Continuous-Time Portfolio Management:
Minimizing the Expected Time to Reach a Goal

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1. Introduction.

In 1956, Kelly [5] introduced a plan for portfolio management based on the criterion of maximizing, at each stage, the expected value of the logarithm of the portfolio value. This "Kelly criterion" was further studied by Breiman [3] in 1961, who established certain asymptotic optimality properties of it. More recently, Bell and Cover [2] constructed a natural model in which the Kelly criterion yields optimal performance in a single period.

The above results are all for discrete-time models. We consider here a continuous-time version in which the stochastic process $\{X_t, t \geq 0\}$, where X_t represents the fortune of an investor at time t, has continuous sample paths. If the object is to reach a fixed goal in minimum expected time, there are problems for discrete-time models of overshooting the goal when the Kelly criterion is followed, and thus it is not optimal in general. However, there are no overshoot problems for our continuous models and an infinitesimal form of the Kelly criterion is proved optimal in section 4 below.

The continuous-time portfolio selection problem is formulated in the next section. Its solution uses results on continuous-time stochastic control from Pestien and Sudderth [7]. These results are explained in section 3 and are then applied to solve the portfolio selection problem in section 4.

2. Portfolio selection.

Consider the problem of managing a portfolio consisting of stocks, bonds and cash to minimize the expected time to reach a given total worth. More precisely, consider a gambler, whose fortune at time t is denoted by X_t, who can choose gambles (make investments) from a set of gambles depending in a linear way on his current fortune. Suppose that trading costs are negligible, and that the trader can buy or sell securities at the same price. In addition, suppose that the use of margin and short sales is not allowed.

For a simple model, suppose that there is one stock and one bond and that the price of the stock at time t, S_t, is governed by a stochastic differential equation:

$$dS_t = \sigma_S S_t dW_t + r_S S_t dt,$$

where W is a standard Brownian motion, while the price of the bond at time t, B_t, satisfies:

$$dB_t = r_B B_t dt.$$

(An accessible introduction to stochastic differential equations is provided by Arnold's book [1]. A general and up-to-date presentation is in Ikeda and Watanabe [4]. A recent paper by Malliaris [6] explains the use of stochastic differential models in finance, and has numerous references to the financial literature.)

If the gambler invests amounts A_t^S and A_t^B in stocks and bonds, respectively, then:

$$dX_t = A_t^S (\sigma_S dW_t + r_S dt) + A_t^B (r_B dt)$$

where the A's must be non-negative and

$$A_t^S + A_t^B \le X_t.$$

Of course we must also require that the gambler choose his gambles without knowledge of the future. To do this we require that A_t^S and A_t^B be non-anticipative with respect to the filtration of the Brownian motion.

Suppose that the gambler has a fixed goal for his fortune, which without loss of generality we take to be 1. Let T denote the first time at which $X_t = 1$; we seek to find the A's which minimize $E[T]$. We shall show that there are optimal A's which are fixed fraction policies, i.e., for which $A_t^S = f_S X_t$ and $A_t^B = f_B X_t$. Moreover, we shall find that the expected time to reach the goal as a function of initial position, x, is of the form $c \log(x)$ and that the Kelly criterion (when interpreted infinitesimally) produces optimal policies.

It should be clear that these techniques apply (almost symbol-for-symbol) to the case of many stocks if one regards r_S as a vector, σ_S a matrix, W as a vector-valued Brownian motion, and A^S as a vector. Moreover, the results apply equally to models which allow short sales and margin.

3. Continuous-time stochastic control.

The terminology and the exposition of stochastic control in this section are borrowed from Pestien and Sudderth [7].

A continuous-time gambling problem is a triple (F,Σ,u) where

(3.1) the state space F is Polish (we shall use a Borel subset of ordinary Euclidean space),

(3.2) the gambling house Σ is a mapping which assigns to each $x \in F$ a non-empty collection of processes $X = \{X_t, t \geq 0\}$ with state space F such that $X_0 = x$ and X has right-continuous paths with left-limits,

(3.3) the utility function u is a Borel function from F to the real line.

A process $X \in \Sigma(x)$ is said to be available at x. Each available X is defined on some probability space $(\Omega,\mathcal{F},P)$ and is adapted to an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete sub-sigma fields of $\mathcal{F}$. The probability space and filtration may depend on X.

A player, starting at position $x \in F$, selects a process $X \in \Sigma(x)$ and receives payoff $u(X)$ defined by

$$(3.4) \qquad u(X) = E[\limsup_{t\to\infty} u(X_t)].$$

The expectation occurring on the right is assumed to be well-defined for every available process X.

The <u>value function</u> V is defined by

$$V(x) = \sup\{u(X): X \in \Sigma(x)\}$$

for every $x \in F$. A process $X \in \Sigma(x)$ is <u>optimal</u> at x if

$$u(X) = V(x).$$

From now on we shall require that F be a Borel subset of the Euclidean space $\mathbb{R}^d$ having non-empty interior, and each process $X = \{X_t\}$ under consideration will be an <u>Ito process</u> of the form

$$(3.5) \qquad X_t = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW_s$$

where $W = \{W_t\}$ is a standard m-dimensional Brownian motion process on $(\Omega,\mathcal{F},P)$ adapted to $\{\mathcal{F}_t\}$, and $\mathcal{F}_t$ is independent of $\{W_{t+s}-W_t, s \geq 0\}$. The function $\alpha = \alpha(t,\omega)$ is to be $\mathbb{R}^d$-valued, jointly measurable, adapted to $\{\mathcal{F}_t\}$ and such that

$$(3.6) \qquad \int_0^t |\alpha(s)|ds < \infty \text{ a.s. for all } t.$$

The function $\beta = \beta(t,\omega)$ has as values real dxm matrices, is jointly measurable, adapted to $\{\mathcal{F}_t\}$, and satisfies

$$(3.7) \qquad E\int_0^t |\beta(s)|^2 ds < \infty \text{ for all } t.$$

For each pair (a,b), where $a \in \mathbb{R}^d$ is a dx1 vector and b is a dxm real-valued matrix, define the differential operator $D(a,b)$ for sufficiently smooth functions $Q:\mathbb{R}^d \to \mathbb{R}$ by

$$D(a,b)Q(y) = Q_x(y)a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{x_i x_j}(y)(bb')_{ij}$$

where

$$Q_x(y) = (\frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial Q}{\partial x_d}),$$

$$Q_{x_i x_j} = \frac{\partial^2 Q}{\partial x_i \partial x_j},$$

and b' is the transpose of b.

We now specify $\Sigma(x)$ by specifying the possible values of α and β. To this end, let $C(x)$ be, for each $x \in F$, a non-empty set of pairs (a,b), where $a \in \mathbb{R}^d$ and b is a real $d \times m$ matrix. (The idea is that $C(x)$ is the set from which a player at state x may choose the value of (α,β).) Assume also that every available process X is absorbed at the time T_X of its first exit from F^o, the interior of F. These conditions define a function Σ_C on F where $\Sigma_C(x)$ is the collection of all processes X having paths in F and satisfying (3.5), (3.6), and (3.7) together with

(3.8) $$(\alpha(t,\omega),\beta(t,\omega)) \in C(X_t(\omega)) \quad \text{for all } (t,\omega),$$

(3.9) $$(\alpha(t,\omega),\beta(t,\omega)) = (0,0) \quad \text{for } t \geq T_X(\omega),$$

(3.10) $$C(x) = \{(0,0)\} \quad \text{for } x \in F - F^o.$$

In the following lemma, which follows directly from lemmas 1 and 2 of [7], G is assumed to be an open set in $\mathbb{R}^d$ which contains F and $\Sigma \subset \Sigma_C$.

Lemma. Suppose $Q:G \to \mathbb{R}$ has continuous second-order derivatives, and that for every available X, $Q(X)$ is well-defined and $Q(X) \geq u(X)$. Assume the following conditions for every $x \in F^o$ and every $X \in \Sigma(x)$:

(i) $D(a,b)Q(x) \leq 0$ for all $(a,b) \in C(x)$,

(ii) $E\int_0^t |Q_x(X_s)\beta(s)|^2 ds < \infty$ for all $t \geq 0$,

(iii) there is an integrable random variable Y such that $Q(X_t) \geq E[Y|\mathcal{F}_t]$ for all $t \geq 0$.

Then $Q \geq V$.

4. Application to portfolio selection.

The problem of section 2 can be formulated as a continuous-time gambling problem in $\mathbb{R}^2$. The first coordinate, x_1, of a state vector x will correspond to portfolio value while the second coordinate, x_2, will represent time. It is convenient to allow negative as well as positive times and define:

(4.1) $F = \{x \in \mathbb{R}^2 : 0 < x_1 \leq 1\}$

Because the object is to minimize expected time, let

(4.2) $u(x) = -x_2$

(4.3) To define Σ, first let $\mathcal{S} \subset \mathbb{R} \times [0,\infty)$ and set

$$C(x) = \left\{ \left[\begin{bmatrix} x_1\mu \\ 1 \end{bmatrix}, \begin{bmatrix} x_1\sigma \\ 0 \end{bmatrix} \right] : (\mu,\sigma) \in \mathcal{S} \right\} \quad \text{if } x_1 < 1,$$

$$= \left\{ \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] \right\} \quad \text{if } x_1 = 1.$$

(For the application to the problem of section 2, we will later take

(4.4) $\mathcal{S} = \{(\mu,\sigma) : \mu = f_S r_S + f_B r_B,\ \sigma = f_S \sigma_S$ such that $f_S \geq 0$, $f_B \geq 0$ and $f_B + f_S \leq 1\}$.)

Every $X \in \Sigma_C(x)$ can be specified by a stochastic differential equation:

$$X(0) = x \tag{4.5}$$

$$dX_t = \begin{bmatrix} X_1(t)\ (\mu(t)dt + \sigma(t)dW_t) \\ 1_{[T>t]}\ dt \end{bmatrix}$$

where μ and σ are non-anticipating and

$$(\mu(t),\sigma(t)) \in \mathscr{S}. \tag{4.6}$$

Notice that for every $X \in \Sigma_C(x)$ the second coordinate process $\{X_2(t)\}$ increases at rate 1 up to time T where

$$T = \inf\{t \geq 0 : X_1(t) = 1\}, \tag{4.7}$$

and, by (3.4) and (4.2)

$$u(X) = -x_2 - ET. \tag{4.8}$$

Now let

$$\Sigma(x) = \{X \in \Sigma_C(x) : u(X) > -\infty\}$$
$$= \{X \in \Sigma_C(x) : ET < \infty\}.$$

To formulate a continuous-time Kelly criterion, consider the process $\{\log(X_1(t))\}$ corresponding to the log of the portfolio value at time t. A simple calculation based on Ito's lemma [1,4] shows that the infinitesimal mean of this process is $\mu(t) - \sigma^2(t)/2$. Thus one should choose μ and σ to achieve

$$M = \sup\{\mu - \sigma^2/2 : (\mu,\sigma) \in \mathscr{S}\}.$$

Call the problem <u>monotone</u> if $C(x) \subseteq C(y)$ for $0 < x_1 \leq y_1 < 1$. (This is certainly true if $\mathscr{S}$ is given by (4.4).)

<u>Theorem.</u> Suppose the problem is monotone and $0 < M < \infty$. Then

$$V(x) = \frac{\log(x_1)}{M} - x_2.$$

If $M = \mu_0 - \sigma_0^2/2$ for some $(\mu_0,\sigma_0) \in \mathcal{S}$, then the process for which $\mu(t) \equiv \mu_0$, $\sigma(t) \equiv \sigma_0$ is optimal.

Theorem 4 of [7] corresponds to the special case where $C(x)$ is a constant set.

Proof: Set $Q(x) = \frac{\log(x_1)}{M} - x_2$. The hard part is showing $Q \geq V$.

The opposite inequality and the statement of optimality follow from the fact that, if $\mu(t) \equiv \mu$ and $\sigma(t) \equiv \sigma$ and $\mu - \sigma^2/2 > 0$ then $ET = \frac{\log(x_1)}{\mu - \sigma^2/2}$ (see [7] for details).

To show $Q \geq V$:

Step 1. For $x \in F$ and

$$(a,b) = \left[\begin{bmatrix} x_1\mu \\ 1 \end{bmatrix}, \begin{bmatrix} x_1\sigma \\ 0 \end{bmatrix} \right] \in C(x),$$

$D(a,b)\, Q(x) \leq 0$.

Proof: $D(a,b)\, Q(x) =$

$$\left[\frac{1}{x_1 M}, -1 \right] \begin{bmatrix} x_1\mu \\ 1 \end{bmatrix} - \frac{1}{2} \frac{x_1^2 \sigma^2}{x_1^2 M} =$$

$$\frac{\mu - \sigma^2/2}{M} - 1 \quad .$$

which is non-positive by the definition of M.□

Step 2. $Q(X) \geq u(X)$ for all available X.

Proof: $Q(X) = E[\limsup_{t \to \infty} (\frac{\log X_1(t)}{M} - X_2(t))]$

$$= E[\ (\log(1))/M - T - x_2\ ]$$

$$= -ET - x_2$$

$$= u(X).\quad \square$$

We could now apply the lemma of section 3 if we could check conditions (ii) and (iii). Not knowing how to do this directly, we shall take another route.

For $0 < \varepsilon < 1$, introduce a new problem $(F_\varepsilon, \Sigma_\varepsilon, u_\varepsilon)$ where

(4.9) $F_\varepsilon = \{x \in F : x_1 \geq \varepsilon\}$,

(4.10) $\Sigma_\varepsilon(x) = \{X \in \Sigma(x) : \inf_{t \geq 0} X_1(t) \geq \varepsilon\}$ for $x \in F$,

(4.11) $u_\varepsilon(x) = u(x)$ for $x \in F$. (We'll write u rather than u_ε from now on.)

Let V_ε be the value function for this new problem. It follows from the lemma that

$$Q(x) \geq V_\varepsilon(x) \tag{4.12}$$

for $x \in F_\varepsilon$. (Condition (i) is a consequence of step 1 above. Condition (ii) follows from (3.7) and the boundedness of Q_x on F_ε. To verify (iii), notice

$$Q(X(t)) = \frac{\log(X_1(t))}{M} - X_2(t)$$

$$\geq \frac{\varepsilon}{M} - x_2 - T.$$

Now use step 1.)

The next step is to check the inequality

$$(4.13) \qquad V_\varepsilon(x^\varepsilon) \geq V(x)$$

where $x^\varepsilon = \begin{bmatrix} x_1+\varepsilon \\ x_2 \end{bmatrix}$ and $x_1 + \varepsilon < 1$.

To see this, define, for each $X \in \Sigma(x)$, the process $X^\varepsilon \in \Sigma_\varepsilon(x^\varepsilon)$ by:

$$X_1^\varepsilon(t) = \begin{cases} (X_1(t) + \varepsilon) & \text{if } \sup_{0\leq s\leq t} \{X_1(s) + \varepsilon\} \leq 1, \\ 1 & \text{if not,} \end{cases}$$

$$X_2^\varepsilon(t) = x_2 + t\wedge T^\varepsilon,$$

where T^ε is the first time at which $X_1^\varepsilon(t) = 1$. To see that $X^\varepsilon \in \Sigma_\varepsilon(x^\varepsilon)$, use the hypothesis that $C(x) \subseteq C(y)$ for $x_1 \leq y_1 < 1$. Now X_1^ε reaches 1 no later than X_1 so that

$$V_\varepsilon(x^\varepsilon) \geq u(X^\varepsilon)$$
$$\geq u(X).$$

Take the supremum over $X \in \Sigma(x)$ to get (4.13).

Finally, by (4.12) and (4.13), $Q(x^\varepsilon) \geq V(x)$. Let $\varepsilon\downarrow 0$ to get $Q \geq V$. This completes the proof.

To apply this result to our portfolio selection problem, we take $\mathcal{S}$ as in (4.4). The theorem then applies directly. For example, supposing $r_S \geq r_B \geq 0$ and $\sigma_S > 0$, M (of the theorem) is given by

$$M = \begin{cases} r_B + \dfrac{(r_S - r_B)^2}{2\,\sigma_S^2} & \text{if } r_S \leq r_B + \sigma_S^2, \\ r_S - \sigma_S^2/2 & \text{otherwise.} \end{cases}$$

The corresponding optimal policies are $f_B = 1 - f_S$ and

$$f_S = \begin{cases} \dfrac{r_S - r_B}{\sigma_S^2} & \text{if this is less than 1,} \\ 1 & \text{otherwise.} \end{cases}$$

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