The 'Potential' of Population Density and Positive Rent Gradients[+]

by

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and

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I. Introduction

For many years theories[1] of intraurban residential location have been put forth which are based on the economic trade-off between housing and transportation costs. In these well-known models, transportation costs are determined solely by accessibility to a monocentric city[2] or CBD (central business district) which contains all employment. Hence, these models assume the journey-to-work as the single most important determinant of household location. This emphasis on the journey-to-work has not been without its critics[3] who have attacked the importance placed on economic variables (in particular, accessibility to the CBD), and suggested consideration to be given to social relations between groups, neighborhood amenities, and environmental factors.

It might be suggested that some of these criticisms were silenced, in part, when Muth [13] was able to extend the trade-off models to derive the conditions under which urban population densities and rents are given by a negative exponential function of distance from the CBD which takes the form:

$$(1)\quad f(x,y) = c_o e^{-cr}$$

where, $f(x,y)$ is density (or rent) at point (x,y) in region, r is distance of point from the CBD, and c is the gradient. Muth's analysis was significant because it provided the missing theoretical economic basis for a long standing empirical result. Over 25 year ago, Colin Clark [5] discovered that urban population densities tend to decline exponentially and this has been verified by researchers in various disciplines[4] for different cities and years. However, economists did not embrace this empirical regularity until Muth provided his theoretical support. The Muth model is nonetheless still based solely on the housing and transportation trade-off and considers only accessibility to a monocentric city where all employment is located.

A corollary of the declining density result of the trade-off theorists is that rents must decline with distance from the CBD. In the Muth [13] model rents are negative exponential, as was the density function, though other functional forms are possible (e.g., Beckmann [3] and Montesano [12] consider a power function for both rent and density). While according to the trade-off theorist the rent and density functions should be of same form, (or at least both should be declining with respect to the CBD), the empirical evidence for rent is much weaker than for density. In general, attempts to fit negative exponential functions for rent have resulted in poorer fits than for density. These attempts have also resulted in generally flatter (i.e., c, the gradient, is smaller), and in some cases even positive[5] rent gradients. In a recent article, Richardson [15] attempts to provide a theoretical explanation for positive rent gradients by introducing what he calls an "externality rent".[6] However, his is still basically a trade-off theory since he assumes location rent is based solely on accessibility to the CBD where all employment is located.

It is intended in the remainder of this paper to provide an alternative explanation, albeit, perhaps, not as theoretically elegant as the trade-off theorists for reasons to be noted later. The approach, which is motivated by the relative strength of the empirical results for density and rents already alluded to, will be to propose that rents are based on general accessibility. While it would be desirable to eventually relax the monocentric employment assumption and, hence propose that households consider accessibility to multiple workplaces, this will not be attempted in this paper. Instead, it is proposed that households consider accessibility to both the CBD (for employment) and to the rest of the urban population. That is, locations which are accessible[7] to people in other areas of the city, for what might be termed the purpose of "visiting" or social interaction, are more desirable than areas without such accessibility. It will be shown that such accessibility for the purpose of "visits" is not synonomous with accessibility to the CBD. In fact, such "visiting" accessibility declines less rapidly than accessibility to the CBD and in certain circumstances can even be positive (i.e., "visits" accessibility can increase with distance from the CBD).[8]

While the ultimate goal is to develop a model along the lines of the one to be presented which would "explain" both rent and density together, in this paper an attempt will be made to explain rent or general accessibility[9] (i.e., to CBD and for "visits") assuming that the density of population is given. Since the negative exponential density is so widely accepted and empirically supported, it will be used though the approach could be generalized to other density specifications.

It is necessary to provide some support for the decision to develop a model of rent assuming density to be given as a negative exponential function, since this contradicts the trade-off theorists who suggest both rent and density are negative exponential. One reason for retaining the density specification (i.e., as negative exponential)without the corresponding rent specification would be the weak empirical support for the latter noted earlier.

A more theoretical rationale may be provided by the recent emphasis[10] among urban economists on the need to consider the historical stock of housing as a determinant of urban residential location. One interpretation of this, which is implicit in the decision to "explain" rent given density, is that the existing housing stock determines in large part the density, but not the rent, of urban areas. It could be argued that at some time in the past the trade-off theorists were correct and both rents and density were in long-run equilibrium, and at this same time in the past (for which no data exists) the empirical fits for both density and rent were good. Such a time might have been when the trade-off theorists assumptions of a monocentric CBD containing all employment (the accessibility to which is all important in trade-off theories) were more true than today.

It might be further conjectured that over time changes occurred (e.g., changes in tastes, the transportation system, or the location of employment in the CBD) that disturbed the original "long-run equilibrium". The trade-off theorists would confront such a change by determining a new "long-run equilibrium" solution for density and rent. It seems more reasonable that the responsiveness to such a change would not be the same for density and rent. Specifically, the existing housing stock would limit the possible changes in density that could occur over time. On the other hand, rent is much more likely to be the equilibrating or responsive variable for such changes. This, then, provides the basis for the approach of this paper, which is to explain rent given density. The focus on the rent (or value) attached to accessibility for the purpose of "visits" could also be justified on the historical grounds in that over time people in urban areas have come to have more leisure time and so place greater emphasis on social interaction or "visits". As a result, rents have adjusted to reflect this emphasis, while density has remained more or less fixed (i.e., conforming to the negative exponential function) because of the permanence of the existing housing stock.

II. The Mathematics of "Potential" for a Negative Exponential Density

Since the intent is to consider the rental value of general accessibility for a given or assumed negative exponential population density it is necessary to first measure such accessibility. Once this has been done in this section using the concept of "potential" it will be possible in the next section to equate potential with rental value (i.e., areas with high potential will have high rental value). Potential, which orginated as a physics analogue, has been used often in urban and regional studies to measure accessibility to some spatial variable (e.g., income, employment, or population) defined over a bounded region or urban area.

In these instances potential has been calculated in discrete form (i.e., for a finite number of areas or zones) as a summuation. The continuous counterpart of potential based on integration, not summation, will indicate the accessibility of any (x,y) point in the urban area to the entire urban population (i.e., to the prescribed or given negative exponential population density). This continuous form of potential will be derived assuming that accessibility is determined by time and that travel speeds vary linearly with distance from the CBD.[11]

The potential, U, of a plane region, R, of density, f, at a point (x,y) is given by

$$(2) \quad U(x,y) = \iint_R \frac{f(w,n)\, dwdn}{d(x,y,w,n)} \quad .^{12}$$

The interest here is in the case where R is a unit circle centered at the origin, (x,y) is interior to R, d is given by

$$(3) \quad d(x,y,w,n) = \sqrt{(x-w)^2 + (y-n)^2}$$

and the density is

$$(4) \quad f(w,n) = K(c)e^{\left(-c\sqrt{w^2 + n^2}\right)}$$

with a positive constant and K(c) chosen so that $\iint_R f(w,n)\, dwdn = \pi$. In particular,

$$(5) \quad K(c) = c^2/2(1-ce^{-c}-e^{-c})$$

There is an interest in the extreme case $c = \infty$, since as $c \to \infty$ we approach the case in which all the "mass" (e.g., employment) is concentrated at the origin (e.g., Central Business District or CBD).[13] In this instance, the resulting potential (e.g., Pc) is

$$(6) \quad U(x,y) = \pi/\sqrt{x^2 + y^2} = \pi/r$$

where $r = \sqrt{x^2 + y^2}$ = distance of point (x,y) from the origin.

It is possible to generalize (2) by replacing distance d by time t and thus creating a time potential given by

$$(7) \quad U(x,y) = \iint_R \frac{f(w,n)\,dw\,dn}{t(x,y,w,n)}$$

in which $t(x,y,w,n)$ is the time to travel between points (x,y) and (w,n). It is further assumed that the travel speed is given by the linear function

$$(8) \quad S = a + br$$

where $a>0$, and $b\geq 0$ are constants and $r = \sqrt{x^2 + y^2}$. If the travel speed is given by (8), then

$$(9) \quad t(x,y,w,n) = \int_0^1 \frac{d(x,y,w,n)\,d\delta}{a + b\sqrt{[x(1-\delta) + \delta w]^2 + [y(1-\delta) + \delta n]^2}}.$$

Note that in the case of a constant travel speed (b=0) (2) and (7) are proportional (since distance = speed x time) and further that if, in addition, a = 1, (7) reduces to (2). For the case $c = \infty$, the equation (e.g., for Pc) corresponding to (6) is

$$(10) \quad U(x,y) = \frac{\pi a}{\sqrt{x^2 + y^2}}$$

when b = 0, and

$$(11) \quad U(x,y) = \frac{\pi b}{\ln\left(1 + \frac{b}{a}\sqrt{x^2 + y^2}\right)}$$

when $b > 0$.

Since equation (7) is not easily solved in closed form, it will be evaluated using numerical integration. These calculations were performed for various values of c and for both constant and linearly increasing travel speeds using FORTRAN programs developed by the authors and contained in the Appendix. These programs perform numerical integration of (7) using Simpson's Rule (Note that due to symmetry the potential only need to be calculated along the X (or r) axis.)[14] While these programs could be employed to generate several data points (i.e., potential for several values of r), only a few graphs will be presented which demonstrate the nature or shape of such potential functions for various assumed travel speeds in the region.

III. Using "Potential" to Measure the Rental Value of Accessibility

In this section various cases are considered (i.e., values of c, the density gradient, and travel speeds) and potential will be given for both for "visits", Pv, and for accessibility to the CBD, Pc. The latter calculations are based on equations (10) or (11) since using $c = \infty$ in negative exponential is equivalent to assuming all density (i.e., employment) is located in the CBD, whereas Pv has been calculated using equation (7). When Pv and Pc are found for various cases, this will be done assuming the mass is the same in both cases.[15]

It would be most desirable to directly show that rent will be flatter when the value of accessibility for "visits" is taken into account in addition to accessibility to the CBD since it has already been noted that the trade-off theorists solution for rent is based solely on the latter type of accessibility. One possible comparison would be to develop a "revised" negative exponential rent function for both "visits" and CBD accessibility with a lower value of c (i.e., flatter) than the rent function based solely on accessibility to the core. Instead, "potential for visits", Pv, will be found and compared to the corresponding potential, Pc, based only on accessibility to the CBD. If, as will be shown to be the case, the Pv function is flatter than Pc for the same case (i.e., travel speed and density gradient, c) this will indirectly prove that a rent function based on accessibility for "visits" will be flatter than one which is not. An attempt could be made to estimate from Pv values what negative exponential function (i.e., what value of c below the original assumed density) corresponds to it, but this is not necessary since in general a flatter negative exponential density results in a flatter potential for "visits", Pv.[16]

Consequently, it is appropriate to proceed with presentation of potential values for access to the CBD (employment), Pc, and potential values for accessibility to the population for the purpose of "visits", Pv. It will be shown that in all cases, Pv is flatter (and in certain instances positive or increasing) than Pc which demonstrates that trade-off theorists rent function, based solely on CBD accessibility, will be steeper than one which takes into account "visit" accessibility, Pv. In each case Pv, Pc and Pm = 1/2 Pv + 1/2 Pc will be graphed. The latter suggests only one possible weighting (1/2, 1/2) which might be given to the two types of accessibility.[17]

The first case (Figure 1) to be considered, City A, is one where c = .1 and travel speed, s = .25 + .25r. This travel speed function assumes travel is twice as fast at the perimeter (r = 1), .50 vs. .25, than in the CBD (r = 0). Later, a "faster" travel speed function will be considered where speed is six times as fast at the perimeter (r = 1). In each case, a corresponding[18] constant travel speed graph (Figure 2) of Pc, Pv and Pm will be presented in a separate table. The reason for considering a linear travel speed is based on the assumption that congestion leads to slower travel speed near the CBD. Also, it will be shown that only in the case of linear travel speed can Pv be positive.[19]

The potentials for City A (c = .1) in Figure 1 (linear travel speed s = .25 + .25r) and Figure 2 (constant travel speed S = .36067) will be examined first. Taking Figure 1, it is observed that Pv, Pc, and Pm are decreasing though Pv is flatter than Pc while Pm is intermediate to both, since it is the average of Pv and Pc. This basic difference, Pv flatter than Pc, holds in Figure 2 as well,

FIGURE 1

POTENTIALS FOR CITY A (c = .1)

LINEAR SPEED S = .25 + 25r

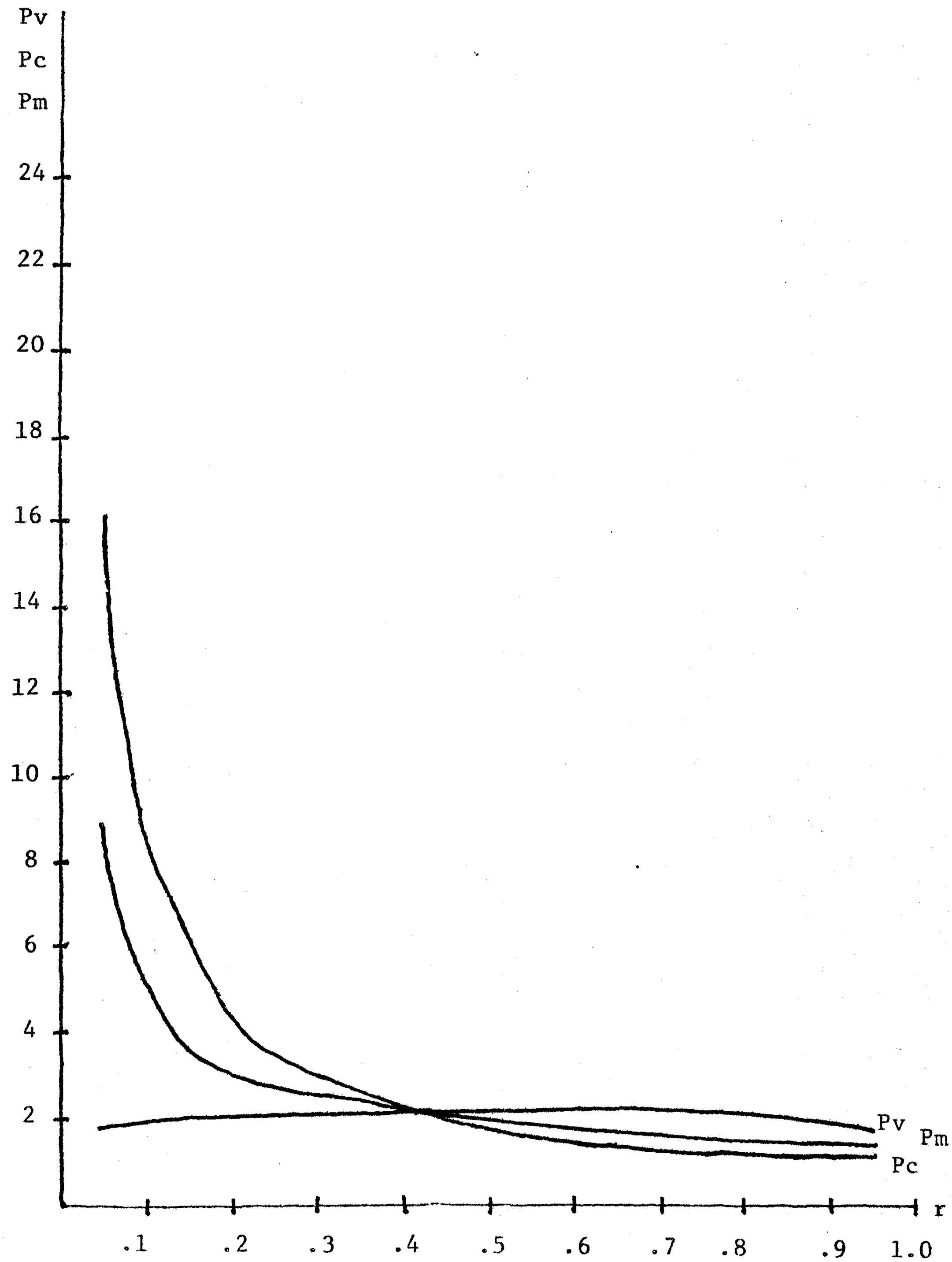
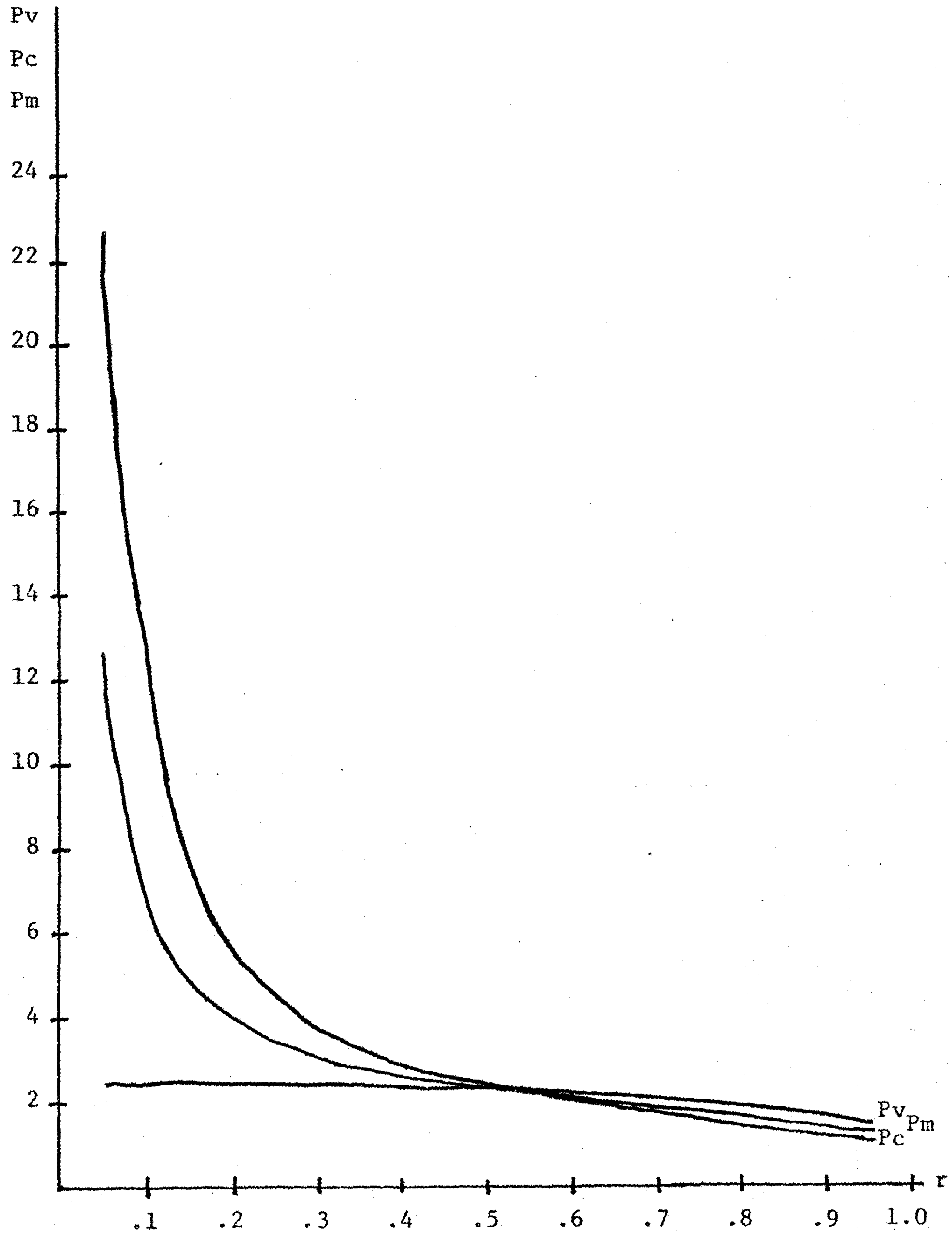


FIGURE 2

POTENTIALS FOR CITY A (c = .1)

CONSTANT SPEED = .36067



though it can be seen that Pv is flatter in Figure 1 than Figure 2. It will be apparent in the rest of the figures that with higher congestion (i.e., as we consider a "faster" speed situation), Pv will be flatter and eventually increasing while Pc is always decreasing. While changes in c will also be considered, these are of secondary importance since as long as travel speed is constant, Pv will be decreasing for all values of c.

Turning to City B, the same c (=.1) is retained, but a "faster" linear travel speed (s = .25 + 1.25r) is used. Again results for this speed (Fig. 3) and the equivalent constant speed (i.e., s = .697 which is the average speed for s = .25 + 1.25r and is higher than City A) case (Fig. 4) are provided. The points made earlier hold here and furthermore, it is observed that Pv is increasing while Pc is decreasing. This is the result of the "faster" travel speed being considered.

The final case, City C, which will illustrate further the nature of potentials, employs the same travel speed as City B, but a steeper[20] density (c = .5). Figure 5 shows the linear travel speed potentials and it can be seen that with the same speed as Figure 3, Pv is not as steeply increasing. This demonstrates that given a travel speed which results in increasing Pv functions, Pv will be less positive (or decreasing) as c approaches zero. Likewise, for a given c the same is true as the travel speed becomes "faster" ($b \to \infty$) with respect to distance from the CBD. A final comment would be that Pc is the same in Figures 3 and 5 and in Figures 4 and 6 since it is not affected by changes in c, only changes in speed.

IV. Conclusion

Hopefully, with these illustrative cases, or hypothetical cities, it has been possible to demonstrate the basic premise which is that Pv, accessibility to the entire population, which is assumed to be distributed according to a negative exponential density for the purpose of "visits", is flatter than accessibility to the

FIGURE 3

POTENTIALS FOR CITY B (c = .1)

LINEAR SPEED S = .25 + 1.25r

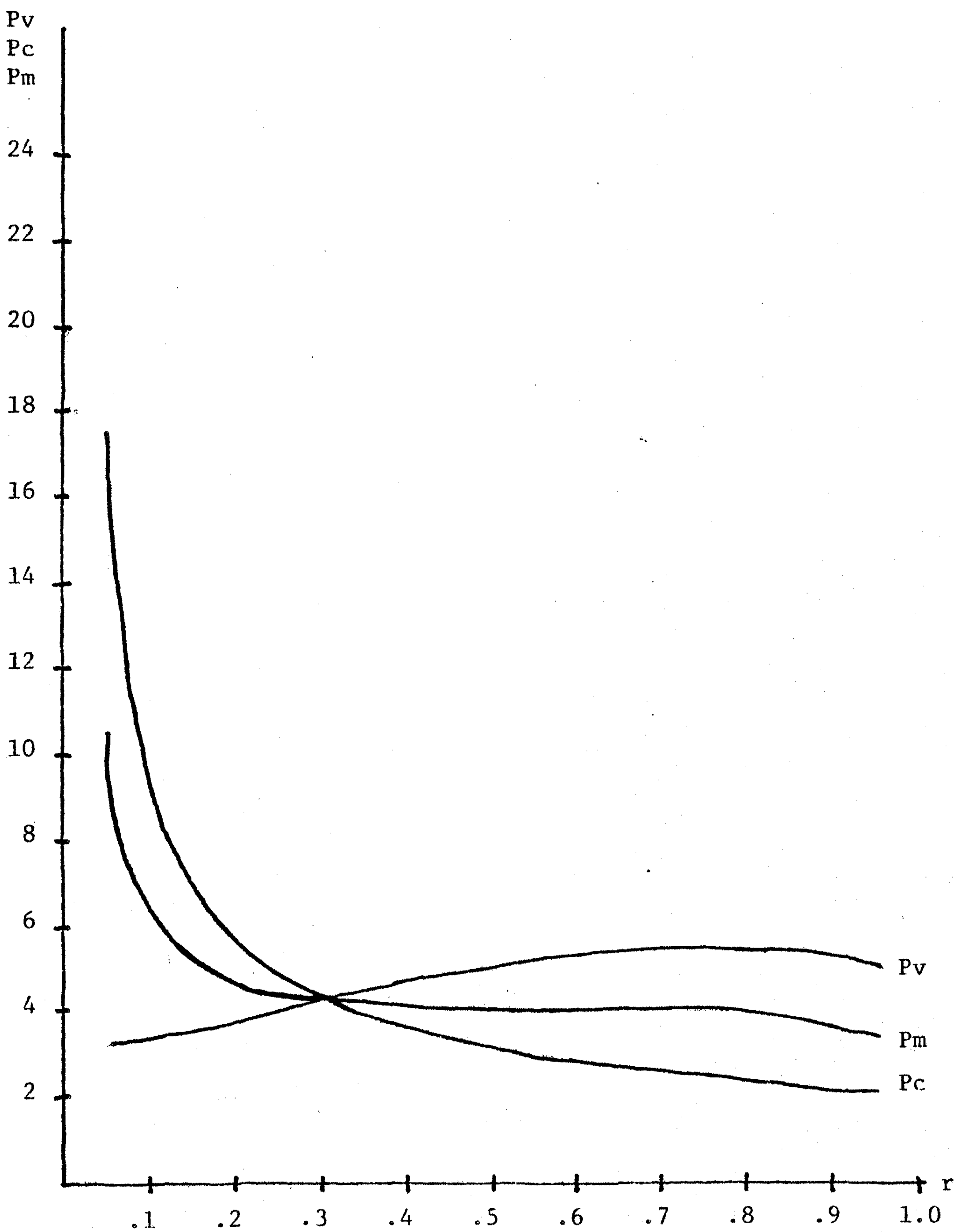


FIGURE 4

POTENTIALS FOR CITY B (c = .1)

CONSTANT SPEED = .69764

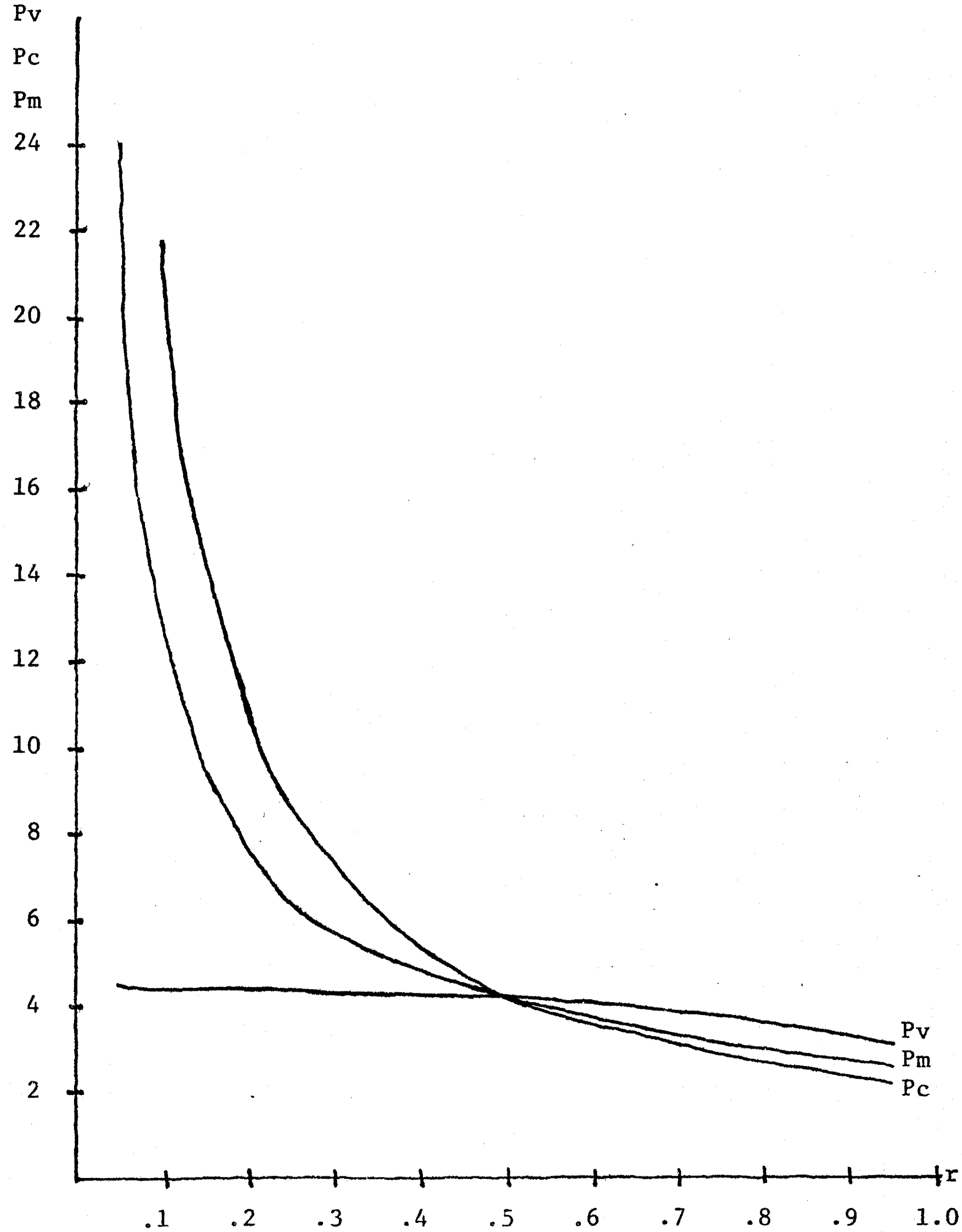


FIGURE 5

POTENTIALS FOR CITY C (c = .5)

LINEAR SPEED S = .25 + 1.25r

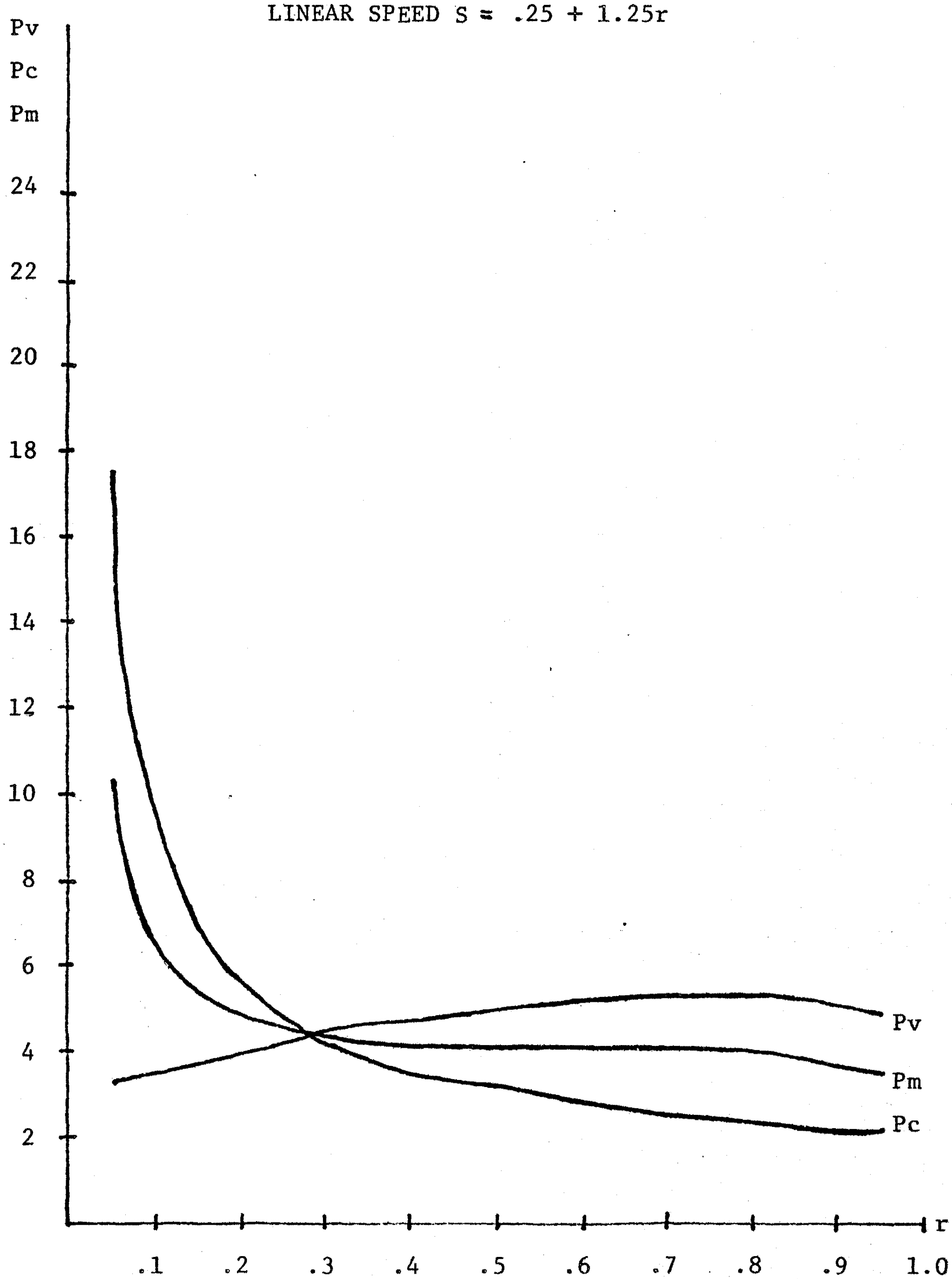
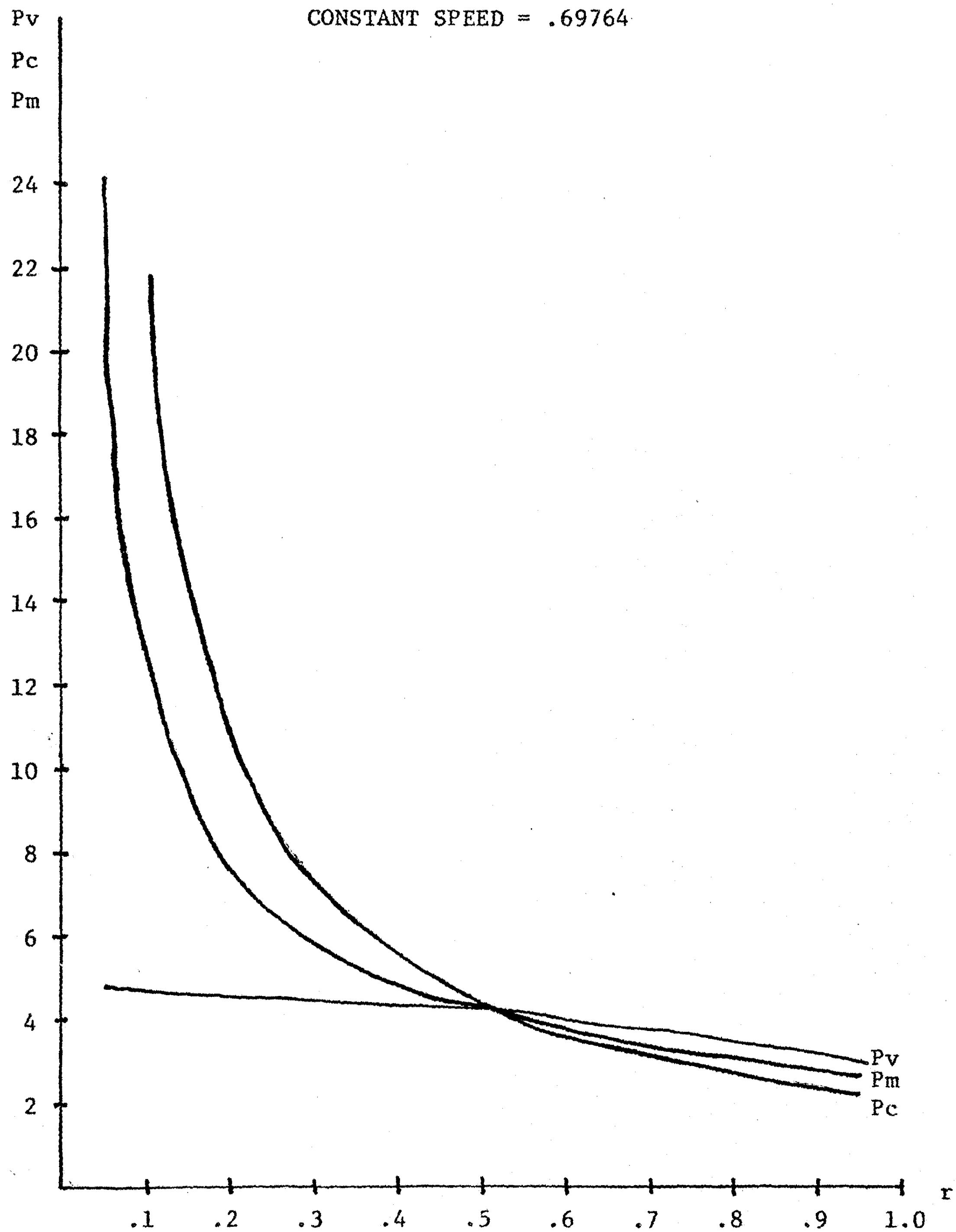


FIGURE 6

POTENTIALS FOR CITY C (c = .5)

CONSTANT SPEED = .69764



CBD wherein all employment opportunities are located. It follows that if both forms of accessibility (Pv and Pc) are valued by households, then rents for land will reflect both values. Therefore, rents will be flatter when Pv is taken into account[21] than when rents are based solely on CBD accessibility (Pc) as in the trade-off theories.

It has been shown that rents may even be positive since Pv can be positive when travel speed is linear. These speeds have been chosen arbitrarily to illustrate different hypothetical situations, though one might consider estimating such speed functions for different cities. The theory presented here would suggest that the rent gradient will tend to be flatter or positive in cities with a "faster" (i.e., $b > 0$ in $s = a + br$) speed function. Such cities might be those without congestion near the CBD and/or those with a "fast" transportation system near the perimeter.

To elaborate on the last point, it may turn out that the model would prove useful for simulating the effects on rent of changes in the transportation system of a city. For instance, a new system which reduced congestion in the core (e.g., because of constructing freeways through a city, as has been the case in recent years) would reduce b in the travel speed function and tend to make rents steeper because Pv would be less flat. Such a simulation would be assuming that it will be rents, more so than density, that would adjust or equilibrate to an exogenous change such as a modification of the transportation system. The rationale for considering rents as more responsive than density, which is the major departure of this paper's approach from the trade-off theories, is based, in part, on the permanence of the housing stock as was detailed earlier.

Whether this appraoch is superior to trade-off theories is at this time uncertain. Definite differences have been suggested with respect to the rent gradient that can be empirically tested. Hopefully, at some point it will be possible to derive a general theory of intraurban residential location which would locate households in "long-run equilibrium" based on both accessibility to the CBD (for employment) and to the rest of the population. The trade-off theories by considering only the former are able to reduce the problem to one dimension (distance from the CBD) and use differential calculus to obtain a solution. When the second type of accessibility is introduced the problem can no longer be reduced to one dimension. It would seem that any analagous solution would require a spatial system of partial differential equations. While such a "long-run equilibrium" theoretical solution is being sought, the question remains whether the simple, but elegant, trade-off theories are to be preferred and accepted. It is believed that the weight of the empirical evidence with respect to rent has been, and will continue to be, more supportive of the approach taken in this paper.

FOOTNOTES

[1]For example, Alonso [1], Kain [7], and Muth [13].

[2]Some attempts to consider two employment centers have been made (e.g., Muth [13] and Romanos [17])but a complete solution for n-centers has not been offered. One of the authors has proposed a model (Steinnes and Fisher [19]) which allows for residence and employment to be simultaneously determined, but it does not emphasize the housing and transportation cost trade-off.

[3]For example, Anderson [2] and Stegman [18].

[4]For example, Berry, Simmons, and Tennant [4], Harrison and Kain [6], Mills [10 and 11], and Muth [13].

[5]For example, Richardson, Vipond, and Furbey [16] and Wilkinson [21].

[6]This rent is a premium paid for land or a neighborhood with low density. That is, he assumes density to be given rather than simultaneously determined with rent as in other trade-off models.

[7]In the next two sections the potential function of a negative exponential density function will be derived and an explanation of how it may be used to measure accessibility to population in the urban area will be provided. Furthermore, since such accessibility has value, if follows that rent will be higher in areas with high potential.

[8]These cases include cities which are not circular and those where the transportation system is such that speed of travel is slower near the CBD.

[9]Others,(e.g., Lowry [9] and Romanos [17])have considered general accessibility (i.e., other than accessibility to the CBD) but not as a means of determining rent, given density.

[10]For example, Harrison and Kain [6], Quigley [14], and Romanos [17].

[11]More precisely, the intent is to consider the case where travel speed is slower near the CBD because of, perhaps, congestion. In an earlier paper [20] the authors developed the continuous potential of a negative exponential density using distance, rather than time, to measure accessibility. In this paper this is equivalent to the special case, which will be considered, where travel speed is constant throughout the region.

[12]For a more detailed treatment of potential than will be provided in this section, the interested reader is referred to Kellogg [8].

[13]This case will be useful to measure accessibility to the CBD, which will be referred to as the potential of CBD, Pc. On the other hand, accessibility to the population, as assumed to be given by (4), will be measured by (2) and termed potential for "visits", Pv.

[14]This is true for a "full" city (i.e., R is defined for a unit circle). In an earlier study [20] the authors have presented the potential, equation (7), for the case of cities which are less than "full", (e.g., half circle for cities like Chicago or Duluth, which are located on a lake.)

[15]This mass, as explained in the last section, is π. That is, it is being assumed that there are the same number of jobs in the core as there are people located in the region, though this could easily be changed.

[16]For more details on the relation between the shape of density (c) and potential, see Steinnes and Snow [20]. This connection is revealed, in part, in Figures 4 and 6 which demonstrate tha Pv is flatter for c = .1 than for c = .5. While Steinnes and Snow [20] have developed proxies for potential based on c, the density gradient, the reverse, estimating c from a set of potential values, has not been done.

[17]At a later date an empirical regression study might be conducted using actual rents for a particular city and fitting bv, bc of Pm = bv Pv + bc Pc. Furthermore, such a function might be fitted for different groups (e.g., income, race, or occupation) in order to determine if each weighs the two types of accessibility differently.

[18]The constant travel speed chosen is the average speed in going from CBD ($r = 0$) to perimeter ($r = 1$) using the linear travel speed function ($a + br$). That is, the time elapsed in traveling radially from CBD to perimeter will be the same for constant and linear travel speed in each City (A, B, or C).

[19]It is possible for Pv to be positive, with respect to distance from the CBD, for the constant speed case when the city is less than full as detailed in Steinnes and Snow [20]. However, for a full city, Pv can only be positive when travel speeds are rising with respect to r, (i.e., travel is faster toward the perimeter of the CBD).

[20]The values of c chosen are among those found most often for U.S. cities in empirical studies (e.g., Mills [11] and Muth [13]).

[21]The extent to which Pv is taken into account will depend on the weights (bv, bc) of Pm. As noted earlier, the determination of such weights is something most appropriately determined empirically.

APPENDIX

This program will perform integration of (2), Pv, and (6), Pc, for various values of c and a, b in linear travel speed function a + br. The version presented will compute potentials for c = .1, .5, and 1, a = .25, b = .25, .5, .75, 1.0, and 1.25; and for equivalent constant speed cases of each a, b pair. Potential values are found for .05 increments of r (i.e., r = .05, .1, .15, ..., .95), but this could be increased by increasing N and M. However, an increase would result in a rise in computer cost.

```
      DIMENSION X(11),FW(11),CA(11),CB(11),CW(11),CDW(11)
      N=10
      M=10
      AM=N
      AM=N
      MM=M-1
      NM=N-1
      MP=M+1
      NP=M+1
      XA=3.1415927/AN
      Z=0.0
      DO 30 IY=1,5
      YII=IY
      YINT=.25
      YISL=.25*YII
      AVGSP=YISL/ALOG(1.0 + (YISL/YINT))
      DO 30 ISW=1,2
      DO 30 IC=10,28,9
      IF (IC-19) 10,12,14
   12 C=0.5
      GO TO 15
   14 C=1.0
      GO TO 15
   10 C=0.1
   15 Z=(C*C)/(1.0-C*(EXP(-C))-EXP(-C))
      DO 30 IR=1,19
      AR=IR
      R=.05 + (AP-1.0)*.05
      IF (ISW-1) 70,70,90
   70 PI=(AVGSP*3.1415927)/R
      GO TO 100
   90 PI=(YISL*3.1415927)/(ALOG(1.0 + (YISL/YINT)*R))
  100 BS=0.0
      DO 135 K=1,NP
      AK=K
      AI=(AK-1.0)*XA
   11 GW=-R*COS(AI) +SQRT(1.0-R*R*SIN(AI)*SIN(AI))
      WW=GW/AM
```

```
   13 DO 220 J=1,MP
      SUM=0.0
      AJ=J
      DW=(AJ-1.0)*WW
      SQW=SQRT(R*R +DW*DW+2.0*R*DW*COS(AI))
      IF (ISW-1) 370,370,970
  970 DO 600 JJ=1,WP
      AVGSP=0.0
      AVGSP=0.0
      AJJ=JJ
      CA(JJ)=.1*(AJJ-1.0)*R + (1.0-.1*(AJJ-1.0))*(R+DW*COS(AI))
      CB(JJ)=(1.0-.1*(AJJ-1.0))*(DW*SIN(AI))
      CW(JJ)=1.0/(YINT+YISL*SQRT(CA(JJ)*CA(JJ)+CB(JJ)*CB(JJ)))
  600 CONTINUE
      DEVEN=0.0
      DO 601 JS=2,M,2
      DEVEN=DEVEN+CW(JS)
  601 CONTINUE
      DODD=0.0
      DO 602 JR=3,MM,2
      DODD=DODD+CW(JR)
  602 CONTINUE
      CDW(J)= (.1/3.0)*(CW(1)+4.0*DEVEN+2.0*DODD+CW(MP))
      FW(J)=(EXP(-C*SQW))/CDW(J)
      GO TO 220
  370 FW(J)=(EXP(-C*SQW))*AVGSP
  220 CONTINUE
      XEVEN=0.0
      DO 500 J=2,M,2
      XEVEN=XEVEN+FW(J)
  500 CONTINUE
      XODD=0.0
      DO 501 J=3,MM,2
  501 XODD=XODD+FW(J)
      X(K)=(WW/3.0)*(FW(1)+4.0*XEVEN+2.0*XODD+FW(MP))
  135 CONTINUE
      HXEVEN=0.0
      DO 520 K=2,N,2
      HXEVEN=HXEVEN+X(K)
  520 CONTINUE
      HXODD=0.0
      DO 530 K=3,NM,2
      HXODD=HXODD+X(K)
  530 CONTINUE
  531 BS=(XA/3.0)*(X(1)+4.0*HXEVEN+2.0*HXODD+X(NP))
  532 P=BS*Z
      P2=(P1 + P)/2.0
  533 WRITE(6,20) P,C,R,YINT,YISL,AVGSP,P1,P2
   26 FORMAT(8F10.5)
   30 CONTINUE
      STOP
      END
```

REFERENCES

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