{W}'_v\}_{v\in V_\pi}$ *on a representation* $(\pi, V_\pi) \in$ *Rep*$(\mathbf{GL}_r(F))$ *is **Whittaker-type** if it satisfies the following conditions*

1. *(Modernate growth)*

   (a) When F is Archimedean, there are constants $C, N > 0$ and a continuous semi-norm ν of V_π such that

   $$|\mathrm{W}'_v(g)| \leqslant C\|g\|^N\nu(v), \quad g \in GL_r, v \in V_\pi. \tag{4.1.1}$$

   (b) When F is non-Archimedean, for every $v \in \pi$, there is a constant $C_v > 0$ such that

   $$|\mathrm{W}'_v(g)| \leqslant C_v\|g\|^N, \quad g \in GL_r, v \in V_\pi.$$

2. (Whittaker equivariance), we have

$$\mathrm{W}'_v(ngg') = \psi_n(n)\mathrm{W}'_{\pi(g')(v)}(g), \quad v \in V_\pi, n \in N_m, g, g' \in GL_m. \tag{4.1.2}$$

In Definition 5.5.2, we will see examples of Whittaker-type functionals defined by Bessel and Fourier-Jacobi models.

Let $(\pi, V_\pi) \in \mathit{Rep}_{\mathrm{adm}}(GL_{r+1})$, $(\pi', V_{\pi'}) \in \mathit{Rep}(GL_r)$ and π is generic. Fix a Whittaker functional λ of π and a family of Whittaker-type functions $\{\mathrm{W}'_{v'}\}_{v' \in V_{\pi'}}$, the **Rankin-Selberg type integral** is defined

$$\mathrm{J}_{v,v'}(s) = \int_{N_r \backslash GL_r} \lambda(\pi(g)v)\overline{\mathrm{W}'_{v'}(g)}|\det(g)|^s dg, \quad v \in V_\pi, v' \in V_{\pi'}. \tag{4.1.3}$$

Using a stronger estimate of the restriction to GL_r of the Whittaker functions of GL_{r+1}, one can prove the absolute convergence of the integral.

Theorem 4.1.3. *Given Whittaker functional λ and a family of Whittaker-type function $\{\mathrm{W}'_{v'}\}_{v' \in V_{\pi'}}$, the integral $\mathrm{J}_v(s)$ defined in* (4.1.3) *is absolutely convergent when* $\mathrm{Re}(s)$ *is large enough.*

Proof. Let $\mathbf{A}_r$ be the subgroup of $\mathbf{GL}_r$ consisting of diagonal matrices. We consider the Iwasawa decomposition

$$GL_r = N_r A_r K_r,$$

where K_r is the maximal compact subgroup of GL_r and $A_r = \mathbf{A}_r(F_0)$ for

$$F_0 = \begin{cases} \mathbb{R} & \text{when } F \text{ is Archimedean,} \\ F & \text{when } F \text{ is non-Archimedean.} \end{cases} \tag{4.1.4}$$

Then

$$\begin{aligned} \mathrm{J}_{v,v'}(s) &= \int_{N_r \backslash GL_r} \lambda(\pi(\mathrm{diag}(g,1))v)\mathrm{W}'_{v'}(g)|\det(g)|^s dg \\ &= \int_{A_r K} \int_{N_r \backslash GL_r} \lambda(\pi(\mathrm{diag}(g,1))v)\mathrm{W}'_{v'}(g)|\det(g)|^s dg \end{aligned}$$

- When F is Archimedean, following [Jac09, Section 3.2], we define

$$\xi_h(g) = \prod_{i=1}^{r-1}(1 + (a_i a_{i+1}^{-1})^2),$$

where $a = \mathrm{diag}(a_1, \cdots, a_r) \in A_r$. Then from [Jac09, Proposition 3.1], there exists $C, M, N > 0$, such that

$$|\lambda(\pi(\mathrm{diag}(ak, 1))v)| \leqslant C \cdot \xi_h(a)^{-M} \|a\|^N, \quad a \in A_r, k \in K$$

From (4.1.1), there exists seminorm ν, constants $C', N' > 0$ such that

$$|\mathrm{W}'_{v'}(\mathrm{diag}(ak, 1))| \leqslant C' \|a\|^{N'} \nu(v) \quad a \in A_r, k \in K.$$

When $\mathrm{Re}(s)$ is large enough, the absolute convergence of the integral follows from [Jac09, Lemma 3.5].

- When F is non-Archimedean, from [JPSS79, Lemma 2.3.5], there is a positive quasi-character χ of A_r and a positive element ϕ in $C_c^\infty(F^r)$ such that

  $$|\mathrm{W}_v(\mathrm{diag}(ak, 1))| \leq \chi(a)\phi(a_1 a_2^{-1}, \cdots, a_{r-1} a_r^{-1}, a_r), \quad a = \mathrm{diag}(a_1, a_2, \cdots, a_r).$$

  From (4.1.1), for every $v' \in V_{\pi'}$, there is constants $C'_{v'}, N' > 0$ such that

  $$|\mathrm{W}'_{v'}(\mathrm{diag}(ak, 1))| \leqslant C'_{v'} \|a\|^N, \quad a \in A_r, k \in K.$$

  When $\mathrm{Re}(s)$ is large enough, the absolute convergence of the integral follows from the proof for [JPSS83, Theorem 2.7].

□

Theorem 4.1.4. *When $\lambda, \mathrm{W}'_{v'}$ are nonzero, then there exists $v \in \pi$, such that the integral $\mathrm{J}_{v,v'}$ defined in* (4.1.3) *is nonzero.*

Proof. We take a submanifold U of GL_r such that

$$N_r \times U \to GL_r$$

is an open embedding. For a smooth compactly supported function ϕ on A_r, we set

$$\mathrm{W}_\phi(g) = \begin{cases} \psi_r(n)\phi(u), & \text{if for } n \in N_r, u \in U \\ 0, & \text{otherwise.} \end{cases}$$

Then W_ϕ is a smooth function on GL_r with compact support modulo N_r such that

$$\mathrm{W}_\phi(nm) = \psi_r(n)\mathrm{W}_\phi(m), n \in N_r, m \in GL_r.$$

Then, from [JS81, Section 3] (see also [Co04, Section 4]), there exists $v_\phi \in V_\pi$ such that

$$\lambda(\pi(\mathrm{diag}(g,0))v_\phi) = \mathrm{W}_\phi(g), g \in GL_r.$$

The integral

$$\begin{aligned}\mathrm{J}_{v_\phi,v'}(s) &= \int_{N_r\backslash GL_r} \lambda(\pi(\mathrm{diag}(g,1))v_\phi)\overline{\mathrm{W}'_{v'}(g)}|\det(g)|^s dg \\ &= \int_U \phi(u)\mathrm{W}'_{v'}(u)|\det(u)|^s du\end{aligned}$$

Therefore, we can find a $\phi \in C_c^\infty(U)$ such that $\mathrm{J}_{v_\phi,v'}(s) \neq 0$. □

4.1.2 Meromorphic continuation

This section introduces the admissibility of Whittaker-type functionals and proves the meromorphic continuation of the integral following [JPSS83] [JS90].

Definition 4.1.5. *Let F_0 be the field defined in* (4.1.4). *We call a Whittaker-type functional* W'_v *is* ***admissible*** *if* W'_v *has an expansion*

$$\mathrm{W}'_v(a) = \sum_{\xi\in\Sigma'} \phi_{\xi,v}(a_1,\cdots,a_n)\xi(a_1,\cdots,a_n), \tag{4.1.5}$$

where Σ' is a finite set of finite functions of F_0^r that is only dependent on π_V, π_W, and ϕ_ξ are Schwartz functions in $\mathcal{S}(F_0^r)$.

Theorem 4.1.6. *When $\mathrm{W}'_{v'}$ is admissible for all $v' \in V_{\pi'}$, the integral $\mathrm{J}_{v,v'}$ defined in* (4.1.3) *admits a meromorphic continuation to the whole complex plane $\mathbb{C}$.*

The proof for the theorem uses the results on the Mellin transforms of $F_0^\times$-finite functions, we refer the readers to [I78, Chapter 1] for details. The following lemma is a result of the discussion in [I78, §1.4,§1.5].

Lemma 4.1.7. *1. Every $(F_0^\times)^n$-finite function is a linear combinations of*

$$\prod_{i=1}^m \chi_i(a_i)|a_i|^{s_i}(\log|a_i|)^{n_i}, a_i \in F_0^\times,$$

where χ_i are unitary characters of $F^\times$, $s_i \in \mathbb{R}$ and $n_i \in \mathbb{N}$.

2. *A finite product of $(F_0^\times)^m$-finite functions is $(F_0^\times)^m$-finite.*

Definition 4.1.8. 1. *For every finite function ξ, we denote by $\mathfrak{o}(\xi)$ the tuple $\{(\chi_i, s_i, n_i)\}_{1\leqslant i\leqslant r}$ such that*

$$\xi(a_1, \cdots, a_r) = \prod_{i=1}^{m} \chi_i(a_i)|a_i|^{s_i}(\log|a_i|)^{n_i}.$$

2. *With the notations above, we denote by p_ξ the polynomial*

$$p_\xi(s) = \prod_{i=1}^{m}(s+s_i)^{n_i}$$

when F is Archimedean and the polynomial

$$p_\xi(s) = \prod_{i=1}^{m}(1-q^{s+s_i})^{n_i}$$

when F is non-Archimedean.

For a Schwartz function $\Phi \in \mathcal{S}((F_0^\times)^m)$ and a tuple $\{(\chi_i, s_i, n_i)\}_{1\leqslant i\leqslant r}$, we define the following Mellin transform

$$\mathcal{J}^{\{(\chi_i,s_i,n_i)\}_{1\leqslant i\leqslant m}}(s,\Phi) = \int_{(F_0^\times)^m} \Phi(a_1,\cdots,a_m)\prod_{i=1}^{m}\chi_i(a_i)|a_i|^{s+s_i}(\log|a_i|)^{n_i}da. \tag{4.1.6}$$

Here

$$da = da_1^\times \cdots da_m^\times = \frac{da_1}{|a_1|}\cdots\frac{da_m}{|a_m|}. \tag{4.1.7}$$

Proposition 4.1.9. *For a Schwartz function $\Phi \in \mathcal{S}((F^\times)^m)$ and a tuple $\{(\chi_i, s_i, n_i)\}_{1\leqslant i\leqslant r}$, the integral*

$$p_\xi(s)\mathcal{J}^{\{(\chi_i,s_i,n_i)\}_{1\leqslant i\leqslant r}}(s,\Phi)$$

has an analytic continuation to an entire function.

Proof. When Re(s) large enough, $\prod_{i=1}^{m}\chi_i(a_i)|a_i|^{s+s_i}(\log|a_i|)^{n_i}$ is a function of modernate growth on $(F_0^\times)^r$. Hence, the integral (4.1.6) defines a continuous functional on $\mathcal{S}((F_0^\times)^r)$, so we may assume

$$\Phi(a_1,\cdots,a_r) = \prod_{i=1}^{r}\Phi_i(a_i),$$

then

$$\mathcal{J}^{\{(\chi_i,s_i,n_i)\}_{1\leqslant i\leqslant r}}(s,\Phi)=\prod_{1\leqslant i\leqslant r}\mathcal{J}^{\{(\chi_i,s_i,n_i)\}}(s,\Phi_i).$$

From [I78, §1.4.2], for every $1\leqslant i\leqslant r$, $(s+s_i)^{n_i}\mathcal{J}^{\{(\chi_i,s_i,n_i)\}}(s,\Phi_i)$ can be extended to an entire function on $s\in\mathbb{C}$. From [I78, §1.5.3], for every $1\leqslant i\leqslant r$, $(1-q^{s+s_i})^{n_i}\mathcal{J}^{\{(\chi_i,s_i,n_i)\}}(s,\Phi_i)$ can be extended to an entire function on $s\in\mathbb{C}$. This proves the existence of analytic continuation of $p_\xi(s)\mathcal{J}^{\{(\chi_i,s_i,n_i)\}_{1\leqslant i\leqslant r}}(s,\Phi)$ to the whole complex plane $\mathbb{C}$. □

From [JS90, Proposition 4.1], we have an expansion of the Whittaker functions on $A_r\times 1\subset GL_{r+1}$:

$$\mathrm{W}_v(\mathrm{diag}(a,1))=\sum_{\xi\in\Sigma}\phi_{\xi,v}(a_1,\cdots,a_r)\xi(a_1,\cdots,a_r),\quad a=\mathrm{diag}(a_1,\cdots,a_r),\tag{4.1.8}$$

where Σ is a finite set of $(F_0^\times)^r$-finite functions that is only dependent on π, and $\phi_{\xi,v}$ are Schwartz functions in $\mathcal{S}((F_0^\times)^r)$.

Theorem 4.1.10. $\mathrm{J}_{v,v'}$ *has an meromorphic continuation to* $s\in\mathbb{C}$.

Proof. We denote by $\widetilde{\mathrm{J}}_{v,v'}(s)$ the integral

$$\int_{A_r}\mathrm{W}_v(\mathrm{diag}(a,1))\mathrm{W}'_{v'}(a))|\det(a)|^s da.$$

Then we expand the Whittaker functional and the Bessel functional using (4.1.8) and (4.1.5) and obtain that

$$\widetilde{\mathrm{J}}_{v,v'}(s)=\sum_{\xi\in\Sigma,\xi'\in\Sigma'}\int_{A_r}\phi_{\xi,v}\phi_{\xi',v'}\xi\xi'(a_1,\cdots,a_r)|\det(a)|^s da.\tag{4.1.9}$$

From Proposition 4.1.9, the function

$$p_{\xi\xi'}(s)\int_{A_r}\phi_{\xi,v}\phi_{\xi',v'}\xi\xi(a_1,\cdots,a_r)|\det(a)|^s da$$

has an analytic continuation and we denote it by $\Psi_{\xi,\xi',v,v'}(s)$, then

$$\widetilde{\mathrm{J}}_{v,v'}(s)=\sum_{\xi\in\Sigma,\xi'\in\Sigma'}p_{\xi\xi'}(s)^{-1}\Psi_{\xi,\xi',v,v'}(s)$$

From the Fubini's theorem, when $\mathrm{Re}(s)$ is large enough, we have

$$\begin{aligned}\mathrm{J}_{v,v'}(s) &= \int_{K_r}\int_{A_r} \mathrm{W}_v(\mathrm{diag}(ak,1))\mathrm{W}'_{v'}(ak)|\det(ak)|^s dadk \\ &= \int_{K_r} \widetilde{\mathrm{J}}_{\pi(\mathrm{diag}(k,1))v,\pi'(k)v'}(s)dk \\ &= \sum_{\xi\in\Sigma,\xi'\in\Sigma'} p_{\xi_{X^+},\xi_{V,W}}(s)^{-1}\int_{K_r} \Psi_{\xi,\xi',\pi(\mathrm{diag}(k,1))v,\pi'(k)v'}(s)dk.\end{aligned}$$

Since K_r is compact, and, for $k \in K_r$, the functions

$$\Psi_{\xi,\xi',\pi(\mathrm{diag}(k,1))v,\pi'(k)v'}$$

are entire, the integral

$$\int_{K_r} \Psi_{\xi,\xi',\pi(\mathrm{diag}(k,1))v,\pi'(k)v'}(s)dk$$

gives an entire function of s. Then $\mathrm{J}_{v,v'}$ has an meromorphic continuation to the whole complex plane. $\square$

4.2 Igusa-type integrals

This section introduces results in [GSS19] on the Igusa-type integrals. In this section, we assume F is an Archimedean local field.

4.2.1 Formulation

Let $\mathbf{G}$ be an algebraic group acting on an algebraic variety $\mathbf{X}$. Suppose the $\mathbf{G}$-action on $\mathbf{X}$ has a Zariski open orbit $\mathbf{U}$ and the complement $Z = X - U$ is the zero set of a regular function f on $\mathbf{X}$ that is (G, χ)-equivariant for a character χ of G. The integral

$$\int_U \Phi(x)dx, U \in \mathcal{S}(U)$$

is a continuous functional on $\mathcal{S}(U)$ and the integral extends to $\mathcal{S}(X)$ when $Re(s)$ is large enough using the following lemma (take $\Psi = \int_U dx$).

Lemma 4.2.1. *For every continuous functional Ψ on $\mathcal{S}(U)$,*

$$\mathrm{J}_{s,\Psi}(\varphi) = \Psi(\varphi|f|^s), \varphi \in \mathcal{S}(U)$$

can be extended to $\mathcal{S}(X)$ when $Re(s)$ is large enough.

Proof. From [AG08], we have

$$\mathcal{S}(U) = \varprojlim_n \mathcal{S}_{Z,n}(X),$$

where the space $\mathcal{S}_{Z,n}(X)$ consists of Schwartz functions on X such that all their k-th derivatives vanish on $\mathcal{Z}$ for $k \leqslant n$.

Hence,

$$\Psi \in \mathcal{S}(U)^* = \varinjlim_n \mathcal{S}_{Z,n}(X).$$

and then there is $n \geqslant 0$ such that $\Psi \in \mathcal{S}_{Z,n}(X)$. $\square$

Definition 4.2.2. *For a Fréchet space V, we denote by $\mathcal{S}(X, V)$ the space of V-valued Schwartz functions on X. We define $\mathcal{S}^*(X, V)$ to be its continuous linear dual space. From [GSS19, Proposition 2.15], we have*

$$\mathcal{S}^*(X, V) = \mathcal{S}^*(X)\widehat{\otimes}V^*$$

4.2.2 Results on holonomic D-modules

This section introduces results on holonomic D-modules and we refer the reader to [HT07] for details.

Definition 4.2.3.

- *The D-**module** on a smooth affine algebraic variety $\mathbf{X}$ is a module of the algebra of differential operators $D(\mathbf{X})$.*
- *The algebra $D(\mathbf{X})$ has a natural graded structure that induces graded structure on any D-module M. Define the **singular support** $SS(M)$ to be the support of the associated graded module.*
- $\dim SS(M) \geqslant n$ *and we call M holonomic if* $\dim SS(M) = n$.

Definition 4.2.4. *A distribution $\xi \in \mathcal{S}^*(X, V)$ is called holonomic if the submodule $\xi D(X) \subset \mathcal{S}^*(X, V)$ generated by ξ is holonomic.*

Theorem 4.2.5 (Theorem of [GSS19]). *If ξ is holonomic, then the family (X, V) defined by $\xi_\lambda = \xi p^\lambda$ for $\lambda \in \mathbb{C} > N$ for some $N > 0$ has a meromorphic continuation to the entire complex plane. Moreover, all the distributions in the extended family and all the coefficients of the Laurent expansion at any $\lambda \in \mathbb{C}$ are holonomic.*

The following lemma helps determining whether a distribut

Lemma 4.2.6 (Lemma 3.3 of [GSS19]). *Let $\mathbf{Z} \subset \mathbf{X}$ be a closed smooth subvariety, let $\xi \in S^*(Z, V)$, and let $\eta \in S^*(X, V)$ be the extension of ξ to X by zero. Then*

- *η is holonomic if and only if ξ is holonomic.*
- *Let an algebraic group $\mathbf{G}$ act transitively on $\mathbf{Z}$, and its $\mathbb{R}$-points G act linearly on V. Suppose that ξ is G-equivariant and V is finite-dimensional.*

Then ξ is holonomic.

4.2.3 Meromorphic continuation

Then we state results in [GSS19] that prove the meromorphic continuation of the Igusa zeta integrals.

Let $QIF(G)$ be the full subcategory of $Rep^{\mathrm{adm}}(G)$ consisting of objects σ satisfying

(4.2.1) There is a subgroup $H \in Grp$ of G and a finite-dimensional representation ρ such that there is a sujection

$$\mathrm{Ind}_H^G \rho \twoheadrightarrow \sigma$$

Here Ind is the Schwartz induction as in Section 3.2.

Example 4.2.7. *1. Let $G \in Grp$ be a reductive Lie group. For every irreducible $\sigma \in Rep^{\mathrm{adm}}(G)$, there is a finite-dimensional representation ρ_0 of the minimal parabolic subgroup P_0 with a surjection*

$$\mathrm{Ind}_{P_0}^G \rho_0 \twoheadrightarrow \sigma.$$

Therefore, $\sigma \in QIF(G)$.

2. *Let $G = G_V^J = \mathbf{G}_V^J(F)$ for a Jacobi group $\mathbf{G}_V^J$ in Section 2.1. Given an irreducible representation $\sigma^J = \sigma \otimes \omega_{V,\psi}$, where $\sigma \in Rep^{\mathrm{adm}}(G_V)$, from part 1 and Lemma 3.2.3, there is a subgroup P_0^J of G_V^J surjection*

$$\mathrm{Ind}_{P_0^J}^{G_V^J}(\rho_0 \otimes \psi) \twoheadrightarrow \sigma^J.$$

Therefore, $\sigma^J \in QIF(G)$.

The following theorem is a simple generalization of [GSS19, Proposition 4.9].

Theorem 4.2.8. *Let F be an Archimedean local field, $\mathbf{G}$ be a linear algebraic F-groups and $\mathbf{H}, \mathbf{P} \subset \mathbf{G}$ be F-subgroups. Let $C = H \cap P$ and $\sigma \in QIF(P)$ with a sujection*

$$\mathrm{Ind}_Q^P \rho_Q \twoheadrightarrow \sigma. \tag{4.2.2}$$

1. *$G \backslash HP$ is the zero set of an $(H \times P, 1 \times \psi_F)$-invariant polynomial f of G;*
2. *$Q\backslash G/H$ has finitely many double cosets;*
3. *σ has a non-zero $(C, \delta_C \delta_H^{-1})$-functional.*

Then $\mathrm{Ind}_P^G(\sigma)$ admits a non-zero H-invariant continuous linear functional.

The proof follows from the proof for [GSS19, Proposition 4.7] that uses Bernstein's proof for the meromorphic continuation of a holonomic family of distributions.

Proof. From the surjection (4.2.2), we define an integral operator

$$\begin{aligned} I : \mathcal{S}(P, V) &\twoheadrightarrow \mathrm{Ind}_Q^P(\rho_Q) \\ I(a)(p) &:= \int_Q \rho_Q(q) a(pq) dq. \end{aligned}$$

Let τ be the kernel of I. We denote by τ to be the kernel of I and by tensoring with the nuclear Fréchet space $\mathcal{S}(G)$, we obtain an exact sequence ([CHM00, Lemma A.3])

$$0 \to \mathcal{S}(G)\widehat{\otimes}\tau \to \mathcal{S}(G)\widehat{\otimes}\mathcal{S}(P, \rho_Q) \to \mathcal{S}(G)\widehat{\otimes}\sigma \to 0.$$

Here we have

$$\mathcal{S}(G)\widehat{\otimes}\mathcal{S}(P, \rho_Q) = \mathcal{S}(G \times P, \rho_Q).$$

By dualizing the exact sequence, we obtain an exact sequence

$$0 \to \mathcal{S}^*(G,\sigma) \xrightarrow{\Phi} \mathcal{S}^*(G \times P, \rho_Q) \to \mathcal{S}^*(G,\tau) \to 0.$$

Then we obtain

$$\Phi : \mathcal{S}^*(G,\sigma) \hookrightarrow \mathcal{S}^*(G \times P, \rho_Q)$$

from the exact sequence.

By the Frobenius reciprocity, one has that

$$\mathrm{Hom}_H(\mathrm{Ind}_C^H(\sigma|_C), 1_H) \simeq \mathrm{Hom}_C(\sigma, \delta_C^{-1}\delta_H),$$

and the representation $\mathrm{Ind}_C^H(\sigma|_C) \subset \mathrm{Ind}_P^G(\sigma)$ can be regarded as the space of left (P,σ)-equivariant functionals on $\mathcal{S}(PH,\sigma)$, assumption 3 ensures that there is a nonzero element $\mathrm{Hom}_H(\mathrm{Ind}_C^H(\sigma|_C), 1_H)$ on this space. For $\varphi \in \mathcal{S}(G,\sigma)$,

$$\Psi(g) = \mu^+(\int_P \sigma(p)\varphi(pg)dp) \tag{4.2.3}$$

defines a continuous functional on $\mathcal{S}(PH,\sigma)$. Use Lemma 4.2.1, we define the Igusa zeta integral

$$\mathrm{J}_{s,\Psi}(\varphi) = \Psi(|f|^s\varphi)$$

when $Re(s)$ is large enough and we regard the integral as a family of elements in $\mathcal{S}^*(G,\sigma)$. We prove the meromorphic continuation of $\mathrm{J}_{s,\Psi}$ to $\mathbb{C}$ by showing that

1. The family $\Phi(\mathrm{J}_{s,\Psi})$ has a meromorphic continuation;
2. The meromorphic family lies in $\mathrm{Im}(\Phi)$.

From the assumption, $Q\backslash G/H$ has finitely many double cosets

$$Qx_iH, \quad i = 0, 1, \cdots, k$$

and we may assume Qx_0H is open. Then from Lemma 4.2.6(1), $\Phi(\mathrm{J}_{s,\Psi} \circ I)|_{Qx_0H}$ is holonomic, and then one can prove by induction from Lemma 4.2.6(2) that $\Phi(\mathrm{J}_{s,\Psi} \circ I)$ is holonomic. Therefore, Point (1) follows from Theorem 4.2.5.

Point (2) can be easily verified as in [GSS19, Corollary 4.9]. □

Chapter 5

Multiplicity Formula

This section aims to show how the methods in Chapter 3 and Chapter 4 can be applied to the multiplicity formula.

Let F be a local field of characteristic zero, and $\{\mathbf{G}_V\}_{V\in\mathcal{I}_1}, \{\mathbf{H}_W\}_{W\in\mathcal{I}_2}$ be a pair of towers defined in Section 2.1. The multiplicity formula aims to connect the multiplicities of representations in generic Vogan packets and the multiplicities of tempered representations. The following theorem suggests that representations in generic Vogan packets can be expressed as certain parabolic induction. The proof is in Section 6.3.1.

Theorem 5.0.1. *An irreducible representation π_V of G_V is in a generic Vogan packet if and only if there is a decomposition $V = V_0 \perp (X \oplus X^\vee)$ and*

$$\pi_V = \sigma_V \ltimes \pi_{V_0}$$

for $\pi_{V_0} \in \Pi_{F,\mathrm{temp}}(\mathbf{G}_{V_0})$ and $\sigma_V = |\cdot|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\cdot|^{s_{V,l_V}}\sigma_{V,l_V} \in \Pi_F(\mathbf{GL}(X))$, where $X = \bigoplus_{i=1}^{l_V} X_i$, $\sigma_{V,l_V} \in \Pi_{F,\mathrm{temp}}(\mathbf{GL}(X_i))$,$s_{V,i} \in \mathbb{C}$ and

$$\mathrm{Re}(s_{V,1}) \geqslant \cdots \geqslant \mathrm{Re}(s_{V,l_V}) > 0.$$

The multiplicity formula is designed to relate the multiplicity of parabolic inductions and that of their inducing data. Generally speaking, for representations of general linear groups

$$\sigma_V = |\cdot|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\cdot|^{s_{V,l_V}}\sigma_{V,l_V} \quad \sigma_W = |\cdot|^{s_{W,1}}\sigma_{W,1} \times \cdots \times |\cdot|^{s_{W,l_W}}\sigma_{W,l_W} \tag{5.0.1}$$

where $\sigma_{V,i}, \sigma_{W,i}$ are irreducible tempered representations of $\mathbf{GL}_{n_{V,i}}(E), \mathbf{GL}_{n_{W,i}}(E)$ and $s_{V,i}, s_{W,i} \in \mathbb{C}$ satisfying

$$\mathrm{Re}(s_{V,1}) \geqslant \cdots \geqslant \mathrm{Re}(s_{V,l_V}) > 0, \mathrm{Re}(s_{W,1}) \geqslant \cdots \geqslant \mathrm{Re}(s_{w,l_W}) > 0.$$

Let

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \pi_W = \sigma_W \ltimes \pi_{W_0},$$

where π_{V_0} is an irreducible tempered representations of G_{V_0} and π_{W_0} is

1. an irreducible tempered representation of H_{W_0} in Bessel cases;
2. equal to $\widetilde{\pi}_{W_0} \otimes \omega_{W_0,\psi}$, where $\widetilde{\pi}_{W_0}$ is a tempered representation in $Rep(H_{W_0})$ in Fourier-Jacobi cases.

Notice that, for Fourier-Jacobi cases, the multiplicity

$$m(\pi_V, \pi_W) = m(\pi_V \boxtimes (\widetilde{\pi}_W \otimes \omega_{W,\psi}))$$

was only defined in situations when $W \subset V$ in Section 2.1. In Section 5.1, we introduce a model to define the multiplicity $m(\pi_V \boxtimes (\widetilde{\pi}_W \otimes \omega_{W,\psi}))$ when $V \subset W$ such that

$$m(\pi_V \boxtimes (\widetilde{\pi}_W \otimes \omega_{W,\psi})) = m((\widetilde{\pi}_W \otimes \omega_{W,\psi}) \boxtimes \pi_V).$$

The distinguish these Fourier-Jacobi cases defined in different ways. We denote by (FJ 2) the Fourier-Jacobi cases when $V \subset W$ and denote by (FJ 1) the Fourier-Jacobi cases when $W \subsetneq V$. We also abbreviate the Bessel cases as (B).

The multiplicity formula can be stated uniformly for all cases as follows.

Theorem 5.0.2. *In the above setup, we have*

$$m(\pi_V \boxtimes \pi_W) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

We do not need to assume the irreducibility of π_V, π_W in the multiplicity formula. In [MW12], the multiplicity formula was proved using a mathematical induction using a mathematical induction from the basic forms.

Given a relevant pair (V, W), we set X^+ as a totally isotropic space

- such that the space $W^+ = W \perp (X^+ \oplus (X^+))^\vee$ contains V and $\dim_E W^+ - \dim_E V = 1$;
- such that the space $W^+ = W \perp (X^+ \oplus (X^+))^\vee$ is equal to V;
- such that the space $V^+ = V \perp (X^+ \oplus (X^+))^\vee$ contains W and $\dim_F V^+ - \dim_F W = 2$.

We associate to (V, W) another relevant pair

$$\begin{cases} (W^+, V) & \text{in (B) cases,} \\ (V, W^+) & \text{in (FJ 1) cases,} \\ (V^+, W) & \text{in (FJ 2) cases.} \end{cases} \tag{5.0.2}$$

We call the model defined by the pair in (5.0.2) **the basic case associated to** the model defined by (V, W).

The basic forms include an equality to reduce to basic cases and two inequalities that compare the multiplicity of a model and the multiplicity of the basic cases associated with it. More precisely, in most cases, we compare the multiplicity $m(\pi_V \boxtimes \pi_W)$ with the multiplicity for

$$\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W) = \begin{cases} (\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_V \boxtimes (\sigma_{X^+} \ltimes \pi_W) & \text{in (FJ 1) cases,} \\ (\sigma_{X^+} \ltimes \pi_V) \boxtimes \pi_W & \text{in (FJ 2) cases.} \end{cases}$$

Due to the significant difference between representation theory over Archimedean and non-Archimedean local fields, the basic form differs between the two types of local fields.

Definition 5.0.3. *We define* $\mathrm{LI}(\pi_V) = \mathrm{Re}(s_{V,1})$ *for* π_V *in the form of* $\sigma_V \ltimes \pi_{V_0}$, *where*

$$\sigma_V = |\cdot|^{s_{V,1}} \sigma_{V,1} \times \cdots \times |\cdot|^{s_{V,l_V}} \sigma_{V,l_V},$$

$\pi_{V,i}$ *tempered and* $\mathrm{Re}(s_{V,1}) \geqslant \cdots \geqslant \mathrm{Re}(s_{V,l_V}) > 0$. *From Theorem 5.0.1,* $\mathrm{LI}(\pi_V)$ *is well-defined for* π_V *in generic Vogan packets.*

Theorem 5.0.4 (non-Archimedean)**.** *When F is non-Archimedean, the basic form is the following.*

1. **(Reduction to basic cases)** *For a suppercuspidal representation* σ_{X^+} *of* $\mathrm{GL}(X^+)$ *such that* $\pi_V^\vee$ *does not belong to the Bernstein component associated to* $\sigma_{X^+} \boxtimes \pi_W$, *we have*

$$m(\pi_V \boxtimes \pi_W) = m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$$

2. **(Basic forms of the first inequality)** *For a tempered representation* σ_{X^+} *of* $\mathrm{GL}(X^+)$, *we have*

$$m(\pi_V \boxtimes \pi_W) \geqslant m(\mathrm{I}(|\det|^s \sigma_{X^+}, \pi_V, \pi_W))$$

when $\mathrm{Re}(s)$ *is larger than or equal to a parameter* $\mathrm{LI}(\pi_V)$ *of* π_V.

3. **(Basic forms of the second inequality)** *For a generic* σ_{X^+} *of* $Rep_{\mathrm{adm}}(\mathrm{GL}(X^+))$, *we have*

$$m(\pi_V \boxtimes \pi_W) \leqslant m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$$

Theorem 5.0.5 (Archimedean)**.** *When* F *is Archimedean, the basic form is the following.*

1. **(Reduction to basic cases)** *For spherical principal series representations* $|\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{\dim_E X}}$ *of* $\mathrm{GL}(X^+)$, *we have*

$$m(\pi_V \boxtimes \pi_W) = m(\mathrm{I}(\sigma_{X^+}, \pi_W, \pi_V))$$

for $(s_1, \cdots, s_{\dim_E X}) \in \mathbb{C}^{\dim_E X}$ *in general positions.*

2. **(Basic form of the first inequality)**

   (a) When $\dim_E X^+ = 1$,

$$m(\pi_V \boxtimes \pi_W) \geqslant m(\mathrm{I}(|\cdot|^s, \pi_W, \pi_V))$$

   when $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$.

   (b) When $E = \mathbb{R}$ *and* $\dim_E X = 2$,

$$m(\mathrm{I}'(|\cdot|^{s+\frac{m}{2}} \mathrm{sgn}^{m+1}, \pi_V, \pi_W)) \geqslant m(\mathrm{I}(|\det|^s D_m, \pi_V, \pi_W))$$

   when $\mathrm{Re}(s) + \frac{m}{2} \geqslant \mathrm{LI}(\pi_V)$, *where* I' *is defined in Definition 5.3.2.*

3. **(Basic form of the second inequality)** *For a generic σ_{X^+} of $Rep_{\mathrm{adm}}(\mathrm{GL}(X^+))$, we have*
$$m(\pi_V \boxtimes \pi_W) \leqslant m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$$

Due to the different formulations of the basic forms over two types of local fields. The proof of non-archimedean local fields can be established parallelly and has been elaborated in [MW12] [GI16] [CJLZ23] for all cases.

5.1 Some notions for the uniform formulation

In [Ch23a], the author pointed out the Fourier-Jacobi cases could be treated parallelly as the Bessel cases by changing one of the towers into a tower of Jacobi groups. Let $\mathcal{T}_1 = \{\mathbf{G}_V\}_{V \in \mathcal{I}_1}, \mathcal{T}_2 = \{\mathbf{H}_W\}_{W \in \mathcal{I}_2}$ be towers in Section 2.1. In Fourier-Jacobi cases, we have $\mathcal{I}_1 = \mathcal{I}_2$, and we denote it by $\mathcal{I}$. We consider the pair of towers as
$$\mathcal{T}_1 = \{\mathbf{G}_V\}_{V \in \mathcal{I}}, \mathcal{T}_1^J = \{\mathbf{G}_V^J\}_{W \in \mathcal{I}},$$
or
$$\mathcal{T}_2 = \{\mathbf{H}_V\}_{V \in \mathcal{I}}, \mathcal{T}_2^J = \{\mathbf{H}_W^J\}_{W \in \mathcal{I}}.$$

Without loss of generality, we may assume G_V is not metaplectic, in this case, we consider towers $\mathcal{T}_1, \mathcal{T}_1^J$. Then the groups in the towers are algebraic groups over F.

For $\pi_V \in \Pi_F(\mathbf{G}_V)$ and $\pi_W \in \Pi_F(\mathbf{G}_V^J)$, the multiplicity $m(\pi_V \boxtimes \pi_W)$ is defined in Section 2.1 when $W \subset V$. Notice that the situation when $W = V$ is defined differently. We extend the situation to the cases when $V \subset W$ by introducing a family of models.

For $V \subset W \in \mathcal{I}$ with decomposition $V = W \perp (X \oplus X^\vee)$. Choose a F-basis $\{z_i\}_{i=1}^r$ of X and its dual basis $\{z_{-i}\}_{i=1}^r$ of $X^\vee$, we have
$$\langle z_i, z_j \rangle_V = \mathrm{sgn}(i)\delta_{i,-j}, \quad \forall i,j \in \{\pm 1, \dots, \pm r\}.$$

As in Section 2.1, we let $\mathbf{P}_{V,r} = \mathbf{L}_{V,r} \cdot \mathbf{N}_{V,r-1}$ be the parabolic subgroup of $\mathbf{G}_W$ stabilizing
$$X_1 \subset X_2 \subset \cdots \subset X_{r-1}$$
and let $\mathbf{N}^J = \mathbf{N}_{W,r-1} \ltimes X_{r-1}$.

We let

$$\mathbf{G} = \mathbf{G}_V \times \mathbf{G}_W^J$$

and identify $\mathbf{N}^J$ as a subgroup of $\mathbf{G}$ via $\mathbf{G}_W^J \hookrightarrow 1 \times \mathbf{G}_W^J$. In these cases, $\Delta\mathbf{G}_V$, the image of the diagonal embedding $\mathbf{G}_V \hookrightarrow \mathbf{G}_V \times (\mathbf{G}_W \ltimes 1) \subset \mathbf{G}_V \times (\mathbf{G}_W \ltimes \mathcal{H}(W))$, acts on $\mathbf{N}^J$ by adjoint action, respectively, and we set

$$\mathbf{H} = \Delta\mathbf{G}_V \ltimes \mathbf{N}^J.$$

We define an algebraic character $\lambda_{\mathbf{N}^J} : \mathbf{N}^J = \mathbf{N} \ltimes X_{r-1} \to \mathrm{Res}_{E/F}\mathbb{G}_a$ by

$$\lambda_{\mathbf{N}^J}(n \ltimes z_W) = \sum_{i=1}^{r-1} \langle z_{-i}, nz_{i+1}\rangle + \langle z_{-r}, z_W\rangle, \quad n \in \mathbf{N}, \quad z_W \in X_{r-1}.$$

Let $\lambda_{\mathbf{H},F} : \mathbf{H}(F) \to \mathrm{Res}_{E/F}\mathbb{G}_a(F) = E$ be the induced morphism on F-rational points. We define an unitary character of $\mathbf{H}(F)$ by

$$\xi(h) = \psi_E(\lambda_{\mathbf{H},F}(h)), \quad h \in \mathbf{H}(F),$$

The following lemma suggests that the multiplicities defined for representations of pairs of groups in $\mathcal{T}_1$ and $\mathcal{T}_1^J$ are the same as that in $\mathcal{T}_2$ and $\mathcal{T}_2^J$.

Lemma 5.1.1. *For $V, W \in \mathcal{I}$, we have*

$$m(\pi_V \boxtimes (\widetilde{\pi}_W \otimes \omega_{W,\psi_F})) = m((\pi_V \otimes \omega_{V,\psi_F}) \boxtimes \widetilde{\pi}_W)$$

Proof. The proof over archimedean cases [Ch23a, Lemma 2.2.13]. The proof over non-archimedean local fields follows verbatim using [GKT, Lemma 3.55]. □

Hence, the multiplicity defined in this section is compatible with the definition in Section 2.1. As the groups $\mathrm{G}_V, \mathrm{G}_V^J$ are algebraic groups, we only need to work in the setting of algebraic group.

5.2 Mackey theory

The basic idea to prove the basic forms is to use the Mackey theory. For $H_1, H_2 \subset G \in Grp$ and $\pi_{H_2} \in Rep(H_2)$, Mackey theory uses the structure of double cosets $H_2\backslash G/H_1$ to study the space

$$\mathrm{Hom}_{H_1}(\mathrm{Ind}_{H_2}^{G}\pi_{H_2}, 1_{H_1})$$

Back to the setting of Section 2.1, we define algebraic groups $\mathbf{G}^+, \mathbf{H}^+, \mathbf{P}^+$ over F and apply the Mackey theory in the set up of $G = \mathbf{G}^+(F)$, $H_1 = \mathbf{H}^+(F)$, $H_2 = \mathbf{P}^+(F)$.

Basic GGP triple associated to a relevant pair

We (G^+, H^+, ξ^+) be the Gan-Gross-Prasad triple associated to the relevant pair in (5.0.2). More precisely, as we have reduced the problem to the algebraic setting in Section 5.1, we have $G^+ = \mathbf{G}^+(F)$, $H^+ = \mathbf{H}^+(F)$, where

$$\mathbf{G}^+ = \begin{cases} \mathbf{G}_{W^+} \times \mathbf{G}_V & \text{in (B) cases,} \\ \mathbf{G}_V \times \mathbf{G}^J_{W^+} & \text{in (FJ 1) cases,} \\ \mathbf{G}_{V^+} \times \mathbf{G}^J_W & \text{in (FJ 2) cases,} \end{cases}$$

$$\mathbf{H}^+ = \begin{cases} \Delta\mathbf{G}_V & \text{in (B) cases,} \\ \Delta\mathbf{G}^J_{W^+} & \text{in (FJ 1) cases,} \\ \Delta\mathbf{G}_W & \text{in (FJ 2) cases} \end{cases}$$

and ξ^+ is the trivial character of $H^+ = \mathbf{H}^+(F)$.

We let $\mathbf{P}^+$ be the closed subgroup of $\mathbf{G}^+$ defined by

- $\mathbf{P}^+ = \mathbf{P}_{W^+,X^+} \times \mathbf{G}_V$ in (B) cases;
- $\mathbf{P}^+ = \gamma^{-1}(\mathbf{G}_V \times \mathbf{P}^J_{W^+,X^+})\gamma = \mathbf{G}_V \times \gamma^{-1}\mathbf{P}^J_{W^+,X^+}\gamma$ in (FJ 1) cases, where $\gamma = 1 \times (1 \ltimes z^+_{-1}) \in \mathbf{G}_V \times (\mathbf{G}_{W^+} \ltimes \mathcal{H}(W^+)) = \mathbf{G}^+$;
- $\mathbf{P}^+ = \gamma^{-1}(\mathbf{P}_{V^+,X^+} \times \mathbf{G}^J_W)\gamma = \gamma^{-1}\mathbf{P}_{V^+,X^+}\gamma \times \mathbf{G}^J_W$ in (FJ 2) cases, where $\gamma \in \mathbf{G}^+_V$ is the element satisfying

$$\gamma z_W = z_W, \quad \gamma z^+_1 = z^-_1, \quad \gamma z^{-1}_1 = -z^+_1.$$

Based on the Levi decomposition of $\mathbf{P}_{W^+,X^+}, \mathbf{P}_{V^+,X^+}$ and the pseudo-Levi decomposition of $\mathbf{P}^J_{W^+}$, we set the pseudo-Levi subgroup $\mathbf{L}^+ \subset \mathbf{P}^+$ as

$$\begin{cases} (\mathrm{Res}_{E/F}\mathbf{GL}(X^+) \times \mathbf{G}_W) \times \mathbf{G}_V & \text{in (B) cases,} \\ \gamma(\mathbf{G}_V \times (\mathrm{Res}_{E/F}\mathbf{GL}(X^+) \times \mathbf{G}^J_W))\gamma^{-1} & \text{in (FJ 1) cases,} \\ \gamma((\mathrm{Res}_{E/F}\mathbf{GL}(X^+) \times \mathbf{G}_V) \times \mathbf{G}^J_W)\gamma^{-1} & \text{in (FJ 2) cases.} \end{cases} \tag{5.2.1}$$

and, in each situation, we denote by $p_{\mathbf{GL}}$ the morphism from $\mathbf{L}^+$ to $\mathrm{Res}_{E/F}\mathbf{GL}(X^+)$.

(5.2.2) 1. In (B) cases, we identify the H^+-action on $P^+\backslash G^+$ as the G_V action on $P_{W^+,X^+}\backslash G_{W^+}$, which consists of totally isotropic subspaces $X_{W^+} = X^+g \subset W^+(g \in G_{W^+})$ with $\dim_E X_{W^+} = r+1$ and the open orbit consists of those not contained in V.

2. In (FJ 1) cases, we identify H^+-action on $P^+\backslash G^+$ as the G_V-action on $P^J_{W^+,X^+}\backslash G^J_{W^+}$, which consists of the totally isotropic subspaces $X^+\gamma g$ $(g \in G^J_{W^+})$ of $W^{++} = W^+ \oplus^\perp Z^{++}$ where $Z^{++} = X_1^{++} \oplus Y_1^{++}$ is a hyperbolic plane over E. The open orbit consists of those not contained in W.

3. In (FJ 2) cases, we identify H^+-action on $P^+\backslash G^+$ as the G^J_W-action on $P_{V^+,X^+}\backslash G_{V^+}$, which consists of the totally isotropic subspaces $X^+\gamma g$ $(g \in G^J_{V^+})$ of V^+ and the open orbit consists of those not contained in $W \oplus X_1^+$.

Open double coset

The structure or open double cosets have been computed explicitly in [Ch23a, §2.3]. We introduce the following notions to describe the structure of the stabilizer.

Definition 5.2.1. *1. (Mirabolic subgroups) Let $X' \subset X''$ be vector spaces over E such that $\dim_E X'+1 = \dim_E X''$, we denote by $\mathbf{R}_{X',X''}$ the mirabolic subgroup of $\mathbf{GL}(X^+)$ stabilizing X' and invariant on X''/X', that is,*

$$\mathbf{R}_{X',X''} = \{g \in \mathbf{GL}(X'') : gX' \subset X',\ g \textit{ acts trivially on } X''/X'\}.$$

The isomorphism class of $\mathbf{R}_{X',X''}$ is only determined by on $r = \dim_E X'$ and we denote this isomorphism class by $\mathbf{R}_{r,1}$.

2. In (B) cases, let $\mathbf{N}'_V$ be the subgroup of $\mathbf{N}_V$ that fixes all points on $X \oplus D$, then

(5.2.3)

$$(1 \times \mathbf{G}_W) \ltimes \mathbf{N}'_V \subset (\mathrm{Res}_{E/F}\mathbf{GL}(X) \times \mathbf{G}_{W\oplus D}) \ltimes \mathbf{N}_V$$

is the subgroup of $\mathbf{P}_V$ that fixes all points on $X \oplus D$.

Lemma 5.2.2. *If $\mathbf{G}_V$ is not metaplectic, P^+H^+ is an open double coset and $\mathbf{P}^+ \cap \mathbf{H}^+ = \Delta\mathbf{S}_{\mathrm{open}}$, where*

$$\mathbf{S}_{\mathrm{open}} = \begin{cases} \mathbf{P}_{W^+,X^+} \cap \mathbf{G}_V & \text{in (B) cases} \\ \gamma^{-1}\mathbf{P}^J_{W^+,X^+}\gamma \cap \mathbf{G}_V & \text{in (FJ 1) cases} \\ \gamma^{-1}\mathbf{P}_{V^+,X^+}\gamma \cap \mathbf{G}^J_W & \text{in (FJ 2) cases} \end{cases}$$

Moreover, there is a decomposition

$$\mathbf{S}_{\mathrm{open}} = \begin{cases} (\mathrm{Res}_{E/F}\mathbf{R}_{X',X^+} \times \mathbf{G}_{V_0}) \ltimes \mathbf{N}'_V & \text{in (B) cases,} \\ (\mathrm{Res}_{E/F}\mathbf{R}_{X',X^+} \times \mathbf{G}_W) \ltimes \mathbf{N}_V & \text{in (FJ 1) cases,} \\ (\mathrm{Res}_{E/F}\mathbf{R}_{X',X^+} \times \mathbf{G}^J_V) \ltimes \mathbf{N}_W & \text{in (FJ 2) cases,} \end{cases}$$

where

$$X' = \begin{cases} X & \text{in (B)(FJ 2) cases,} \\ \mathrm{Span}\{z_2^+, \cdots, z_r^+\}, & \text{in (FJ 1) cases.} \end{cases}$$

Corollary 5.2.3. *We have $\mathbf{H} \subset \mathbf{P}^+ \cap \mathbf{H}^+$ and*

$$\mathbf{H}(F)\backslash\mathbf{P}^+ \cap \mathbf{H}^+(F) = \mathbf{N}_{0,X^+}(E)\backslash\mathbf{R}_{X',X^+}(E),$$

where X', X^+ are chosen in Lemma 5.2.2 when $\mathbf{G}_V$ is not metaplectic; and are chosen to be the X', X^+ associate to $\underline{\mathbf{G}}_V$ in Lemma 5.2.2 when $\mathbf{G}_V$ is metaplectic, where

$$\underline{\mathbf{G}}_V = \begin{cases} \mathbf{GL}(V) & \text{when } \mathbf{G}_V = \widetilde{\mathbf{GL}}(V), \\ \mathbf{Sp}(V) & \text{when } \mathbf{G}_V = \widetilde{\mathbf{Sp}}(V). \end{cases}$$

Closed double cosets

We also use the structure of the closed double cosets. Here are some results in [Ch23a, §2.3].

Lemma 5.2.4. *We can choose appropriate $\gamma' \in G^+$ such that*

1. *In (B) cases,*
    - *(a) when $\mathbf{G}_V = \mathbf{U}(V)$, $\mathcal{Z}$ has one H^+-orbit $[\gamma']$;*
    - *(b) when $\mathbf{G}_V = \mathbf{SO}(V)$, $\mathcal{Z}$ has two H^+-orbits $[\gamma']$ and $[\gamma' g'']$ for an element $g'' \in \Delta\mathrm{O}(V) - \Delta\mathrm{SO}(V)$, and both are closed orbits.*

2. In (FJ 1) cases, $\mathcal{Z}$ has one H^+-orbit $[\gamma']$;

3. In (FJ 2) cases,

  (a) When $\dim X^+ > 1$, $\mathcal{Z}$ has one H^+-orbit $[\gamma']$;

  (b) When $\dim X^+ = 1$, $\mathcal{Z}$ has two orbits $[\gamma^{-1}]$ and $[\gamma']$, where $[\gamma^{-1}]$ is a single point and $[\gamma']$ is not.

Lemma 5.2.5. *Following the notations in Lemma 5.2.4, the stabilizer group*

$$\mathbf{S}_{\gamma'} = \begin{cases} \Delta(\mathrm{Res}_{E/F}\mathbf{GL}(X_c) \times \mathbf{G}_{V_0}) \ltimes \mathbf{N}_V & \text{in (B) (FJ 1) cases,} \\ \Delta(\mathrm{Res}_{E/F}\mathbf{GL}(X_c) \times \mathbf{G}^J_{W_0}) \ltimes \mathbf{N}^J_W & \text{in (FJ 2) cases,} \end{cases}$$

where the notations are defined as following

- *We denote by X'_c the totally isotropic subspace of $V, W^+ = V, W \oplus X_1^+$ in (5.2.2) corresponding the representative γ', respectively. We denote by*

$$X_c = \begin{cases} X'_c & \text{in (B) (FJ 1) cases,} \\ W \cap X'_c & \text{in (FJ 2) cases.} \end{cases}$$

- *There are decompositions*

$$V = X_c \oplus Y_c \oplus V_0 \text{ in (B)(FJ 1) cases,}$$

$$W = X_c \oplus Y_c \oplus W_0 \text{ in (FJ 2) cases,}$$

  respectively, satisfying $X_c \oplus Y_c$ is a nondegenerate ϵ-hermitian space.

- *$\mathbf{N}_{V,X_c}, \mathbf{N}^J_{W,X_c}$ are the pseudo-unipotent part of the pseudo-parabolic subgroup $\mathbf{P}_{V,X_c}, \mathbf{P}^J_{W,X_c}$ of $\mathbf{G}_V, \mathbf{G}^J_W$ stabilizing X_c.*

Lemma 5.2.6. *As a representation of the stabilizer group $\mathbf{S}_{\gamma'}$, the fiber of the conormal bundle $\mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}}$ at γ' is*

$$\mathrm{Fib}_{\gamma'}(\mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}}) = \begin{cases} \mathrm{std}_{\mathrm{X_c}} \oplus \overline{\mathrm{std}_{\mathrm{X_c}}} & \text{when } \mathbf{G}_V = \mathbf{U}(V), \\ \mathrm{std}_{\mathrm{X_c}} & \text{when } \mathbf{G}_V = \mathbf{SO}(V), \mathbf{Sp}(V), \mathbf{Mp}(V), \end{cases}$$

where $\mathrm{std}_{\mathrm{X_c}}$ is the standard representation of $\mathrm{GL}(X_c)$.

Lemma 5.2.7. *The complement $G^+ - H^+P^+$ is the zero set of a polynomial f^+ on G^+ that is left H^+-invariant and right (P^+, ψ_{P^+})-equivariant for the algebraic character $|\det \circ p_{\mathbf{GL}}|^2$ of P^+, where $p_{\mathbf{GL}}$ is the the projection from P^+ to the GL-part of L^+ of P^+ (5.2.1).*

5.3 Proof for the basic form of the first inequality

In this section, we prove Theorem 5.0.5(2) using a refined distributional analysis. We first define $\mathrm{LI}(\pi_V)$ and $\mathrm{I}'(\chi, \pi_V, \pi_W)$ in Theorem 5.0.5(2).

Definition 5.3.1. *For a representation*

$$\pi_V = |\cdot|^{s_1}\sigma_{V,1} \times \cdots \times |\cdot|^{s_r}\sigma_{V,r} \ltimes \pi_{V_0},$$

as in Theorem 5.6.1, we define $\mathrm{LI}(\pi_V)$ to be the supermum of

1. $\mathrm{Re}(s_i)$*, when $\sigma_{V,i}$ is a unitary character of* $\mathrm{GL}_1(E)$*;*
2. $\mathrm{Re}(s_i) + \frac{m_i}{2}$*, when $\sigma_{V,i}$ is the discrete series D_{m_i} of* $\mathrm{GL}_2(\mathbb{R})$*.*

Definition 5.3.2. *When $\dim X^+ = 2$, let χ be a character of $\mathbb{R}^\times$, then we define*

$$\mathrm{I}'(\chi, \pi_V, \pi_W) = \begin{cases} (\chi \ltimes \pi_W) \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_W \boxtimes (\chi \ltimes \pi_V) & \text{in (FJ 1) cases,} \\ (\chi \ltimes \pi_V) \boxtimes \pi_W & \text{in (FJ 2) cases.} \end{cases}$$

Since $\mathbf{G}^+(F)$ is unimodular, by definition,

$$\begin{aligned} \mathrm{I}(|\det|^s\sigma_{X^+}, \pi_V, \pi_W) &= \mathrm{Ind}_{P^+}^{\mathcal{S}, G^+}(\delta_{P^+}^{1/2} \otimes l(|\det|^s\sigma_{X^+}, \pi_V, \pi_W)) \\ &= \Gamma^{\mathcal{S}}(P^+\backslash G^+, \mathcal{E}_s), \end{aligned}$$

where $\mathcal{E}_s$ the bundle

$$P^+\backslash(G^+ \times \delta_{P^+}^{1/2} \otimes l(|\det|^s\sigma_{X^+}, \pi_V, \pi_W)).$$

Let $\mathcal{X} = P^+\backslash G^+$ and $\mathcal{U}$ be the open H^+-orbit $P^+\backslash P^+H^+$ (Lemma 5.2.2) and $\mathcal{Z}$ be the complement of $\mathcal{U}$ in $\mathcal{X}$. Then $\mathcal{Z} \in Top$.

The extension by zero gives an embedding of $\Gamma^S(\mathcal{U}, \mathcal{E}_s)$ into $\Gamma^S(\mathcal{X}, \mathcal{E}_s)$, and we define

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s) = \Gamma^S(\mathcal{X}, \mathcal{E}_s)/\Gamma^S(\mathcal{U}, \mathcal{E}_s).$$

To prove Theorem 5.0.5(2), it suffices to show that

Lemma 5.3.3. *Under the conditions on* $\mathrm{Re}(s)$ *in Theorem 5.0.5(2), we have*

1. $\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}_s), 1_{H^+}) \geqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}_s), 1_{H^+})$;
2. (a) $m(\pi_V \boxtimes \pi_W) \geqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}_s), 1_{H^+})$ *when* $\sigma_{X^+} = |\cdot|^s \chi$;
   (b) $m(\mathrm{I}'(|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}, \pi_V, \pi_W)) \geqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}_s), 1_{H^+})$ *when* $\sigma_{X^+} = |\det|^s D_m$.

The proof for Lemma 5.3.3 is based on the following properties of the Hom-functor in the category *Rep*.

Lemma 5.3.4. *With the notations above, the* Hom*-functor in the category Rep enjoys the following properties.*

1. **(Left exactness)** *For an exact sequence* $0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0$ *in Rep*(H^+)*, there is an exact sequence*

$$0 \to \mathrm{Hom}_{H^+}(\pi_3, 1_{H^+}) \to \mathrm{Hom}_{H^+}(\pi_2, 1_{H^+}) \to \mathrm{Hom}_{H^+}(\pi_1, 1_{H^+});$$

2. **(Link between direct limit and inverse limit)** *Given a projective system* $\{\pi_\alpha\}_{\alpha \in I}$ *in Rep*(H^+)*, we have*

$$\mathrm{Hom}_{H^+}(\varprojlim_{\alpha} \pi_\alpha, 1_{H^+}) = \varinjlim_{\alpha} \mathrm{Hom}_{H^+}(\pi_\alpha, 1_{H^+});$$

3. **(Vanishing results for reductive groups)** *For tempered representations* σ_i $(1 \leqslant i \leqslant l)$*, tempered representation* $\pi_{V_0'}$ *of* $G_{V_0'}$*, and* $\mathrm{Re}(s_1) \geqslant \cdots \geqslant \mathrm{Re}(s_l) > 0$*, we let*

$$\pi_V = |\det|^{s_1}\sigma_1 \times \cdots \times |\det|^{s_r}\sigma_r \ltimes \pi_{V_0},$$

then we have

$$\mathrm{Hom}_{\mathbf{G}_V}((|\det|^s \sigma_V \ltimes \pi_{V_0'}) \widehat{\otimes} \pi_V, 1_{\mathbf{G}_V}) = 0$$

for $\pi_{V_0'} \in$ *Rep*$(\mathbf{G}_{V_0})$ *when* $\mathrm{Re}(s) > \mathrm{Re}(s_1)$;

4. **(Vanishing results for Jacobi groups)** *When* $\mathbf{G}_V = \mathbf{Sp}(V), \widetilde{\mathbf{Sp}}(V), \mathbf{U}(V)$*, let* $\pi_V^J = \widetilde{\pi}_V \otimes \omega_{V,\psi_F} \in Rep^{\psi_F}(G_V^J)$ *(see Example 3.1.5), where*

$$\widetilde{\pi}_V = |\det|^{s_1}\sigma_1 \times \cdots \times |\det|^{s_r}\sigma_r \ltimes \widetilde{\pi}_{V_0},$$

where s_i, σ_i *are as in (3) and* $\widetilde{\pi}_{V_0}$ *is an irreducible tempered representaiton of* $\widetilde{G}_{V_0}$*, then we have*

$$\mathrm{Hom}_{G_V^J}((|\det|^s\sigma_V \ltimes \pi^J_{V_0'})\widehat{\otimes}\pi^J_V, 1_{G_V^J}) = 0$$

for $\pi^J_{V_0'} = \widetilde{\pi}_{V_0'} \otimes \omega_{V_0,\psi_F^{-1}} \in Rep^{\psi_F^{-1}}(G^J_{V_0})$ *when* $\mathrm{Re}(s) > \mathrm{Re}(s_1)$.

Points (1)(2) are well-known.

Theorem 5.3.5. *In the above setting, for*

$$\pi_V = |\det|^{s_1}\sigma_1 \times \cdots \times |\det|^{s_l}\sigma_l \ltimes \pi_{V_0},$$

we have

1. $\mathrm{Hom}_{G_V}(\pi_{V'}, \mathrm{LQ}(\pi_V)) = 0$ *when* $s' > s_1$*, and*
2. $\mathrm{Hom}_{G_V}(\pi_{V'}, \pi_V) = 0$ *when* $s' > s_1$.

Proof for Lemma 5.3.4. From the reciprocity theorem (see [Ch23a, Lemma 2.22]), we have

$$\begin{aligned}
&\mathrm{Hom}_{G_V}((|\det|^s\sigma_V \ltimes \pi_{V_0'})\widehat{\otimes}\pi_V, 1_{G_V})\\
=&\mathrm{Hom}_{G_V}((|\det|^s\sigma_V \ltimes \pi_{V_0'}), \pi_V^\vee)\\
=&\mathrm{Hom}_{G_V}(|\det|^s\sigma_V \ltimes \pi_{V_0'}, (|\det|^{s_1}\sigma_1)^\tau \times \cdots \times (|\det|^{s_r}\sigma_l)^\tau \ltimes \pi_{V_0})
\end{aligned}$$

where τ is the complex conjugation when $E \neq F$ and τ is the trivial action when $E = F$. From Theorem 5.3.5(2), this space is equal to zero, then we proved Point (3). From Lemma 3.2.3,

$$|\det|^s\sigma_V \ltimes \pi^J_{V_0'} = (|\det|^s\sigma_V \ltimes \widetilde{\pi}_{V_0'}) \otimes \omega_{V,\psi_F}.$$

By computing the coinvariant of tensor product of Weil representations using mixed model ([Ch23a, Lemma 2.2.3])

$$(\omega_{V,\psi_F} \otimes \omega_{V,\psi_F^{-1}})_{\mathcal{H}(V)(F)} = \mathbb{C}.1.$$

Then

$$\mathrm{Hom}_{G_V^J}((|\det|^s\sigma_V \ltimes \widetilde{\pi}_{V_0'})\widehat{\otimes}\pi_V\widehat{\otimes}(\omega_{V,\psi_F}\otimes\omega_{V,\psi_F^{-1}}), 1_{G_V^J}) = \mathrm{Hom}_{G_V}((|\det|^s\sigma_V \ltimes \pi_{V_0'})\widehat{\otimes}\pi_V, 1_{G_V}).$$

This reduces Point (4) to Point (3). □

Proof for Lemma 5.3.3(1). We prove Point (1) by analyzing the complement $\mathcal{Z} = \mathcal{X} - \mathcal{U}$ of the open orbit. From the left exactness (Lemma 5.3.4), there is an exact sequence

$$0 \to \mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s), 1_{H^+}) \to \mathrm{Hom}_{H^+}(\Gamma^{S}(\mathcal{X}, \mathcal{E}_s), 1_{H^+}) \to \mathrm{Hom}_{H^+}(\Gamma^{S}(\mathcal{U}, \mathcal{E}_s), 1_{H^+}).$$

Hence, to prove the inequality in (1), it suffices to prove that

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s), 1_{H^+}) = 0 \tag{5.3.1}$$

under the given conditions.

One can study the structure of $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s)$ with the Borel's lemma in [CS20].

Lemma 5.3.6. *There is a complete descending filtration* $\{\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s)_k\}_{k \geqslant 0}$ *of* $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s)$ *such that the graded pieces*

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s)_k / \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}_s)_{k+1} = \Gamma^{\mathcal{S}}\left(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}|\mathcal{X}} \otimes \mathcal{E}_s\big|_{\mathcal{Z}}\right), \quad k \in \mathbb{N}.$$

where $\mathcal{N}_{\mathcal{Z}|\mathcal{X}}$ *is the conormal bundle of* $\mathcal{Z}$ *in* $\mathcal{X}$*.*

Recall that in Lemma 5.2.5, we computed

$$\mathbf{S}_{\gamma'} = \begin{cases} \Delta(\mathrm{Res}_{E/F}\mathbf{GL}(X_c) \times \mathbf{G}_{V_0}) \ltimes \mathbf{N}_V & \text{in (B)(FJ 1) cases,} \\ \Delta(\mathrm{Res}_{E/F}\mathbf{GL}(X_c) \times \mathbf{G}^J_{V_0^+}) \ltimes \mathbf{N}^J_W & \text{in (FJ 2) cases,} \end{cases}$$

where V_0, W_0^+, V_0^+ are given as in Lemma 5.2.4, and we may assume (1) $V_0 \subset W$ and $\dim_E W = \dim_E V_0 + 1$ in (B) cases; (2) $W = V_0$ in (FJ 1) cases; (3) $V_0^+ \subset V$ and $\dim_E V = \dim_E V_0^+ + 2$ in (FJ 2) cases.

From Lemma 5.2.6, we compute the fiber $\mathrm{Fib}_{\gamma'}(\mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}|\mathcal{X}} \otimes \mathcal{E}_s|_{\mathcal{Z}})$ of the bundle at γ' as a representation of $\mathbf{S}_{\gamma'}$, which is equal to

$$\begin{cases} \delta_{P^+}^{1/2} \otimes ((|\det|^{s+1/2}\sigma_{X^+} \otimes \mathrm{Sym}^k\rho) \boxtimes \pi_W^{\gamma'}|_{G_{V_0}}) \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_V \boxtimes \delta_{P^+}^{1/2} \otimes ((|\det|^{s+1/2}\sigma_{X^+} \otimes \mathrm{Sym}^k\rho) \boxtimes \pi_W^{\gamma'}|_{G_{W_0^+}}) & \text{in (FJ 1) cases,} \\ \delta_{P^+}^{1/2} \otimes ((|\det|^{s+1/2}\sigma_{X^+} \otimes \mathrm{Sym}^k\rho) \boxtimes \pi_V^{\gamma'}|_{G^J_{V_0^+}}) \boxtimes \pi_W & \text{in (FJ 2) cases,} \end{cases}$$

where $\rho = \mathrm{Fib}_{\gamma'}(\mathcal{N}^{\vee}_{\mathcal{Z}|\mathcal{X}})$ is the fiber of the conormal bundle $\mathcal{N}^{\vee}_{\mathcal{Z}|\mathcal{X}}$ at γ' given in Lemma 5.2.6. The representations obtained from restriction are Fréchet of moderate growth but not necessarily Casselman-Wallach.

We first detailed the cases when $\mathcal{Z}$ has a single orbit $[\gamma']$ in Lemma 5.2.4. By comparing the unipotent part, we obtain that

$$\delta_{P_{W^+,X^+}}(a)\delta^{-1}_{P_{V,X_c}}(a^{\gamma'}) = |\det(a)| \text{ in (B) cases,}$$

$$\delta_{P^J_{W^+,X^+}}(a)\delta^{-1}_{P_{V,X_c}}(a^{\gamma'}) = |\det(a)| \text{ in (FJ 1) cases,}$$

$$\delta_{P_{V^+,X^+}}(a)\delta^{-1}_{P^J_{W,X_c}}(a^{\gamma'}) = |\det(a)| \text{ in (FJ 2) cases,}$$

for $a \in \mathrm{GL}(X^+)$. Then we have

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_k/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_{k+1} = \begin{cases} (|\det|^{s+1/2}\sigma_{X^+}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_W|_{G_{V_0}} \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_V \boxtimes (|\det|^{s+1/2}\sigma_{X^+}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_W|_{G_{W_0^+}} & \text{in (FJ 1) cases,} \\ (|\det|^{s+1/2}\sigma_{X^+}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_V|_{G^J_{V_0^+}} \boxtimes \pi_W & \text{in (FJ 2) cases.} \end{cases}$$

Therefore, from Lemma 5.3.4(3)(4) and the assumptions on $\mathrm{Re}(s)$, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_k/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_{k+1}, 1_{H^+}) = 0, \quad k = 0,1,\cdots$$

From Lemma 5.3.4(1)(2), we have (5.3.1), so we completed the proof for Point (1).

When $\mathcal{Z}$ has two orbits, from Lemma 5.2.4, it is either the special orthogonal Bessel cases or almost equal-rank Fourier-Jacobi cases. In (B) cases when $\mathbf{G}_V = \mathbf{SO}(V)$, it was computed in [Ch21] that

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_k = ((|\det|^{s+1/2}\sigma_V\otimes \mathrm{Sym}^k\rho)\ltimes \pi_W|_{G_{V_0}}\boxtimes \pi_V)^{\oplus 2}.$$

In (FJ 2) cases when $\dim X^+ = 1$, notice $\mathcal{Z}$ has an open orbit $[\gamma']$ and a closed orbit $[\gamma]$, which is a single point.

$$0 \to \Gamma^{\mathcal{S}}([\gamma'], \mathrm{Sym}^k\mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}}\otimes \mathcal{E}_s\big|_{\mathcal{Z}}) \to \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_s)_k \to \Gamma^{\mathcal{S}}_{[\gamma^{-1}]}(\mathcal{Z},\mathcal{E}_s) \to 0.$$

Using the method for single orbit situation, one can show that

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}([\gamma'], \mathrm{Sym}^k\mathcal{N}^\vee_{\mathcal{Z}|\mathcal{X}}\otimes \mathcal{E}_s\big|_{\mathcal{Z}}), 1_{H^+}) = 0, \quad \mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{[\gamma^{-1}]}(\mathcal{Z},\mathcal{E}_s), 1_{H^+}) = 0.$$

This special case is detailed in [CCZ]. □

To prove Point (2), we introduce the following property of discrete series representations.

Lemma 5.3.7 (Lemma 5.4.9 of [Ch21])**.** $|\det|^s D_m|_{R_{1,1}}$ *has a subrepresentation (the underlying space is not necessarily closed) isomorphic to*

$$\mathrm{Ind}_{\mathbb{R}^\times \times 1}^{\mathcal{S}, R_{1,1}}(|\cdot|^{s+\frac{m+1}{2}}\mathrm{sgn}^{m+1}).$$

Moreover, there is a complete descending filtration with graded pieces isomorphic to

$$|\det|^{k+s+\frac{m}{2}}\mathrm{sgn}(\det)^k|_{R_{1,1}}, \text{ for } k = 1, 2, \cdots .$$

Then from the exactness of Schwartz induction ([CS20, Proposition 7.1]) and exactness of tensor product with nuclear Fréchet space ([CHM00, Lemma A.3]), the embedding

$$\mathrm{Ind}_{\mathbb{R}^\times \times 1}^{\mathcal{S}, R_{1,1}}(|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}) \hookrightarrow D_m|_{R_{1,1}}$$

induces an embedding

$$\mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S}, H^+}(\delta_{P^+}^{1/2}\otimes \mathrm{Ind}_{\mathbb{R}^\times \times 1}^{R_{1,1}}(\chi)\boxtimes\pi_V\boxtimes\pi_W) \hookrightarrow \mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S}, H^+}(\delta_{P^+}^{1/2}\otimes|\det|^s D_m\boxtimes\pi_V\boxtimes\pi_W) = \Gamma^S(\mathcal{U}, \mathcal{E}_s).$$

(5.3.2) The quotient

$$\Gamma^S(\mathcal{U}, \mathcal{E}_s)/\mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S}, H^+}(\delta_{P^+}^{1/2}\otimes \mathrm{Ind}_{\mathbb{R}^\times \times 1}^{R_{1,1}}(\chi)\boxtimes\pi_V\boxtimes\pi_W)$$

has a complete descending filtration with graded pieces isomorphic to

$$\mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S}, H^+}(\delta_{P^+}^{1/2}\otimes |\det|^{k+s+\frac{m}{2}}\mathrm{sgn}(\det)^k|_{R_{1,1}}\boxtimes\pi_V\boxtimes\pi_W).$$

Lemma 5.3.8. *For a character* χ *of* $\mathbb{R}^\times$*, we have*

$$\mathrm{I}'(\chi, \pi_V, \pi_W) = \mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S}, H^+}(\delta_{P^+}^{1/2}\otimes \mathrm{Ind}_{\mathbb{R}^\times \times 1}^{R_{1,1}}(|\cdot|^{-1/2}\chi))\boxtimes\pi_V\boxtimes\pi_W).$$

Proof. Since both G^+, H^+ are unimodular, $H^+\backslash G^+$ has a G^+-invariant measure. Since H^+P^+ is open in G (Lemma 5.2.5), this measure induces an P^+-invariant measure on $P^+\cap H^+\backslash P^+$. Hence,

$$\delta_{P^+}|_{P^+\cap H^+} = \delta_{P^+\cap H^+}.$$

Then the equality follows from the fact that

$$\delta_{R_{1,1}}(\mathrm{diag}(a, 1)) = |a|.$$

□

Definition 5.3.9. *For a character χ of $\mathbb{R}^\times$, we have*

$$\mathrm{I}''(\chi, \pi_V, \pi_W) = \mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S},H^+}(\delta_{P^+}^{1/2}\chi(\det)\boxtimes\pi_V\boxtimes\pi_W)$$

Lemma 5.3.10.

$$\mathrm{I}''(\chi,\pi_V,\pi_W) = \begin{cases} (\chi\ltimes\mathrm{Ind}_{G_W}^{\mathcal{S},G_{W\oplus Ez_0}}(\pi_W))\boxtimes\pi_V & \textit{in (B) cases},\\ \pi_V\boxtimes(\chi\ltimes\mathrm{Ind}_{G_W^J}^{\mathcal{S},G_{W\oplus H_1}}(\pi_W)) & \textit{in (FJ 1) cases},\\ (\chi\ltimes\mathrm{Ind}_{G_V}^{G_V^J}(\pi_V))\boxtimes\pi_W & \textit{in (FJ 2) cases.}\end{cases}$$

Proof. This follows from the fact that $\mathbf{G}_{W\oplus Ez_0}$, $\mathbf{G}_W$ (in (B) cases), $\mathbf{G}_{W\oplus H_1}$, $\mathbf{G}_W^J$ (in (FJ 1) cases), $\mathbf{G}_V^J$, $\mathbf{G}_V$ (in (FJ 2) cases) are unimodular. □

Proof for Lemma 5.3.3(2). We prove Point (2) by analyzing the open orbit $\mathcal{U}$.

When $\sigma_{X^+} = |\cdot|^s\chi$, from Corollary 5.2.3, we have $\mathcal{U} = H\backslash P^+\cap H^+ = \{1\}$, so

$$\dim\mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E}_s), 1_{H^+}) = m(\pi_V\boxtimes\pi_W)$$

When $\sigma_{X^+} = |\det|^s D_m$, from Corollary 5.2.3, $\mathcal{U} = H\backslash P^+\cap H^+ = \{N_{X^+}\backslash R_{X',X^+}\} = \mathbb{R}^\times$. From Lemma 5.3.10 and Lemma 5.3.3(3)(4),

$$\mathrm{Hom}_{H^+}(\mathrm{I}''(|\det|^{s+k+\frac{m}{2}}\mathrm{sgn}(\det)^k,\pi_V,\pi_W)),\mathbb{C}) = 0, \quad \text{for } k = 1,2,\cdots$$

then from (5.3.2) Lemma 5.3.4(1)(2), we have

$$\dim\mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E}),1_{H^+}) \leqslant m(\pi_V\boxtimes\mathrm{I}'(|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1},\pi_V,\pi_W)).$$

□

5.4 Reduction to basic cases

Theorem 5.0.5(1) was proved in [X2, Proposition 6.1] for unitary Bessel cases, and in [Ch21, Lemma 5.1.3] for special orthogonal Bessel cases using a Schwartz homology analog of the proof for Theorem 5.0.5(2).

As in the previous section, we have

$$\mathrm{I}(\sigma_{\underline{s}},\pi_V,\pi_W) = \Gamma^{\mathcal{S}}(P^+\backslash G^+,\mathcal{E}_{\underline{s}}),$$

where $\mathcal{E}_{\underline{s}}$ the bundle

$$P^{+}\backslash(G^{+} \times \delta_{P^{+}}^{1/2} \otimes l(\sigma_{\underline{s}}, \pi_V, \pi_W)).$$

The following lemma is the counterpart of Lemma 5.3.3

Lemma 5.4.1. *We have*

1. $H_i(H^{+}, \Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}_{\underline{s}})) = H_i(H^{+}, \Gamma^{\mathcal{S}}(\mathcal{X}, \mathcal{E}_{\underline{s}}))$ *for* $\underline{s}$ *in general positions;*
2. $H_i(H^{+}, \Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}_s)) = H_i(H, \pi_V \boxtimes \pi_W \otimes \xi^{-1})$ *for* $\underline{s}$ *in general positions.*

The idea for Lemma 5.4.1 proof is to replace Lemma 5.3.4 in the proof for Theorem 5.0.5(2) with properties of the Schwartz homology in Lemma 5.4.2.

Lemma 5.4.2. *The following properties of the Schwartz homology hold.*

1. **(Long exact sequence)** *For an exact sequence*

$$0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0$$

of H^{+}*-representations, there is a long exact sequence*

$$\to H_i^{\mathcal{S}}(H^{+}, \pi_1) \to H_i^{\mathcal{S}}(H^{+}, \pi_2) \to H_i^{\mathcal{S}}(H^{+}, \pi_3) \to H_{i-1}^{\mathcal{S}}(H^{+}, \pi_1) \to \cdots$$

2. **(Commutes with inverse limit)** *Given a projective system* $\{\pi_\alpha\}_I$ *with surjective* $\pi_\alpha \to \pi_\beta$ *for* $\alpha > \beta$*, we have*

$$H_i^{\mathcal{S}}(H^{+}, \varprojlim_{\alpha} \pi_\alpha) = \varprojlim_{\alpha} H_i^{\mathcal{S}}(H^{+}, \pi_\alpha).$$

3. **(Vanishing result for reductive groups)** *In the setting of Lemma 5.3.4(3), we have*

$$H_i^{\mathcal{S}}(\Delta G_V, (|\det|^s \sigma_V \ltimes \pi_{V_0}) \boxtimes \pi_V) = 0$$

for $s \in \mathbb{C}$ *in general positions.*

4. *(Vanishing results for Jacobi groups) In the settings of Lemma 5.3.4(4), we have*

$$H_i^{\mathcal{S}}(\Delta G_V^J(F), ((|\det|^s \sigma_V \ltimes \pi_{V_0}) \widehat{\otimes} \pi_V, 1_{G_V^J}) = 0$$

for $s \in \mathbb{C}$ *in general positions.*

Proof. Point (1) is a property of derived functors. Point (2) was proved in [X2, Proposition 2.13]. Point (3) follows from the vanishing criterion on infinitesimal characters ([X3]) using the arguments in [X3, §3] or [Ch23a, Lemma 5.2.7]. Using the arguments in the proof for 5.3.4, Point (4) follows from Point (3). □

Proof for Lemma 5.4.1(1). Following the computation complete descending filtration $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_k$ of $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})=\Gamma^{\mathcal{S}}(\mathcal{X},\mathcal{E}_{\underline{s}})/\Gamma^{\mathcal{S}}(\mathcal{U},\mathcal{E}_{\underline{s}})$ with

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_k/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_{k+1}=\begin{cases}(|\det|^{s+1/2}\sigma_{\underline{s}}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_W|_{G_{V_0}}\boxtimes \pi_V & \text{in (B) cases,}\\ \pi_V\boxtimes(|\det|^{1/2}\sigma_{\underline{s}}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_W|_{G_{W_0^+}} & \text{in (FJ 1) cases,}\\ (|\det|^{1/2}\sigma_{\underline{s}}\otimes \mathrm{Sym}^k\rho)\ltimes \pi^{\gamma'}_V|_{G^J_{V_0^+}}\boxtimes \pi_W & \text{in (FJ 2) cases.}\end{cases}$$

Then from Lemma 5.4.2(3)(4),

$$H_i(H^+,\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_k/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_{k+1})=0,\quad i=0,1,\cdots,\quad k=0,1,\cdots$$

Hence, from Lemma 5.4.2(1), we have

$$H_i(H^+,\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})/\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}})_k)=0,\quad i=0,1,\cdots,\quad k=0,1,\cdots$$

Therefore, from Lemma 5.4.2(2), we have

$$H_i(H^+,\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X},\mathcal{E}_{\underline{s}}))=0,\quad i=0,1,\cdots$$

Then from Lemma 5.4.2(1), we obtain Lemma 5.4.1(1). □

The idea for the proof for Lemma 5.4.1(2) is also parallel to that for Lemma 5.3.3(2). We need to study the structure of $\sigma_{\underline{s}}/\mathrm{Ind}^{\mathcal{S},R_{X',X^+}}_{N_{0,X^+}}(\psi^{-1}_{0,X^+})$ with Lemma 5.4.3.

We first introduce some notations.

- We denote by $P_{0,n}$ a Borel subgroup of $GL_n(F)$ and denote by $N_{0,n}$ the unipotent radical of $P_{0,n}$;
- We denote by $P_{a,b,c}$ the parabolic subgroup of GL_{a+b+c} stabilizing a filtration

$$X_a\subset X_{a+b}\subset X_{a+b+c},$$

where $\dim X_i=i$, and we denote by $L_{a,b,c}$ its Levi subgroup.

Lemma 5.4.3 (Proposition 5.1 of [X2]; Proposition 5.3.4 of [Ch21])**.** *There is a R_{X',X^+}-equivariant embedding from $\mathrm{Ind}_{N_{0,X^+}}^{\mathcal{S},R_{X',X^+}}(\psi_{0,X^+}^{-1})$ to $\sigma_{\underline{s}}$. The quotient $\sigma_{\underline{s}}/\mathrm{Ind}_{N_{0,X^+}}^{\mathcal{S},R_{X',X^+}}(\psi_{0,X^+}^{-1})$ admits an R_{X',X^+}-stable complete filtration whose graded pieces have the shape*

$$\mathrm{Ind}_{P_{a,b,c}}^{\mathcal{S},R_{\dim X',1}}(\tau_a \boxtimes \tau_b \boxtimes \tau_c),$$

where $a+b+c = \dim X^+$, $a+b \neq 0$ and the $P_{a,b,c}$-representation $\tau_a \boxtimes \tau_b \boxtimes \tau_c$ is regarded as the inflation from $L_{a,b,c}$-representation $\tau_a \boxtimes \tau_b \boxtimes \tau_c$.

1. *$\tau_a = \mathrm{Ind}_{P_{0,a}}^{\mathcal{S},\mathrm{GL}_a}(\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1} \boxtimes \cdots \boxtimes \mathrm{sgn}^{m_a}|\cdot|^{s_{i_a}+k_a})$ where $1 \leqslant i_1, \cdots, i_a \leqslant t+1$ are integers, $l_1, \cdots, l_a \in \mathbb{Z}$ and $k_1, \cdots, k_a \in \frac{1}{2}\mathbb{Z}$;*

2. *$\tau_b = \tau_b' \otimes \rho$ where τ_b' is a representation of the same form as τ_a and ρ is a finite-dimensional representation of $\mathrm{GL}_b(\mathbb{R})$;*

3. *$\tau_c = \mathrm{Ind}_{N_{0,c}}^{R_{c-1,1}}(\psi_c^{-1})$.*

Proof. Using this lemma and the exactness of Schwartz induction and the exactness of tensor product with nuclear Fréchet space, we obtain a complete descending filtration of

$$\Gamma_o = \Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}_{\underline{s}})/\mathrm{Ind}_{P^+\cap H^+}^{\mathcal{S},H^+}(\mathrm{Ind}_{N_{0,X^+}}^{\mathcal{S},R_{X',X^+}}(\psi_{0,X^+}^{-1}) \boxtimes \pi_V \boxtimes \pi_W)$$

with graded pieces isomorphic to

$$\begin{cases} (\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1'} \ltimes \pi) \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_V \boxtimes (\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1'} \ltimes \pi) & \text{in (FJ 1) cases,} \\ (\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1'} \ltimes \pi) \boxtimes \pi_W & \text{in (FJ 2) cases} \end{cases}$$

for certain Fréchet representation π of moderate growth and $k_1' \in \frac{1}{2}\mathbb{Z}$, then from Lemma 5.4.2(3)(4), we have the Schwartz homology of the graded piece all equal to zero. Then Lemma 5.4.2(1)(2) implies Lemma 5.4.1. □

Proof for Theorem 5.0.5(1). Then we prove Theorem 5.0.5(1) from Lemma 5.4.1. By definition,

$$m(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) = \dim \mathrm{Hom}_{H^+}(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W), 1_{H^+}) = \dim \mathrm{Hom}_{\mathbb{C}}(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)_{H^+}^{\mathrm{Haus}}, \mathbb{C}),$$

and

$$m(\pi_V \boxtimes \pi_W) = \dim \mathrm{Hom}_H(\pi_V \boxtimes \pi_W, \xi^{-1}) = \dim \mathrm{Hom}_{\mathbb{C}}((\pi_V \boxtimes \pi_W \otimes \xi)_H^{\mathrm{Haus}}, \mathbb{C}).$$

Hence, it suffices to show that

$$\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)_{H^+}^{\mathrm{Haus}} = (\pi_V \boxtimes \pi_W \otimes \xi^{-1})_H^{\mathrm{Haus}}. \tag{5.4.1}$$

By definition,

$$\mathrm{H}_0^{\mathcal{S}}(H^+, \mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) = \mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)_{H^+}, \quad \mathrm{H}_0^{\mathcal{S}}(H, \pi_V \boxtimes \pi_W \otimes \xi^{-1}) = (\pi_V \boxtimes \pi_W \otimes \xi^{-1})_H.$$

Then from Lemma 5.4.1,

$$\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)_{H^+} = (\pi_V \boxtimes \pi_W \otimes \xi^{-1})_H$$

as topological spaces, then their maximal Hausdorff quotients are the same, so we have (5.4.1). □

5.5 Proof for the basic form of the second inequality

This section proves the basic form of the second inequality using the integral method in Chapter 4. Let (V, W) be a relevant pair with the GGP triple (G, H, ξ), and $\pi_V \in$ $Rep^{\mathrm{adm}}(G_V), \pi_W \in Rep^{\mathrm{adm}}(H_W)$. We follow the notions in Section 5.2. Suppose $m(\pi_V, \pi_W) \neq 0$, we fix a nonzero functional

$$\mu \in \begin{cases} \mathrm{Hom}_H(\pi_V \boxtimes \pi_W, \xi) & \text{in (B) caes,} \\ \mathrm{Hom}_H(\pi_V \boxtimes \pi_W^J, \xi) & \text{in (FJ 1)(FJ 2) cases} \end{cases} \tag{5.5.1}$$

of (G, H, ξ). Taking Bessel cases as example, following [JSZ10], we construct a family of integrals

$$\mathrm{I}_{s,\mu,\lambda}(\varphi_s, v) = \int_{H\backslash H^+} \mu(v, \Lambda(\varphi_s(h)))d(h,h) \quad \varphi_s \in |\det|^s \sigma \ltimes \pi_W,\ v \in \pi_V,$$

where $\Lambda : \sigma \boxtimes \pi_W \to \pi_W$ constructed from a fixed nonzero Whittaker functional $\lambda : \sigma \to \mathbb{C}$. The integral method gives a nonzero element in $\dim \mathrm{Hom}_{\mathbf{H}^+}(\mathrm{I}(|\det|^s, \pi_V, \pi_W), 1_{H^+})$. This proves Theorem 5.0.5(2).

We also construct similar integrals for Fourier-Jacobi models. We treat these integrals over $H\backslash H^+$ as the composition of integral operators over $H^+ \cap P^+ \backslash H^+$ and over $H\backslash H^+ \cap P^+$ and prove the meromorphic continuation for each integral operator.

5.5.1 The integral over $H\backslash P^+ \cap H^+$

In this section, we use the integral method for Rankin-Selberg-type integrals to prove the following theorem.

Theorem 5.5.1. *Suppose there is a nonzero* μ *in* (5.5.1)*, we have a nonzero functional in*

$$\mathrm{Hom}_{P^+\cap H^+}(l(|\det|^{s_0}\sigma_{X^+}, \pi_V, \pi_W), 1_{P^+\cap H^+})$$

for every $s_0 \in \mathbb{C}$*.*

Let $\mathbf{P}^+ = \mathbf{L}^+ \ltimes \mathbf{N}^+$ be the subgroup of $\mathbf{G}^+$ defined in Section 5.2 and $\mathbf{L}^+$ is given in (5.2.1).

For a generic representation $\sigma_{X^+} \in \mathit{Rep}^{\mathrm{adm}}(\mathrm{GL}(X^+))$ and irreducible representations $\pi_V \in \mathit{Rep}^{\mathrm{adm}}(G_V)$, and (i) $\pi_W \in \mathit{Rep}^{\mathrm{adm}}(H_W)$, we define a P^+-representation $l(\sigma_{X^+}, \pi_V, \pi_W)$ inflated from the L^+-representation

$$\begin{cases} (\sigma_{X^+} \boxtimes \pi_W) \boxtimes \pi_V & \text{in (B) cases,} \\ \pi_V \boxtimes (\sigma_{X^+} \boxtimes \pi_W^J) & \text{in (FJ 1) cases,} \\ (\sigma_{X^+} \boxtimes \pi_V) \boxtimes \pi_W^J & \text{in (FJ 2) cases.} \end{cases}$$

and denote by $\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W)$ the normalized Schwartz induction

$$\mathrm{Ind}_{P^+}^{\mathcal{S},G^+}(\delta_{P^+}^{1/2} \otimes l(\sigma_{X^+}, \pi_V, \pi_W)).$$

Definition 5.5.2. *Fix a functional* μ *as in* (5.5.1)*.*

1. *For* $v_{\pi_V} \in \pi_V$ *and* $v_{\pi_W} \in \pi_W$*, we denote by* $\mathrm{B}_{v_{\sigma_V}, v_{\sigma_W}}$ *the* ***Bessel functional*** *defined by*

   $$\mathrm{B}_{v_{\pi_V}, v_{\pi_W}}(g_V, g_W) = \mu(\pi_V(g_V)v_{\pi_V}, \pi_W(g_W)v_{\pi_W}), \quad g_V \in \mathbf{G}_V(F), \quad g_W \in \mathbf{G}_W(F).$$

   We denote by $\mathrm{FJ}_{v_{\sigma_V}, v_{\sigma_W}}$ *the* ***Fourier-Jacobi functional***

   $$\mathrm{FJ}_{v_{\pi_V}, v_{\pi_W}}(g_V, g_W) = \mu(\pi_V(g_V)v_{\pi_V}, \pi_W^J(g_W)v_{\pi_W}), \quad g_V \in \mathbf{G}_V(F), \quad g_W \in \mathbf{G}_W^J(F)$$

2. *Following the notion in Section 5.2, for* $g \in \mathbf{R}_{X',X^+}(E) \subset \mathbf{P}^+ \cap \mathbf{H}^+(F)$*, we define the functional*

$$\mathrm{W}'_{v_{\pi_V},v_{\pi_W}}(g) = \begin{cases} \mathrm{B}_{v_{\pi_V},v_{\pi_W}}(\mathrm{diag}(g, 1_{W\oplus D}), 1_W) & \textit{in (B) cases,} \\ \mathrm{B}_{v_{\pi_V},v_{\pi_W}}(1_W, \mathrm{diag}(g, 1_W)) & \textit{in (FJ 1) cases,} \\ \mathrm{FJ}_{v_{\pi_V},v_{\pi_W}}(\mathrm{diag}(g, 1_V), 1_V) & \textit{in (FJ 2) cases.} \end{cases} \tag{5.5.2}$$

It is not difficult to verify that the functionals $\mathrm{W}'_{v_{\pi_V},v_{\pi_W}}$ defined above are Whittaker-type functionals of $\mathbf{GL}(X')$ over E.

For fixed Whittaker functional λ of σ_{X^+} and , when $\mathrm{Re}(s)$ is large enough, we construct $\mathrm{J}_{v,v'}(s)$ following Section 4.1 (the ground field is E, $r = \dim X^+ - 1$) for

$$v \in V_{|\det|^s\sigma_{X^+}} = V_{\sigma_{X^+}}, v' \in \begin{cases} V_{\pi_V \boxtimes \pi_W} & \text{in (B) cases,} \\ V_{\pi_V \boxtimes \pi_W^J} & \text{in (FJ 1)(FJ 2) cases.} \end{cases}$$

From Corollary 5.2.3, we have

$$H \backslash P^+ \cap H^+ = \mathbf{N}_{r+1}(E) \backslash \mathbf{R}_{r,1}(E) = \mathbf{N}_r(E) \backslash \mathbf{GL}_r(E)$$

We set $\mathrm{T}_s(v, v') := \mathrm{J}_{v,v'}(s)$, then we have a family of elements

$$\mathrm{T}_s \in \mathrm{Hom}_{P^+ \cap H^+}(l(|\det|^s\sigma_{X^+}, \pi_V, \pi_W), \mathbb{C}).$$

From Theorem 4.1.4, T_s is a nonzero meromorphic family. Suppose the Whittaker-type functions defined (5.5.2) are admissible, following Section 4.1.2, T_s extends to a nonzero meromorphic family on $s \in \mathbb{C}$ in the continuous dual $(V_{l(\sigma_{X^+} \boxtimes \pi_V \boxtimes \pi_W)})^*$, without abusing the notions, we denote it by T_s, $s \in \mathbb{C}$. Then, from Lemma 4.0.1, for every $s_0 \in \mathbb{C}$, the principal term at $s = s_0$ gives a nonzero element in $\mathrm{Hom}_{P^+ \cap H^+}(l(|\det|^{s_0}\sigma_{X^+}, \pi_V, \pi_W), \mathbb{C})$. This proves Theorem 5.5.1.

It suffices to verify that the Whittaker-typed functional defined by μ is an admissible Whittaker-typed functional.

Lemma 5.5.3. *The Bessel and Fourier-Jacobi functionals are admissible Whittaker-type functionals.*

When F is non-Archimedean the proof follows from an analog to [JPSS79, Proposition 2.2]. We do not elaborate the details here as our focus in this chapter is the Archimedean case. When F is Archimedean, in [JS90], Jacquet and Shalika used properties of Jacquet models of the reductive groups.

Definition 5.5.4. *1. For a representation π of $G = \mathbf{G}(F)$ for a reductive group $\mathbf{G}$ over F, we denote by π^{alg} the $(\mathfrak{g}_{\mathbb{C}}, K)$-module associated to π. Here $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}(F) \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified Lie algebra of G.*

2. For a parabolic subgroup $P = L \ltimes N$ of G, we define the Jacquet module

$$\mathrm{Jac}_P^{\mathrm{alg}}(\pi^{\mathrm{alg}}) = \pi^{\mathrm{alg}}/\pi(\mathfrak{n}_{\mathbb{C}})\pi^{\mathrm{alg}}.$$

Definition 5.5.5. *Let (G, H, ξ) be a Bessel or Fourier-Jacobi triple associated to (V, W) and $\pi = \pi_V \boxtimes \pi_W \in Rep^{\mathrm{adm}}(G)$ in (B) cases and $\pi = \pi_V \boxtimes \pi_W^J \in Rep^{\mathrm{adm}}(G)$ in (FJ 1)(FJ 2) cases.*

For $1 \leqslant j \leqslant r$, we define

$$\mathbf{P}_j = \begin{cases} \mathbf{P}_{V,j} \times \widetilde{\mathbf{P}}_W & \text{in (B) and (FJ 1) cases,} \\ \mathbf{P}_V \times \widetilde{\mathbf{P}}_{W,j} & \text{in (FJ 2) cases,} \end{cases}$$

where $P_{V,j}, \widetilde{P}_{W,j}$ is defined as in Section 2.1, then the Jacquet module

$$\mathrm{Jac}_{P_j}^{\mathrm{alg}}((\pi_V \boxtimes \pi'_W)^{\mathrm{alg}}) = \begin{cases} \mathrm{Jac}_{P_{V,j}}^{\mathrm{alg}}(\pi_V^{\mathrm{alg}}) \boxtimes \pi_W^{\prime,\mathrm{alg}} & \text{in (B) and (FJ 1) cases,} \\ \pi_V^{\mathrm{alg}} \boxtimes \mathrm{Jac}_{\widetilde{P}_{W,j}}^{\mathrm{alg}}(\pi_W^{\prime,\mathrm{alg}}) & \text{in (FJ 2) cases.} \end{cases}$$

Proof for Lemma 5.5.3. Take $\mathbf{P}_j = \mathbf{L}_j \ltimes \mathbf{N}_j$ be the Levi decomposition of $\mathbf{P}_j$, then

$$\mathbf{L}_j(F) = (E^\times)^j \times \mathbf{G}_j(F).$$

for some reductive $\mathbf{G}_j$. We consider the generalized eigenspace decomposition

$$\mathrm{Jac}_{P_j}^{\mathrm{alg}}((\pi_V \boxtimes \pi'_W)^{\mathrm{alg}}) = \bigoplus_{\mu \in \Sigma_j} \mathrm{Jac}_{P_j}^{\mathrm{alg}}((\pi_V \boxtimes \pi'_W)^{\mathrm{alg}})_\mu,$$

with respect to $(E^\times)^j$, where Σ_j is a finite set of characters of $(E^\times)^j$.

From [Wal88, Lemma 4.3.1], $\mathrm{Jac}_{P_j}^{\mathrm{alg}}((\pi_V \boxtimes \pi_W)^{\mathrm{alg}})$ is a finite-length $\mathcal{U}((\mathfrak{l}_j)_{\mathbb{C}})$-module, then there exists an integer k such that

$$(\pi_{V,W}(a) - \mu(a))^k \mathrm{Jac}_{P_j}^{\mathrm{alg}}((\pi_V \boxtimes \pi'_W)^{\mathrm{alg}})_\mu = 0, \quad a \in (E^\times)^j, \mu \in \Sigma_j.$$

Define $\mathcal{K}$ the space of functions φ on $(\mathbb{R}^\times)^n$ in to form of

$$\varphi(a_1, \cdots, a_n, 1) = \begin{cases} \mathrm{W}_{v_{\pi_V}^{\mathrm{alg}}, v_{\pi_W}^{\mathrm{alg}}}(\mathrm{diag}(a_1, \cdots, a_n, 1)) & \text{in (B) cases,} \\ \mathrm{W}_{v_{\pi_V}^{\mathrm{alg}}, v_{\pi_W}^{\mathrm{alg}} \otimes \Phi}(\mathrm{diag}(a_1, \cdots, a_n, 1)) & \text{in (FJ 1)(FJ 2) cases,} \end{cases}$$

where $v_{\pi_V}^{\mathrm{alg}} \in \pi_V^{\mathrm{alg}}$, $v_{\pi_W}^{\mathrm{alg}} \in \pi_W^{\mathrm{alg}}$, $\Phi \in \omega_{W,\psi_F}$.

It follows verbatim from [JS90, §4.1] when $E = \mathbb{C}$ and from [Sou93, §3.3] when $E = \mathbb{R}$ that $\mathcal{K}$ has the following properties.

1. Every function $\varphi \in \mathcal{K}$ is smooth on $(E^\times)^r$ and is bounded by

$$C \cdot \prod_{j=1}^{r} (1 + |a_j|^2 + |a_j^{-1}|^2)^N;$$

for some $C > 0$ and a positive integer N.

2. The space $\mathcal{K}$ is closed under
   (a) Multiplication by a_j, $\bar{a}_j$;
   (b) Euler operators

$$D_j = \begin{cases} a_j \frac{\partial f}{\partial a_j} & \text{when } E = \mathbb{R}, \\ a_j \frac{\partial f}{\partial a_j} + \bar{a}_j \frac{\partial f}{\partial \bar{a}_j} & \text{when } E = \mathbb{C}; \end{cases}$$

   (c) The difference between the holomorphic and antiholomorphic derivatives

$$R_j = \frac{\partial f}{\partial a_j} - \frac{\partial f}{\partial \bar{a}_j}.$$

3. Each $\varphi \in \mathcal{K}$ has a decomposition

$$\varphi = \sum_{\mu \in \Sigma_j} \varphi_\mu$$

such that

(a) When $E = \mathbb{R}$, there exists $\theta \in \mathcal{K}$ such that

$$(D_j - \mu)^k \varphi_x = a_j \theta.$$

(b) When $E = \mathbb{C}$, there exists $\theta_1, \theta_2 \in \mathcal{K}$ such that

$$(D_j - \mu)^k \varphi_x = a_j \theta_1 + \bar{a}_j \theta_2.$$

Following the general situations discussed in [JS90, §4.2], there is a finite set $\Sigma_{V,W}$ of finite functions that is only dependent on Σ_j's (thus only dependent on $\pi_V^{\mathrm{alg}}, \pi_W^{\mathrm{alg}}$) such that $\mathrm{W}'_{v_{\pi_V}^{\mathrm{alg}} \otimes v_{\pi_W}^{\mathrm{alg}}}$ has the expansion

$$\sum_{\xi \in \Sigma_{V,W}} \phi_{\xi . v_{\pi_V}^{\mathrm{alg}} \otimes v_{\pi_W}^{\mathrm{alg}}}(a_1, \cdots, a_n) \xi(a_1, \cdots, a_n).$$

Following the computation of [JS90, §4.3], the method in [Cas89] can be applied to extend the results to $\mathrm{W}'_{v_{\pi_V} \otimes v_{\pi_W}}$ for $v_{\pi_V} \in \pi_V$ and $v_{\pi_W} \in \pi_W$.

□

5.5.2 The integral over $P^+ \cap H^+ \backslash H^+$

In this section, we follow the work of D. Gourevitch, S. Sahi, and E. Sayag to generalize their results so that this approach also applies to (FJ 1)(FJ 2) cases.

In all situations, we have $\delta_{H^+} = 1$, and from Lemma 5.2.2, we can compute $\delta_{P^+ \cap H^+}$ to obtain the following lemma.

Lemma 5.5.6. *There exists $s_0 \in \mathbb{R}$ such that*

$$\delta_{P^+ \cap H^+} \delta_{H^+}^{-1} \otimes l(\sigma_{X^+}, \pi_V, \pi_W) = l(|\det|^{s_0} \sigma_{X^+}, \pi_V, \pi_W).$$

Theorem 5.5.7. *Suppose there is a nonzero element*

$$\mu^+ \in \mathrm{Hom}_{P^+ \cap H^+}(l(\sigma_{X^+}, \pi_V, \pi_W), 1_{P^+ \cap H^+}),$$

then there is a nonzero element in

$$\mathrm{Hom}_{H^+}(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W), 1_{H^+}).$$

Proof. From Lemma 5.2.7, the complement $G^+ - H^+P^+$ is the zero set of a polynomial f^+ on G^+ and f^+ is left H^+-invariant and right (P^+, ψ_{P^+})-equivariant for an algebraic character ψ_{P^+} of P^+, where $\psi_{P^+} = |\det \circ p_{\mathbf{GL}}|^2$.

Given $\mu^+ \in \mathrm{Hom}_{P^+ \cap H^+}(l(|\det|^s \sigma_{X^+}, \pi_V, \pi_W), 1_{P^+\cap H^+})$, from results in Section 4.2, we define $\mathrm{J}_{s^+,\Phi}$ as in Lemma 4.2.1, where Φ is defined in (4.2.3). Then from Theorem 4.2.8, there is a meromorphic family

$$\mathrm{J}_{s^+,\Phi} \in \mathcal{S}^*(G^+, l(\sigma_{X^+}, \pi_V, \pi_W))^{(P^+, l(\sigma_{X^+}, \pi_V, \pi_W))}, s^+ \in \mathbb{C}.$$

Then, by Lemma 4.0.1, the principal term at $s^+ = -s_0$ is a nonzero element in

$$\mathrm{Hom}_{H^+}(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W), 1_{H^+}).$$

□

Therefore, in all cases, suppose $m(\pi_V, \pi_W) \neq 0$, then there is a nonzero function μ in (5.5.1). From Theorem 5.5.1 and 5.5.7, there is a nonzero element, we have a nonzero element in $\mathrm{Hom}_{H^+}(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$. Together with the multiplicity one theorem $m(\pi_V, \pi_W) \leqslant 1$, we have

$$m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W)) \geqslant m(\pi_V, \pi_W).$$

This proves Theorem 5.0.5(2).

5.6 Prove the inequalities from the basic forms

5.6.1 The first inequality

In this section, we prove the first inequality

$$m(\pi_V \boxtimes \pi_W) \geqslant m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$$

in Theorem 5.0.2 from Theorem 5.0.5(1)(2) using a modification of the mathematical induction in [MW12].

Every tempered representation σ of a general linear group is in the form of

$$|\det|^{s_1}\sigma_1 \times \cdots \times |\det|^{s_l}\sigma_l$$

where $s_i \in \mathbb{C}$, σ_i is either equal to an one-dimensional unitary $\mathrm{GL}_1(E)$-representation, or a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ when $E = \mathbb{R}$. Then we can prove the first inequality in the basic cases with the following theorem.

The proof follows from a modification of the proof in [MW12, §1.4-§1.6], which proves the basic form of the first inequality, and then applies mathematical induction to reduce the general inequalities to those in the basic forms.

Theorem 5.6.1. *Let*

$$\sigma_V = |\cdot|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\cdot|^{s_{V,l_V}}\sigma_{V,l_V}, \quad \sigma_W = |\cdot|^{s_{W,1}}\sigma_{W,1} \times \cdots \times |\cdot|^{s_{W,l_W}}\sigma_{W,l_W},$$

where $\sigma_{V,i}, \sigma_{W,i}$ are either equal to unitary characters of $\mathrm{GL}_1(E)$, or discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ when $E = \mathbb{R}$, and $\mathrm{Re}(s_{V,i}), \mathrm{Re}(s_{W,i}) \geqslant 0$, then for irreducible tempered representations π_{V_0}, π_{W_0} and $\pi_V = \sigma_V \ltimes \pi_{V_0}, \pi_W = \sigma_W \ltimes \pi_{W_0}$

$$m(\pi_V \boxtimes \pi_W) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Proof for "the first inequality" (Theorem 5.6.1). Then we prove "the first inequality" follows from the basic forms. From Theorem 5.0.5, we may assume (V, W) is a basic relevant pair. The mathematical induction can be completed following a modification of the proof in [Ch21, §5.4]. To put more emphasis on the induction, we abuse to notions and take $m(\pi_V \boxtimes \pi_W) = m(\pi_W \boxtimes \pi_V)$. The precise orders are given in a similar mathematical induction in the next section.

We may assume

$$\mathrm{Re}(s_{V,1}) + \frac{m_{V,1}}{2} \geqslant \cdots \geqslant \mathrm{Re}(s_{V,l_V}) + \frac{m_{V,l_V}}{2} \geqslant 0, \text{and}$$
$$\mathrm{Re}(s_{W,1}) + \frac{m_{V,l_V}}{2} \geqslant \cdots \geqslant \mathrm{Re}(s_{W,l_W}) + \frac{m_{V,l_V}}{2} \geqslant 0.$$

We prove the inequality in the theorem by mathematical induction on the lexicographical order of $(N(\sigma_V, \sigma_W), M(\sigma_V, \sigma_W)) \in \mathbb{N}^2$, where

$$N(\sigma_V, \sigma_W) = \sum_{i=1}^{l_V} n_{V,i} + \sum_{i=1}^{l_W} n_{W,i},$$

and $M(\sigma_V, \sigma_W)$ is equal to the largest i such that

$$\mathrm{Re}(s_{W,i}) + \frac{m_{W,i}}{2} > \mathrm{Re}(s_{V,1}) + \frac{m_{V,1}}{2}$$

when $\mathrm{LC}(\pi_W) > \mathrm{LC}(\pi_V)$, the largest i such that

$$\mathrm{Re}(s_{V,i}) + \frac{m_{V,i}}{2} > \mathrm{Re}(s_{W,1}) + \frac{m_{W,1}}{2}$$

when $\mathrm{LC}(\pi_V) > \mathrm{LC}(\pi_W)$, and 0 when $\mathrm{LC}(\pi_V) = \mathrm{LC}(\pi_W)$.

We set

$$\sigma'_V = |\cdot|^{s_{V,2}}\sigma_{V,1} \times \cdots \times |\cdot|^{s_{V,l_V}}\sigma_{V,l_V}, \quad \sigma'_W = |\cdot|^{s_2}\sigma_{W,1} \times \cdots \times |\cdot|^{s_{W,l_W}}\sigma_{W,l_W}$$

$$\pi'_V = \sigma'_V \ltimes \pi_{V_0}, \quad \pi'_W = \sigma'_W \ltimes \pi_{W_0}$$

If $N(\sigma_V, \sigma_W) = 0$, $\pi_V = \pi_{V_0}$ and $\pi_W = \pi_{W_0}$, then

$$m(\pi_V \boxtimes \pi_W) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

When $N(\sigma_V, \sigma_W) > 0$,

1. If $\mathrm{LC}(\pi_V) > \mathrm{LC}(\pi_W)$,
   (a) When $n_{V,1} = 1$, from Theorem 5.0.5(2), we have

   $$m(\pi'_V, \pi_W) \geqslant m(\pi_V \boxtimes \pi_W)$$

   and

   $$N(\sigma'_V, \sigma_W) = N(\sigma_V, \sigma_W) - 1.$$

   (b) When $n_{V,1} = 2$, from Theorem 5.0.5(2), we have

   $$m((|\cdot|^{s_{V,1}+\frac{m_{V,1}}{2}}\mathrm{sgn}^{m_{V,1}+1} \ltimes \pi'_V) \boxtimes \pi_W) \leqslant m(\pi_V \boxtimes \pi_W)$$

   and

   $$N(|\cdot|^{s_{V,1}+\frac{m_{V,1}}{2}}\mathrm{sgn}^{m_{V,1}+1} \ltimes \sigma'_V, \sigma_W) = N(\sigma_V, \sigma_W) - 1.$$

2. If $\mathrm{LC}(\pi_V) < \mathrm{LC}(\pi_W)$, from Theorem 5.0.5(1), we can fix $s' \in \mathbb{C}$ with $\mathrm{Re}(s') = 0$ such that

   $$m(\pi_V \boxtimes (|\cdot|^{s'} \ltimes \pi_W)) = m(\pi_V, \pi_W).$$

(a) If $M(\sigma_V, \sigma_W) > 1$ and $n_{W,1} = 1$, by applying Theorem 5.0.5(2), we obtain

$$m(\pi_V, |\cdot|^{s'} \ltimes \pi'_W) \leqslant m(\pi_V \boxtimes |\cdot|^{s'} \ltimes \pi'_W)$$

and we have

$$N(\sigma_V, |\cdot|^{s'} \ltimes \sigma'_W) = N(\sigma_V, \sigma_W), \quad M(\sigma_V, |\cdot|^{s'} \ltimes \sigma'_W) = M(\sigma_V, \sigma_W) - 1.$$

(b) If $M(\sigma_V, \sigma_W) = 1$ and $n_{W,1} = 1$, we can apply Theorem 5.0.5(2) twice and obtain that

$$m(\pi_V \boxtimes (|\cdot|^{s_1} \ltimes \pi_W)) \leqslant m(\pi'_V \boxtimes (|\cdot|^{s'} \ltimes \pi'_W))$$

and we have

$$N(\sigma'_V, |\cdot|^{s'} \ltimes \sigma'_W) = N(\sigma_V, \sigma_W) - 1.$$

(c) When $n_{W,1} = 2$, from Theorem 5.0.5(2), we have

$$m(\pi_V \boxtimes (|\cdot|^{s'} \ltimes \pi_W)) \leqslant m(\pi_V \boxtimes (|\cdot|^{s'} \ltimes |\cdot|^{s_{W,1}+\frac{m_{W,1}}{2}} \mathrm{sgn}^{m_{W,1}+1} \ltimes \pi'_W))$$

and $\pi_V \boxtimes (|\cdot|^{s'} \ltimes |\cdot|^{s_{W,1}+\frac{m_{W,1}}{2}} \mathrm{sgn}^{m_{W,1}+1} \ltimes \pi'_W)$ is in case (a)(b) with

$$N(\sigma_V \boxtimes (|\cdot|^{s'} \times |\cdot|^{s_{W,1}+\frac{m_{W,1}}{2}} \mathrm{sgn}^{m_{W,1}+1} \ltimes \sigma'_W)) = N(\sigma_V, \sigma_W), \text{ and}$$

$$M(\sigma_V \boxtimes (|\cdot|^{s'} \ltimes |\cdot|^{s_{W,1}+\frac{m_{W,1}}{2}} \mathrm{sgn}^{m_{W,1}+1} \ltimes \sigma'_W) \leqslant M(\sigma_V, \sigma_W)$$

□

5.6.2 The second inequality

Then we prove the counterpart of the "second inequality" in [MW12, §1.6], that is, $m(\pi_V \boxtimes \pi_W) \leqslant m(\mathrm{I}(\sigma_{X^+}, \pi_V, \pi_W))$ in Theorem 5.0.2. It is not hard to imagine that the proof for the second inequality is essentially applying Theorem 5.0.5 twice. Still, there are two details we need to pay attention to:

1. We need to ensure the pair of representations always associates to a basic relevant pair every time we apply Theorem 5.0.5(1).

2. Since we have not proven the inequality for finite-length π_V and π_W, we need to ensure our representation on the $\mathbf{G}_V$-type block of the inducing datum is always irreducible when applying Theorem 5.0.5(3).

For point (1), we follow the idea in [JSZ10] [LS13] and consider the parabolic induction with a spherical principal series representation (Theorem 5.0.5). For point (2), we choose an appropriate parameter of the spherical principal series representation to ensure the irreducibility of the parabolic induction.

Theorem 5.6.2. *Given irreducible Casselman-Wallach representations* π_{V_0}, π_{W_0} *and generic Casselman-Wallach representations* σ_V, σ_W*, we set*

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \pi_W = \sigma_W \ltimes \pi_{W_0},$$

then

$$m(\pi_V \boxtimes \pi_W) \geqslant m(\pi_{V_0} \boxtimes \pi_{W_0})$$

Proof. We consider $W^+ = (X^+ \oplus Y^+) \oplus^\perp W$ as in Section 5.2. Since π_W is Casselman-Wallach, from Theorem 5.0.5(1), we can find a spherical principal series representation $\sigma_{\underline{s}}$ such that

$$m(\pi_V \boxtimes \pi_W) = m(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) \tag{5.6.1}$$

From Theorem 5.0.5(3), we have

$$\begin{aligned}
&m(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) = m((\sigma_{\underline{s}} \ltimes \pi_W) \boxtimes \pi_V) \geqslant m(\pi_V \boxtimes \pi_{W_0}), \text{ in (B) cases;}\\
&m(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) = m(\pi_V \boxtimes (\sigma_{\underline{s}} \ltimes \pi_W)) \geqslant m(\pi_V \boxtimes \pi_{W_0}), \text{ in (FJ 1) cases;}\\
&m(\mathrm{I}(\sigma_{\underline{s}}, \pi_V, \pi_W)) = m((\sigma_{\underline{s}} \ltimes \pi_V) \boxtimes \pi_W) \geqslant m(\pi_{V_0} \boxtimes \pi_W), \text{ in (FJ 2) cases.}
\end{aligned} \tag{5.6.2}$$

From Theorem 5.0.5(1), we can find a spherical principal series representation $\sigma_{\underline{s}'}$ such that

$$\begin{aligned}
m(\pi_V \boxtimes \pi_{W_0}) &= m((\sigma_{\underline{s}} \ltimes \pi_{W_0}) \boxtimes \pi_V), \text{ in (B) cases;}\\
m(\pi_V \boxtimes \pi_{W_0}) &= m(\pi_V \boxtimes (\sigma_{\underline{s}} \ltimes \pi_{W_0})), \text{ in (FJ 1) cases;}\\
m(\pi_{V_0} \boxtimes \pi_W) &= m((\sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_W), \text{ in (FJ 2) cases.}
\end{aligned}$$

From Theorem 5.0.5(1), we can find $s \in \mathbb{C}$ such that

$$\begin{aligned}
&m((\sigma_{\underline{s}} \ltimes \pi_{W_0}) \boxtimes \pi_V) = m((|\cdot|^s \ltimes \pi_V) \boxtimes (\sigma_{\underline{s}} \ltimes \pi_{W_0})), \text{ in (B) cases;}\\
&m(\pi_V \boxtimes (\sigma_{\underline{s}} \ltimes \pi_{W_0})) = m(\pi_V \boxtimes (|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{W_0})), \text{ in (FJ 1) cases (RHS is a (FJ 2) case);}\\
&m((\sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_W) = m((|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_W), \text{ in (FJ 2) cases (RHS is a (FJ 1) case).}
\end{aligned}$$

Similarly,

$$m(\pi_{V_0} \boxtimes \pi_{W_0}) = m(\mathrm{I}(\sigma_{\underline{s}'}, \pi_{V_0}, \pi_{W_0})). \tag{5.6.3}$$

From [SV80, Theorem 1.1] and Langlands classification, we may also assume $\sigma_{\underline{s}'} \ltimes \pi_{W_0}$ is irreducible.

From Theorem 5.0.5(3), we have

$$\begin{aligned} m((|\cdot|^s \ltimes \pi_V) \boxtimes (\sigma_{\underline{s}'} \ltimes \pi_{W_0})) &\geqslant m((\sigma_{\underline{s}'} \ltimes \pi_{W_0}) \boxtimes \pi_{V_0}), \text{ in (B) cases;} \\ m(\pi_V \boxtimes (|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{W_0})) &\geqslant m(\pi_{V_0} \boxtimes (|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{W_0}), \text{ in (FJ 1) cases;} \\ m((|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_W) &\geqslant m((|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_{W_0}), \text{ in (FJ 2) cases.} \end{aligned} \tag{5.6.4}$$

Recall that our choice of s and $\underline{s}'$ are in general positions, then we may assume

$$\begin{aligned} m((\sigma_{\underline{s}'} \ltimes \pi_{W_0}) \boxtimes \pi_{V_0}) &= m(\pi_{V_0}, \pi_{W_0}), \text{ in (B) cases;} \\ m(\pi_{V_0} \boxtimes (|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{W_0}) &= m(\pi_{V_0}, \pi_{W_0}), \text{ in (FJ 1) cases;} \\ m((|\cdot|^s \times \sigma_{\underline{s}} \ltimes \pi_{V_0}) \boxtimes \pi_{W_0}) &= m(\pi_{V_0}, \pi_{W_0}), \text{ in (FJ 2) cases.} \end{aligned} \tag{5.6.5}$$

In conclusion,

$$m(\pi_V \boxtimes \pi_W) \geqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

This completes the proof for ”the second inequality”. □

Chapter 6

Uniform Approach

In this section, we explain how the proof for the local Gan-Gross-Prasad conjecture can be formulated following the diagram below for all cases.

$$\begin{tikzcd}
\text{Uniqueness of generic repn in Vogan } L\text{-packet} \arrow[d, Rightarrow, "\text{Local trace formula+Endoscopy}"'] & \\
\text{tempered basic Bessel cases} \arrow[r, Leftrightarrow, "\text{Theta correspondence}"] \arrow[d, Rightarrow, "\text{Multiplicity formula}"'] & \text{tempered basic Fourier-Jacobi cases} \arrow[d, Rightarrow, "\text{Multiplicity formula}"] \\
\text{Bessel cases} & \text{Fourier-Jacobi cases}
\end{tikzcd}$$

6.1 Local trace formula and endoscopy

This section briefly introduces the approach in [Wald10] [Wald12a] [Wald12b] [Wald12c] using local trace formula and endoscopy. This approach was developed to prove the other cases in [BP14] [BP16] [BP20] [Luo21] [CL22].

Harish-Chandra's study on trace characters lays the foundation of the trace formula approach. Let $\mathbf{G}$ be a reductive group over a local field F with characteristic zero, for any $\pi \in \Pi_F(\mathbf{G})$, Harish-Chandra proved that the trace distribution

$$\Theta_\pi(f) = \mathrm{Tr}(\int_G f(g)\pi(g)dg), f \in C_c^\infty(\mathbf{G}(F))$$

is a locally integral function that is smooth on the regular semisimple locus $\mathbf{G}(F)_{\mathrm{reg,ss}}$. This function is known as the **Harish-Chandra character** and we write it as Θ_π.

We denote by D^G is the Weyl discriminant given by

$$D^G(x) = |\det(1 - \mathrm{Ad}(x)|_{\mathfrak{g}/\mathfrak{g}_x})|, x \in \mathbf{G}(F)_{\mathrm{ss}}$$

The growth of the Harish-Chandra character is controlled by the property that $D^G(x)^{1/2}\Theta_\pi$ is locally bounded. Moreover, for admissible representations $\pi_i (1 \leqslant i \leqslant l)$ that are nonequivalent (infinitesimally nonequivalent in Archimedean cases), their Harish-Chandra characters Θ_{π_i} are linearly independent. The trace formula approach aims to study spectral properties of the representation π via the geometric properties Θ_π.

For a reductive Lie algebra $\mathfrak{g}$ over F, we denote by $\mathcal{N}(\mathfrak{g}(F))$ the set of nilpotent orbits on $\mathfrak{g}(F)$ and denote by $\mathcal{N}_{\mathrm{reg}}(\mathfrak{g}(F))$ the subset of regular elements in it. By Harish-Chandra character expansion, for every $x \in \mathbf{G}(F)_{\mathrm{ss}}$, there exists complex numbers $c_{\mathcal{O}_x,\Theta_\pi}$, indexed by $\mathcal{O}_x \in \mathcal{N}(\mathfrak{g}_x(F))$, such that

$$\Theta_\pi(x\exp(Y)) = \sum_{\mathcal{O}_x \in \mathcal{N}(\mathfrak{g}_x(F))} c_{\mathcal{O}_x,\Theta_\pi}\widehat{\mu_{\mathcal{O}_x}}(Y), \text{ when } F \text{ is non-Archimedean, and}$$
$$\Theta_\pi(x\exp(Y)) = \sum_{\mathcal{O}_x \in \mathcal{N}(\mathfrak{g}_x(F))} c_{\mathcal{O}_x,\Theta_\pi}\widehat{\mu_{\mathcal{O}_x}}(Y) + O(|Y|_{\mathfrak{g}_x}), \text{ when } F = \mathbb{R},$$

for $Y \in \mathfrak{g}_x(F)$ sufficiently near zero. Here $\widehat{\mu_{\mathcal{O}_x}}(Y)$ is the Fourier transform of the orbital integrals on $\mathcal{O}_x$ and $|\cdot|_{\mathfrak{g}_x}$ is a norm on the Lie algebra $\mathfrak{g}(F)$. The numbers $c_{\mathcal{O}_x,\Theta_\pi}$ are called **germs** in the Harish-Chandra character expansion of Θ_π at x. Smooth functions on the regular semisimple locus satisfying this property called **quasi-characters** ([BP20, §4]).

In the context of special orthogonal Bessel cases over non-Archimedean local fields, Waldspurger defined a geometric multiplicity for quasi-characters by

$$m_{\mathrm{geom}}(\Theta) = \int_{\Gamma(G,H)} D^G(x)^{1/2} c_\Theta(x)\Delta(x)^{-1/2}dx \tag{6.1.1}$$

where $\Gamma(G,H)$ the space of semisimple G-conjugacy classes that intersect H satisfying some other properties, $\Delta(x) = D^G(x)D^H(x)^{-2}$ and $c_\Theta(x) = c_{\mathcal{O}_x,\Theta}$ for uniformly chosen $\mathcal{O}_x \in \mathcal{N}(\mathfrak{g}_x(F))$. By studying the local trace formula concerning the triple (G, H, ξ), Waldspurger proved that, for a tempered representation π of G,

$$m(\pi) = m_{\mathrm{geom}}(\Theta_\pi). \tag{6.1.2}$$

Then one can study $m(\pi)$ via Θ_π.

Beuzart-Plessis generalized this result to tempered unitary cases over any local fields in [BP16] [BP20]. In unitary cases, the definition of geometric multiplicity is

$$m_{\text{geom}}(\Theta) = \lim_{s \to 0^+} \int_{\Gamma(G,H)} D^G(x)^{1/2} c_\Theta(x) \Delta(x)^{s-1/2} dx.$$

Therefore, the limit process needs to be taken into consideration. Luo proved this special orthogonal cases over Archimedean local fields in [Luo21].

6.1.1 Multiplicity-one in Vogan L-packets

This section introduces Waldspurger's approach for multiplicity-one part of the conjecture. Let (G, H, ξ) be a GGP triple of Bessel type. For a tempered L-parameter φ of $\mathbf{G}$, set the quasi-character

$$\Theta_{\Pi_{F,\varphi}(\mathbf{G})} = \Sigma_{\pi \in \Pi_{F,\varphi}(\mathbf{G})} \Theta_\pi.$$

Then, from (6.1.2), the multiplicity-one part of the local GGP conjecture reduces to show that

$$\sum_{\alpha \in H^1(F,\mathbf{H})} m_{\text{geom}}(\Theta_{\Pi_{F,\varphi}(\mathbf{G}_\alpha)}) = 1. \tag{6.1.3}$$

Subtle computation of the integral expansion of the left-hand side of (6.1.3) implies that it is equal to the number of generic representations in $\Pi_{F,\varphi}(\mathbf{G}_{qs})$ with respective to the fixed Whittaker datum (see [Wald12c] [BP20, §12.6]), which is equal to one by results in the local Langlands correspondence.

6.1.2 Epsilon-dichotomy

This section introduces Waldspurger's approach towards epsilon-dichotomy and its simplification over $\mathbb{R}$.

To prove the epsilon-dichotomy, Waldspurger expressed the geometric multiplicity in terms of the stable geometric multiplicity of endoscopic groups using endoscopic character relation. Then, using the local trace formula on twisted reductive groups and twisted endoscopy, Waldspurger computed the stable geometric multiplicity of endoscopic groups using twisted multiplicities, which can be computed directly using the functional equation of Rankin-Selberg integrals.

The **stable geometric multiplicity** is defined as

$$m_{\mathrm{geom}}^{\mathrm{Stab}}(\Theta) = \int_{\Gamma(G,H)} D^G(x)^{1/2} c_\Theta^{\mathrm{Stab}}(x)\Delta(x)^{-1/2}dx$$

where the stable germ $c_\Theta^{\mathrm{Stab}}(x) = \frac{1}{|\mathcal{N}_{\mathrm{reg}}(\mathfrak{g}_x(F))|}\sum_{\mathcal{O}\in\mathcal{N}_{\mathrm{reg}}(\mathfrak{g}_x(F))} c_{\Theta,\mathcal{O}}(x)$.

For every tempered parameter φ_V of $\mathbf{SO}(V)$, an element $s_V \in \mathcal{S}_{\varphi_V}$ gives a decomposition

$$\varphi_V^{\mathrm{ss}} = \varphi_{V_+}^{\mathrm{ss}} \oplus \varphi_{V_-}^{\mathrm{ss}},$$

where $\varphi_{V_+}^{\mathrm{ss}} = (\varphi_V^{\mathrm{ss}})^{s=+1}$, $\varphi_{V_-}^{\mathrm{ss}} = (\varphi_V^{\mathrm{ss}})^{s=-1}$ and $\varphi_{V_\pm}$ are L-parameters of $\mathbf{SO}(V_\pm)$.

Consider the endoscopic group $\mathbf{SO}(V_+)\times\mathbf{SO}(V_-)$ such that the underlying spaces of $\varphi_{V_\pm}$ is those the standard representations of ${}^L\mathbf{SO}(V_\pm)$. For every $\alpha \in H^1(F,H)$, let

$$\Theta^{s_V}_{\Pi_{F,\varphi_W}(\mathbf{SO}(W_\alpha))} = \sum_{\pi_\alpha\in\Pi_{F,\varphi_W}(\mathbf{SO}(W_\alpha))} \chi_{\pi_{V_\alpha}}(s_V)\Theta_{\pi_{V_\alpha}}.$$

By computing geometric multiplicity using endoscopic character relation, i.e., $e(G_{V_\alpha})\Theta_{\Sigma_{V_\alpha}^{s_\alpha}}$ is the endoscopic transfer of the stable character $\Theta_{\Sigma^1_{V_+}\times\Sigma^1_{V_-}}$, we have the following relation between geometric multiplicity and stable geometric multiplicity of endoscopic groups

$$\begin{aligned}
&\sum_{\alpha\in H^1(F,H)} m_{\mathrm{geom}}(\Theta^{s_V}_{\Pi_{F,\varphi_V}(\mathbf{SO}(V_\alpha))}, \Theta^{s_W}_{\Pi_{F,\varphi_W}(\mathbf{SO}(W_\alpha))})\\
=&m^S_{\mathrm{geom}}(\Theta_{\Pi_{F,\varphi_{V_-}}(\mathbf{SO}(V_-))}, \Theta_{\Pi_{F,\varphi_{W_+}}(\mathbf{SO}(W_+))})\\
&\cdot m^S_{\mathrm{geom}}(\Theta_{\Pi_{F,\varphi_{V_+}}(\mathbf{SO}(V_+))}, \Theta_{\Pi_{F,\varphi_{W_-}}(\mathbf{SO}(W_-))}).
\end{aligned} \tag{6.1.4}$$

When $F = \mathbb{C}$, there is exactly one representation in the Vogan packet, so epsilon-dichotomy is automatic. Let $F = \mathbb{R}$. When $\dim V > 3$, the L-parameter φ_V of $\mathbf{SO}(V)$ can be classified as **parabolic type** or **endoscopic type**.

The L-parameter φ_V is of parabolic type if there is a decomposition

$$\varphi_V^{\mathrm{ss}} = \varphi_V^{\mathbf{GL}} \oplus \varphi_{V_0}^{\mathrm{ss}} \oplus (\varphi_V^{\mathbf{GL}})^\vee, \quad V = V_0 \perp (X_V \oplus X_V^\vee)$$

where $X_V \neq 0$ and the L-parameter $\varphi_V^{\mathbf{GL}} : \mathrm{WD}_{\mathbb{R}} \to {}^L\mathbf{GL}(X_V)$ does not contain any self-dual irreducible components. In this case, let $\Pi_{F,\varphi_V^{\mathbf{GL}}}(\mathbf{GL}(X_V)) = \{\sigma\}$, and then the map

$$\pi_0 \mapsto \sigma \ltimes \pi_0 = \mathrm{Ind}_{\mathbf{P}_{X_V}(F)}^{\mathbf{G}_V(F)}(\delta_{P_{X_V}}^{1/2}\sigma\widehat{\otimes}\pi_0) \tag{6.1.5}$$

defines an isomorphism

$$\Pi^{\text{Vogan}}_{F,\varphi_{V_0}} \to \Pi^{\text{Vogan}}_{F,\varphi_V},$$

where $\mathbf{P}_{X_V}$ is the parabolic subgroup of $\mathbf{G}_V$ stabilizing X_V.

The L-parameter φ_V is of endoscopic type if there exists $s \in \mathcal{S}_{\varphi_V}$ such that the endoscopic group $\mathbf{G}'_V = \mathbf{SO}(V_+) \times \mathbf{SO}(V_-)$ of $\mathbf{G}_V = \mathbf{SO}(V)$ determined by s is smaller than $\mathbf{SO}(V)$, that is,

$$^L\mathbf{G}'_V \subsetneq {}^L\mathbf{G}_V, \quad \mathrm{Cent}_{\widehat{\mathbf{G}}_V}(s)^o = \widehat{\mathbf{G}}'_V. \tag{6.1.6}$$

We used properties of geometric multiplicity in [Luo21] to prove the equality

$$m((\sigma_V \ltimes \pi_V) \boxtimes (\sigma_W \ltimes \pi_W)) = m(\pi_V \boxtimes \pi_W),$$

for tempered representations $\sigma_V, \sigma_W, \pi_V, \pi_W$. This allows us to reduce the conjecture to smaller cases when φ_V is of parabolic type.

We set

$$m^S_{V,W} := \sum_{\alpha \in H^1(\mathbb{R},\mathbf{H})} \sum_{\substack{\pi_V \in \Pi_{F,\varphi_V}(\mathbf{SO}(V_\alpha)) \\ \pi_W \in \Pi_{F,\varphi_W}(\mathbf{SO}(W_\alpha))}} \chi_\pi(1_{\varphi_V}, -1_{\varphi_W}) m(\pi_V, \pi_W).$$

Using (6.1.2) and (6.1.4), we obtain that

$$m^S_{V,W} = m^S_{\text{geom}}(\Theta_{\Pi_{F,\varphi_V}(\mathbf{SO}(V))}, \Theta_{\Pi_{F,\varphi_W}(\mathbf{SO}(W))})$$

and

$$\sum_{\alpha \in H^1(\mathbb{R},\mathbf{H})} \sum_{\substack{\pi_V \in \Pi_{F,\varphi_V}(\mathbf{SO}(V_\alpha)) \\ \pi_W \in \Pi_{F,\varphi_W}(\mathbf{SO}(W_\alpha))}} \chi_{\pi_{V_\alpha}}(s_V)\chi_{\pi_{W_\alpha}}(s_W) m(\pi_{V_\alpha}, \pi_{W_\alpha}) = m^S_{V_-,W_+} m^S_{V_+,W_-}. \tag{6.1.7}$$

The left-hand side is equal to

$$\chi_{\pi^0_V}(s_V)\chi_{\pi^0_W}(s_W), \tag{6.1.8}$$

where $\chi_{\pi^0_V}, \chi_{\pi^0_W}$ are the characters corresponding to the unique pair (π^0_V, π^0_W) in $\Pi^{\text{Vogan}}_{F,\varphi_V \times \varphi_W}$ with multiplicity equal to one. Assuming the local Gan-Gross-Prasad conjecture for small cases, (6.1.7) implies (6.1.8) is equal to $\chi^W_{\varphi_V}(s_V)\chi^V_{\varphi_W}(s_W)$ when s satisfy (6.1.6). When φ_V is of endoscopic type, $s_V \in \mathcal{S}_{\varphi_V}$ not satisfying (6.1.6) is a nontrivial subgroup of $\mathcal{S}_{\varphi_V}$, so

$$\chi_{\pi^0_V} \times \chi_{\pi^0_W} = \chi^W_{\varphi_V} \times \chi^V_{\varphi_W}, \text{ for all } s_V \times s_W \in \mathcal{S}_{\varphi_V} \times \mathcal{S}_{\varphi_W}.$$

This allows us to reduce the conjecture to smaller cases. Parallel arguments can also be established for unitary Bessel cases to obtain a uniform proof of epsilon-dichotomy for tempered Bessel cases over Archimedean local fields.

6.2 Bridges between basic tempered cases

The first basic tempered Fourier-Jacobi cases was proved by Gan and Ichino for unitary groups over non-Archimedean local fields. They proved from the basic tempered Bessel cases using the seesaw identity in the theory of theta correspondence. Then Atobe proved the results for symplectic-metaplectic groups over non-Archimedean local fields, which involved more techinical details. This section takes the symplectic-metaplectic cases as an example and show how the basic tempered Bessel and Fourier-Jacobi cases are related to each other.

6.2.1 Theta correspondence

This section reviews some tools in theta correspondence. We refer to [GKT] for the definitions of **reductive dual pairs** and **theta lifts**. The reductive dual pairs used in the proof are

1. $(\mathrm{U}(V), \mathrm{U}(W))$, for hermitian spaces V, W;
2. $(\mathrm{Sp}(V), \mathrm{O}(W))$, for a symplectic space V and an even-dimensional quadratic space W;
3. $(\mathrm{Mp}(V), \mathrm{O}(W))$, for a symplectic space V and an odd-dimensional quadratic space W.

For reductive dual pairs (G, H) in (1)-(3), the theta lift $\Theta_\psi(\pi)$ of $\pi \in \Pi_F(G)$ is the unique representation such that

$$\omega_{V\otimes W,\psi} / \bigcap_{f \in \mathrm{Hom}_{\widetilde{G}(F)}(\omega_{V\otimes W,\psi}|_{\widetilde{G}(F)}, \pi)} \ker(f) = \pi \otimes \Theta_\psi(\pi)$$

Here $\widetilde{G}(F)$ is an appropriate cover of G. Moreover, $\Theta_\psi(\pi)$ has a unique irreducible quotient $\theta_\psi(\pi)$.

The diagram

$$\begin{tikzcd}
G \arrow[d, dash] \arrow[dr, dash] & G' \arrow[d, dash] \arrow[dl, dash] \\
H & H'
\end{tikzcd}$$

is called seesaw diagram if and only if $H \subset G$, $H' \subset G'$ and (G, H'), (G', H) are reductive dual pairs. Under consistency conditions of the reductive dual pairs, for every $\pi \in \Pi_F(H)$ and $\sigma \in \Pi_F(H')$, there is seesaw identity

$$\mathrm{Hom}(\Theta_\psi(\sigma), \pi) = \mathrm{Hom}(\Theta_\psi(\pi), \sigma) \tag{6.2.1}$$

We write the diagram for the seesaw identity as

$$\begin{tikzcd}
\Theta_\psi(\sigma) & G \arrow[d, dash] \arrow[dr, dash] & G' \arrow[d, dash] \arrow[dl, dash] & \Theta_\psi(\pi) \\
\pi & H & H' & \sigma
\end{tikzcd}$$

6.2.2 Bridges by seesaw identity: non-Archimedean cases

Then we introduces the bridge between basic tempered Bessel and Fourier-Jacobi cases built by the seesaw identities in [GI16] [At18]. This section reviews the bridge between special orthogonal Bessel cases and symplectic-metaplectic Fourier-Jacobi cases over non-Archimedean cases. The bridges in unitary cases can be established.

Non-Archimedean. Let F be a non-Archimedean local field. For a quadratic space W over F satisfying $\dim W = 2n+1$ and $\mathrm{disc}(W) = 1$. The pair $(\mathbf{SO}(W), \mathbf{Sp}(V_{2n}))$ is a reductive dual pair, where V_{2n} is the $2n$-dimensional symplectic spaces over F. Let $\sigma \in \Pi_F(\mathbf{SO}(W))$, then $\mathrm{Ind}_{\mathrm{SO}(W)}^{\mathrm{O}(W)}(\sigma)$ decomposes into two representations $\sigma', \sigma' \circ \mathrm{sgn} \in \Pi_F(\mathbf{O}(W))$.

$$\theta_{V_{2n}} : \coprod_{\substack{\dim V = \dim W + 1 \\ \mathrm{disc}(W)=1}} \Pi_F(\mathbf{SO}(W)) \to \Pi_F^{\mathrm{genu}}(\mathbf{Mp}(2n)),$$

is defined as $\theta_{V_{2n}}(\sigma) = \theta_\psi(\sigma')$.

Consider the seesaw diagram

$$\begin{tikzcd}
\Theta_\psi(\sigma') \boxtimes \omega_{V_{2n},\psi^{-1}} & \mathrm{Mp}(2n) \times_{\pm 1} \mathrm{Mp}(2n) \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(W') \arrow[d, dash] \arrow[dl, dash] & \Theta_\psi(\pi) \\
\pi & \mathrm{Sp}(2n) & \mathrm{O}(W) \times \mathrm{O}(L) & \sigma' \boxtimes 1
\end{tikzcd} \tag{6.2.2}$$

where $\dim W = 2n+1$, $\mathrm{disc}(W) = 1$, L is an anisotropic line $W' = W \perp L$. Then, the seesaw identity reads

$$\mathrm{Hom}_{\mathrm{O}(W)}(\Theta_{\psi_F}(\pi), \sigma') = \mathrm{Hom}_{\mathrm{Sp}(2n)}(\Theta_{\psi_F}(\sigma') \widehat{\otimes} \omega_{V_{2n},\psi_F}, \pi). \tag{6.2.3}$$

It is known over non-Archimedean local fields that $\Theta_\psi(\sigma') = \theta_\psi(\sigma')$, then

$$\mathrm{Hom}_{\mathrm{Sp}(2n)}(\Theta_{\psi_F}(\sigma') \widehat{\otimes} \omega_{V_{2n},\psi_F}, \pi) = \mathrm{Hom}_{\mathrm{Sp}(2n)}(\theta_{\psi_F}(\sigma') \widehat{\otimes} \omega_{V_{2n},\psi_F}, \pi)$$

Hence, the dimension of the right-hand side of (6.2.3) is related to the multiplicity

$$m(\theta_{\psi_F}(\sigma'), \pi^\vee)$$

of the basic tempered Fourier-Jacobi case for $(\mathbf{Sp}(V_{2n}), \mathbf{Mp}(V_{2n}))$.

Then, we relate the dimension of the left-hand side of (6.2.3) to the multiplicity of a basic tempered Bessel case. $\theta_{\psi_F}(\pi)$ is the unique quotient of $\Theta_{\psi_F}(\pi)$,

$$\mathrm{Hom}_{\mathrm{O}(W)}(\Theta_{\psi_F}(\pi), \sigma') = \mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi_F}(\pi), \sigma')$$

Since $\theta_{\psi_F}(\sigma' \circ \mathrm{sgn}) = 0$, from the seesaw identity for a similar diagram as (6.2.2),

$$\mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi_F}(\pi), \sigma' \circ \mathrm{sgn}) = 0.$$

Using the Frobenius reciprocity, we obtain that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi_F}(\pi), \sigma') =& \mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi_F}(\pi), \mathrm{Ind}_{\mathrm{SO}(W)}^{\mathrm{O}(W)}(\sigma'|_{\mathrm{SO}(W)})) \\ =& \mathrm{Hom}_{\mathrm{SO}(W)}(\theta_{\psi_F}(\pi)|_{\mathrm{SO}(W)}, \sigma'|_{\mathrm{SO}(W)}).\end{aligned}$$

The dimension of this space is equal to the multiplicity

$$m(\theta_{\psi_F}(\pi)|_{\mathrm{SO}(W')}, (\sigma'|_{\mathrm{SO}(W)})^\vee)$$

of the basic tempered Bessel case $(\mathbf{SO}(W'), \mathbf{SO}(W))$ if

$$\theta_{\psi_F}(\pi)|_{\mathrm{SO}(W')}$$

is an irreducible tempered representation. From Prasad's conjecture, the semisimplification of the L-parameter of $\theta_{\psi_F}(\pi)$ has a summand of the trivial representation. This implies the irreducibility of $\theta_{\psi_F}(\pi)|_{\mathrm{SO}(W')}$.

6.2.3 Bridges by seesaw identity: Archimedean cases

Over Archimedean local fields, the main difficulty for generalizing this approach is that $\Theta_{\psi_F}(\sigma') = \theta_{\psi_F}(\sigma')$ was only established in stable-range situations. An ongoing work of R. Chen, J.-L. Zou and the author aims to refine this method using multiplicity inequalities so that we only use this result in stable-range cases.

In the ongoing work of Chen-Zou-myself, we modified this approach to only use "big theta=small theta" in stable-range cases. We first treat the seesaw identity for the diagram

$$\begin{tikzcd}
\Theta_{\psi_F,W+H_2^{\oplus k},V_{2n}}(\sigma_1^k)\boxtimes\omega_{V_{2n},\psi_F^{-1}} & \mathrm{Mp}(2n)\times_{\pm 1}\mathrm{Mp}(2n) \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(W'+H_2^{\oplus k}) \arrow[d, dash] \arrow[dl, dash] & \Theta_{\psi,V_{2n},W'+H_2^{\oplus k}}(\pi) \\
\pi & \mathrm{Sp}(2n) & \mathrm{O}(W+H_2^{\oplus k})\times\mathrm{O}(L) & \sigma_1^k\boxtimes 1
\end{tikzcd}$$

where H_2 is a hyperbolic plane and we choose σ_1^k to be the Langlands quotient of the parabolic induction $|\cdot|^{k-\frac{1}{2}}\times\cdots\times|\cdot|^{\frac{1}{2}}\ltimes\sigma_1$. This choice of σ_1^k guarantees the equality $\theta_{\psi,W+H_2^{\oplus k},V_{2n}}(\sigma_1^k) = \theta_{\psi,W,V_{2n}}(\sigma_1)$. Then we proved a multiplicity inequality by modifying Mœglin-Waldspurger's proof for the first inequality in [MW12] to obtain that

$$\dim \mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi,V_{2n},W'}(\pi)|_{\mathrm{SO}(W)},\sigma_1|_{\mathrm{SO}(W)}) \geq \dim \mathrm{Hom}_{\mathrm{O}(W)}(\theta_{\psi,V_{2n+2k},W'}(\pi)|_{\mathrm{SO}(W)},\sigma_1^k|_{\mathrm{SO}(W)}).$$

This implies

$$\begin{aligned}
m(\pi^\vee\boxtimes(\theta_{\psi,W,V_{2n}}(\sigma_1)\otimes\overline{\omega_{V_{2n},\psi}})) &= \dim\mathrm{Hom}_{\mathrm{Sp}(2n)}(\theta_{\psi,W,V_{2n}}(\sigma_1)\otimes\omega_{V_{2n},\psi},\pi)\\
&= \dim\mathrm{Hom}_{\mathrm{Sp}(2n)}(\theta_{\psi,W+H_2^{\oplus k},V_{2n}}(\sigma_1^k)\boxtimes\omega_{V_{2n},\psi},\pi)\\
&\leq m(\theta_{\psi,V_{2n},W'}(\pi)|_{\mathrm{SO}(W)}\boxtimes\sigma_1|_{\mathrm{SO}(W)}^\vee)
\end{aligned}$$

Similarly, we obtain the other side of the inequality by adding hyperbolic spaces to the left part of the seesaw diagram. By comparing the L-parameters of π and $\theta_{\psi,V_{2n},W'}(\pi)$, the equality

$$m(\pi^\vee\boxtimes\theta_{\psi,W,V_{2n}}(\sigma_1)\otimes\overline{\omega_{2n,\psi}}) = m(\theta_{\psi,V_{2n},W'}(\pi)|_{\mathrm{SO}(W)}\boxtimes\sigma_1|_{\mathrm{SO}(W)}^\vee)$$

reduces the basic tempered Fourier-Jacobi cases to the basic tempered Bessel cases as in the non-Archimedean situations.

6.3 Reduction to basic tempered using multiplicity formula

This section proves the local Gan-Gross-Prasad conjecture from the basic tempered cases using the multiplicity formula in Theorem 5.0.2.

6.3.1 Structure theorem

This section proves the structure theorem (5.0.1) for the reduction to tempered basic cases.

Let $\mathbf{G}_V = \mathbf{U}(V), \mathbf{SO}(V), \mathbf{Sp}(V), \mathbf{Mp}(V)$. For a parameter $\varphi_V : \mathcal{W}_{\mathbb{R}} \to {}^L\mathbf{G}_V$, its semisimplification $\varphi_V^{\mathrm{ss}} = \mathrm{Std}_{\widehat{\mathbf{G}}_V} \cdot \varphi_V$ can be decomposed as

$$\varphi_V^{\mathrm{ss}} = \bigoplus_{i=1}^{d_1} \varphi_1(s_{1,i}, \epsilon_i) \oplus \bigoplus_{j=1}^{d_2} \varphi_2(s_{2,j}, l_j)$$

where $s_{1,i}, s_{2,j} \in \mathbb{C}$, $\epsilon_i = 0, 1$, $l_j = 1, 2, \cdots$, $\varphi_1(s_{1,i}, \epsilon_i) : \mathcal{W}_{\mathbb{R}} \to \mathbf{GL}_1(\mathbb{C})$ is the L-parameter for the character $|\cdot|^{s_{1,i}}\mathrm{sgn}^{\epsilon_i}$ of $\mathbf{GL}_1(\mathbb{R})$ and $\varphi_2(s_{2,j}, \epsilon_j) : \mathcal{W}_{\mathbb{R}} \to \mathbf{GL}_2(\mathbb{C})$ is the L-parameter of the essentially discrete series representation $|\det|^{s_{2,j}} D_{l_j}$.

Since φ_V^{ss} factors through ${}^L\mathbf{G}_V$, it preserves an ϵ-hermitian form, therefore, there is a decomposition $V = V_0 \oplus (X \oplus X^\vee)$ and one can decompose φ_V^{ss} as

$$\varphi_V^{\mathrm{ss}} = \rho \oplus \rho^\vee \oplus \varphi_{V_0}^{ss},$$

where φ_{V_0} is a tempered L-parameter of $\mathbf{G}_{V_0}$ and ρ is a generic L-parameter of $\mathbf{GL}(X)$.

It is easy to verify (see [GGP12, §4]) that there are isomorphism

$$\mathcal{S}_{\varphi_V} \simeq \mathcal{S}_{\varphi'_V} \tag{6.3.1}$$

The following structure theorem characterizing representations in $\Pi_{F,\varphi_V}^{\mathrm{Vogan}}$ was proved from the standard module conjecture [CS98] when F is non-Archimedean and from [Vog78] when F is Archimedean. The proof has been elaborated in [MW12] [GI16] [At18] [X2] [Ch21] [Ch23a] for all cases.

Theorem 6.3.1. *There is a bijection*

$$\Pi_{F,\varphi_{V_0}}^{\mathrm{Vogan}} \to \Pi_{F,\varphi_V}^{\mathrm{Vogan}}$$
$$\pi_{V_0} \mapsto \sigma_V \ltimes \pi_{V_0}$$

Proof.

- When $\mathbf{G}_V$ is not metaplectic, the theorem follows from [Vog78, Theorem 6.2] over Archimedean local fields and the standard model conjecture [CS98] [Mu00] using the arguments in [MW12] over non-Archimedean local fields.
- When $\mathbf{G}_V$ is metaplectic, the theorem follows from the corresponding special orthogonal case and properties of theta correspondence.

□

Then the structure theorem follows naturally from Theorem 6.3.1.

6.3.2 Reduction with multiplicity formula

In the setting of Section 2.1, using the structure theorem (Theorem 5.0.1), given generic parameters φ_V, φ_W and $\pi_V \in \Pi_{F,\varphi_V}(\mathbf{G}_V), \pi_W \in \Pi_{F,\varphi_W}(\mathbf{H}_W)$ in generic L-packets, we have

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \pi_W = \sigma_W \ltimes \pi_{W_0}$$

for $\pi_{V_0} \in \Pi^{\text{Vogan}}_{F,\varphi_{V_0}}$, $\pi_{W_0} \in \Pi^{\text{Vogan}}_{F,\varphi_{W_0}}$ and generic σ_V, σ_W. From Theorem 5.0.2, we have

$$m(\pi_V, \pi_W) = m(\pi_{V_0}, \pi_{W_0}).$$

This multiplicity formula connects the generic cases and a tempered cases. From Theorem 5.0.5(1) and Theorem 5.0.4(1), we an find an appropriate σ'_W such that $\pi_{V_0}, \sigma'_W \ltimes \pi_{W_0}$ are representations correspond to a basic pair and

$$m(\pi_{V_0}, \pi_{W_0}) = m(\pi_{V_0}, \sigma'_W \ltimes \pi_{W_0})$$

for all $\pi_{V_0} \in \Pi^{\text{Vogan}}_{F,\varphi_{V_0}}, \pi_{W_0} \in \Pi^{\text{Vogan}}_{F,\varphi_{W_0}}$. These reduces the local Gan-Gross-Prasad conjecture to the basic tempered cases.

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