

(1980 or 1981)

## THE CANNIBALS' URN

Richard F. Green

Department of Mathematical Sciences

University of Minnesota, Duluth

Duluth, Minnesota 55812

### INTRODUCTION

Cannibalism is a widespread phenomenon in the natural world (Fox 1975). Cannibalism may be important in regulating population size and in influencing community structure. For certain animals conspecifics provide a substantial proportion of food intake. For example, in his study of the desert <sup>Scorpion</sup> scorpion Paruroctonus mesaensis, Polis (1979) found that 28% of the biomass ingested by scorpions was from other scorpions of the same species.

In order to produce and maintain a unit of biomass in one animal it is necessary to consume several units of biomass. The ratio of the biomass produced to the biomass consumed is referred to as the "growth efficiency" of the animal (see Slobodkin 1961, Macfadyen 1963). The growth efficiency of the desert scorpion is not known precisely (but see Yokota 1979 for data on incorporation efficiency). According to Macfadyen, Gere (1957) gives a growth efficiency value of around 15% for terrestrial invertebrates. Scorpions are "sit and wait" predators with low metabolic costs and

might have a greater growth efficiency than some other terrestrial invertebrates, but since the scorpions mature slowly their growth efficiency probably is not much greater than 15%.

Since about 28% of the biomass consumed by the desert scorpion is from conspecifics most adult scorpions will have eaten at least one and perhaps more of their fellows. This situation resembles that of the apocryphal economy in which everyone makes his living by taking in his neighbor's laundry. The question is this: If we start with a given large number of young scorpions and they eat each other (and other food as well) until they are adults, how many scorpions will survive to adulthood (ignoring other sources of predation on scorpions)?

#### AN URN MODEL

It is possible to construct and analyze an urn model which roughly represents the cannibalism found in desert scorpions. In the following model it is assumed that each animal (or ball) will eat exactly one other, or will be eaten itself. The victims of each act of cannibalism will be chosen at random, with individuals that have already fed being as likely to be eaten as those that have not fed.

The urn model for the cannibalism process may be described as follows:

1. An urn originally contains  $N$  balls. These balls are all "innocent", white balls; they have not yet eaten anyone.

2. A white ball is taken from the urn and painted black, <sup>(the victim)</sup> indicating it has become a cannibal, and another ball <sub>A</sub> is chosen from the urn and discarded. Then the painted ball is returned to the urn. The urn now contains  $N - 1$  balls, one black and  $N - 2$  white.

3. A white ball is taken from the urn, painted black, and another ball (which may be either black or white) is <sup>at random</sup> drawn <sub>A</sub> from the <sub>A</sub> balls remaining in the urn and discarded. Then the painted ball is returned to the urn. After the second draw the urn contains  $N - 2$  balls, one black and  $N - 3$  white if the discarded ball was black, or two black and  $N - 4$  white if the discarded ball was white.

4. This process continues: a white ball is drawn and painted black, another ball is drawn at random and discarded and the painted ball is returned to the urn. The process continues until all the balls in the urn are black (have eaten someone). When this happens there will be some number,  $Z$ , of black balls in the urn. The question is, what will  $Z$  be?

The value  $Z$  will be a random variable. The smallest possible value will be  $Z = 1$ , which will have probability  $1/(N-1)!$ . The largest possible value will be  $Z = N/2$  if  $N$  is even, or  $(N-1)/2$  if  $N$  is odd. For  $N$  even  $P(Z = N/2) = 1/2^{N/2-1}$ . For  $N$  odd  $P(Z = (N-1)/2) = 1/3 \cdot 2^{(N-1)/2-1}$ .

THE DISTRIBUTION OF Z, THE NUMBER OF CANNIBALS  
IN THE URN AFTER EVERYONE'S EATEN

I do not know any simple, explicit expression for the distribution of Z, but there exists a simple iterative expression which may be used to compute the distribution numerically.

Let  $x(t)$  denote the number of black balls and  $y(t)$  the number of white balls in the urn at time  $t$  (after the  $t$ th draw). Denote the probability of ever having  $x$  black balls and  $y$  white balls in the urn by  $P(x,y)$ . Since  $x + y = N - t$ , it is not necessary to indicate explicitly that  $x$  and  $y$  are functions of  $t$ . For  $N \geq 2$  after one draw we have

$$(1) \quad \begin{aligned} P(1, N-2) &= 1 \text{ and} \\ P(x, N-x-1) &= 0 \text{ for all values of } x \text{ other than } 1. \end{aligned}$$

Then we have the iterative expression

$$(2) \quad P(x,y) = \frac{x}{x+y} \cdot P(x,y+1) + \frac{y+1}{x+y} \cdot P(x-1,y+2).$$

Notice that (1) and (2) depend on  $N$ , the number of balls originally in the urn.

In order to find the distribution of Z we start with (1) and then use (2) to find all the non-zero values of  $P(x,y)$ , first for  $x + y = N - 2$ , then for  $x + y = N - 3$ , then for  $x + y = N - 4$  and so on until  $x + y = 1$ . The distribution of Z will be given by  $P(Z = x) = P(x,0)$  for  $x = 1, 2, \dots, N/2$  or  $(N-1)/2$  depending on whether  $N$  is

even or odd, respectively.

The distribution of  $Z$  is given in Table 1 for  $N = 5, 10, 15, 20, 25, 30, 35, 40$ . Table 1 also gives values for the mean and variance of  $Z$  and values for the mean and variance divided by  $N$ . It is seen that the mean divided by  $N$  rapidly converges to  $e^{-1}$  as  $N$  increases, while the variance divided by  $N$  rapidly converges to  $3e^{-2} - e^{-1}$ .

It is easy to see that a result like the law of large numbers holds for the cannibals' urn. The idea is this. For some large number of balls  $N$  originally in the urn, the number of black balls in the urn after  $t$  draws,  $X(t)$ , is equal to the number of white balls drawn as "victims" in the  $t$  draws. On each draw the "victim" is a white ball or not and  $X(t)$  may be thought of as the sum of  $t$  random variables, each taking value 0 or 1 with some probability which depends on the number of the draw. These 0-1 random variables will be negatively correlated with each other and the variance of  $X(t)$  will thus be less than  $.25t$ . Chebychev's inequality may be applied in the same way as in the Bernoulli law of large numbers.

While it is clear that a version of the law of large numbers must hold, it is necessary to do a calculation to determine what the limit of  $Z/N$  will be. Since it is the limit that is of interest we can assume that  $N$  is large and consider a continuous, deterministic version of the process.

Let  $t$  represent the proportion of balls discarded from the urn ("eaten") and let  $u(t)$  and  $v(t)$  represent the proportion of black and white balls, respectively, in the urn at the time when proportion  $t$  balls have been discarded.

We have  $u(0) = 0$ ,  $v(0) = 1$ , and

$$\frac{du(t)}{dt} = \frac{v(t)}{u(t) + v(t)}, \text{ and}$$

$$(3) \quad \frac{dv(t)}{dt} = -1 - \frac{du(t)}{dt},$$

which can be solved numerically and evaluated at  $v(t) = 0$ . The desired value is the value of  $u(t)$  when  $v(t) = 0$ . Calculations show this value to be close to  $e^{-1}$  and the value becomes closer as the calculation becomes more precise.

Not only does a type of law of large numbers hold for the cannibals' urn but it seems that a central limit theorem will hold as well. It is not yet clear how to prove this, but numerical calculations of the first six cumulants of  $Z$  strongly suggest that these cumulants are all of order  $N$ . Table 2 gives values for the first six cumulants divided by  $N$  for  $N = 5, 10, 15, 20, 25, 30, 35, 40$ . Even for such small values of  $N$  it seems that the cumulants divided by  $N$  converge rapidly to constants.

If the central limit theorem suggested by the numerical calculations is true, then for large  $N$  the number of cannibals left after everyone has become a cannibal or a victim will have approximately a normal distribution with mean  $= e^{-1}N$  and variance  $= (3e^{-2} - e^{-1})N$ .

## DISCUSSION

The cannibals' urn described in this paper provides only a rough representation of the process of cannibalism seen in the desert sand scorpion, which it was originally intended to model. The model does illustrate a number of interesting mathematical points, however. One of these is that if we consider the sequence of urn sizes:  $N = 2, 3, 4, 5, 6, \dots$  and for each we look at the total number of possible outcomes, characterized by the sequence of colors of the balls discarded, we see that these numbers of possible outcomes for urns of the various sizes forms a Fibonacci series. For example, for  $N = 6$  the possible outcomes are: WWW, WWBB, WBWB, WBBW, WBBBB. If we represent the number of possible outcomes for an urn of size  $N$  by  $F(N)$ , then we have:  $F(2) = F(3) = 1$ ,  $F(4) = 2$ ,  $F(5) = 3$ ,  $F(6) = 5$ ,  $F(7) = 8$ , and so on.

There are several reasons the cannibals' urn is an unrealistic model.

1. The model assumes that all balls are equally likely to be chosen as "victims". Among the scorpions the older age classes tend to prey upon the younger, although cannibalism sometimes occurs within an age class. Since larger scorpions eat smaller ones, it would take several victims to equal the biomass of the cannibal.

2. The model assumes that each ball "eats" exactly

one other ball, or is eaten itself (and possibly both). This is unrealistic for two reasons.

a) If the proportion of the diet made up by conspecifics is not equal to the growth efficiency then the average adult will not have to eat his own weight's worth of conspecifics. In the desert sand scorpion he may have to eat more.

b) The number of other scorpions eaten does not have to be the same for all surviving individuals.

The cannibals' urn is not appropriate for the desert sand scorpion because of 1. However, if each individual has more than one victim, as suggested by 2a), this can be incorporated in the model. A calculation similar to that using expressions (3) may be done. If each cannibal eats two victims then the number of survivors will be approximately  $e^{-2N}$ .

It is not clear whether 2b) will have any effect on the general conclusion of the model.

## REFERENCES

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Table 2: Cumulants/N for various values of N.

	K1/N	K2/N ( $\times 10^{-2}$ )	K3/N ( $\times 10^{-3}$ )	K4/N ( $\times 10^{-3}$ )	K5/N ( $\times 10^{-4}$ )	K6/N ( $\times 10^{-3}$ )
5	.36666667	2.7777778	-18.518519	4.6296296	123.45679	-23.662551
N 10	.36787946	3.8132613	-3.5643453	-1.3570281	81.952175	3.6173583
15	.36787944	3.8126408	-3.7580174	-3.3030914	8.1788067	.044041609
20	.36787944	3.8126409	-3.7579457	-3.2987432	9.2749404	1.4141451
25	.36787944	3.8126409	-3.7579457	-3.2987445	9.2737142	1.4085611
30	.36787944	3.8126409	-3.7579457	-3.2987445	9.2737146	1.4085801
35	.36787944	3.8126409	-3.7579457	-3.2987445	9.2737121	1.4085723
40	.36787944	3.8126409	-3.7579457	-3.2987445	9.2737116	1.4085752