LORENZ EQUATIONS PART II: "RANDOMLY" ROTATED HOMOCLINIC ORBITS AND CHAOTIC TRAJECTORIES

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Lorenz Equations Part II:
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Abstract. The Lorenz equations are a system of ordinary differential equations

$$x' = s(y - x), \quad y' = Rx - y - xz, \quad z' = xy - qz$$

where s, R, and q are positive parameters. We show that for each non-negative integer N, there are positive parameters s, q, and R such that the Lorenz system has homoclinic orbits associated with the origin (i.e., orbits that tend to the origin as $t \to \pm\infty$) which can rotate around the z-axis $N/2$ times; namely, the x-component changes sign exactly N times, the y-component changes sign exactly $N+1$ times, and the zeros of x and y are simple and interlace. Also, we show the existence of "randomly" rotated homoclinic orbits (i.e., homoclinic orbits that rotate around the z-axis for arbitrary prescribed number of times, and during each time interval in which trajectories do not cross the $\{x = 0\}$ plane, they rotate around the vertical line $x = y = \sqrt{q(R-1)}$ or $x = y = -\sqrt{q(R-1)}$ certain number of times determined by arbitrary prescribed finite integer sequences). Furthermore, we establish the existence of chaotic trajectories (i.e., trajectories that rotate around the z-axis infinitely many times and during each time interval in which the trajectories do not cross the $\{x = 0\}$ plane, the trajectories rotate around the vertical line $x = y = \sqrt{q(R-1)}$ or $x = y = -\sqrt{q(R-1)}$ certain number of times determined by arbitrary prescribed bounded integer sequences).

Keywords. Lorenz equations, homoclinic orbits, chaotic behavior.

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1. Introduction.

The Lorenz equations we studied here are a system of ordinary differential equations

$$\begin{cases} x' = s(y - x), \\ y' = Rx - y - xz, \\ z' = xy - qz \end{cases} \tag{1.1}$$

where $' = \frac{d}{dt}$ and s, R, and q are positive parameters. This system was first presented in 1963 by E. N. Lorenz [11] in studying fluid convection in a two dimensional layer heated from below. In the last decades, there has been an immense amount of interest generated by these equations due to the fact that for some parameter values, numerical computed solutions oscillate in the pseudo-random way which people call "chaotic". For

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more detailed description of the observed "chaotic" behavior and mathematical theories built upon (1.1) such as the geometric models of the Lorenz equations, Sil'nikov–type bifurcations, and averaging methods, see [3, 5, 15, 13, 17, 14], a review book of Sparrow [16], a mathematical text book of Guckenheimer & Holmes [4], and the references therein.

Generally speaking, to apply certain well–developed theories to the concrete example of (1.1), certain hypotheses have to be verified. One way to achieve this would be by a computer simulation, but there are few rigorous results. Recently, Hastings & Troy [8, 9], Hassard, Hastings, Troy, & Zhang [7], and Mischaikow & Mrozek [12] built up mathematical theories for certain characterizations of the chaotic behavior of (1.1) and implemented rigorous arithmetic numerical schemes to verify the validity of their hypotheses, and hence lead to affirmative conclusions on the chaotic behavior they studied for the solutions of (1.1) for certain parameter values (s, R, q).

Among all the solutions of (1.1), a very special one is a homoclinic orbit associated with the origin, which is a solution having the property that it approaches the origin as $t \to \pm\infty$. An example of its fascination is a homoclinic explosion which refers to the appearance of various chaotic trajectories when the parameters are perturbed from the values where there is a homoclinic orbit. See Sparrow [16] for more detailed description of the homoclinic explosion.

The existence of homoclinic orbits associated with the origin for (1.1) can be seen from strong numerical evidence. The first pure mathematical proof was given by Hastings and Troy [10]. Using a shooting argument and a pioneer, though tedious, mathematical analysis on the nature of the solution when $(s, R, q) = (10, 1000, 1)$, they were able to show that for each (s, q) in some neighborhood of the point $(10, 1)$ there is an R in the interval $(1, 1000)$ such that the Lorenz system has a homoclinic orbit associated with the origin. A rigorous numerical implementation of a similar method for the existence of a homoclinic orbit was done by Hassard and Zhang [6] to pin down the value of the parameter R. They showed that when $s = 10$ and $q = 8/3$, R is between 13.9265 and 13.927.

In [1], we established the following result:

Proposition (Theorem 1.1 of [1]) *Assume that s and q are given positive constants. Then the Lorenz system (1.1) has at least one homoclinic orbit associated with the origin for some $R \in (0, \infty)$ if and only if $s > (2q + 1)/3$.*

The existence part of the proposition was proven by a shooting argument first used by Hastings and Troy [10] and later by Hassard and Zhang [6], whereas the non–existence part was proven by construction of Liaponov functions.

It is worthy mentioning that all the homoclinic orbits established in [10, 6, 1] are the simplest homoclinic orbits of (1.1) in the sense that x does not change sign and x' changes sign only once. In this paper, we shall establish the existence of more complicated homoclinic orbits, as well as chaotic trajectories.

Observed from the Proposition that the line $s = (2q + 1)/3$ is the border line of the existence and non–

existence of any homoclinic orbits associated with the origin, we anticipate that at least a small portion of the chaotic behavior of the Lorenz system bifurcates from this border line. After studied the behavior of the unstable manifold of the origin when R is very large and $s-(2q+1)/3$ is positive but very small, we found that there are not only complicated homoclinic orbits, but also chaotic trajectories and homoclinic explosions. For the existence of homoclinic orbits, we report the following results:

Theorem 1.1. **(Rotated Homoclinic Orbits)** *For any given positive number q and non–negative integer N, there exists a large positive constant $R_0(N,q)$ such that for each $R > R_0$, there is a positive number $s = s(N,q,R) = (2q+1)/3 + O(R^{-1/2})$ such that the Lorenz system (1.1) has a homoclinic orbit associated with the origin which rotates around the z-axis $N/2$ times; more precisely, x changes sign exactly N times, y changes sign exactly $N+1$ times, and the zeros of x and y are simple and interlace.*

Theorem 1.2. **("Randomly" rotated homoclinic orbits)** *Let q be a given positive constant and N be a given positive integer. Then there exists a positive constant $R_0(q,N)$ such that for every $R > R_0$, every integer $L \geq 2$, and every integer sequence $\{k^l\}_{l=2}^{L}$ satisfying $k^l \in \{0,1,\cdots,N-1\}$ for all $l = 2,\cdots,L$, there exists a positive number $s = (2q+1)/3+O(R^{-1/2})$ such that the Lorenz system (1.1) has a homoclinic orbit associated with the origin which has the following properties:*

(1) all the zeros of x, x', and y are simple;

(2) x has exactly L zeros;

(3) if denote by $\{T^l\}_{l=1}^{L}$ all the zeros of x in the increasing order, then x' (or y) changes sign once in $(-\infty,T^1)$, $2N+3$ times in (T^1,T^2), and $2k^l+1$ times in (T^l,T^{l+1}) for all $l = 2,\cdots,L$ $(T^{L+1} \equiv \infty)$.

Geometrically, the homoclinic orbit rotates around the z-axis $L/2$ times, and in each time interval where x does not change sign, the orbit rotates k^l+1 times around the vertical line $x = y = \sqrt{q(R-1)}$ (if $x < 0$) or $x = y = -\sqrt{q(R-1)}$ (if $x < 0$) for all $l = 0,1,\cdots,L$ $(k^0 \equiv 0, k^1 \equiv N+1)$.

Concerning the existence of chaotic trajectories of the Lorenz system, Hastings and Troy first developed a shooting argument to prove the existence of chaotic trajectories [9]. In their approach, they need two ansatz conditions, which were verified for certain parameters by a numerical implementation [7]. In this paper, we shall prove the following existence result:

Theorem 1.3. **(One Way Chaotic Orbits)** *Let q be any given positive number and N be any given positive integer. Then there exists a large positive constant $R_0(q,N)$ such that for every $R > R_0$ and any integer sequence $\{k^l\}_{l=2}^{\infty}$ satisfying $k^l \in \{0,\cdots,N-1\}$ for all $l \geq 2$, there exists a positive constant $s = (2q+1)/3 + O(R^{-1/2})$ such that the trajectory γ^+, the unstable manifold of the origin that initially enters the first octant, has the following properties:*

(a) all the zeros of x, x', and y are simple;

(b) x has infinitely many zeros;

(c) if denote by $\{T^l\}_{l=1}^{\infty}$ all the zeros of x in the increasing order, then x' (or y) changes sign once in $(-\infty,T^1)$, $2N+3$ times in (T^1,T^2), and $2k^l+1$ times in (T^l,T^{l+1}) for all $l \geq 2$.

In terms of the phase plane, the trajectory approaches the origin from the first octant as $t \to -\infty$. *It rotates around the* z *axis infinitely many times, and during each time interval where* x *does not change sign, the trajectory rotates* $k^l + 1$ *times around the vertical line* $x = y = \sqrt{q(R-1)}$ *(if* $x > 0$*) or* $x = y = -\sqrt{q(R-1)}$ *(if* $(x < 0)$*; Here* $k^0 \equiv 0$ *and* $k^1 \equiv N + 1$.

Theorem 1.4 (Two Way Chaotic Orbits) *Given positive number* q *and positive integer* N, *there exists a positive number* $R_0(q, N)$ *such that for every* $R > R_0$ *and every sequence* $\{k^l\}_{l=-\infty}^{\infty}$ *satisfying* $k^l \in \{0, 1 \cdots, N-1\}$ *for all* l, *there exist positive constant* $s = (2q+1)/3 + O(R^{-1/2})$ *and initial data* $(x(0), y(0), z(0))$ *satisfying* $x(0) = y(0) = 2\sqrt{s(R-1)}[1 + o(1)]$ *such that the solution of (1.1) has the following properties:*

(a) All the zeros of x, x', *and* y *are simple;*

(b) x *has infinitely many zeros for both* $t > 0$ *and* $t < 0$*;*

(c) if denote by $\{T^l\}_{l=-\infty}^{\infty}$ *all the zeros of* x *in the increasing order with* $T^0 < 0 < T^1$, *then the number of zeros of* x' *in* (T^l, T^{l+1}) *is* $2k^l + 1$ *for all* $l = 0, \pm 1, \pm 2, \cdots$.

Geometrically, the trajectory cross the $\{x = 0\}$ *plane infinitely times in both the positive and negative time directions and during each time interval where* x *do not change sign, the trajectory rotates around the vertical line* $x = y = \sqrt{q(R-1)}$ *(if* $x > 0$*) or the line* $x = y = -\sqrt{q(R-1)}$ *(if* $x < 0$*) certain number of times indicated by the given integer sequence* $\{k^l\}_{l=-\infty}^{\infty}$.

As shall be seen in the following sections, our proof is purely analytic; namely, it does not need any kind of computer assistance.

More recently, Mischaikow & Mrozek [12] showed that under certain topological conditions, a Poincare map on its invariant set is equivalent to, under a modulation, the shift map on binary sequences. Also, with a computer assistance, they showed that their topological conditions are satisfied by the Lorenz system for certain parameter values. In a forthcoming paper [2], we shall show the existence of homoclinic explosions near the homoclinic orbit established in Theorem 1.1 with $N = 0$; namely, we shall show the existence of a Poincare map on certain set which is homeomorphic to the shift map on binary sequences.

We organize our paper as follows. In §2, we transfer (1.1) into a system which is a perturbation of the Duffing equation. Also, we introduce a function which serves as a guider for finding the location of a trajectory when it is away from the origin. In §3, we provide some local analysis near the origin, and in §4, we prove Theorem 1.1. Finally, in §5, we prove Theorems 1.2–1.4.

2. An equivalent form of the Lorenz system.

Interested in only the case when $R >> 1$ and $0 < s - (2q+1)/3 = O(R^{-1/2})$, we introduce the following change of variables from $(t, x(t), y(t), z(t))$ to $(\tau, u(\tau), v(\tau), w(\tau))$:

$$\left\{\begin{array}{l} \tau = t[s(R-1)]^{1/2}, \\ x(t) = 2[s(R-1)]^{1/2}u(\tau), \\ y(t) = 2(R-1)\Big\{v(\tau) + [s/(R-1)]^{1/2}u(\tau)\Big\}, \\ z(t) = 2(R-1)u^2(\tau) + [4s-2q][(R-1)/s]^{1/2}w(\tau). \end{array}\right. \tag{2.1}$$

Under this transformation, the Lorenz equations become the equivalent system

$$\left\{\begin{array}{l} \dot{u} = v, \\ \dot{v} = u(1-2u^2) - 2\varepsilon[v + (1+a\varepsilon)uw], \\ \dot{w} = u^2 - \varepsilon bw \end{array}\right. \tag{2.2}$$

where $\dot{} = \frac{d}{d\tau}$ and

$$\varepsilon = \frac{1+s}{2[s(R-1)]^{1/2}}, \qquad b = \frac{2q}{1+s}, \qquad a = \frac{3s-2q-1}{\varepsilon(s+1)}. \tag{2.3}$$

A similar transformation for large R and fixed s and q was first introduced by Robbins [13] to establish the existence of periodic solutions of (1.1) for large R. Note that (2.2) is invariant under the transformation from (u, v, w) to $(-u, -v, w)$. Also, both the positive and negative w–axis are trajectories.

In the sequel, we shall always assume that N (in Theorems 1.1–1.4) is a given fixed positive integer, b is a given fixed positive number, ε is a positive number as small as we need, and a is a positive parameter varying in the range $[b, (3N+1)b]$. Whenever we say ε is small, we mean that $\varepsilon \in (0, \varepsilon_0]$ where $\varepsilon_0 = \varepsilon_0(b, N)$ depends only on b and N. Also, all $O(\varepsilon)$ stands for a quantity or a function (in τ) whose magnitude is no bigger than $C\varepsilon$ where $C = C(b, N)$ is a constant depending only on b and N.

Note that when $\varepsilon = 0$, the first two equations in (2.2) become the Duffing equation $\ddot{u} = u - 2u^2$, and one explicit solution is

$$u = \text{sech }\tau, \qquad v^2 = u^2 - u^4, \qquad [w - (B+1)]^2 + u^2 = 1$$

where B is an arbitrary constant.

The following lemma is crucial in our analysis:

Lemma 2.1. *Let (u, v, w) be any solution of (2.2). Define*

$$H = v^2 - u^2 + u^4 + 2\varepsilon[uv + (1+a\varepsilon)u^2w]. \tag{2.4}$$

Then

$$\begin{aligned} \dot{H} &= -2\varepsilon H + 2\varepsilon^2u^2[au^2 - b(1+a\varepsilon)w], && (2.5) \\ v &= -\varepsilon u \pm \Big\{u^2 - u^4 + \varepsilon u^2[\varepsilon - 2(1+a\varepsilon)w] + H\Big\}^{1/2}, && (2.6) \\ H(\tau_2) &= H(\tau_1)e^{2\varepsilon(\tau_1-\tau_2)} + 2\varepsilon^2\int_{\tau_1}^{\tau_2} e^{2\varepsilon(\tau-\tau_2)}u^2(\tau)[au^2(\tau) - b(1+a\varepsilon)w(\tau)]\,d\tau, && (2.7) \\ w(\tau_2) &= w(\tau_1)e^{\varepsilon b(\tau_1-\tau_2)} + \int_{\tau_1}^{\tau_2} e^{\varepsilon b(\tau-\tau_2)}u^2(\tau)\,d\tau && (2.8) \end{aligned}$$

for all $\tau_1 < \tau_2$.

Proof. The identity (2.5) follows by differentiating H and using (2.2), whereas the identity (2.6) follows by solving v from the definition equation of H in (2.4). The last two identities follow by using the integrating factors and the differential equations for H and w. □

3. Some local analysis.

Since a homoclinic orbit associated with the origin initially lies on the unstable manifold of the origin, and the system (2.2) is invariant under the transformation $(u, v, w) \to (-u, -v, w)$, we begin with the study of the trajectory corresponding to the unstable manifold of the origin which initially lie in $\{u > 0\}$.

Lemma 3.1 *There exists a unique trajectory* γ^+ *which approaches the origin as* $\tau \to -\infty$ *and initially lies in the first octant. In addition, there exists a first time such that* γ^+ *intersects the* u–w *plane. More precisely, if* $(u(\cdot), v(\cdot), w(\cdot))$ *is a solution of (2.2) representing* γ^+, *then for every positive and small* ε, *there exists a* τ^0 *such that* $v = \dot{u} > 0$ *in* $(-\infty, \tau^0)$ *and*

$$u(\tau^0) = 1 + O(\varepsilon), \quad v(\tau^0) = 0, \quad w(\tau^0) = 1 + O(\varepsilon), \quad \varepsilon^{-2} H(\tau^0) = [4a/3 - b] + O(\varepsilon).$$

Remark 3.1. By the definition equation of H in (2.4), the $O(\varepsilon^3)$ estimate for $H(\tau^0)$ can lead to an $O(\varepsilon^2)$ estimate for $u(\tau^0)$.

Proof. Since the matrix of the linearized system (2.2) at the origin has two negative eigenvalues and one positive eigenvalue $\lambda^+ := -\varepsilon + \sqrt{1 + \varepsilon^2}$ with eigenvector $(1, \lambda^+, 0)$, the existence of γ^+ follows from a standard theory of autonomous system. In addition, when τ is sufficiently negative large, $v(\tau) = \lambda^+ u(\tau) + o(u(\tau))$.

Define $T := \sup\{\tau \geq -\infty \bigm| 0 < u < 4/5 \text{ and } v > u/2 \text{ in } (-\infty, \tau)\}$. Since $\lambda^+ > 1/2$, $T > -\infty$. In addition, at $\tau = T$, either $u = 4/5$ or $v = u/2$.

Note that $dt = du/v \leq 2du/u$ in $(-\infty, T]$ (since $v > u/2$), so that $\int_{-\infty}^{\tau} u^2(\hat{\tau}) d\hat{\tau} \leq 2 \int_0^{u(\tau)} u\, du = u^2(\tau)$ for all $\tau \in (-\infty, T]$. It then follows from (2.8) and (2.7) with $\tau_1 = -\infty$ that

$$0 \leq w(\tau) \leq u^2(\tau), \qquad \varepsilon^{-2} |H(\tau)| \leq [a + b(1 + a\varepsilon)] u^4(\tau), \quad \forall \tau \in (-\infty, T].$$

Consequently, from (2.6), $v = V_0(u)[1 + O(\varepsilon)]$ in $(-\infty, T]$ where $V_0(u) = u\sqrt{1 - u^2}$. Therefore, by the definition of T, we must have $u(T) = 4/5$.

Since $0 \leq w \leq u^2$ in $(-\infty, T]$, the differential equation $\dot{w} = -\varepsilon b w + u^2$ implies that $\dot{w} = [1 + O(\varepsilon)] u^2$, which, in turn, implies that

$$w(\tau) = \int_0^{u(\tau)} \frac{[1 + O(\varepsilon)] u^2}{V_0(u)[1 + O(\varepsilon)]} du = W_0(u(\tau))[1 + O(\varepsilon)] \qquad \forall\, \tau \in (-\infty, T] \tag{3.1}$$

where $W_0(u) := \int_0^u u^2 / V_0(u) du = 1 - \sqrt{1 - u^2}$. Similarly, for all $\tau \in (-\infty, T]$,

$$\varepsilon^{-2} H(\tau) = 2 \int_0^{u(\tau)} \frac{u^2 [a u^2 - b W_0(u)]}{V_0(u)} du + O(\varepsilon u^4(\tau)) \qquad \forall\, \tau \in (-\infty, T]. \tag{3.2}$$

Now when $\tau \in [T, T+1]$, γ^+ is within a distance of order ε to the solution of (2.2) with $\varepsilon = 0$ and initial data $(4/5, 3/5, W(T))$; that is $v^2 = u^2 - u^4 + O(\varepsilon)$. It then follows that there is a $\tau^0 \in (T, T+1)$ such that $v(\tau^0) = 0$ and in the interval $[T, \tau^0]$, $u(\cdot) = U_0(v(\cdot))[1 + O(\varepsilon)]$ and $\dot{v}(\cdot) = U_0(v(\cdot)) - 2U_0^3(v(\cdot)) + O(\varepsilon)$ where $U_0(v) = [\frac{1}{2} + (\frac{1}{4} - v^2)^{1/2}]^{1/2}$. Hence, writing $dt = \frac{1}{\dot{v}}dv = \frac{1+O(\varepsilon)}{U_0(v)-2U_0^3(v)}dv$ in $[T, \tau^0]$, we can use the identities (2.8) and (2.7) with $\tau_2 = \tau^0$ and $\tau_1 = T$ and the estimates (3.1) and (3.2) with $\tau = T$ to computer that

$$\begin{aligned}
w(\tau^0) &= \int_0^{5/4} \frac{u^2}{V_0(u)}\, du + \int_{V_0(4/5)+O(\varepsilon)}^{0} \frac{U_0^2(v)}{U_0(v) - 2U_0^3(v)} dv + O(\varepsilon) \\
&= \int_0^1 \frac{u^2}{V_0(u)} du + O(\varepsilon) = W_0(1) + O(\varepsilon) = 1 + O(\varepsilon), \\
\varepsilon^{-2} H(\tau^1) &= \cdots = 2\int_0^1 \frac{u^2(au^2 - bW_0(u))}{u\sqrt{1-u^2}} du + O(\varepsilon) = (4a/3 - b) + O(\varepsilon)
\end{aligned}$$

where in the second equality in calculating $w(\tau^0)$, the identity $\int_{V_0(4/5)}^{0} \frac{U^0(v)}{U_0(v)-2U_0^3(v)} dv = \int_{4/5}^{1} \frac{u^2}{V_0(u)} du$ has been used, whereas in the calculation of $H(\tau^0)$, "$\cdots$" represents a similar procedure as in the calculation of $w(\tau^0)$. This completes the proof of the lemma. □

We continue to follow the trajectory γ^+ after it intersects the u-w plane. For later applications, we study general trajectories starting from the u–w plane near $u = 1$.

Lemma 3.2 *Let $M \geq 2(a + b + 1)$ be any fixed constant and $(u(\cdot), v(\cdot), w(\cdot))$ be a solution of (2.2) with initial data $(u(0), v(0), w(0))$ satisfying*

$$v(0) = 0, \quad u(0) = 1 + O(\varepsilon), \quad w(0) \in [0, M], \quad \varepsilon^{-2}H(0) \in [-M, M].$$

Then, for any positive and small ε, there exists $\hat{T} > 0$ such that $v < 0$ in $(0, \hat{T}]$ and

$$\begin{aligned}
&u(\hat{T}) = M^2\varepsilon, \qquad w(\hat{T}) = w(0) + 1 + O(\varepsilon|\ln\varepsilon|), \\
&\varepsilon^{-2}H(\hat{T}) = [\varepsilon^{-2}H(0) - 2bw(0)] + [4a/3 - b] + O(\varepsilon|\ln\varepsilon|).
\end{aligned}$$

where $|O(\varepsilon|\ln\varepsilon|)| \leq C\varepsilon|\ln\varepsilon|$ for some C depending only on a, b, and M.

Proof. Before u drops to $1/2$, the trajectory will stay within a distance of order ε to the surface $v^2 = u^2 - u^4$. It then follows that there exists $T \in (0, 1)$ such that $u(T) = 4/5$ and $v < 0$ in $(0, T]$. In addition, $u = U_0(v) + O(\varepsilon)$ and $\dot{v} = U_0(v) - 2U_0^3(v) + O(\varepsilon)$ so that one can calculate, in a way similar to that in the proof of Lemma 3.1, that

$$w(T) = w(0)e^{-\varepsilon bT} + \int_0^{v(T)} e^{2b\varepsilon(\tau - T)} \frac{u^2}{\dot{v}} dv = w(0) + 1 - W_0(4/5) + O(\varepsilon), \tag{3.3}$$

$$\varepsilon^{-2}H(T) = \varepsilon^{-2}H(0) + 2\int_{4/5}^{1} \frac{u\{au^2 - b[w(0) + 1 - W_0(u)]\}}{\sqrt{1-u^2}} du + O(\varepsilon). \tag{3.4}$$

Define $\hat{T} = \sup\{\tau > T \mid -v > u/2 \text{ and } M^2\varepsilon < u < 4/5 \text{ in } (T, \tau)\}$. Then in $[T, \hat{T})$, $dt = du/v < -2du/u$, so that, from (2.8),

$$0 \leq w(\tau) \leq w(T) - \int_{4/5}^{u(\tau)} 2u du \leq w(0) + 2.$$

Similarly, from (2.7),

$$\varepsilon^{-2}|H(\tau)| \le \varepsilon^{-2}|H(T)| - 4\int_{4/5}^{u(\tau)} u[au^2 + b(1+a\varepsilon)(w(0)+2)]du \le M^2$$

for all $\tau \in [T, \hat{T}]$. It then follows from (2.6) that

$$v = -u\sqrt{1-u^2}\Big\{1 + O(\varepsilon) + \theta M^2\varepsilon^2/u^2\Big\} \qquad \forall\, \tau \in [T, \hat{T}]$$

where $\theta \in [-1, 1]$. Since $u \ge M^2\varepsilon$ and $\dot{u} = v < 0$ in $[T, \hat{T}]$, by the definition of $\hat{T}$, we must have $u(\hat{T}) = M^2\varepsilon$. In addition,

$$\hat{T} - T = \int_{4/5}^{M^2\varepsilon} \frac{du(\tau)}{v(\tau)} = \int_{M^2\varepsilon}^{4/5} \frac{1 + O(\varepsilon) + \theta M^2\varepsilon^2/u^2}{u\sqrt{1-u^2}} du = O(|\ln\varepsilon|).$$

Hence, from (2.8),

$$\begin{aligned} w(\tau) &= w(T)(1 + O(\varepsilon|\ln\varepsilon|)) + \int_{4/5}^{u(\tau)} (1 + O(\varepsilon|\ln\varepsilon|))u^2 \frac{1 + O(\varepsilon) + \theta M^2\varepsilon^2/u^2}{-u\sqrt{1-u^2}} du \\ &= w(0) + 1 - W_0(u(\tau)) + O(\varepsilon|\ln\varepsilon|) \qquad \forall\tau \in [T, \hat{T}] \end{aligned}$$

by substituting the estimate for $w(T)$ in (3.3). Consequently, $w(\hat{T}) = w(0) + 1 + O(\varepsilon\ln\varepsilon|)$ since $W_0(M^2\varepsilon) = O(\varepsilon^2)$. Similarly, from (2.7), we have that

$$\begin{aligned} \varepsilon^{-2}H(\hat{T}) &= \varepsilon^{-2}H(T)[1 + O(\varepsilon|\ln\varepsilon|)] + 2\int_{M^2\varepsilon}^{4/5} \frac{1 + O(\varepsilon) + \theta M^2\varepsilon^2/u^2}{u\sqrt{1-u^2}} [1 + O(\varepsilon|\ln\varepsilon|)]u^2 \times \\ &\qquad \times\Big(au^2 - b[1 + a\varepsilon][w(0) + 1 - W_0(u) + O(\varepsilon|\ln\varepsilon|)]\Big)\, du \\ &= \varepsilon^{-2}H(0) + 2\int_0^1 \frac{u[au^2 - b(w(0) + 1 - W_0^+(u))]}{\sqrt{1-u^2}} du + O(\varepsilon|\ln\varepsilon|) \\ &= \varepsilon^{-2}H(0) + 4a/3 - b[1 + 2w(0)] + O(\varepsilon|\ln\varepsilon|). \end{aligned}$$

The assertion of the lemma thus follows. □

After $\hat{T}$, the projection of the trajectory onto the u–v plane can experience one of the following: (1) approaching the origin without touching the u–axis or the v–axis, (2) intersecting the u–axis and then entering the first quadrant, or (3) intersecting the v–axis and then entering the third quadrant. The following lemma provides some necessary conditions and estimates for each of these cases to happen.

Lemma 3.3 *Let $M \ge 2(a+b+1)$ be any fixed constant. Assume that $u(0) = M^2\varepsilon$, $w(0) \in [0, M]$, $\varepsilon^{-2}H(0) \in [-M, M]$, and $v(0) < 0$. Then the following holds:*

(1) If $H(0) \ge \varepsilon^3$, then there exists $T > 0$ such that $v(\tau) < 0$ for all $\tau \in [0, T]$ and

$$u(T) = 0, \qquad w(T) = w(0) + O(\varepsilon|\ln\varepsilon|), \qquad H(T) = H(0)(1 + O(\varepsilon\ln\varepsilon)) + O(\varepsilon^4). \tag{3.5}$$

(2) If $H(0) \le -\varepsilon^3$, then there exists $T > 0$ such that $u > 0$ in $[0, T]$, $v < 0$ in $[0, T)$, $v(T) = 0$, and the same estimates for $w(T)$ and $H(T)$ as in (3.5) hold.

(3) If $H(0) \in [-\varepsilon^3, \varepsilon^3]$*, then there exists* $T \in (0,\infty) \cup \{\infty\}$ *such that* $u > 0$ *and* $v < 0$ *in* $(0,T)$ *and at* T, $u(T)v(T) = 0$ *and*

$$|H(T)| \leq |H(0)| + O(\varepsilon^4), \qquad 0 \leq w(T) \leq w(0) + O(\varepsilon).$$

(4) There exists $h^* \in (-\varepsilon^3, \varepsilon^3)$ *such that if* $H(0) = h^*$*, then* $u > 0$ *and* $v < 0$ *in* $(0,\infty)$ *and that* $(u,v,w) \to (0,0,0)$ *as* $\tau \to \infty$*.*

Proof. (1) Set $T = \sup\{\tau > 0 \,\big|\, 0 < u < M^2\varepsilon, v < 0, \text{ and } H > H(0)/2 \text{ in } (0,\tau)\}$. Then, since $H > 0$, by (2.6),

$$-v = [u^2 + H]^{1/2}(1 + O(\varepsilon)) \geq [u^2 + H(0)/2]^{1/2}[1 + O(\varepsilon)] \text{ in } (0,T).$$

Hence, $\int_0^T u^2\, d\tau \leq [1 + O(\varepsilon)] \int_0^{M^2\varepsilon} u^2[u^2 + H(0)/2]^{-1/2}\, du = O(\varepsilon^2)$, so that from (2.8) and (2.7), $w(T) = w(0)e^{-\varepsilon bT} + O(\varepsilon^2)$ and $\varepsilon^{-2}H(T) = \varepsilon^{-2}H(0)e^{-2b\varepsilon T} + O(\varepsilon^2)$. Note that

$$T = \int_0^T \frac{du(\tau)}{v(\tau)} \leq [1 + O(\varepsilon)] \int_0^{M^2\varepsilon} [u^2 + H(0)/2]^{-1/2}\, du = O(|\ln(\varepsilon^{-2}H(0))|).$$

Since $H(0) \geq \varepsilon^3$, it then follows that $T = O(|\ln \varepsilon|)$ and $H(T) = H(0)[1 + O(\varepsilon \ln \varepsilon)] + O(\varepsilon^4) \geq \frac{3}{4}H(0)$. Hence, by the definition of T, we must have $u(T) = 0$. The first assertion of the lemma thus follows.

(2) Define $T = \sup\{\tau > 0 \,\big|\, M^2\varepsilon > u > |v|/2, v < 0, H < H(0)/2, \text{ and } 0 \leq w \leq w(0) + 1 \text{ in } (0,\tau)\}$. Then in$(0,T)$, one can derive from the equation for $\dot{v}$ that $\dot{v} = u(1 + O(\varepsilon))$, and from the definition equation of H that $u = [v^2 - H]^{1/2}[1 + O(\varepsilon)]$. Using the same argument as in (1) with the roles of u and v being interchanged, one obtains the second assertion of the lemma.

(3) Let $T \in (0,\infty) \cup \{\infty\}$ be the first time such that the trajectory hits the u–w plane or the v–w plane. Then, in $(0,T)$, $v = \dot{u} < 0$ so that $u \in (0, M^2\varepsilon]$. From the estimate $\dot{w} = u^2 - \varepsilon bw \leq M^4\varepsilon^2 - \varepsilon bw$, it follows that $0 \leq w \leq \max\{w(0), \varepsilon M^4/b\}$ in $(0,T)$. Consequently, from the equation for $\dot{v}$, we have that $\dot{v} = u(1 + O(\varepsilon)) - 2\varepsilon v \geq u(1 + O(\varepsilon))$ in $(0,T)$. Therefore, $\int_0^T u^2 = \int_{v(0)}^{v(T)} u^2 dv/\dot{v} = O(\varepsilon^2)$. The third assertion of the lemma thus follows from (2.8) and (2.7).

(4) For each $h = H(0) \in [-\varepsilon^3, \varepsilon^3]$, let $T(h)$ be defined as in (3). Then $u(T(h))v(T(h)) = 0$. In addition, if $T(h) < \infty$, then $u^2(T(h)) + v^2(T(h)) > 0$ (since the positive w–axis is a trajectory).

If $T(h) < \infty$ and $u(T(h)) = 0$, then $\dot{u}(T(h)) = v(T(h)) < 0$, so that, by the Implicit Function Theorem, for all $\tilde{h}$ near h, $T(\tilde{h})$ is finite and $u(T(\tilde{h})) = 0$. Similarly, if $T(h) < \infty$ and $v(T(h)) = 0$, then $\dot{v}(T(h)) = u(T(h))(1 + O(\varepsilon)) > 0$, so that for all $\tilde{h}$ near h, $T(\tilde{h}) < \infty$ and $v(T(\tilde{h})) = 0$.

Now define $\mathcal{H}^+ = \{h \in (-\varepsilon^3, \varepsilon^3) \mid T(h) < \infty \text{ and } u(T(h)) = 0\}$ and $\mathcal{H}^- = \{h \in (-\varepsilon^3, \varepsilon^3) \mid T(h) < \infty \text{ and } v(T(h)) = 0\}$. From the proceeding analysis, both $\mathcal{H}^+$ and $\mathcal{H}^-$ are open and non-empty. Thus, there exists $h^* \in (-\varepsilon^3, \varepsilon^3)$ which belongs neither to $\mathcal{H}^+$ nor to $\mathcal{H}^-$. Clearly, for this h^*, $T(h^*) = \infty$. Hence, by the definition of $T(h^*)$, for all $\tau \in (0,\infty)$, $\dot{u} = v < 0$, $u \in (0, M^2\varepsilon)$, and $w \in (0, \max\{w(0), M^4\varepsilon/b\})$. From this, one can derive that $(u,v,w) \to (0,0,0)$ as $\tau \to \infty$. The assertion of the lemma thus follows. $\square$

Remark 3.2. In fact, by a detailed study of the stable manifold of the origin, one can actually show that h^* in the Lemma is unique. In addition, when $H(0) \in (h^*, M\varepsilon^2]$, u reaches zero before v, and when $H(0) \in [-M\varepsilon^2, h^*)$, v reaches zero before u. For simplicity, we shall not pursue it here.

Assume that γ^+ does not approach the origin in the way described in Lemma 3.3 (4). We now continue to study the behavior of the solution after it intersects the v–w plane or the u–w plane. Since the system is invariant under the transformation $(u, v, w) \to (-u, -v, w)$, we need only consider trajectories starting from the boundary of the first octant.

Lemma 3.4 *Assume that* $(u(0), v(0), w(0)) \in \Big(\{0\} \times (0, M\varepsilon] \cup (0, M\varepsilon] \times \{0\}\Big) \times [0, M]$. *Then there exists a constant* $\tau_1 = -\frac{1}{2}[1 + O(\varepsilon)] \ln |H(0)| + O(1)$ *such that* $v > 0$ *in* $(0, \tau_1)$, *and*

$$v(\tau_1) = 0, \qquad u(\tau_1) = 1 + O(\varepsilon), \qquad w(\tau_1) = w(0)e^{-2b\varepsilon\tau_1} + 1 + O(\varepsilon),$$
$$\varepsilon^{-2}H(\tau_1) = \varepsilon^{-2}H(0)e^{-2\varepsilon\tau_1} - 2bw(0)e^{-\varepsilon b\tau_1} + [4a/3 - b] + O(\varepsilon).$$

Note that if $u(0) = 0$, then $H(0) = v^2(0) > 0$, and if $v(0) = 0$, then $H(0) = u^2(0)[-1 + O(\varepsilon)] < 0$.

Proof. Define $T = \sup\{\tau > 0 \,\big|\, u(0) < u < 1/2, v > 0, 0 < w < M + 2 \text{ in } (0, \tau)\}$.

Since $\frac{d}{d\tau}[v^2 - u^2 + u^4] = -4\varepsilon[v^2 + (1 + a\varepsilon)uvw] \le 0$ in $[0, T]$, $v^2(\cdot) < u^2(\cdot) - u^4(\cdot) + v^2(0) - u^2(0) + u^4(0)$ in $[0, T]$. Also, for all $\tau \in [0, T)$,

$$\begin{aligned} &\frac{d}{d\tau}\Big\{v^2 - [1 - 4M\varepsilon]u^2 + u^4 + 4\varepsilon[u - u(0)]v\Big\} \\ = \;& 4\varepsilon u[u - u(0)][1 - 2u^2 - 2\varepsilon(1 + a\varepsilon)w] + 4\varepsilon uv\Big[2M - (1 + a\varepsilon)w - 1 + \tfrac{u(0)}{u}\Big] > 0. \end{aligned}$$

It then follows that for all $\tau \in (0, T]$, $v^2 - [1 - 4M\varepsilon]u^2 + u^4 + 4\varepsilon[u - u(0)]v > v^2(0) - [1 - 4M\varepsilon]u^2(0) + u^4(0)$; that is, in $[0, T]$,

$$v > -2\varepsilon[u - u(0)] + \sqrt{[2\varepsilon(u - u(0))]^2 + v^2(0) + (1 - 4M\varepsilon)[u^2 - u^2(0)] - [u^4 - u^4(0)]}.$$

Hence, $v = \sqrt{v^2(0) + [u^2 - u^2(0)][1 - u^2 + u^2(0)]}(1 + O(\varepsilon))$.

With the estimate on v, it is easy to see that $\int_0^T u^2 \le 1$, so that $w(\tau) \le w(0) + 1$ in $(0, T]$. Hence, by the definition T, $u(T) = 1/2$. In addition,

$$\begin{aligned} T &= \int_{u(0)}^{1/2} du/v = (1 + O(\varepsilon)) \int_{u(0)}^{1/2} \Big\{v^2(0) + [u^2 - u^2(0)][1 - u^2 + u^2(0)]\Big\}^{-1/2} du \\ &= (1 + O(\varepsilon))\Big[-\tfrac{1}{2}\ln|H(0)| + O(1)\Big] \end{aligned}$$

by directly calculation. Also, one can show that $T - \tau = O(|\ln u(\tau)|)$ for all $\tau \in [0, T]$.

Finally, from (2.8) and (2.7), we have, for all $\tau \in (0, T]$,

$$\begin{aligned} w(\tau) &= w(0)e^{-\varepsilon b\tau} + \int_0^{u(\tau)} u[1 + O(\varepsilon|\ln u|)][1 - u^2]^{-1/2}\, du + O(\varepsilon) \\ &= w(0)e^{-\varepsilon b\tau} + W_0(u) + O(\varepsilon), \\ \varepsilon^{-2}H(T) &= \varepsilon^{-2}H(0)e^{-2\varepsilon T} + 2\int_0^{1/2} u[1 + O(\varepsilon \ln u)][1 - u^2]^{-1/2} \times \\ &\qquad \Big\{au^2 - bw(0)e^{-\varepsilon bT}[1 + O(\varepsilon \ln u)] + W_0(u) + O(\varepsilon)\Big\}\, du + O(\varepsilon). \end{aligned}$$

The rest of the proof then follows from the same lines as in the proof of Lemma 3.1. $\Box$

Lemma 3.5 *Assume that* $(u(0),v(0),w(0)) \in \Big(\{0\}\times[1/M,M]\cup[1/M,M]\times\{0\}\Big)\times[0,M]$. *Then there exists* $\tau^1 > 0$ *such that* $v > 0$ *in* $(0,\tau^1)$ *and that*

$$v(\tau^1) = 0, \qquad u(\tau^1) = 1 + O(\varepsilon), \qquad w(\tau^1) = w(0) + 1 + O(\varepsilon|\ln\varepsilon|),$$
$$\varepsilon^{-2}H(\tau^1) = \varepsilon^{-2}H(0) - 2bw(0) + [\tfrac{4}{3}a - b] + O(\varepsilon|\ln\varepsilon|).$$

Proof. Since in the current case $|H(0)| \geq \varepsilon^2/(2M^2)$, the assertion of the lemma follows from Lemma 3.4. $\Box$

In the sequel, we shall take $M = 4(N+1)(a+b+1)$ and $a \in [b,(3N+1)b]$ so that all $O(\varepsilon|\ln\varepsilon|)$ in this section are bounded by $C\varepsilon|\ln\varepsilon|$ for some constant C depending only on b and N.

We end this section with a few observations on the consequences of the Lemmas 3.1–3.5. These observations constitute the basic idea of the proof of our main theorems.

Lemma 3.3 shows that when a trajectory passes by the w axis, whether its u coordinate changes sign or not (i.e., whether it hits the $\{u=0\}$ plane or the $\{v=0\}$ plane) is mostly determined by $\varepsilon^{-2}H$.

The estimates in Lemmas 3.2 and 3.4 reveal that if the value of $\varepsilon^{-2}H$ is A and the value of w is B when a trajectory leaves the $M^2\varepsilon$–neighborhood of the w–axis, then when it visits the w–axis in the next time, w will increase to $B + 2 + O(\varepsilon\ln\varepsilon)$ and $\varepsilon^{-2}H$ will become $A - 4bB + 8a/3 - 4b + O(\varepsilon\ln\varepsilon)$. Hence, depending on the sign of $B - [2a/(3b) - 1]$, H can either increase or decrease. Anyway, the amount of change of $\varepsilon^{-2}H$ is mostly determined by B.

If a trajectory passes by the w–axis not very close, say, outside of an ε^2–neighborhood, then when it leaves the $M^2\varepsilon$–neighborhood of the w–axis, w changes only $O(\varepsilon\ln\varepsilon)$ and H changes only $O(\varepsilon^3\ln\varepsilon)$. However, if the trajectory passes by the w-axis very closely, then when it enters the $M^2\varepsilon$–neighborhood of the w–axis, $\varepsilon^{-2}H$ must be necessarily small ($\leq \varepsilon$) and when it leaves the $M^2\varepsilon$ neighborhood of w–axis, w may drop a significant amount.

Hence, if initially $\varepsilon^{-2}H$ and w are of order 1, then after the trajectory made a few number of "far away" visits to the w-axis, w becomes bigger than $2a/(3b) - 1$, and therefore H begins to decrease. In addition, as long as $\varepsilon^{-2}H$ remains not too small, the visit will be not "close" and hence w will keep increase in each visit. Finally, after a finite number of visit, either $\varepsilon^{-2}H$ drops from positive to sufficiently negative (say, $\leq -O(\varepsilon\ln\varepsilon|)$), or $\varepsilon^{-2}H$ becomes just sufficiently small.

If the first case happens, the trajectory leaves the $M^2\varepsilon$–neighborhood of the w–axis with w being bigger than than $2a/(3b) - 1$ and the value of $\varepsilon^{-2}H$ being lesser than $-O(\varepsilon|\ln\varepsilon|)$, so that in the next several visits to the w–axis, H keeps sufficiently negative ($\leq -O(\varepsilon\ln\varepsilon|)$ and the particle does not cross the $u = 0$ plane.

If the second case happens, the trajectory has to experience a "very close" visit to the w–axis for a very long time, during which w decreases a significant amount, so that depending on the amount that w lose, at the next time when the trajectory visits the w-axis, $\varepsilon^{-2}H$ can be "large", "negative large", or just small.

Our main idea of the construction of the trajectories in Theorems 1.2–1.4 is to adjust the parameter a such that trajectories always have a "very" close visit to the w-axis. After each close visit, w will lose certain amount which will be recovered during the trajectories travel outside of the $M^2\varepsilon$–neighborhood of the w-axis. In addition, the adjustment is made so that in the next close visit to the w–axis, the trajectories can pass the $u = 0$ plane (in case $\varepsilon^{-2}H$ is large), or pass the $v = 0$ plane (in case $\varepsilon^{-2}H$ is negative large), or even enter the origin (in case $\varepsilon^{-2}H$ is just equal to certain small value). Thus, the "chaotic" trajectories can be obtained by a shooting argument.

4. Rotated Homoclinic Orbits.

With the basic calculation in §3, we can now establish the existence of rotated homoclinic orbits.

Theorem 4.1 *Let N be a positive integer and b be a positive number. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(b, N)$ and $C = C(b, N)$ such that for every $\varepsilon \in (0, \varepsilon_0]$, the following holds:*

(1) For every $a \in [\frac{3}{2}(N - \frac{1}{2})b, \frac{3}{2}(N + \frac{1}{2})b]$, let (u, v, w) be the solution of (2.2) which corresponds to γ^+. Then there exists $\hat{T}^N \in (-\infty, \infty)$ such that in $(-\infty, \hat{T}^N)$, u vanishes exactly $N - 1$ times, v vanishes exactly N times, and all the zeros of u and v are simple and interlace. In addition, for all $\tau \in (-\infty, \hat{T}^N]$, $0 \leq w(\tau) \leq 2N + O(\varepsilon|\ln\varepsilon|)$ and $\varepsilon^{-2}H(\tau) \leq \frac{8N}{3}a + 4Nb + O(\varepsilon|\ln\varepsilon|)$, and at $\hat{T}^N$,

$$\begin{aligned} u(\hat{T}^N) &= (-1)^{N-1}M^2\varepsilon, \qquad (-1)^N v(\hat{T}^N) > 0, \\ w(\hat{T}^N) &= 2N + O(\varepsilon|\ln\varepsilon|), \\ \varepsilon^{-2}H(\hat{T}^N) &= \tfrac{8}{3}N(a - \tfrac{3}{2}Nb) + O(\varepsilon|\ln\varepsilon|) \end{aligned}$$

where $|O(\varepsilon|\ln\varepsilon|)| \leq C(b, N)\varepsilon|\ln\varepsilon|$.

(2) There exists $a^* = a(b, N, \varepsilon)$ satisfying

$$a^* = \tfrac{3}{2}Nb + O(\varepsilon|\ln\varepsilon|)$$

such that γ^+ with parameters (a^*, b, ε) is a homoclinic orbit associated with the origin and its u component changes sign $N - 1$ times, v component changes sign N times, and the zeros of u and v are simple and interlace. Furthermore, for all $\tau \in (-\infty, \infty)$, $v^2(\tau) = u^2(\tau) - u^4(\tau) + O(\varepsilon)$ where $|O(\varepsilon)| \leq C(b, N)\varepsilon$.

Proof. (1) Let τ^0 be as in Lemma 3.1 and $\hat{T}$ be the constant in Lemma 3.2 with initial data $(u(\tau_0), 0, w(\tau_0))$. Set $\hat{T}^1 = \hat{T} + \tau^0$. Then by the estimates in Lemmas 3.1 and 3.2, $v(\hat{T}^1) < 0$, and

$$u(\hat{T}^1) = M^2\varepsilon, \quad w(\hat{T}^1) = 2 + O(\varepsilon\ln\varepsilon), \quad \varepsilon^{-2}H(\hat{T}^1) = \left[\tfrac{8}{3}a - 4b\right] + O(\varepsilon\ln\varepsilon).$$

The first assertion of the theorem for the case $N = 1$ thus follows.

Next we consider the case $N \geq 2$. Since $a \geq \frac{3}{2}(N - \frac{1}{2})b$, $\frac{8}{3}a - 4b \geq 4b(N - 3/2) \geq 2b$, so that, by the first assertion of Lemma 3.3, there exists a constant $T^1 > \hat{T}^1$ such that $v < 0$ in $(\tau^0, T^1]$ and

$$u(T^1) = 0, \quad w(T^1) = 2 + O(\varepsilon\ln\varepsilon), \quad \varepsilon^{-2}H(T^1) = \left[\tfrac{8}{3}a - 4b\right] + O(\varepsilon\ln\varepsilon).$$

Now we shall use the Mathematical Induction to show that there exists $\{T^i\}_{i=1}^{N-1}$ such that $T^1 < T^2 < \cdots < T^{N-1}$ and for each $i = 1, \cdots, N-1$,

$$\left\{ \begin{array}{rl} u(T^i) = & 0, \qquad (-1)^i v(T^i) > 0, \\ w(T^i) = & 2i + O(\varepsilon \ln \varepsilon), \\ \varepsilon^{-2} H(T^i) = & 2i\left[\frac{4}{3}a - 2ib\right] + O(\varepsilon \ln \varepsilon). \end{array} \right. \tag{4.1}$$

Clearly, T^1 exists. Assume that for some $i < N-1$, T^i exists. We want to show that T^{i+1} exists also. By Lemma 3.5, there exists $\tau^i > T^i$ such that $(-1)^i v > 0$ in $[T^i, \tau^i)$, and that

$$\begin{aligned} v(\tau^i) &= 0, \qquad u(\tau^i) = (-1)^i + O(\varepsilon), \\ w(\tau^i) &= 1 + w(T^i) + O(\varepsilon \ln \varepsilon) = 2i + 1 + O(\varepsilon \ln \varepsilon), \\ \varepsilon^{-2} H(\tau^i) &= [\varepsilon^{-2} H(T^i) - 2bw(T^i)] + [\tfrac{4}{3}a - b] + O(\varepsilon \ln \varepsilon) \\ &= (2i+1)\left\{\tfrac{4}{3}a - (2i+1)b\right\} + O(\varepsilon \ln \varepsilon). \end{aligned}$$

Consequently, Lemma 3.2 implies there exists $\hat{T}^{i+1} > \tau^i$ such that $(-1)^{i+1} v > 0$ in $(\tau^i, \hat{T}^{i+1}]$ and

$$\begin{aligned} u(\hat{T}^{i+1}) &= (-1)^i M^2 \varepsilon, \\ w(\hat{T}^{i+1}) &= 1 + w(\tau^i) + O(\varepsilon \ln \varepsilon) = 2(i+1) + O(\varepsilon \ln \varepsilon), \\ \varepsilon^{-2} H(\hat{T}^{i+1}) &= \varepsilon^{-2} H(\tau^i) + 4a/3 - b(1 + 2w(\tau^i) + O(\varepsilon \ln \varepsilon) \\ &= 2(i+1)\left\{\tfrac{4}{3}a - 2(i+1)b\right\} + O(\varepsilon \ln \varepsilon). \end{aligned}$$

By the assumption on a and b, $\frac{4}{3}a - 2(i+1)b \geq (2N - 2i - 3)b \geq b$ since $i \leq N-2$. Hence $\varepsilon^{-2} H(\hat{T}^{i+1}) > (i+1)b + O(\varepsilon \ln \varepsilon) \geq \varepsilon$ and therefore by Lemma 3.3 (1), there exists $T^{i+1} > \hat{T}^{i+1}$ such that $(-1)^{i+1} v > 0$ in $[\hat{T}^{i+1}, T^{i+1}]$, $u(T^{i+1}) = 0$ and the estimates for $w(\hat{T}^{i+1})$ and $H(\hat{T}^{i+1})$ remain unchanged for $w(T^{i+1})$ and $H(T^{i+1})$. Therefore, by the Mathematical Induction, there exists $T^1 < \cdots < T^{N-1}$ such that the estimates in (4.1) with $i = 1, \cdots, N-1$ hold. Also, from the induction proof, the first assertion of the theorem follows.

(2) From the first assertion of the theorem, for all $a \in [\frac{3}{2}(N - \frac{1}{2})b, \frac{3}{2}(N + \frac{1}{2})b]$ and for all $i = 1, \cdots, N-1$, $\dot{v}(\tau^i) \neq 0$ and $\dot{u}(T^i) \neq 0$ for all $i = 1, \cdots, N-1$. This implies, by the Implicit Function Theorem, that for all $i = 1, \cdots, N-1$, τ^i and T^i are smooth functions of a in $[\frac{3}{2}(N - \frac{1}{2})b, \frac{3}{2}(N + \frac{1}{2})b]$ (subject to the normalization $\tau^0 = 0$). In addition, $\varepsilon^{-2} H(\hat{T}^N) > \varepsilon$ if $a \in [\frac{3}{2}Nb + M\varepsilon |\ln \varepsilon|, \frac{3}{2}(N + \frac{1}{2})b]$ and $\varepsilon^{-2} H(\hat{T}^N) < -\varepsilon$ if $a \in [\frac{3}{2}(N - \frac{1}{2})b, \frac{3}{2}Nb - M\varepsilon |\ln \varepsilon|]$. The existence of a^* thus follows from a similar argument as in the proof of Lemma 3.3 (4). Finally, since $H = O(\varepsilon^2)$ and $w = O(1)$, we know from the definition equation of H, that $v^2 = u^2 - u^4 + O(\varepsilon)$ for all $\tau \in (-\infty, \infty)$. □

Remark 4.1. One may notice from (2.3) that fixing b and ε and varying a is not equivalent to fixing q and R and varying s. However, note that in the proof of Theorem 4.1, one may let a and b vary along certain continuous curves and still get the existence of the homoclinic orbits needed. Hence, following a similar shooting argument as in the proof of Theorem 4.1 (or Lemma 3.3 (4)), one can show that for each q,

N, and large R, there exists $s = (2q+1)/3 + O(R^{-1/2})$ such that for (b, ε, a) given by (2.3), the trajectory γ^+ of (2.2) is a homoclinic orbits satisfying all the assertions in Theorem 4.1.

Clearly, to prove Theorem 1.1, we need only to show that in the original variables (x, y, z), the zeros of x and y of the homoclinic orbits established in Theorem 4.1 interlace. This follows from the following lemma.

Lemma 4.2 *Assume that a and b are positive constant of order 1 and ε is a small positive constant. Let (u, v, w) be any solution of (2.2). Assume that there exists $\tau_1 \geq -\infty$ and $\tau_2 \in (\tau^1, \infty]$ such that $u(\tau_1)v(\tau_1) = u(\tau_2)v(\tau_2) = 0$, $u > 0$ in (τ_1, τ_2), and v changes sign exactly once in (τ_1, τ_2). Also assume that $0 \leq w(\tau_1) = O(1), \varepsilon^{-2}H(\tau_1) = O(1)$. Let $(x(t), y(t), z(t))$ and $t = t(\tau)$ be the functions obtained via the transformation (2.1). Then in $(t(\tau_1), t(\tau_2))$, all the zeros of y are simple and y vanishes exactly once if $u(\tau_2) = 0$ and exactly twice if $v(\tau_2) = 0$ and $u(\tau_2) \neq 0$. Also, the trajectory rotates around the vertical line $x = y = \sqrt{q(R-1)}$ once.*

Proof. First of all, by the proof of Lemmas 3.2–3.4, $v^2 = u^2 - u^4 + O(\varepsilon)$ uniformly in (τ_1, τ_2). In addition, u is bounded by 2 and w is bounded by $w(0) + 3$ in (τ_1, τ_2).

Since $v = 0$ if and only if $x = y$, and $x = \sqrt{q(R-1)}$ if and only if $u = \sqrt{q/(4s)} = \sqrt{3q/(8q + 4 + O(\varepsilon))}$, we know that the trajectory in the xyz coordinates never touches the vertical line $x = y = \sqrt{q(R-1)}$, so that the trajectory rotates around the vertical line $x = y = \sqrt{q(R-1)}$.

Note that y is proportional to $v + \frac{2s}{1+s}\varepsilon u$. It then follows that $y = 0$ only if $v = O(\varepsilon)$. Also, when $y = 0$, $v = -\frac{2s}{1+s}\varepsilon u$, so that y' is proportional to the quantity $u\Big[1 - 2u^2 - \varepsilon u(1 + a\varepsilon)w + \frac{4s\varepsilon^2}{(1+s)^2}\Big]$. Hence, we conclude the following:

(1) if $u \geq 3/4$ and $y = 0$, then $y' < 0$;

(2) if $1/4 \leq u \leq 3/4$, then $y \neq 0$;

(3) if $0 < u \leq 1/4$ and $y = 0$, then $y' > 0$.

The assertion of the Lemma thus follows. □

Clearly, Theorem 1.1 follows from Theorem 4.1, Remark 4.1, and Lemma 4.2.

5. "Randomly" rotated homoclinic orbits and chaotic trajectories.

Now we shall prove Theorems 1.2–1.4.

In this section, b is a fixed positive constant, $N \geq 1$ is a fixed integer, $a \in [b, 4b]$ is a parameter, ε is a positive number which can be as small as we need, and M is a large constant depending only on b and N.

Lemma 5.1 *Let $(u(\cdot), v(\cdot), w(\cdot))$ be a solution of (2.2) with initial data*

$$u(0) = 0, \qquad v(0) = v_0 \in (0, M\varepsilon], \qquad w(0) = w_0 \in [0, M].$$

Let $T \in (0, \infty]$ be the first time at which u vanishes and let σ be the number of zeros of v in $(0, T)$ (σ can be ∞). Denote by $\tau_1, \cdots, \tau_\sigma$ the zeros of v in $(0, T)$, ordered in an increasing order. Then the following holds:

(1) $\sigma \geq 1$ *and for each positive integer* $j \leq \min\{\sigma, 2N+4\}$,

$$\begin{cases} 0 < w(\tau_j) \leq j + w(0) + O(\varepsilon), \\ \varepsilon^{-2}|H(\tau_j)| \leq \varepsilon^{-2}|H(0)| + 4aj/3 + j^2 b + jw(0) + O(\varepsilon), \\ u(\tau_j) = 1 + O(\varepsilon) \ \text{ if } j \text{ is odd and } \ u(\tau_j) = O(\varepsilon) \ \text{ if } j \text{ is even}, \\ \dot{v}(\tau_j) \neq 0. \end{cases} \tag{5.1}$$

(2) If in addition assume that $T < \infty$, $\sigma \leq 2N+4$, *and*

$$(\varepsilon^{-2}H(0), w(0)) \in D \equiv \Big\{(h,w) \ \Big| \ 0 < h \leq f(w), 0 \leq w \leq 2(1+a/b)\Big\}$$

where

$$f(w) := b\Big[(2+a/b)^2 - (w - a/b)^2\Big].$$

Then σ *is odd and* $(\varepsilon^{-2}H(T), w(T)) \in D$. *Moreover,* $v^2 = u^2 - u^4 + O(\varepsilon)$ *in* $[0,T]$.

Proof. (1) Clearly, by Lemma 3.4, $\sigma \geq 1$ and (5.1) holds for $j = 1$. Assume that (5.1) holds for $j = 2k+1$ for some $k \in \{0, 1, \cdots, N\}$. Then, starting from τ_{2k+1}, we can apply Lemma 3.2 to conclude that there exists $\hat{T} > \tau_{2k+1}$ such that

$$v < 0 \text{ in } (\tau_{2k+1}, \hat{T}], \qquad u(\hat{T}) = M^2\varepsilon, \qquad w(\hat{T}) \leq w(\tau_{2k+1}) + 1 + O(\varepsilon),$$
$$\varepsilon^{-2}|H(\hat{T})| \leq \varepsilon^{-2}|H(\tau_{2k+1})| + 3a/4 + b + 2bw(\tau_{2k+1}) + O(\varepsilon).$$

Consequently, if $\sigma \geq 2k+2$, then by Lemma 3.3,

$$\begin{aligned} |w(\tau_{2k+2})| &\leq |w(\hat{T})| + O(\varepsilon) \leq w(0) + 2k + 2 + O(\varepsilon), \\ \varepsilon^{-2}|H(\tau_{2k+2})| &\leq \varepsilon^{-2}|H(\hat{T})| + O(\varepsilon) \\ &\leq 4(2k+2)a/3 + (2k+2)bw(0) + (2k+2)^2 b + O(\varepsilon). \end{aligned}$$

Hence, (5.1) holds for $j = 2k+2$.

Now applying Lemma 3.4 for a trajectory starting from $(u(\tau_{2k+2}), 0, w(\tau_{2k+2}))$, we can conclude that $\sigma \geq 2k+3$ and that (5.1) holds also for $j = 2k+3$. Hence, by the Mathematical Induction, (5.1) holds for all $j \leq \min\{\sigma, 2N+4\}$. This proves the first assertion of the lemma.

(2) The oddness of σ follows from Lemma 3.4. It remains to prove $(\varepsilon^{-2}H(T), w(T)) \in D$. We write $\sigma = 2K+1$. Then $K \in \{0, \cdots, N+1\}$.

First we consider the case either $K \geq 1$ or $\{K = 0, \ H(0) \leq \varepsilon^3\}$. In case of $K \geq 1$, $\varepsilon^{-2}H(\tau_{2K}) = -u^2[1 - 2u^2 - \varepsilon(1 + a\varepsilon)w]\Big|_{\tau=\tau_{2K}} < 0$. Hence, no matter $K \geq 1$ or $\{K = 0, H(0) \leq \varepsilon^3\}$, Lemma 3.4 yields that

$$\varepsilon^{-2}H(\tau_{2K+1}) \leq 4a/3 - b + O(\varepsilon).$$

Therefore, by Lemma 3.2, there exists a constant $\hat{T} > \tau_{2K+1}$ such that in $v < 0$ in $(\tau_{2K+1}, \hat{T})$, $u(\hat{T}) = M^2\varepsilon$, and

$$\begin{aligned} H(\hat{T}) &\le H(\tau_{2K+1}) + 4a/3 - b - 2bw(\tau_{2K+1}) + O(\varepsilon|\ln\varepsilon|) \\ &\le 8a/3 - 2b - 2bw(\tau_{2K+1}) + O(\varepsilon\ln\varepsilon). \end{aligned} \tag{5.2}$$

Since $T < \infty$ and $\sigma = 2K+1$, by Lemma 3.3, we must have $H(\hat{T}) > -\varepsilon^3$, which implies that $w(\tau_{2K+1}) \le 4a/(3b) - 1 + O(\varepsilon\ln\varepsilon)$. Hence, applying Lemma 3.3 (3) and Lemma 3.2, we have the estimate

$$w(T) \le w(\hat{T}) + O(\varepsilon) \le w(\tau_{2K+1}) + 1 + O(\varepsilon\ln\varepsilon) \le 4a/(3b) + O(\varepsilon\ln\varepsilon).$$

Also, using (5.2) and applying Lemma 3.3 (3) we obtain

$$\varepsilon^{-2}H(T) \le \varepsilon^{-2}H(\hat{T}) + O(\varepsilon\ln\varepsilon) \le 8a/3 - 4b + O(\varepsilon\ln\varepsilon).$$

Noting that $H(T) = v^2(T) > 0$ and $\min_{\{w\in(0,4a/3b)\}} f(w) = b\Big[(a/b+2)^2 - (a/b)^2\Big] = 4a + 4b > \varepsilon^{-2}H(T)$, we hence conclude that $(\varepsilon^{-2}H(T), w(T)) \in D$.

It remains to consider the case $\{K = 0,\ H(0) > \varepsilon^3\}$. It follows from Lemma 3.4 that

$$\varepsilon^{-2}H(\tau_1) = \varepsilon^{-2}H(0) + 4a/3 - b - 2bw(0) + O(\varepsilon|\ln\varepsilon|), \qquad w(\tau_1) = w(0) + 1 + O(\varepsilon|\ln\varepsilon|),$$

and, from Lemma 3.2 that there exists $\hat{T} \in (\tau_1, T)$ such that

$$u(\hat{T}) = M^2\varepsilon, \quad w(\hat{T}) = w(0) + 2 + O(\varepsilon\ln\varepsilon),$$

$$\varepsilon^{-2}H(\hat{T}) = \varepsilon^{-2}H(0) + 8a/3 - 4b - 4bw(0) + O(\varepsilon\ln\varepsilon). \tag{5.3}$$

Since $\sigma = 1$ and $T < \infty$, Lemma 3.3 implies that $H(\hat{T}) > -\varepsilon^3$, which yields

$$w(0) \le \Big\{\varepsilon^{-2}H(0) + 8a/3 - 4b\Big\}/(4b) + O(\varepsilon\ln\varepsilon). \tag{5.4}$$

We consider two cases:
(i) $w(0) \le \Big\{\varepsilon^{-2}H(0) + 8a/3 - 4b\Big\}/(4b) - \sqrt{\varepsilon}$, and
(ii) $w(0) \ge \Big\{\varepsilon^{-2}H(0) + 8a/3 - 4b\Big\}/(4b) - \sqrt{\varepsilon}$.
In case (i), (5.3) yields $\varepsilon^{-2}H(\hat{T}) \ge 4b\sqrt{\varepsilon} - O(\varepsilon\ln\varepsilon) > \varepsilon$, so that by Lemma 3.3 (1),

$$\begin{aligned} \varepsilon^{-2}H(T) &= \varepsilon^{-2}H(\hat{T}) + O(\varepsilon\ln\varepsilon) = \varepsilon^{-2}H(0) + 8a/3 - 4b - 4bw(0) + O(\varepsilon|\ln\varepsilon|), \\ w(T) &= w(\hat{T}) + O(\varepsilon\ln\varepsilon) = w(0) + 2 + O(\varepsilon\ln\varepsilon). \end{aligned}$$

Since $\varepsilon^{-2}H(0) \le f(w(0))$, it follows that

$$\begin{aligned} \varepsilon^{-2}H(T) - f(w(T)) &\le f(w(0)) + 8a/3 - 4b - 4bw(0) - f(w(0)+2) + O(\varepsilon|\ln\varepsilon|) \\ &= -4a/3 + O(\varepsilon|\ln\varepsilon|) < 0. \end{aligned}$$

Moreover, $H(T) > 0$ yields $f(w(T)) > 0$, which implies $w(T) < 2(a/b+1)$. Thus $(\varepsilon^{-2}H(T), w(T)) \in D$.

In case (ii) , we have, by (5.4),

$$[\varepsilon^{-2}H(0) + 8a/3 - 4b]/(4b) - w(0) = O(\sqrt{\varepsilon}). \tag{5.5}$$

Hence, from (5.3), $\varepsilon^{-2}H(\hat{T}) = O(\sqrt{\varepsilon})$, and from Lemma 3.3 (1) and (3),

$$\varepsilon^{-2}H(T) = O(\sqrt{\varepsilon}), \qquad w(T) \le w(\hat{T}) + O(\varepsilon) \le w(0) + 2 + O(\varepsilon|\ln\varepsilon|).$$

Clearly, if $w(T) \le a/b$, then $\varepsilon^{-2}H(T) - f(w(T)) < 0$. If $w(T) > a/b$, then $-f(w(T)) \le -f(w(0)+2) + O(\varepsilon|\ln\varepsilon|)$, so that, by (5.5) and the assumption $\varepsilon^{-2}H(0) \le f(w(0))$,

$$\begin{aligned}
\varepsilon^{-2}H(T) - f(w(T)) &\le O(\sqrt{\varepsilon}) - f(w(0)+2) + \left\{\varepsilon^{-2}H(0) + 8a/3 - 4b - 4bw(0)\right\} \\
&\le O(\sqrt{\varepsilon}) - f(w(0)+2) + \left\{f(w(0)) + 8a/3 - 4b - 4bw(0)\right\} \\
&= O(\sqrt{\varepsilon}) - 4a/3 < 0.
\end{aligned}$$

In summary, we have that $(\varepsilon^{-2}H(T), w(T)) \in D$, thereby completing the proof of the lemma. □

Remark 5.1. (1) In Lemma 5.2, all $O(\varepsilon)$ are quantities whose absolute values do not exceed $C(M, b, N)\varepsilon$.

(2) Due to the accumulation of $O(\varepsilon)$ term, we cannot let N in the lemma go to infinite.

In the sequel, we shall denote by γ_a^+ the trajectory of the unstable manifold of the origin which initially lies in $\{u > 0\}$. By Lemma 3.1, it will first hit the u–w plane near $u = 1$ at τ^0. Unless specified, (u, v, w) will always be the solution corresponding to γ_a^+. Also, we normalize the solution so that τ_0 in Lemma 3.1 is zero. We shall use $T^i(a)$ to denote the i^{th} sign change of u and if u change sign only $i-1$ times then $T^i = \infty$. Also, when $T^i(a) < \infty$, we use $\sigma^i(a)$ to denote the number sign changes of v in $(T^i(a), T^{i+1}(a))$. If the sign change of v in $(T^i(a), T^{i+1}(a))$ is infinite, then $\sigma^i(a) = \infty$. Finally, we use τ_j^i to denote the j^{th} sign change of v in the interval $(T^i(a), T^{i+1}(a))$.

Our proof of Theorems 1.2–1.4 follows a similar shooting argument used in [9] where two ansatz conditions need to be checked by a computer. However, here in our case, the following lemma replaces their ansatz conditions.

Lemma 5.2 *1. For every $a \in [2b, 4b]$, $T^1(a) < \infty$ and $\sigma^0(a) = 1$;*

2. *For every $a \in [3b + M\varepsilon|\ln\varepsilon|, 4b]$, $T^2(a) < \infty$, $\sigma^0 = 1$, and $\sigma^1(a) = 1$;*

3. *For every $a \in [2b, 3b - M\varepsilon|\ln\varepsilon|]$, $\sigma^0(a) = 1$, $T^1(a) < \infty$, and $\sigma^1(a) \ge 2N + 4$.*

4. *Assume that for some $l \ge 1$ and $a_0 \in (2b, 4b)$, $T^l(a_0) \le \infty$ and $\max_{1\le j\le l}\{\sigma^j(a_0)\} \le 2N + 4$. Then, the following holds:*

   (a) for any $i = 1, \cdots, l$, $(\varepsilon^{-2}H(T^i), w(T^i)) \in D$;

(b) $T^l(\cdot)$, $\tau_1^l, \cdots, \tau^l_{\min\{\sigma^l(a_0),2N+4\}}$ *are all continuous in a neighborhood of* a*;*

(c) Let $(\alpha,\beta) \in (2b, 4b)$ *be an interval such that* $T^l(\cdot) < \infty$ *in* (α,β)*, then for all* $j = 1, \cdots, l-1$, σ^j *is a constant in* (α,β) *and* T^j, $\tau_1^j, \cdots, \tau^j_{\sigma^j}$ *are all continuous in* (α,β)*.*

Proof. The first and second assertions of the lemma follows from Theorem 4.1 (1).

To prove the third assertion of the lemma, note that from the first assertion of Theorem 4.1, at $\hat{T}^2$, $w(\hat{T}^2) = 4 + O(\varepsilon\ln\varepsilon)$ and $\varepsilon^{-2}H(\hat{T}^2) \le -M\varepsilon|\ln\varepsilon|/2$. Hence, by Lemma 3.3 (3), $\sigma^2(a) \ge 2$, and $w(\tau_2^2(a)) = 4 + O(\varepsilon\ln\varepsilon|)$ and $\varepsilon^{-2}H(\tau_2^2(a)) = \frac{16}{3}(a - 3b) + O(\varepsilon\ln\varepsilon)$. Repeatedly applying the estimates in Lemma 3.4, Lemma 3.2, and Lemma 3.3 (2), we can conclude that $\sigma^2(a) \ge 2N+4$, and at each $j = 1, 2, \cdots, N+1$, $w(\tau^2_{2j}(a)) = 2 + 2j + O(\varepsilon\ln\varepsilon)$ and $\varepsilon^{-2}H(\tau^{2j}) = \frac{16}{3}j(a - 3jb) + O(\varepsilon\ln\varepsilon)$ (cf. the proof of Theorem 4.1). The third assertion of the lemma thus follows.

By Lemma 5.1, we have 4(a) and $\dot v \ne 0$ at τ_i^j and $\dot u \ne 0$ at T^j for all $j = 1, \cdots, l$ and $i = 0, \cdots, \min\{\sigma^j, N+4\}$. The last assertion of the lemma thus follows from the Implicit Function Theorem. □

The following two lemmas are the key ingredient of the proof of Theorem 1.2.

Lemma 5.3 *Let* $\alpha^0 = 3b - M\varepsilon|\ln\varepsilon|$, $\beta^0 = 3b + M\varepsilon|\ln\varepsilon|$*, and set* $I^0 = (\alpha^0, \beta^0)$*. Then there exist* $N+1$ *intervals* $I_1^0 = (\alpha_1^0, \beta_1^0), \cdots, I_{N+1} = (\alpha^0_{N+1}, \beta^0_{N+1})$ *such that*

$$\begin{aligned}
&\alpha^0 < \alpha^0_{N+1} < \beta^0_{N+1} \le \alpha^0_N < \beta^0_N \le \cdots < \beta^0_2 \le \alpha^0_1 < \beta^0_1 < \beta^0, &(5.6)\\
&T^2(\cdot) \in C^0(\cup_{k=1}^{N+1} I_k^0),\\
&\sigma^1(a) = 2k+1 \qquad \forall\, a \in [\alpha_k, \beta_k) \qquad \forall\, k = 1, \cdots, N+1,\\
&\sigma^1(\beta_k) = 2k-1 \qquad \forall\, k = 1, \cdots, N+1.
\end{aligned}$$

In addition, for all $k = 1, \cdots, N+1$, $\gamma^+_{\beta^0_k}$ *and* $\gamma^+_{\alpha^0_k}$ *are homoclinic orbits associated with the origin.*

Lemma 5.4 *Let* $l \ge 1$ *be any given integer and assume that* I^l *is an interval having the following properties:*

1. *If* $l = 1$ *then* $I^1 = I^0_{N+1}$ *and if* $l \ge 2$ *then* $\bar{I}^l \subset I^0_{N+1}$*;*
2. $T^l \in C^0(\bar{I}^l)$, $T^{l+1} \in C^0(I^l)$*, and* $\max_{1\le j\le l}\sigma^j \le 2N+4$*;*
3. *Set* $\sigma^l = 2K^l + 1$*. If* $K^l \ge 1$*, then both* γ_{α^l} *and* γ_{β^l} *are homoclinic orbits with* $T^{l+1} = \infty$ *and*

   $$\min\{\sigma^l(\alpha^l), \sigma^l(\beta^l)\} = 2K^l - 1, \qquad \max\{\sigma^l(\alpha^l), \sigma^l(\beta^l)\} = 2K^l + 1.$$

   If $K^l = 0$*, then* $\sigma^l(\alpha^l) = \sigma^l(\beta^l) = 1$*, and at one end of* I^l, γ^+ *is a homoclinic orbit with* $T^{l+1} = \infty$ *and at the other end of* I^l, $T^{l+1} < \infty$ *and* $\sigma^{l+1} \ge 2N+1$*.*

Then there exist N *disjoint intervals* $I_0^l = (\alpha_0^l, \beta_0^l)$, $\cdots$, $I^l_{N-1} = (\alpha^l_{N-1}, \beta^l_{N-1})$ *having the following properties:*

1. *$\bar{I}^l_0, \cdots, \bar{I}^l_{N-1}$ are all contained in I^l and they are distributed either in a increasing or a decreasing order in I^l;*

2. *For each $k = 0, \cdots, N-1$, $T^{l+2} \in C^0(I^l_k)$ and $\sigma^{l+1}(a) = 2k+1$ for all $a \in I^l_k$;*

3. *For each $k = 1, \cdots, N-1$, $\gamma^+_{\alpha^l_k}$ and $\gamma^+_{\beta^l_k}$ are homoclinic orbits with $T^{l+2} = \infty$ and*

$$\min\{\sigma^{l+1}(\alpha^l_k), \sigma^{l+1}(\beta^l_k)\} = 2k-1, \qquad \max\{\sigma^{l+1}(\alpha^l_k), \sigma^{l+1}(\beta^l_k)\} = 2k+1.$$

For $k = 0$, $\sigma^{l+1}(a) = 1$ for all $a \in \bar{I}^l_0$, and at one end of I^l_0, γ^+ is a homoclinic orbit with $T^{l+2} = \infty$ and at the other end of I^l_0, $T^{l+2} < \infty$ and $\sigma^{l+2} \geq 2N+1$.

Proof of Lemma 5.3. Define, for each $k = 1, \cdots, N+1$,

$$\beta^0_k = \sup\{a \in I^0 \,\Big|\, \sigma^1(a) \geq 2k+1 \text{ in } (\alpha^0, a)\}$$

Since by Lemma 5.2, $\sigma^1 \geq 2N+4$ for every a near α^0, and $\sigma^1 = 1, T^2 < \infty$ for all a near β^0, β^0_k is well-defined, and $\alpha_0 < \beta^0_{N+1} \leq \cdots \leq \beta^0_1 < \beta_0$.

For each $k \in \{1, \cdots, N+1\}$, by Lemma 5.2 and the definition of β^0_k, we can deduce that $\sigma^1(\beta^0_k) < 2k+1$ and $\liminf_{a \nearrow \beta^0_k} \tau^1_{2k}(a) = \infty$. Let j be the smallest positive integer such that $\limsup_{a \nearrow \beta^0_k} \tau^1_{2j}(a) = \infty$ and let $\{a^m\}^\infty_{m=1}$ be a sequence such that $a^m \nearrow \beta^0_k$ and $\tau^1_{2j}(a^m) \to \infty$ as $m \to \infty$. Then, upon taking a subsequence if necessary, we can assume that, as $m \to \infty$, $\gamma^+_{a^m}(\tau^1_{2j+1}(a^m)) \to (-1 + \varepsilon U^1, 0, W^1)$ and $\gamma_{a^m}(\tau^1_{2j}(a^m)) \to (0, 0, W^2)$ for some $U^1, W^1, W^2 \in [0, M]$. Since ε is fixed, an argument similar to the proof of Lemma 3.3 shows that $W^2 = 0$. It then follows that the trajectory $\gamma(\tau) := \gamma^+_{a^m}(\tau - \tau^1_{2j+1}(a^m))$ will tend to $\gamma^-_{\beta^0_k}(\tau)$ as $m \to \infty$. Since $T^1(\beta^0_k) < \infty$ and $\sigma^0(\beta^0_k) = 1$, we mush have $T^2(a^m) - \tau_{2j-1}(a^m) = T^1(\beta^0_k) + o(1)$, and $\sigma^1(a^m) = 2j+1$ for all m large enough. Therefore, by the definition of β^0_k, we must have $k = j$. Consequently, $\limsup_{a \nearrow \beta^0_k} \tau^1_{2k-2}(a) < \infty$, which also implies that $\limsup_{a \nearrow \beta^0_k} \tau^1_{2k-1}(a) < \infty$. Hence $\sigma(\beta^0_k) \geq 2k-1$. Since $\sigma^1(\beta^0_k) < 2k+1$ and $\sigma^1(\beta^0_k)$ is odd, we must have $\sigma^1(\beta^0_k) = 2k-1$ for all $k = 1, \cdots, N+1$. Consequently, $\beta^0_{N+1} < \beta^0_N < \cdots < \beta^0_1$.

Similarly, from $\liminf_{a \nearrow \beta^0_{2k}} \tau^1_{2k}(a) = \infty$, we can conclude that as $a \nearrow \beta^0_k$, $\gamma^+_a(\tau - \tau^1_{2k+1}(a))$ approaches $\gamma^-(\beta^0_k)$. It then follows that there exists $\delta > 0$ satisfying

$$T^2(a) < \infty \text{ and } \sigma^1(a) = 2k+1 \quad \forall\, a \in (\beta^0_k - \delta, \beta^0_k) \text{ for all } k = 1, \cdots, N+1. \tag{5.7}$$

Noting that for each $k = 1, \cdots, N$, σ^1 is discontinuous at β^0_k, we must have $T^2(\beta^k_0) = \infty$, which implies that $\gamma^+_{\beta^0_k}$ is a homoclinic orbit associated with the origin.

Next we define

$$\alpha^0_k := \inf\{a \in (\alpha^0, \beta_k) \,\Big|\, T^2 < \infty \text{ in } (a, \beta^0_k)\}, \qquad k = 1, \cdots, N+1.$$

By (5.7), α^0_k is well-defined and $\alpha^0_k < \beta^0_k$ for all $k = 1, \cdots, N+1$. In addition, $T^2 < \infty$ in (α^0_k, β^0_k) so that $\sigma^1 = 2k+1$ in (α^0_k, β^0_k) for all $k = 1, \cdots, N+1$.

Since $T^2 = \infty$ on $\beta_1^0, \cdots, \beta_{N+1}^0$ and $\sigma^1(a_0) \geq 2N+4$, we must have $\alpha_{N+1} \in (\alpha^0, \beta_{N+1}^0)$ and $\alpha_k \in [\beta_{k+1}^0, \beta_k^0)$ for all $k = 1, \cdots, N$. That is, (5.6) holds.

From the definition of β_k^0, $\sigma^1 \geq 2k+1$ in (a_0, β_k^0). It follows that $\sigma^1(\alpha_k^0) \geq 2k+1$. Also, since $\sigma^1 = 2k+1$ in (α_k^0, β_k^0), we must have $\sigma^1(\alpha_k^0) \leq 2k+1$. This implies that $\sigma^1(\alpha_k^0) = 2k+1$. In addition, by the definition of α_k^0, we must have $T^2(\alpha_k^0) = \infty$, which implies that $\gamma_{\alpha_k^0}$ is a homoclinic orbit associated with the origin. This completes the proof of Lemma 5.3. □

Proof of Lemma 5.4.

First we consider the case when γ_{α^l} is a homoclinic orbit and $\sigma^l(\alpha^l) \geq \sigma^l(\beta^l)$.

We begin with showing that

$$\sigma^{l+1}(a) \geq 2N+1 \qquad \text{if } \ 0 < \beta^l - a << 1. \tag{5.8}$$

In fact, when $K^l = 0$, by the assumption, $T^{l+1}(\beta^l) < \infty$, and $\sigma^{l+1}(\beta^l) \geq 2N+1$, so that from Lemma 5.2 (4), (5.8) holds.

When $K^l \geq 1$, γ_{β^l} is a homoclinic orbit with $\sigma^l(\beta^l) = 2K^l - 1$ and $T^{l+1}(\beta^l) = \infty$. Since $T^{l+1} < \infty$ and $\sigma^l = 2K^l + 1$ in (α^l, β^l), by Lemma 5.2 (4), $\tau_{2K^l}^l$ is smooth in (α^l, β^l). Hence, we must have , $\liminf_{a \nearrow \beta^l} \tau_{2K^l}^l = \infty$. Consequently, $\gamma(\tau) := \gamma_a(\tau - \tau_{2K^l+1}^l(a))$ approaches to either $\gamma_{\beta^l}^+(\tau)$ or $\gamma_{\beta^l}^-(\tau)$ as $a \nearrow \beta^l$. Since $\beta^l \in \bar{I}_{N+1}^0$, we know that $T^1(\beta^l) < \infty$ and either $\sigma^1(\beta^l) = 2N+3$ (if $\beta^l \in [\alpha_{N+1}^0, \beta_{N+1}^0)$ or $\sigma^1(\beta^l) = 2N+1$ (if $\beta^l = \beta_{N+1}^0$). It then follows that for all a smaller than and sufficiently close to β^l, $\sigma^{l+1}(a) \geq 2N+1$. Thus, (5.8) holds.

By assumption, γ_{α^l} is a homoclinic orbit and $T^{l+1} \in C^0(I^l)$. It follows that $\lim_{a \searrow \alpha^l} T^{l+1}(a) = \infty$. Hence $\gamma(\tau) := \gamma_a(\tau - \tau_1^{l+1})$ will tend to either $\gamma_{\alpha^l}^+(\tau)$ or $\gamma_{\alpha^l}^-(\tau)$ when $a \searrow \alpha^l$. Since $\alpha^l \in [\alpha_{N+1}^0, \beta_{N+1}^0)$, $\sigma^0(\alpha^l) = 1$, $T^1(\alpha^l) < \infty$ and $\sigma^1(\alpha^l) = 2N+3$. It then follows that for all a bigger than and sufficiently close to α^l,

$$\sigma^{l+1}(a) = 1, \qquad T^{l+2}(a) < \infty, \qquad \sigma^{l+2}(a) \geq 2N+3.$$

We let α_0^l be one of these points.

Now we can define for all $k = 1, \cdots, N$,

$$\alpha_k^l = \inf\{a \in I^l \;\Big|\; \sigma^{l+1} \geq 2k+1 \text{ in } (a, \beta^l)\}.$$

Then, by the previous discussion near α^l and β^l, we know that α_k^l is well–defined and

$$\alpha^l < \alpha_0^l \leq \alpha_1^l \leq \cdots \leq \alpha_{N-1}^l < \beta^l.$$

Now using a similar argument as in Lemma 5.3, we can prove the following:

(1) $\sigma^{l+1}(a) \geq 2k+1$ in (α_k^l, β^l);

(2) $\sigma^{l+1}(\alpha_k^l) = 2k-1$ and $\gamma_{\alpha_k^l}$ is a homoclinic orbit with $T^{l+2}(\alpha_k^l) = \infty$ for all $k = 1, \cdots, N$;

(3) $\alpha_k < \alpha_{k+1}$ for all $k = 0, \cdots, N-1$;

(4) there exists $\delta > 0$ such that $T^{l+2}(a) < \infty$ and $\sigma^{l+1}(a) = 2k+1$ in $(\alpha_k^l, \alpha_k^l + \delta)$ for every $k = 0, \cdots, N$.

Now we can define, for each $k = 0, \cdots, N-1$,

$$\beta_k^l := \sup\{a > \alpha_k^l \,\Big|\, T^{l+2} < \infty \ \text{ in } \ (\alpha_k^l, a)\},$$

Then, by the properties of α_k^l, β_k^l is well–defined and $\beta_k^l \in [\alpha_k^l + \delta, \alpha_{k+1}^l]$ for all $k = 0, \cdots, N-1$. Similar to the proof in Lemma 5.3, one can show that for each $k = 0, \cdots, N-1$, $\gamma_{\beta_k^l}$ is a homoclinic orbit with $\sigma^{l+1}(\beta_k^l) = 2k+1$ and $T^{l+2}(\beta_k^l) = \infty$. This proves the assertion of the Lemma.

The case when the roles of α^l and β^l are reversed can be similar treated. This completes the proof of the Lemma. □

Remark 5.2. If $l \geq 2$, then $\bar{I}^l \subset\subset I^1$, so that $\sigma^1(a) = 2N+3$ for all $a \in \bar{I}^l$. Hence, for all $l \geq 2$, a subinterval I_N^l having the corresponding properties stated in Lemma 5.4 can be similarly constructed. That is, k^l $(l \geq 3)$ in Theorem 1.2 and 1.3 can be taken from $\{0, \cdots, N\}$ (instead from $\{0, \cdots, N-1\}$). For simplicity, we shall not pursue it here.

Proof of Theorem 1.2.

Take $I^0 = (\alpha^0, \beta^0)$ and $I^1 = (\alpha_{N+1}^0, \beta_{N+1}^0)$. We construct $I^2, I^3, \cdots, I^L$ by Mathematical Induction as follows.

Let $l \geq 1$ be an arbitrary positive integer. Assume that $I^0, I^1, \cdots, I^l$ have been constructed such that $\bar{I}^j \subset I^{j+1}$ and $\sigma^j = 2k^j + 1$ in I^j for all $j = 1, \cdots, l$. In addition, I^l satisfies the assumption in Lemma 5.4. Let $I_0^l, \cdots, I_{N-1}^l$ be the intervals obtained in Lemma 5.4. Define $I^{l+1} = I_{k^{l+1}}^l$. Then we have that $\bar{I}^{l+1} \subset I^l$, $\sigma^{l+1} = 2k^{l+1} + 1$ in I^{l+1}, and I^{l+1} satisfies all the assumption of Lemma 5.4 with l replaced by $l+1$. Therefore, by Mathematical Induction, we can construct $I^0, I^1, \cdots, I^L$ such that $\bar{I}^{l+1} \subset I^l$ for all $l = 0, \cdots, L-1$, $\sigma^l = 2k^l + 1$ in I^l and I^l satisfies the assumption of Lemma 5.4 for all $l = 2, \cdots, L$.

Since $\bar{I}^l \subset I^{l-1}$ for all $l = 1, \cdots, L$, we conclude that for all $a \in [\alpha^L, \beta^L]$, $T^L(a) \leq \infty$ and $\sigma^l(a) = 2k^l + 1$ for all $l = 1, \cdots, L-1$. In addition, either $\gamma_{\alpha^L}^+$ or $\gamma_{\beta^L}^+$ is a homoclinic orbit with $T^{L+1} = \infty$ and $\sigma^l = 2k^l + 1$ for all $l = 0, \cdots, L$ $(k^0 \equiv 0,\ k^1 \equiv N+1)$. The assertion of Theorem 1.2 thus follows from Remark 5.2 and Lemma 4.2. □

Proof of Theorem 1.3. Construct $I^0, I^1, \cdots, I^l, \cdots$ inductively as in the proof of Theorem 1.2 for the sequence $\{k^l\}_{l=2}^{\infty}$. Set $I^\infty = \cap_{l=0}^{\infty} \bar{I}^l$. Since $\bar{I}^{l+1} \subset I^l$ for all $l \geq 0$, I^∞ is non–empty. Clearly, taking any $a \in I^\infty$, γ_a^+ will have the property needed by the assertion of the theorem. □

Proof of Theorem 1.4. For each positive even integer j and all integers $l \geq 2$, let $k_j^l = k^{l-j}$. Let a_j be the constant in Theorem 1.3 with the sequence $\{k_j^l\}_{l=2}^{\infty}$. Set $\gamma_j(\tau) = \gamma_{a_j}^+(\tau - \tau_1^j)$. Then $\gamma_j(0) = (1 + \varepsilon O(1), 0, O(1))$ where $O(1)$ are bounded independent of j. In addition, $\sigma^i = 2k^i + 1$ for all $i \geq 2 - j$ where σ^i is the number of zeros of v in (T^i, T^{i+1}) and $\{T^i\}_{i=-j+1}^{\infty}$ are all the zeros of u in the increasing order. Let j_m be a subsequence such that as $m \to \infty$, $j_m \to \infty$, a_{j_m} has a limit $a^* \in \cup_{k=0}^{N-1} \bar{I}_k^1 \subset (\alpha_{N+1}^0, \beta_{N+1}^0)$ and $\gamma_{j_m}(0)$ has a limit $(1 + \theta^* \varepsilon, 0, w^*)$. We claim that for the parameter (b, a^*, ε), the solution γ^* of (2.2) with initial data $(1 + \theta^* \varepsilon, 0, w^*)$ is the trajectory we want.

Assume that l is the minimum positive integer such that $T^l = \infty$. If $l < \infty$, then since as $m \to \infty$, γ_{j_m} approaches γ^*, we must have that $T^l_m \to \infty$ and $\gamma_{j_m}(T^l_m) \to (0,0,0)$ as $m \to \infty$. This implies that after T^l_m, the trajectory γ_{j_m} will stay close either to $\gamma^+_{a^*}$ or $\gamma^-_{a^*}$, which implies that $\sigma^{l+2}_{j_m} = 2N+3$. This is impossible since $\sigma^{l+2}_{j_m} = 2k^{l+2}+1 < 2N-1$. Hence we must have $l = \infty$. That is, $T^l < \infty$ for all $l \geq 1$. Consequently, by Lemma 5.1, for any fixed $l \geq 1$, σ^l remains unchanged if we make small perturbation on a from a^* and on the initial data from $(1+\theta^*\varepsilon, 0, Q^*)$. Hence, by the way we obtaining γ^*, $\sigma^l = 2k^l + 1$ for all $l \geq 1$. Similarly, we can show that $T^l > -\infty$ and $\sigma^l = 2k^l + 1$ for all $l \leq 0$. This completes the proof of Theorem 1.4.

Remark 5.3. For arbitrary fixed $a \in [2b, 3b - M\varepsilon|\ln\varepsilon|]$, setting $(u(0), v(0), w(0)) = (1, 0, Q)$ as the initial data for (2.2), one can actually take Q as a shooting parameter and use similar argument as above to prove the existence chaotic trajectories. Details are omitted.

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