On solutions to Navier-Stokes equations in critical spaces

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Abstract

In this thesis, we consider solutions to the incompressible Navier-Stokes equations in three spatial dimensions in the critical homogenous Sobolev space. We attempt to unify the theory of mild solutions and the theory of suitable weak solutions to show that assuming the existence of initial data leading to a finite time singularity the set of such initial is closed in the weak topology and sequentially compact modulo translations and dilations. This result is motivated by a theorem of Gallagher, Iftimie and Planchon which states that this set is closed in the strong topology.

We present two proofs of our result. The first one is based on the theory of suitable weak solutions and partial regularity for the Navier-Stokes equaitons. The second approach is rooted in the profile decomposition developed by Gallagher.

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Chapter 1

Introduction

1.1 Statement of the problem

We consider the incompressible Navier-Stokes equations in three spatial dimensions

$$\begin{aligned} &\partial_t v - \nu\Delta v + v\cdot\nabla v + \nabla p = 0 && \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ &\operatorname{div} v = 0 && \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ &v(0,x) = \phi(x) && \text{in } \mathbb{R}^3, \end{aligned} \tag{NSE}$$

where $v(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}^3$ is the velocity, $p(t,x):\mathbb{R}^+\times\mathbb{R}^3\to\mathbb{R}$ is the associated pressure and $\phi(x):\mathbb{R}^3\to\mathbb{R}^3$ is given divergence-free initial data, $\nu>0$ is the viscosity coefficient and for our convenience, without loss of generality, we set $\nu=1$. The divergence free condition on v represents the incompressibility of the fluid.

The Cauchy problem for (NSE), although widely studied, is still far from being fully understood. There are essentially two branches of approaching it. In [20] J. Leray introduced the concept of *weak solution* and proved global existence for initial data $\phi(x) \in L^2(\mathbb{R}^3)$. Uniqueness of these solutions (and other basic properties that one would desire from a solution) has remained an open problem. In [7] H. Fujita and T. Kato proved existence of solutions for data $\phi(x) \in \dot{H}^{1/2}(\mathbb{R}^3)$ by semi-group methods, known as *mild solutions*. The theory of mild solutions is mostly associated with Kato and Fujita, but it is worth mentioning that some version of it was already contained in the work of Leray. Mild solutions are unique but only local in time: $v(t,x) \in C([0,T^*),\dot{H}^{1/2}(\mathbb{R}^3))$, unless one is willing to make an assumption on smallness of initial data. This branch of results has been subsequently extended and the most recent result ([17]) states global well-posedness of (NSE) for small data in BMO^{-1}. In the attempt to fill the gap between weak L^2 solutions and mild L^3 (or $\dot{H}^{1/2}$) solutions, C. Caldèron in [2] proved existence of global weak solutions with data $\phi(x) \in L^p(\mathbb{R}^3)$, $2 \leq p \leq 3$. These results have been later extended by P.G. Lemarié-Rieusset to uniformly locally L^2 data in [19].

The theory of weak solutions takes advantage of the specific structure of the Navier-Stokes system, and in particular of the energy estimate. Kato's approach, on the other hand, is more general in its nature and applies to many other parabolic (or dispersive) semilinear problems. It is very illuminating to relate the use of different spaces for initial data and resulting solutions to scaling considerations: the energy estimate is based on the L^2 norm, which is "below" the scale-invariant norms for (NSE), namely $L^3(\mathbb{R}^3)$ (or $\dot{H}^{1/2}(\mathbb{R}^3)$). Thus (NSE) is said to be *supercritical* with respect to scaling. This might be a reason which precludes attempts to use the energy estimate to derive information for the mild solutions. To give an accurate description of the local regularity of weak solutions, V. Scheffer introduced in [25] a local version of the energy inequality. Weak solutions satisfying the local energy estimate are called following the work of L. Caffarelli, R. Kohn and L. Nirenberg *suitable solutions.* Notice that in three spatial dimensions for initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$ one has both - a local mild solution in the sense of T. Kato and a global weak solution in the sense of J. Leray. The issue of their coincidence has been addressed by J. Serrin in [26] and H. Sohr and W. von Wahl in [30]. The latter result has been then extended by P.G. Lemarié-Rieusset to data locally in L^2 (see [19]).

In this thesis we study the Navier-Stokes equations with initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$. It is known that, regardless of initial data, $v(t,x)$ and the corresponding pressure $p(t,x)$ become instantly smooth until possibly (see [6]) a singularity occurs. However, as mentioned above, if one accepts smallness restrictions of initial data, solutions remain smooth globally in time. Let us thus define

$$\rho_{max} := \sup \left\{ \begin{array}{c} \rho > 0 : \text{for } \|\phi\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \rho, \\ \text{corresponding } v(t,x) \text{ is a global strong solution of (NSE)} \end{array} \right\}.$$

Global existence for any initial data would imply $\rho = +\infty$, this problem however remains unsolved (see [6]) and throughout the following we will assume that blow up in (NSE) is possible i.e. $\rho < +\infty$. In principle ρ_{max} could be finite for various reasons, which depend on the exact notion of the solution. However, one can show (see Lemma 3.2.3) that with the natural definition of the mild solution, the only reason ρ_{max} could be finite is the appearance of finite-time singularities in the solution $v(t,x)$ for some initial data $\phi(x)$.

The purpose of this thesis is to investigate the following question:

If ρ_{max} *is finite, does there exist an initial datum* $\phi(x) \in \dot{H}^{1/2}(\mathbb{R}^3)$ *with* $\|\phi\|_{\dot{H}^{1/2}} = \rho_{max}$*, such that the solution* $v(t,x)$ *of the Cauchy problem (NSE) develops a singularity in finite time?*

We show that the answer to this question is affirmative. The initial data $\phi(x)$ with $\|\phi\|_{\dot{H}^{1/2}} = \rho_{max}$ leading to a singularity will be called $\dot{H}^{1/2}$*-minimal singularity-generating data*. We show that, if singularities exist, the set of the $\dot{H}^{1/2}$-minimal singularity-generating data is in fact a (nonempty) subset of $\dot{H}^{1/2}$ which is compact, modulo the action of the scalings $\phi(x) \to \lambda\phi(\lambda x)$ and translations $\phi(x) \to \phi(x - x_0)$.

In the chapters that follow, we begin with an introduction to known results and recall different notions of solutions. We then establish our results. Before we begin, we give a brief summary of each chapter.

1.2 Chapter summaries

Chapter 2: In this chapter we briefly review the scaling properties of the Navier-Stokes equations as well as different notions of solutions to the Cauchy problem (NSE). In particular we describe *mild solutions*, *Leray solutions* and *suitable weak solutions*.

Chapter 3: In this chapter we first recall some auxiliary results, needed in the latter part of our considerations. Then, we proceed to studying the posed problem in the setting of suitable weak solutions and Leray solutions with initial data in $\dot{H}^{1/2}(R^3)$. The strategy is to consider a sequence of initial data in $\phi^k \in \dot{H}^{1/2}(\mathbb{R}^3)$ with the property $\|\phi^k\|_{\dot{H}^{1/2}} \searrow \rho_{max}$. We first show existence of Leray solutions in this case with some appropriate energy estimates - Theorem 3.2.1. By a "weak-strong uniqueness" theorem - Theorem 3.2.2, these Leray solutions have to coincide with the mild solutions on the maximal time of existence of the latter ones. Next, we demonstrate that the only reason for a mild solution to be local is a finite-time singularity - Proposition 3.2.3. Using a version of localized a-priori estimates due to Lemarié-Rieusset - Lemma 3.2.5 we prove the main theorem of Chapter 2 - Theorem 3.2.6, namely that in the setting of weak converging initial data and a weakly converging sequence of corresponding Leray solutions, the limit solutions corresponds to the weak limit of initial data. It is notable that the proof relays strongly on suitability of these solutions, in particular uses the local energy inequality together with the fact that the embedding $\dot{H}^{1/2} \hookrightarrow L^2$ is compact on compact subsets of $\mathbb{R}^3$. The answer to the problem considered in this thesis is derived from this theorem as Corollary 3.

Chapter 4: In this chapter we treat the problem posed in the introduction using a different approach. First, we recall a decomposition of bounded sequences in $\dot{H}^{1/2}$, due to P. Gérard, and a how this decomposition is propagated by the Navier-Stokes equations - a result by I. Gallagher. The next section contains a proof of our main result relying completely on this theory.

Chapter 2

Background and auxiliary results

2.1 Symmetry invariance

An important observation about the Cauchy problem (NSE) is that it is invariant under spatial translations and dilations. In other words, if $(v(t,x), p(t,x))$ is a solution of (NSE) with data $\phi(x)$, then for any $\lambda > 0$ and $x_0 \in \mathbb{R}^3$

$$v^{\lambda,x_0}(t,x) = \lambda v(\lambda^2 t, \lambda x - x_0), \qquad p^{\lambda,x_0}(t,x) = \lambda^2 p(\lambda^2 t, \lambda x - x_0) \tag{2.1}$$

is also a solution with initial data

$$\phi^{\lambda,x_0}(x) = \lambda\phi(\lambda x - x_0). \tag{2.2}$$

In particular, the dilation invariance plays a very important role in the analysis of (NSE) and underlays the classification of functional spaces into sub-/super-/critical depending on how the scaling affects the norm.

2.2 Mild solutions

In this section we review the results we need about the so-called *mild solutions* of the problem (NSE). This approach was introduced by H. Fujita and T. Kato, [7], see also [13], although the terminology was introduced later. Let us first recall some basic facts about the linear Stokes problem

$$\begin{aligned} &\partial_t v - \nu\Delta v + \nabla p = \partial_{x_k} f_k && \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ &\operatorname{div} v = 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ &v(0,x) = \phi(x) && \text{in } \mathbb{R}^3. \end{aligned} \tag{S}$$

Here $f_k = (f_{1k}, \dots, f_{3k})$ for $k = 1, \dots, 3$. The framework of mild solutions is set up as follows. Consider a Banach space of initial data $X \subset \mathcal{S}'(\mathbb{R}^3)$, and a Banach space $Y_T \subset L^2_{loc}((0,T) \times \mathbb{R}^3)$ in which we will look for a solution. Furthermore, let $S(t)$ be the solutions operator of the heat equation and $\mathbb{P}$ be the Helmholtz projection of vector fields onto the divergence-free vector fields. By definition, the solution of the linear problem (S) is given by the representation formula

$$v(t,x) = S(t)\phi(x) + \int_0^t S(t-s)\mathbb{P}\partial_{x_k} f_k \, ds. \tag{2.3}$$

Let us now clarify the above equation. Consider the heat equation with the forcing term $f(t,x)$ and initial data $\phi(x)$

$$\begin{aligned} &\partial_t v - \Delta v = f && \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ &v(0,x) = \phi(x) && \text{in } \mathbb{R}^3. \end{aligned} \tag{2.4}$$

We are looking for a solution $v(t,x)$ as being a continuous in time variable with values in a functional Banach space Y. We can assume that Y is separable and

$v(t,x) \in C([0,\infty), Y)$ has the usual meaning. Let us consider a second Banach space Y' with the following two properties: $\Delta : Y \to Y'$ is continuous and if $\varphi \in \mathcal{S}(\mathbb{R}^3)$ then the convolution operator $v \mapsto v * \varphi$ continuously maps Y' to Y. Then with the proceeding notations, a *mild solution* $v(t,x)$ to the heat equation is a solution belonging to $C([0,\infty), Y)$ when $f \in C([0,\infty), Y')$. If such a mild solutions exists, it is uniquely given by

$$v(t,x) = S(t)\phi(x) + \int_0^t S(t-s)f(s)\,ds. \tag{2.5}$$

The integral on the right-hand side belongs to $C([0,\infty), Y')$ and in general does not belong to $C([0,\infty), Y)$.

Moreover, one can very often give up on continuous dependence on time and substitute it with some measurability/integrability conditions. This way, we obtain solutions as Lebesgue measurable variables (with some integrability) with values in Lebesgue spaces.

In order to solve the nonlinear problem (NSE) we notice that due to the divergence-free condition we can write

$$v \cdot \nabla v = \operatorname{div}(v \otimes v), \tag{2.6}$$

and regard the nonlinearity as the right-hand side in the problem (S). This allows us to view the Cauchy problem (NSE) as a fixed point problem in some appropriate Banach spaces X and Y_T. We define a bilinear form $B(\cdot,\cdot) : Y_T \times Y_T \to Y_T$ by

$$B(u,v) = -\int_0^t S(t-s)\mathbb{P}\operatorname{div}(u \otimes v)\,ds, \tag{2.7}$$

then the Cauchy problem for the Navier-Stokes equations can be rewritten as

$$v = S(t)\phi + B(v,v). \tag{2.8}$$

The key-property for obtaining a mild solution is the continuity of the form $B(\cdot,\cdot)$ as a map from $Y_T \times Y_T \to Y_T$. This is equivalent to

$$\|B(u,v)\|_{Y_T} \leq \eta \|u\|_{Y_T} \, \|v\|_{Y_T}. \tag{2.9}$$

The scaling invariance of (NSE) plays a crucial role in what kind of existence results one can expect. Roughly speaking, if the scaling does not affect $\|\cdot\|_{Y_T}$, i.e. we are in the *critical* case, one can expect global existence (at least for small initial data). In this scenario, estimate (2.9) satisfies the same invariance. In the case of *subcritical* spaces one can only hope for local existence. The theory of mild solutions breaks down for *supercritical* spaces.

Once continuity of $B(\cdot,\cdot)$ is given, we use the following abstract lemma to construct a fixed point

Lemma 2.2.1 *Let Y_T be a Banach space and $B(\cdot,\cdot) : Y_T \times Y_T \to Y_T$ be a continuous bilinear form such that estimate (2.9) is satisfied. Let $0 < \delta < \frac{1}{4\eta}$. If*

$$\|S(t)\phi\|_{Y_T} \leq \delta \tag{2.10}$$

then equation (2.8) has a solution in Y_T. This solution is the only one such that

$$\|v\|_{Y_T} \leq 2\delta. \tag{2.11}$$

We will also need a modified version of Lemma 2.2.1 to deal with equations containing some extra linear terms.

Lemma 2.2.2 *Let* Y_T *be a Banach space. Let* $G : Y_T \to Y_T$ *be linear with* $\|G(f)\|_{Y_T} \leq \delta\|f\|_{Y_T}$ *for all* $f \in Y_T$*, and let* $B : Y_T \times Y_T \to Y_T$ *be bilinear with* $\|B(f,g)\|_{Y_T} \leq C\|f\|_{Y_T}\|g\|_{Y_T}$ *for all* $f, g \in Y_T$*. Let* $h \in Y_T$ *with* $\|h\|_{Y_T} \leq \frac{(1-\delta)^2}{4C}$*. Let* $0 < \xi_1 < \xi_2$ *be the two roots of the equation* $\xi = \|h\|_{Y_T} + \delta\xi + C\xi^2$*, i.e.* $\xi_{1,2} = \frac{(1-\delta)\pm\sqrt{(1-\delta)^2-4C\|h\|_{Y_T}}}{2C}$*. Then the equation*

$$f + G(f) + B(f,f) = h \tag{2.12}$$

has a solution $\bar{f}$ *satisfying* $\|\bar{f}\|_{Y_T} \leq \xi_1$*. Moreover, the solution* $\bar{f}$ *is unique in the open ball* $\{f \in Y_T : \|f\| < \xi_2\}$*.*

Summarizing, we have the following

Definition 1 (Mild solution) *Consider a Banach space* $X \subset \mathcal{S}'(\mathbb{R}^3)$ *and let* $Y_T \subset L^2_{loc}((0,T)\times\mathbb{R}^3)$ *be such that if* $\phi(x) \in X$ *then* $S(t)\phi(x) \in Y_T$ *for* $0 < t < T$*. Let* $\phi(x) \in X$ *be such that* $\operatorname{div}\phi = 0$*. We call* $v(t,x) \in Y_T$ *a mild solution on the time interval* $(0,T)$ *if it satisfies the integral identity*

$$v(t,x) = S(t)\phi(x) - \int_0^t S(t-s)\mathbb{P}\operatorname{div}(v\otimes v)(s)\,ds. \tag{2.13}$$

The status of the integral on the right-hand side is not immediately obvious and strongly depends on the choice of functional spaces. For instance, recalling the above discussion of the heat equation and the functional spaces pairs Y and Y', if $\operatorname{div}(v\otimes v)$ belongs to $C([0,\infty),Y')$ then our integral is a Y' valued Bochner integral. If $v(t,x)$ is a mild solution to (NSE) then the integral belongs to $C([0,\infty),Y)$. This, however, cannot be always guaranteed as demonstrated by the choice of $Y = L^3(\mathbb{R}^3)$.

The choice of spaces X and Y_T is intimately tied to the scaling of (NSE) described in previous section. There exists a vast literature containing results about existence of milds solutions in various spaces, see [17, 19, 14, 4]. In this thesis we shall be concerned mainly with the spaces $\dot{H}^{1/2}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$ which are the unique homogenous Sobolev and Lebesgue critical spaces on $\mathbb{R}^3$.

2.3 Leray solutions

The modern approach to the Cauchy problem (NSE) has started with the pioneering work [20] of J. Leray. We consider the Cauchy problem (NSE) with initial data ϕ in the Lebesgue space $L^2(\mathbb{R}^3)$. This allows us to prove the energy estimate

$$\|v(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|v(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)}\, ds \le \|\phi\|^2_{L^2(\mathbb{R}^3)}. \tag{2.14}$$

Leray's strategy was to consider a mollified version of (NSE). Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ be such that $\omega \ge 0$ and $\int_{\mathbb{R}^3} \omega\, dx = 1$ and let

$$\omega_\epsilon = \epsilon^{-3}\omega(x/\epsilon). \tag{2.15}$$

Consider

$$\begin{aligned} &\partial_t v - \nu\Delta v + \operatorname{div}((\omega_\epsilon * v)\otimes v) + \nabla p = 0 && \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ &\operatorname{div} v = 0 && \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ &v(0,x) = \phi(x) && \text{in } \mathbb{R}^3. \end{aligned} \tag{NSE$_\epsilon$}$$

These equations satisfy an energy estimate analogous to (2.14). Although not explicitly stated, Leray constructed local solutions of (NSE$_\epsilon$) by methods precursory to the theory of mild solutions. The energy estimate for the mollified systems allows us to prolong these local solutions to global ones. Then, the energy estimate (2.14) lets us chose a weak limit as $\epsilon \to 0$. More precisely, Leray obtained the following result.

Theorem 2.3.1 *For all* $\phi \in L^2(\mathbb{R}^3)$ *so that* $\operatorname{div} \phi = 0$, *there exists a weak solution* $v(t,x) \in L^\infty((0,\infty), L^2(\mathbb{R}^3)) \cap L^2((0,\infty), \dot{H}^1(\mathbb{R}^3))$ *for the Navier-Stokes equations (NSE) so that* $\lim_{t\to 0} \|v(t,\cdot) - \phi(\cdot)\|_{L^2(\mathbb{R}^3)} = 0$. *Moreover, we may chose this solution so that it satisfies the estimate (2.14).*

The energy estimate (2.14) can be furthermore improved. For almost all $t_0 > 0$ and all $t > t_0$ we have

$$\|v(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + 2\int_{t_0}^{t} \|v(s,\cdot)\|^2_{\dot{H}^1}\, ds \le \|v(t_0,\cdot)\|^2_{L^2(\mathbb{R}^3)}. \tag{2.14'}$$

Leray solutions constructed in a way described above satisfy this *strong energy inequality.*

As opposed to the mild solutions, Leray solutions are global, however their uniqueness has remained an open problem. Uniqueness of Leray solutions can be shown if one allows for some additional integrability properties. For instance we have the following variant of a result due to J. Serrin, see [26, 19].

Theorem 2.3.2 *Let* $\phi \in L^2(\mathbb{R}^3)$ *with* $\operatorname{div} \phi = 0$. *Assume that there exists a Leray solution* $v(t,x)$ *on* $(0,T) \times \mathbb{R}^3$ *(for some* $T \in (0,\infty]$*)) for the Cauchy problem (NSE) with the initial value* ϕ *so that for some* $q \in (3,\infty]$ *we have* $v(t,x) \in L^p((0,T), L^q(\mathbb{R}^3))$, *with* $\frac{1}{p} = \frac{1}{2} - \frac{3}{2q}$. *Then* $v(t,x)$ *is the unique Leray solution associated with* ϕ *on* $(0,T)$.

It is worth mentioning that the $L^2(\mathbb{R}^3)$ framework of Leray solutions and the energy estimates (2.14) and (2.14') in the 3-dimensional case do not satisfy the scaling symmetry described in Section 2.1. Leray's approach thus seems to be somehow separated from the theory of mild solutions described in previous section.

2.4 Suitable solutions

Leray's theorem, described in previous section is based on an energy inequality. This allowed J. Leray to prove that the Lebesgue measure of the set of time instants at which a flow may become singular is zero (this can be generalized to the case of the Hausdorf measure $\mathcal{H}^{1/2}$). To give an accurate description of the local regularity of a Leray solution, V. Scheffer introduced a local version of this inequality, see [25]. Weak solutions satisfying the local energy inequality are termed *suitable weak solutions.*

Definition 2 *Let* $v(t,x)$ *be a weak solution for the Navier-Stokes equation on* $(0,T)\times\mathbb{R}^3$. $v(t,x)$ *is suitable on the cylinder* $Q=(a,b)\times B_{x_0,r}$ *if* $v(t,x)$ *satisfies*

i) $\sup_{a<t<b}\int_{B_{x_o,r}}|v|^2\,dx<\infty$

ii) $\iint_Q|\nabla v|^2\,dxdt<\infty$

iii) $\iint_Q|p|^{3/2}\,dxdt<\infty$

iv) for all $\varphi\in\mathcal{D}(Q)$ *such that* $\varphi\geq 0$ *we have*

$$2\iint|\nabla v|^2\varphi\,dxdt\leq\iint|v|^2(\partial_t\varphi+\Delta\varphi)\,dxdt+\iint(|v|^2+2p)(v\cdot\nabla)\varphi\,dxdt \quad (2.16)$$

$v(t,x)$ *is suitable on* $(0,T)\times\mathbb{R}^3$ *if* v *is suitable on all cylinders* $Q\subset(0,T)\times\mathbb{R}^3$.

Global existence of suitable weak solutions has been established by L. Caffarelli, R. Kohn and L. Nirenberg in [1] who were also able to improve Leray's regularity result and showed that the parabolic Hausdorf measure (in space-time) $\mathcal{P}^1$ of a set of possible singularities is zero. Further properties and results regarding suitable weak solutions, important for our reasoning, will be described in Section 3.1.

Chapter 3

Leray solutions with data in $\dot{H}^{1/2}(\mathbb{R}^3)$

In this chapter we prove our main result using methods based on combining the mild and weak solutions to (NSE). We start by recalling some properties of suitable weak solutions and then proceed with considerations of the problem posed in the introduction. It is worth mentioning that methods used in this chapter rely on the choice of functional spaces, in particular on the compactness of the local embedding $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ which enables us to pass to the limit in the local energy inequality. At this stage it is not obvious how to generalize this approach to other spaces of initial data (for instance $L^3(\mathbb{R}^3)$). We prove that, under the assumption that singularities in (NSE) can occur, the set of minimal initial data generating a singularity is non-empty and compact modulo symmetries of the Navier-Stokes equations.

3.1 Auxiliary results

In what follows we will use standard notation for euclidean balls centered at $x_0 \in \mathbb{R}^3$ and parabolic cylinders $Q_{z_0,r}$ entered at $z_0 = (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$:

$$\begin{aligned} B_{x_0,r} &= \{x \in \mathbb{R}^3 : |x - x_0| < r\} \\ Q_{z_0,r} &= (t_0 - r^2, t_0] \times B_{x_0,r}. \end{aligned} \tag{3.1}$$

Given a suitable weak solution (v, p), a point $z_0 = (t_0, x_0) \in \mathcal{O}$ is called a *regular point* of (v, p) if v is Hölder continuous in a neighborhood of z_0. A *singular point* $z_0 \in \mathcal{O}$ of (v, p) is any point which is not regular. We will use the following two propositions, the various versions of which can be found in [1, 21, 24, 18]. The version below contains some quantitative estimates which are often not explicitly stated in the literature, although they are implicit in the proofs. A sketch of the proof of the spatial derivatives estimates can be found for instance in [23]

Proposition 3.1.1 (ϵ-regularity criterion) *There exists $\epsilon_0 > 0$ such that the following statement is true:*

If (v, p) is a suitable weak solution in $\mathcal{O}$, such that

$$\frac{1}{r^2} \iint_{Q_{z_0,r}} (|v|^3 + |p|^{3/2})\, dxdt < \epsilon_0 \tag{3.2}$$

for some $Q_{z_0,r}$ compactly contained in $\mathcal{O}$, then all points of $Q_{z_0,r/2}$ are regular points of (v, p). Moreover, in $Q_{z_0,r/2}$ one has

$$|\nabla^k v| \leq C_k r^{-1-k} \qquad k = 0, 1, \ldots \quad \textit{and} \tag{3.3}$$

$$|v(t, x) - v(t', x)| \leq C' |t - t'|^{1/3}. \tag{3.4}$$

Proposition 3.1.2 (Compactness) *Let* (v^k, p^k), $k = 1, 2, \dots$ *be a sequence of suitable weak solutions such that* v^k *are uniformly bounded in the energy space* $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ *on compact subsets of* $\mathcal{O}$ *and* p^k *are uniformly bounded in* $L^{3/2}_t L^{3/2}_x$ *on compact subsets of* $\mathcal{O}$*. Then the sequence* v^k *is compact in* $L^3_t L^3_x$ *on compact subsets of* $\mathcal{O}$*. Moreover, if* $v^k \to v$ *in* $L^3_t L^3_x$ *on compact subsets of* $\mathcal{O}$ *and* $p^k \rightharpoonup p$ *in* $L^{3/2}_t L^{3/2}_x$ *on compact subsets of* $\mathcal{O}$*, then* (v, p) *is again a suitable weak solution.*

The two previous propositions imply the following lemma, which will be important for the proof of our main results.

Lemma 3.1.3 (Stability of singularities) *In the situation of Proposition 3.1.2, assume that* $z^k \in \mathcal{O}$ *are singular points of* (v^k, p^k), $k = 1, 2, \dots$, *and that* $z^k \to z_0 \in \mathcal{O}$*. Then* z_0 *is a singular point of* (v, p)*.*

Proof. If the regularity criterion in Proposition 3.1.1 did not contain the pressure p, the statement of the Lemma would follow immediately: indeed, if z_0 is a regular point of v, then $r^{-2} \iint_{Q_{z_0,r}} |v|^3 \; dxdt = O(r^3)$ as $r \to 0^+$. Choosing a sufficiently small r, one sees that $r^{-2} \iint_{Q_{z_0,r}} |v^k|^3 \; dxdt$ is small for large k by the strong convergence of v^k in $L^3_t L^3_x$. However such argument cannot be applied to the pressure term, since the sequence p^k may not have a subsequence which is compact in $L^{3/2}_t L^{3/2}_x$. To deal with this difficulty we use an approach which can be found in many forms in the proofs of partial regularity [1, 21, 24, 18]. The pressure p^k solves the equation

$$-\Delta p^k = \partial_i \partial_j (v^k_i v^k_j). \tag{3.5}$$

Recall that the term $v_i^k v_j^k$ is compact in $L_t^{3/2} L_x^{3/2}$ on compact subsets of $\mathcal{O}$. Therefore we can invert the Laplacian in (3.5) using a suitable boundary condition, (or just taking the Riesz transforms $\tilde{p}^k = R_i R_j (v_i^k v_j^k \mathbf{1}_{B_{x_0,r}})$) and decompose p^k as

$$p^k = \tilde{p}^k + h^k \tag{3.6}$$

with $\tilde{p}^k$ compact in $L_t^{3/2} L_x^{3/2}(Q_{z_0,r})$ (by Caldéron-Zygmund estimates) and h^k bounded in $L_t^{3/2} L_x^{3/2}$ and harmonic in x on $Q_{z_0,r}$. The terms $\tilde{p}^k$ can be dealt with in the same way as the terms v^k. The term h^k is handled by using classical estimates for harmonic functions: let $\gamma \geq 1$ and let $h \in L_x^\gamma(B_{x_0,r})$ be harmonic in $B_{x_0,r}$. We denote $(h)_{r'} = |B_{x_0,r'}|^{-1} \int_{B_{x_0,r'}} h \, dx$. For $r' \leq r/2$ and $x' \in B_{x_0,r'}$ we have

$$|h(x') - h(x)|^\gamma \leq C_\gamma \left(\frac{r'}{r} \right)^\gamma r^{-3} \int_{B_{x_0,r}} |h|^\gamma \, dx. \tag{3.7}$$

We recall that we can change the pressure by any function depending only on t. Therefore we can use (3.7) with $h = h^k$, and integrating over $Q_{z_0,r'}$, we get the required smallness of the term $(r')^{-2} \iint_{Q_{z_0,r'}} |h^k - (h^k(\cdot,t))_{r'}|^{3/2}$ dxdt. $\square$

We will also need a modified version of Lemma 3.1.3 which describes what happens outside of a compact sets confining all singularities.

Lemma 3.1.4 *Under the assumptions of Proposition 3.1.2, let K be a compact subset of $\mathcal{O}$. If each point of K is a regular point of v, then for sufficiently large k, each point of K is also a regular point of v^k, and on the set K the functions v^k converge to v, together with all spatial derivatives.*

Proof. The statement follows from the proof of Lemma 3.1.3 and the estimates in Proposition 3.1.1. ☐

3.2 Proof of the main theorem and other results

In his pioneering work [20] J. Leray proved the existence of the weak solutions to the Cauchy problem (NSE) with initial data in $L^2(\mathbb{R}^3)$. A key ingredient in his approach is the energy inequality (2.14). This inequality is the only known a-priori bound for general solutions. At the first glance it would seem that for its application it is crucial that $\phi \in L^2(\mathbb{R}^3)$, which would rule out using Leray's techniques in the setting of mild solutions, where the basic assumption is $\phi \in \dot{H}^{1/2}$, which is not a subset of $L^2(\mathbb{R}^3)$.

However, P.G. Lemarié-Rieusset [19] found a generalization of (2.14) to the situation when the energy is only (uniformly) locally finite, and this makes it possible to extend the theory of Leray's weak solutions to a much more general setting. Here we will not need the full version of Lemarié-Rieusset's local theory, but we will need a version of his inequality for local energy, see Lemma 3.2.5. To account for the fact that our initial data are not square-integrable, we will use a trick introduced by C. Caldéron, which allows one to decompose a $\dot{H}^{1/2}$ vector field into an L^2 part and a small $\dot{H}^{1/2}$ perturbation.

We first prove existence of suitable solutions with initial data in $\dot{H}^{1/2}$.

Theorem 3.2.1 (Existence of Leray solutions) *Let $\phi \in \dot{H}^{1/2}(\mathbb{R}^3)$. There exists a global weak solution (v, p) to (NSE) which is suitable in the sense of*

Caffarelli-Kohn-Nirenberg and such that $v(t,x) \to \phi(x)$ *in* L^2 *as* $t \to 0$ *on every compact subset of* $\mathbb{R}^3$*. Such solution will be called a Leray solution with* $\dot{H}^{1/2}$ *initial data.*

Proof. Consider the decomposition of initial data (see [2, 10]) as

$$\phi = a_0 + u_0 \tag{3.8}$$

where both $\operatorname{div} a_0 = \operatorname{div} u_0 = 0$, a_0 is smooth and small in $\dot{H}^{1/2}$ and $u_0 \in L^2(\mathbb{R}^3)$. We will construct the solution to the problem (NSE), $v(t,x)$ as

$$v(t,x) = a(t,x) + u(t,x) \tag{3.9}$$

where $a(t,x)$ is a solution to (NSE) (but with small initial data!) and $u(t,x)$ is a new unknown function satisfying the equation

$$\partial_t u + a \cdot \nabla u + u \cdot \nabla a + u \cdot \nabla u + \nabla q - \Delta u = 0 \tag{3.10}$$

with the initial condition $u(0,x) = u_0$ (recall that $u_0 \in L^2(\mathbb{R}^3)$). We will do this in several steps.

Step 1: We first construct a global mild solution to (NSE) $a(t,x)$ in the space $L^4_t \dot{H}^1_x$ with initial data $a(0,x) = a_0$ and recall a global estimate useful in the proceeding part of the proof.

To obtain $a(t,x)$ we use the framework of Lemma 2.2.1. First, notice that by the energy estimate for the heat equation, we have

$$\|S(t)a_0\|_{L^4_t \dot{H}^1_x} \leq \|a_0\|_{\dot{H}^{1/2}}. \tag{3.11}$$

Next, we use the estimate for the product rule in Sobolev spaces

$$\|fg\|_{\dot{H}^{1/2}} \leq C\|f\|_{\dot{H}^1}\|g\|_{\dot{H}^1} \tag{3.12}$$

which implies that for $f, g \in L_t^4\dot{H}_x^1$ we have $fg \in L_t^2\dot{H}_x^{1/2}$. Recall also the energy inequality for the linear problem (S)

$$\| \, |\nabla|^s v\|^2_{L_t^\infty L_x^2} + \| \, |\nabla|^s v\|^2_{L_t^2 \dot{H}_x^1} \leq \| \, |\nabla|^s a_0\|^2_{L_x^2} + \| \, |\nabla|^s f\|^2_{L_t^2 L_x^2}. \tag{3.13}$$

Combining (3.12) and (3.13) we get the continuity estimate for $B(\cdot,\cdot)$

$$\begin{aligned} \|B(f,g)\|^2_{L_t^4\dot{H}_x^1} &\leq \|B(f,g)\|^2_{L_t^\infty \dot{H}_x^{1/2}} + \|B(f,g)\|^2_{L^2\dot{H}_x^{3/2}} \\ &\leq \|f\otimes g\|^2_{L^2\dot{H}_x^{1/2}} \leq \eta\|f\|^2_{L_t^4\dot{H}_x^1}\|g\|^2_{L_t^4\dot{H}_x^1}. \end{aligned} \tag{3.14}$$

In fact, estimate (3.14) shows that B maps $L_t^4\dot{H}_x^1 \times L_t^4\dot{H}_x^1 \to C_t\dot{H}_x^{1/2} \cap L_t^2\dot{H}_x^{3/2}$.

Furthermore, if $\|a_0\|_{\dot{H}^{1/2}}$ is small, $a(t,x)$ satisfies the following global estimate (see [3])

$$\|a\|^2_{L_t^\infty \dot{H}_x^{1/2}} + \|a\|^2_{L_t^2\dot{H}_x^{3/2}} \leq \|a_0\|^2_{\dot{H}^{1/2}}, \tag{3.15}$$

thus by interpolation we have the following

$$\|a\|_{L_t^4\dot{H}_x^1} \leq \|a_0\|_{\dot{H}^{1/2}} \tag{3.16}$$

for initial data small enough.

Step 2: Given $a(t,x)$ constructed in the previous step, we can now construct the remaining part of $v(t,x)$, namely $u(t,x)$ which satisfies equation (3.10).

Mollified equations. Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ be such that $\omega \geq 0$ and $\int_{\mathbb{R}^3} \omega \, dx = 1$ and let

$$\omega_\epsilon = \epsilon^{-3}\omega(x/\epsilon). \tag{3.17}$$

We will consider equations

$$\begin{aligned}
&\partial_t u + \operatorname{div}((a * \omega_\epsilon) \otimes u + u \otimes (a * \omega_\epsilon)) + \operatorname{div}((u * \omega_\epsilon) \otimes u) + \nabla q - \Delta u = 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
&\operatorname{div} u = 0 && \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
&u(0,x) = u_0(x) && \text{in } \mathbb{R}^3.
\end{aligned} \tag{3.10$_\epsilon$}$$

Recall that $u_0(x) \in L^2(\mathbb{R}^3)$. The problem of solving (3.10$_\epsilon$) reduces to a fixed point problem

$$u(t,x) = S(t)u_0(x) + G_\epsilon(u) + B_\epsilon(u,u), \tag{3.18}$$

where the linear operator G_ϵ is given by

$$G_\epsilon(f) = -\int_0^t S(t-s)\mathbb{P}\operatorname{div}((a * \omega_\epsilon) \otimes f + f \otimes (a * \omega_\epsilon))\, dxds \tag{3.19}$$

and the bilinear form B_ϵ is given by

$$B_\epsilon(f,g) = -\int_0^t S(t-s)\mathbb{P}\operatorname{div}((f * \omega_\epsilon) \otimes g)\, dxds. \tag{3.20}$$

The energy estimate for the heat equation gives

$$\|S(t)u_0\|_{L_t^\infty L_x^2} \leq \|u_0\|_{L^2}. \tag{3.21}$$

For the bilinear form, we have the following estimate

$$\begin{aligned}
&\left\| \int_0^t S(t-s)\mathbb{P}\operatorname{div}((f * \omega_\epsilon) \otimes g)\, dxds \right\|_{L^2} \\
&\leq \int_0^t \|\nabla K(t-s) * ((f * \omega_\epsilon) \otimes g)\|_{L^2}\, ds,
\end{aligned} \tag{3.22}$$

where $K(\cdot)$ is the Stokes kernel. By Young's and Hölder's inequalities we get

$$\begin{aligned} &\int_0^t \|\nabla K(t-s) * ((f * \omega_\epsilon) \otimes g)\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|(f * \omega_\epsilon) g\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|(f * \omega_\epsilon)\|_{L^\infty_x} \|g\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|f\|_{L^2} \|\omega_\epsilon\|_{L^2_x} \|g\|_{L^2_x}\, ds. \end{aligned} \tag{3.23}$$

Recall now that

$$\|\nabla K(t-s)\|_{L^1_x} \leq C(t-s)^{-1/2}, \tag{3.24}$$

and

$$\|\omega_\epsilon\|_{L^2_x} \sim \epsilon^{-3/2} \|\omega\|_{L^2_x}. \tag{3.25}$$

This gives the estimate

$$\|B_\epsilon(f,g)\|_{L^\infty_t L^2_x} \leq C T^{1/2} \epsilon^{-3/2} \|f\|_{L^\infty_t L^2_x} \|g\|_{L^\infty_t L^2_x} \tag{3.26}$$

on the time interval $(0,T)$. To treat the linear operator G_ϵ we proceed in a similar manner. We get

$$\begin{aligned} &\int_0^t \|\nabla K(t-s) * ((a * \omega_\epsilon) \otimes f)\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|(a * \omega_\epsilon) f\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|(a * \omega_\epsilon)\|_{L^\infty_x} \|f\|_{L^2_x}\, ds \\ &\leq \int_0^t \|\nabla K(t-s)\|_{L^1_x} \|a\|_{L^3} \|\omega_\epsilon\|_{L^{3/2}_x} \|f\|_{L^2_x}\, ds. \end{aligned} \tag{3.27}$$

Notice that

$$\|\omega_\epsilon\|_{L^{3/2}_x} \sim \epsilon^{-1} \|\omega\|_{L^{3/2}_x}. \tag{3.28}$$

We have

$$\|G_\epsilon(f)\|_{L_t^\infty L_x^2} \leq CT^{1/2}\epsilon^{-1}\|a\|_{L^\infty \dot{H}_x^{1/2}}\|f\|_{L_t^\infty L_x^2}, \tag{3.29}$$

which gives

$$\|G_\epsilon(f)\|_{L_t^\infty L_x^2} \leq CT^{1/2}\epsilon^{-1}\|a_0\|_{\dot{H}^{1/2}}\|f\|_{L_t^\infty L_x^2}, \tag{3.30}$$

by estimate (3.15). To apply Lemma 2.2.2 we first make T small enough (having $\epsilon > 0$ fixed) so that

$$\delta = CT^{1/2}\epsilon^{-1}\|a_0\|_{\dot{H}^{1/2}} < \frac{1}{2}. \tag{3.31}$$

Next, to ensure that the conditions of Lemma 2.2.2 are satisfied, we have (by possibly making T smaller)

$$\frac{T^{1/2}\epsilon^{-3/2}}{(1 - T^{1/2}\epsilon^{-1})^2} < 2T^{1/2}\epsilon^{-3/2} \tag{3.32}$$

and the right hand-side can be made small enough by picking T sufficiently small. Thus we obtain local solutions u_ϵ in the class $L_t^\infty L_x^2$ to the mollified equations (3.10$_\epsilon$).

A uniform energy estimate. We now prove an energy estimate for equations (3.10$_\epsilon$), which is uniform in ϵ. We first notice that the energy identity for this equation is

$$\int_{\mathbb{R}^3} |u_\epsilon(t,x)|^2 \, dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u_\epsilon(t,x)|^2 \, dxds = \int_{\mathbb{R}^3} |u_0|^2 \, dx + 2\int_0^t \int_{\mathbb{R}^3} ((a * \omega_\epsilon) \cdot \nabla u_\epsilon) u_\epsilon \, dxds. \tag{3.33}$$

The last integral on the right-hand side is estimated by

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{R}^3} ((a * \omega_\epsilon)\cdot\nabla u_\epsilon)u_\epsilon \, dxds\right| &\le \int_0^t \|a*\omega_\epsilon\|_{L^6_x}\|\nabla u_\epsilon\|_{L^2_x}\|u_\epsilon\|_{L^3_x}\, ds \\ \le C\int_0^t \|a\|_{L^6_x}\|\nabla u_\epsilon\|_{L^2_x}\|u_\epsilon\|_{\dot H^{1/2}_x}\, ds &\le C\int_0^t \|a\|_{L^6_x}\|\nabla u_\epsilon\|^{3/2}_{L^2_x}\|u_\epsilon\|_{L^2_x}\, ds \qquad (3.34)\\ \le C\|a\|_{L^4_t\dot H^1_x}\|u_\epsilon\|^{1/2}_{L^\infty_t L^2_x}\|\nabla u_\epsilon\|^{3/4}_{L^2_t L^2_x}\end{aligned}$$

and if $\|a\|_{L^4_t\dot H^1_x}$ is sufficiently small (which can be achieved by choosing sufficiently small $\|a_0\|_{\dot H^{1/2}}$ - see estimate (3.16)), this term can be incorporated using Young's inequality in the left-hand side of (3.33) giving a good energy estimate for equation (3.10), namely

$$\|u_\epsilon\|^2_{L^\infty_t L^2_x} + \|u_\epsilon\|^2_{L^2_t\dot H^1_x} \le C\|u_0\|^2_{L^2}. \qquad (3.35)$$

Estimate (3.35) shows also that solutions to the mollified equations (3.10_ϵ) are in fact global and belong to the space $L^\infty_t L^2_x \cap L^2_t \dot H^1_x$.

The limiting process. First, notice that thanks to the energy estimate (3.35) we control the associated pressure q_ϵ and $\partial_t u_\epsilon$. For any $T > 0$ we have the following: $\sup_\epsilon \int_0^T \|a_\epsilon \otimes u_\epsilon + u_\epsilon \otimes a_\epsilon\|^2_{L^2_x}\, ds < \infty$, where $a_\epsilon = a * \omega_\epsilon$. Since u_ϵ are uniformly bounded in $L^\infty_t L^2_x \cap L^2_t \dot H^1_x$ by interpolation we also have $\sup_\epsilon \int_0^T \|u_\epsilon \otimes u_\epsilon\|^2_{L^{3/2}_x}\, ds < \infty$. We get that for any $\phi \in \mathcal{D}(\mathbb{R}^3)$ we have $\sup_\epsilon \int_0^T \|\phi q_\epsilon\|^2_{L^{3/2}_x}\, ds < \infty$ and $\sup_\epsilon \int_0^T \|\phi \partial_t u_\epsilon\|^2_{H^{-3/2}_x}\, ds < \infty$. Thus, possibly passing to a subsequence we have strong convergence $u_\epsilon \to u$ in $(L^2_t L^2_x)_{loc}$: for all $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ we have

$$\lim_{\epsilon\to 0}\iint |\phi(t,x)(u_\epsilon(t,x) - u(t,x))|^2\, dxdt = 0. \qquad (3.36)$$

Moreover, since u_ϵ are uniformly bounded in $L^\infty_t L^2_x$ and $L^2_t L^6_x$ we have (possibly passing to a subsequence) strong convergence in $(L^6_t L^2_x)_{loc}$ and $(L^3_t L^3_x)_{loc}$. Notice that q_ϵ can be computed as

$$q_\epsilon = \sum_{i,j=1}^{3} R_i R_j ((a * \omega_\epsilon)_i u_j^\epsilon + u_i^\epsilon (a * \omega_\epsilon)_j + (u^\epsilon * \omega_\epsilon)_i u_j^\epsilon) \tag{3.37}$$

thus by estimates (3.15) and (3.35) we control q^ϵ in the space $L_t^2 L_x^{3/2}$, uniformly in ϵ. Hence we can assume that the sequence q_ϵ converges weakly to q. Upon this, we can pass to the limit in equation (3.10_ϵ) noting that the product terms pose no problem due to strong convergence of u^ϵ in $(L_t^2 L_x^2)_{loc}$. The pair (u, p) is a weak solution of equation (3.10).

Step 3: We will now show that solutions $v(t,x) = a(t,x) + u(t,x)$ are in fact suitable weak solutions. Let $Q = (t_1, t_2) \times B_r(x_0)$. First, notice that since $a(t,x)$ is smooth on Q and $u \in L_t^\infty L_x^2$, we have

$$\sup_{t_1 < t < t_2} \int_{B_r(x_0)} |v(t, \cdot)|^2 \, dx < \infty. \tag{3.38}$$

Also, since $a(t,x)$ is smooth on the considered space-time domain and $u \in L_t^2 \dot{H}_x^1$, we have

$$\iint_Q |\nabla v(t,x)|^2 \, dxdt < \infty. \tag{3.39}$$

Furthermore, since $a \in L_t^\infty L_x^3$ and $u \in L_t^\infty L_x^2 \cap L_t^2 L_x^6 \subset L_t^4 L_x^3$, for all finite T we have

$$\int_0^T \int_{\mathbb{R}^3} |u(t,x)|^3 \, dxdt < \infty \tag{3.40}$$

and thus

$$\int_0^T \int_{\mathbb{R}^3} |p(t,x)|^{3/2} < \infty, \tag{3.41}$$

since p may be computed as $p = \sum_{i,j=1}^{3} R_i R_j (u_i u_j)$. What remains is to prove that $v(t,x)$ satisfies the local energy inequality. We consider the approximations

$$v^\epsilon(t,x) = a(t,x) + u^\epsilon(t,x). \tag{3.42}$$

Putting together equations for $a(t,x)$ and $u^\epsilon(t,x)$ we get

$$\begin{gathered}\partial_t(a+u^\epsilon) + \operatorname{div}((a*\omega_\epsilon)\otimes u^\epsilon + u^\epsilon \otimes (a*\omega_\epsilon)) + \operatorname{div}((u^\epsilon * \omega_\epsilon)\otimes u^\epsilon) + \operatorname{div}(a\otimes a) \\ +\nabla(q^\epsilon + p) - \Delta(a+u^\epsilon) = 0.\end{gathered} \tag{3.43}$$

Notice that passing to the limit, $v(t,x) = a(t,x) + u(t,x)$ is a weak solution of (NSE). Furthermore, equation (3.43) satisfies the following local energy equality

$$\begin{gathered}2\iint |\nabla(a+u^\epsilon)|^2\phi \, dxdt = \iint |a+u^\epsilon|^2(\partial_t\phi + \Delta\phi)\, dxdt \\ + \iint 2(p+q^\epsilon)(a+u^\epsilon)\nabla\phi\, dxdt + \iint ((a*\omega_\epsilon)\otimes u^\epsilon)\nabla a\phi\, dxdt \\ + \iint (a+u^\epsilon)((a*\omega_\epsilon)\otimes u^\epsilon)\nabla\phi\, dxdt + \iint (u^\epsilon\otimes(a*\omega_\epsilon))\nabla(a+u^\epsilon)\phi\, dxdt \\ + \iint (a+u^\epsilon)(u^\epsilon \otimes (a*\omega_\epsilon))\nabla\phi\, dxdt + \iint ((u^\epsilon*\omega_\epsilon)\otimes u^\epsilon)\nabla a\phi\, dxdt \\ + \iint (a+u^\epsilon)((u^\epsilon*\omega_\epsilon)\otimes u^\epsilon)\nabla\phi\, dxdt + \iint (a\otimes a)\nabla u^\epsilon\phi\, dxdt \\ + \iint (a+u^\epsilon)(a\otimes a)\nabla\phi\, dxdt,\end{gathered} \tag{3.44}$$

where $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ and $\phi \geq 0$. Recall that we have the following convergence in $\mathcal{D}'$: $\nabla u^\epsilon \to \nabla u$ (weakly in $(L_t^2 L_x^2)_{loc}$), $u^\epsilon \to u$ (strongly in $(L_t^3 L_x^3)_{loc}$), $(u^\epsilon * \omega_\epsilon) \to u$ (strongly in $(L_t^3 L_x^3)_{loc}$), $(a*\omega_\epsilon) \to a$ (strongly in $(L_t^3 L_x^3)_{loc}$), $q^\epsilon \to q$ (weakly in $L_t^2 L_x^{3/2}$). We use the fact that

$$\iint (f\otimes f)\nabla f\, dxdt = 0, \tag{3.45}$$

to eliminate some terms and pass to the limit

$$\begin{aligned}
2\iint |\nabla(a+u)|^2\phi \, dxdt &\leq \iint |a+u|^2(\partial_t\phi + \Delta\phi)\, dxdt \\
&+ \iint 2(p+q)(a+u)\nabla\phi \, dxdt + \iint (a\otimes u)\nabla a\phi \, dxdt \\
&+ \iint (a+u)(a\otimes u)\nabla\phi \, dxdt + \iint (u\otimes a)\nabla(a+u)\phi \, dxdt \\
&+ \iint (a+u)(u\otimes a)\nabla\phi \, dxdt + \iint (u\otimes u)\nabla a\phi \, dxdt \\
&+ \iint (a+u)(u\otimes u)\nabla\phi \, dxdt + \iint (a\otimes a)\nabla u\phi \, dxdt \\
&+ \iint (a+u)(a\otimes a)\nabla\phi \, dxdt.
\end{aligned} \tag{3.46}$$

Thus we have

$$\begin{aligned}
2\iint |\nabla(a+u)|^2\phi \, dxdt &\leq \iint |a+u|^2(\partial_t\phi + \Delta\phi)\, dxdt \\
&+ \iint 2(p+q)(a+u)\nabla\phi \, dxdt + \iint (a\otimes u)\nabla a\phi \, dxdt \\
&+ \iint (a+u)((a+u)\otimes(a+u))\nabla\phi \, dxdt + \iint (u\otimes a)\nabla(a+u)\phi \, dxdt \\
&+ \iint (u\otimes u)\nabla a\phi \, dxdt \\
&+ \iint (a\otimes a)\nabla u\phi \, dxdt
\end{aligned} \tag{3.47}$$

Using the fact that

$$\iint ((a+u)\otimes(a+u))\nabla(a+u)\, dxdt = 0, \tag{3.48}$$

we get

$$\begin{aligned}
2\iint |\nabla(a+u)|^2\phi \, dxdt &\leq \iint |a+u|^2(\partial_t\phi + \Delta\phi)\, dxdt \\
&+ \iint 2(p+q)(a+u)\nabla\phi \, dxdt + \iint |(a+u)|^2(a+u)\nabla\phi \, dxdt,
\end{aligned} \tag{3.49}$$

thus showing that $v(t,x) = a(t,x) + u(t,x)$ satisfies the local energy inequality. $v(t,x)$ is therefore suitable in the sense of Caffarelli-Kohn-Nirenberg.

Step 4: It remains to show that $v(t,x)$ attains initial data in the L^2 sense on compact subsets of $\mathbb{R}^3$, more precisely $v(t,x) \to \phi(x)$ in L^2 on compact subsets of $\mathbb{R}^3$.

First, notice that $a(t,x) \in C([0,+\infty), \dot{H}^{1/2}(\mathbb{R}^3))$, thus we have

$$\lim_{t\to 0^+} \|a(t,\cdot) - a_0\|_{\dot{H}^{1/2}_x} = 0, \tag{3.50}$$

which implies that we have $a(t,x) \to a_0$ in L^2 on every compact subset of $\mathbb{R}^3$ as $t \to 0^+$. Following Leray, we furthermore notice that $u(t,\cdot)$ converges weakly to u_0 in L^2. We thus have

$$\|u_0\|^2_{L^2_x} \leq \liminf_{t\to 0^+} \|u(t,\cdot)\|^2_{L^2_x}. \tag{3.51}$$

However the energy estimate for u gives us

$$\limsup_{t\to 0+} \|u(t,\cdot)\|^2_{L^2_x} \leq \|u_0\|^2_{L^2_x}. \tag{3.52}$$

This implies that $\lim_{t\to 0^+} \|u(t,\cdot) - u_0\|_{L^2_x} = 0$. Therefore, on every compact subset of $\mathbb{R}^3$ we have $\lim_{t\to 0^+} \|v(t,\cdot) - \phi\|_{L^2_x} = 0$. $\square$

The relation of the Leray solution and the mild solution introduced in Section (constructed in the same way as in Step 1 of the proof of Theorem 3.2.1) is clarified by the following "weak-strong uniqueness" theorem. In the case $\phi \in L^2 \cap \dot{H}^1$ the theorem was proved by J. Leray. Leray's result was generalized in various directions, see for example [27, 30, 19]. we will use the following version which is a special case of Theorem 33.2, p. 354 from Lemarié-Rieusset's book [19].

Theorem 3.2.2 *Let* $v(t,x)$ *be a Leray solution of the initial value problem (NSE) with* $\phi(x) \in \dot{H}^{1/2}$*. Let* $T_{max}(\phi)$ *be the maximal time of existence of the mild solution of (NSE) with the same initial value* $\phi(x)$*. Then the mild solution coincides with* $v(t,x)$ *in* $\mathbb{R}^3 \times [0, T_{max})$*.*

The problem of uniqueness of $v(t,x)$ after $T_{max}(\phi)$ is open. At the time of this writing we cannot rule out that $T_{max}(\phi)$ is finite and that the Leray solution is not unique after $T_{max}(\phi)$. We will denote the set of all Leray solutions with initial data $\phi \in \dot{H}^{1/2}$ by $NS(\phi)$.

The mild solutions have the same regularity as the heat-extension of ϕ since for short times they are just a perturbation of $S(t)\phi$. This can be iterated forward to the time interval where the solution exist. In particular, the mild solutions are smooth in $(0, T_{max}(\phi)) \times \mathbb{R}^3$ However, one has to be cautious with this statement if "regularity" also means decay properties of v as $x \to \infty$. Due to the non-local effect of the constraint $\operatorname{div} u = 0$, the solutions v can have slower decay at infinity than $S(t)\phi$. One obvious reason for $T_{max}(\phi)$ to be finite would be the development of a singularity in the solution v at time $T_{max}(\phi)$. A-priori it is not clear that this is the only reason. One could also imagine a scenario where the $L^4_t \dot{H}^1_x$ norm of the solution would blow up even though the solution would remain smooth on each compact subset. Proposition 3.1.1 can be used to show that the only reason for $T_{max}(\phi) < \infty$ can be a finite time singularity.

Proposition 3.2.3 *Let* $v(t,x)$ *be a Leray solution of the initial value problem (NSE) with* $\phi(x) \in \dot{H}^{1/2}$*. Let* $T_{max}(\phi)$ *be the maximal time of existence of the mild solution of (NSE) with the same initial value* $\phi(x)$*. If* $T_{max}(\phi) < \infty$ *then* $v(t,x)$ *is singular at* $t = T_{max}(\phi)$*.*

Proof. Let us assume that $T = T_{max}(\phi)$ is finite. Let $r = \sqrt{T/2}$. Consider the decomposition

$$v(t,x) = a(t,x) + u(t,x), \tag{3.53}$$

as in the proof of Theorem 3.2.1, where $a(t,x)$ is a solution generated by $a_0(x)$ with $\|a_0\|_{\dot{H}^{1/2}}$ small (and hence $a(t,x)$ satisfies global estimates (3.15)) and $u(t,x)$ satisfies the energy estimate (3.35). Notice that these estimates do not deteriorate as t approaches T. We use these estimates, together with corresponding estimates for the pressure p and considering the complement in $\mathbb{R}^3$ of a ball with sufficiently large radius $R > 0$, the assumptions of Proposition 3.1.1 are satisfied for our solution (v,p) and any $Q_{z_0,r}$ with $z_0 = (x_0,T)$ and $|x_0| > R$.

If $v(t,x)$ does not develop a singularity at time T in the ball $B_{0,R}$, it means that v and ∇v will be bounded in (t_1,T) for any $t_1 > 0$. We can write the Navier-Stokes equations for $v(t,x)$ as

$$\partial_t v - \Delta v + \nabla p = -\operatorname{div}(v \otimes v). \tag{3.54}$$

Using the energy estimates and estimates coming from the perturbation theory for the small solution $a(t,x)$, the energy estimates for $u(t,x)$, together with the pointwise bound for $v(t,x)$ and $\nabla v(t,x)$ we get that the term $v \otimes v = (a+u) \otimes (a+u)$

is bounded in $L^2_tL^2_x((T/2,T)\times\mathbb{R}^3)$ and $L^2_t\dot{H}^1_x((T/2,T)\times\mathbb{R}^3)$, and therefore also in $L^2_t\dot{H}^{1/2}_x((T/2,T)\times\mathbb{R}^3)$. Viewing (3.54) as a linear equation with the right-hand side $-\operatorname{div}(v\otimes v)$, we see by the energy estimate that $v\in L^4_t\dot{H}^1_x((0,T)\times\mathbb{R}^3)$. This is a contradiction since if T is finite, we have

$$\lim_{\bar{T}\to T}\|v\|_{L^4_t\dot{H}^1_x((0,\bar{T})\times\mathbb{R}^3)}=+\infty. \tag{3.55}$$

which means that T could not have been the maximal time of existence of the mild solution. □

The weak solution $u(t,x)$ of equation (3.10) always belongs to the energy space $L^\infty_tL^2_x\cap L^2_t\dot{H}^1_x$. As noticed already by J. Leray in [20], this implies that $u(t,x)$ is smooth and small for large times. In our set-up we can see this from the fact that

$$\|u(t)\|^2_{\dot{H}^{1/2}}\le\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \tag{3.56}$$

and the expression on the right-hand side of this inequality has to be small on a large set of times if $u(t,x)$ is in the energy space. Following the reasoning of J. Leray, ([20], p. 246), we notice that at a time t_0 when $\|v(t_0)\|_{\dot{H}^{1/2}}$ is small, we can apply the theory of mild solutions and the weak-strong uniqueness results to see that after time t_0 the solution $u(t,x)$ coincides with a global mild solution. This can be summarized in the following statement:

Proposition 3.2.4 *Let $v(t,x)$ be a Leray solution of the Cauchy problem (NSE) with $\phi(x)\in\dot{H}^{1/2}$. Then for some compact set $K\subset\mathbb{R}^3\times(0,\infty)$ we have $\nabla v\in L^4_tL^2_x(\mathbb{R}^3\times(0,\infty)\setminus K)$. In particular, v is regular at every point of $\mathbb{R}^3\times(0,\infty)\setminus K$.*

Proof. From the proof of Theorem 3.2.3 and the comments above, we see that all singularities are confined to a compact subset of the space-time, K. On the complement $(0,\infty)\times\mathbb{R}^3\setminus K$ we have $v(t,x)\in L^4_t\dot{H}^1_x$. Regularity of $v(t,x)$ on $(0,\infty)\times\mathbb{R}^3\setminus K$ follows then from the Ladyzhenskaya-Prodi-Serrin criterion. $\square$

The energy estimate for $u(t,x)$ which can be obtained from equation (3.10) depends on the decomposition of the initial data $\phi(x)=a_0(x)+u_0(x)$. For our purposes, we need an energy estimate which has some uniformity. An estimate found by P.G. Lemarié-Rieusset in his work on weak uniformly locally square integrable solutions provides exactly what we need. We will the following notation: for $x_0\in\mathbb{R}^3$ and $r>0$ we will denote by $\bar{Q}_{x_0,r}$ the space-time cylinder $(0,r^2)\times B_{x_0,r}$. We will also use the notation $\|v\|_{\mathcal{E}(\bar{Q}_{x_0,r})}$ to denote the energy norm defined by

$$\|v\|^2_{\mathcal{E}(\bar{Q}_{x_0,r})}=\|v\|^2_{L^\infty_t L^2_x(\bar{Q}_{x_0,r})}+\|\nabla v\|^2_{L^2_t L^2_x(\bar{Q}_{x_0,r})}. \tag{3.57}$$

Lemma 3.2.5 *Let $\phi(x)\in\dot{H}^{1/2}$ and let $v(t,x)$ be a Leray solution of the Cauchy problem (NSE) with initial data $\phi(x)$. Then for each $r>0$ and $x_0\in\mathbb{R}^3$*

$$\|u\|^2_{\mathcal{E}(\bar{Q}_{x_0,r})}\le C(\|\phi\|_{\dot{H}^{1/2}})r \tag{3.58}$$

and, for a suitable function $p_{x_0,r}(t)$ of t,

$$\iint_{\bar{Q}_{x_0,r}}|p-p_{x_0,r}(t)|^{3/2}\,dxdt\le C(\|\phi\|_{\dot{H}^{1/2}})r^2. \tag{3.59}$$

Proof. The first estimate can be derived from Proposition 32.1, p. 342 and its proof in [19], and the second estimate follows from Lemma 32.2, p. 343 in the same book. There are two crucial points in the proof of these estimates. One is that the energy flux in the localized energy estimate 2.16 is bounded by $\sim |u|^3$, if we count pressure as $p \sim |u|^2$. The energy itself controls $\sim |u|^{10/3}$, and it is important for the proof that this is a higher power than the energy flux. This is no longer the case in higher dimensions and therefore a possible generalization to higher dimensions would not be straightforward. Similar issue arises in the proof of partial regularity. The second point is that the non-local effects of the pressure are under control so that the heuristics $p \sim |u|^2$ is valid, at least as far as the estimates are concerned. To see this, we notice that the kernel of the pressure equation

$$-\Delta p = \partial_i \partial_j (F_{ij}), \qquad \text{with } F_{ij} = u_i u_j \tag{3.60}$$

is

$$G_{ij} = \partial_i \partial_j G, \qquad \text{with } G(x) = \frac{1}{4\pi |x|}. \tag{3.61}$$

Therefore the kernel for expressing the gradient ∇p in terms of F_{ij} decays as $|x|^{-4}$ as $x \to \infty$, and is integrable near ∞. This fast decay makes it possible to estimate the contributions to ∇p from far away. This would not be the case for p, and hence we have to work with ∇p, which is the reason for the appearance of the function $p_{x_0,r}$ in the estimate (3.59). This part of the argument would work in higher dimensions as well. $\square$

Our main idea of tackling the problem considered in this thesis is to consider a sequence of initial data ϕ^k with $\|\phi^k\|_{\dot{H}^{1/2}} \searrow \rho_{max}$ togethter with a sequence of corresponding solutions $v^(t,x)$ such that these become singular at $(t,x) = (1,0)$. The main difficulty lays in the fact that the sequence of initial data converges only weakly, say to ϕ_0, in $\dot{H}^{1/2}$ and a-priori we cannot guarantee that either $\|\phi_0\|_{\dot{H}^{1/2}} = \rho_{max}$ or that the solution (v, p) obtained as a strong $(L^3_t L^3_x)_{loc}$ and a weak $L^{3/2}_t L^{3/2}_x$ limit, respectively corresponds to initial data ϕ_0. This is answered by the following result which is one of the main results of this section.

Theorem 3.2.6 *Let $\phi^k(x)$ be a bounded sequence of initial conditions in $\dot{H}^{1/2}$ converging weakly in $\dot{H}^{1/2}$ to $\phi(x)$. Let $v^k(t,x) \in NS(\phi^k(x))$ be Leray solutions of the Cauchy problem (NSE) with initial conditions $\phi^k(x)$. Assume that $v^k(t,x)$ converge weakly to $v(t,x)$ in distributions. Then $v(t,x) \in NS(\phi(x))$, i.e. $v(t,x)$ is a Leray solution of the Cauchy problem with initial condition $\phi(x)$.*

Proof. Using Lemma 3.2.5, Proposition 3.1.2 and Theorem 3.2.3, we see that it is enough to show that $v(t) \to \phi$ in L^2 on every compact subset of $\mathbb{R}^3$. This can be seen as follows. We take a nonnegative smooth test function $\varphi(x,t)$ compactly supported in $[0,\infty) \times \mathbb{R}^3$. Note that we are taking the interval $[0,\infty)$, which is closed at zero, and $\varphi(0,x)$ does not have to vanish everywhere. We write a version of the local energy inequality with such test functions in the following form.

$$\begin{gathered}\int_0^\infty \int_{\mathbb{R}^3} [-|v^k|^2 \varphi_t + 2|\nabla v^k|^2 \varphi] \, dxdt \leq \\ \int_{\mathbb{R}^3} |\phi^k|^2 \varphi(0,x) \, dx + \int_0^\infty \int_{\mathbb{R}^3} [|v^k|^2 \Delta\varphi + (|v^k|^2 + 2p^k) v^k \nabla\varphi] \, dxdt.\end{gathered} \tag{3.62}$$

Since for every compactly supported smooth test function ψ the sequence $\phi^k \psi$ is compact in $L^2(\mathbb{R}^3)$, we see that in the limit $k \to \infty$ the inequality (3.62) will be preserved. Hence

$$\begin{gathered}\int_0^\infty \int_{\mathbb{R}^3} [-|v|^2 \varphi_t + 2|\nabla v|^2 \varphi] \; dxdt \leq \\ \int_{\mathbb{R}^3} |\phi|^2 \varphi(0,x) \; dx + \int_0^\infty \int_{\mathbb{R}^3} [|v|^2 \Delta\varphi + (|v|^2 + 2p) v \nabla\varphi] \; dxdt,\end{gathered} \tag{3.63}$$

where p is a suitable pressure corresponding to the solution v. The last inequality implies that the required local L^2-continuity property at time $t = 0$ for the solution v. The key points, known in the theory of weak solutions and going back to Leray's paper [20] are that

(a) the equation implies that $v(t) \to \phi$ weakly in L^2 on compact subsets as $t \to 0$ and

(b) inequality (3.63) implies that for every compactly supported smooth test function ψ one has $\limsup_{t\to 0^+} \|v(t)\psi\|_{L^2} \leq \|\phi\psi\|_{L^2}$. $\square$

Theorem 3.2.6 has several important corollaries.

Corollary 1 *Let* $\phi^k(x), \phi(x), v^k(t,x)$ *be as in Theorem 3.2.6. Let* $(0, T_{max}(\phi))$ *be the maximal interval of existence of the mild solution* $v(t,x)$ *starting at* $\phi(x)$. *Then for any compact set* $K \subset \mathbb{R}^3 \times (0, T_{max}(\phi))$ *and any* $k \geq k_0 = k_0(K)$ *the solutions* $v^k(t,x)$ *are regular at all points of* K *and converge uniformly to* $v(t,x)$ *in* K, *together with all spatial derivatives.*

Proof. From Lemma 3.2.5 we obtain uniform estimates in the energy space which imply strong convergence of v^k in $L^3_t L^3_x$ on compact subsets of $(0, T_{max}(\phi))$ and weak convergence of $p^k \rightharpoonup p$ in $L^{3/2}_t L^{3/2}_x$ on such subsets. Proposition 3.1.2 implies that the resulting limit is also a suitable weak solution and Theorem 3.2.6 guarantees that it corresponds to appropriate initial data, namely $\phi(x)$. We obtain the claim applying Lemma 3.1.4. $\square$

Corollary 2 *Let* $\phi^k(x), \phi(x), v^k(t,x)$ *be as in Theorem 3.2.6. Assume that* $T_{max}(\phi^k) = T < +\infty$ *for each* k *and that the singular points* z_k *of* $v^k(t,x)$ *at* $t = T$ *(which exist by Proposition 3.2.4) stay in a compact subset of* $\mathbb{R}^3 \times \{T\}$*. Then* $T_{max}(\phi) \leq T$.

Proof. Apply Theorem 3.2.6, together with Lemma 3.2.5, Proposition 3.1.2 and Lemma 3.1.1. $\square$

We now are equipped with all the necessary tools to answer the problem stated in the introduction. Let us recall that

$$\rho_{max} = \sup\{\rho : T_{max}(\phi) = +\infty \text{ for every } \phi \in \dot{H}^{1/2} \text{ with } \|\phi\|_{\dot{H}^{1/2}} < \rho \}. \tag{3.64}$$

Let us also define

$$\mathcal{M} = \{\phi \in \dot{H}^{1/2} : T_{max}(\phi) < \infty, \|\phi\|_{\dot{H}^{1/2}} = \rho_{max}\}. \tag{3.65}$$

The following corollary answers the main problem considered in this thesis.

Corollary 3 *The set* $\mathcal{M}$ *is non-empty. Moreover,* $\mathcal{M}$ *is compact modulo scalings and translations, i.e. if* $\phi^k \in \mathcal{M}$ *is a sequence in* $\mathcal{M}$*, then there exist* $\lambda_k > 0$ *and* $x_0^k \in \mathbb{R}^3$ *such that the sequence* $\varphi^k(x) \in \dot{H}^{1/2}$ *defined by* $\varphi^k(x) = \lambda^k \phi^k(x - x_0^k)$ *is compact in* $\dot{H}^{1/2}$.

Proof. Let $\phi^k(x) \in \dot{H}^{1/2}$ be a sequence of initial data with $T_{max}(\phi^k)$ finite and $\|\phi^k\|_{\dot{H}^{1/2}} \to \rho_{max}$. Find $\lambda_k > 0$ and x_0^k so that the functions given by $\varphi^k(x) = \lambda_k \phi^k(x - x_0^k)$ develop their first singularity at time $t = 1$ and that $(t, x) = (1, 0)$ is a singular point of $v^k(t, x)$. We can assume that the functions $v_0^k(x) = v^k(0, x)$ converge weakly in $\dot{H}^{1/2}$ to $\phi_0 \in \dot{H}^{1/2}$. By Corollary 2 we know that $T_{max}(\phi_0) \leq 1$, and by definition of ρ_{max} this means that $\|\phi_0\|_{\dot{H}^{1/2}} = \rho_{max}$. This shows that $\mathcal{M}$ is non-empty. We also see that $\|v_0^k\|_{\dot{H}^{1/2}} \to \|\phi_0\|_{\dot{H}^{1/2}}$ and hence $v_0^k \to \phi_0$ strongly. ☐

The following last corollary can be thought of as a variant of results in [8] and [29].

Corollary 4 *Assume that every solution of the Cauchy problem (NSE) with* $\phi(x) \in \dot{H}^{1/2}$ *is regular, i.e.* $T_{max}(\phi) = +\infty$ *for each* $\phi \in \dot{H}^{1/2}$. *Then, for* $l = 0, 1, 2, \ldots$ *there exist functions* $F_l : [0, \infty) \to [0, \infty)$ *such that*

$$t^{(l+1)/2} \sup_x |\nabla^l v(t, x)| \leq F_l(\|\phi\|_{\dot{H}^{1/2}}). \tag{3.66}$$

Proof. Let us prove the case $l = 0$, the proof for the derivatives being the same. By scaling invariance, it is enough to prove the statement for $t = 1$. If the statement fails we can assume by the translational invariance that there exists a sequence of initial data ϕ^k bounded in $\dot{H}^{1/2}$ such that for the corresponding solutions $v^k(t, x)$ one has $|v^k(1, 0)| \geq k$. Let ϕ_0 be a weak limit of ϕ^k. By our assumption, the solution $v(t, x)$ corresponding to ϕ_0 is regular at $(t, x) = (1, 0)$ and by Theorem 3.2.6 and Lemma 3.1.4 we get a contradiction. ☐

Chapter 4

An alternative proof via the profile decomposition

In this chapter we approach the problem of minimal initial data generating a singularity in a different way, trying to use a setup that may accommodate more spaces of initial data than the one used in Chapter 3. Instead of suitable weak solutions, we work with the profile decomposition of the sequence of initial data together with the sequence of corresponding solutions. The framework for this chapter is provided by the work of I. Gallagher, see [8]. This setting may provide a way to generalize our results in particular to the Lebesgue space $L^3(\mathbb{R}^3)$. We start by introducing the notation and recall some useful results, necessary in the later part of this chapter.

4.1 Auxiliary results

Let us start by defining the setting underlying the results presented in this chapter. Let us recall the existence result by T. Kato and H. Fujita. Following [3] we can formulate it in the following way: if $\phi(x) \in \dot{H}^{1/2}(\mathbb{R}^3)$, then there exists a unique maximal time $T_{max}(\phi) > 0$ and a unique solution $v(t,x) = NS(\phi)$ associated with $\phi(x)$ such that

$$v(t,x) \in C([0,T], \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0,T], \dot{H}^{3/2}(\mathbb{R}^3)) \quad \text{for all } T < T_{max}(\phi). \tag{4.1}$$

Moreover, if $T_{max}(\phi) < +\infty$, then we have

$$\lim_{T \to T_{max}(\phi)} \|v\|_{L^2([0,T],\dot{H}^{3/2}(\mathbb{R}^3))} = +\infty. \tag{4.2}$$

Furthermore, there exists a universal constant $C_E > 0$ such that

$$\|\phi\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < C_E \implies T_{max}(\phi) = +\infty, \tag{4.3}$$

and we have for any $t \geq 0$

$$\|v(t)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + \int_0^t \|v(s)\|^2_{\dot{H}^{3/2}(\mathbb{R}^3)}\, ds \leq C\|\phi\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)}. \tag{4.4}$$

One can think of C_E as a bound on the initial data such that we obtain a solution together with an a-priori estimate. If a solution $v(t,x)$ satisfies the estimate (4.4), this of course implies that $T_{max}(\phi) = +\infty$. However, it is not clear that we cannot have $T_{max}(\phi) = +\infty$ without (4.4), thus we set

$$C_G := \sup \left\{ \begin{array}{c} \rho > 0 : \text{ for } \|\phi\|_{\dot{H}^{1/2}} < \rho, \\ \\ v(t,x) = NS(\phi) \text{ is a global strong solution of (NSE)} \end{array} \right\}. \tag{4.5}$$

Evidently we have $C_G \geq C_E$ - in other words - it is conceivable to have global solutions without an a-priori estimate.

In her paper [8], I. Gallagher considered a decomposition of a bounded sequence of initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$ and analyzed how this decomposition is propagated by the Navier-Stokes equations. In consistency with that result we define the spaces

$$\begin{aligned} E_T &= C([0,T], \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0,T], \dot{H}^{3/2}(\mathbb{R}^3)), && (4.6) \\ E_\infty &= C_b(\mathbb{R}^+, \dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^{3/2}(\mathbb{R}^3)), && (4.7) \end{aligned}$$

where C_b denotes the set of bounded, continuous functions. We norm these spaces respectively by

$$\begin{aligned} \|v\|_{E_T} &= \sup_{0\leq t\leq T} \Big(\|v(t)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + 2\|v\|^2_{L^2([0,t],\dot{H}^{3/2}(\mathbb{R}^3))} \Big)^{1/2} && (4.8) \\ \|v\|_{E_\infty} &= \Big(\|v\|^2_{L^\infty(\mathbb{R}^+,\dot{H}^{1/2}(\mathbb{R}^3))} + 2\|v\|_{L^2(\mathbb{R}^+,\dot{H}^{3/2}(\mathbb{R}^3))} \Big)^{1/2}. && (4.9) \end{aligned}$$

Gallagher considers also the set $D_\infty \subset \dot{H}^{1/2}(\mathbb{R}^3)$ defined by

$$\mathcal{D}_\infty = \{\phi \in \dot{H}^{1/2}(\mathbb{R}^3) : NS(\phi) \in E_\infty\}. \qquad (4.10)$$

Following a more general framework, Gallagher defines the constant

$$C_{NS} = \sup\{\rho > 0 : \bar{B}_\rho \subset \mathcal{D}_\infty\}. \qquad (4.11)$$

Clearly one has the relation $C_E \leq C_{NS} \leq C_G$ and initially it is not clear that $C_{NS} = C_G$. If this were false, it would mean that for data with norms between C_{NS} and C_G the second component of $\|\cdot\|_{E_T}$ becomes infinite as $T \to \infty$ resulting in a global solution which does not belong to the space E_∞ (see the remark on

p. 289 in [8]). However in [9] I. Gallagher, D. Iftimie and F. Planchon have shown that any global solution automatically is in E_∞, thus proving $C_{NS} = C_G$.

In this thesis we are concerned with possible singular solutions. In order to utilize the above described functional framework, we have to address the connection between the loss of regularity and the mentioned spaces. There are several reasons for which a solution $v(t,x)$ may not be in E_∞. One obvious cause is the blow-up of the $\dot{H}^{1/2}$-norm. This however does not immediately imply the existence of a singularity. Again, one could imagine a situation in which the solution ceases to have sufficient decay in space. Recall also that for global solutions one has the decay of the $\dot{H}^{1/2}$-norm to zero as $t \to \infty$ thus the $\dot{H}^{1/2}$-norm cannot grow as $t \to \infty$. Another scenario is that the second component of the E_∞-norm becomes infinite in finite time but the solution stays bounded in $\dot{H}^{1/2}$. This however is eliminated by Theorem 2 in [9]. Theorem 1 in the same paper takes care of the case where for $t \to \infty$, the $\| \cdot \|_{L^2([0,t],\dot{H}^{3/2})}$-norm becomes infinite. One thus has to address the issue that the only possible way for $v(t,x)$ not to be in E_∞ is by developing a singularity in finite time. This has been done in the previous chapter - see Proposition 3.2.3.

We now briefly recall some tools used in the proof of our result together with their references.

First, notice that it is well known that the embedding $\dot{H}^{1/2}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$ is not compact. Assuming for a moment that the embedding was compact, in order to prove our result, one could consider a sequence of initial data $\phi_n(x) \in \dot{H}^{1/2}(\mathbb{R}^3)$, leading to singular solutions, and such that $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \to C_G$ while

$\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \geq C_G$. One could extract a subsequence weakly converging to $\phi^0(x)$ in $\dot{H}^{1/2}(\mathbb{R}^3)$. Then, compactness of the emebding would imply strong convergence of $\phi_n(x)$ to $\phi^0(x) \in L^3(\mathbb{R}^3)$. This combined with the stability result by I. Gallagher, D. Iftimie and F. Planchon (see [10]) would give the desired conclusion. Thus, in order to prove the theorem, we have to consider the defect of compactness of the embedding of $\dot{H}^{1/2}(\mathbb{R}^3)$ into $L^3(\mathbb{R}^3)$, which was very precisely studied by P. Gérard in [12]. Let us recall this result.

Theorem 4.1.1 *Let (ϕ_n) be a bounded sequence of functions in $\dot{H}^{1/2}(\mathbb{R}^3)$. Then up to the extraction of a subsequence, it can be decomposed in the following way:*

$$\forall\, \ell \in \mathbb{N} \setminus \{0\}, \qquad \phi_n(x) = \phi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j} \phi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \psi_n^\ell(x), \tag{4.12}$$

where the functions ϕ^j are in $\dot{H}^{1/2}(\mathbb{R}^3)$ for all $j \in \mathbb{N}$, where (ψ_n^ℓ) is a bounded sequence in $\dot{H}^{1/2}(\mathbb{R}^3)$ uniformly in $\ell \in \mathbb{N} \setminus \{0\}$, and satisfies

$$\lim_{\ell \to \infty} \limsup_{n \to \infty} \|\psi_n^\ell\|_{L^3(\mathbb{R}^3)} = 0, \tag{4.13}$$

and where for any $j \in \mathbb{N} \setminus \{0\}$, (h_n^j, x_n^j) is a sequence in $(\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^3)^{\mathbb{N}}$ with the following orthogonality property: for every integers (j, k) such that $j \neq k$, we have

$$\begin{cases} \textit{either} \quad \lim_{n\to\infty} \left(\frac{h_n^j}{h_k^j} + \frac{h_k^j}{h_n^j} \right) = +\infty \\ \\ \textit{or} \qquad h_n^j = h_k^j \quad \textit{and} \quad \lim_{n\to\infty} \frac{|x_n^j - x_k^j|}{h_n^j} = +\infty. \end{cases} \tag{4.14}$$

Finally we have for every $\ell \in \mathbb{N} \setminus \{0\}$,

$$\|\phi_n\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} = \sum_{j=0}^{\ell} \|\phi^j\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + \|\psi_n^\ell\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + o(1), \tag{4.15}$$

as n goes to infinity.

Under some very special circumstances, it is possible to reduce the situation to a sequence of initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$ for which the weak convergence actually turns out to be strong. In order to prove our result in this case, we will use a stability result obtained by I. Gallagher, D. Iftimie and F. Planchon in the setting of the $\dot{H}^{1/2}(\mathbb{R}^3)$ data. We have the following

Theorem 4.1.2 *Let $v \in C(\mathbb{R}^+, \dot{H}^{1/2}(\mathbb{R}^3))$ be a mild solution of (NSE) with initial data $v_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$. Then*

$$\lim_{t\to+\infty} \|v(t,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = 0, \tag{4.16}$$

$$v(t,x) \in L^2(\mathbb{R}^+, \dot{H}^{3/2}(\mathbb{R}^3)). \tag{4.17}$$

Moreover, there exists $\epsilon = \epsilon(v)$ such that if $\|v_0 - u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \epsilon(v)$ then the mild solution $u(t,x)$ with initial data $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$ is also global, that is belongs to $C(\mathbb{R}^+, \dot{H}^{1/2}(\mathbb{R}^3))$ and

$$\begin{aligned} &\|(u-v)(t,\cdot)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + \int_0^t \|\nabla(u-v)(s,\cdot)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)}\, ds \\ &\le C\|v_0 - u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \exp\left(C\int_0^t \|v\|^2_{\dot{H}^{3/2}(\mathbb{R}^3)}\, ds + \int_0^t \|v\|^4_{\dot{H}^1(\mathbb{R}^3)}\, ds\right). \end{aligned} \tag{4.18}$$

The non-linear character of the system forces us to introduce an error term due to the interaction between profiles while analyzing a solution corresponding to initial data composed of a set of profiles, in terms of solutions corresponding to separate profiles. To control this interaction we will use a method based on perturbation techniques.

Proposition 4.1.3 *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$ be fixed. There exists a constant C independent of T, such that the following is true. Let (f_n) and (g_n) be two families of vector fields, bounded in E_T and in $L^2([0,T], \dot{H}^{-1/2}(\mathbb{R}^3))$ respectively. If*

$$\sup_{n\in\mathbb{N}} \|g_n\|_{L^2([0,T],\dot{H}^{-1/2}(\mathbb{R}^3))} \leq C \exp\left(-2C \sup_{n\in\mathbb{N}} \|f_n\|_{E_T}^4\right), \tag{4.19}$$

then there exists a unique solution in E_T to the following system:

$$\begin{aligned} &\partial_t r_n + \mathbb{P}(r_n \cdot \nabla r_n) - \Delta r_n + \mathcal{Q}(r_n, f_n) = g_n, \\ &r_n(0,x) = 0, \end{aligned} \tag{4.20}$$

where $\mathbb{P}$ is the Leray projector onto divergence free vector fields, and where

$$\mathcal{Q}(a,b) = \mathbb{P}(a \cdot \nabla b + b \cdot \nabla a).$$

Moreover, we have

$$\|r_n\|_{E_T} \leq C \|g_n\|_{L^2([0,T],\dot{H}^{-1/2}(\mathbb{R}^3))} \left(1 + \exp\left(C\|f_n\|_{E_T}^4\right)\right). \tag{4.21}$$

The crucial part of applying Proposition 4.1.3 is to ensure that condition (4.19) is satisfied, at least for sufficiently large n. Thus we have to know to what extent the orthogonality properties of the profiles are preserved by the evolution according to the system (NSE). This is the merit of the following

Proposition 4.1.4 *Let $T \in \mathbb{R}^+ \cup \{+\infty\}$ be given, and let ϕ^1 and ϕ^2 be two divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3)$. Consider two sequences (h_n^1, x_n^1) and (h_n^2, x_n^2) in $(\mathbb{R}^+\backslash\{0\}\times\mathbb{R}^3)^{\mathbb{N}}$ orthogonal in the sense of (4.14). Suppose for instance that $h_n^1 \leq h_n^2$. For $j = 1, 2$ define*

$$\phi_n^j(x) = \frac{1}{h_n^j}\phi^j\left(\frac{x - x_n^j}{h_n^j}\right). \tag{4.22}$$

Let $NS(\phi^1)$ and $NS(\phi^2)$ be the solutions of (NSE) in the class E_T. Then we have the following

$$\lim_{n\to\infty} \sup_{t\in[0,(h_n^1)^2T]} \left(NS(\phi_n^1)(t,\cdot)|NS(\phi_n^2)(t,\cdot)\right)_{\dot{H}^{1/2}(\mathbb{R}^3)} = 0, \tag{4.23}$$

$$\lim_{n\to\infty} \left(NS(\phi_n^1)|NS(\phi_n^2)\right)_{L^2([0,(h_n^1)^2,\dot{H}^{3/2}(\mathbb{R}^3)])} = 0, \tag{4.24}$$

$$\lim_{n\to\infty} \|NS(\phi_n^1)NS(\phi_n^2)\|_{L^4([0,(h_n^1)^2T],L^2(\mathbb{R}^3))} = 0. \tag{4.25}$$

The proof of Proposition 4.1.4 can be found in [8]. One of the issues that should be addressed at this point is that as $h_n^1 \to 0$ the orthogonality result of Proposition 4.1.4 is valid on intervals that possibly shrink to zero. This may be the case if our solutions $NS(\cdot)$ are only local. Following Gallagher, we incorporate the convention that if $T = +\infty$ (in other words, considered solutions are global) we have $(h_n^1)^2 \cdot T = +\infty$ even if $h_n^1 \to 0$.

4.2 Proof of the main theorem

Let us recall the setup of Corollary 3 from Chapter 3. Let $T_{max}(\phi)$ be the maximal time of existence of a mild solution with initial data $\phi \in \dot{H}^{1/2}$. We have

$$\rho_{max} = \sup\{\rho : T_{max}(\phi) = +\infty \text{ for every } \phi \in \dot{H}^{1/2} \text{ with } \|\phi\|_{\dot{H}^{1/2}} < \rho \}. \tag{4.26}$$

and

$$\mathcal{M} = \{\phi \in \dot{H}^{1/2} : T_{max}(\phi) < \infty, \|\phi\|_{\dot{H}^{1/2}} = \rho_{max}\}. \tag{4.27}$$

We are now ready to prove our main result of this chapter, which is

Theorem 4.2.1 *Assume that $C_G < +\infty$. Then $\mathcal{M} \neq \emptyset$.*

Proof. Let $\phi_n(x) \in \dot{H}^{1/2}(\mathbb{R}^3)$ be a sequence of initial data such that the corresponding mild solutions $v_n(t,x)$ become singular in finite time at (T_n, x_n) and $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \searrow C_G$, $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \geq C_G$. If for some n we have $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = C_G$ then we are done, thus we may assume that $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} > C_G$. Let $\lambda_n = \sqrt{T_n}$. For every n consider the rescaled functions

$$(v_n(t,x))_{\lambda_n} = \lambda_n v_n(\lambda_n^2 t, \lambda_n x + x_n). \tag{4.28}$$

Notice that $(v_n(t,x))_{\lambda_n}$ are again mild solutions of (NSE) and their $\dot{H}^{1/2}$-norm is preserved by the scaling. This way, without loss of generality, we can assume that all $v_n(t,x)$ become singular at $(1,0)$. Possibly passing to a subsequence, we can further assume that $\phi_n(x) \rightharpoonup \phi^0(x)$ in $\dot{H}^{1/2}(\mathbb{R}^3)$. Since ϕ_n converges weakly in $\dot{H}^{1/2}(\mathbb{R}^3)$, it is bounded and Theorem 4.1.1 yields that we can decompose the sequence as

$$\phi_n(x) = \phi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j} \phi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \psi_n^\ell(x), \tag{4.29}$$

with the properties (4.13) – (4.15). By lower semi-continuity of the norm with respect to weak convergence we have

$$\|\phi^0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq C_G. \tag{4.30}$$

Furthermore, notice that the functions ϕ^j are weak limit points in $\dot{H}^{1/2}(\mathbb{R}^3)$ of the sequence $\phi_n^j(x) = h_n^j \phi_n(h_n^j x + x_n^j)$. Due to the scaling invariance of the $\dot{H}^{1/2}$-norm we have

$$\|\phi_n^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = \|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \tag{4.31}$$

thus by the lower semi-continuity of the norm with respect to weak convergence, we have

$$\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \leq C_G. \tag{4.32}$$

Note, that since $\phi_n(x)$ are divergence free and $\phi^j(x)$ is the weak limit point of the sequence $\phi_n^j(x) = h_n^j \phi_n(h_n^j x + x_n^j)$, this implies that $\phi^j(x)$ are also divergence free. Thus all terms in (4.29) are divergence free.

If we pass to the limit $n \to \infty$, we have the following two cases.

Case 1: There exist $j \in \{0, \ldots, \ell\}$ such that

$$\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = C_G. \tag{4.33}$$

First, notice that since $\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \searrow C_G$ then due to (4.15), there can be at most only one such j.

If $j = 0$ then assume that the corresponding solution $v(t, x)$ does not become singular. Notice that $\|\phi^0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = C_G$ implies that the sequence $\phi_n \rightharpoonup \phi^0$ actually converges strongly in $\dot{H}^{1/2}(\mathbb{R}^3)$, $\phi_n \to \phi^0$. This is in contradiction with Theorem 4.1.2, since for n large we have $\|\phi^0 - \phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \epsilon(v)$ thus the solutions $v_n(t, x)$ belong in particular to E_2 but then by the Ladyzhenskaya-Prodi-Serrin condition, since they are in $L^5((0, 2) \times \mathbb{R}^3)$, they are regular on $(0, 2)$ contradicting the assumption that they become singular at time $T = 1$. This implies that $\phi^0(x)$ is the minimal singularity-generating data.

Thus we may assume $j \neq 0$. Property (4.15) implies that the decomposition of the initial data has the form

$$\phi_n(x) = \frac{1}{h_n^j}\phi^j\left(\frac{x - x_n^j}{h_n}\right) + \psi_n(x). \tag{4.34}$$

Since we have only one profile, we may drop the index j. We rescale equation (4.34) and obtain

$$h_n\phi_n(h_n x + x_n) = \phi(x) + h_n\psi_n(h_n x + x_n). \tag{4.35}$$

Notice that the rescaling preserves the $\dot{H}^{1/2}$-norm. Without loss of generality, we may denote the rescaled sequences $h_n\phi_n(h_n x + x_n)$ by $\phi_n(x)$ and $h_n\psi_n(h_n x + x_n)$ by $\psi_n(x)$, respectively. We are in a situation, where the mild solutions $v_n(t,x)$, corresponding to initial data $\phi_n(x)$ become singular at (h_n^2, x_n).

Possibly passing to a subsequence we have several cases:

(i): $h_n \to +\infty$.

This is the hardest of the three cases to deal with. The following reasoning follows the one in [10]. We have a sequence of solutions $v_n(t,x)$ regular on $(0, h_n^2)$ and developing a singularity at $T = h_n^2$. Since the $\dot{H}^{1/2}$-norm is preserved by the scaling, property (4.15) implies that $\psi_n(x)$ converges strongly to 0 in $\dot{H}^{1/2}(\mathbb{R}^3)$. Let $\epsilon > 0$. Consider the decomposition of $\phi(x) = u_0(x) + w_0(x)$ with $\operatorname{div} u_0 = \operatorname{div} w_0 = 0$, $\|w_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \epsilon/2$ and $u_0 \in L^2(\mathbb{R}^3)$. This decomposition is independent of the index n since we scaled equation (4.34) to fix the profile ϕ. Let $w_n(t,x)$ be the solution of (NSE) with data $w_0(x) + \psi_n(x)$. For n large enough

$w_n(t,x)$ is a global solution and satisfies the a-priori estimate

$$\|w_n(t,\cdot)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + \int_0^t \|\nabla w_n(s,\cdot)\|^2_{\dot{H}^{1/2}}\, ds \le \|w_0+\psi_n\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)}. \tag{4.36}$$

Since $\psi_n \to 0$ in $\dot{H}^{1/2}(\mathbb{R}^3)$, we may choose n sufficiently large, so that

$$\|w_n(t,\cdot)\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)} + \int_0^t \|\nabla w_n(s,\cdot)\|^2_{\dot{H}^{1/2}}\, ds \le \epsilon^2, \tag{4.37}$$

which implies that for n large we have

$$\sup_{t\ge 0} \|w_n(t,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \le \epsilon. \tag{4.38}$$

Notice that $u_n(t,x) = v_n(t,x) - w_n(t,x)$ satisfy the equation

$$\begin{aligned}
&\partial_t u_n - \Delta u_n + u_n\cdot\nabla u_n + w_n\cdot\nabla u_n + u_n\cdot\nabla w_n - \nabla q = 0 && \text{in } (0,h_n^2)\times\mathbb{R}^3,\\
&\operatorname{div} u_n = 0 && \text{in } (0,h_n^2)\times\mathbb{R}^3,\\
&u_n(0,x) = u_0(x) && \text{in } \mathbb{R}^3.
\end{aligned} \tag{4.39}$$

For every $T < h_n^2$, since

$$v_n(t,x) \in C([0,T],\dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0,T],\dot{H}^{3/2}(\mathbb{R}^3)) \tag{4.40}$$

and $w_n(t,x)$ are global, we have

$$u_n(t,x) \in C([0,T],\dot{H}^{1/2}(\mathbb{R}^3)) \cap L^2([0,T],\dot{H}^{3/2}(\mathbb{R}^3)). \tag{4.41}$$

Furthermore, for every $t < h_n^2$ we have the following energy estimate

$$\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)}\, ds \le \|u_0\|^2_{L^2(\mathbb{R}^3)} + 2\left|\int_0^t\int_{\mathbb{R}^3} (u_n\cdot\nabla w_n)\cdot u_n\, dxds\right|. \tag{4.42}$$

To estimate the second term on the right-hand side of (4.42) due to Hölder's inequality and Sobolev imbeddings we have the following

$$\begin{aligned}\left|\int_0^t \int_{\mathbb{R}^3} (u_n \cdot \nabla w_n) \cdot u_n \, dxds\right| &= \left|\int_0^t \int_{\mathbb{R}^3} (u_n \otimes w_n) \cdot \nabla u_n \, dxds\right| \\ &\le C \int_0^t \|w_n(s,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds\end{aligned} \tag{4.43}$$

Because of (4.38) we have

$$\int_0^t \|w_n(s,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds \le C\epsilon \int_0^t \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds. \tag{4.44}$$

Thus we have

$$\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds \le \|u_0\|^2_{L^2(\mathbb{R}^3)} + 2C\epsilon \int_0^t \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds. \tag{4.45}$$

Take $\epsilon < \min(1/2C, C_G/2)$. We obtain for all $t < h_n^2$

$$\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|u_n(s,\cdot)\|^2_{\dot{H}^1(\mathbb{R}^3)} \, ds \le \|u_0\|^2_{L^2(\mathbb{R}^3)}. \tag{4.46}$$

Interpolating between the spaces $L^\infty([0,T], L^2(\mathbb{R}^3))$ and $L^2([0,T], \dot{H}^1(\mathbb{R}^3))$ we obtain that $u_n(t,x) \in L^4([0,T], \dot{H}^{1/2}(\mathbb{R}^3))$ for all $T < h_n^2$. Assume for a moment that for every n and every $t \in [0, h_n^2/2]$ we have $\|u_n(t,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \ge \epsilon$. This implies

$$\epsilon^4 h_n^2/2 \le \int_0^{h_n^2/2} \|u_n(s,\cdot)\|^4_{\dot{H}^{1/2}(\mathbb{R}^3)} \, ds \le \|u_0\|^4_{L^2(\mathbb{R}^3)}. \tag{4.47}$$

Finally notice that for n sufficiently large, inequality (4.47) cannot hold, thus for n large enough there exist $0 < t_n < h_n^2$ such that $\|u_n(t_n,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \epsilon$. In this case however, because of $v_n(t,x) = u_n(t,x) + w_n(t,x)$ and (4.38), we have

$$\|v_n(t_n,\cdot)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < C_G, \tag{4.48}$$

thus $v_n(t,x)$ is a global solution. Hence we have obtained a contradiction.

(ii): $h_n \to 0$

Since ψ_n converges to zero strongly in $\dot{H}^{1/2}(\mathbb{R}^3)$ we actually have strong convergence $\phi_n \to \phi$ in $\dot{H}^{1/2}(\mathbb{R}^3)$ and solutions corresponding to data ϕ_n become singular at times $h_n^2 \to 0$. Thus assuming that $\phi(x)$ leads to a global solution $v(t,x)$, for n large enough we have $\|\phi_n - \phi\|_{\dot{H}^{1/2}} < \epsilon(v)$. However inequality (4.18) does not hold. This implies that $\phi(x)$ is the minimal singularity-generating data.

(iii): $h_n \to h$ with $h \neq 0$ and $h < +\infty$.

As in the case above, we have strong convergence of initial data in $\dot{H}^{1/2}(\mathbb{R}^3)$, however the stability estimate is violated. This implies that $\phi(x)$ is the minimal singularity-generating data.

Case 2: For all $j \in \{0, \ldots, \ell\}$ we have $\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < C_G$ and

$$\phi_n(x) = \phi^0(x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j} \phi^j \left(\frac{x - x_n^j}{h_n^j} \right) + \psi_n^\ell(x). \tag{4.49}$$

Let V^j be the mild solutions of (NSE) corresponding to data ϕ^j for $j \in \{0, \ldots, \ell\}$. Observe that due to the assumption that $\|\phi^j\|_{\dot{H}^{1/2}} < C_G$, these solutions are global and in fact $V^j \in E_\infty$. Consider the decomposition analogous to the one in Theorem 2 in [8]. We set

$$v_n(t,x) = V^0(t,x) + \sum_{j=1}^{\ell} \frac{1}{h_n^j} V^j \left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j} \right) + w_n^\ell(t,x) + r_n^\ell(t,x), \tag{4.50}$$

where $w_n^\ell(t,x)$ solve the homogenous heat equation with initial data $\psi_n^\ell(x)$ and $r_n^\ell(t,x)$ satisfy

$$\begin{aligned}&\partial_t r_n^\ell + \mathbb{P}(r_n^\ell \cdot \nabla r_n^\ell) - \Delta r_n^\ell + \mathcal{Q}(r_n^\ell, f_n^\ell) = g_n^\ell,\\ &r_n^\ell(0,x) = 0,\end{aligned} \tag{4.51}$$

with

$$f_n^\ell = \sum_{j\le\ell} v_n^j + w_n^\ell, \tag{4.52}$$

and

$$g_n^\ell = -\frac{1}{2}\sum_{j\neq k} \mathcal{Q}(v_n^j, v_n^k) - \sum_{j\le\ell} \mathcal{Q}(v_n^j, w_n^\ell) - \mathbb{P}(w_n^\ell \cdot \nabla w_n^\ell), \tag{4.53}$$

and v_n^j are defined as

$$v_n^j(t,x) = \frac{1}{h_n^j} V^j\left(\frac{t}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right). \tag{4.54}$$

Recall that all terms on the right-hand side of (4.50) but possibly $r_n^\ell(t,x)$ are in E_∞. The issue here though is that $v_n^j(t,x)$ do not necessarily satisfy the a-priori estimate (4.4) thus possibly creating an obstacle for applying Proposition 4.1.3. This is only known for sufficiently small data (in other words, the norm of ϕ^j has to be sufficiently small) and it is conceivable that $C_E < C_G$. As it will become visible later, absence of an estimate like (4.4) may potentially lead to lack of control over $\|f_n^\ell\|_{E_\infty}$ with increasing ℓ (which would make it impossible to apply Proposition 4.1.3). However, due to (4.15) there exists j_0 depending only on C_E and C_G such that

$$\|\phi^j\|_{\dot{H}^{1/2}} < C_E \quad \text{for} \;\; j \ge j_0.$$

Hence there is only a finite number of V^j which possibly do not satisfy an a-priori estimate. This, combined with property (4.15) and the scaling invariance of the norms, allows us to deduce that

$$\sum_{j=0}^{\infty} \|V^j\|_{E_\infty}^2 < +\infty, \tag{4.55}$$

which is going to play an important role later in the proof and takes care of that issue.

If we can show that for sufficiently large ℓ, n we have $r_n^\ell(t,x) \in E_2$, this shows that $v_n(t,x) \in E_2$ thus v_n belong to $L^5((0,2)\times\mathbb{R}^3)$ and thus by the Ladyzhenskaya-Prodi-Serrin condition are regular up to $T = 2$. This will contradict the assumption that solutions corresponding to initial data ϕ_n become singular at $T = 1$. Since all terms but $r_n^\ell(t,x)$ on the left hand-side of (4.50) are global, we turn our attention to this particular term. To control $r_n^\ell(t,x)$ we will make use of Proposition 4.1.3.

It is enough to show that $\|f_n^\ell\|_{E_2}$ is bounded uniformly in ℓ, n and that for sufficiently large ℓ, n we have $\|g_n^\ell\|_{L^2([0,2],\dot{H}^{-1/2}(\mathbb{R}^3))}$ small enough so that condition (4.19) is satisfied. To avoid issues with scaling of the length of the time intervals on which we work we chose to consider $\mathbb{R}^+$. This is consistent with our remarks following Proposition 4.1.4.

Theorem 4.1.1 yields that the sequence (ψ_n^ℓ) is bounded in $\dot{H}^{1/2}(\mathbb{R}^3)$ uniformly in ℓ thus we have w_n^ℓ bounded in E_∞ uniformly in ℓ by $\|\psi_n^\ell\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$. Hence we also have a bound in E_2. Due to the a-priori estimate in $\dot{H}^{1/2}$ for sufficiently small data and the scale invariance of involved norms we have the estimate

$$\|v_n^j\|_{E_\infty} \leq C\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \quad \text{for } j \geq j_0. \tag{4.56}$$

Notice also that Proposition 4.1.4 gives us

$$\Big\|\sum_{j\leq\ell} v_n^j\Big\|_{E_\infty}^2 = \sum_{j\leq\ell}\|v_n^j\|_{E_\infty}^2 + o(1), \qquad \text{as } n\to\infty, \tag{4.57}$$

thus property (4.15) toghether with the fact that $\|v_n^j\|_{E_\infty} = \|V^j\|_{E_\infty}$ gives us

$$\Big\|\sum_{j\leq\ell} v_n^j\Big\|_{E_2}^2 \leq \Big\|\sum_{j\leq\ell} v_n^j\Big\|_{E_\infty}^2 \leq \sum_{j=0}^{j_0}\|V^j\|_{E_\infty} + 2\|\phi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)}^2 \leq \sum_{j=0}^{j_0}\|V^j\|_{E_\infty} + 2C_G \tag{4.58}$$

for sufficiently large n. Recall that j_0 does not depend neither on ℓ nor on n. Thus in particular we have an estimate of $\|f_n^\ell\|_{E_\infty}$ uniform in ℓ which implies a uniform estimate of $\|f_n^\ell\|_{E_2}$.

To show that g_n^ℓ is small in $L^2([0,2],\dot{H}^{-1/2}(\mathbb{R}^3))$ for n large enough we proceed as follows. Notice that all terms v_n^j, w_n^ℓ belong to E_∞ due to the assumption on initial data and recall that $\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < C_G$ for $j\in\{j_0,\ldots,\ell\}$. By interpolation between $L^\infty(\mathbb{R}^+,\dot{H}^{1/2}(\mathbb{R}^3))$ and $L^2(\mathbb{R}^+,\dot{H}^{3/2}(\mathbb{R}^3))$ we obtain that all v_n^j and w_n^ℓ belong to $L^{8/3}(\mathbb{R}^+,\dot{H}^{5/4}(\mathbb{R}^3))$ and for $j\geq j_0$ we have estimates by the norms of initial data $\|\phi^j\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$. For w_n^ℓ we have an a-priori estimate by $\|\psi_n^\ell\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$. We recall the product rule in Sobolev spaces:

$$\forall s,t<3/2,\ s+t>0, \qquad \|ab\|_{\dot{H}^{s+t-\frac{3}{2}}(\mathbb{R}^3)} \leq C(s,t)\|a\|_{\dot{H}^s(\mathbb{R}^3)}\|b\|_{\dot{H}^t(\mathbb{R}^3)}. \tag{4.59}$$

All building blocks of g_n^ℓ are of the form $\mathbb{P}(a_n\cdot\nabla b_n)$ where we have $a_n, b_n \in L^{8/3}(\mathbb{R}^+,\dot{H}^{5/4}(\mathbb{R}^3))$ thus we obtain

$$\|\mathbb{P}(a_n\cdot\nabla b_n)\|_{L^{4/3}(\mathbb{R}^+,L^2(\mathbb{R}^3))} \leq C\|a_n\|_{L^{8/3}(\mathbb{R}^+,\dot{H}^{5/4}(\mathbb{R}^3))}\|b_n\|_{L^{8/3}(\mathbb{R}^+,\dot{H}^{5/4}(\mathbb{R}^3))}. \tag{4.60}$$

Hence we have

$$\|\mathcal{Q}(v_n^j, v_n^k)\|_{L^{4/3}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C\|v_n^j\|_{E_\infty}\|v_n^k\|_{E_\infty}, \tag{4.61}$$

$$\|\mathcal{Q}(v_n^j, w_n^\ell)\|_{L^{4/3}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C\|v_n^j\|_{E_\infty}\|\psi_n^\ell\|_{\dot{H}^{1/2}(\mathbb{R}^3)}, \tag{4.62}$$

$$\|\mathbb{P}(w_n^\ell \cdot \nabla w_n^\ell)\|_{L^{4/3}(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq C\|\psi_n^\ell\|^2_{\dot{H}^{1/2}(\mathbb{R}^3)}. \tag{4.63}$$

Estimates (4.60)–(4.63) show that g_n^ℓ is bounded in $L^{4/3}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ with a bound uniform in ℓ due to the inequality $2ab \leq a^2 + b^2$, the a-priori estimate for almost all v_n^j and property (4.15) (summing over j, k). These estimates are also uniform in n because of the scaling invariance of involved norms. If we can make g_n^ℓ small in $L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))$ uniformly in ℓ for large n then by interpolation between $L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))$ and $L^{4/3}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ we will obtain that g_n^ℓ is small in $L^2(\mathbb{R}^+, \dot{H}^{-1/2}(\mathbb{R}^3))$ thus also in $L^2([0,2], \dot{H}^{-1/2}(\mathbb{R}^3))$. Therefore the result amounts to showing that for sufficiently large n the following quantities

$$\|\mathcal{Q}(v_n^j, v_n^k)\|_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))}, \tag{4.64}$$

$$\Big\|\mathcal{Q}(\sum_{j\leq\ell} v_n^j, w_n^\ell)\Big\|_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))}, \tag{4.65}$$

and

$$\|\mathbb{P}(w_n^\ell \cdot \nabla w_n^\ell)\|_{L^4(\mathbb{R}^+, \dot{H}^{-1}(\mathbb{R}^3))}, \tag{4.66}$$

become sufficiently small (for fixed ℓ ! Here essentially lies the reason why the limits have to be taken in the order first ℓ then n for a fixed ℓ large enough).

Notice that due to the divergence free condition this reduces to showing that

$$\forall j \neq k, \qquad \lim_{n\to\infty} \|v_n^j v_n^k\|_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0, \tag{4.67}$$

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \Big\| w_n^\ell \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0, \tag{4.68}$$

and

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \|w_n^\ell w_n^\ell\|_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} = 0. \tag{4.69}$$

Let us first take care of (4.69). We have

$$\|w_n^\ell w_n^\ell\|_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq \|w_n^\ell\|_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.70}$$

By the imbedding $\dot{H}^1 \hookrightarrow L^6(\mathbb{R}^3)$ we get

$$\|w_n^\ell\|_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \leq C \|w_n^\ell\|_{L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))}. \tag{4.71}$$

Interpolation between $L^\infty(\mathbb{R}^+, \dot{H}^{1/2}(\mathbb{R}^3))$ and $L^2(\mathbb{R}^+, \dot{H}^{3/2}(\mathbb{R}^3))$ gives the bound of w_n^ℓ in $L^4(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3))$ and combined with the a-priori estimate of the heat equation we have an estimate uniform in n. We also have an estimate of w_n^ℓ in $L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))$ by $\|\psi_n^\ell\|_{L^3(\mathbb{R}^3)}$ which for large ℓ and n can be made as small as needed. This of course implies an appropriate estimate in $L^4([0,2], \dot{H}^{-1}(\mathbb{R}^3))$ of $\mathbb{P}(w_n^\ell \cdot \nabla w_n^\ell)$.

Estimate (4.68) follows essentially the same pattern. We have

$$\Big\| w_n^\ell \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+, L^2(\mathbb{R}^3))} \leq \Big\| \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+, L^6(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+, L^3(\mathbb{R}^3))} \tag{4.72}$$

and by the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ we have

$$\Big\| \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+,L^6(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^3(\mathbb{R}^3))} \leq C \Big\| \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))} \|w_n^\ell\|_{L^\infty(\mathbb{R}^+,L^3(\mathbb{R}^3))}. \tag{4.73}$$

By interpolation, we have

$$\Big\| \sum_{j\leq \ell} v_n^j \Big\|_{L^4(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))} \leq \Big\| \sum_{j\leq \ell} v_n^j \Big\|^{1/2}_{L^\infty(\mathbb{R}^+,\dot{H}^{1/2}(\mathbb{R}^3))} \Big\| \sum_{j\leq \ell} v_n^j \Big\|^{1/2}_{L^2(\mathbb{R}^+,\dot{H}^{3/2}(\mathbb{R}^3))}. \tag{4.74}$$

Notice that the on the right hand-side we take advantage of $C_{NS} = C_G$. We get

$$\Big\| \sum_{j\leq \ell} v_n^j \Big\|^{1/2}_{L^\infty(\mathbb{R}^+,\dot{H}^{1/2}(\mathbb{R}^3))} \Big\| \sum_{j\leq \ell} v_n^j \Big\|^{1/2}_{L^2(\mathbb{R}^+,\dot{H}^{3/2}(\mathbb{R}^3))} \leq \Big\| \sum_{j\leq \ell} v_n^j \Big\|_{E_\infty}. \tag{4.75}$$

Estimate (4.58) gives us a bound uniform in ℓ and n. From (4.13) applied to the second factor on the right hand-side of (4.73) we obtain (4.68) and thus an estimate of $\mathcal{Q}(\sum_{j\leq\ell} v_n^j, w_n^\ell)$ in $L^2([0,2], \dot{H}^{-1/2}(\mathbb{R}^3))$.

Estimate (4.67) is a direct consequence of (4.25) in Proposition 4.1.4. Notice that in our application, we sum this estimate over $j, k \in \{0, \dots, \ell\}$, $j \neq k$ thus possibly have no control as we increase ℓ. However, once ℓ is fixed to make $\|\psi_n^\ell\|_{L^3(\mathbb{R}^3)}$ small enough, we can consider n large enough so that the sum over $j \neq k$ is still arbitrarily small. Thus we have, for a fixed ℓ

$$\lim_{n\to\infty} \Big\| \sum_{\substack{j\neq k \\ j,k\in\{0,\dots,\ell\}}} \mathcal{Q}(v_n^j, v_k^j) \Big\|_{L^4([0,2],\dot{H}^{-1}(\mathbb{R}^3))} = 0. \tag{4.76}$$

Hence, we obtain

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \|g_n^\ell\|_{L^2([0,2],\dot{H}^{-1/2}(\mathbb{R}^3))} = 0. \tag{4.77}$$

This, combined with the above shown uniform estimate of $\|f_n^\ell\|_{E_2}$ and Proposition 4.1.3 gives us a uniform bound of $\|r_n^\ell\|_{E_2}$ thus also a bound of $\|v_n\|_{E_2}$ uniform in n. Applying the Ladyzhenskaya-Prodi-Serrin condition we see that for n large enough our solution $v_n(t,x)$ is regular up to time $T = 2$ which contradicts the assumption that solutions v_n become singular at the time $T = 1$. This rules out case 2. This concludes the proof. $\square$

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