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Marcel K. Richter and Kam-Chau Wong

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Department of Economics
University of Minnesota
Minneapolis, MN 55455

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Marcel K. Richter

Department of Economics, University of Minnesota, Minnesota MN 55455, U.S.A.

and

Kam-Chau Wong*

Department of Economics, Chinese University of Hong Kong, Shatin, Hong Kong[†]

Abstract: Representing binary ordering relations by numerical functions is a basic problem of the theory of measurement. It has applications in many fields of science, and arises in psychology and economics as the problem of establishing utility functions for preference relations.

We obtain definable utility representations for (both continuous and upper semicontinuous) definable preferences in o-minimal expansions of real closed ordered fields. Such preferences have particular significance for modeling “bounded rationality” [18]. Our proofs are based on geometric theorems for definable sets, and provide new alternatives to the classical tools of separability (Debreu [5], Rader [16]) and metric-completeness (Arrow and Hahn [1]).

The initial application of these ideas in economics was made by Blume and Zame (1992). Our results extend their Theorem 1 in several directions (see Remark 1a below).

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[†]E-mail: kamchauwong@cuhk.hk; phone: (852) 2609-7053; fax: (852) 2603-5805.

1. Introduction

Measurement — the representation of relations by numerical functions, is a basic topic in many sciences ([19], [13]). We address one of its most elementary problems: representation of orderings by order preserving (“utility”) functions. The orderings lie in cartesian products of o-minimal structures, and the utility functions take their values in the same structures.

Since the papers of van den Dries [7], Pillay and Steinhorn [15], and Knight, Pillay, and Steinhorn, [12], the theory of o-minimal structures has grown into a wide ranging generalization of semialgebraic and subanalytic topology (cf. [9], [10]). Its tools have also been found useful in economic theory [2], [18]. To facilitate future applications of o-minimality, this paper provides a basic utility representation for preferences that are definable in the first order languages of o-minimal structures. (Cf. footnote 7.) In [18] we have noted that restricting attention to such preferences is a useful tool in modeling “bounded rationality” of agents.

In o-minimal expansions of the real number field \mathbb{R} , Blume and Zame [2] obtained definable utility functions for a special class of definable preferences. Since their proof was based on the metric method of Arrow and Hahn [1], it required the preferences to be continuous, and the definable consumption sets X to be connected, closed, convex, and bounded-below subsets of the complete metric space \mathbb{R}^n . Basing our proofs on fundamental geometric theorems for definable sets, our main Theorem extends [2]’s result in three directions: i) we allow o-minimal expansions of arbitrary real closed fields⁽¹⁾ (rather than just \mathbb{R}), where the underlying spaces need not be topologically complete; ii) we relax all the technical assumptions on X except connectedness; iii) we allow upper semicontinuous (rather than just continuous) preferences. In other words, we extend [2]’s result to a definable analogue of the classical theorems of Debreu [5] and Rader [16].

Although our theorem parallels the Debreu-Rader theorems, our proofs are necessarily based on different tools. Roughly speaking, while their methods rested on separability, ours rest on definability.⁽²⁾ By separability their utility constructions can use dense countable (hence not necessarily definable, cf. footnote 3) bases of the consumption set; however, in general they fail to give definable functions. By contrast, our base is a set of representatives from the indiffer-

⁽¹⁾ For economic applications of o-minimal real closed field expansions M , see [18].

⁽²⁾ Since real closed ordered fields need not be Archimedean, our spaces need not be separable.

ence classes, which can be decomposed into finitely many preference-monotone paths, and gives definable utility functions.

2. Results

We begin by reviewing some notation from [15], [12], [2], and [18]. Standard notions in first order predicate logic are referred to [11], [3], etc. For surveys of o-minimal structures, we recommend [9] and [10].

Recall (cf. [11], [3]) that for any structure $\mathcal{M} = (M, \dots)$, a set $X \subset M^n$ is *parametrically definable* if there is a first order formula $\phi(x_1, \dots, x_n) \in L(\mathcal{M})$ (the first order predicate language of \mathcal{M} together with names for the elements of M) such that X is the set of all $\bar{b} \in M^n$ satisfying $\phi(\cdot)$ in \mathcal{M} , i.e. $X = \{\bar{b} \in M^n : \mathcal{M} \models \phi[\bar{b}]\}$. For brevity, we drop mention of parameters, and refer simply to “definable.”

An *o-minimal structure* ([15], cf. [7], [12]) is an ordered structure $\mathcal{M} = (M, <, \dots)$ in which every definable (with parameters) subset of M is a finite union of points in M and intervals (a, b) where $a, b \in M \cup \{-\infty, +\infty\}$.⁽³⁾ Our main results apply to those \mathcal{M} that are expansions of real closed ordered field, so $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$.

As usual, we endow M with the interval topology, i.e. intervals (a, b) form a basis; and M^n is given the product topology.

Let $X \subseteq M^n$ be definable. A definable set Y is *open in* (*closed in*) X if $Y = X \cap Y'$ for some open (*closed*) set $Y' \subseteq M^n$.⁽⁴⁾ ⁽⁵⁾

We say X is *definably connected* (cf. [15], [12]) if there do not exist non-empty definable sets Y_1, Y_2 both open in X such that $X = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$.

A function $f : X \rightarrow M$ is *upper semicontinuous* if the upper contour set $f^{-1}([s, +\infty))$ is closed in X for all $s \in M$.

A function $f : X \rightarrow M^m$ is *definable* (cf. [15],[12]) if its graph $\{(x, f(x)) : x \in X\} \subseteq M^{n+m}$ is a definable set.

A *preference* \succsim on X is a reflexive, transitive, and total binary relation on X .⁽⁶⁾

⁽³⁾ Thus in many o-minimal structures (e.g., $(\mathbb{R}, <, \dots)$) all infinite countable sets in M are undefinable.

⁽⁴⁾ By o-minimality it is equivalent that Y' also be definable, as shown in [12], Proposition 2.1.

⁽⁵⁾ Clearly, a definable Y is closed in X if its complement $X \setminus Y$ is open in X .

⁽⁶⁾ As usual, $x \succ x'$ means $\neg x' \succsim x$, and $x \sim x'$ means $x \succsim x' \ \& \ x' \succsim x$.

A preference \succsim on X is *upper semicontinuous* (*lower semicontinuous*) if the weakly-preferred set $\{y \in X : y \succsim x\}$ (weakly-worse set $\{y \in X : y \preceq x\}$) is closed in X for all $x \in X$. And \succsim is *continuous* if it is both upper semicontinuous and lower semicontinuous.

A preference \succsim on X is *linear* if $x \not\sim y$ for all $x, y \in X$. When X is an interval, we say a preference \succsim on X is *constant (increasing) (decreasing)* if for all $x, y \in X$: $x > y$ implies $x \sim y$ ($x \succ y$) ($x \prec y$).

A preference \succsim on X is *definable* (cf. [2], [18]) if the set $\{(x, y) : x \succsim y\} \subseteq M^{2n}$ is definable.⁽⁷⁾

We say a preference \succsim on X is *represented* by a function $u : X \rightarrow M$ if $x \succsim x' \Leftrightarrow u(x) \geq u(x')$ for all $x, x' \in X$.

Our main result is the following definable analogue of the Debreu-Rader theorems ([5], [16]).

Theorem. *Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ be an \mathcal{o} -minimal expansion of a real closed ordered field. Let $X \subseteq M^n$ be a definably-connected definable set, and \succsim be a definable preference on X .*

- a) *If \succsim is upper semicontinuous, then \succsim can be represented by an upper semicontinuous definable function $u : X \rightarrow M$.*
- b) *If \succsim is continuous, then \succsim can be represented by a continuous definable function $u : X \rightarrow M$.*

a) A predecessor of our theorem is [2], Theorem 1, which shows that in an arbitrary \mathcal{o} -minimal expansion of the real number field $(\mathbb{R}, <, +, \cdot, 0, 1, \dots)$, a continuous definable preference \succsim on a definable, closed, convex, below-bounded set $X \subseteq \mathbb{R}^n$ can be represented by a continuous definable utility function $u : X \rightarrow \mathbb{R}$. Our theorem extends it by permitting \mathcal{o} -minimal expansions of arbitrary real closed ordered fields, relaxing all the technical assumptions on X except definable connectedness, and allowing upper semicontinuous preferences (i.e. obtaining Part (b)).

b) Our theorem relaxes the classical separability assumptions used in [16], [5].

c) The assumption of upper semicontinuity is still indispensable, since the lexicographic preference is definable and the classical arguments (cf. [6], pp. 72-73) still apply to show that no utility representation exists.

⁽⁷⁾ The class of definable preferences is wide, containing most preferences economists use in applications, Cobb-Douglas, lexicographic, piecewise linear, semialgebraic, subanalytic, etc. For implications of, and intuition about definability, see [2], [18].

The Theorem is an immediate corollary of the following two propositions.

Proposition 1. Consider any \mathcal{M} , X and \succsim where:

$\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ is an ω -minimal expansion of a real closed ordered field, $X \subseteq M^n$ is a definable set, and \succsim is a definable preference on X . (1)

Then \succsim is represented by some definable utility function $u : X \rightarrow M$.

Proposition 2. Let \mathcal{M} , X and \succsim satisfy (1). Let X be definably connected and let \succsim be represented by some definable function $U : X \rightarrow M$.

- a) If \succsim is upper semicontinuous, then \succsim is represented by some continuous function $u : X \rightarrow M$.
- b) If \succsim is continuous, then \succsim is represented by some continuous definable function $u : X \rightarrow M$.

Notice that connectedness is not required in Proposition 1 for obtaining definable representations. It is used only in Proposition 2 for obtaining continuity and semicontinuity for the representations.

The rest of this paper is devoted to proving Propositions 1 and 2.

3. Proofs

The mathematical background given in Section 2 is sufficient for proving Proposition 2. For the proof of Proposition 1, we require several basic theorems (e.g. cell-decomposition) in ω -minimal structures, which are collected in Theorems A.1-3 and Proposition A.1 in the Appendix.

Proof of Proposition 2. (Part b) If the representation U is not continuous, then we transform it along the lines of ([2], proof of Theorem 1, paragraph 3), as follows. By ω -minimality,

the definable set $U(X) \subseteq M$ is a finite union of disjoint intervals I_i whose left and right boundary points a_i and b_i are in $M \cup \{-\infty, +\infty\}$. (2)

The continuity of \succsim implies that each pair (I_i, I_{i+1}) corresponds to neither a jump nor a gap (i.e. either $(b_i \in I_i \ \& \ a_{i+1} \notin I_{i+1})$ or $(b_i \notin I_i \ \& \ a_{i+1} \in I_{i+1})$ holds); otherwise by carrying over standard arguments (cf. [17], proof of Theorem 1)) to our definable context it is easy to show that continuity of the definable \succsim would separate X into two disjoint definable open sets, contradicting the definable connectedness of X . Therefore, we can shift the disjoint intervals I_i , pasting

them together into a single interval I , so we can transform U to a definable representation u from X onto an interval I .⁽⁸⁾ As usual, continuity of u follows easily, since $u(X) = I$ and \succ is continuous.

(Part a) If the representation U is not upper semicontinuous, we modify it along the above lines, as follows. First, let $I_1, a_1, b_1, \dots, I_n, a_n, b_n$ be as given in (2). Replacing the definable analogue of [17] by a definable analogue of [16], we see that the upper semicontinuity of the definable \succ ensures for every $c \in A$, if the pre-image $U^{-1}([c, \infty))$ is not closed, then there is some i with $c \in (b_i, a_{i+1}]$ & $b_i \in I_i$ & $a_{i+1} \notin I_{i+1}$. Again by shifting, we can past together all such consecutive intervals I_i and I_{i+1} ; this gives a definable representation U that has such $U^{-1}[c, \infty)$ closed for all $c \in M$, so is upper semicontinuous. Q.E.D.

It remains to prove Proposition 1. We will apply extensively the techniques of decomposing a definable set into finitely many cells. The notion of a cell is standard (cf. [4], [7],[15],[12]), defined as follows:

If $X = \{a\}$, where $a \in M$, then X is a cell, and $\dim(X) = 0$.

If X is an interval (a, b) , where $a, b \in M \cup \{+\infty, -\infty\}$, then X is a cell, and $\dim(X) = 1$,

If $Y \subseteq M^n$ is a cell and $\dim(Y) = k$, then:

if $f : Y \rightarrow M$ is definable and continuous, then the set $X = \{(y, f(y)) : y \in Y\} \subseteq M^{n+1}$ is a cell, and $\dim(X) = k$;

if $f_1, f_2 : Y \rightarrow M$ are definable and continuous, and such that $f_1(y) < f_2(y)$ for all $y \in Y$, then the set $X = \{(y, s) : y \in Y \text{ \& } f_1(y) < s < f_2(y)\}$ is a cell, and $\dim(X) = k + 1$.

Nothing else is a cell of dimension k or $k + 1$ on M^n .

Remark 2.

a) By induction on n , it is easy to show that if a cell $X \subseteq M^n$ has $\dim(X) = 1$, then there exists an interval (a, b) and a definable, continuous, and one-to-one function f mapping (a, b) onto X .

b) More generally, for all n, k with $k \leq n$, if a cell $X \subseteq M^n$ has $\dim(X) = k$, then there exists a cell $Y \subseteq M^k$ with $\dim(Y) = k$, and there exists a definable, continuous, and one-to-one function f mapping Y onto X .

c) For every densely ordered o-minimal structure $\mathcal{M} = (M, <, \dots)$, every

⁽⁸⁾ E.g. we can define $u : X \rightarrow M$ by $u(y) = U(y)$ for $y \in U^{-1}(I_0)$, and $u(y) = U(y) - \sum_{j=1}^{i-1} a_{j+1} - b_j$ for $y \in U^{-1}(I_i)$ & $i = 1, \dots, m$.

cell in M^n is definably connected (cf. [12], Proposition 2.4).

We will also use extensively the notion of a preference-jump point, which is defined as follows. For any X , any preference \succsim on X , and any $K \subseteq X$, we say x is a (\succsim, X, K) -jump point if $x \in K$ and x has a separating point $y \in X$ (i.e. $x \succ y$ and x is in the boundary of the weakly worse set $\{z \in K : y \succsim z\}$). Clearly, if $K \subseteq Y \subseteq X$, then a $(\succsim|_Y, Y, K)$ -jump point is also a (\succsim, X, K) -jump point. Also, if there are no (\succsim, X, X) -jump points, then \succsim is continuous.

Proof of Proposition 1. Let \mathcal{M} , M and \succsim satisfy (1).

First, by the Definable Skolem Function Theorem (Thm. A.2), we can take a definable set $Y \subseteq X$ of representatives; i.e. $Y \subseteq X$ and for all $x \in X$, there exists exactly one $y \in Y$ with $x \sim y$. Then \succsim is linear on Y .

Next, by the Cell Decomposition Theorem (Thm. A.3), we can decompose the definable set Y into disjoint cells C_1, \dots, C_n . Since $\succsim|_Y$ is definable, linear, and upper semicontinuous, Lemma 2 below ensures that $\dim(C_i) < 2$ for all i . Then Lemma 1 below ensures that each C_i has at most only finitely many $(\succsim|_Y, Y, C_i)$ -jump points. By separating out these jump points, we can assume that for each C_i :

$$\begin{array}{ll} \text{either} & \text{a) } \dim(C_i) = 0 \text{ (i.e. } C_i \text{ is a singleton)} \\ \text{or} & \text{b) } \dim(C_i) = 1 \text{ and } C_i \text{ has no } (\succsim|_Y, Y, C_i)\text{-jump points.} \end{array} \quad (3)$$

Then we order the cells C_i as follows. For any distinct cells C_i, C_j , we define $C_i \succ C_j$ if there exists $x \in C_i$ and $y \in C_j$ such that $x \succ y$. Since \succsim is linear on $Y = \cup C_i$, all pairs of cells are related by \succ . We now show that

$$C_i \succ C_j \Rightarrow \neg C_j \succ C_i. \quad (4)$$

To prove (4), by linearity of $\succsim|_Y$ it suffices to show that it is impossible to have some $x, z \in C_i$ and $y \in C_j$ with $x \succ y \succ z$. Suppose by contradiction that such x, y, z exist. Then C_i is not a singleton, so C_i satisfies (3b). By applying Remark 2a, we pick a continuous definable function g from the interval $[0, 1]$ into the 1-dimensional cell C_i such that $g(0) = z$ & $g(1) = x$. Notice that for all $t \in [0, 1]$ we have $g(t) \neq y$, so $g(t) \not\sim y$ (by linearity of $\succsim|_Y$). The set $\{t \in [0, 1] : g(t) \succ y\}$ is definable, so it is a finite union of intervals and has an infimum $\bar{t} \in [0, 1]$. By continuity of g and upper semicontinuity of \succsim we have $g(\bar{t}) \succsim y$, so $g(\bar{t}) \succ y$ and $\bar{t} > 0$. For $\bar{t} > t \in [0, 1]$ we have $y \succ g(t)$; so as $t \nearrow \bar{t}$, we have $g(t) \rightarrow g(\bar{t}) \succ y \succ g(t)$, so $g(\bar{t})$ is a $(\succsim|_Y, Y, C_i)$ -jump point, contradicting (3b).

Now by (4), to obtain a definable utility representation U on the finite union Y of disjoint C_i , it suffices to obtain a definable utility representation for each

$\succsim|_{C_i}$. By Remark 2a, for each 1-dimensional cell C_i , we can pick an interval (a_i, b_i) , a continuous definable bijection $f_i : (a_i, b_i) \rightarrow C_i$. By (3b) C_i has no $(\succsim|_{C_i}, C_i, C_i)$ -jump points, so the preference $\succsim|_{C_i}$ is continuous on C_i . Let the definable preference \succsim_i on (a_i, b_i) be defined by $x \succsim_i y \Leftrightarrow f_i(x) \succsim f_i(y)$. Then \succsim_i is continuous and linear; so by Lemma 3, \succsim_i is either increasing or decreasing. Consequently each $\succsim|_{C_i}$ can be represented either by the definable function $f_i^{-1}(x)$ or the function $-f_i^{-1}(x)$. So we can find a definable representation $U : Y \rightarrow M$ of $\succsim|_Y$.

Finally, it is easy to extend U to all of X ; e.g., for each $x \in X$ define $u(x) = U(y)$, where y is the unique $y \in Y$ with $x \sim y$. Q.E.D.

Lemma 1. *Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ be an ω -minimal expansion of a real closed ordered field. Let $Y \subseteq M^n$ be a definable set, and \succsim be a linear, upper semicontinuous, and definable preference on Y . Then for every cell $C \subseteq Y$ with $\dim(C) < 2$, there exist at most finitely many (\succsim, Y, C) -jump points.*

PROOF. Suppose not. Then there is a 1-dimensional cell $C \subseteq Y$ containing infinitely many (\succsim, Y, C) -jump points. By the Cell Decomposition Theorem (Thm. A.3), we can obtain a 1-dimensional cell $D \subseteq C$ containing only (\succsim, Y, C) -jump (hence (\succsim, Y, D) -jump) points. Since D is 1-dimensional, we can identify it with an interval (a, b) . Taking a subinterval if necessary, by the Monotone Preference Proposition (Prop. A.1) we can assume \succsim is either increasing or decreasing on the interval $D = (a, b)$. Since the argument is identical for either alternative, we assume \succsim is increasing. By the Definable Skolem Function Theorem (Thm. A.2), we can pick a definable function $f : D \rightarrow Y$ that assigns to each $x \in D$ a separating point $f(x) \in Y$, so by increasingness of $\succsim|_D$ we have: $x \succ f(x) \succ z$ for $x < z \in D$. By picking a subinterval if necessary, by the Cell Decomposition Theorem again we can assume f is continuous. Now let $z \in D$, $x \in D$ and $x \searrow z$, then we have $f(x) \succ z \succ f(z) \leftarrow f(x)$, contradicting the upper semicontinuity of \succsim . Q.E.D.

Lemma 2. *Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ be an ω -minimal expansion of a real closed ordered field. Let $Y \subseteq M^n$ be a definable set, and let \succsim be a linear, upper semicontinuous and definable preference on Y . Then each cell $C \subseteq Y$ has $\dim(C) < 2$.*

PROOF. Suppose not. Then replacing Y by any 2-dimensional cell $C' \subseteq Y$, we can assume Y itself is a 2-dimensional cell. For by Remark 2b, $Y = f(X)$ for some 2-dimensional cell $X \subseteq M^2$ and some continuous and definable bijection $f : X \rightarrow Y$. Then f and \succsim together clearly induce an upper semicontinuous,

linear, and definable preference on $X \subseteq M^2$. Therefore, we can pretend that the two dimensional cell $Y \subseteq M^2$, and from that we will derive a contradiction.

Let X_1 be the set of $(a, b) \in Y$ such that there exists some open interval $I \ni a$ with $I \times \{b\} \subset Y$ and $I \times \{b\}$ has no $(\succ, Y, I \times \{b\})$ -jump points. Similarly, let X_2 be the set of $(a, b) \in Y$ such that there exists some open interval $I' \ni b$ with $\{a\} \times I' \subset Y$ and⁽⁹⁾

$$\begin{aligned} \{a\} \times I' \text{ has no } (\succ, Y, \{a\} \times I')\text{-jump points, and } \succ|_{\{a\} \times I'} \\ \text{is constant, increasing, or decreasing on the } I'. \end{aligned} \quad (5)$$

By the Cell Decomposition Theorem (Thm. A.3) we can pick a finite decomposition \mathcal{P} of cells that partitions X_1 and X_2 . We will show that:

$$\begin{aligned} \text{any 2-dimensional (hence open) cell } \tilde{C} \in \mathcal{P} \text{ has no} \\ (\succ, Y, \tilde{C})\text{-jump points.} \end{aligned} \quad (6)$$

Before that, we derive a contradiction from (6). First, since Y is 2-dimensional, we can pick a 2-dimensional $\tilde{C} \in \mathcal{P}$, and by (6) the linear relation $\succ|_{\tilde{C}}$ is continuous. We can pick any $x, y, z \in \tilde{C}$ with $x \succ y \succ z$, and pick any continuous definable function $g : [0, 1] \rightarrow \tilde{C}$ avoiding y (i.e. $y \notin g([0, 1])$) such that $g(0) = z$ & $g(1) = x$. As in the above proof of (4), we define $\bar{t} = \inf\{t \in [0, 1] : g(t) \succ y\}$, so $\bar{t} > 0$; and let $t \in [0, 1]$ and $t \nearrow \bar{t}$, then $g(t) \prec y \prec g(\bar{t}) \leftarrow g(t)$, contradicting the continuity of $\succ|_{\tilde{C}}$.

To prove (6), we will follow the lines of [12], pp. 602-603, proof of Proposition 5.3. We consider any 2-dimensional cell $\tilde{C} \in \mathcal{P}$. Since $\tilde{C} \subseteq Y \subseteq M^2$ has $\dim(\tilde{C}) = 2$, we can pick an open interval $\tilde{I} \subseteq M$, and continuous definable functions $f_1, f_2 : \tilde{I} \rightarrow M$ with $f_1 < f_2$ and $\tilde{C} = \{(a, b) : a \in \tilde{I} \text{ \& } f_1(a) < b < f_2(a)\}$. We claim:

- i) $\tilde{C} \subseteq X_1$,
- ii) $\tilde{C} \subseteq X_2$, and for every $a \in \tilde{I}$, the interval $I' = (f_1(a), f_2(a))$ satisfies (5). (7)

To prove (6) from (7), we consider any point $(\bar{a}, \bar{b}) \in \tilde{C}$. To show that (\bar{a}, \bar{b}) is not a (\succ, Y, \tilde{C}) -jump point, we assume $z \in Y$ with $(\bar{a}, \bar{b}) \succ z$. It suffices to find an open box $B \subseteq \tilde{C}$ with $(\bar{a}, \bar{b}) \in B$ and $B \succ z$.⁽¹⁰⁾ First, by (7ii) we can

⁽⁹⁾ Of course, in (5) by $\succ|_{\{a\} \times I'}$ being constant (increasing, decreasing) on I' we mean that the induced preference $\tilde{\succ}$ defined by $b \tilde{\succ} b' \Leftrightarrow (a, b) \succ (a, b')$ is constant (increasing, decreasing) on I' .

⁽¹⁰⁾ For any set K , we write $K \succ z$ to mean $x \succ z$ for all $x \in K$.

pick an interval $[b_1, b_2] \subseteq (f_1(\bar{a}), f_2(\bar{a}))$ with $b_1 < \bar{b} < b_2$ and $\{\bar{a}\} \times [b_1, b_2] \succ z$. By (7i) we can pick open intervals $I_1, I_2 \ni \bar{a}$ such that

$$I_1 \times \{b_1\}, I_2 \times \{b_2\} \subset \tilde{C}, \text{ and } I_1 \times \{b_1\}, I_2 \times \{b_2\} \succ z. \quad (8)$$

Now we can pick any open box $B = I \times I' \subseteq \tilde{C}$ of open intervals I, I' such $\bar{a} \in I \subseteq I_1 \cap I_2$ and $\bar{b} \in I' \subseteq [b_1, b_2]$. Then for any $(a', b') \in B = I \times I'$, by (8) we have $(a', b_1), (a', b_2) \succ z$, so by (7ii) we have $(a', b') \succ z$ (since $b_1 < b' < b_2$).

It remains to prove (7). To see (7i), we consider any $(a, b) \in \tilde{C}$. Then we can pick an open interval I with $I \times \{b\} \subset \tilde{C}$. By Lemma 1, we can pick an open interval $\hat{I} \subseteq I$ so that $\hat{I} \times \{b\}$ has no (\succ, Y, \hat{I}) -jump points. Let $a' \in \hat{I}$; then $(a', b) \in X_1$. Recall that \mathcal{P} partitions X_1 and $\tilde{C} \in \mathcal{P}$, so we have $\tilde{C} \subseteq X_1$, i.e. (7i) holds. Similarly, it follows from Prop. A.1 and Lemma 1 that $\tilde{C} \subseteq X_2$. Now let $a \in \tilde{I}$. Again by Lemma 1 and Prop. A.1, there is an m such that there exist $b_0, \dots, b_m \in M$ with $f_1(a) = b_0 < b_1 < \dots < b_m = f_2(a)$ and all the intervals $I'_i = (b_i, b_{i+1})$ satisfy property (5) with $I' = I'_i$. If the least of such m is 1, then we are done. Otherwise, we see that every open interval $I \ni b_1$ cannot satisfy (5), contradicting that the point $(a, b_1) \in \tilde{C} \subseteq X_2$. This proves (7ii), completing the proof of Lemma 2. Q.E.D.

Lemma 3. *Let $\mathcal{M} = (M, <, \dots)$ be a densely ordered o-minimal structure. Let $I = (a, b)$ be an interval in M , and let \succ be a definable preference on (a, b) . If \succ is linear and continuous, then \succ is either increasing or decreasing.*

PROOF. Suppose \succ is neither increasing nor decreasing. Then by linearity, there exist elements $w < u < v < z \in I$ such that either $w \succ u$ & $z \succ v$, or $u \succ w$ & $v \succ z$. Since the argument is similar in either alternative, we assume $u \succ w$ & $v \succ z$. By linearity of \succ , either $w \succ z$ or $z \succ w$. Since the argument is similar in either alternative, we assume $w \succ z$. Then we can pick an $\bar{x} \in (w, z)$ with $\bar{x} \succ u$ & $\bar{x} \succ v$ (e.g., pick the appropriate $\bar{x} \in \{u, v\}$), so $\bar{x} \succ w \succ z$. By continuity, the nonempty definable sets $U = \{x \in [\bar{x}, z] : x \succ w\}$ and $W = \{x \in [\bar{x}, z] : w \succ x\}$ are closed; since the interval $[\bar{x}, z]$ is definably connected, there exist some $x' \in U \cap W$, so $x' \neq w$ (since $x' \in [\bar{x}, z] \not\prec w$) and $x' \sim w$, contradicting the linearity of \succ . Q.E.D.

Appendix: basic tools in o-minimal structures

Here we discuss four tools in o-minimal structures $\mathcal{M} = (M, \dots)$. Theorem A.1 and Proposition A.1 apply to all o-minimal structures, Theorem A.2 applies

to \mathcal{o} -minimal \mathcal{M} where the order $<$ is dense,⁽¹¹⁾ and Theorem A.2 applies to \mathcal{o} -minimal expansions of real closed ordered fields.

The following theorem asserts that every definable unary function is piecewise constant or strictly monotone.

Theorem A.1 (Monotone Function Restriction [15]). *Let $\mathcal{M} = (M, <, \dots)$ be an \mathcal{o} -minimal structure. Then for any interval (a, b) in M , and any definable function $f : X \rightarrow M$, there are points $a = a_0 < a_1 < \dots < a_n = b$ such that on each interval (a_i, a_{i+1}) , the function f is either constant (i.e. $f(x) = f(x')$ for all $x, x' \in (a_i, a_{i+1})$) or an isomorphism (i.e. $f|_{(a_i, a_{i+1})}$ is either an order-preserving or order-reversing mapping from (a_i, a_{i+1}) onto some interval (c, d) in M).⁽¹²⁾*

Extending Theorem A.1 to definable preferences, we obtain the following proposition.

Proposition A.1 (Monotone Preference Restriction). *Let $\mathcal{M} = (M, <, \dots)$ be an \mathcal{o} -minimal structure. Then for any interval (a, b) in M , and any definable preference \succsim on (a, b) , there are points $a = a_0 < a_1 < \dots < a_n = b$ such that on each interval (a_i, a_{i+1}) , the preference \succsim is constant, increasing, or decreasing.*

The next theorem asserts definable Skolem functions for definable sets.

Theorem A.2 (Skolem Functions [8]). *Let $\mathcal{M} = (M, <, +, \cdot, 0, 1, \dots)$ be an \mathcal{o} -minimal expansion of a real closed ordered field. Then for any definable set $X \subseteq M^{n+m}$, there exists a definable Skolem function $f : M^n \rightarrow M^m$, i.e. for all $x \in M^n$, if there is an $y \in M^m$ with $(x, y) \in X$, then $(x, f(x)) \in X$.*

The next theorem decomposes the domain of any definable function into finitely many cells C_1, \dots, C_n , where on each cell the function is continuous. Moreover, given any finite family of definable sets Y_j , the decomposition $\{C_i\}_{i=1}^n$ can be chosen so that it partitions each Y_j .

Theorem A.3 (Cell Decomposition [12]). *Let $\mathcal{M} = (M, <, \dots)$ be an \mathcal{o} -minimal structure where $<$ is a dense order. Then for any definable set $X \subseteq M^n$,*

⁽¹¹⁾ I.e. for any $x, z \in M$ with $x < z$, there is a $y \in M$ with $x < y < z$. Of course the order of an ordered field is dense.

⁽¹²⁾ Notice that when $f|_{(a_i, a_{i+1})}$ is an isomorphism, then it and its (definable) inverse is also continuous.

and any definable function $f : X \rightarrow M^m$, there exists a finite family of disjoint cells $C_1, \dots, C_n \subseteq M^n$ such that $M = \cup_{i=1}^n C_i$ and $f|_{C_i}$ is continuous for all i . Moreover, given any finite family $\{Y_j\}$ of definable sets the decomposition $\{C_i\}$ can be chosen so that it partitions each Y_j i.e. $C_i \subseteq Y_j$ or $C_i \cap Y_j = \emptyset$ for all C_i .

Theorem A.1 is proved in [15], Theorem 4.2, and Proposition A.1 follows from easy modifications of those arguments.⁽¹³⁾ Theorem A.2 is proved in [8]; see also [14], Proposition 2.14. Theorem A.3 is a restatement of facts (3.5) and (3.6) in [12], p. 598.

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⁽¹³⁾ E.g. replacing inequality " $f(x) > f(y)$ " by " $x > y$," replacing " $(\exists t)[f(s) < t]$ " (see the fourth display in [15], p. 581) by " $(\exists t)[s < t]$."

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