Weighted quasimap wall-crossing via localization

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Dedication

For my parents, who never once doubted I could.

Abstract

The aim of this paper is to prove a wall-crossing formula at the level of virtual fundamental classes of weighted stable quasimaps to a class of GIT quotients via localization. The idea of the proof is to use virtual localization on a larger moduli space, referred to as the "master space", the fixed-point loci of which are in one-to-one correspondence with the terms in the wall-crossing formula. As an immediate application, we obtain the modified string and dilaton equations for stable quasimaps to GIT quotients.

Contents

Chapter 1

Introduction

1.1 Overview

Let W be a complex affine variety acted upon by a reductive algebraic group $\mathbf{G}$. Fix a character θ of $\mathbf{G}$ such that the action of $\mathbf{G}$ on the semistable locus with respect to θ is free. For a positive rational number ε, the theory of ε-stable quasimaps to the GIT quotient $W/\!\!/_\theta\mathbf{G}$ was developed in [CFKM14], generalizing and unifying many previous constructions [Kon95, MOP11, CFK10, MM07, Tod11].

For each stability parameter $\varepsilon \in \mathbb{Q}_{>0}$, the moduli stack of ε-stable quasimaps carries a perfect obstruction theory and comes equipped with evaluation maps to $W/\!\!/_\theta\mathbf{G}$ and tautological cotangent classes at the markings. As with Gromov-Witten invariants, one can define quasimap descendant invariants by pulling back cohomology classes under the evaluation maps and integrating against the virtual fundamental class. Explicitly, let $Q^{\varepsilon}_{g,k}(W/\!\!/_\theta\mathbf{G},\beta)$ denote the moduli space of ε-stable quasimaps of class β. When $W/\!\!/_\theta\mathbf{G}$ is projective, its (descendant) ε-quasimap invariants are defined by

$$\langle \gamma_1\psi_1^{a_1},\dots,\gamma_k\psi_k^{a_k}\rangle^{\varepsilon}_{g,k,\beta} := \int_{[Q^{\varepsilon}_{g,k}(W/\!\!/_\theta\mathbf{G},\beta)]^{\mathrm{vir}}} \prod_{j=1}^{k} \mathrm{ev}_j^*(\gamma_j)\psi_j^{a_j}\,,$$

for integers $a_j \geq 0$ and cohomology clases $\gamma_j \in H^*(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$. If $W/\!\!/_\theta \mathbf{G}$ is only quasiprojective but has a torus action with good properties, one may still define a good theory of *equivariant* quasimap invariants using the virtual localization formula. Again paralleling Gromov-Witten theory, these invariants are arranged into generating functions. For example, in genus zero one has the J^ε-function, $J^\varepsilon(q, \mathbf{t}, z)$, where q is a formal parameter keeping track of the degree (or numerical class), $\mathbf{t} \in H^*(W/\!\!/_\theta \mathbf{G})$ and z is a certain equivariant parameter.

There is a wall-and-chamber structure on the stability parameter space $\varepsilon \in \mathbb{Q}_{>0}$ with walls at $\frac{1}{k}$ for k a positive natural number. In the asymptotic case $\varepsilon \to 0$, denoted by $\varepsilon = 0+$, $J^{0+}(q, \mathbf{t}, z)$ becomes the (big) I-function from [CFK13], whose specialization at $\mathbf{t} = 0$ is Givental's (small) I-function in the toric case (up to an exponential factor). On the other end, when $\varepsilon \geq 1$, denoted by $\varepsilon = +\infty$, ε-stable quasimaps are just stable maps and $J^{+\infty}(q, \mathbf{t}, z)$ is the usual big J-function, and object of great interest in Gromov-Witten theory, which is, in general, difficult to compute.

As ε varies, these theories are expected to be related via wall-crossing theorems. The genus zero picture is analyzed in [CFK13], where it is shown than the I-function and J-function are related by quasimap wall-crossing theorems, thereby generalising and giving a rigorous and geometrically meaningful interpretation to Givental's toric mirror theorems. Treatments for the higher genus case can be found in [CFK20, CFK17, CJR17a, Zho19], the latter two of which employ the masterspace technique used in the main proof of this paper.

A consequence of these results is that one can compute the I-function in place of the J-function. When dealing with invariants with 1 or 0 insertions, the advantage here is that the I-function can be easier to compute or written down as a closed formula. However, there are other interesting cases in which the I-function is still hard to compute. To remedy this problem, I.Ciocan-Fontanine and B.Kim introduced a new big $\mathbb{I}$-function in [CFK16] which is conjectured to lie on the Lagrangian cone of the Gromov-Witten theory of $W/\!\!/\mathbf{G}$. In the same paper, it was shown that not only does that conjecture hold in certain situations but also that the big $\mathbb{I}$-function can be expressed in terms of the (usual) small I-function.

For example, if $l_1, \ldots, l_r$ are the degrees of the hypersurfaces with intersection $X \subset \mathbb{P}^N$ of codimension $r \leq N$, then the big-$\mathbb{I}$-function of X is given by

$$\exp\left(\frac{1}{z}\sum_{j=0}^{\infty}\sum_{i=0}^{r} t_{ij}\left(zq\frac{d}{dq}+H\right)^{i}(-z)^{j}\right) I_X^{small}(q,z)\,,$$

where the small I-function of X is given by the formula (see [Giv96])

$$I_X^{small}(q,z) = \mathbb{1} + \sum_{d\geq 1} q^d \frac{\prod_{k=1}^{r}\prod_{m=1}^{l_k d}(l_k H + mz)}{\prod_{m=1}^{d}(H+mz)^{N+1}}\,, \tag{1.1}$$

and where H denotes the restriction to X of the hyperplane class on $\mathbb{P}^N$. Similar formulas exist for the big-$\mathbb{I}$-functions of toric varieties or complete intersections in toric varieties (see 5.3 in [CFK16] and 7.2.8 in [CFK10] for more details).

The big $\mathbb{I}$-function is defined as a generating function of *weighted* (0+)-stable quasimap invariants, where "light" markings are allowed to coincide with basepoints. Let

$$Q_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta)$$

denote the moduli stack of weighted (0+)-stable quasimaps with m heavy (proper) markings and n infinitesimally weighted markings. These are (relatively) proper DM stacks and come equipped with evalution maps to $W/\!\!/_\theta \mathbf{G}$ and $[W/\mathbf{G}]$ at the heavy and light markings respectively, as well as a virtual fundamental class. One of the benefits of introducing light markings is that there exist maps $Q_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \to Q_{g,m|n-1}(W/\!\!/_\theta \mathbf{G}, \beta)$ forgetting one marking which are identified with the universal curve, contrary to the case of usual quasimaps with only heavy markings. A similar property was exploited by Y.Zhou in the appendix of [CJR17b] to complete the main proof of the paper. Other notable examples in which both light markings and wall-crossing theorems have been used, are [KL18] and [LP18]. In the former, B.Kim and H.Lho prove the genus-1 mirror theorem for the quintic threefold, while in the latter, H.Lho and R.Pandharipande prove the holomorphic anomaly equation for local $\mathbb{P}^2$.

1.2 Statement of main results

The main result of this paper is a wall-crossing theorem at the level of virtual classes comparing the moduli stacks of weighted quasimaps when one heavy marking is changed to a light one. In order to state it, we need to introduce some further notation.

The GIT set-up gives a natural morphism $\iota : [W/\mathbf{G}] \to [\mathbb{C}^{M+1}/\mathbb{C}^*]$ for some positive integer M, inducing a closed immersion $\iota : W/\!\!/_\theta \mathbf{G} \to \mathbb{P}^M$ and a morphism (denoted by the same letter)

$$\iota : Q_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \to Q_{m|n}(\mathbb{P}^M, d(\beta))\,,$$

where $d(\beta) := \iota_*(\beta) \in \mathrm{Hom}(\mathrm{Pic}[\mathbb{C}^{M+1}/\mathbb{C}^*], \mathbb{Z}) \cong \mathbb{Z}$. Let m, n be positive integers. There exists a morphism

$$c : Q_{g,m+1|n-1}(\mathbb{P}^M, d(\beta)) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

which converts the last heavy marking into a light one and stabilizes. Next, for any subset $J \subseteq \{1, \ldots, n\}$ with $1 \in J$ and numerical class $\beta' \in \mathrm{Hom}(\mathrm{Pic}[W/\mathbf{G}], \mathbb{Z})$ such that $0 < d(\beta') \leq d(\beta)$, there is a morphism

$$b_{\beta',J} : Q_{g,m|y_1,(y_j)_{j\notin J}}(\mathbb{P}^M, d(\beta')) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

which places $|J| - 1$ light markings and a basepoint of length $d(\beta - \beta')$ at the first light marking.

Let L be a line bundle on $[W/\mathbf{G}]$ of the form $L = [W \times \mathbb{C}_{\theta_0}/\mathbf{G}]$ for some character $\theta_0 \in \chi(\mathbf{G})$ in the same GIT chamber as θ (see 5.2 in [CFK16] for more details). Let

$\hat{H} := c_1(L)$ and let $\mathbb{I}(q, \mathbf{t}(\psi), z)$ denote the (big) I-function where we restrict $\mathbf{t}(\psi)$ to

$$\mathbf{t}(\psi) = \sum_{ij} t_{ij} \hat{H}^i \psi^j \,.$$

We are now ready to state the main result.

Theorem 1.2.1 (Heavy-to-light wallcrossing formula)**.** *Let m, n be positive integers and i, j be (possibly zero) non-negative integers. For each $J \subseteq \{1, \ldots, n\}$, let $\lambda_J(z) = (-1)^{|J|-1} z^{2-|J|}$. We have the following relation in the homology group $H_*(Q_{g,m|n}(\mathbb{P}^M, d(\beta)))_{\mathbb{Q}}$*

$$\iota_* \left(\hat{ev}^*(\hat{H}^i)\hat{\psi}^j \cap [Q_{g,m|n,\beta}]^{vir} \right) = c_* \iota_* \big(ev_\bullet^*(H^i)\psi_\bullet^j \cap [Q_{g,m+\bullet|n-1,\beta}]^{vir} \big)$$
$$+ \sum_{0<\beta'\leq\beta} \sum_J (b_{\beta',J})_* c_* \iota_* \Bigg((ev_\bullet)^* \Bigg(\Bigg[\lambda_J(z) \frac{\partial}{\partial t_{i,j}} \mathbb{I}(q, \mathbf{t}, z) \Bigg|_{t=0} \Bigg]_{+,\beta'} \Bigg) \Bigg|_{z=-\psi_\bullet} \cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \Bigg) \tag{1.2}$$

where $\hat{ev}$ and $\hat{\psi}$ are at the first light marking, $H = \kappa(\hat{H})$ for $\kappa : H^*([W/\mathbf{G}], \mathbb{Q}) \to H^*(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$ the Kirwan map, and the notation $[\ ]_{+,\beta'}$ indicates taking non-negative powers of z and then taking the coefficient of $q^{\beta'}$.

More generally, for arbitrary classes $\delta_1, \ldots, \delta_m \in H^*(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$, there is an equality

$$\iota_* \left(\prod_{k=1}^m ev_k^*\delta_k \, \hat{ev}^*(\hat{H}^i)\hat{\psi}^j \cap [Q_{g,m|n,\beta}]^{vir} \right) = c_* \iota_* \big(\prod_{k=1}^m ev_k^*\delta_k \; ev_\bullet^*(H^i)\psi_\bullet^j \cap [Q_{g,m+\bullet|n-1,\beta}]^{vir} \big)$$
$$+ \sum_{0<\beta'\leq\beta} \sum_J (b_{\beta',J})_* c_* \iota_* \Bigg(\prod_{k=1}^m ev_k^*\delta_k \, (ev_\bullet)^* \Bigg(\Bigg[\lambda_J(z) \frac{\partial}{\partial t_{i,j}} \mathbb{I}(q, \mathbf{t}, z) \Bigg|_{t=0} \Bigg]_{+,\beta'} \Bigg) \Bigg|_{z=-\psi_\bullet}$$
$$\cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \Bigg) \tag{1.3}$$

Note that the case of complete intersections in projective space, this was proven by I. Ciocan-Fontanine and B.Kim (see §1.6 in [CFK20]). The equivalent wall-crossing formula for weighted Fan-Jarvis-Ruan-Witten (FJRW) invariants was proven in [Zho17].

1.3 Outline

We begin by reviewing the necessary background and definitions in Chapter 2. Chapter 3 is devoted to the construction of the master space, which admits a torus action. In Chapter 4, we identify the $\mathbb{C}^*$-fixed loci and their contributions to the virtual localization formula. This yields a relation in the Chow group of the master space. In Chapter 5 we define some stabilization maps and explain how pushing forward this relation along these maps gives us the wall-crossing formula. We conclude by discussing some immediate applications in Chapter 6.

Chapter 2

Background

2.1 A class of GIT targets

Throughout, the base field is $\mathbb{C}$. Consider a triple $(W, \mathbf{G}, \theta)$, where $W = \mathrm{Spec}(A)$ is an affine algebraic variety, $\mathbf{G}$ is reductive algebraic group acting on W and $\theta \in \chi(\mathbf{G})$ is a character of $\mathbf{G}$. There are three quotients associated with this triple; the quotient stack $[W/\mathbf{G}]$, the affine quotient $W/_{\mathrm{aff}}\mathbf{G} = \mathrm{Spec}(A^{\mathbf{G}})$, and the GIT quotient

$$W/\!\!/_{\theta}\mathbf{G} = \mathrm{Proj}(\oplus_{n \geq 0} \Gamma(W, L_{\theta}^{\otimes n}))^{\mathbf{G}}$$

with respect to the linearization $L_\theta = W \times \mathbb{C}_\theta$ defined by θ. The GIT quotient has a projective morphism

$$W/\!\!/_{\theta}\mathbf{G} \to W/_{\mathrm{aff}}\mathbf{G}\,,$$

to the affine quotient and is projective precisely when $W/_{\mathrm{aff}}\mathbf{G}$ is a point, i.e. the only $\mathbf{G}$-invariant functions are the constant functions. Let

$$W^s := W^s(\theta) \quad \text{and} \quad W^{ss} := W^{ss}(\theta)$$

denote the open subsets of stable (respectively semi-stable) points determined by L_θ (see [New09] for definitions). From now on, we'll assume the following:

(i) $\emptyset \neq W^s = W^{ss}$;

(ii) W^s is nonsingular;

(iii) $\mathbf{G}$ acts freely on W^s.

(iv) W has at worst l.c.i. singularities

It follows from Luna's slice theorem that $W^s \to W/\!\!/_\theta\mathbf{G}$ is a principal $\mathbf{G}$-bundle in the étale topology. Hence $W/\!\!/_\theta\mathbf{G}$ is nonsingular variety which coincides with $[W^s/\mathbf{G}]$. In particular, it is naturally an open substack in $[W/\mathbf{G}]$.

2.2 Maps from curves to $[W/\mathbf{G}]$

Let C be a (possibly disconnected) reduced, projective, at worst nodal curve. By the definition of quotient stacks, a map $[u] : C \to [W/\mathbf{G}]$ corresponds to a pair $(P, \tilde{u})$ of a principal $\mathbf{G}$-bundle

$$P \to C$$

and a $\mathbf{G}$-equivariant morphism

$$\tilde{u} : P \to W .$$

Equivalently, we'll consider the data (P, u), with

$$u : C \to P \times_{\mathbf{G}} W$$

a section of the fibre bundle $P \times_{\mathbf{G}} W \to C$. Here, $P \times_{\mathbf{G}} W$ denotes the "mixed construction" stack quotient $[P \times W/\mathbf{G}]$ with $\mathbf{G}$ acting diagonally on $P \times W$. Denote by $\mathrm{Map}(C, [W/\mathbf{G}])$ the stack parametrizing such pairs (P, u). It follows from general results in [Lie06] and [AOV10] that this is an Artin stack (e.g. see Lemma 2.8 in [CCFK14]).

2.3 Induced line bundles and the numerical class of maps

Let $\mathrm{Pic}^{\mathbf{G}}(W)$ denote the group of $\mathbf{G}$-linearized line bundles on W and recall that $L \to [L/\mathbf{G}]$ gives an identification $\mathrm{Pic}^{\mathbf{G}}(W) = \mathrm{Pic}([W/\mathbf{G}])$. Let (C, P, u) be as above. For a $\mathbf{G}$-equivariant line bundle L on W, we have a cartesian diagram

$$\begin{array}{ccc} P \times_{\mathbf{G}} L & \longrightarrow & P \times_{\mathbf{G}} W \\ \downarrow & & \downarrow \\ [L/\mathbf{G}] & \longrightarrow & [W/\mathbf{G}] \end{array}$$

Hence we get an induced line bundle $u^*(P \times_{\mathbf{G}} L) = [u]^*([L/\mathbf{G}])$ on C.

Definition 2.3.1. The *numerical class* β of $(P, u) \in \mathrm{Map}(C, [W/\mathbf{G}])$ is the homomorphism

$$\beta : \mathrm{Pic}([W/\mathbf{G}]) \to \mathbb{Z}, \quad \beta(L) = \deg_C(u^*(P \times_{\mathbf{G}} L)) .$$

2.4 Weighted quasimaps to $W/\!\!/_\theta \mathbf{G}$

Fix non-negative integers m, n and a numerical class $\beta \in \mathrm{Hom}(\mathrm{Pic}([W/\mathbf{G}], \mathbb{Z})$. Given a tuple $(\theta, \lambda) = \chi(\mathbf{G}) \times \mathbb{Q}_{>0}$ with $\lambda << 1$ sufficiently small, we'll be concerned with the following special case[1] of stable weighted quasimaps to $[W/\mathbf{G}]$.

Denote by L_θ the $\mathbb{Q}$-line bundle on $[W/\mathbf{G}]$ associated to θ, namely, $L_\theta = W \times_{\mathbf{G}} \mathbb{C}_\theta$. Given a map $[u] : C \to [W/\mathbf{G}]$, let

$$\mathcal{L}_\theta := [u]^*(L_\theta) .$$

[1]More generally, there is a theory of θ-stable weighted quasimaps where θ is allowed to be rational and the weights of the markings are arbitrary rational numbers in (0,1] (see [CFK16] for more details).

Definition 2.4.1 ([CFK16]). An *$(m|n)$-weighted stable quasimap to $W/\!\!/_\theta \mathbf{G}$ of class β* is a tuple $\big((C, x_1, \ldots, x_m, y_1, \ldots, y_n), P, u\big)$ where

(1) $\big($*$(m|n)$-weighted prestable map to $W/\!\!/_\theta \mathbf{G}$*$\big)$

   (a) C is a genus g, prestable curve over the field $\mathbb{C}$

   (b) x_i and y_j are smooth points on C with

$$\sum_i \delta_{x_i,p} + \lambda \sum_j \delta_{y_j,p} \leq 1$$

   for every smooth point p of C.

   (c) (P, u) is morphism from C to $[W/\mathbf{G}]$ as explained in (2.2).

(2) (*θ-quasimap*) $[u]^{-1}([W^{un}(\theta)/\mathbf{G}])$ is pure 0-dimensional

(3) (*θ-prestability*) $[u]^{-1}(W/\!\!/_\theta \mathbf{G})$ contains all nodes of C.

(4) $\big($*$(m|n)$-stability*$\big)$

   (a) The $\mathbb{Q}$-line bundle

$$\omega_C\left(\sum_{i=1}^{n} x_i + \lambda \sum_{j=1}^{n} y_j\right) \otimes \mathcal{L}_\theta^\epsilon$$

   is ample for every rational number $\epsilon > 0$.

   (b) For all $1 \leq i \leq m$, x_i is not a basepoint

(5) (numerical class β) The quasimap has numerical class β.

Remark 1. Note that condition 1(c) above is equivalent to requiring that the heavy markings x_i are pairwise distinct and do not coincide with any of the light markings y_j. Note also that we are allowing the markings y_j to be basepoints.

Remark 2. In the notation of [CFK16] (see Definition 2.10), the $(m|n)$-weighted stable quasimaps defined here are called $(0+, 0+)$-stable maps to $[W/\mathbf{G}]$ of type $(g, m|n, \beta)$. Since we are only considering this special case here, we drop the $(0+, 0+)$ notation.

Definition 2.4.2. An *isomorphism* between two weighted quasimaps

$$(C, p_1, \ldots, p_k, P, u)\,,$$

and

$$(C', p'_1, \ldots, p'_k, P', u')\,,$$

consists of an isomorphism $f : C \to C'$ of the underlying curves, along with an isomorphism $\sigma : P \to f^*P'$, such that the markings and the section are preserved:

$$f(p_j) = p'_j,\ \sigma_W(u) = f^*(u')\,,$$

where $\sigma_W : P \times_{\mathbf{G}} W \to P' \times_{\mathbf{G}} W$ is the isomorphism of fibre bundles induced by σ.

Definition 2.4.3. A *family* of $(m|n)$-weighted stable quasimaps of genus g and numerical class β over a base scheme S consists of the data

$$(\pi : \mathcal{C} \to S, x_1, \ldots, x_m, y_1, \ldots, y_n, \mathcal{P}, u)$$

where

- $\pi : \mathcal{C} \to S$ is a flat family of curves over S, that is, a flat proper morphism of relative dimension one,
- x_i, and y_j are sections of π,
- $\mathcal{P}$ is a principal $\mathbf{G}$-bundle on $\mathcal{C}$,
- $u : \mathcal{C} \to \mathcal{P} \times_{\mathbf{G}} W$ is a section of the natural projection $\mathcal{P} \times_{\mathbf{G}} W \to \mathcal{C}$.

such that the restriction to every geometric fibre $\mathcal{C}_s$ of π is an $(m|n)$-weighted stable quasimap of genus g and class β. An isomorphism between two such families $(\mathcal{C} \to S, \ldots)$ and $(\mathcal{C}' \to S, \ldots)$ consists of an isomorphism of S-schemes $f : \mathcal{C} \to \mathcal{C}'$, and an isomorphism of $\mathbf{G}$-bundles $\sigma : \mathcal{P} \to f^*\mathcal{P}'$, which preserve the markings and the section.

Remark 3. Stability of the weighted quasimap $(C, \mathbf{x}, \mathbf{y}, P, u)$ implies the following properties

- Every rational component of the underlying curve C has at least two nodal or weight 1 marked points. On any such component, deg $([u]^*(\mathcal{L}_\theta)) > 0$ or there is at least one more special point (node, heavy or light marking).
- The automorphism group of a stable $(m|n)$-weighted quasimap is reduced and finite.
- $\beta(L_\theta) \geq 0$ with equality if and only if $\beta = 0$. Furthermore, the same holds for any subcurve C' and the induced quasimap (this is Lemma 3.2.1 in [CFKM14]).

Definition 2.4.4. An element $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}[W/\mathbf{G}], \mathbb{Z})$ is called *θ-effective* if it can be realized as a finite sum of classes of θ-quasimaps.

We denote by $\mathrm{Eff}(W, \mathbf{G}, \theta)$ the subset of θ-effective classes. It is a semigroup with no notrivial invertible elements, i.e. $\beta_1 + \beta_2 = 0$ for $\beta_i \in \mathrm{Eff}(W, \mathbf{G}, \theta)$ implies that $\beta_1 = \beta_2 = 0$.

In [CFK16] it was observed that by treating each marking as an effective divisor on C, $(m|n)$-weighted quasimaps can be viewed as an particular instance of 0-pointed ε-stable quasimaps to $[W/G] \times [\mathbb{C}/\mathbb{C}^*]^{m+n}$. Hence, Theorem 7.1.6 in [CFKM14] applies and we get a well-behaved moduli space.

Theorem 2.4.1 ([CFK16])**.** *The moduli space $Q_{g,m|n}(W/\!\!/_\theta\mathbf{G}, \beta)$ of $(m|n)$-weighted stable quasimaps to $W/\!\!/_\theta\mathbf{G}$ is a Deligne-Mumford stack, proper over the affine quotient, with a canonical perfect obstruction theory.*

Note that

$$2g - 2 + m + \lambda n + \beta(L_\theta) > 0$$

is a necessary condition for the moduli stack to be non-empty.

A remark on notation. We also use the shorthand notation $Q_{g,m|n,\beta}$ instead of $Q_{g,m|n}(W/\!\!/_\theta\mathbf{G}, \beta)$ in cases where the notation would be too dense otherwise.

2.5 Perfect obstruction theories

The wall-crossing formula is expressed at the (virtual) cycle level rather than the numerical level, so we include here a review of the theory of virtual classes.

Definition 2.5.1 ([BF97])**.** Let X be a scheme (or DM stack) and $L_X^{\geq -1}$ its truncated cotangent complex. A *perfect obstruction theory* for X is a complex $E_X^\bullet \in D^{[-1,0]}(X)$ which is locally quasi-isomorphic to a complex of locally free sheaves and admits a morphism

$$\phi : E_X^\bullet \to L_X^{\geq -1} ,$$

such that

(i) $h^0(\phi)$ is an isomorphism; and

(ii) $h^{-1}(\phi)$ is surjective.

Given a perfect obstruction theory for X, the construction of [BF97] gives rise to a virtual funcamental class $[X]^{vir} \in A_d(X)$ where $d = \mathrm{rk} E^\bullet$ is the virtual dimension[2] of X with respect to the obstruction theory $E^\bullet$.

Example 2.5.1. Consider a regular embedding $i : Y \to W$ fitting into a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{j} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & W \end{array}$$

where V, W are smooth. Then, the complex

$$E^\bullet = [g^* N_{V/W}^\vee \to j^*\Omega_V]$$

[2]If $E^\bullet$ is written locally as a complex of vector bundles $[E^{-1} \to E^0]$, then this is $\mathrm{rk} E^\bullet = \dim E^0 - \dim E^{-1}$. This is a well-defined locally constant function on X, which is assumed to be constant.

in degrees [-1,0] forms a perfect obstruction theory for X. The map comes from

$$g^*N^\vee_{V/W} \to g^*i^*\Omega_W = j^*f^*\Omega_W \to j^*\Omega_V\,.$$

This is a perfect obstruction theory because the complex comes equipped with a map to $L^\bullet_X$ coming from the maps $g^*L^\bullet_Y \to L^\bullet_X$ and $j^*L^\bullet_V \to L^\bullet_X$. Note that the associated virtual fundamental class is $[X, E^\bullet] = i^![V]$. For a projective variety $Y \in \mathbb{P}^n$ we can take $V = W$, in which case the perfect obstruction theory is

$$[N^\vee_{X/\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}]$$

and the associated virtual fundamental class is

$$[X, E^\bullet] = i^![\mathbb{P}^n]\,.$$

In particular, if Y is a local complete intersection then the perfect obstruction theoy is the cotangent complex (which is the same as the truncated cotangent complex). Note that this construction also works with DM stacks.

Example 2.5.2 (Spaces of Morphisms [BF97])**.** . Let C and V be projective k-schemes. Let $X = \mathrm{Mor}(C, V)$ be the k-scheme of morphisms from C to V. Let $f : C \times X \to V$ be the universal morphism and $\pi : C \times X \to X$ be the projection. By the functorial properties of the cotangent complex we get a homomorphism

$$f^*L^\bullet_V \to L^\bullet_{C\times X} \to L^\bullet_{C\times X/C}$$

and a homomorphism

$$\pi^*L_X \to L^\bullet_{C\times X}\,.$$

The latter is an isomorphism so that we get an induced homomorphism

$$e : f^* L_V^\bullet \to \pi^* L_X^\bullet \, .$$

Assume that C has a dualizing complex ω_C. Then we get a homomorphism

$$e \otimes \omega_C : f^* L_V^\bullet \otimes^L \omega_C \to \pi^* L_X^\bullet \otimes^L \omega_C = \pi^! L_X^\bullet$$

and by adjunction a homomorphism

$$\pi_*(e \otimes \omega_C) : R\pi_*(f^* L_V^\bullet \otimes^L \omega_C) \to L_X^\bullet \, .$$

By duality we have

$$R\pi_*(f^* L_V^\bullet \otimes^L \omega_C) = (R\pi_*(f^* T_V^\bullet))^\vee \, .$$

Let us denote the resulting homomorphism by

$$\pi_*(e^\vee)^\vee : (R\pi_*(f^* T_V^\bullet))^\vee \to L_X^\bullet \, .$$

Proposition 2.5.1 ([BF97] Prop.6.2). *Assume that C is Gorenstein. Then the homomorphism $\phi := \pi_*(e^\vee)^\vee$ is an obstruction theory for X. If C is a curve and V is smooth, then this obstruction theory is perfect.*

In our case, we'll define an absolute perfect obstruction theory via a *relative* perfect obstruction theory. As before, a perfect relative obstruction theory for X over Y gives us a virtual fundamental class $[X]^{vir}$.

Definition 2.5.2. Let Y be a smooth equidimensional algebraic k-stack. Given an algebraic stack X over Y which is of relative DM type over Y, we define a perfect *relative* obstruction theory $E_{X/Y}^\bullet$ for X over Y just as in Definition 2.5.1, but with $L_X^\bullet$ replaced by $L_{X/Y}^\bullet$. If $X \xrightarrow{f} Y$ is the given map, we also write $E_f^\bullet$ and $L_f^\bullet$ for $E_{X/Y}^\bullet$ and

$L^\bullet_{X/Y}$.

Proposition 2.5.2 (Appendix B in [GP99])**.** *Let M be a finite type DM stack and $\tau : M \to \mathfrak{M}$ a morphism to a smooth Artin stack $\mathfrak{M}$ which is locally of finite type and of pure dimension. Let $\phi : E^\bullet_{M/\mathfrak{M}} \to L^\bullet_{M/\mathfrak{M}}$ be a relative perfect obstruction theory. If h denotes the composite $E^\bullet_{M/\mathfrak{M}} \to L^\bullet_{M/\mathfrak{M}} \to \tau^* L_{\mathfrak{M}}[1]$ and we set $E^\bullet_M = c(h)[-1]$, then the induced $\psi : E^\bullet_M \to L^\bullet_M$ is an (absolute) perfect obstruction theory, fitting into a distinguished triangle*

$$\tau^* L^\bullet_{\mathfrak{M}} \to E^\bullet_M \to E^\bullet_{M/\mathfrak{M}} \to \tau^* L^\bullet_{\mathfrak{M}}[1] \tag{2.1}$$

Moreover, ϕ and ψ determine the same virtual class in $A_* M$.

Definition 2.5.3 ([Man08])**.** Let $F \xrightarrow{f} G \xrightarrow{g} \mathfrak{M}$ be a Deligne-Mumford type morphism of stacks. If we are given a distinguished triangle of perfect relative obstruction theories

$$f^* E^\bullet_{G/\mathfrak{M}} \xrightarrow{\phi} E^\bullet_{F/\mathfrak{M}} \to E^\bullet_{F/G} \to f^* E^\bullet_{G/\mathfrak{M}}[1]$$

with a morphism to the distinguished triangle

$$f^* L^\bullet_{G/\mathfrak{M}} \to L^\bullet_{F/\mathfrak{M}} \to L^\bullet_{F/G} \to f^* L^\bullet_{G/\mathfrak{M}}[1]\,,$$

then we call $(E^\bullet_{F/G}, E^\bullet_{G/\mathfrak{M}}, E^\bullet_{F/\mathfrak{M}})$ a compatible triple.

Remark. (see Construction 3.13 in [Man08]) If there is a morphism $E^\bullet_{F/G} \xrightarrow{\psi} f^* E^\bullet_{G/\mathfrak{M}}[1]$ compatible with the corresponding morphism between the cotangent complexes, then the cone of ψ determines a complex $E^\bullet_{F/\mathfrak{M}}$ which fits in a distinguished triangle as above. Moreover, $E^\bullet_{F/\mathfrak{M}}$ defines a relative obstruction theory. If $E^\bullet_{F/G}$ and $E^\bullet_{G/\mathfrak{M}}$ are perfect, then $E^\bullet_{F/\mathfrak{M}}$ is perfect.

Next, we "rewrite" the canonical obstruction theory of $Q_{g,m|n}(W/\!\!/_{\theta}\mathbf{G},\beta)$ in a form that is more convenient for our purposes. Recall that by treating each marking as an effective divisor on C, $(m|n)$-weighted stable quasimaps of numerical class β can be identified with

$$Q^{\vartheta}_{g,0}([W/\mathbf{G}]\times[\mathbb{C}/\mathbb{C}^*]^{m+n}),(\beta,1\ldots,1))\,,$$

where ϑ is the rational character

$$\vartheta := \theta\oplus\underbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}_{m\text{ times}}\oplus\underbrace{\lambda\mathrm{id}\oplus\cdots\oplus\lambda\mathrm{id}}_{n\text{ times}}\in\chi(\mathbf{G}\times(\mathbb{C}^*)^{m+n})_{\mathbb{Q}}\,,$$

for $\lambda << 1$ sufficiently small. This gives rise to a canonical relative obstruction theory E_{μ} where μ is the forgetful map in the diagram

$$Q_{g,m|n}(W/\!\!/_{\theta}\mathbf{G},\beta)\xrightarrow{\mu}\mathfrak{B}un_{\mathbf{G}\times(\mathbb{C}^*)^{m+n}}\xrightarrow{\nu}\mathfrak{M}_{g,0} \tag{2.2}$$

and $\mathfrak{M}_{g,0}$ is the Artin stack of (unmarked) prestable curves of genus g. We can instead define a perfect relative obstruction theory $E_{\tilde{\mu}}$ where $\tilde{\mu}$ is the forgetful map in the diagram

$$Q_{g,m|n}(W/\!\!/_{\theta}\mathbf{G},\beta)\xrightarrow{\tilde{\mu}}\mathfrak{B}un_{\mathbf{G}}\xrightarrow{\tilde{\nu}}\widetilde{\mathfrak{M}}_{g,m+n} \tag{2.3}$$

and $\widetilde{\mathfrak{M}}_{g,m+n}$ denotes the smooth Artin stack of genus g nodal curves with $m+n$ not necessarily distinct markings and at most $2g-2+m+n$ irreducible components. By Proposition 2.5.2 both these perfect relative obstruction theories give rise to absolute perfect obstruction theories for $Q_{g,m|n}(W/\!\!/_{\theta}\mathbf{G},\beta)$.

Lemma 2.5.1. *The two (absolute) perfect obstruction theories (2.2) and (2.3) described above give rise to the same virtual fundamental class $[Q_{g,m|n}(W/\!\!/_{\theta}\mathbf{G},\beta)]^{vir}$.*

Proof. First note that since the right arrow in (2.2) is a smooth map between Artin stacks, we also have an induced perfect relative obstruction theory $E_{\nu\circ\mu}$ fitting in a distinguished triangle

$$\mu^*\mathbb{L}_\nu^\bullet \to E_\mu^\bullet \to E_{\nu\circ\mu}^\bullet \to \mu^*\mathbb{L}_\nu^\bullet[1]\,.$$

By Proposition 2.5.2, all three perfect obstruction theories determine the same virtual fundamental class. The same holds for (2.3), so it suffices to show that $E_{\nu\circ\mu}$ and $E_{\tilde{\nu}\circ\tilde{\mu}}$ determine the same virtual fundamental class. To see this, observe that there is forgetful morphism $f : \widetilde{\mathfrak{M}}_{g,m+n} \to \mathfrak{M}_{g,0}$ forgetting the markings. The relative obstruction theory $E_{\tilde{\nu}\circ\tilde{\mu}}$ is constructed as the cone of the map $E_{\tilde{\nu}\circ\tilde{\mu}} \to (\tilde{\mu}\circ\tilde{\nu})^* E_f^\bullet[1]$, so the triple $(E_{\nu\circ\mu}^\bullet, E_{\tilde{\nu}\circ\tilde{\mu}}^\bullet, E_f^\bullet)$ is compatible. The result then follows from [Man08, 3.2]. □

2.6 Virtual Localization

One of the main tools for computing virtual classes is virtual localization [GP99], a technique used to simplify the calculation of the virtual class of spaces admitting a torus action. It is a generalisation to the virtual setting of Atiyah-Bott localization, which reduces integrals over a space X with a torus action to integrals over the fixed loci X_i. Here we explain the set-up and state the result, omitting proofs.

Let X be an algebraic Deligne-Mumford stack with a torus action carrying a perfect obstruction theory $\phi : E^\bullet \to L_X^\bullet$ equipped with an equivariant T-action. This yields an equivariant virtual fundamental class in the equivariant Chow group $A_d^T(X)$, where d is the virtual dimension of $E^\bullet$.

Let X_i denote the irreducible components of the fixed locus and let $E_i^\bullet$ denote the restriction of the complex $E^\bullet$ to X_i. The complex $E_i^\bullet$ decomposes as a direct sum by T-characters:

$$E_i^\bullet = E_i^{\bullet,f} \oplus E_i^{\bullet,m},$$

where the first summand corresponds to the trivial character (the T-fixed part) and the second summand corresponds to all the non-trivial characters (the moving part). The fixed part of ϕ induces a perfect obstruction theory

$$\phi : E_i^{\bullet,f} \to L_{X_i}^{\bullet}.$$

We define the virtual structure on X_i to be the one given by ϕ_i. Since X_i may be singular, it might not have a normal bundle. Instead, we replace this with the moving part of $E_{\bullet,i}$ and call this the *virtual normal bundle* N_i^{vir} to X_i. These constructions yield the following natural formulation of the virtual localization formula

$$[X]^{vir} = \iota_* \sum_i \frac{[X_i]^{vir}}{e(N_i^{vir})}\,, \tag{2.4}$$

where the Euler class of $N_i^{vir} = [E_{0,i}^m \to E_{1,0}^m]$ is defined to be the ratio $e(E_{0,i}^m)/e(E_{1,i}^m)$. This is well-defined since N_i^{vir} is a complex of bundles with non-zero T-weights, so the Euler class $e(N_i^{vir})$ becomes invertible in the localized ring

$$A_*^{\mathbb{C}^*}(X)_t = A_*^{\mathbb{C}^*}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}\left[t, \frac{1}{t}\right].$$

(2.4) implies the virtual Bott residue formula for any isobaric polynomial in the Chern classes of vector bundles on X of degree equal to the virtual dimension of X

$$\int_{[X]^{vir}} P(c_{k_1}(E_1), \ldots, c_{k_m}(E_m)) = \sum_i \int_{[X_i]^{vir}} \frac{P(c_{k_1}(E_{1,i}), \ldots, c_{k_m}(E_{m,i}))}{e(N_i^{vir})}, \tag{2.5}$$

where the Euler classes on the right-hand side are equivariant classes.

2.7 The (big) $\mathbb{I}$-function

Let $\mathbb{Q}[\mathrm{Eff}(W, \mathbf{G}, \theta)]$ be the semigroup ring. Write q^β for the element corresponding to the $\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)$. Define the *Novikov ring* associated to the triple $(W, \mathbf{G}, \theta)$,

$$\Lambda := \left\{ \sum_{\beta \in \mathrm{Eff}(W,\mathbf{G},\theta)} \alpha_\beta q^\beta : \alpha_\beta \in \mathbb{Q} \right\} ,$$

the $\mathfrak{m}$-adic completion with respect to the ideal $\mathfrak{m}$ generated by $\{q^\beta | \beta \neq 0\}$. Throughout, we will consider the *even* cohomology of $W/\!\!/\mathbf{G}$ and $[W/\mathbf{G}]$ with coefficients in $\mathbb{Q}$ and Λ. In what follows, we assume that $W/\!\!/\mathbf{G}$ is projective; the extension to quasi-projective targets is discussed in the 2.7.4.

2.7.1 Weighted graph spaces

Definition 2.7.1. Given a tuple (θ, λ) define the (θ, λ)-*stable quasimap graph space* as follows:

$$QG^{\theta,\lambda}_{g,m|n,\beta}([W/\mathbf{G}]) := Q^{\theta\oplus 3\mathrm{id},\lambda}_{g,m|n}([W \times \mathbb{C}^2/\mathbf{G} \times \mathbb{C}^*], (\beta, 1)) ,$$

where $\theta \oplus 3\mathrm{id}$ is a rational character of $\mathbf{G} \times \mathbb{C}^*$. A $\mathbb{C}$-point of the graph space is described by the data

$$((C, \mathbf{x}, \mathbf{y}), (f, \phi) : C \to [W/\mathbf{G}] \times \mathbb{P}^1) ,$$

where ϕ is a regular map to $\mathbb{P}^1$ of class 1. Therefore, the domain curve C has a distinguished irreducible component C_0 canonically isomorphic to $\mathbb{P}^1$ via ϕ.

The "standard" $\mathbb{C}^*$-action,

$$t \cdot [\xi_0, \xi_1] = [t\xi_0, \xi_1] \qquad \text{for } t \in \mathbb{C}^*,\ [\xi_0, \xi_1] \in \mathbb{P}^1,$$

induces a $\mathbb{C}^*$-action on the graph space. With this convention, the $\mathbb{C}^*$-equivariant first Chern class of the tangent line $T_0\mathbb{P}^1$ at $0 \in \mathbb{P}^1$ is $c_1^{\mathbb{C}^*}(T_0\mathbb{P}^1) = z$, where z denotes the equivariant parameter, i.e. $H^*_{\mathbb{C}^*}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Q}[z]$.

The graph space comes equipped with evaluation morphisms

$$\hat{\mathrm{ev}}_j : QG^{\theta,\lambda}_{g,m|n,\beta}([W/\mathbf{G}]) \to [W/\mathbf{G}]\,, \quad j = 1,\dots,n$$
$$\mathrm{ev}_i : QG^{\theta,\lambda}_{g,m|n,\beta}([W/\mathbf{G}]) \to W/\!\!/_\theta\mathbf{G}\,, \quad i = 1,\dots,m$$

and pull-back homomorphisms

$$\hat{\mathrm{ev}}^*_j : H^*([W/\mathbf{G}],\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z] \to H^*_{\mathbb{C}^*}(QG^{\theta,\lambda}_{g,m|n,\beta}([W/\mathbf{G}],\mathbb{Q})\,.$$

Now fix $(\theta,\lambda) = (0+,0+)$ and consider the graph spaces

$$QG^{0+,0+}_{0,0|k,\beta}([W/\mathbf{G}]) \cong QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}]) \times (\mathbb{P}^1)^k\,.$$

The space $QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}])$ coincides with $QG_{0,0}(W/\!\!/_\theta\mathbf{G},\beta;\mathbb{P}^1)$ defined in [CFKM14, §7.2]. The description of the fixed loci for the $\mathbb{C}^*$-action is parallel to the one given in [CFK13, §4.1]. In particular, we have the part $F_{k,\beta}$ of the $\mathbb{C}^*$-fixed locus for which the markings and the entire class are over $0 \in \mathbb{P}^1$:

$$F_{k,\beta} = F_\beta \times 0^k\,, \tag{2.6}$$

where $F_\beta = F_{0,\beta}$ is the distinguished $\mathbb{C}^*$-fixed locus in $QG^{0+,0+}_{0,0,\beta}([W/\mathbf{G}])$. This F_β parametrizes quasimaps of class β

$$f : \mathbb{P}^1 \to [W/\mathbf{G}]$$

with a base-point of length $\beta(L_\theta)$ at $0 \in \mathbb{P}^1$. The restriction of f to $\mathbb{P}^1 \setminus \{0\}$ is a constant map to $W/\!\!/_\theta\mathbf{G}$ and this defines a *proper* evaluation map at the generic point

$$\mathrm{ev}_\bullet : F_{k,\beta} \to W/\!\!/_\theta\mathbf{G}\,.$$

2.7.2 The big $\mathbb{I}$-function

The (big) $\mathbb{I}$-function is defined as a formal sum over all $k \geq 0$ and $\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)$ of localization residues over these distinguished components of the fixed loci, pushed forward to $W /\!\!/ \mathbf{G}$ by evaluation maps. Note that this is the big $\mathbb{I}$-function defined in [CFK16] which differs from the one defined in [CFKM14].

Definition 2.7.2. For $\mathbf{t} \in H^*([W/\mathbf{G}], \mathbb{Q}) \subset H^*([W/\mathbf{G}], \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[z]$, the *big* $\mathbb{I}$*-function* is

$$\mathbb{I}(q, \mathbf{t}, z) = \sum_{\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)} \sum_{k \geq 0} \frac{q^\beta}{k!} (\mathrm{ev}_\bullet)_* \left(\frac{\iota_\beta^* \big(\prod_{i=1}^k \hat{\mathrm{ev}}_i^*(\mathbf{t}) \big) \cap [F_{k,\beta}]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_{k,\beta}})} \right), \tag{2.7}$$

where $\iota_\beta : F_{k,\beta} \hookrightarrow QG^{0+,0+}_{0,0|k,\beta}([W/\mathbf{G}])$ is the inclusion, $N^{\mathrm{vir}}_{F_{k,\beta}}$ is the virtual normal bundle and $e^{\mathbb{C}^*}$ denotes the equivariant Euler class. It is a formal function of $\mathbf{t}$ taking values in

$$H^*(W /\!\!/ \mathbf{G}, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda \left\{ \left\{ \frac{1}{z} \right\} \right\}$$

The evaluation maps

$$\hat{ev}_\beta = \hat{ev}_j : F_\beta \to [W/\mathbf{G}] \,,$$

do not depend on the choice of j, since all marked points are concentrated on $0 \in \mathbb{P}^1$. It follows from this and (2.6) that the big $\mathbb{I}$-function (2.7) becomes

$$\sum_{\beta \in \mathrm{Eff}(W, \mathbf{G}, \theta)} \sum_{k \geq 0} q^\beta (\mathrm{ev}_\bullet)_* \left(\exp\left(\frac{\hat{ev}_\beta^*(\mathbf{t})}{z} \right) \cap \frac{[F_\beta]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta})} \right), \tag{2.8}$$

where $\hat{\mathrm{ev}}_\beta : F_\beta \to [W/\mathbf{G}]$ is the evaluation map (independent of the marking), $N^{\mathrm{vir}}_{F_\beta}$ is the virtual normal bundle of F_β in $QG_{0,0,\beta}(W /\!\!/ \mathbf{G})$ and $e_{\mathbb{C}^*}$ denotes the equivariant Euler class. The specialization

$$\mathbb{I}(q, 0, z) = \sum_\beta q^\beta \mathbb{I}_\beta(z)$$

is called the small I-function of $W/\!\!/_{\theta}\mathbf{G}$. In this case, the terminology does agree with the one in [CFKM14].

Usually, we will only be concerned with the restriction of $\mathbf{t}$ to a finite dimensional subspace of $H^*([W/\mathbf{G}],\mathbb{Q}) \to H^*(W/\!\!/_{\theta}\mathbf{G},\mathbb{Q})$ as follows: Let

$$\kappa : H^*([W/\mathbf{G}],\mathbb{Q}) \to H^*(W/\!\!/\mathbf{G},\mathbb{Q})$$

denote the Kirwan map (surjective by [Kir84]) induced from the open immersion $W/\!\!/_{\theta}\mathbf{G} = [W^{ss}/\mathbf{G}] \subset [W/\mathbf{G}]$. Fix a homogeneous basis $\{\gamma_i\}_i$ of $H^*(W/\!\!/_{\theta}\mathbf{G})$ and choose homogeneous lifts $\tilde{\gamma}_i \in H^*([W/\mathbf{G}],\mathbb{Q})$ with $\kappa(\tilde{\gamma}_i) = \gamma_i$. After restricting to

$$\mathbf{t} := \sum_i t_i \tilde{\gamma}_i\,,$$

the big $\mathbb{I}$-function becomes a formal function of the finitely many variables $\{t_i\}$.

Now suppose that for some $\gamma_{i,\beta}(z) \in H^*(W/\!\!/_{\theta}\mathbf{G}) \otimes \mathbb{Q}[z]$,

$$(\mathrm{ev}_\bullet)^*\gamma_{i,\beta}(z) = \hat{\mathrm{ev}}_\beta^*(\tilde{\gamma}_i)\,.$$

Then, by the projection formula, the big $\mathbb{I}$-function (2.8) becomes

$$\mathbb{I}(q,\mathbf{t},z) = \sum_\beta \exp\left(\sum_i t_i\gamma_{i,\beta}(z)/z\right) q^\beta \mathbb{I}_\beta(z) \quad \text{for} \quad \mathbf{t} = \sum t_i\tilde{\gamma}_i\,.$$

From this we see that whenever the small I-function is known, in order to obtain an explicit formula for $\mathbb{I}$ it remains to find explicitly such classes $\gamma_{i,\beta}(z)$.

Let L be a line bundle on $[W/\mathbf{G}]$ of the form $L_j = [W \times \mathbb{C}_{\theta_0}/\mathbf{G}]$ for some character $\theta_0 \in \chi(\mathbf{G})$ in the same GIT chamber as θ. We can then take $\hat{H} = c_1(L)$ and restrict to $\tilde{\gamma}_{i,j}(\psi) = \hat{H}^i\psi^j$ where $\hat{\psi}$ is the insertion at the first light marking. Note that here are we are considering the natural extension to descendant invariants: in the definition of the big-$\mathbb{I}$ function, for $\mathbf{t}(\psi) = \mathbf{t}_0 + \mathbf{t}_1\psi + \mathbf{t}_2\psi^2 + \ldots \in H^*(W/\!\!/\mathbf{G})[\psi]$, let $\hat{ev}_i^*(\mathbf{t}(\psi)) =$

$\sum_k ev_i^*(\mathbf{t}_k)\psi_i^k$. By Lemma 5.2 in [CFK16], we can take $\gamma_{i,j,\beta}(\psi)(z) = (H+\beta(L)z)^i(-z)^j$, where $H = \kappa(\hat{H})$, to obtain

$$\mathbb{I}(q,\mathbf{t}(\psi),z) = \sum_{\beta} \exp\left(\frac{1}{z}\sum_{ij} t_{ij}(H+\beta(L)z)^i(-z)^j\right) q^\beta \mathbb{I}_\beta(z) \quad \text{for} \quad \mathbf{t}(\psi) = \sum_{i,j} t_{i,j}\hat{H}^i\hat{\psi}^j \tag{2.9}$$

2.7.3 Examples.

Let H denote the hyperplane class of $\mathbb{P}^n = \mathbb{C}^{n+1}/\!\!/\mathbb{C}^*$. In this case, we can take $\tilde{\gamma}_i^i = \hat{H}^i$ for $i = 0, \ldots, n$ where $\hat{H}$ is the hyperplane class on $[\mathbb{C}^{n+1}/\mathbb{C}^*]$ and $\gamma_{i,d}(z) = (H+dz)^i$. Combining this with the formula for the small I-function for projective space, we obtain

$$\mathbb{I}_{\mathbb{P}^n}(q,\mathbf{t},z) = \sum_{d=0}^{\infty} q^d \frac{\exp\left(\sum_{i=0}^n t_i(H+dz)^i/z\right)}{\prod_{k=1}^d (H+kz)^{n+1}}.$$

More generally, let $l_1, \ldots, l_r$ be the degrees of the hypersurfaces whose intersection is X, and let H denote the restiction of X to the hyperplane class on $\mathbb{P}^n$. Then,

$$\mathbb{I}_X(q,\mathbf{t},z) = \sum_{d=0}^{\infty} q^d \frac{\exp\left(\sum_{i=0}^n t_i(H+dz)^i/z\right)}{\prod_{k=1}^d (H+kz)^{n+1}} \prod_{m=1}^{r}\prod_{n=1}^{l_m d}(l_m H + nz) \tag{2.10}$$

2.7.4 Equivariant theory for non-compact targets

When $W/\!\!/_\theta\mathbf{G}$ is not projective, the integrals against the virtual class are not well-defined due to the lack of properness of the quasimap moduli spaces. However, as is the case with Gromov-Witten theory, there are many situations for which one can get an interesting theory of *equivariant* invariants via virtual localization.

More explicitly, suppose that W admits an additional action by an algebraic torus $\mathbf{T} \cong (\mathbb{C}^*)^r$ which commutes with the action of $\mathbf{G}$. There are induced actions on $[W/\mathbf{G}]$, on $W/\!\!/_\theta \mathbf{G}$ and on $W/_{\text{aff}}\mathbf{G}$. The proper morphism

$$W/\!\!/_\theta \mathbf{G} \to W/_{\text{aff}}\mathbf{G}$$

is $\mathbf{T}$-equivariant. Further, there are induced $\mathbf{T}$-actions on the moduli spaces of $Q_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta)$ and the associated graph spaces (c.f. 6.3 in [CFKM14]).

Assume that the $\mathbf{T}$-fixed locus in the affine quotient is proper. It follows that the $\mathbf{T}$-fixed loci in $W/\!\!/\mathbf{G}$ and in all the moduli spaces of quasimaps are also proper. Let

$$H^*_{\mathbf{T}}(\text{Spec}(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}[\lambda_1, \dots, \lambda_r]\,,$$

and

$$H^*_{\mathbf{T},\text{loc}}(\text{Spec}(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}(\lambda_1, \dots, \lambda_r)\,,$$

denote the equivariant cohomology and the localized equivariant cohomology of a point respectively. The cohomology ring of $W/\!\!/_\theta \mathbf{G}$ (and $[W/\mathbf{G}]$) is replaced by the localized equivariant cohomology

$$H^*_{\mathbf{T},\text{loc}}(W/\!\!/_\theta \mathbf{G}, \mathbb{Q}) = H^*_{\mathbf{T}}(W/\!\!/_\theta \mathbf{G}, \mathbb{Q}) \otimes \mathbb{Q}(\lambda_1, \dots, \lambda_r)$$

while the Novikov ring is now $\Lambda = H^*_{\mathbf{T},\text{loc}}(\text{Spec}(\mathbb{C}), \mathbb{Q})[[q]]$. The pairing is defined by the localization formula

$$\langle \delta, \gamma \rangle = \int_{W/\!\!/\mathbf{G}^{\mathbf{T}}} \frac{i^*(\delta\gamma)}{e_{\mathbf{T}}(N)}\,,$$

where $i : W/\!\!/_\theta \mathbf{G}^{\mathbf{T}}$ is the inclusion of the fixed point locus and N is the $e_{\mathbf{T}}(N)$ is the equivariant Euler class of the normal bundles. Quasimap invariants are defined by similar localization formulas. Both the pairing and the invariants take values in the field $H^*_{\mathbf{T},\text{loc}}(\text{Spec}(\mathbb{C}), \mathbb{Q})$.

Note that if $W/\!\!/_\theta \mathbf{G}$ is projective, and we take all insertions in the non-localized equivariant cohomology $H^*_{\mathbf{T}}(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$, then the $\mathbf{T}$-equivariant invariants may be defined without localization and take values in the ring $\mathbb{Q}[\lambda_1, \dots, \lambda_r]$. Specializing at $\lambda_1 = \cdots = \lambda_r = 0$ recovers the non-equivariant theory from the previous subsections.

The results present in the rest of the paper hold in the equivariant quasiprojective case, with no change in the arguments. To keep notation light, we will restrict the presentation to the case of projective targets $W/\!\!/_\theta \mathbf{G}$.

Chapter 3

The master space

Fix once and for all a triple $(W, \mathbf{G}, \theta)$ satisfying the assumptions in Section 2.1 and a positive rational number $0 < \lambda \ll 1$. From now on, we drop θ and λ from the notation.

Our goal is to compare the virtual cycle of the moduli space $Q_{g,m|n}(W/\!\!/_\theta\mathbf{G}, \beta)$ with m heavy and n light markings with the virtual cycle of the moduli space $Q_{g,m+1|n-1}(W/\!\!/_\theta\mathbf{G}, \beta)$ with one less light marking and one extra heavy one. The method used here is virtual localization on a larger moduli space, referred to as the "masterspace", parametrizing quasimaps with the additional data of a tangent vector at one of the markings. The idea of using this type of masterspace is due to Yang Zhou and first appeared in the appendix of [CJR17b] and in [Zho17].

3.1 The construction of $\widetilde{Q}_{g,m|n}(W/\!\!/_\theta\mathbf{G}, \beta)$

Definition 3.1.1. Let S be a scheme. An *S-family of genus-g, $(m|n)$-weighted stable quasimaps to $W/\!\!/\mathbf{G}$ of class β with mixed marking* consists of the data

$$(\pi : C \to S, x_1, \ldots, x_m, y_1, \ldots, y_n; P, u, N, v_1, v_2)\,,$$

where

(1) $(\pi : C \to S, \mathbf{x}, \mathbf{y}, P, u)$ is a family of $(m|n)$-prestable quasimaps of genus g to $W/\!\!/_\theta \mathbf{G}$;

(2) N is a line bundle on S;

(3) $v_1 \in H^0(S, T_{y_1} \otimes N)$ and $v_2 \in H^0(S, N)$ are sections without common zeros, where T_{y_1} is the line bundle on S formed by the relative tangent spaces at y_1.

This data is required to satisfy the following conditions on the geometric fibres:

(a) (*Nondegeneracy*) There are no basepoints at the x_i and each x_i doesn't intersect any other marking.

(b) (*Generic Stability*) The $\mathbb{Q}$-line bundle

$$\omega_C\left(\sum_{i=1}^n x_i + y_1 + \lambda \sum_{j=2}^n y_j\right) \otimes \mathcal{L}_\theta^\varepsilon$$

is ample for all positive rational numbers $\varepsilon > 0$.

(c) When $v_1 = 0$, y_1 isn't a basepoint and is separate from the rest of the markings.

(d) When $v_2 = 0$, the $\mathbb{Q}$-line bundle

$$\omega_C\left(\sum_{i=1}^n x_i + \lambda \sum_{j=1}^n y_j\right) \otimes \mathcal{L}_\theta^\varepsilon$$

is ample for every positive rational numbers $\varepsilon > 0$.

Remark 4. If v_2 is nowhere vanishing, then it gives a trivialization $N \cong \mathcal{O}_S$ which sends v_2 to 1. That is, the data (N, v_1, v_2) is equivalent to $v_1/v_2 \in H^0(S, T_{y_1})$. When $v_2 = 0$, v_1 is non-zero so (locally) gives an isomorphism $N \cong T_{y_1}^\vee$ sending v_1 to 1. Therefore, at every closed point $s \in S$, we can view (N, v_1, v_2) as a point $[v_1 : v_2] \in \mathbb{P}(T_{y_1} \oplus \mathcal{O}_s)$.

In other words, a family over a point consists of a prestable quasimap to $W/\!\!/_\theta \mathbf{G}$ together with a tangent vector $v_1/v_2 \in T_{y_1} \cup \{\infty\}$, where y_1 behaves as a heavy marking when $v_1 = 0$ and as a light marking when $v_2 = 0$.

Theorem 3.1.1. *There is a Deligne-Mumford stack of finite type $\widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta)$ parametrizing genus-g, $(m|n)$-weighted quasimaps of class β with mixed marking.*

Proof. This is a straightforward modification of the argument in [CFKM14]. Denote by $\mathfrak{M}$ the smooth Artin stack of genus-g nodal curves C with $m+n$ not necessarily distinct markings $x_1, \ldots x_m, y_1, \ldots y_n$ in the smooth locus of C, such that C has at most $2g-2+m+n$ irreducible components. This is a finite type smooth Artin stack. Let

$$\mathfrak{Bun}_{\mathbf{G}} \to \mathfrak{M} \quad \text{and} \quad \mathfrak{T}_{y_1} \to \mathfrak{M}$$

be the relative moduli stack of principal $\mathbf{G}$-bundles and the line bundle formed by the relative tangent spaces at y_1 respectively. Given a weighted quasimap with mixed marking $(C, \mathbf{x}, \mathbf{y}, P, u, N, v_1, v_2)$, recall the identification of the data (N, v_1, v_2) (at a geometric point) with a tangent vector in $T_{y_1} \cup \infty$ (see Remark 4). We have the forgetful morphisms

$$\widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \xrightarrow{\mu} \mathfrak{Bun}_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}}),$$

and

$$\widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \xrightarrow{\nu} \mathfrak{M}.$$

By Theorem 2.3.5 in [CFKM14] there is an open substack of finite type $\mathfrak{Z} \subset \mathfrak{Bun}_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}})$ fitting into the commutative diagram

$$\begin{array}{ccc} \mathfrak{Z} & \hookrightarrow & \mathfrak{Bun}_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}}) \\ & \searrow & \downarrow \\ & & \mathfrak{M} \end{array}$$

such that μ factors through $\mathfrak{Z}$. It then suffices to show that the morphism μ is representable and of finite type. Granted this it follows that the master space is an Artin stack of finite type. By stability it is, in fact, a Deligne-Mumford stack as required.

To show this, the same arguments as in [CFKM14] can be used to obtain an Artin stack with schematic morphism of finite type

$$\mathfrak{Hilb}_{\beta,g}(\mathfrak{B} \times_{\mathbf{G}} \overline{W}/\mathfrak{Z}) \times_{\mathfrak{Z}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{C}}) \xrightarrow{\mu} \mathfrak{Z}\,,$$

where $\overline{W}$ is a projective $\mathbf{G}$-equivariant completion of W, $\mathfrak{C}$ is the universal curve over $\mathfrak{Z}$ and $\mathfrak{B}$ is the universal principal $\mathbf{G}$-bundle over $\mathfrak{C}$. Its points parametrize tuples $(Y, [v_1 : v_2])$ where Y is a closed subscheme in the the fibres of the projection

$$q : \mathfrak{B} \times_{\mathbf{G}} \overline{W} \to \mathfrak{Z}\,,$$

and $[v_1 : v_2] \in T_{y_1} \cup \infty$. We then define a substack $\mathcal{H}$ of $\mathfrak{Hilb}_{\beta,g}(\mathfrak{B} \times_{\mathbf{G}} \overline{W}/\mathfrak{Z}) \times_{\mathfrak{Z}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{C}})$ as follows. First impose the following requirements succesively:

(i) the subscheme Y lies inside $P \times_{\mathbf{G}} W$

(ii) Y is the image of a section $u : C \to P \times_{\mathbf{G}} W$

(iii) the section u maps the generic points of the components of C, the nodes and the heavy markings to $P \times_G W^s$

(iv) (generic) stability holds i.e. $\omega_C(\sum x_i + y_1 + \varepsilon \sum_{j=2}^{n} y_j)$ is ample

Next, take $\mathcal{H}$ to be the union of the three substacks defined by the following conditions

(1) $v_1 = 0$ and u maps the marking y_1 to $P \times_{\mathbf{G}} W^s$ and y_1 doesn't intersect any other markings

(2) $v_2 = 0$ and $\omega_C(\sum x_i + \varepsilon \sum y_j)$ is ample

(3) $v_1 \neq 0$ and $v_2 \neq 0$

Since each of these is either an open or closed condition, $\mathcal{H}$ is a locally closed substack. Further, it is identified with $\widetilde{Q}_{g,m|n}(W/\!\!/_{\theta}\mathbf{G}, \beta)$.

□

Theorem 3.1.2. *The stack $\widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta)$ is proper.*

We split the argument in two parts: separatedness and properness. Both parts are proved using the respective valuative criteria and standard semistable reduction techniques.

3.2 Separatedness

Let R be a DVR over $\mathbb{C}$ with quotient field K. Let $\Delta = \mathrm{Spec}(R)$ and let $0 \in \mathrm{Spec}(R)$ be the closed point. Put $\Delta^\circ := \Delta \setminus \{0\} = \mathrm{Spec}(K)$. Assume we have two flat families of $(m|n)$-weighted quasimaps with mixed marking

$$(\pi : C \to \Delta, \mathbf{x}, \mathbf{y}, P, u, N, v_1, v_2) \quad \text{and} \quad (\pi : C' \to \Delta, \mathbf{x}', \mathbf{y}', P', u', N', v_1', v_2'),$$

which are isomorphic away from the central fibre. We will show that the isomorphism extends over 0. This is a straightforward modification of the argument in [CFKM14].

Proof. By semistable reduction, we can find a third family $\mathcal{C}$ of prestable-pointed curves over Δ which dominates both C and C' and preserves the sections. We may assume that the central fiber C_0 of $\mathcal{C}$ has automorphism group at most 1-dimensional, since the central fibres of C and C' both have the same property. This implies that $q : \mathcal{C} \to C$ and $q' : \mathcal{C} \to C'$ are obtained by blowing up either nodes in the central fibre or a point where at least two heavy markings coincide[1].

Pull back the rest of the data to $\mathcal{C}$ to obtain two families over Δ of *prestable* $(m|n)$-weighted quasimaps, i.e. families satisfying the nondegeneracy condition, but possibly not the stability condition on some components of the central fibre $\mathcal{C}_0$.

Over Δ°, the two families are isomorphic. Let Z_1, $Z_2 \subset \mathcal{C}$ be the base loci of the two families. Their union intersects $\mathcal{C}_0$ in a finite subset Z_0 of nonsingular points, so we can think of either family as giving a regular map from $\mathcal{C} \setminus \{Z_1 \cup Z_2) \to W/\!\!/\mathbf{G}$. These

[1]Need at least two special points on any rational component of $\mathcal{C}$

maps agree over Δ°, and therefore must be the same: either $W/\!\!/\mathbf{G}$ is projective, or we use the argument that if two principal $\mathbf{G}$-bundles on a regular complex surface S agree outside finitely many points, then they agree on S. It follows that the isomorphism between the two families extends to an isomorphism on $\mathcal{C} \setminus \{Z_0\}$. Since Z_0 is a finite set of smooth points on a surface, the isomorphism extends to all of $\mathcal{C}$.

Note that there cannot be any component of $\mathcal{C}_0$ which is contracted to a node by, say, the map $q : \mathcal{C} \to C$ but not by $q' : \mathcal{C} \to C'$. Indeed, over such a component, $q^*([u]^*L_\theta)$ must be trivial, while by stability $q'^*([u]^*L_\theta)$ is not trivial. Finally, if $\tilde{E}$ is a smooth rational subcurve of $\mathcal{C}$ containing y_1 and only one other special point, then $\tilde{E}$ cannot be contracted unless $\deg[u]^*L_\theta|_{\tilde{E}} = 0$, since contracting $\tilde{E}$ takes v_1 to 0 (so y_1 cannot coincide with any other markings). It follows that the initial families C and C' are isomorphic. □

3.3 Properness

Let R be a DVR over $\mathbb{C}$ and let K be its field of fractions. It suffices to show that given family of stable $(m|n)$-weighted quasimaps with mixed marking over $\Delta^\circ := \mathrm{Spec}(K)$

$$\xi^\circ = (\pi^\circ : C^\circ \to \Delta^\circ, \mathbf{x}^\circ, \mathbf{y}^\circ, P^\circ, u^\circ, N^\circ, v_1^\circ, v_2^\circ)$$

there exists, possibly after a finite base change, an extension

$$\xi = (\pi : C \to \Delta, \mathbf{x}, \mathbf{y}, P, u, N, v_1, v_2)$$

of π° to a family of stable $(m|n)$-weighted quasimaps with mixed marking over $\Delta = \mathrm{Spec}(R)$. By separatedness, this extension will be unique up to unique isomorphism.

Proof. Given a family $C \to \Delta$ of weighted quasimaps over Δ, denote by C_0 the central fibre and by $x_i(0)$ and $y_j(0)$ the markings on C_0.h

We begin by making a couple of reductions. If $v_1^\circ = 0$ or $v_2^\circ = 0$, then ξ° is a family of weighted quasimaps and the theorem then follows from the properness of the moduli spaces $Q_{g,m|n}(W/\!\!/\mathbf{G}, \beta)$. Hence, we may assume that $v_1^\circ \neq 0$ and $v_2^\circ \neq 0$ on Δ°. Note that this gives us a trivialization $N \cong \mathcal{O}_{\Delta^\circ}$ so that v_1/v_2 is equivalent to a tangent vector at y_1.

Next, we reduce to the case where $\pi^\circ : C^\circ \to \Delta^\circ$ is smooth. By standard stable reduction, possibly after finite base change, the normalization of C° is a disjoint union $\coprod_{k=0}^N C_k^\circ$ of smooth curves over Δ°. We view the preimages of $\mathbf{y}^\circ$ as light markings on the normalization, while the preimages of $\mathbf{x}^\circ$ and the nodes of C° are viewed as heavy markings. Assume that the preimage of y_1 is on C_0°. Then, for each $k > 0$, C_k° together with the markings (and the pulled-back principal $\mathbf{G}$-bundle and section) forms a family ξ_k° of weighted quasimaps in some $Q_{g_k,m_k|n_k}(W/\!\!/\mathbf{G}, \beta_k)(\Delta^\circ)$, where β_k is the numerical class of the u° restricted to the irreducible component C_k of C. Since these moduli spaces are proper, possibly after finite base change, ξ_k° extends uniquely to a Δ-family $\xi_k \in Q_{g_k,m_k|n_k}(W/\!\!/\mathbf{G}, \beta_k)(\Delta)$. For $i = 0$, the map $C_0^\circ \to C^\circ$ induces an isomorphism of relative tangent sheaves near (the preimage of) y_1. Therefore, the pointed curve C_0° together with the rest of the pulled-back data is a family in $\widetilde{Q}_{g_0,m_0|n_0}(W/\!\!/\mathbf{G}, \beta_0)(\Delta_0)$ with smooth total space and therefore, possibly after finite base-change, also extends to a family ξ_0 over Δ by assumption. By gluing together the ξ_i along the preimages of the nodes we get the required extension ξ of ξ°. Hence, we can assume that π° is smooth for the remainder of the proof.

We consider two cases: First we consider the case where $2g-2+n+\varepsilon m+\deg > 0$. In this case, when viewing y_1 as a light marking, the generic fibre is a family of stable $(m|n)$-weighed quasimaps so we can extend to an $(m|n)$-weighted family of stable quasimaps with light marking y_1. Since $v_1^\circ \neq 0$ gives us us a trivialization of $N \cong \mathcal{O}_\Delta$ there is a unique extension (N, v_1, v_2) to this family where v_1 and v_2 have no common zeroes.

There are now two possible situations that violate stability:

$$v_1(0) = 0 \text{ and, either } u(y_1(0)) = 0\,, \text{ or } y_1(0) = y_j(0) \text{ for } j \neq 1\,, \tag{3.1}$$

in the special fibre. If this happens, we blow up the total space of the family at the marking $y_1(0)$ of the special fibre. We then extend the quasimap $[u]$ to this new family over Δ. Since the vanishing order of v_1 at 0 drops by one after each step, we can repeat this procedure finitely many times until (3.1) does not hold. Finally, we contract the rational components on which $[u]^*(L_\theta)$ is trivial and the maximal sub-chains of exceptional divisors that do not contain any of the markings $y_j(0)$. Note that u^*L_θ has nontrivial degree on the exceptional divisor that contains y_1, so we obtain a stable family as required.

Suppose, now, that $2g-2+n+\varepsilon m+\deg \leq 1$. Due to the generic stability condition, this is only possible if $g = 0, n = 1$ and deg $[u]^*L_\theta < 1$. In this case we can find an isomorphism $C^\circ \cong \Delta^\circ \times \mathbb{P}^1$ that identifies $0 \times \Delta^\circ$ with y_1°, $\infty \times \Delta^\circ$ with x_1° and v_1°/v_2° with the tangent vector $\partial/\partial w$ where w is the coordinate on $\mathbb{P}^1$.

Begin by taking $C = \Delta \times \mathbb{P}^1$ and $N = \mathcal{O}_\Delta$. Set $v_1 = 1$, $v_2 = \partial/\partial w$. Again, extend the markings y_j° for $j \neq 1$ and the pre-stable quasimap to C. In this case, stability can only break in the following two ways

$$x_1(0) = y_j(0) \text{ for some } j \neq 1 \quad \text{ or } \quad u(x_1(0)) = 0\,. \tag{3.2}$$

If this happens, we blow-up C repeatedly at $x_1(0)$ until (3.2) does not hold. This only takes finitely many steps, since the order of vanishing of v_1 goes down by one at each step, and x_1 does not intersect any other markings in the generic fibre. Finally, we blow-down the unstable components to obtain the required extension. □

3.4 The Perfect Obstruction Theory

Denote by $\mathfrak{M}$ the smooth Artin stack of genus g nodal curves C with $m+n$ not necessarily distinct markings $x_1, \dots, x_m, y_1, \dots, y_n$ in the smooth locus of C, such that C has at most $2g-2+m+n$ irreducible components. Let $\widetilde{Q}$ denote the masterspace $\widetilde{Q}_{g,m|n}(W/\!\!/\mathbf{G}, \beta)$. Let

$$\mathfrak{B}un_{\mathbf{G}} \to \mathfrak{M} \quad \text{and} \quad \mathfrak{T}_{y_1} \to \mathfrak{M}$$

bet the relative moduli stack of principal $\mathbf{G}$-bundles and the line bundle formed by the relative cotangent spaces at y_1 respectively. There is a forgetful morphism $\widetilde{Q} \to \mathfrak{B}un_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}})$ to the smooth Artin stack parametrizing $(C, \mathbf{x}, \mathbf{y}, P, [v_1 : v_2])$. Consider the following diagram

$$\begin{array}{c} \mathfrak{B} \times_{\mathbf{G}} W \\ \downarrow \uparrow u \\ \mathfrak{C} \\ \downarrow \pi \\ \widetilde{Q} \\ \downarrow \mu \\ \mathfrak{B}un_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}}) \\ \downarrow \nu \\ \mathfrak{M} \end{array}$$

where $\mathfrak{C}$ is the universal curve, $\mathfrak{B}$ is the universal principal $\mathbf{G}$-bundle,u the universal section and μ, ν are the forgetful maps. We have a canonical perfect relative obstruction theory

$$E_\mu = R\pi_*(\mathcal{H}\mathrm{om}(\mathbb{L}_u, \mathcal{O}_{\mathfrak{C}}))[1]\,, \tag{3.3}$$

where $\mathbb{L}_u$ denotes the cotangent complex of u. Since $\mathfrak{B}un_{\mathbf{G}} \times_{\mathfrak{M}} \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}})$ and $\mathfrak{M}$ are smooth Artin stacks, E_μ defines an absolute perfect obstruction theory $E_{\widetilde{Q}}$ for the masterspace.

3.5 Two distinguished triangles

The two following distinguished triangles will be our primary tools for analysing the fixed and moving parts of the obstruction theory for the master space. Both arise as (dual versions of) (2.1). Denote by $\mathfrak{P}$ the smooth Artin stack $\mathfrak{B}un_{\mathbf{G}} \times \mathbb{P}(\mathfrak{T}_{y_1} \oplus \mathcal{O}_{\mathfrak{M}})$ and recall that $E_{\widetilde{Q}}$ denotes the (absolute) perfect obstruction theory of the masterspace. First, we have the distinguished triangle

$$\mu^* T_{\mathfrak{P}}[-1] \to E_{\mu}^{\vee} \to E_{\widetilde{Q}}^{\vee} \xrightarrow{+1}, \tag{3.4}$$

associated to the composition $\widetilde{Q} \to \mathfrak{P} \to \mathrm{Spec}(k)$. Similarly, associated the composition $\mathfrak{P} \to \mathfrak{M} \to \mathrm{Spec}(k)$ we have the distinguished triangle

$$\nu^* T_{\mathfrak{m}}[-1] \to T_{\nu}^{\vee} \to T_{\mathfrak{P}} \xrightarrow{+1} \tag{3.5}$$

Chapter 4

Virtual Localization on the Master Space

Suppose we have a $\mathbb{C}^*$-action on a Deligne-Mumford stack X and a $\mathbb{C}^*$-equivariant perfect obstruction theory $E_X^\bullet$. Then the fixed part of its restriction to a connected component X_i of the fixed locus determines a perfect obstruction theory for X_i and thus gives rise to virtual fundamental class $[X_i]^{vir} \in A_*(X_i)$. On the other hand, the moving part of $E_X^\bullet|_{X_i}$ determines the virtual normal bundle N_i^{vir} to X_i. The virtual localization formula [GP99] is then

$$[X]^{vir} = \sum_i (\iota_i)_* \left(\frac{[X_i]^{vir}}{e(N_i^{vir})} \right) \in A_*^{\mathbb{C}^*}(X) \otimes \mathbb{Q}\big[t, \frac{1}{t}\big]\,, \tag{4.1}$$

where ι_i is the inclusion of X_i into X and t is the generator of the $\mathbb{C}^*$-equivariant ring of a point. The master space $\widetilde{Q}_{g,m|n}([W/\mathbf{G}], \beta)$ admits a torus action

$$t \cdot (C, \mathbf{x}, \mathbf{y}, P, u, N, v_1, v_2) = (C, \mathbf{x}, \mathbf{y}, P, u, N, tv_1, v_2) \quad \text{for } t \in \mathbb{C}^*$$

In the following section, we identify the fixed loci along with their virtual classes and virtual normal bundles. Recall that we denote by $z \in A^1(B\mathbb{C}^*)$ the first Chern class of

$\mathbb{C}_1$, the standard action with weight 1.

4.1 Fixed loci and their contributions

Set-theoretically, the fixed loci are given by

(i) $\widetilde{F}_0 = \{v_1 = 0\}$;

(ii) $\widetilde{F}_\infty = \{v_2 = 0\}$;

(iii) for each $0 < \beta' \leq \beta$ and $J \subseteq \{1, \ldots, n\}$ for which $1 \in J$, we have the fixed locus $\widetilde{F}_{\beta',J}$ for which the underlying curve C consists of two subcurves attached at a node:

- C_0: a rational component with one (distinct) special point $y_1 = y_j$, $j \in J$ which is a basepoint of length $\beta'(L_\theta)$;
- C_g: a genus-g subcurve with the rest of the markings such that the restriction of the quasimap to C_g has class $\beta - \beta'$.

To see this, suppose v_1 and v_2 are both non-zero. Let C_0 be the irreducible component of C that contains y_1. Then the only way $t \in \mathbb{C}^*$ can scale (tv_1, v_2) back to (v_1, v_2) is if there is a non-trivial automorphism of C_0 that fixes y_1 as well as all the rest of the markings and nodes, so C_0 must be a rational curve that meets the rest of C at a single node. Hence, C_0 contains no heavy markings, and all other light markings on C_0 (if any) must coincide with y_1. Moreover, this non-trivial automorphism must fix the quasimap, so the quasimap is constant outside of the basepoint y_1, which carries all of the class β', where $0 < \beta' \leq \beta$.

Lemma 4.1.1. *The virtual localization contributions of the first two fixed loci are given by*

$$\frac{[\widetilde{F}_0]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{\widetilde{F}_0})} = \frac{[Q_{g,m+1|n-1,\beta}]^{vir}}{z - \psi_{y_1}} \quad \textit{and} \quad \frac{[\widetilde{F}_\infty]^{vir}}{e_{\mathbb{C}^*}(N^{vir}_{\widetilde{F}_\infty})} = \frac{[Q_{g,m|n,\beta}]^{vir}}{-z + \psi_{y_1}}$$

Proof. We prove this for $\widetilde{F}_0$. The other case is similar. Since v_1 vanishes identically on $\widetilde{F}_0$, y_1 does not intersect any other markings there so we have $\widetilde{F}_0 \cong Q_{g,m+1|n-1,\beta}$. Consider the restriction of the distinguished triangle (3.4) to $\widetilde{F}_0$. The complex $E_\mu^\vee|_{\widetilde{F}_0}$ only has fixed part since the action on the section is trivial over $\widetilde{F}_0$. By Lemma 2.5.1, it is identified with the relative perfect obstruction theory of $Q_{g,m+1|n-1,\beta}$ over $\mathfrak{B}un_{\mathbf{G}}$ as defined in [CFKM14]. The fixed and moving parts of $\mu^* T_{\mathfrak{P}}|_{\widetilde{F}_0}$ are identified with $\mu^* T_{\mathfrak{B}un_{\mathbf{G}}}$ and $T_{y_1} \otimes \mathbb{C}_1$. To see this, observe that since v_2 is never vanishing near $\widetilde{F}_0$, it gives a trivialization of N so that (N, v_1, v_2) is equivalent to a section v_1/v_2 of T_{y_1}. □

Proposition 4.1.1. *Stack-theoretically, we have*

$$\widetilde{F}_{\beta',J} \cong Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'} \times_{W/\!\!/_\theta \mathbf{G}} F_{\beta',|J|}\,, \tag{4.2}$$

where $F_{\beta',|J|}$ is the distinguished fixed locus in the graph space $QG^{0+,0+}_{g,0||J|}([W/G],\beta')$ for which the markings and the entire class β' are over $0 \in \mathbb{P}^1$ and the fibre product is taken over the proper evaluation maps to the GIT quotient $W/\!\!/_\theta \mathbf{G}$.

Proof. Given an element of $Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}(W/\!\!/\mathbf{G}) \times F_{\beta',|J|}$ glue the two quasimaps along the markings $\bullet$ and ∞. For the line bundle we can take $\mathcal{O}_S$ and for the tangent vector $[v_1 : v_2] \in T_{y_1} \cup \infty$ take $v_1 = \partial/\partial w$ and $v_2 = 1$, where w is the coordinate near 0 (i.e. on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$). Note that this choice is consistent with the placement of y_1 at 0 in the parametrized $\mathbb{P}^1$.

Conversely, suppose we are given a family

$$(\pi : C \to S, \mathbf{x}, \mathbf{y}, P, u, N, v_1, v_2) \in \widetilde{F}_{\beta',J}(S)\,.$$

By the definition of $\widetilde{F}_{\beta',J}$, the restriction of such a family over a closed point $s \in S$ arises uniquely by gluing as above, since there is a unique (up to quasimap isomorphism) parametrization $\phi : C_{0,s} \cong \mathbb{P}^1$ of the rational component sending y_1 to 0, the node to ∞ and the tangent vector $(v_1/v_2)_s$ to $\partial/\partial w$. We now need to show that we can split the node in families. To see this note that, over each closed point $s \in S$, $\mathbb{C}^*$ acts non-trivially on the tangent space to $C_{0,s}$ at the node $C_{0,s} \cap C_{g,s}$. On the other hand, since $\mathbb{C}^*$ acts trivially on $C_{g,s}$, it acts non-trivially on the first order deformation smoothing the node. Since the family is fixed by $\mathbb{C}^*$, the S-family of curves C can be decomposed as C_g and C_0 glued at the a pair of markings. Finally, to recover the gluing data, restrict the family of quasimaps to C_g and C_0, glued at the pair of markings.

□

Proposition 4.1.2. *For $0 < \beta' \leq \beta$ and $J \subset \{1, \dots, n\}$, we have*

$$[\widetilde{F}_{\beta',J}]^{vir} = \Delta^! \big(\mathrm{pr}_1^*([Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir}) \otimes \mathrm{pr}_2^*([F_{\beta',|J|}]^{vir})\big)$$

and

$$\frac{1}{e_{\mathbb{C}^*}\big(N^{vir}_{\widetilde{F}_{\beta',J}}\big)} = \frac{1}{\widetilde{\mathrm{pr}}_1^*(z+\psi_{x_\bullet})\,\widetilde{\mathrm{pr}}_2^*\big(e_{\mathbb{C}^*}(N^{vir}_{F_{\beta'}/QG_{0,0,\beta'}(W/\!\!/\mathbf{G})})(-z)^{|J|-1}\big)}$$

where Δ is the diagonal in the cartesian diagram

$$\begin{array}{ccc} Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'} \times_{W/\!\!/_\theta \mathbf{G}} F_{\beta',|J|} & \longrightarrow & Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'} \times F_{\beta',|J|} \\ \downarrow & & \downarrow {\scriptstyle ev_\bullet \times ev_\infty} \\ W/\!\!/_\theta \mathbf{G} & \xrightarrow{\quad\Delta\quad} & W/\!\!/_\theta \mathbf{G} \times W/\!\!/_\theta \mathbf{G} \end{array},$$

pr_1 *and* pr_2 *are the projections of the fibre product over* $\mathrm{Spec}(k)$ *and* $\widetilde{pr}_1$ *and* $\widetilde{\mathrm{pr}}_2$ *are the projections of the fibre product over $W/\!\!/_\theta \mathbf{G}$.*

Proof. Consider the restriction of the distinguished triangle (3.4) to $\widetilde{F}_{\beta',J}$

$$\mu^* T_{\mathfrak{P}}[-1]\big|_{\widetilde{F}_{\beta',J}} \to E_\mu^\vee\big|_{\widetilde{F}_{\beta',J}} \to E_{\widetilde{Q}}^\vee\big|_{\widetilde{F}_{\beta',J}} \xrightarrow{+1} . \tag{4.3}$$

From this we see that it suffices to compute the fixed and moving parts of the two terms on the left. First we examine the relative (dual) perfect obstruction theory in the middle. Let $\mathcal{C}_{\beta',J} \to \widetilde{F}_{\beta',J}$ be the pullback of the universal curve under the inclusion

$i_{\beta',J} : \tilde{F}_{\beta',J} \hookrightarrow \widetilde{Q}_{g,m|n,\beta}$. Then $\mathcal{C}_{\beta',J} \cong \mathcal{C}_g \cup \mathcal{C}_0$ where $\mathcal{C}_g$ and $\mathcal{C}_0$ are the pullbacks of the universal curves under $\widetilde{\mathrm{pr}}_1$ and $\widetilde{\mathrm{pr}}_2$ respectively. Let $\Delta_0 \in \mathcal{C}_0$ and $\Delta_g \in \mathcal{C}_g$ be the node $\mathcal{C}_0 \cap \mathcal{C}_g$. Denote by $\widetilde{E}_\bullet$ the restriction of the sheaf

$$\mathcal{H}\mathrm{om}(\mathbb{L}_u, \mathcal{O}_{\mathfrak{C}})[1]$$

to $\mathcal{C}_{\beta'}$. That is, $R\pi_*(\widetilde{E}_\bullet)$ is the relative (dual) perfect obstruction theory $E_\mu^\vee|_{\widetilde{F}_{\beta'}}$ in (3.3). It follows from the normalization exact sequence that we have the following short exact sequence

$$0 \longrightarrow \widetilde{E}_\bullet|_{\mathcal{C}_g}(-\Delta_0) \longrightarrow \widetilde{E}_\bullet \longrightarrow \widetilde{E}_\bullet|_{\mathcal{C}_0} \longrightarrow 0\,,$$

where the first and third terms are pushed forward along the inclusions. This gives us in turn the distinguished triangle

$$R\pi_*(\widetilde{E}_\bullet|_{\mathcal{C}_g}(-\Delta_0)) \longrightarrow R\pi_*(\widetilde{E}_\bullet) \longrightarrow R\pi_*(\widetilde{E}_\bullet|_{\mathcal{C}_0}) \xrightarrow{+1} \tag{4.4}$$

The term on the left is fixed and is identified with the pullback under pr_1 of the relative (dual) perfect obstruction theory of $Q_{g,m+\bullet,\beta-\beta'}$. Similarly, the fixed part of the term on the right is is identified with the pullback under pr_2 of the relative (dual) perfect obstruction theory of $F_{\beta',|J|}$ in the graph space $QG^{0+,0+}_{0,0||J|,\beta'}([W/\mathbf{G}])$. On the other hand, the moving part of the term on the right is exactly the (pullback of the) virtual normal bundle of $F_{\beta',|J|}$.

Next, we need to look at $\mu^* T_{\mathfrak{P}}[-1]|_{\widetilde{F}_{\beta',J}}$. Let $\mathcal{D} \in \mathfrak{M}$ be the closed substack where the curve is the union of a genus-g curve C_g with $m+n-|J|$ markings and a rational component C_0 carrying the markings $y_1 = y_j$, $j \in J$. We have the identification

$$\mu^* T_{\mathfrak{P}}\big|_{\widetilde{F}_{\beta'}} = \mu^*\Big(T_{\mathfrak{P}}\big|_{\nu^{-1}(\mathcal{D})}\Big) \tag{4.5}$$

The restriction of the distinguished triangle (3.5) to $\nu^{-1}(\mathcal{D})$ is

$$\nu^* T_{\mathfrak{m}}[-1]\big|_{\nu^{-1}(\mathcal{D})} \to T_\nu^\vee\big|_{\nu^{-1}(\mathcal{D})} \to T_{\mathfrak{P}}\big|_{\nu^{-1}(\mathcal{D})} \xrightarrow{+1} \tag{4.6}$$

The moving part of the middle term comes from deformations of the section $[v_1 : v_2]$ and this cancels with infinitesimal automorphisms from the first term. What remains is the (pullback) of the normal bundle of $\mathcal{D}$ in $\mathfrak{M}$, which is identified with $(T_{x_\bullet} \boxtimes T_\infty) \oplus (\mathcal{O}_{\mathcal{D}} \otimes \mathbb{C}_{-1})^{|J|-1}$. Putting it all together and by [BF97], Proposition 7.2., we obtain the required equality. $\square$

Chapter 5

The Wall-crossing Formula

Let i and j be non-negative integers and denote by $\hat{ev}$ and $\hat{\psi}$ the evaluation map and cotangent class at the mixed marking respectively. Let $\hat{H} = c_1(L)$ for a line bundle L on $[W/\mathbf{G}]$ as described in (2.7.2). By applying virtual localization (4.1) and by the fixed loci computations in the previous chapter, we obtain the following relation in $A_*^{\mathbb{C}}(\widetilde{Q}) \otimes \mathbb{Q}[z, z^{-1}]$

$$\hat{ev}^*(\hat{H}^i)\hat{\psi}^j \cap [\widetilde{Q}_{g,m|n,\beta}]^{vir} = \sum_{\widetilde{F}} (\alpha_{\widetilde{F}})_* \left(\frac{(\alpha_{\widetilde{F}})^*(\hat{ev}^*(\hat{H}^i)\hat{\psi}^j) \cap [\widetilde{F}]^{vir}}{e_{\mathbb{C}}^*(N_{\widetilde{F}}^{vir})} \right), \tag{5.1}$$

where $\widetilde{F}$ ranges over the fixed loci and $\alpha_{\widetilde{F}}$ is the inclusion $\alpha_{\widetilde{F}} : \widetilde{F} \hookrightarrow \widetilde{Q}$. In what follows, we push forward this relation along a suitable stabilization map to obtain a vanishing that ultimately leads to the desired wall-crossing formula.

The GIT set-up gives rise (see [CFK13], 3.1 for more details) to a natural embedding $\iota : W/\!\!/_\theta \mathbf{G} \hookrightarrow \mathbb{P}^M$ for some positive integer M, and morphisms (denoted by the same letter)

$$\iota : Q_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

where $d(\beta) := \iota_*(\beta) \in \mathrm{Hom}(\mathrm{Pic}([\mathbb{C}^{M+1}/\mathbb{C}^*]), \mathbb{Z}) \cong \mathbb{Z}$. Similarly for the masterspace.

Next, we have a stabilization map

$$c : \widetilde{Q}_{g,m|n}(\mathbb{P}^M, d(\beta)) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta)) \tag{5.2}$$

which forgets the data (N, v_1, v_2) and stabilizes if necessary by contracting "rational tails" while keeping track of the degree. More explicitly, let

$$(\widetilde{C}, \tilde{x_1}, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y_n}, \widetilde{L}, (\tilde{s}_i)_{i=0,\ldots,M}, \widetilde{N}, \tilde{v}_1, \tilde{v}_2)$$

be a geometric point in $\widetilde{Q}_{g,m|n}(\mathbb{P}^M, d(\beta))$, where $\widetilde{L}$ is a line bundle on $\widetilde{C}$ and the $(\tilde{s}_i)$ are $M+1$ sections. Let T the maximal connected tree of rational curves in the domain curve $\widetilde{C}$ satisfying the following properties (if it exists):

(i) T contains the mixed marking y_1 and does not contain any of the heavy markings;

(ii) T meets the rest of the curve C at a single point p.

Let d_T be the degree of the restriction to T of the quasimap. Denote by C the closure of $\widetilde{C} \setminus T$. We then have the line bundle

$$L := \tilde{L}|_C \otimes \mathcal{O}_C\left(d_T\, p\right)\ ,$$

on C, with sections $s_0, \ldots, s_M$ obtained as the compositions

$$s_j : \mathcal{O}_C \xrightarrow{\tilde{s_j}} \tilde{L}|_C \to \tilde{L}|_C \otimes \mathcal{O}_C\left(d_T\, p\right)\ .$$

The image of the quasimap with mixed marking under the morphism (5.2) is the stable $(m|n)$-weighted quasimap

$$(C, x_1, \ldots, x_m, y_1, \ldots, y_n, L, (s_i)_{i=0,\ldots,M})$$

where we have modified the light markings as follows:

(i) If $\tilde{y_j} \in T$, then $y_j = p$;

(ii) otherwise, $y_j = \tilde{y}_j$ remains unchanged.

For a proof that the above construction can be done functorially in families, look at [LLY00] and the discussion in Section 3.1 of [CFK13].

To lighten notation in what follows, denote by

$$\rho : \widetilde{Q}_{g,m|n}(W/\!\!/_{\theta}\mathbf{G}, \beta) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

the composition $\widetilde{Q}_{g,m|n}(W/\!\!/_{\theta}\mathbf{G}, \beta) \xrightarrow{\iota} \widetilde{Q}_{g,m|n}(\mathbb{P}^M, d(\beta)) \xrightarrow{c} Q_{g,m|n}(\mathbb{P}^M, d(\beta))$. Almost identically to the construction above, we have a morphism (denoted by the same letter)

$$c : Q_{g,m+1|n-1}(\mathbb{P}^M, d(\beta)) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

converting the last heavy marking into a light one. Note that c fits into the diagram

$$\begin{array}{ccc} \widetilde{F}_0 \cong Q_{g,m+1|n-1,\beta} & \xhookrightarrow{\alpha_{\widetilde{F}_0}} & \widetilde{Q}_{g,m|n,\beta} \\ \downarrow \iota & & \downarrow \rho \\ Q_{g,m+1|n-1}(\mathbb{P}^M, d(\beta)) & \xrightarrow{c} & Q_{g,m|n}(\mathbb{P}^M, d(\beta)) \end{array}$$

Note that $\rho \circ \alpha_{\widetilde{F}_\infty} = \iota$. Next, for $0 < \beta' \leq \beta$ and $J \subseteq \{1, \ldots, n\}$ with $1 \in J$, we have a morphism

$$b_{\beta',J} : Q_{g,m|y_1,(y_j)_{j \notin J}}(\mathbb{P}^M, d(\beta - \beta')) \to Q_{g,m|n}(\mathbb{P}^M, d(\beta))$$

that places the markings $(y_j)_{j \in J, j \neq 1}$ and a basepoint of length $d(\beta - \beta')$ at y_1. More explicitly, given stable $(m|n - |J| + 1)$ weighted quasimap to $\mathbb{P}^M$

$$(C, x_1, \ldots, x_m, y_1, (y_j)_{j \notin J}, L, (s_i)_{i=0,\ldots,M})$$

over a geometric point, the data

$$\big(C, x_1, \ldots, x_m, y_1, \ldots, y_n, L \otimes \mathcal{O}_C\big(d(\beta-\beta')y_1\big), (\hat{s}_i)_{i=0,\ldots,N}\big)$$

determine an $(m|n)$-weighted quasimap to $\mathbb{P}^M$, where $y_j = y_1$ for $j \in J$, and, as before, the sections $\hat{s}_i$ are obtained as compositions

$$\hat{s}_i : \mathcal{O}_C \xrightarrow{s_i} L \to L \otimes \mathcal{O}_C\big(d(\beta-\beta')y_1\big)\,.$$

In this case, $b_{\beta',J}$ fits into the following commutative diagram

$$\begin{array}{ccc}
\widetilde{F}_{\beta',J} & \xrightarrow{\alpha_{\widetilde{F}_{\beta',J}}} & \widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta) \\
\big\downarrow \widetilde{\mathrm{pr}}_1 & & \\
Q_{g,m+1|(y_j)_{j\notin J}}(W/\!\!/_\theta \mathbf{G}, \beta-\beta') & & \big\downarrow \rho \\
\big\downarrow \iota & & \\
Q_{g,m+1|(y_j)_{j\notin J}}(\mathbb{P}^M, d(\beta-\beta')) & \xrightarrow{b_{\beta',J}} & Q_{g,m|n}(\mathbb{P}^M, d(\beta))
\end{array}$$

Lemma 5.0.1. *Let $\Delta : W/\!\!/_\theta \mathbf{G} \to W/\!\!/_\theta \mathbf{G} \times W/\!\!/_\theta \mathbf{G}$ denote the diagonal map. We have*

$$(\widetilde{pr}_1)_*\big(\Delta^!([Q_{g,J,\beta'}]^{vir} \otimes [F_{\beta',|J|}]^{vir})\big) = [Q_{g,J,\beta'}]^{vir} \cap (ev_\bullet)^*\big((ev_0)_*[F_{\beta',|J|}]^{vir}\big)\,,$$

where all the quasimap moduli spaces have target $W/\!\!/_\theta \mathbf{G}$ and we have used the shorthand notation $Q_{g,J,\beta'}$ for the moduli stack $Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}$.

Proof. Consider the fibre product diagram

$$\begin{array}{ccc}
Q_{g,J,\beta'} \times_{W/\!\!/_\theta \mathbf{G}} F_{\beta',|J|} & \longrightarrow & Q_{g,J,\beta'} \times F_{\beta',|J|} \\
\big\downarrow \widetilde{pr}_1 & & \big\downarrow \mathrm{id}\times \mathrm{ev}_\infty \\
Q_{g,J,\beta'} & \xrightarrow{(\mathrm{id},\ \mathrm{ev}_\bullet)} & Q_{g,J,\beta'} \times W/\!\!/_\theta \mathbf{G} \\
\big\downarrow \mathrm{ev}_\bullet & & \big\downarrow \mathrm{ev}_\bullet \times \mathrm{id} \\
W/\!\!/_\theta \mathbf{G} & \xrightarrow{\Delta} & W/\!\!/_\theta \mathbf{G} \times W/\!\!/_\theta \mathbf{G}
\end{array}$$

The middle map is a regular embedding since it is the embedding of the graph of the evaluation map $\mathrm{ev}_\bullet$. Then, by standard properties of refined Gysin pullbacks we have

$$\begin{aligned}(\widetilde{\mathrm{pr}}_1)_*\big(\Delta^!([Q_{g,J,\beta'}]^{\mathrm{vir}}\otimes[F_{\beta',|J|}]^{\mathrm{vir}})\big) &= \Delta^!\big((\mathrm{id}\times\mathrm{ev}_\infty)_*([Q_{g,J,\beta'}]^{\mathrm{vir}}\otimes[F_{\beta',|J|}]^{\mathrm{vir}})\big)\\ &= (\mathrm{id},\,\mathrm{ev}_\bullet)^*([Q_{g,J,\beta'}]^{\mathrm{vir}}\otimes(\mathrm{ev}_\infty)_*[F_{\beta',|J|}]^{\mathrm{vir}})\\ &= \Delta^!([Q_{g,J,\beta'}]^{\mathrm{vir}}\times(\mathrm{ev}_\infty)_*[F_{\beta',|J|}]^{\mathrm{vir}}\\ &= [Q_{g,J,\beta'}]^{\mathrm{vir}}\cap(ev_\bullet)^*\big((ev_\infty)_*[F_{\beta',|J|}]^{\mathrm{vir}})\end{aligned}$$

□

We are now ready to prove the main result. As before, let $\mathbb{I}(q,\mathbf{t},z)$ denote the big $\mathbb{I}$-function (2.9) (where we have suppressed ψ from the notation $\mathbf{t}(\psi)$).

Theorem 1.2.1 (Heavy-to-light wallcrossing formula). *Let m,n be positive integers and i,j be (possibly zero) non-negative integers. For each subset $J\subseteq\{1,\dots,n\}$, let $\lambda_J(z)=(-1)^{|J|-1}z^{2-|J|}$. We then have the following relation in the homology group $H_*(Q_{g,m|n}(\mathbb{P}^M,d(\beta)),\mathbb{Q})$.*

$$\iota_*\left(\hat{ev}^*(\hat{H}^i)\hat{\psi}^j\cap[Q_{g,m|n,\beta}]^{vir}\right)=c_*\iota_*\big(ev_\bullet^*(H^i)\psi_\bullet^j\cap[Q_{g,m+\bullet|n-1,\beta}]^{vir}\big)$$
$$+\sum_{0<\beta'\leq\beta}\sum_J(b_{\beta',J})_*c_*\iota_*\left((ev_\bullet)^*\left(\left[\lambda_J(z)\frac{\partial}{\partial t_{i,j}}\mathbb{I}(q,\mathbf{t},z)\Big|_{t=0}\right]_{+,\beta'}\right)\Big|_{z=-\psi_\bullet}\cap[Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir}\right)$$

where $\hat{ev}$ and $\hat{\psi}$ are at the first light marking, $H=\kappa(\hat{H})$ for κ the Kirwan map, and the notation $[\ ]_{+,\beta'}$ indicates taking non-negative powers of z and then taking the coefficient of $q^{\beta'}$.

More generally, for arbitrary classes $\delta_1, \ldots, \delta_m \in H^*(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$, there is an equality

$$\iota_* \left(\prod_{k=1}^{n} ev_k^* \delta_k \, \hat{ev}^*(\hat{H}^i) \hat{\psi}^j \cap [Q_{g,m|n,\beta}]^{vir} \right) = c_* \iota_* \Big(\prod_{k=1}^{n} ev_k^* \delta_k \, ev_\bullet^*(H^i) \psi_\bullet^j \cap [Q_{g,m+\bullet|n-1,\beta}]^{vir} \Big)$$
$$+ \sum_{0<\beta'\leq\beta} \sum_J (b_{\beta',J})_* c_* \iota_* \Big(\prod_{k=1}^{n} ev_k^* \delta_k \, (ev_\bullet)^* \Big(\Big[\lambda_J(z) \frac{\partial}{\partial t_{i,j}} \mathbb{I}(q,\mathbf{t},z) \Big|_{t=0} \Big]_{+,\beta'} \Big) \Big|_{z=-\psi_\bullet}$$
$$\cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \Big)$$

Proof. First note that the equivariant ψ-classes on the masterspace $\widetilde{Q}_{g,m|n}(W/\!\!/_\theta \mathbf{G}, \beta)$ restrict to the non-equivariant ψ-classes on the fixed point components except in the following cases: for each $j \in J$, the class ψ_{y_j} restricts to $-z$ on $\widetilde{F}_{\beta',J}$. Further $\hat{H}$ restricts to $ev_\bullet^*(H) + \beta'(L)z$ on $\widetilde{F}_{\beta',J}$.

Next, consider the trivial $\mathbb{C}^*$-action on $\widetilde{Q}_{g,m|n}(W/\!\!/\mathbf{G}, \beta)$. The stabilization map ρ is $\mathbb{C}^*$-equivariant. Pushing forward the relation (5.1) along ρ we get a relation in $A_*^{\mathbb{C}}(Q_{g,m|n}(\mathbb{P}^M, d(\beta)) \otimes \mathbb{Q}[z, z^{-1}]$. The left hand side has no factors of negative powers of z so neither does the right hand side. In particular, the vanishing of the coefficient of z^{-1} gives us the required formula: After pushing foward the relation (5.1) along ρ, substituting the fixed loci contributions (4.1.1) and (4.1.2), and by (5.0.1) the right-hand-side becomes

$$\frac{i_* \left(\hat{ev}^*(\hat{H}^i) \hat{\psi}^j \cap [Q_{g,m|n,\beta}]^{vir} \right)}{z - \hat{\psi})} + c_* i_* \left(\frac{ev_\bullet^*(H^i) \psi_\bullet^j \cap [Q_{g,m+\bullet|n-1,\beta}]^{vir}}{-z + \psi_\bullet} \right)$$
$$+ \sum_{0<\beta'\leq\beta} \frac{1}{(-z)^{|J|-1}(z+\hat{\psi})} (b_{J,\beta'})_* c_* i_* \left(ev_\bullet^* \left((H + \beta'(L)z)^i (-z)^j \cap (ev_\infty)_* \left(\frac{[F_{\beta'}]^{vir}}{e_{\mathbb{C}^*}(N_{F_{\beta'}}^{vir})} \right) \right) \right.$$
$$\left. \cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \right) \tag{5.3}$$

By (2.9), the last summand can be written in terms of the big-$\mathbb{I}$-function as follows

$$\sum_{0<\beta'\leq\beta} \frac{1}{(-z)^{|J|-1}(z+\hat{\psi})}(b_{J,\beta'})_* c_* i_* \left(ev_\bullet^* \left(\left[z\frac{\partial}{\partial t_{ij}}\mathbb{I}\Big|_{\mathbf{t}=0} \right]_{\beta'} \right) \cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \right)$$

where the notation $[\,]_{\beta'}$ indicates taking the coefficient of $q^{\beta'}$. Expand this as a power series of z as follows

$$\sum_{0<\beta'\leq\beta} \frac{(-1)^{|J|-1}}{z^{|J|-1}} \left(1 - \frac{\hat{\psi}}{z} + \left(\frac{\hat{\psi}}{z}\right)^2 - \left(\frac{\hat{\psi}}{z}\right)^3 - \dots \right)(b_{J,\beta'})_* c_* i_* \Bigg(ev_\bullet^* \Bigg(\left[\frac{\partial}{\partial t_{ij}}\mathbb{I}\Big|_{\mathbf{t}=0} \right]_{\beta'} \Bigg) \\ \cap [Q_{g,m+\bullet|(y_j)_{j\notin J},\beta-\beta'}]^{vir} \Bigg). \tag{5.4}$$

After substituting this into (5.3) and expanding the rest as a power series of z, the vanishing of the coefficient of z^{-1} proves the first equality (1.2) of the theorem.

To obtain the more general version, substitute $\hat{ev}(\hat{H}^i)\,\hat{\psi}^j$ for $\prod_{k=1}^m ev_k^*\delta_k\,\hat{ev}(\hat{H}^i)\,\hat{\psi}^j$ in (5.1). The rest of the proof follows almost identically.

□

Remark 5. It should be stated that in the case where X is a hypersurface or complete intersection in projective space, Theorem 1.2.1 has been proven by I.Ciocan-Fontanine and Bumsig Kim in [CFK20] (see Remark 1.6). We discuss this explicit case in the following chapter.

Chapter 6

An application; the string and dilaton equations for stable quasimaps

An immediate application of the wall-crossing formula yields the modified versions of the string and dilaton equations for (0+)-stable quasimaps, which can in turn be used to prove other identities in (0+)-theory.

Recall that for arbitrary classes $\delta_i \in H^*(W/\!\!/_\theta \mathbf{G}, \mathbb{Q})$ and $\eta_j \in H^*([W/\mathbf{G}], \mathbb{Q})$, we define descendant $(m|n)$-weighted quasimap invariants as

$$\left\langle \delta_1\psi_1^{a_1}, \ldots, \delta_m\psi_m^{a_m} | \eta_1\hat{\psi}_1^{b_1}, \ldots, \eta_k\hat{\psi}_n^{b_n} \right\rangle_{g,m|n,\beta} = \int_{[Q_{g,m|n}(W/\!\!/_\theta \mathbf{G},\beta)]^{vir}} \prod_{i=1}^{m} ev_i^*(\delta_i)\psi_i^{a_i} \prod_{j}^{n} \hat{ev}_j^*(\eta_j)\hat{\psi}_j^{b_j} ,$$

where ψ_i is the psi-class associated to the ith heavy marking, $\hat{\psi}_j$ is the psi-class associated to the jth light marking, and a_i, b_j are non-negative integers. We begin by proving the following auxiliary result.

Proposition 6.0.1. *Let $\delta_1, \ldots, \delta_m$ be arbitrary classes in $H^*(W/\!\!/_\theta \mathbf{G})$ and i, j be non-negative integers. Then*

$$\sum_\beta q^\beta \langle \delta_1 \psi_1^{a_1}, \ldots, \delta_m \psi_m^{a_m} | \hat{H}^i \hat{\psi}^j \rangle_{g,m|1,\beta} =$$
$$\sum_\beta q^\beta \Big\langle \delta_1 \psi_1^{a_1}, \ldots, \delta_m \psi_m^{a_m}, \left[z \frac{\partial}{\partial t_{ij}} \mathbb{I} \right]_+ \Big|_{t=0, z=-\psi_{n+1}} \Big\rangle_{g,m+1,\beta}$$

Proof. Apply (1.3) of the main theorem (1.2.1) and sum over β. □

As mentioned earlier, in contrast to quasimaps without light markings, there exist forgetful morphisms

$$Q_{g,m|n+1}(W/\!\!/\mathbf{G}, \beta) \to Q_{g,m|n}(W/\!\!/\mathbf{G}, \beta)$$

forgetting the last light marking, which are identified with the universal curve. In particular, we have the forgetful morphism

$$\tau : Q_{g,m|\bullet}(W/\!\!/\mathbf{G}, \beta) \to Q_{g,m}(W/\!\!/\mathbf{G}, \beta)$$

to the moduli space of (the usual) m-pointed (0+)-stable quasimaps. As in Gromov-Witten theory, we have a relation

$$\tau^* \psi_i = \psi_i + D_i \,,$$

where D_i is the boundary divisor corresponding to curves with one genus 0, degree-zero component carrying the marked points i and $\bullet$. From this we can derive an analogue of the *string equation*

$$\langle \delta_1 \psi^{a_1}, \ldots, \delta_m \psi^{a_m}, \mathbb{1} | \rangle_{g,m|\bullet,\beta} = \sum_{i=1}^{m} \langle \delta_1 \psi^{a_1}, \ldots \delta_i \psi^{a_i - 1}, \ldots, \delta_m \psi^{a_m} \rangle_{g,m,\beta} \,, \qquad (6.1)$$

and the *dilaton* equation

$$\langle \delta_1\psi^{a_1}, \dots, \delta_m\psi^{a_m}, |\hat{\psi}\mathbb{1}\rangle_{g,m|\bullet,\beta} = (2g-2+m)\langle \delta_1\psi^{a_1}, \dots, \delta_m\psi^{a_m}\rangle_{g,m,\beta} \tag{6.2}$$

where $\mathbb{1}$ denotes the unit in $H^*(W/\!\!/\mathbf{G})$.

Write the small I-function of $W/\!\!/_\theta\mathbf{G}$ as follows

$$\mathbb{I}_{W/\!\!/_\theta\mathbf{G}}(q,0,z) = \sum_{k\in\mathbb{Z}} z^{-k} I_k(q)\,.$$

Corollary 6.0.1 (String Equation for (0+)-stable quasimaps)**.** *Let $\delta_1, \dots, \delta_m \in H^*(W/\!\!/\mathbf{G})$ and $a_1, \dots, a_m$ be non-negative integers. Then, for all $m \geq 2$,*

$$\sum_\beta q^\beta \langle \delta_1\psi_1^{a_1}, \dots, \delta_m\psi_m^{a_m}, I_0(q)\mathbb{1}\rangle^{0+}_{g,m+1,\beta} =$$
$$\sum_\beta q^\beta \sum_{j=1}^m \langle \delta_1\psi_1^{a_1}, \dots, \delta_{j-1}\psi_{j-1}^{a_{j-1}}, \delta_j\psi_j^{a_j-1}, \delta_{j+1}\psi_{j+1}^{a_{j+1}}, \dots, \delta_m\psi_m^{a_m}\rangle^{0+}_{g,m,\beta}$$

Proof. By the string equation (6.1), the right-hand side of the equation above is equal to

$$\sum_\beta q^\beta \langle \delta_1\psi_1^{a_1}, \dots, \delta_m\psi_m^{a_m}, \mathbb{1}\rangle^{0+}_{g,m|\bullet,\beta}\,.$$

On the other hand, from (2.9) we have

$$\left[z\frac{\partial}{\partial t_{0,0}}\mathbb{I}\right]_+ \Bigg|_{\mathbf{t}=0,z=-\psi_\bullet} = I_0(q)\mathbb{1}\,.$$

The result then follows from (6.0.1). □

Corollary 6.0.2 (Dilaton equation for (0+)-stable quasimaps)**.** *With the same notation* *as above, we have*

$$\sum_{\beta} q^{\beta} \langle \delta_1 \psi_1^{a_1}, \ldots, \delta_m \psi_m^{a_m}, (I_0(q)\mathbb{1})\psi - I_1(q) \rangle_{g,m+1,\beta}^{0+} =$$

$$(2g-2+n) \sum_{\beta} q^{\beta} \langle \delta_1 \psi_1^{a_1}, \ldots, \delta_m \psi_m^{a_m} \rangle_{g,m,\beta}^{0+},$$

where the sum if over all θ-effective β.

Proof. By the dilaton equation (6.2), the right-hand side of the above equation is equal to

$$\sum_{\beta} q^{\beta} \langle \delta_1 \psi_1^{a_1}, \ldots, \delta_m \psi_m^{a_m}, \psi \mathbb{1} \rangle_{g,m|\bullet,\beta}^{0+}.$$

On the other hand, from (2.9) we have

$$\left[z \frac{\partial}{\partial t_{0,1}} \mathbb{I} \right]_+ \Bigg|_{\mathbf{t}=0, z=-\psi_\bullet} = \psi I_0(q) - I_1(q).$$

The result now follows from (6.0.1). □

Remark 6. The corollaries (6.0.1) and (6.0.2) were conjectured for all genera and proven for semi-positive targets and $g = 0$ in [CFK17, 3.4].

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