Lattice QCD near the light-cone

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Near light-cone coordinates

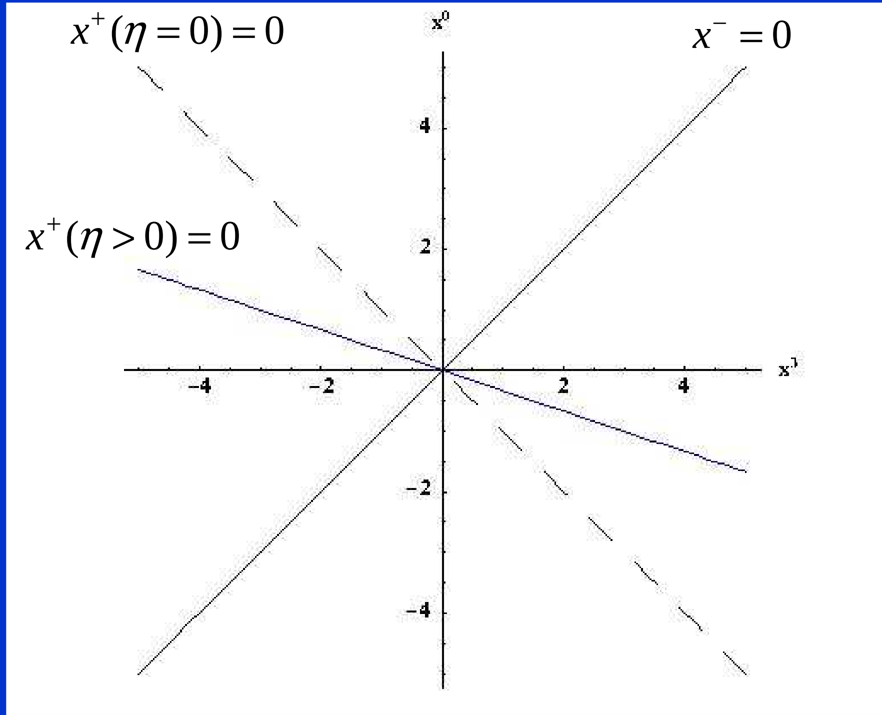
$$x^t = x^+ = \frac{1}{\sqrt{2}}\left\{\left(1+\frac{\eta^2}{2}\right)x^0 + \left(1-\frac{\eta^2}{2}\right)x^3\right\}$$

$$x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^3\right) .$$

x^1, x^2 are unchanged

- $\Leftrightarrow$ Boost with $\beta = \frac{1-\eta^2/2}{1+\eta^2/2}$

+ linear transformation



Motivation

- Near light-cone coordinates keep a direct link to equal time theories
- Near light-cone QCD has a nontrivial vacuum which cannot be neglected even in the light cone limit
- Near light-cone coordinates seem to be a promising tool in order to investigate high energy scattering on the lattice:
  - Nachtmann (**Eur.Phys.J.C7:459**): Meson-meson scattering amplitude governed by the correlation of two Wegner-Wilson loops near the light cone

- Intention: Measure the correlation function of two Wegner-Wilson loops on the lattice
- Normal way of doing: Go over to Euclidean time
  Write the action in terms of links and plaquettes
  Sample the path integral by a Monte-Carlo
- Euclidean gluonic Lagrange density

$$x^+ = -\mathrm{i}\, x_E^+ \qquad S = \mathrm{i}\int d^4x_E \mathcal{L}_E \equiv \mathrm{i} S_E \qquad Z = \int DA e^{-S_E}$$

$$\mathcal{L}_E \equiv \frac{1}{2} F^a_{+-} F^a_{+-} + \sum_k \left(\frac{\eta^2}{2} F^a_{+k} F^a_{+k} - \mathrm{i} F^a_{+k} F^a_{-k} \right) + \frac{1}{2} F^a_{12} F^a_{12}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\, f^{abc} A_\mu{}^b A_\nu{}^c$$

- a complex action remains $\Longrightarrow$ Ordinary MC sampling of the eucledian path integral is ruled out

$$\langle O \rangle = \left\langle e^{-i\,\mathrm{Im}(S_E)} O \right\rangle_{\mathrm{Re}(S_E)} \Big/ \left\langle e^{-i\,\mathrm{Im}(S_E)} \right\rangle_{\mathrm{Re}(S_E)}$$

- Possible way out: Hamiltonian formulation
  $\Longrightarrow$ Sampling of the ground state wavefunctional with guided diffusion quantum Monte-Carlo

Continuum Hamiltonian

- Perform Legendre transformation of the Lagrange density for $A_+ = 0$:

$$\mathcal{L} = \sum_a \left[\frac{1}{2} F^a_{+-} F^a_{+-} + \sum_{k=1}^{2} \left(F^a_{+k} F^a_{-k} + \frac{\eta^2}{2} F^a_{+k} F^a_{+k} \right) - \frac{1}{2} F^a_{12} F^a_{12} \right]$$

$$\Pi^a_k = \frac{\delta \mathcal{L}}{\delta \partial_+ A^a_k} = \frac{\delta \mathcal{L}}{\delta F^a_{+k}} = F^a_{-k} + \eta^2 F^a_{+k}$$

$$\Pi^a_- = \frac{\delta \mathcal{L}}{\delta \partial_+ A^a_-} = \frac{\delta \mathcal{L}}{\delta F^a_{+-}} = F^a_{+-}$$

- Then, the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} \sum_a \left[\Pi^a_- \Pi^a_- + F^a_{12} F^a_{12} + \sum_{k=1}^{2} \frac{1}{\eta^2} \left(\Pi^a_k - F^a_{-k} \right)^2 \right]$$

- The Hamiltonian shows electro-magnetic duality in the transverse fields

$$\Pi^a_k \longleftrightarrow F^a_{-k}$$

- The Hamiltonian has to be supplemented by Gauss law (EOM for A_+) :

$$G = D_- \Pi_- + \sum_{k=1}^{2} D_k \Pi_k$$

$$[\mathcal{H}, G] = 0$$

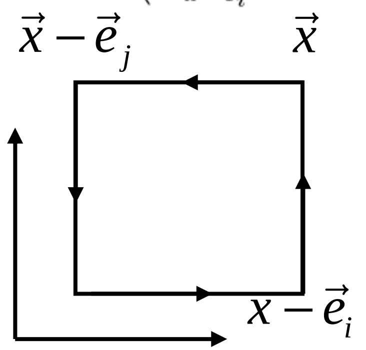
$$G \,|\Psi\rangle = 0$$

The lattice Hamiltonian

- Derivation of the lattice Hamiltonian from the path integral formulation with the transfer matrix-method (Creutz Phys. Rev. D 15, 1128):
  - Observation: Lattice action connects only two time slices
    $\Longrightarrow$Path Integral factorizes in time (transfer matrix)
  - Define Hilbert space
  - Write transfer matrix as operator (Introduce momenta as generator of translations). This operator induces time translations -> Hamiltonian

$$
\begin{aligned}
S_{lat} \; = \; & \frac{2}{g^2}\sum_x \Big\{ -\frac{a_\perp^2}{a_+ a_-} Tr\left[Re\left(U_-(x-e_+)U_-^\dagger(x)\right)\right] \\
& + \frac{a_+ a_-}{a_\perp^2} Tr\left[Re\left(U_{12}(x)\right)\right] \\
& + \sum_k Tr\left[Im\left(U_k(x-e_+)U_k^\dagger(x)\right) Im\left(U_{-k}(x)\right)\right] \\
& - \frac{a_-}{a_+}\eta^2 \sum_k Tr\left[Re\left(U_k(x-e_+)U_k^\dagger(x)\right)\right] \Big\} \\
& + \frac{2}{g^2}\sum_x Tr\left[2\eta^2 \frac{a_-}{a_+} + \frac{a_\perp^2}{a_+ a_-} - \frac{a_+ a_-}{a_\perp^2}\right]
\end{aligned}
$$

$$
U_i(x) \; \equiv \; \mathcal{P}\exp\left(ig \int_{x-\hat{e}_i}^{x} dy^\mu A_\mu^a(y)\lambda_a \right)
$$



$$
U_{ij}(x) = U_i(x)U_j(x-\hat{e}_i)U_i^\dagger(x-\hat{e}_j)U_j^\dagger(x)
$$

$$\widehat{H} = \sum_{\vec{x}} \left[\!\left[\frac{1}{2} g^2 \frac{1}{a_-} \sum_{k,a} \frac{1}{\eta^2} \left\{ \widehat{\Pi}_k^a(\vec{x}) - \frac{2}{g^2} Tr \left[\lambda^a Im \left(\widehat{U}_{-k}(\vec{x}) \right) \right] \right\}^2 + \frac{1}{2} g^2 \frac{a_-}{a_\perp^2} \sum_a \widehat{\Pi}_-^a(\vec{x})^2 + \frac{2}{g^2} \frac{a_-}{a_\perp^2} Tr \left[\mathbb{1} - Re \left(\widehat{U}_{12}(\vec{x}) \right) \right] \right]\!\right]$$

- Dominant part is similar to a particle coupled to a vector potential in ordinary QM
- Guided QMC is not applicable: Local energy $E_L(\{U\}) = \hat{H}\Phi(\{U\})$ is complex (branching process) $\Longrightarrow$ large fluctuations once again
- $\Longrightarrow$ variational optimization of the ground state wavefunctional
  - Try to solve the dominant part analytically as good as possible
  - Variationally optimize the full Hamiltonian with respect to the total energy

One plaquette solution of the dominant part

- Dominant part of the single site Hamiltonian:

$$\widehat{H}_0^k(\vec{x}) = \frac{1}{2}\frac{1}{\eta^2}\frac{g^2}{a_-}\sum_a \left(\widehat{\Pi}_k^a(\vec{x}) - \frac{2}{g^2} Tr\left[\lambda^a Im\left(\widehat{U}_{-k}(\vec{x})\right)\right]\right)^2$$

- polar representation of SU(2)

$$U_{-k}(\vec{x}) = \cos\left(\frac{1}{2}B_p\right) + \mathrm{i}\widehat{n}_p^a \tau^a \sin\left(\frac{1}{2}B_p\right)$$

- Hamilton operator in terms of the SU(2) parameters:

$$\widehat{H}_0\Psi(B_p) = \frac{1}{\eta^2}\frac{1}{a_-\beta'}\left[-\frac{\partial^2}{\partial B_p^2} - \cot\left(\frac{1}{2}B_p\right) + \frac{3}{2}\mathrm{i}\beta'\cos\left(\frac{1}{2}B_p\right) + 2\mathrm{i}\beta'\sin\left(\frac{1}{2}B_p\right)\frac{\partial}{\partial B_p} + \beta'^2\sin\left(\frac{1}{2}B_p\right)^2\right]\Psi(B_p)$$

$$\beta' = \frac{2}{g^2}$$

- Schrödinger equation in

$$\widehat{H}_0 \Psi(B_p) = \epsilon \Psi(B_p)$$

- Ground state wavefunctional

$$\Psi_0(B_p) = e^{-2i\beta' \cos\left(\frac{1}{2}B_p\right)} = e^{-i\frac{2}{g^2}Tr\left[Re\left(\widehat{U}_{-k}(\vec{x})\right)\right]} \qquad \epsilon_0 \;=\; 0$$

- Interpret ground state wavefunctional as unitary transformation:

$$\frac{1}{2}\frac{1}{\eta^2} e^{-i\frac{2}{g^2}Tr\left[Re\left(\widehat{U}_{-k}(\vec{x})\right)\right]}\; \widehat{\Pi}^a_k(\vec{x})^2\; e^{i\frac{2}{g^2}Tr\left[Re\left(\widehat{U}_{-k}(\vec{x})\right)\right]} \;=\; \widehat{H}_0(\vec{x})$$

$\Longrightarrow$ Single plaquette Hamiltonian is unitarily equivalent to a free particle

<u>But:</u> Product Ansatz

$$\Psi_0 \;=\; \exp\left\{-i\beta \sum_{\vec{y},k} Tr\left[Re\left(U_{-k}(\vec{y})\right)\right]\right\}$$

does not apply

$$\langle \widehat{H}_0 \rangle \;=\; 0 \qquad \langle \widehat{H}_0^2 \rangle \;=\; \frac{9}{64}\frac{2V_{lat}}{\eta^4} \qquad (\Delta E)^2 \;=\; \langle \widehat{H}_0^2 \rangle - \langle \widehat{H}_0 \rangle^2$$

Solution of the discrete non-compact Hamiltonian in $\partial_- A_-(\vec{x})=0$ gauge

- Discrete non compact Hamiltonian for $A_-(\vec{x}) = A_-^3(\vec{x}_\perp)\lambda^3$

$$\widehat{H}_0 = \frac{1}{2\eta^2}\sum_{\vec{x},k,a}\left[\widehat{\Pi}_k^a(\vec{x}) - \widehat{F}^a_{-k}(\vec{x})\right]^2$$

$$\left[\widehat{\Pi}_k^a(\vec{x}), \widehat{A}_{k'}^b(\vec{x}')\right] = -\mathrm{i}\delta^{ab}\delta_{kk'}\delta_{\vec{x}\vec{x}'}$$

- Discrete field strength tensor

$$\begin{aligned}\widehat{F}^a_{-k}(\vec{x}) &= \partial_-\widehat{A}_k^a(\vec{x}) - \partial_k\widehat{A}^a_-(\vec{x}_\perp) + gf^{abc}\widehat{A}^b_-(\vec{x}_\perp)\widehat{A}_k^c(\vec{x}) \\ &= \frac{1}{2a_-}\left(\widehat{A}_k^a(\vec{x}+\hat{e}_-) - \widehat{A}_k^a(\vec{x}-\hat{e}_-)\right) - \partial_k\widehat{A}^a_-(\vec{x}) \\ &\quad + gf^{abc}\widehat{A}^b_-(\vec{x}_\perp)\widehat{A}_k^c(\vec{x}) + \mathcal{O}(a_-^2)\end{aligned}$$

$$\begin{aligned}\left[\widehat{\Pi}_k^a(\vec{x}), \widehat{F}^b_{-k'}(\vec{x}')\right] &= -\mathrm{i}\delta^{ab}\delta_{kk'}\frac{1}{2a_-}\left(\delta_{\vec{x},\vec{x}'+\hat{e}_-} - \delta_{\vec{x},\vec{x}'-\hat{e}_-}\right) \\ &\quad -\mathrm{i}gf^{abc}A^c_-(\vec{x}_\perp)\delta_{kk'}\delta_{\vec{x},\vec{x}'}\end{aligned}$$

- Mode expansion of the fields in longitudinal direction:

$$\widehat{\Pi}^a_k(\vec{x}) = \frac{1}{\sqrt{L}} \sum_{q_-} \widetilde{\Pi}^a_k(q_-, \vec{x}_\perp) e^{-\mathrm{i}q_- x_-}$$
$$\widehat{F}^a_{-k}(\vec{x}) = \frac{1}{\sqrt{L}} \sum_{q_-} \widetilde{F}^a_{-k}(q_-, \vec{x}_\perp) e^{\mathrm{i}q_- x_-}$$

$$q_- = \frac{2\pi}{L} n$$

$$\left[\widetilde{\Pi}^a_k(q_-, \vec{x}_\perp), \widetilde{F}^b_{-k'}(q'_-, \vec{x}'_\perp)\right] = \left(\delta^{ab}\omega(q_-) - \mathrm{i}g f^{abc} A^c_-(\vec{x}_\perp)\right) \delta_{kk'} \delta_{\vec{x}_\perp, \vec{x}'_\perp} \delta_{q_-, q'_-}$$

- Lattice derivative

$$\omega(q_-) = \frac{\sin(q_- a_-)}{a_-}$$

- Hamiltonian

$$\widehat{H}_0 = \frac{1}{\eta^2} \sum_{\vec{x}_\perp, k, a} \sum_{q_- > 0} \Big[\widetilde{\Pi}^a_k(q_-, \vec{x}_\perp)\widetilde{\Pi}^a_k(-q_-, \vec{x}_\perp) - \widetilde{F}^a_{-k}(q_-, \vec{x}_\perp)\widetilde{\Pi}^a_k(q_-, \vec{x}_\perp)$$
$$-\widetilde{F}^a_{-k}(-q_-, \vec{x}_\perp)\widetilde{\Pi}^a_k(-q_-, \vec{x}_\perp) + \widetilde{F}^a_{-k}(q_-, \vec{x}_\perp)\widetilde{F}^a_{-k}(-q_-, \vec{x}_\perp)\Big]$$

- Perform a transformation of Bogolubov type

$$\begin{aligned}\widetilde{\Pi}_k^a(q_-,\vec{x}_\perp) &= \gamma(q_-)B_k^a(-q_-,\vec{x}_\perp)+\gamma(q_-)^* B_k^{a\dagger}(q_-,\vec{x}_\perp)\\ \widetilde{F}^a_{-k}(q_-,\vec{x}_\perp) &= \alpha(q_-)B_k^a(q_-,\vec{x}_\perp)-\alpha(q_-)^* B_k^{a\dagger}(-q_-,\vec{x}_\perp)\end{aligned}$$

- Choose coefficients:

$$\begin{aligned}\alpha(q_-) &= \frac{1}{\sqrt{2}}\mathrm{sign}(q_-)\sqrt{|\omega(q_-)|}\\ \gamma(q_-) &= -\frac{1}{\sqrt{2}}\sqrt{|\omega(q_-)|}\end{aligned}$$

- Then:
  - $\left[B_k^a(q_-,\vec{x}_\perp),B_{k'}^b(q'_-,\vec{x}'_\perp)\right] \stackrel{!}{=} 0$
  - $\left[B_k^a(q_-,\vec{x}_\perp),B_{k'}^{b\dagger}(q'_-,\vec{x}'_\perp)\right] = \left(\delta^{ab}+\mathrm{i}gf^{abc}\frac{A_-^c(\vec{x}_\perp)}{\omega(q_-)}\right)\delta_{k,k'}\delta_{\vec{x}_\perp,\vec{x}'_\perp}\delta_{q_-,q'_-}$
  - $\widehat{H}_0$ is only a functional of the operator $B_k^{a\dagger}(q_-,\vec{x}_\perp)B_k^a(q_-,\vec{x}_\perp)$

- Then, the Hamiltonian is given by

$$\widehat{H}_0 = \frac{2}{\eta^2} \sum_{k,a} \sum_{\vec{x}_\perp, q_- > 0} \omega(q_-) \left(B_k^{a\dagger}(q_-, \vec{x}_\perp) B_k^a(q_-, \vec{x}_\perp) + \frac{1}{2} \right)$$

- Ground state

$$B_k^a(q_-, \vec{x}_\perp) \left| \Psi_0 \right\rangle = 0 \qquad E_0 = \frac{3}{\eta^2} V \frac{\cot(\pi / N_-)}{N_-}$$

$$\Psi_0 = \exp\left\{ -\frac{1}{2} \sum_k \sum_{a,b} \sum_{\vec{x},\vec{x}'} \widehat{F}^a_{-k}(\vec{x}) g^{ab}(\vec{x}, \vec{x}') \widehat{F}^b_{-k}(\vec{x}') \right\}$$

$$g^{ab}(\vec{x}, \vec{x}') \equiv \frac{1}{L} \sum_{q_- \neq 0, \pi} e^{\mathrm{i} q_- (x_- - x'_-)} \left(\delta^{a3} \delta^{b3} \frac{1}{|\omega(q_-)|} + (f^{ab3})^2 \mathrm{sign}(q_-) \frac{\delta^{ab} \omega(q_-) - \mathrm{i} g f^{ab3} A^3_-(\vec{x}_\perp)}{|\omega(q_-)|^2 - g^2 A^3_-(\vec{x}_\perp)^2} \right) \delta_{\vec{x}_\perp, \vec{x}'_\perp}$$

- Invariant under residual gauge transformations

Conclusions:

- Near light cone coordinates seem to be a promising tool in order to describe high energy scattering on the lattice (Nachtmann)
- Eucledian path integral as well as Diffusion Quantum Monte Carlo treatments of the theory are inefficient due to complex phases during the update process
- We have analytically computed the light-cone ground state wave-functional in the $\partial_- A_- = 0$ gauge, compatible with periodic boundary conditions
- This should be a nice starting point in order to perform perturbation theory in eta