Steenrod Squares on Khovanov Homology and an Implementation

A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Sarah Milstein

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

Tyler Lawson

November, 2022

Acknowledgements

I want to express my sincere thanks to the following people for all of their support throughout my time in graduate school:

My advisor Tyler Lawson for his help debugging the code for this project, as well as his conceptual and detailed feedback on this paper and my defense talk. In addition, his mentoring and support throughout this entire process have been invaluable. Gilad Lerman, Sasha Voronov, and Craig Westerland for serving on my committee. Gilad Lerman also for helping me to develop as a data scientist. Helen Wong and Deanna Haunsperger for supporting me early in my mathematical development. Jennifer Kloke for being one of my first mathematical mentors.

To my friends, whose support has meant the world to me. Harini Chandramouli for always being willing to listen. Katy Weber for all of the Sunday mornings we have spent together. Jorin Schug for always having the wrong answer for a crossword puzzle clue and often having the right answer, too. Cora Brown for being willing to spend way too much time on data science projects with me. Ryan Coopergard, Daniel Hess, Carolynn Johnson, Michelle Mastrianni, Greg Michel, and Tara Palnitkar for their friendship both inside and outside of mathematics. Marielle Bricker, Laur Freymiller, Anne Halvorsen, Anne Hochberg, Tiffany Lee, Chance Rodriguez, and Brian Shank for supporting me from near and far. Athelas Christley, Thomas Erickson, Nancy Gaschott, Ann Luce, Don Luce, Lill Luce, Logan Luce, Mark Ritchie, Tim Thompson, and Linda Vieira for the delicious food and the warm community. Dan Eldridge, Becca Eldridge, Doug Johnson, and Kathryn Sikkink for always cheering me on.

To my family, I could not have done this without you. My siblings Brian Milstein and Rachel Milstein for reminding me not to take myself too seriously. My parents Carol Singer and Hal Milstein for their unwavering support and their constant encouragement.

My wonderful partner Matt Sikkink Johnson for always believing in me, for helping me appreciate my progress, and for reminding me to have a snack.

Dedication

For teaching me the value of hard work and for supporting me every step of the way, this work is dedicated to my parents, Carol Singer and Hal Milstein.

Abstract

Link invariants allow us to distinguish links. Often, the more complicated or involved a knot invariant is to calculate the more links it can successfully help us distinguish. In this paper, we start with the Jones polynomial, which is a relatively straightforward invariant to compute. We build upon to that to define Khovanov homology, a more involved but also more informative invariant. From there, we describe Morán's construction of the Steenrod squares $\mathfrak{Sq}^i$ on Khovanov homology [Mor20]. These operations allow us to further distinguish knots with identical Khovanov homology groups. We present a few by-hand calculations of these operations, which quickly become unwieldy.

Then, we describe our software implementation of Steenrod squares on Khovanov homology in Sagemath. This forms a package called `KhovanovSteenrodSquare` and is available at `https://github.com/samilstein/KhovanovSteenrodSquare`. As long as a link does not contain a component that is the unknot or the Hopf link, this code can compute the Steenrod square on any of the Khovanov homology groups for that link. We used this code to compute $\mathfrak{Sq}^1$, $\mathfrak{Sq}^2$, $\mathfrak{Sq}^3$, and $\mathfrak{Sq}^4$ on the disjoint union of one, two, three, and four trefoils respectively. We also engaged in a systematic search for examples of nontrivial $\mathfrak{Sq}^4$ for knots with up to 15 crossings. This implementation allows us to capture the usefulness of this invariant while decreasing the effort required to compute it.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

We often think of a knot as any piece of string that is tangled in some way, like shoe laces or a headphone cable. However, in topology, objects remain equivalent as we push and pull them around just as long as we do not tear them. Thus, these tangled strings can all be untangled, leaving us with nothing to study. In order to preserve these tangles, we must glue the ends of the string together. That is, to construct a mathematical knot, we take a piece of string, tangle it around, and fuse the ends together. Then to study knots in a mathematical way, we define knot invariants. These are well-defined assignments from equivalence classes of knots to any mathematical set. This set can be as simple as the set {yes, no}, or as we will see, it can be as involved as cohomology groups or topological spaces.

In 1984, Jones defined a knot invariant, which we call the Jones polynomial, that assigns a polynomial in one variable to each equivalence class of knots. This invariant can distinguish more knots than other, previous knot invariants such as the Alexander polynomial. In particular, the Jones polynomial can distinguish handedness of knots, where applicable. Recall that to swap handedness of a knot, we change all over-crossings to under-crossings and vice versa. Sometimes taking the mirror image of a knot gives us a different knot, in which case the Jones polynomial of these two knots is always different. The Alexander polynomial can not always distinguish such distinct pairs.

In 2000, Khovanov was trying to figure out if there was a homology theory whose Euler characteristic was the Jones polynomial. In this work, he defined a process of assigning a cochain complex to a knot. The cohomology groups of this complex, which

we call the Khovanov homology groups, are a knot invariant, and their graded Euler characteristic is the Jones polynomial [Kho00]. This relationship between the Jones polynomial and Khovanov homology is called *categorification*.

Then in 2014, Lipshitz and Sarkar constructed a spectrum assigned to a knot whose cohomology groups are the Khovanov homology groups of that knot [LS14a]. We call this spectrum the Khovanov spectrum, and its homotopy type is a knot invariant. The following year, along with Lawson, they provided a more systematic construction of the Khovanov spectrum [LLS20].

Since there is a spectrum underlying Khovanov homology, there is more structure available to these cohomology groups, such as cohomology operations, and in particular the Steenrod operations. In particular, since spectra with isomorphic cohomology groups can have different cohomology operations, we can use these operations to distinguish knots with the same Khovanov homology. In 2014, Lipshitz and Sarkar gave formulas for the Steenrod operations Sq^1 and Sq^2 on the Khovanov homology groups of a knot. Later in 2014, Lipshitz and Sarkar, as well as Seed, implemented this work in code. Seed found pairs of knots and links with between 12 and 14 crossings with isomorphic Khovanov homology but different cohomology operations [LS14b, See12]. Then, in 2019 Morán gave definitions of the Steenrod operations $\mathfrak{sq}^i$ for all i, generalizaing the formulas of Lipshitz and Sarkar [Mor20].

Our continuation of this work was to convert Morán's Steenrod square construction to a computer algorithm that could compute $\mathfrak{sq}^i$ for any oriented link. Using our package `KhovanovSteenrodSquare` written in Sagemath and available at `https://github.com/samilstein/KhovanovSteenrodSquare`, one can enter the PD or DT code of any oriented link, which is a standardized format for describing a link, and the desired Steenrod square. Then, the code will return the definition of this operation on cohomology classes, as well as the map's rank. In our initial testing, we evaluated some Steenrod operations on smaller links. We computed $\mathfrak{sq}^1$ on a single left-handed trefoil, which took about 0.09 seconds. We computed $\mathfrak{sq}^2$ on the disjoint union of two trefoils, which took about 0.2 seconds. We computed $\mathfrak{sq}^3$ on the disjoint union of three trefoils, which took about 22 seconds.

However, because Lipshitz and Sarkar as well as Seed had already implemented Sq^1

and Sq^2 and because we can use the Adem relations to compute Sq^3 from

$$Sq^3 = Sq^1 Sq^2,$$

our next Steenrod operation of interest was Sq^4. As a proof of concept, we computed $\mathfrak{sq}^4$ on the disjoint union of four trefoils and found it to be nontrivial. This computation took about 9 hours. This map is expected to be nonzero due to a Cartan formula for the Khovanov homology of disjoint links.

Continuing our search, we then computed the Khovanov homology groups or used existing computations for all prime knots up to 14 crossings and many knots with 15 crossings. However, we were unable to find another knot in our search with a spot where a nontrivial $\mathfrak{sq}^4$ might exist. That is, none of these knots had a quantum degree with nonzero Khovanov homology groups that were four cohomological degrees apart. Thus, in this initial search we did not find another nontrivial $\mathfrak{sq}^4$.

1.1 Outline of the Paper

We now describe the outline for the rest of the paper. In Section 2 we provide some general background on knot theory. We also describe the Jones polynomial and Khovanov homology. In Section 3 we give additional background, including describing the Burnside category in general, as well as the particular object in the Burnside category that we will use in the Steenrod square formulation. We will also define the augmented semi-simplicial category. Then, in Section 4 we present Morán's construction of the Steenrod operations $\mathfrak{sq}^i$ defined on Khovanov homology. After that, in Section 5 we present worked-out examples for the following Steenrod operations

$$\mathfrak{sq}^1 : H^{-3,-7}(T; \mathbb{Z}/2\mathbb{Z}) \to H^{-2,-7}(T; \mathbb{Z}/2\mathbb{Z})$$

and

$$\mathfrak{sq}^2 : H^{-6,-14}(T; \mathbb{Z}/2\mathbb{Z}) \to H^{-4,-14}(T; \mathbb{Z}/2\mathbb{Z})$$

where T is the left-handed trefoil. Finally, in Section 6 we describe our implementation in Sagemath of Morán's Steenrod square construction that can be applied to any oriented

link that does not have an unlink or Hopf link component. We also describe some initial results from running this code.

1.2 Notational Notes

First, throughtout this paper, we will refer to links. This includes knots, which are one-component links, as well as links with more than one component. Also, very importantly,in order to ensure that our knot invariants are invariants for links, we will need to fix an orientation on each link in order to determine the number of positive and negative crossings. Thus, when we say "link" throughout this paper, we will mean "oriented link". For knots, the number of positive and negative crossings will not change when we swap the orientation, and so the orientation of the knots will not change the invariants we discuss in this paper. However, this is not the case for all links.

Second, we will only consider cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients, unless otherwise stated. Thus, if there are no coefficients specified, they are assumed to be $\mathbb{Z}/2\mathbb{Z}$. Additionally, we will use the notation $\mathbb{Z}/2$ to refer to $\mathbb{Z}/2\mathbb{Z}$.

Third, throughout this paper, we will discuss the Steenrod squares Sq^i. In particular, we will explore the construction of Morán's operations $\mathfrak{sq}^i$ [Mor20]. In the paper where Morán constructs these maps, he shows that the maps $\mathfrak{sq}^i$ satisfy many of the properties that define the Steenrod squares Sq^i, including satisfying a Cartan formula, having the first square $\mathfrak{sq}^1$ be the Bockstein homomorphism, and being invariant under Reidemeister moves. However, he notes that he does not explicitly prove that the maps $\mathfrak{sq}^i$ coincide with the Steenrod squares Sq^i defined on the Khovanov spectrum. This is expected in a forthcoming paper with Gutiérrez called "Singular chain functors for spectra". In the rest of this paper, we will refer to both $\mathfrak{sq}^i$ and Sq^i as Steenrod squares. We will use $\mathfrak{sq}^i$ when specifically describing a Steenrod square constructed following Morán's paper, and we will use Sq^i to refer generally to Steenrod squares.

Chapter 2

Knot Theory and Khovanov Homology Background

We start this section with a very brief introduction of some relevant knot theory concepts. Then, we will discuss two well-known and well-studied invariants of oriented links, the Jones polynomial and Khovanov homology, which will help us distinguish links. This will set us up so that in Section 4 to define Steenrod operations on the Khovanov homology of a link. These are another invariant that can provide more information about a link and gives us more to study through the lens of algebraic topology.

The Jones polynomial assigns a Laurent polynomial to a link diagram. Khovanov homology gives us a way to build a cochain complex from a link diagram, whose cohomology is an invariant of the link. Neither is a complete invariant, meaning that neither is able to distinguish every pair of distinct links. However, both are fairly straightforward, if labor-intensive, to compute. Thus, both allow us to distinguish many links.

2.1 Knot Theory Background

We start with a bit of notation. Recall that we consider any link diagram to be an oriented link diagram. We will write n for the number of crossings in a link diagram L. We will often order the crossings in the diagram from 1 to n arbitrarily, but once we choose an ordering on the crossings, it is fixed. We say that n_+ is the number of right-handed, or positive, crossings as shown in Figure 2.1a, and n_- is the number

of left-handed, or negative, crossings as shown in Figure 2.1b. The total number of crossings is the sum of the positive and negative crossings. That is, we have

$$n = n_+ + n_-.$$

(a) Right-handed, or positive, crossing

(b) Left-handed, or negative, crossing

Figure 2.1: Orientation of crossings

Now, we briefly discuss a process of *resolving* a crossing in a link diagram, which we will use throughout this paper when defining link invariants. The idea here is that we would like to undo a crossing in a link while leaving the strands of the link connected. We will then look at the resulting figures and use them to help us understand the original link.

Suppose we have zoomed in on one crossing of a link as shown in Figure 2.2, and the remainder of the link lives outside the dotted circle. We can undo the crossing of the strands while still keeping the strands of the link connected in two possible ways.

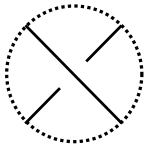


Figure 2.2: Link Crossing

If we imagine walking along the over-strand starting at the bottom right edge in Figure 2.2, suppose we turn left and connect to the under-strand. We follow the same procedure if we start walking from the top right edge of the over-strand. This procedure is called a 0-smoothing, and the result is shown in Figure 2.3a. Alternatively, we could make a right turn as we walk along the over-strand from either edge to connect to the under-stands. We call this procedure a 1-smoothing, and the result is shown in Figure 2.3b. Notice that the strands in the figure are still connected but there is now no longer a crossing in this spot. This process of replacing Figure 2.2 by either Figure 2.3a via a

0-smoothing or Figure 2.3b via a 1-smoothing is what we mean by *resolving* a crossing.

(a) 0-Smoothing (b) 1-Smoothing

Figure 2.3: The Possible Smoothings of a Crossing

We can imagine doing this resolving process for all the crossings in a link, which results in what we call a *complete smoothing* or *resolution* of the link. This complete smoothing has no crossings in it and is made up of some number of unknots, which we will also refer to as cycles. Note that each link has multiple complete smoothings since each crossing can be resolved via either a 0-smoothing or a 1-smoothing.

When computing the Jones polynomial or Khovanov homology, we will first arbitrarily order the crossings in a link. Then, we will resolve all crossings of a link each via both a 0-smoothing and a 1-smoothing and record all of the resulting complete smoothings. Note that, when we resolve each crossing of the link, this corresponds to an ordered tuple of 0s and 1s. That is, if a link has n crossings, each complete smoothing corresponds to a vertex of the n-dimensional cube $\{0,1\}^n$, where the value in coordinate i of the vertex, which is either 0 or 1, tells us whether crossing i of the link was resolved via a 0-smoothing or a 1-smoothing in this particular complete resolution. As an example, we can see all complete smoothings of the trefoil in Figure 2.4. We will use this correspondence between smoothings of a link with n crossings and the n-dimensional cube throughout the rest of this paper to compute a number of link invariants.

2.2 Jones Polynomial

In this section we will describe the Jones polynomial of an oriented link. In order to compute the Jones polynomial for a link L, we will begin by computing the Kauffman bracket $\langle L \rangle$, for which we will describe two equivalent methods. One standard approach

is to use the relations:

$$\begin{aligned} \langle \emptyset \rangle &= 1 \\ \langle \bigcirc L \rangle &= (q + q^{-1})\langle L \rangle \\ \langle \times \rangle &= q^{-1}\langle \asymp \rangle - q\langle) (\rangle \end{aligned}$$

The first relation says that the Kauffman bracket of an empty knot is 1. The second relation says the Kauffman bracket of the disjoint union of the unknot with any link L is equal to the Kauffman bracket of that link L multiplied by the factor $(q + q^{-1})$. The third relation says that we can compute the Kauffman bracket of a link with at least one crossing by resolving the crossing shown on the left-hand side in both possible ways and computing the Kauffman bracket of the two resulting links, as shown on the right-hand side of this relation. In this first method for computing the Jones polynomial, we use these relations, resolving crossings and removing unknots until we are left with a Laurent polynomial only in q.

An alternate method for computing the Jones polynomial starts with resolving all n crossings in a link diagram L at once in all possible ways instead of resolving the crossings one at a time. Each complete smoothing S_α corresponding to vertex $\alpha \in \{0,1\}^n$ is the union of some number k_α of cycles and was created by applying some number r_α of 1-smoothings.

Definition 2.1. We call this number r_α of 1-smoothing the *height* of α. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we will also write the height of α as $|\alpha|$, where we have

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Then, to each smoothing we assign the term $q^{r_\alpha}(q + q^{-1})^{k_\alpha}$. From this, we take an alternating sum of these terms to get, the Kauffman bracket of L, given by the following formula:

$$\langle L \rangle = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha} q^{r_\alpha} (q + q^{-1})^{k_\alpha}.$$

From that, we compute the *unnormalized Jones polynomial* by multiplying the

Kauffman bracket by the factor $(-1)^{n_-}(q^{n_+-2n_-})$ to get:

$$\hat{J}(L) = (-1)^{n_-}(q^{n_+-2n_-})\langle L\rangle.$$

Finally, to get the *normalized Jones polynomial* $J(L)$, or simply the *Jones polynomial*, we divide the above expression by $(q+q^{-1})$ to get:

$$J(L) \coloneqq \hat{J}(L)/(q+q^{-1}).$$

This last scaling factor adjusts the Jones polynomial, so that the unknot has Jones polynomial equal to 1. That is, this scaling is not mathematically impactful. Rather it is simply a convention so that the simplest link has the most basic polynomial value for its Jones polynomial.

We now go through an example of computing the Jones polynomial of the right-handed trefoil knot to clarify this process.

Example 2.2. Following an example from [BN02], we let L be the diagram of the right-handed trefoil knot shown in the upper left corner of Figure 2.4. We order the crossings of L counterclockwise as shown. As was previously mentioned because of L is a knot, orientation does not change its Jones polynomial, so we simply choose one option in order to compute n_+ and n_-, which are 3 and 0 respectively.

Next we resolve all crossings of L in all possible ways. Recall that we can associate each complete smoothing of L to a vertex of the cube $\{0,1\}^3$. We also recall that the height r_α of a vertex α is the number of 1-smoothing in that resolution. We draw the cube so that vertices of the same height appear in a single column as shown in Figure 2.4.

Now, we compute the Kauffman bracket of the trefoil. To do this, we first assign the term $q^{r_\alpha}(q+q^{-1})^{k_\alpha}$ to each vertex $\alpha \in \{0,1\}^3$, where k_α is the number of cycles in the smoothing at α. For example, the complete smoothing at vertex (0,0,0) has 2 cycles, so we have $k_{(0,0,0)} = 2$. The height of this vertex, which is equal to the sum of its coordinates, is 0, and so we have $r_{(0,0,0)} = 0$. Thus, we assign the monomial

$$q^0(q+q^{-1})^2 = (q+q^{-1})^2$$

to the vertex (0,0,0). The term assigned to each vertex is shown in the upper right corner of each vertex in Figure 2.4. Next, we multiply each term by $(-1)^{r_\alpha}$ and add them all together to get the Kauffman bracket $\langle L \rangle$:

$$\langle L \rangle = q^{-2} + 1 + q^2 - q^6,$$

which is the first term under the cube diagram in Figure 2.4.

Next, we multiply the Kauffman bracket by $(-1)^{n_-} q^{n_+ - 2n_-}$ where $(n_+, n_-) = (3, 0)$. This gives us the unnormalized Jones polynomial, which is the second term in the row at the bottom of Figure 2.4 and is

$$\hat{J}(L) = q + q^3 + q^5 - q^9.$$

Finally to get the normalized Jones polynomial which is the last term under the diagram in Figure 2.4, we divide by $(q + q^{-1})$ to get

$$J(L) = q^2 + q^6 - q^8.$$

The Jones polynomial can distinguish many links. For example, it can tell us that the left-handed trefoil is a distinct link from the right-handed trefoil. However, it is not a complete invariant. There are still many links it cannot distinguish. For example, Thistlethwaite discovered a nontrivial link with Jones polynomial equal to that of the two-component unlink [Thi01]. Furthermore, it is an open problem whether there is a nontrivial link with Jones polynomial equal to that of the unknot. Thus, we seek to improve upon the Jones Polynomial.

2.3 Khovanov Homology

To help us distinguish even more links than we could with the Jones polynomial, we turn to a more involved invariant to get out more information. Khovanov described a new invariant, which we call Khovanov homology, that assigns a cochain complex and its cohomology groups to a link [Kho00]. He showed that the graded Poincaré polynomial of a link's Khovanov homology groups is the Jones polynomial of that link. Additionally,

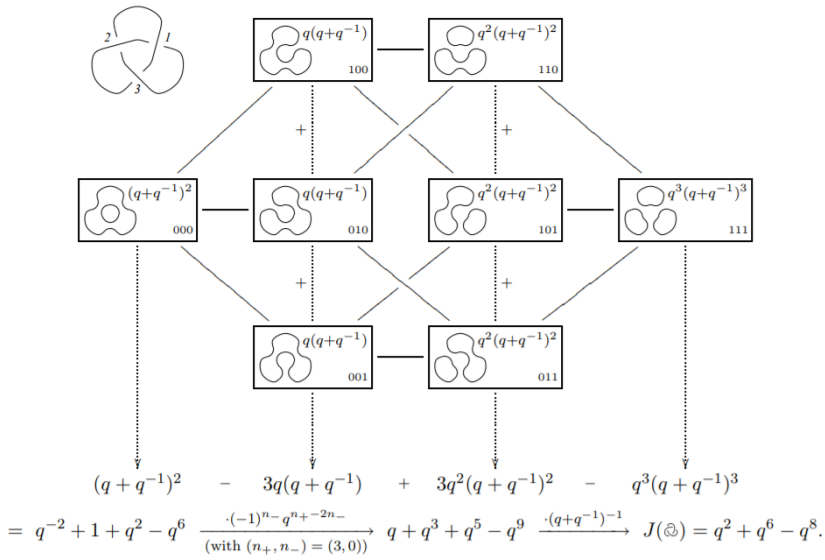


Figure 2.4: Computing the Jones polynomial of the right-handed trefoil, where the first polynomial along the bottom row is the Kauffman bracket, the middle polynomial is the unnormalized Jones polynomial, and the last polynomial is the normalized Jones polynomial [BN02]

where two links related by a Reidemeister move or a smoothing may have completely unrelated Jones polynomials, the Khovanov homology groups of these links are related in a predictable way. These smoothings and relations on links give us functions on their Khovanov homology.

As was mentioned above, it is not yet known whether the Jones polynomial can completely determine whether or not a link is the unknot. However, in 2011 Kronheimer and Mrowka proved that Khovanov homology is an unknot detector [KM11]. That is, any link with Khovanov homology equal to that of the unknot must itself be the unknot.

In this section we describe Khovanov homology. We start by constructing the Khovanov complex of a link. Then we compute the cohomology groups of this complex, which are the Khovanov homology groups of the link. The structure and content in this section follows Bar-Natan's paper [BN02].

2.3.1 Vector Space Background

Over the next two sections, we construct the Khovanov complex. We start by associating cochain groups to a link diagram in this section and then associating differentials in the following section.

We begin with some relevant definitions in quick succession.

Definition 2.3. [BN02, Definition 3.1] A $\mathbb{Z}$-*graded vector space* W is a vector space, which can be decomposed as a direct sum over $\mathbb{Z}$:

$$W = \bigoplus_{m \in \mathbb{Z}} W^m$$

where each W^m is, itself, a vector space. We call the elements of W^m the *homogeneous* elements with degree m. Tthe *graded dimension* of W is the formal sum

$$q \dim W := \sum_m q^m \dim W^m.$$

In the Khovanov homology construction, we will use vector spaces built out of the following graded vector space.

Example 2.4. Let V be a graded vector space with exactly two basis elements, v_+ in

degree 1, and v_- in degree -1. Then, we have $q \dim V = q + q^{-1}$.

In particular, we will be interested in tensor products and direct sums of copies of this vector space V. To help with our computations later, we discuss how the degrees of elements in vector spaces. For the tensor product of two graded vector spaces V and W, we have

$$(V \otimes W)^n = \bigoplus_{\{(i,j)|\, i+j=n\}} V^i \otimes W^j.$$

This means that the degree n homogeneous component of a tensor product of V and W is made up of homogeneous components of V and W whose degrees add to n. Then if we have $a \in V^i$ and $b \in W^j$, their tensor product lies in degree $i + j$, that is

$$a \otimes b \in (V \otimes W)^{i+j}.$$

In contrast, for the direct sum of two graded vector spaces V and W, we have

$$(V \oplus W)^n = V^n \oplus W^n.$$

Here, the degree n homogeneous component of the direct sum is made up of the degree n component of V and the degree n component of W. That is, if $a \in V^i$ and $b \in W^j$, we say that

$$(a, b) \in (V \oplus W)^i$$

if $i = j$. If $i \neq j$, then this element is not in any particular dimension. Instead, we could write it as the sum of two homogeneous elements, i.e.,

$$(a, b) = (a, 0) + (0, b),$$

where $(a, 0) \in (V \oplus W)^i$ and $(0, b) \in (V \oplus W)^j$.

As we mentioned above, the Khovanov complex will be a bigraded vector space, which we define here.

Definition 2.5. A *bigraded vector space* W has the structure

$$W = \bigoplus_n \bigoplus_m W^{n,m} = \bigoplus_{n,m} W^{n,m},$$

where $W^{n,m}$ is a vector space. Here, W is a graded vector space of graded vector spaces, meaning that the vector space in each degree n also has the structure of a graded vector space. That is, we have

$$W^n = W^{n,*} = \bigoplus_m W^{n,m}.$$

We can consider a general cochain complex $\mathcal{C}$ as a graded space where the degree i term $\mathcal{C}^i$ is the i^{th} cochain group. That is, we have $\mathcal{C} = \bigoplus_i \mathcal{C}^i$. We call i the *cohomological degree*. The Khovanov complex has a second grading since each cochain group will, itself, be a graded vector space. We call this second grading the *quantum degree*.

Now that we have two gradings, we will give different names to the analogous processes of shifting the degree within each grading so that we can keep track of where these shifts are happening. We will use these later to normalize the Khovanov complex similar to how we normalized the Jones polynomial above. The first definition defines how we shift the quantum degree of the graded pieces within one cochain group.

Definition 2.6. [BN02, Definition 3.2] Let $\cdot\{l\}$ be the *degree shift* operation on graded vector spaces. If $W = \bigoplus_m W_m$ is a graded vector space, we set

$$W\{l\}_m \coloneqq W_{m-l},$$

or equivalently

$$W_m = W\{l\}_{m+l}.$$

Here, we have $q \dim W\{l\} = q^l q \dim W$.

This next definition describes how we shift the cohomological degree of the cochain groups across the cochain complex as a whole.

Definition 2.7. [BN02, Definition 3.3] Let $\cdot[s]$ be the *height shift* operation on cochain complexes. If $\mathcal{C}$ is the cochain complex

$$\cdots \to \mathcal{C}^r \xrightarrow{d^r} \mathcal{C}^{r+1} \to \dots$$

of (possibly graded) vector spaces, then we have

$$\mathcal{C}[s]^r = \mathcal{C}^{r-s},$$

or equivalently

$$\mathcal{C}^r = \mathcal{C}[s]^{r+s}.$$

We call r the *height* of a group $\mathcal{C}^r$ in the complex, thus giving us the name "height shift."

As mentioned above, both the degree shift and the height shift describe the process of shifting the degree of the graded components within a vector space but each is for a different grading. Degree shift is an operation on the individual cochain groups, while height shift is an operation on the entire cochain complex.

2.4 Defining the Khovanov Cochain Groups

We now begin defining Khovanov homology for a link L. We start by defining the cochain groups. Then, we will provide an example of this construction for the right-handed trefoil.

As in the computation of the Jones polynomial, we will need to choose an orientation on L in order to compute its Khovanov homology. Recall that for knots, changing the orientation will not change the resulting value for the invariant. However, because this is not the case for all links, we will specify the orientation we use where needed. We also recall some notation from the previous section, which we will use again here: n is the number of crossings of L, and n_+ and n_- are the numbers of positive and negative crossings of L respectively.

Now, we start to construct the cochain groups of the Khovanov complex $\mathcal{C}(L)$. We begin the same way we did when defining the Jones polynomial of a link. First, suppose we have chosen an orientation on L. Next, we fix an arbitrary ordering of the crossings of L. Next, we resolve all the crossings of L in all possible ways giving us 2^n different complete smoothings of L. We assign each complete smoothing to a vertex of the n-dimensional cube $\{0,1\}^n$, as we did when computing the Jones polynomial. Recall that this correspondence between vertices and smoothings assigns a vertex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0,1\}^n$ to the smoothing that has crossing i resolved by an α_i-smoothing.

Next, we assign to each vertex α the graded vector space

$$V_\alpha(L) \coloneqq V^{\otimes k_\alpha}\{r_\alpha\},$$

where again k_α is the number of cycles in the smoothing of L associated to α and r_α is the height of the smoothing given by the sum of the components of α. We can think of $V_\alpha(L)$ as assigning one copy of V to each cycle in the smoothing at α, and shifting the degree by the factor r_α.

Now, we collect together these $V_\alpha(L)$ vector spaces, grouping them by height r_α. We write $[\![L]\!]^r$ for the group of $V_\alpha(L)$ for all vertices of height r. That is, we let

$$\begin{aligned}[\![L]\!]^r &\coloneqq \bigoplus_{\alpha\,:\,|\alpha|=r} V_\alpha(L) \\ &= \bigoplus_{\alpha\,:\,|\alpha|=r} V^{\otimes k_\alpha}\{r\}\end{aligned}$$

Note that since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vertex of the cube $\{0,1\}^n$, then all the α_i are either 0 or 1. The minimal height of a vertex α is 0, which occurs if all the α_i are 0. Thus, we have $[\![L]\!]^r = 0$ for $r < 0$. Similarly, the maximal height of a vertex α is n, which occurs if all the α_i are 1. Thus, we have $[\![L]\!]^r = 0$ for $r > n$. In summary, $[\![L]\!]^r$ can only be nonzero for $0 \leq r \leq n$, although $[\![L]\!]^r$ may still be zero for some values of r in this range.

Now, taking the direct sum of these groups $[\![L]\!]^r$ over this range $0 \leq r \leq n$ gives us the cochain groups:

$$\begin{aligned}[\![L]\!] &= \bigoplus_{r=0}^{n} [\![L]\!]^r \\ &= \bigoplus_{r=0}^{n} \bigoplus_{\alpha:|\alpha|=r} V_\alpha(L).\end{aligned}$$

This is a bigraded vector space with cohomological grading on the complex and quantum grading on each cochain group.

These are not quite the Khovanov cochain groups. Instead, similar to how we multiplied the Kauffman bracket by a particular factor in terms of n_+ and n_- to get

the Jones polynomial, we must shift the cochain groups $[\![L]\!]$ in order to end up with the Khovanov cochain groups here.

Definition 2.8. We define the *Khovanov cochain groups* $\mathcal{C}(L)$ for a link L to be

$$\mathcal{C}(L) \coloneqq [\![L]\!][-n_-]\{n_+ - 2n_-\},$$

where the degree shift of $n_+ - 2n_-$ adjusts the quantum degree, and the height shift of $-n_-$ adjusts the cohomological degree.

We will define the differentials on this complex in the next section.

To help us understand this cochain group construction, we now go through an example.

Example 2.9. Consider the right-handed trefoil shown in the top left corner of Figure 2.5. In this example, we will construct its Khovanov cochain groups.

First, we fix an arbitrary ordering on its crossings as shown in the link diagram in Figure 2.5. Next, we resolve all of the crossings of this link in all possible ways. We assign each complete smoothing to a vertex of the 3-cube as we did in the computation of the Jones polynomial. This is also shown in Figure 2.5. As before, the sequence of 0s and 1s in the bottom right corner of each box tells us which vertex of the 3-cube the smoothing corresponds to, and it tell us how each crossings of the trefoil is resolved to get the figure at each vertex.

We then assign $V_\alpha(L) = V^{\otimes k_\alpha}\{r_\alpha\}$ to the smoothing at each vertex. The graded vector space in the upper right corner at each vertex in Figure 2.5 tells us the $V_\alpha(L)$ term for that particular α. For example, consider the smoothing at vertex $(1,1,0)$, which is the top smoothing in the second column from the right. Its corresponding vector space is

$$V^{\otimes k_{(1,1,0)}}\{r_{(1,1,0)}\} = V^{\otimes 2}\{2\},$$

where we have $k_{(1,1,0)} = 2$ because there are 2 cycles in this smoothing, and we have $r_{(1,1,0)} = 2$ because 2 of the crossings in this smoothing are resolved by 1-smoothings. That is, the sum of the coordinates of the vertex α is $1 + 1 + 0 = 2$, and so this vertex has height 2.

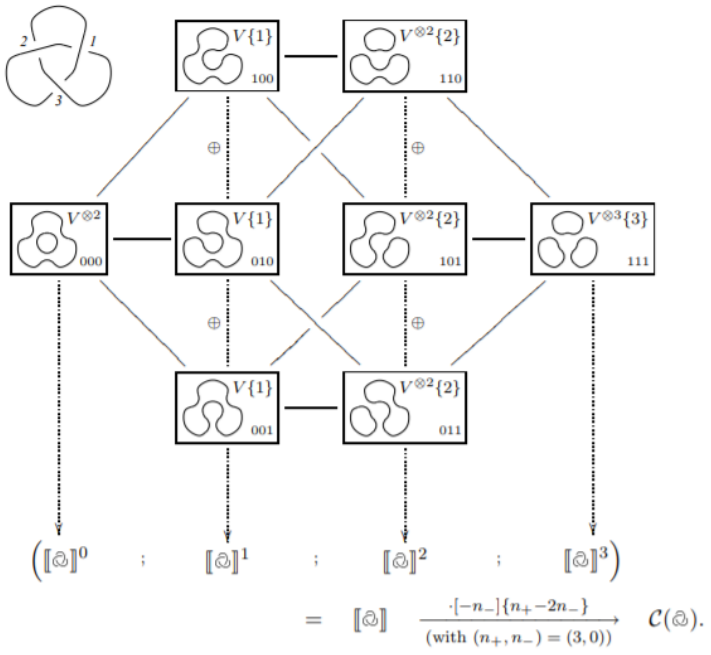


Figure 2.5: Khovanov complex of the right-handed trefoil [BN02]

Now, to get the group $[\![L]\!]^r$, we sum over all vector spaces associated to smoothings at height r. This gives us the groups:

$$
\begin{aligned}
[\![L]\!]^0 &= V^{\otimes 2} \\
[\![L]\!]^1 &= V\{1\} \oplus V\{1\} \oplus V\{1\} \\
[\![L]\!]^2 &= V^{\otimes 2}\{2\} \oplus V^{\otimes 2}\{2\} \oplus V^{\otimes 2}\{2\} \\
[\![L]\!]^3 &= V^{\otimes 3}\{3\}.
\end{aligned}
$$

Recall that each of these groups is, itself, a graded vector space. Let us briefly describe some of the graded pieces of $[\![L]\!]^1$, the cochain group in degree 1 of the complex, to see what this group looks like. We will use notation $[\![L]\!]^{i,j}$. The first superscript i tells us the cohomological degree in the grading on the complex $[\![L]\!]$. The second superscript j tells us the degree in the quantum grading on the cochain group $[\![L]\!]^i$. For example, $[\![L]\!]^{1,2}$ is the degree 2 homogeneous component of the degree 1 cochain group. Now, we present some of the graded pieces of $[\![L]\!]^1$:

$$
\begin{aligned}
[\![L]\!]^{1,2} &= (V\{1\} \oplus V\{1\} \oplus V\{1\})^2 \\
&= V\{1\}^2 \oplus V\{1\}^2 \oplus V\{1\}^2 \\
&= V^1 \oplus V^1 \oplus V^1 \\
&= span_{\mathbb{Z}/2}\{(v_+, 0, 0), (0, v_+, 0), (0, 0, v_+)\} \\
[\![L]\!]^{1,1} &= (V\{1\} \oplus V\{1\} \oplus V\{1\})^1 \\
&= V\{1\}^1 \oplus V\{1\}^1 \oplus V\{1\}^1 \\
&= V^0 \oplus V^0 \oplus V^0 \\
&= 0 \\
[\![L]\!]^{1,0} &= (V\{1\} \oplus V\{1\} \oplus V\{1\})^0 \\
&= V\{1\}^0 \oplus V\{1\}^0 \oplus V\{1\}^0 \\
&= V^{-1} \oplus V^{-1} \oplus V^{-1} \\
&= span_{\mathbb{Z}/2}\{(v_-, 0, 0), (0, v_-, 0), (0, 0, v_-)\}
\end{aligned}
$$

Here, we see that $[\![L]\!]^{1,2}$ is the direct sum of 3 copies of V^1, where $V^1 = \langle v_+ \rangle$ is the $\mathbb{Z}/2$-span of the degree 1 generator of V. Similarly, $[\![L]\!]^{1,0}$ is direct sum of 3 copies of V^{-1}, where $V^{-1} = \langle v_- \rangle$ is the $\mathbb{Z}/2$-span of the degree -1 generator of V. Finally, we see that $[\![L]\!]^{1,1}$ and all other homogeneous components of $[\![L]\!]^1$ are empty.

Now to get from $[\![L]\!]$ to the Khovanov cochain groups $\mathcal{C}(L)$, we apply both a degree shift and a height shift. Recall that for the trefoil, we have $(n_+, n_-) = (3, 0)$. Thus, we have

$$\begin{aligned}\mathcal{C}(L) &= [\![L]\!][-n_-]\{n_+ - 2n_-\} \\ &= [\![L]\!][0]\{3\} \\ &= [\![L]\!]\{3\},\end{aligned}$$

where $[0]$ is the trivial height shift, and $\{3\}$ is the degree shift by 3.

Writing these groups out, we have that the Khovanov cochain groups for the right-handed trefoil are:

$$\begin{aligned}\mathcal{C}(L)^0 &= [\![L]\!]^0\{3\} = V^{\otimes 2}\{3\} \\ \mathcal{C}(L)^1 &= [\![L]\!]^1\{3\} = (V\{1\} \oplus V\{1\} \oplus V\{1\})\{3\} \\ \mathcal{C}(L)^2 &= [\![L]\!]^2\{3\} = (V^{\otimes 2}\{2\} \oplus V^{\otimes 2}\{2\} \oplus V^{\otimes 2}\{2\})\{3\} \\ \mathcal{C}(L)^3 &= [\![L]\!]^3\{3\} = V^{\otimes 3}\{3\}\{3\} = V^{\otimes 3}\{6\}.\end{aligned}$$

Above we described in detail the graded pieces of $[\![L]\!]^1$, so here we describe the graded pieces of $\mathcal{C}(L)^1$ in terms of those pieces:

$$\begin{aligned}\mathcal{C}(L)^{1,5} &= ([\![L]\!]\{3\})^{1,5} \\ &= [\![L]\!]^{1,2} \\ \mathcal{C}(L)^{1,4} &= ([\![L]\!]\{3\})^{1,4} \\ &= [\![L]\!]^{1,1} \\ \mathcal{C}(L)^{1,3} &= ([\![L]\!]\{3\})^{1,3} \\ &= [\![L]\!]^{1,0}\end{aligned}$$

In the next section, we will continue this example by defining the differential on these cochain groups and computing the Khovanov homology groups.

2.4.1 Defining the Differential Maps

Now that we have defined the Khovanov cochain groups, we will define the differentials that map between these groups and make this into a cochain complex. We will see in this section that we can define the differential on the Khovanov cochain groups $\mathcal{C}(L)$ by piecing together maps between the direct summands of each cochain group. Recall that each direct summand corresponds to one complete smoothing of L, or one vertex of the n-cube. That is, we will define one map for each edge in the n-cube diagram of L as in Figure 2.5, and then we can combine these together using an alternating sum to get the entire differential map. We start by defining the maps along the edges of the cube diagram.

Consider an edge of the n-cube $\{0,1\}^n$ that connects vertices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ such that we have $\alpha_j = \beta_j$ for all j except one of them. Call this coordinate i where we have $\alpha_i \neq \beta_i$. We direct this edge from α to β if $\alpha_i = 0$ and $\beta_i = 1$. In this case, we call α the tail of the edge and β the head of the edge, as if this edge is an arrow. We label this edge by the sequence in $\{0, 1, \star\}^n$ given by

$$\xi = (\alpha_1, \ldots, \alpha_{i-1}, \star, \alpha_{i+1}, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_{i-1}, \star, \beta_{i+1}, \ldots, \beta_n),$$

where the j-th coordinate of its label for $j \neq i$ is $\alpha_j = \beta_j \in \{0,1\}$, and the i-th coordinate of its label is $\star$. For example, in Figure 2.5, consider the edge connecting the smoothing at the vertex $\alpha = (0,0,0)$ with the top smoothing in the second column at the vertex $\beta = (1,0,0)$. Then this edge would be labeled by the sequence $\xi = \star 00$, since α and β differ in the first spot and both have a 0 in the second two spots.

Further, we define the height $|\xi|$ of an edge to be the height of the vertex at its tail. For example, for the edge $\xi = \star 00$ we just discussed, it has height $|\xi| = |\alpha| = |(0,0,0)| = 0$.

Now that we have labeled every vertex and edge of the n-cube, we can define a map d_ξ for each edge ξ. First, recall that for each vertex α, we assign to it the vector space $V_\alpha(L) = V^{\otimes k_\alpha}\{r_\alpha\}$, where k_α is the number of cycles in that resolution of L and r_α is

the height of the vertex. Note that if two vertices α and β are connected by an edge ξ, they differ in exactly one coordinate. This means that their associated resolutions of L differ by how exactly one crossing is resolved, either by a 0-smoothing as in the case of α or by a 1-smoothing as in the case of β. This difference between α and β corresponds to either two cycles in α merging into one cycle in β, or one cycle in α splitting into two cycles in β. In terms of vector spaces then, the map $V_\alpha(L) \to V_\beta(L)$ either contains a piece of the form

$$V \otimes V \to V$$

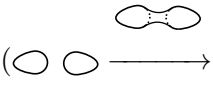
or of the form

$$V \to V \otimes V.$$

We want the map d_ξ to correspond to this merging or dividing of cycles.

Then, to define d_ξ on the vector space $V_\alpha(L)$, we use the maps m, which corresponds to two cycles merging into one, and Δ, which corresponds to one cycle dividing into two. We define these maps below, where the left-hand side of each line shows the inspiration for each map coming from merging or splitting cycles.

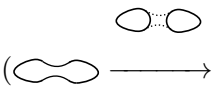
Definition 2.10. We define the map $m : V \otimes V \to V$, which corresponds to merging cycles, as follows:


$\longrightarrow$

$\longrightarrow (V \otimes V \xrightarrow{m} V)$

$$m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$$

We define the map $\Delta : V \to V \otimes V$, which corresponds to dividing a cycle, as follows:


$\longrightarrow$

$\longrightarrow (V \xrightarrow{\Delta} V \otimes V)$

$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

To be precise, we note that $V_\alpha(L)$ and $V_\beta(L)$ may have more terms than just those involved in the splitting or merging. That is, there may be other tensor factors of V that

correspond to cycles left unchanged between α and β. The map $d_\xi : V_\alpha(L) \to V_\beta(L)$ will be the identity on these unchanged factors, meaning that d_ξ will really be one of either m or Δ tensored with some number of identity maps.

Note that, we want this map d_ϵ to be of degree 0, meaning that the quantum degree of an element in its domain has the same quantum degree as the image of that element. Also, recall that the same element increases its quantum degree by one as its cohomological degree increases by one. Thus, to account for all of this, both m and Δ must have degree -1. That is, we can view m as a map:

$$V \otimes V \to V\{1\},$$

and we can view Δ as a map:

$$V \to (V \otimes V)\{1\}.$$

Using this definition of the map d_ξ for each edge ξ, we can define the differential $d^r : \mathcal{C}(L)^r \to \mathcal{C}(L)^{r+1}$ to be:

$$d^r := \sum_{\xi\,:\,|\xi|=r} (-1)^\xi d_\xi,$$

where we have $(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$, and where j is the position of $\star$ in the label of ξ. For detail on showing that d^r is a differential, see Section 3.2 of Bar-Natan's paper [BN02]. This differential turns the Khovanov cochain groups we defined in the last section into the Khovanov cochain complex.

Recall though that when we take cohomology, we will use coefficients in $\mathbb{Z}/2$, and so we will ignore the sign in the definition of d^r. We describe the signs here for completeness. We will define the maps in our trefoil example from the previous section after defining Khovanov homology in the next section.

2.4.2 Khovanov Homology

Now that we have defined the Khovanov complex $\mathcal{C}(L)$ associated to an oriented link L, we can compute its cohomology. We call this the Khovanov homology of the link, denoted $H^{*,*}(\mathcal{C}(L))$, $H^{*,*}(L)$, or sometimes $Kh^{*,*}(L)$ with coefficients in $\mathbb{Z}/2$.

Note that since $\mathcal{C}(L)$ is a bigraded vector space, then each Khovanov homology group will be a graded vector space, where we call its internal grading the quantum

grading. We denote the j-th quantum graded piece of the i-th Khovanov homology group by $H^{i,j}(L)$. We can collect together the j-th quantum graded piece from each of the cohomology groups to get $H^{*,j}(L)$, which is itself a graded vector space, and which we will discuss later.

We can take the graded Poincaré polynomial $P(L)$ of the Khovanov homology, which is defined as follows

$$P(L) \coloneqq \sum_r t^r \ q \dim H^{r,*}(\mathcal{C}(L)).$$

The graded Poincaré polynomial of the Khovanov homology of a link gives us back the Jones polynomial. We describe this relationship and the invariance of Khovanov homology in the following theorem.

Theorem 2.11. [Kho00, Theorem 2] The graded dimensions of the Khovanov homology groups $H^{r,*}(L)$ are link invariants, and hence $P(L)$, the Poincaré polynomial of the Khovanov homology groups, which is a polynomial in the variables t and q, is a link invariant that equals the unnormalized Jones polynomial when we set $t = -1$. Further, the isomorphism classes of the Khovanov homology groups are themselves a link invariant.

For details on the proof of the first statement of Theorem 2.11, see Bar-Natan's paper [BN02]. For details on the second statement, see Khovanov's paper [Kho00].

Now that we have defined Khovanov homology, we can finish our example of computing the Khovanov homology of the right-handed trefoil.

Example 2.12. Continuing Example 2.9 where we computed the cochain groups of the Khovanov complex for the right-handed trefoil, we now define the differential map on the cochain groups and compute the Khovanov homology groups. The ordering of the crossings of the trefoil and the orientation we fixed from the previous example are shown in the top left corner of Figure 2.6.

For notational purposes, we will label the cycles in each smoothing of the trefoil and thereby order the vector spaces in $V_\alpha(L)$. To do this, we start by labeling each segment of an edge in the original trefoil that lies between any two crossings. We pick an arbitrary starting point and order the segments, moving in the direction of the orientation we have chosen. From this, we label each cycle in a smoothing by the

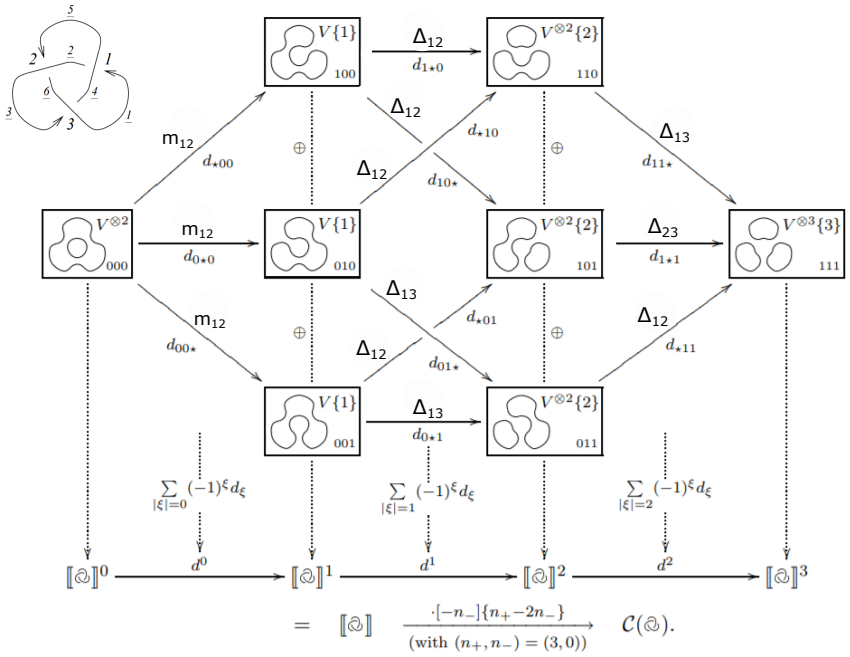


Figure 2.6: Computing the Khovanov homology of the right-handed trefoil [BN02]

minimal label of the segments it contains. We order the vector spaces in the tensor product for each smoothing $V_\alpha(L)$ according to this order. Note that this labeling will not affect the resulting Khovanov homology. It is just to help us keep track of which vector space corresponds to which cycle.

The labeling of the segments is shown as the underlined numbers in the top left corner of Figure 2.6. Then for example, in the smoothing corresponding to vertex $(0,1,1)$, we label the large cycle by 1 and smaller cycle by 3. Thus, in the corresponding vector space $V_{(0,1,1)} = V \otimes V$, the first copy of V corresponds to cycle 1, and the second copy of V corresponds to cycle 3. We will not label these vector spaces by 1 and 3 in order to reduce notation, but using this ordering convention, we can always determine which vector space in each $V_\alpha(L)$ corresponds to which cycle in the complete resolution.

We also want to order the vector spaces $V_\alpha(L)$ in each cochain group $\mathcal{C}(L)^r$, so that we can determine which vector space $V_\alpha(L)$ corresponds to which smoothing. The ordering we use is $V_{\alpha_1}(L) \oplus V_{\alpha_2}(L) \oplus \cdots \oplus C_{\alpha_m}(L)$. Here, α_1 is the top vertex in the column for height r in Figure 2.6. Then, α_2 is the second vertex in that column. We

keep going down along the column until we reach the bottom vertex α_m. For example for $\mathcal{C}(L)^1 = (V\{1\} \oplus V\{1\} \oplus V\{1\})\{3\}$, the first copy of $V\{1\}$ corresponds to the smoothing at $(1,0,0)$, the second copy corresponds to $(0,1,0)$, and the last copy corresponds to $(0,0,1)$.

With this notation in hand, we now define the maps on the edges of the 3-cube in Figure 2.6. Recall that the map $m : V \otimes V \to V$ corresponds to combining two cycles into one in a smoothing, and the map $\Delta : V \to V \otimes V$ corresponds to dividing a cycle into two in a smoothing. Thus, we can label m and Δ with subscripts to indicate which two cycles are either combining or splitting. For example, we denote $d_{\star 00} : V^{\otimes 2} \to V\{1\}$ by $m_{12}^{(0,0,0)}$ since its domain corresponds to the vertex $(0,0,0)$ and since it combines cycles 1 and 2 in the smoothing at vertex $(0,0,0)$ to get a single cycle at vertex $(1,0,0)$. In Figure 2.6, we write along each edge what the map is in terms of this notation, ignoring the superscripts for simplicity.

Using these edge maps, we define the differential on the Khovanov cochain groups to be the sum of these maps on the edges. Recall that we are using $\mathbb{Z}/2$ coefficients, and so we can disregard the sign that was included in the initial definition of the differential. Thus, for example, $d^1 : \mathcal{C}(L)^1 \to \mathcal{C}(L)^2$ is defined by the assignment

$$(x,y,z) \mapsto (\Delta_{12}^{(1,0,0)}(x) + \Delta_{12}^{(1,0,0)}(y),\ \Delta_{12}^{(0,1,0)}(x) + \Delta_{12}^{(0,1,0)}(z),\ \Delta_{13}^{(0,0,1)}(y) + \Delta_{13}^{(0,0,1)}(z)).$$

Now that we have defined the cochain groups and the differential, we can begin to compute the cohomology of this complex. We will first compute the kernel and image of each differential.

To do this, we will write each map as a matrix by specifying a fixed order on the generators of the cochain groups. Again, consider $d^1 : C^1(L) \to C^2(L)$. To write this as a matrix, we start by ordering a basis B_1 for $\mathcal{C}(L)^1$ and a basis B_2 for $\mathcal{C}(L)^2$ as follows, ignoring degree shift for the time being:

$$\begin{aligned} B_1 = &\{(v_+,0,0),(v_-,0,0),(0,v_+,0),(0,v_-,0),(0,0,v_+),(0,0,v_-)\} \\ B_2 = &\{(v_+ \otimes v_+,0,0),(v_+ \otimes v_-,0,0),(v_- \otimes v_+,0,0),(v_- \otimes v_-,0,0), \\ &(0,v_+ \otimes v_+,0),(0,v_+ \otimes v_-,0),(0,v_- \otimes v_+,0),(0,v_- \otimes v_-,0), \\ &(0,0,v_+ \otimes v_+),(0,0,v_+ \otimes v_-),(0,0,v_- \otimes v_+),(0,0,v_- \otimes v_-)\} \end{aligned}$$

Using this, we can write d^1 as the following matrix, remembering that we are working with coefficients in $\mathbb{Z}/2$.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

We can compute the kernel and image of this map, and we get that

$$\ker(d^1) = span_{\mathbb{Z}/2}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= span_{\mathbb{Z}/2}\{(v_+, v_+, v_+), (v_-, v_-, v_-)\}$$

$$\mathrm{im}(d^1) = span_{\mathbb{Z}/2}\left\{ \begin{bmatrix} 0\\1\\1\\0\\0\\1\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\0\\0\\0\\0\\0\\0\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} \right\}$$

$$= span_{\mathbb{Z}/2}\{(v_+ \otimes v_- + v_- \otimes v_+, v_+ \otimes v_- + v_- \otimes v_+, 0), (v_- \otimes v_-, v_- \otimes v_-, 0),$$
$$(v_+ \otimes v_- + v_- \otimes v_+, 0, v_+ \otimes v_- + v_- \otimes v_+), (v_- \otimes v_-, 0, v_- \otimes v_-)\}$$

Similarly, we could write down the matrix representations for the other differentials and compute their kernels and images though we do not include these here. Once we have done that, we can compute the cohomology by taking quotients of these kernels and images. In cohomological degree 0, we have

$$H^{0,*}(\mathcal{C}(L); \mathbb{Z}/2) = \ker(d^0)/\mathrm{im}(d^{-1})$$
$$\cong \langle (v_+ \otimes v_- + v_- \otimes v_+), (v_- \otimes v_-) \rangle.$$

At this point, we want to remember all of those degree shifts that we have been ignoring. For example, when we say $v_- \otimes v_-$, we do not mean the particular element $v_- \otimes v_-$, which normally lives in quantum degree -2. Instead, after applying the degree shift, this element lives in quantum degree 1. Similarly, we do not want the exact term $(v_+ \otimes v_- + v_- \otimes v_+)$, which normally has quantum degree 0, but instead, after applying the degree shift, this element lives in quantum degree 3.

By similar computations, we can compute the other cohomology groups. This gives

us the following Khovanov homology groups for the right-handed trefoil:

$$
\begin{aligned}
&H^{0,1}(L;\mathbb{Z}/2) \cong span_{\mathbb{Z}/2}\langle v_- \otimes v_- \rangle \cong \mathbb{Z}/2 \\
&H^{0,3}(L;\mathbb{Z}/2) \cong span_{\mathbb{Z}/2}\langle (v_+ \otimes v_-) + (v_- \otimes v_+) \rangle \cong \mathbb{Z}/2 \\
&H^{1,*}(L;\mathbb{Z}/2) = 0 \\
&H^{2,5}(L;\mathbb{Z}/2) \cong span_{\mathbb{Z}/2}\langle (v_- \otimes v_+, 0, v_- \otimes v_+) \rangle \cong \mathbb{Z}/2 \\
&H^{2,7}(L;\mathbb{Z}/2) \cong span_{\mathbb{Z}/2}\langle (v_+ \otimes v_+, 0, v_+ \otimes v_+) \rangle \cong \mathbb{Z}/2 \\
&H^{3,9}(L;\mathbb{Z}/2) \cong span_{\mathbb{Z}/2}\langle v_+ \otimes v_+ \otimes v_+ \rangle \cong \mathbb{Z}/2.
\end{aligned}
$$

To sanity check ourselves, we compute the graded Poincaré polynomial $P(L)$ of the Khovanov homology groups, and let $t = -1$. This gives us

$$P(L)_{t=-1} = q + q^3 + q^5 - q^9,$$

which is the unnormalized Jones polynomial of the right-handed trefoil that we computed in Example 2.2 as Theorem 2.11 told us it would be.

Chapter 3

Background for the Steenrod Square Computation

In this section we will provide background necessary for Morán's Steenrod square construction on Khovanov homology. This includes some discussion of particular integer sequences, as well as the Burnside category and the augmented semi-simplicial category.

3.1 Sequences

We start by defining the collection of sequences $P_q(n)$, where q and n are both integers. This collection contains all increasing sequences of length n of the integers $\{0, 1, \ldots, n\}$. For example, we have

$$P_3(4) = \{(0,1,2), (0,1,3), (0,1,4), (0,2,3), (0,2,4), (0,3,4), (1,2,3), (1,2,4), (1,3,4), (2,3,4)\}.$$

We will also use the collection $P(n)$, which we define by

$$P(n) = \bigcup_q P_q(n).$$

Next, for $V \in P_p(n)$, we define the map

$$\psi_V : \{U \in P_q(n) \,|\, U \cap V = \emptyset\} \longrightarrow P_q(n-p)$$

as follows

$$(u_1, \ldots, u_q) \longmapsto (w_1, \ldots, w_q), \qquad \text{where } w_j = u_j - |\{v \in V \mid v < u_j\}.$$

The idea of this map is that for disjoint subsets V and W of $\{0, 1, \ldots, n\}$ we renumber the elements of a set U after removing the elements of V from the set $\{0, 1, \ldots, n\}$. For example, consider

$$\psi_{(1,3,4)} : \{U \in P_4(7) \mid U \cap V = \emptyset\} \to P_4(4)$$

applied to $(0, 2, 5, 7)$. We define the image of each coordinate as follows

$$\begin{aligned} 0 &\longmapsto 0 - |\emptyset| = 0 - 0 = 0 \\ 2 &\longmapsto 2 - |\{1\}| = 2 - 1 = 1 \\ 5 &\longmapsto 5 - |\{1, 3, 4\}| = 5 - 3 = 2 \\ 7 &\longmapsto 7 - |\{1, 3, 4\}| = 7 - 3 = 4. \end{aligned}$$

Thus, we have the assignment $\psi_{(1,3,4)}(0, 2, 5, 7) = (0, 1, 2, 4)$.

We will also be interested in the collection of sequences $\mathcal{P}_q(n)$, where q and n are again both integers. However, this collection $\mathcal{P}_q(n)$ contains all non-decreasing sequences of length q of the integers $\{0, 1, \ldots, n\}$, where each integer can appear at most twice. For example, we have

$$\mathcal{P}_3(2) = \{(0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2), (0, 2, 2), (1, 1, 2), (1, 2, 2)\}.$$

We will also use the collection $\mathcal{P}(n)$, which we define by

$$\mathcal{P}(n) = \bigcup_q \mathcal{P}_q(n).$$

Now, given a sequence $U \in \mathcal{P}_q(n)$, we define the following subsets:

- $\dot{U}$ is the subset containing all of the integers that appear once in U,
- $\ddot{U}$ is the subset containing a single instance of all of the integers that appear twice in U, and

- $\bar{U}$ is the subset of unique integers in U.

For example, suppose we have $U = (1, 2, 2, 3, 4, 4)$. Then, we will have

$$\dot{U} = (1, 3)$$
$$\ddot{U} = (2, 4)$$
$$\bar{U} = (1, 2, 3, 4).$$

3.2 The Burnside Category

In this section, we describe a 2-category called the Burnside category, which we denote by $\mathscr{B}$. Then, we will describe a functor that sends vertices and edges of the n-cube diagram for a link to the objects and morphisms of the Burnside category. This will be used in our construction of the Steenrod operations.

3.2.1 Correspondences and the Burnside Category

Before presenting the definition of the Burnside category, whose morphisms are correspondences, we describe correspondences, in general, along with some of their properties.

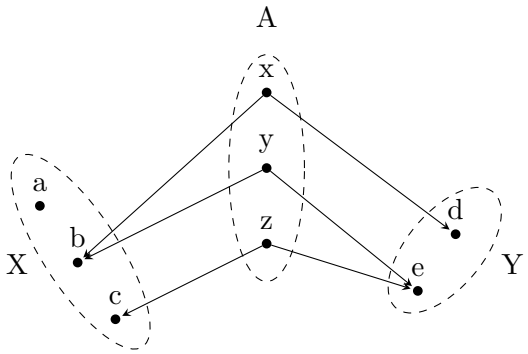
Definition 3.1. [LLS20] Given sets X and Y, a *correspondence* from X to Y is a set A and maps $s : A \to X$ and $t : A \to Y$, sometimes called the *source* and *target* maps, respectively.

Consider the following example.

Example 3.2. Let $X = \{a, b, c\}$ and $Y = \{d, e\}$. One correspondence from X to Y is given by $A = \{x, y, z\}$ with the following source and target map assignments:

$$s(x) = b \qquad t(x) = d$$
$$s(y) = b \qquad t(y) = e$$
$$s(z) = c \qquad t(z) = e$$

We can represent this pictorially as follows:



We can think of A as a multi-valued function that is only defined on a subset of its domain. This would give us the following assignments:

$$A(b) = d \text{ since } s(x) = b,\ t(x) = d,$$
$$A(b) = e \text{ since } s(y) = b,\ t(y) = e,$$
$$A(c) = e \text{ since } s(x) = c,\ t(z) = e.$$

Note that since a is not the source of anything in A, if we are thinking of A as a map, then A would not be defined on $a \in X$. This explains why s and t are called the source and target maps, respectively.

Example 3.3. We can view a function $f : U \to V$ as a correspondence A from U to V. One way to do this is to let $A \coloneqq \{(u, f(u)) \in U \times V\}$. Then, define the source map s to be projection onto the first component, that is

$$s(u, f(u)) = u \in U,$$

and define the target map to be projection onto the second component, that is

$$t(u, f(u)) = f(u) \in V.$$

We will follow Example 3.3 when we define the functor F^q from edges in n-cube diagrams of links to correspondences later in this section.

Now, given two correspondences, we will define what it means to compose them and what it means to have an isomorphism between them.

Definition 3.4. [LLS20] Given two correspondences (A, s_A, t_A) from X to Y and (B, s_B, t_B) from Y to Z, we define their *composition* to be the correspondence

$$(C, s, t) = (B, s_B, t_B) \circ (A, s_A, t_A)$$

from X to Z given by

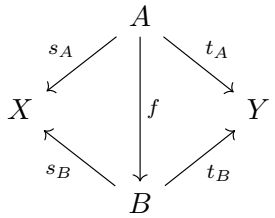
$$C = B \times_Y A = \{(a, b) \in A \times B \mid t(a) = s(b)\},$$

where $t(a) = t_A(a)$ and $s(b) = s_B(b)$.

Remark 3.5. Note that in Lawson, Lipshitz, and Sarkar's paper when they present this definition, they write the tuples in $B \times_Y A$ in the opposite order as:

$$C = B \times_Y A = \{(b, a) \in B \times A \mid t(a) = s(b)\}.$$

Definition 3.6. Given two correspondences (A, s_A, t_A) and (B, s_B, t_B) both from X to Y, we define an *isomorphism of correspondences* to be a bijection of sets $f : A \to B$, which commutes with the source and target maps, meaning that $s_A = s_B \circ f$ and $t_A = t_B \circ f$. In other words, both triangles in the following diagram commute:



In the construction of the Steenrod operations, we will associate sets to each complete resolution of a link. We will define correspondences between these sets based on the edge maps between the resolutions. These associated sets and correspondences will all be finite. Thus, we are interested in the collection of finite sets, finite correspondences, and morphisms between the correspondences. This collection of objects, 1-morphisms, and 2-morphisms forms a category called the Burnside category.

Definition 3.7. [LLS20] The *Burnside category*, which we denote $\mathscr{B}$, is the 2-category whose objects are finite sets, whose 1-morphisms are finite correspondences, and whose 2-morphisms are isomorphisms between the finite correspondences.

3.2.2 A Correspondence from a Link Diagram via the Functor F^q

In this section, we will describe a particular method for associating correspondences to link diagrams that will be useful for defining Steenrod operations on Khovanov homology. To do this for a given link with n crossings and for a given quantum degree q, we define a functor $F^q : \underline{2}^n \to \mathscr{B}$, where $\underline{2}^n$ is the cube category, which we define as follows.

Definition 3.8. The *cube category* is denoted $\underline{2}^n$ where $n \geq 1$. Its collection of objects is $\{0,1\}^n$, which contains 2^n elements. For any pair of objects $u, v \in \{0,1\}^n$, there is either a single morphism $u \to v$ if we have $u \leq v$ or a single morphism $v \to u$ otherwise. By saying $u \leq v$ for $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots v_n)$, we mean that we have $u_i \leq v_i$ for $i = 0, 1, \ldots, n$.

Now, we will define the functor F^q on the objects and morphisms of $\underline{2}^n$, as well as on compositions of morphisms, and then provide a few examples. After that, we will define 2-morphisms between these 1-morphisms in $\mathscr{B}$. It will be sufficient to define only the 2-morphisms in $\mathscr{B}$ between compositions as $F^q(g) \circ F^q(f) \to F^q(g') \circ F^q(f')$ [LLS20]. Thus, we will only present the definition of these 2-morphisms, after which we will provide an example.

3.2.2.1 F^q Defined on Objects and Morphisms

We define F^q on a vertex $a \in \underline{2}^n$ to be the set of generators of the vector space associated to that vertex in quantum degree q. That is, this functor sends a vertex a to the set containing all the pairs (a, l), where:

- l comes from a labeling by v_+ and v_- of the complete smoothing associated to a for the given link, and
- l lives in quantum degree q, after performing all degree shifts relevant to defining the Khovanov complex.

We call this pair (a, l) an *enhanced state* of a. Here, we define F^q to send each vertex to a distinct set in $\mathscr{B}$. We note now that these sets are disjoint, although they need not be. This is only so that down the line when we take a disjoint union of some of these sets for different vertices, we ensure that it will end up being disjoint. We could have instead mapped a vertex a to the set of all elements l described above, rather than tuples (a, l).

Now, we will define F^q on the edges of the n-cube. Suppose we have an edge map $f : a_1 \to a_2$ in $Mor(\underline{2}^n)$. Let $\bar{f} : V_1 \to V_2$ be the map on vector spaces associated to f, where vector spaces V_1 and V_2 are associated to vertices a_1 and a_2 respectively.

Recall that the map $\bar{f}$ on vector spaces is made up of either the map $m : V \otimes V \to V$ or the map $\Delta : V \to V \otimes V$. Recall that these maps are defined as follows:

$$\begin{aligned} m :& v_+ \otimes v_+ \longmapsto v_+ \\ & v_+ \otimes v_- \longmapsto v_- \\ & v_- \otimes v_+ \longmapsto v_- \\ & v_- \otimes v_- \longmapsto 0 \\ \Delta :& v_+ \longmapsto (v_+ \otimes v_-) + (v_- \otimes v_+) \\ & v_- \longmapsto v_- \otimes v_- \end{aligned}$$

The functor F^q on the map m gives us the correspondence:

$$\begin{aligned} \{&(v_+ \otimes v_+, v_+), \\ &(v_+ \otimes v_-, v_-), \\ &(v_- \otimes v_+, v_-)\}, \end{aligned}$$

where the source map is projection onto the first component, and the target map is projection onto the second component. Notice that this correspondence is all pairs $(u, m(u))$ for all u except for the case when we have $m(u) = 0$.

The functor F^q on the map Δ gives us the correspondence:

$$\begin{aligned} \{&(v_+, v_+ \otimes v_-), \\ &(v_+, v_- \otimes v_+), \end{aligned}$$

$$(v_-, v_- \otimes v_-)\},$$

where the source and target maps are again projections.

The image of f under F^q will be one of these correspondences above, possibly with additional tensor terms in each tuple that do not change, coming from vector spaces in V_1 and V_2 that $\bar{f}$ is the identity on.

We can also define F^q on compositions of edge maps $f_n \circ \cdots \circ f_1$ by taking the composition of the correspondences $F^q(f_n), \ldots, F^q(f_1)$. Recall that we will write the element of a composition of correspondences in the opposite order to how we write correspondence composition. For example, we will say that a composition $A \circ B$ contains elements $(b, a) \in B \times A$ for which $t_A(a) = s_B(b)$. In our case, since the individual correspondences, themselves, contain pairs, we are considering pairs of pairs $((a_1, a_2), (b_1, b_2))$, where we have $t(a_1, a_2) = s(b_1, b_2)$. Recall that we have $t(a_1, a_2) = a_2$ and $s(b_1, b_2) = b_1$, and so the elements of the composition of correspondences will have the form $((a_1, a_2), (b_1, b_2))$ where they have $a_2 = b_1$. To simplify notation, we will write these elements of the correspondence as triples (a_1, a_2, b_2).

Below we provide several examples. We start with two examples of defining correspondences from edge maps in link diagrams in Examples 3.9 and 3.10. Then in Example 3.11, we describe the correspondence associated to the composition of edge maps.

Example 3.9. Consider the link diagram of a twisted unknot in Figure 3.1.

Figure 3.1: The Twisted Unknot

It has only one crossing, which we can resolve either using a 0-smoothing or a 1-smoothing. We assign each of these smoothings to a vertex of the 1-cube, $\{0, 1\}$, where the edge between these vertices is a morphism from vertex 0 to vertex 1 as shown in Figure 3.2. For notational purposes, we will assume that the cycles at vertex 0 are ordered from top to bottom. We will need to define an orientation on this knot to determine n_+ and n_- although the choice of orientation will not change these values.

quantum degree	cohomological degree 0	cohomological degree 1
3	$v_+ \otimes v_+$	v_+
1	$v_+ \otimes v_-, v_- \otimes v_+$	v_-
-1	$v_- \otimes v_-$	

Table 3.1: Generators and their Quantum Degrees

For both choices, we have $n_+ = 1$ and $n_- = 0$.

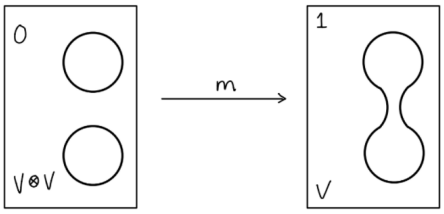


Figure 3.2: The 1-cube Representing the Twisted Unknot

Following our process for computing Khovanov homology, we assign to each smoothing a vector space given by $V^{\otimes k}\{r\}$, where k is given by the number of cycles in the particular resolution and r is the height of the particular vertex and represents the initial degree shift. We assign $V \otimes V$ to the complete resolution at vertex 0, and we assign $V\{1\}$ to the complete resolution at vertex 1. The secondary degree shift is by the factor $n_+ - 2n_-$, which in this case is 1. The height shift is $-n_-$. However, since $n_- = 0$, this is a trivial shift. Thus, the vector space associated to vertex 0 lives in cohomological degree 0, and the vector space associated to vertex 1 lives in cohomological degree 1. In Table 3.1, we present the generators of the vector space associated to each vertex of the 1-cube shown in Figure 3.2 and their cohomological and quantum degrees. We define F^q on the objects of $\underline{2}^1$ for $q = 3, 1, -1$ as follows:

$$F^3(0) = \{v_+ \otimes v_+\} \qquad F^1(0) = \{v_+ \otimes v_-, v_- \otimes v_+\} \qquad F^{-1}(0) = \{v_- \otimes v_-\}$$
$$F^3(1) = \{v_+\} \qquad F^1(1) = \{v_-\} \qquad F^{-1}(1) = \emptyset$$

Note that for simplicity, we drop the vertex from the image of each element. That is, we write the element in $F^3(0)$ as $v_+ \otimes v_+$ rather than $((0), v_+ \otimes v_+)$.

Before defining F^q on the morphism here, we note that there is a single edge in the 1-cube diagram. The one edge is defined by the map $m : V \otimes V \to V$.

Now, we define F^q in each quantum degree using the portion of the map m defined on the generators in that quantum degree. That is, in quantum degree 3, we define a correspondence from $F^3(0) = \{v_+ \otimes v_+\}$ to $F^3(1) = \{v_+\}$ to be

$$F^3(d_*) = \{(v_+ \otimes v_+, v_+)\}.$$

The source and target maps are defined as follows

$$\begin{aligned} s(v_+ \otimes v_+, v_+) &= v_+ \otimes v_+ \\ t(v_+ \otimes v_+, v_+) &= v_+. \end{aligned}$$

While this correspondence is exactly $F^3(0) \times F^3(1)$, this will not always be the case.

Next, in quantum degree 1, we define a correspondence from $F^1(0) = \{v_+ \otimes v_-, v_- \otimes v_+\}$ to $F^1(1) = \{v_-\}$ to be

$$F^1(d_*) = \{(v_+ \otimes v_-, v_-), (v_- \otimes v_+, v_-)\}.$$

The source and target maps are defined as follows

$$\begin{aligned} s(v_+ \otimes v_-, v_-) &= v_+ \otimes v_- & \qquad s(v_- \otimes v_+, v_-) &= v_- \otimes v_+ \\ t(v_+ \otimes v_-, v_-) &= v_- & \qquad t(v_- \otimes v_+, v_-) &= v_-. \end{aligned}$$

Again, we see that this correspondence is exactly $F^1(0) \times F^1(1)$. However, again this will not always be the case.

Finally, we consider the correspondence associated to quantum degree -1. Because we have $m(v_- \otimes v_-) = 0$ and we set F^{-1} to the empty set, we define this correspondence to be the empty set.

In the above example, we saw how we can assign correspondences to an edge map defined by $m : V \otimes V \to V$. In the next example, we explore how we associate correspondences to edge maps defined by $\Delta : V \to V \otimes V$.

Example 3.10. Consider the mirror image of the link from Example 3.9, as shown in

quantum degree	cohomological degree -1	cohomological degree 0
1		$v_+ \otimes v_+$
-1	v_+	$v_+ \otimes v_-, v_- \otimes v_+$
-3	v_-	$v_- \otimes v_-$

Table 3.2: Generators and their Quantum Degrees

Figure 3.3.

Figure 3.3: The Mirror Image of the Twisted Unknot

Again, it has only one crossing, which we can resolve either using a 0-smoothing or a 1-smoothing. We assign each of these complete smoothings to a vertex of the 1-cube, $\{0, 1\}$, where the edge between these vertices is a morphism from vertex 0 to vertex 1.

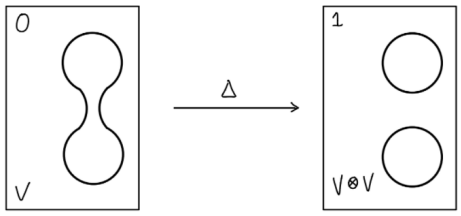


Figure 3.4: The 1-cube Representing the Mirror Image of the Twisted Unknot

Again, we assign to each smoothing a vector space given by $V^{\otimes k}\{r\}$. We assign V to the resolution at vertex 0, and we assign $(V \otimes V)\{1\}$ to the resolution at vertex 1. In this example, the secondary degree shift is by the factor $n_+ - 2n_-$, where here we have $n_+ = 0$ and $n_- = 1$, and so this degree shift is -2. The height shift is $-n_-$, where here we have $n_- = 1$. Thus, the vector space associated to vertex 0 lives in cohomological degree -1, and the vector space associated to vertex 1 lives in cohomological degree 0. In Table 3.2, we present the generators of the vector space associated to each vertex of the 1-cube in Figure 3.4 and their corresponding cohomological and quantum degrees.

We define F^q on the objects of $\underline{2}^1$ for $q = 1, -1, -3$ as follows:

$$F^1(0) = \emptyset \qquad F^{-1}(0) = \{v_+\} \qquad F^{-3}(0) = \{v_-\}$$
$$F^1(1) = \{v_+ \otimes v_+\} \qquad F^{-1}(1) = \{v_+ \otimes v_-, v_- \otimes v_+\} \qquad F^{-3}(1) = \{v_- \otimes v_-\}$$

Before defining F^q on the morphism here, we note that there is a single edge in the 1-cube diagram. The one edge is defined by the map $\Delta : V \to V \otimes V$.

Now, we define F^q in each quantum degree using the portion of the map Δ defined on the generators in that degree. First, in quantum degree -3, we define a correspondence from $F^{-3}(0) = \{v_-\}$ to $F^{-3}(1) = \{v_- \otimes v_-\}$ to be

$$F^{-3}(d_*) = \{(v_-, v_- \otimes v_-)\}.$$

The source and target maps are defined as follows

$$s(v_-, v_- \otimes v_-) = v_-$$
$$t(v_-, v_- \otimes v_-) = v_- \otimes v_-.$$

Next, in quantum degree -1, we define a correspondence from $F^{-1}(0) = \{v_+\}$ to $F^{-1}(1) = \{v_+ \otimes v_-, v_- \otimes v_+\}$ to be

$$F^{-1}(d_*) = \{(v_+, v_+ \otimes v_-), (v_+, v_- \otimes v_+)\}.$$

The source and target maps are defined as follows

$$s(v_+, v_+ \otimes v_-) = v_+ \qquad s(v_+, v_- \otimes v_+) = v_+$$
$$t(v_+, v_+ \otimes v_-) = v_+ \otimes v_- \qquad t(v_+, v_- \otimes v_+) = v_- \otimes v_+.$$

Finally, we consider the correspondence associated to quantum degree 1. Recall that under $\Delta : V \to V \otimes V$, the element $v_+ \otimes v_+$ has no preimage. Thus, we simply define this correspondence to be the empty set.

With this definition of F^q on any single edge map, we can define the correspondence associated to a composition of edge maps as the composition of the correspondence for

each edge map in the given quantum degree. To see this in practice, we present the following example.

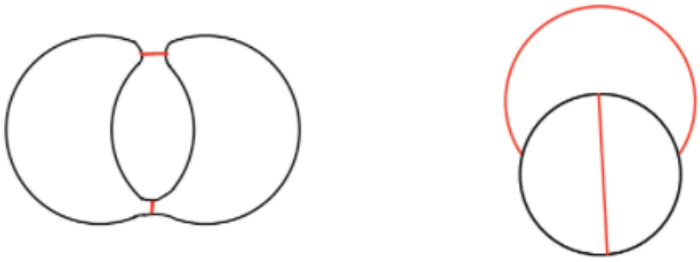
Example 3.11 (The Ladybug Matching)**.** Consider the link consisting of two unknots with one placed partially over the other, as shown in Figure 3.5. Notice that there is no meaningful crossing happening in this link diagram. We could easily slide these two unknots completely apart. However, for example's sake, we consider this link as shown. In this example, we will define correspondences associated to the compositions of the edge maps in the 2-cube diagram for this link. Then, in Example 3.12 we will define a 2-morphism between these correspondences.

Note that regardless of the orientation we choose for each component, one crossing will always be positive, and the other will always be negative. Thus, we have $n_- = 1$ and $n_+ = 1$.

Figure 3.5: Link Diagram of Two Overlapping Unknots

Before we draw the 2-cube diagram for this link, we make a few notational notes, which we will use in Example 3.12 when we define the particular 2-morphism, called the *ladybug matching*. First, to reduce notation, we will not use parentheses for vertices. For example, we write 00 for the vertex $(0, 0)$. Also, we denote 0-smoothings by a red arc connecting the strands of the complete resolution across where the crossing had been, which we can see in the resolution associated to the all 0s vertex of our link shown in Figure 3.6a. Next, in order to be consistent with the diagram from Lipshitz and Sarkar, who defined the ladybug matching, we draw the link diagram in a slightly different way. For example, we present a more natural diagram of the resolution associated to the vertex 00 in Figure 3.6a, and we present the diagram used by Lipshitz and Sarkar for the same resolution in Figure 3.6b. These diagrams differ only by ambient isotopy,

where we stretch the red arcs and reshape the black cycle into a circle. Finally, for this example, we can disregard the numbers 1 and 2 at each resolution in Figure 3.7. This will be important when we revisit this link in Example 3.12.



(a) More Natural Version From Link Diagram

(b) Lipshitz and Sarkar's Version

Figure 3.6: The Resolution Associated to the Vertex 00 for Two Overlapping Unknots

Using this new notation, we draw the 2-cube diagram for this link, as shown in Figure 3.7. We will assume that the cycles are ordered outside to inside and left to right. This will affect how we write the labelings of the cycles but will not affect the correspondence definitions in a meaningful way.

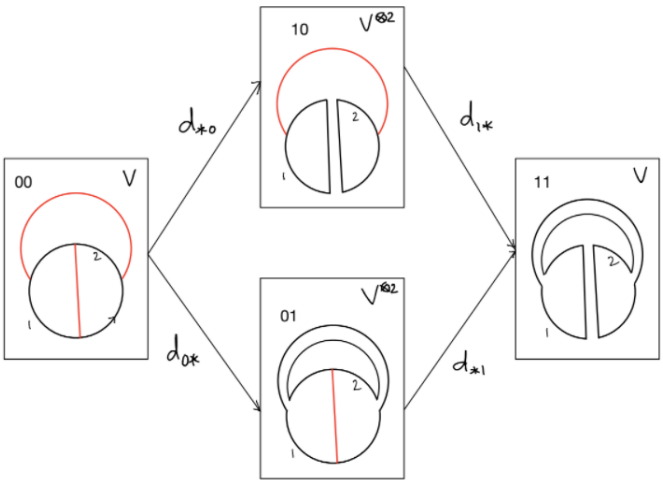


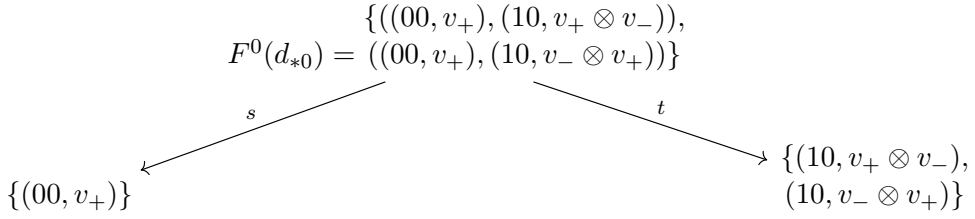
Figure 3.7: 2-Cube Diagram for Two Overlapping Unknots (The Ladybug Matching)

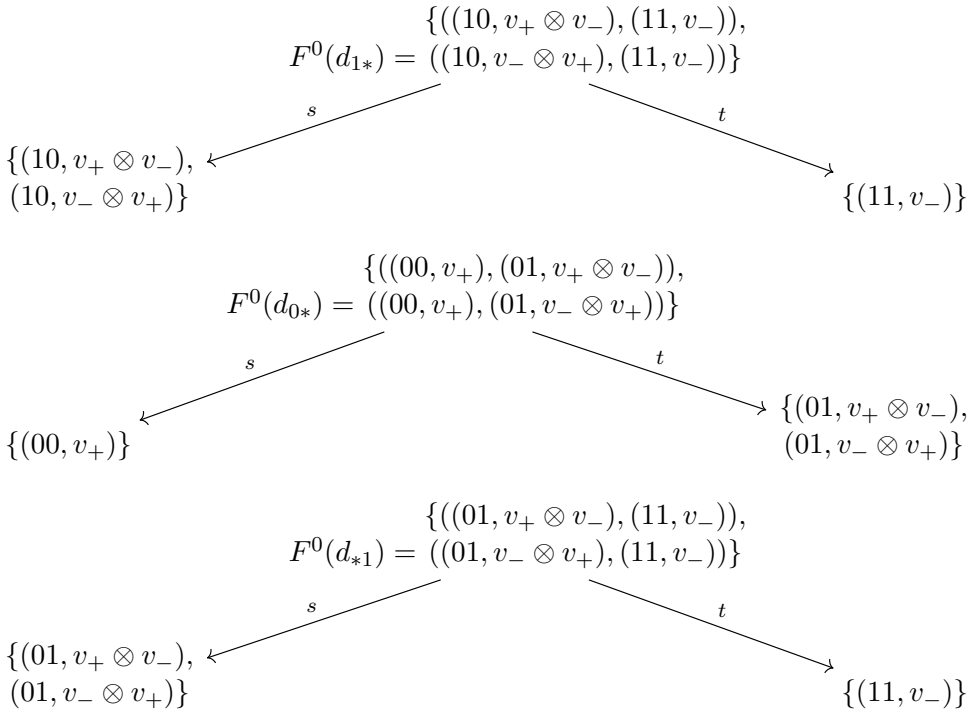
quantum degree	cohomological degree -1	cohomological degree 0	cohomological degree 1
2	$\emptyset$	$(10, v_+ \otimes v_+), (01, v_+ \otimes v_+)$	$(11, v_+)$
0	$(00, v_+)$	$(10, v_+ \otimes v_-), (10, v_- \otimes v_+),$ $(01, v_+ \otimes v_-), (01, v_- \otimes v_+)$	$(11, v_-)$
-2	$(00, v_-)$	$(10, v_- \otimes v_-), (01, v_- \otimes v_-)$	

Table 3.3: Generators and their Quantum Degrees

Next, we assign a vector space to each resolution. In particular, we assign V to the vertex 00, $(V \otimes V)\{1\}$ both to vertex 10 and to vertex 01, and $V\{2\}$ to vertex 11. We apply an additional degree shift to each of these vector spaces by $n_+ - 2n_- = -1$. We also apply a height shift of $-n_- = -1$, meaning that the vector space at vertex 00 lives in cohomological degree -1, the vector spaces at vertices 10 and 01 live in cohomological degree 0, and the vector spaces at vertex 11 live in cohomological degree 1. In Table 3.3, we present the generators of the vector spaces associated to each vertex of the 2-cube above and their cohomological and quantum degrees. Since we have two vertices corresponding to cohomological degree 0 whose generators have the same elements, we specify each generator element as a tuple where the first element is the vertex this element corresponds to and the second is the actual element, itself.

Recall that in this example, we would like to define F^q on the composition of edge maps on the n-cube diagram. Here, we will define two correspondences associated to the map from vertex 00 to vertex 11 in quantum degree 0, corresponding to the two compositions between these vertices. In particular, we will define the correspondence associated to the composition $00 \to 10 \to 11$ and the correspondence associated to the composition $00 \to 01 \to 11$. We start by presenting the correspondence associated to each individual edge map.





As in the previous examples, for an element (a, b) in one of these correspondences, we let

$$s(a, b) = a$$
$$t(a, b) = b.$$

Using the above correspondences, we now define the composition of correspondences $F^0(d_{1*}) \circ F^0(d_{*0})$ and $F^0(d_{*1}) \circ F^0(d_{0*})$. Recall that a composition of correspondences contains the pairs $(a, b) \in F^q(d_A) \times F^q(d_B)$, for which we have $t(a) = s(b)$. Recall that the correspondences will contain pairs $((a_1, a_2), (b_1, b_2))$, where we have $t(a_1, a_2) = s(b_1, b_2)$, which in this case is equivalent to requiring $a_2 = b_1$. Because of this, we will write elements of the correspondence as triples (a_1, a_2, b_2).

First, consider $F^0(d_{1*}) \circ F^0(d_{*0})$. The element $((00, v_+), (10, v_+ \otimes v_-)) \in F^0(d_{*0})$ has target $(10, v_+ \otimes v_-)$, and the element $((10, v_+ \otimes v_-), (11, v_-)) \in F^0(d_{1*})$ has source $(10, v_+ \otimes v_-)$. Thus, the pair of these elements, which in our simplified notation looks like

$$((00, v_+), (10, v_+ \otimes v_-), (11, v_-)),$$

will be in the composition $F^0(d_{1,*}) \circ F^0(d_{*0})$. Similarly, the element

$$((00, v_+), (10, v_- \otimes v_+), (11, v_-))$$

is in the composition $F^0(d_{1*}) \circ F^0(d_{*0})$. For both of these elements, their source is $(00, v_+)$, and their target is $(11, v_-)$. Thus, we have

$$F^0(d_{1*}) \circ F^0(d_{*0}) = \{((00, v_+), (10, v_+ \otimes v_-), (11, v_-)) \\ ((00, v_+), (10, v_- \otimes v_+), (11, v_-))\}.$$

Now, consider $F^0(d_{*1}) \circ F^0(d_{0*})$. The element $((00, v_+), (01, v_+ \otimes v_-)) \in F^0(d_{*0})$ has target $(01, v_+ \otimes v_-)$, and the element $((01, v_+ \otimes v_-), (11, v_-)) \in F^0(d_{1*})$ has source $(01, v_+ \otimes v_-)$. Thus, the pair of these elements, which in our simplified notation looks like

$$((00, v_+), (01, v_+ \otimes v_-), (11, v_-)),$$

will be in the composition. Similarly, the element

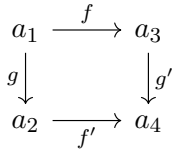
$$((00, v_+), (01, v_- \otimes v_+), (11, v_-))$$

is in the composition $F^0(d_{*1}) \circ F^0(d_{0*})$. For both of these elements, their source is $(00, v_+)$, and their target is $(11, v_-)$. Thus, we have

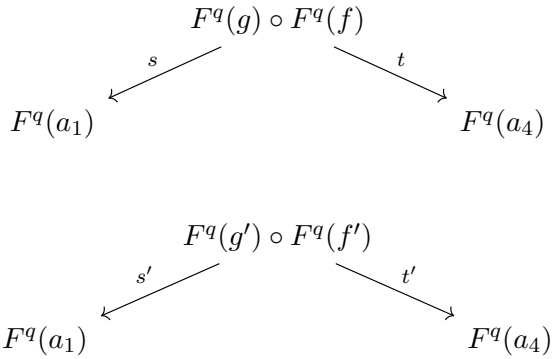
$$F^0(d_{*1}) \circ F^0(d_{0*}) = \{((00, v_+), (01, v_+ \otimes v_-), (11, v_-)) \\ ((00, v_+), (01, v_- \otimes v_+), (11, v_-))\}.$$

3.2.2.2 2-Morphisms in $\mathscr{B}$

Finally, we discuss the 2-morphisms in $\mathscr{B}$ between the images under F^q of edge maps. As mentioned above, it is sufficient to define only 2-morphisms $F^q(g) \circ F^q(f) \to F^q(g') \circ F^q(f')$ between the compositions of images of face maps. Suppose we have the following four face maps in $Mor(\underline{2}^n)$:



Based on our discussion above, we can define the compositions of these correspondences $F^q(g) \circ F^q(f)$ and $F^q(g') \circ F^q(f')$:



Now, we would like to define a 2-morphism

$$h : F^q(g) \circ F^q(f) \to F^q(g') \circ F^q(f').$$

This map h will be defined in one of three ways depending on the values of $F^q(g) \circ F^q(f)$ and $F^q(g') \circ F^q(f')$.

The first option is that h is the identity 2-morphism, which will be the case if we have $f = f'$ and $g = g'$.

The second option is that h is the canonical bijection. This will be the case if for each pair $(x, y) \in F^q(a_1) \times F^q(a_4)$, there is either zero or one element in each of $F^q(g) \circ F^q(f)$ and $F^q(g') \circ F^q(f')$ such that its source is x, and its target is y. In this case, h will send the element of $F^q(g) \circ F^q(f)$ to the element of $F^q(g') \circ F^q(f')$ with the same source and target.

The third option is the special case, in which at least one pair $(x, y) \in F^q(a_1) \times F^q(a_4)$ has exactly two elements in each of $F^q(g) \circ F^q(f)$ and $F^q(g') \circ F^q(f')$ such that their sources are x, and their targets are y. In this case, there is not a canonical way to match up these elements in $F^q(g) \circ F^q(f)$ and $F^q(g') \circ F^q(f')$ to define h. For a given pair (x, y) with two elements in each correspondence, we have to make a choice for h as to which element in $F^q(g) \circ F^q(f)$ gets assigned to which element in $F^q(g') \circ F^q(f')$.

The choice that we will make here was specified originally by Lipshitz and Sarkar and is called the *ladybug matching* [LS14a]. This case comes up when we have one cycle at vertex a_1 labeled by v_+ that splits into two cycles both vertex a_2 and at vertex a_3. At both vertices, we get the two labelings $v_+ \otimes v_-$ and $v_- \otimes v_+$. Then those cycles merge back together into one cycle at vertex a_4 labeled by v_-. The ladybug matching serves to tell us how to match up the cycles between the two intermediate vertices a_2 and a_3.

To describe how the ladybug matching works, we return to our example of two partially overlapping unknots.

We will now go through a concrete example of the ladybug matching.

Example 3.12. Recall the link consisting of two unknots with one placed partially over the other from Example 3.11, as shown again in Figure 3.8. Again, in this example, to reduce notation, we will not use parentheses for vertices. For example, we write 00 for the vertex (0, 0)

Figure 3.8: Link Diagram of Two Overlapping Unknots

In Example 3.11, we defined the correspondences associated to compositions of edge maps $d_{1*} \circ d_{*0}$ and $d_{*1} \circ d_{0*}$. These are the two different ways to map from vertex 00 to vertex 11, as we can see in the 2-cube diagram for this link in Figure 3.9. In this example, we will define the 2-morphism between these correspondences, using the ladybug matching.

Recall from Example 3.11 the two compositions of correspondences, which we list again here:

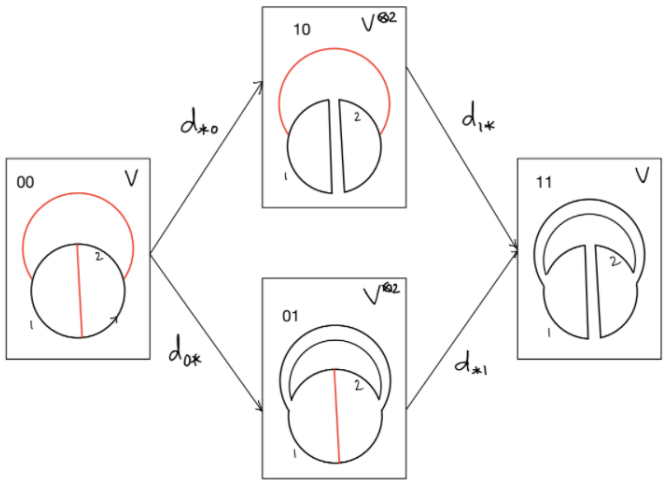


Figure 3.9: 2-Cube Diagram for Two Overlapping Unknots

$$F^0(d_{1*}) \circ F^0(d_{*0})$$
$$\{((00, v_+), (10, v_+ \otimes v_-), (11, v_-)),$$
$$= ((00, v_+), (10, v_- \otimes v_+), (11, v_-))\}$$

$$\{(00, v_+)\} \xleftarrow{s} \quad \xrightarrow{t} \{(11, v_-)\}$$

$$F^0(d_{*1}) \circ F^0(d_{0*}) :$$
$$\{((00, v_+), (01, v_+ \otimes v_-), (11, v_-)),$$
$$((00, v_+), (01, v_- \otimes v_+), (11, v_-))\}$$

$$\{(00, v_+)\} \xleftarrow{s} \quad \xrightarrow{t} \{(11, v_-)\}$$

Again, we define the source and target maps of both correspondences as follows:

$$s(a, b, c) = a$$
$$t(a, b, c) = c.$$

Both correspondences $F^0(d_{1*}) \circ F^0(d_{*0})$ and $F^0(d_{*1}) \circ F^0(d_{0*})$ have two distinct elements with the same source and target. Thus, the 2-morphism we define between

them will be a bijection. It remains to describe how we match up these elements. We can define this choice by deciding how the two cycles in the resolution at vertex 10 are matched up with the two cycles in the resolution at vertex 01 in Figure 3.9.

To define the ladybug matching, we start by drawing arcs in the place of 0-smoothings, but not 1-smoothings, connecting the strands of the complete resolution across where the crossing had been. In the resolution at the left-most vertex 00, one of these arcs is inside the cycle, and the other arc is outside the cycle.

We can think of this cycle as being made up of four strands, where a strand lies between the endpoints of these arcs. If we orient the cycle counter-clockwise, then each strand will either start at the outside arc and end at the inside arc, in which case, we call this a right strand, or the strand will start at the inside arc and end at the outside arc, in which case, we call this a left strand. As we move around the cycle, the strands alternate between these two cases. Following Lipshitz and Sarkar, we will be interested in tracking the right pair of strands. Thus, we label these two right strands by 1 and 2 as shown in Figure 3.9. Note that our choice of ordering between these two strands is arbitrary.

We similarly label those strands in the remaining vertices in the 2-cube diagram. In vertex 10, this means that the left cycle is labeled 1, and the right cycle is labeled 2. In vertex 01, the outside cycle is labeled 1, and the inside cycle is labeled 2. Our 2-morphism h will match up the elements of the correspondences $F^0(d_{1*}) \circ F^0(d_{*0})$ and $F^0(d_{*1}) \circ F^0(d_{0*})$ so that the left cycle in vertex 10 has the same labeling as the outside cycle in vertex 01 and the right cycle in vertex 10 has the same labeling as the inside cycle in vertex 01. Recall that we ordered the cycles in vertex 10 from left to right, and we ordered the cycles in vertex 01 from outside in. Therefore, we are matching up $(10, v_+ \otimes v_-)$ with $(01, v_+ \otimes v_-)$, and we are matching up $(10, v_- \otimes v_+)$ with $(01, v_- \otimes v_+)$. Thus, the 2-morphism $F^0(d_{1*}) \circ F^0(d_{*0}) \to F^0(d_{*1}) \circ F^0(d_{0*})$ makes the following assignments:

$$((00, v_+), (10, v_+ \otimes v_-), (11, v_-)) \longrightarrow ((00, v_+), (01, v_+ \otimes v_-), (11, v_-))$$
$$((00, v_+), (10, v_- \otimes v_+), (11, v_-)) \longrightarrow ((00, v_+), (01, v_- \otimes v_+), (11, v_-))$$

Note that the order of the labelings of the second components is the same between the

domain element and image element. However, this is merely by coincidence and should not be expected to always be the case.

3.2.3 The Augmented Semi-Simplicial Category

In this section, we will define the augmented semi-simplicial category. To do that, we first define Δ_{inj}.

We will present a number of related categories with related and often overlapping morphisms. To keep them all straight, we provide a summary of these morphisms as well as a few other related morphisms in Appendix A.

3.2.3.1 Δ_{inj}

The objects of the category Δ_{inj} are non-empty finite ordinals. That is, the objects are

$$
\begin{aligned}
&[0] = \{0\} \\
&[1] = \{0, 1\} \\
&[2] = \{0, 1, 2\} \\
&\dots \\
&[j] = \{0, 1, \dots, j\} \\
&\dots
\end{aligned}
$$

The morphisms of Δ_{inj} are order-preserving injections. This includes the identity morphism for each object in Δ_{inj}. We also have morphisms, which we call face maps, from $[j]$, a set containing $j+1$ elements, to $[j+1]$, a set containing $j+2$ elements. Since Δ_{inj} contains all order-preserving injections from $[j]$ to $[j+1]$, defining these morphisms is equivalent to choosing one element in $[j+1]$ to not be in the image of this morphism. We write this as ∂_i^{j+1}, where the superscript tells us the ordinal that is the target, and the subscript tells us what element of the target we skip. Note that there are $\binom{j+2}{1} = j+2$ maps ∂_i^{j+1} for each j.

For example, from the ordinal [0] to the ordinal [1], we have the following two face

maps

$$\partial_1^1 : \{0\} \mapsto \{0\}$$
$$\partial_0^1 : \{0\} \mapsto \{1\},$$

and from [1] to [2], we have the following three maps

$$\partial_2^2 : \{0,1\} \mapsto \{0,1\}$$
$$\partial_1^2 : \{0,1\} \mapsto \{0,2\}$$
$$\partial_0^2 : \{0,1\} \mapsto \{1,2\},$$

where the first element on the left gets mapped to the first element on the right, and the second element on the lefts gets mapped to the second element on the right. Finally, the remaining morphisms of Δ_{inj} are compositions of these face maps, which we call generalized face maps.

Definition 3.13 ([Mor20])**.** For a sequence $U \in P_q(p)$, we define the *U-th generalized face map* $\partial_U^p : [p-q] \to [p]$ as the unique order-preserving injective face map that misses the elements of U.

For $U \in P(n)$, $V \in P_q(n)$ and $U = V \cup W$, we have that

$$\partial_{\psi_V(W)}^{n-q} \circ \partial_V^n = \partial_U^n.$$

For example, the generalized face map $\partial_{0,2}^5 : [3] \to [5]$ is defined by the following assignments:

$$\partial_{0,2}^5(0) = 1$$
$$\partial_{0,2}^5(1) = 3$$
$$\partial_{0,2}^5(2) = 4$$
$$\partial_{0,2}^5(3) = 5$$

We can depict Δ_{inj} as the following objects with maps between them:

$$[0] \rightrightarrows [1] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; [2] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; [3] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

3.2.3.2 The Augmented Semi-Simplicial Category Δ_{inj*}

Next, given Δ_{inj}, we define the augmented semi-simplicial category Δ_{inj*} as follows. Its objects are finite ordinals, and its morphisms are order-preserving maps between them. That is, we no longer require the finite ordinals to be non-empty as we did for Δ_{inj}. Thus, the objects of Δ_{inj*} include all the objects of Δ_{inj} with one additional empty object, which we denote by $[-1] = \{\}$. The morphisms of Δ_{inj*}, which we denote $\bar{\partial}^p_U$, include all the morphisms of Δ_{inj}, where we have $\bar{\partial}^p_U = \partial^p_U$. We also include the morphism $\bar{\partial}^0_0 : [-1] \to [0]$, the identity on $[-1]$, as well as a morphism $[-1] \to [n]$ for each $n > 0$. We can visualize this category via the following diagram:

$$[-1] \to [0] \rightrightarrows [1] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; [2] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; [3] \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

3.2.3.3 Δ^{op}_{inj*}

Finally, we define Δ^{op}_{inj*}, which is the opposite category of Δ_{inj*}. This opposite category has the same objects as Δ_{inj*} but its morphisms, which we write as $\bar{\delta}^n_U$, go in the opposite direction. That is, the morphisms in Δ_{inj*} had the form

$$\bar{\partial}^p_U : [p-q] \to [p],$$

where this is the unique order-preserving injective face map that misses the elements of $U \subset [p]$. Now, the morphisms of Δ^{op}_{inj*} have the form

$$\bar{\delta}^p_U : [p] \to [p-q],$$

where we reverse the assignments of $\bar{\partial}^p_U$.

We can visualize this category similarly to Δ_{inj*} using the following diagram:

$$[-1] \leftarrow [0] \leftleftarrows [1] \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; [2] \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; [3] \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots$$

Chapter 4

The Construction of Morán's Steenrod Square $\mathfrak{sq}^i$

Lipshitz and Sarkar construct a space from a link whose cohomology groups are exactly the Khovanov homology groups of that link [LLS20]. In order to ensure that this space is a link invariant, we take suspensions of the space, which gives us a spectrum. Now, recall that the structure of the cup product defines a ring structure on cohomology and allows us to differentiate some spaces with isomorphic cohomology groups. However, the cup product is always 0 when the underlying space is a suspension. Thus, we cannot use the cup product to study links via Khovanov homology.

Steenrod squares can distinguish some spaces with isomorphic cohomology rings. Importantly, Steenrod squares are not always trivial when the underlying space is a suspension. Thus, we would like to define the Steenrod operations on the Khovanov homology of a link, so that we can further distinguish links that may have isomorphic Khovanov homology groups.

Lipshitz and Sarkar gave combinatorial formulas to compute Sq^1 and Sq^2 on the Khovanov homology of a link [LS14b]. They also implemented this method in Sagemath [LS12]. They used this package to compute Sq^1 and Sq^2 for knots with up to 11 crossings.

Building on this work by Lipshitz and Sarkar, Seed implemented these Sq^1 and Sq^2 constructions in C++ [See12, See15]. With this, he found many examples of pairs of

knots with identical integral Khovanov homology but different Sq^1 and Sq^2 maps.

Recall that for $a < 2b$, the Adem relations give us the following identity for Steenrod squares:

$$Sq^a \circ Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-j-1}{a-2j} Sq^{a+b-j} \circ Sq^j,$$

where the binominal coefficient is taken mod 2. Setting $a = 1$ and $b = 2$ in this expression, we get

$$Sq^1 \circ Sq^2 = Sq^3.$$

Thus, using Lipshitz and Sarkar's combinatorial definitions of Sq^1 and Sq^2, we can also determine Sq^3. However, we cannot use Sq^1, Sq^2, and Sq^3 in the Adem relations to determine any further Steenrod operations. Because of this, the next Steenrod operation of interest will be Sq^4.

In an attempt to distinguish more links and gain further understanding of Khovanov homology, we would like to construct $Sq^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2)$, for $i \geq 2$ where $H^{*,*}$ is the Khovanov homology and L is a link. Morán does this by determining $\mathfrak{sq}^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2)$ via a map on augmented semi-simplicial objects instead of on cochain groups or cohomology groups. Thus here, we first describe the particular augmented semi-simplicial objects that our cohomology groups correspond to. Then, we describe Morán's construction of $\mathfrak{sq}^i$.

4.1 Augmented Semi-Simplicial Object in the Burnside Category

In this section, we define a particular augmented semi-simplicial object in the Burnside category $\Delta^{op}_{inj*} \to \mathscr{B}$, using the functor $F^q : \underline{2}^n \to \mathscr{B}$ that we defined above. In the Steenrod square construction, we will be interested in the image of this functor in $\mathscr{B}$.

A general augmented semi-simplicial object in a category $\mathcal{C}$ is a functor $X : \Delta^{op}_{inj*} \to \mathcal{C}$, where the 'augmented' part of the name comes from the fact that the domain category is the augmented semi-simplicial category. Then, an augmented semi-simplicial object in the Burnside category is any functor $X : \Delta^{op}_{inj*} \to \mathscr{B}$. Morán defines the particular augmented semi-simplicial object that we will use in the Steenrod operation construction

via a Kan extension of $F^q : \underline{2}^n \to \mathscr{B}$ with respect to a functor from $\underline{2}^n \to \Delta^{op}_{inj*}$. The interested reader can find more details about this Kan extension in Section 2.8 of Morán's paper [Mor20]. Below, we define this particular augmented semi-simplicial object combining Sections 2.8 and 2.10 of Morán's paper, which matches up ordinals in Δ^{op}_{inj*} with sets of enhanced states, or labelings of link smoothings, in the Burnside category.

Definition 4.1. The augmented semi-simplicial object in the Burnside category

$$X : \Delta^{op}_{inj*} \to \mathscr{B}$$

that we will use to construct the Steenrod operations consists of the following data:

1. For each $r \geq -1$, we have the set

$$X_r \coloneqq X(r) = \bigsqcup_{|A|=r+1} F^q(A),$$

where A is an element in the cube category $\underline{2}^n$ and where F^q is defined as in Section 3.2.2.1.

2. For each $r \geq -1$ and each $U \in \mathcal{P}_q(r)$, we have a correspondence $X_r \leftarrow \delta^r_U \to X_{r-q}$. We define δ^r_U as follows:

$$\delta^r_U \coloneqq X(\bar{\delta}^r_U) = \bigsqcup_{|A|=r+1} F^q(d_{U'} : A' \to A)^*,$$

where the height of vertex A' is given by $|A'| = r + 1 - |U|$. Also, for $U = (u_1, \ldots, u_m)$, A' has the same coordinates as A except that in A' the u_i-th 1-coordinates of A for $i = 1, \ldots, m$ are replaced by 0s. Here, we have $U' \in \{0,1\}^{n-|U|} \cup \{*\}^{|U|}$, where n is the number of crossings in the given link. The sequence U' has 0s and 1s in the coordinates where A' and A match and has $*$ in the coordinates in which A' has a 0 but A has a 1. Also, note that we have defined δ^r_U to be disjoint union of the duals of the morphisms $F^q(d_{U'})$.

We represent δ^r_U by pairs (a, b) where b is in the domain of $F^q(d_{U'})$, and a is in the image of $F^q(d_{U'})(b)$. Then, we define the source map of this correspondence

to send (a, b) to a, and the target map sends (a, b) to b.

3. For each $r \geq -1$, each $U \in \mathcal{P}_q(r)$, and each partition $V_1 \cup V_2 = U$ with $V_1 \in \mathcal{P}_{q_1}(r)$, we have a fiberwise bijection

$$\mu^r_{V_1,V_2} : \delta^r_U \to \delta^r_{V_1} \times_{X_{r-q_1}} \delta^{r-q_1}_{\psi_{V_1}(V_2)}.$$

The correspondences and fiberwise bijections are subject to the following conditions.

4. For each $r \geq -1$, $\delta^r_\emptyset$ is the identity correspondence on X_r.

5. For each $r \geq -1$, and each $V \in \mathcal{P}_q(r)$, $\mu^r_{\emptyset,V}$ and $\mu^r_{V,\emptyset}$ are the identity bijections.

6. For each $r \geq -1$, and each $U \in \mathcal{P}_q(r)$, consider the partition $U = V_1 \cup V_2 \cup V_3$ where we have $V_i \in \mathcal{P}_{q_i}(r)$. For any such partition, the following square commutes:

$$\begin{array}{ccc} \delta^r_U & \xrightarrow{\mu^r_{V_1,V_{23}}} & \delta^{r_1}_{W_{23}} \circ \delta^r_{V_1} \\ \downarrow{\scriptstyle \mu^r_{V_{12},V_3}} & & \downarrow{\scriptstyle Id \times \mu^{r_1}_{W_2,W_3}} \\ \delta^{r_2}_{W_3} \circ \delta^r_{V_{12}} & \xrightarrow{\mu^r_{V_1,V_2} \times Id} & \delta^{r_2}_{W_3} \circ \delta^{r_1}_{W_2} \circ \delta^r_{V_1} \end{array}$$

Here, we use the notation:

$$r_j = r - \sum_{i=1}^{j} q_i, W_2 = \psi_{V_1}(V_2), \qquad W_3 = \psi_{V_1 \cup V_2}(V_3),$$
$$V_{12} = V_1 \cup V_2, V_{23} = V_2 \cup V_3, \qquad W_{23} = \psi_{V_1}(V_2 \cup V_3).$$

We denote the diagonal bijection in the diagram above by

$$\mu^r_{V_1,V_2,V_3} : \partial^r_U \longrightarrow \delta^{r_2}_{W_3} \circ \delta^{r_1}_{W_2} \circ \delta^r_{V_1}.$$

Remark 4.2. We note that for a given r, X_r is the collection of enhanced states corresponding to vertices at height $(r+1)$, which corresponds to cohomological degree $(r - n_- + 1)$. This also ends up matching the ordinal $[r]$, which has $(r+1)$ maps coming out from it in Δ^{op}_{inj*}, with the set of enhanced states corresponding to the vertices of the n-cube that are the target of $(r+1)$ maps. Note that because we will define the

Steenrod operations via a dual map, this is why we are looking for face maps that are the target of $(r+1)$ maps rather than the domain of that many maps.

We will also consider the following morphisms. First, for $V_1 \subset U$ with $V_2 = U \setminus V_1$, we define

$$\lambda^r_{V_1} : \delta^r_U \xrightarrow{\mu^r_{V_1,V_2}} \delta^{r_1}_{W_2} \circ \delta^r_{V_1} \xrightarrow{proj} \delta^r_{V_1}.$$

Second, for disjoint $V_1, V_2 \subset U$ with $V_3 = U \setminus (V_1 \cup V_2)$, we define

$$\lambda^r_{V_1,V_2} : \delta^r_U \xrightarrow{\mu^r_{V_1,V_2,V_3}} \delta^{r_2}_{W_3} \circ \delta^{r_1}_{W_2} \circ \delta^r_{V_1} \xrightarrow{proj} \delta^{r_1}_{W_2} = \delta^{r_1}_{\psi_{V_1}(V_2)},$$

where r_1, r_2, W_2, W_3 are defined as above.

An *ordered* augmented semi-simplicial object has all of the same data as an (unordered) augmented semi-simplicial object with the addition that the elements in its objects X_r and the elements in its correspondences δ^r_U have fixed orders. We will need fixed orderings in the Steenrod operation construction to compare elements. Thus, for the remainder of this paper, when we refer to an augmented semi-simplicial object, we will mean an ordered one.

For an example of this object for a trefoil, see Section 5.1.2, and for an example of this object for the disjoint union of two trefoils, see Section 5.2.2.

4.2 The Map ∇_{r-i}

Suppose we would like to construct $\mathfrak{sq}^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2)$ for some oriented link L. In order to define $\mathfrak{sq}^i : H^{p,q} \to H^{p+i,q}$ on an element $\alpha \in H^{p,q}$, which has semi-simplicial degree r, recall that we have

$$\mathfrak{sq}^i([\alpha]) = [\alpha \smile_{r-i} \alpha].$$

Here, r is given by

$$\begin{aligned} r &= (\text{height}) - 1 \\ &= (p + n_-) - 1. \end{aligned}$$

Recall that "height" is the number of 1-smoothings used in the vertices corresponding to this cohomological degree, and n_- is the number of negative crossings in the link. The n_- term comes from undoing the degree shift used in determining cohomological degree.

The map $\smile_{r-i}$: $C^* \otimes C^* \to C^*$ is defined on $C^* = C^*(X_\bullet; \mathbb{Z}/2)$, where $X_\bullet$ is the augmented semi-simplicial object in $\mathscr{B}$ that we defined above. That is, the degrees of these cochain groups are semi-simplicial degrees. We define $\smile_{r-i}$ using the map dual to it, which we call $\nabla_{r-i} : C_* \to C_* \otimes C_*$, where we have $C_* = C_*(X_\bullet; \mathbb{Z}/2)$. The map ∇_{r-i} defines the Steenrod squaring operations on homology. Then, because homology groups and cohomology groups are dual, we can construct the squaring operations via $\smile_{r-i}$ on cohomology as formally dual to the squaring operations on homology.

In this section, we will define this ∇_{r-i} map and use that to construct $\mathfrak{sq}^i$ on Khovanov cohomology. Then in the next section, we describe two Steenrod squares examples. In section 5.1, we present the construction of

$$\mathfrak{sq}^1 : H^{-3,-7}(T; \mathbb{Z}/2) \to H^{-2,-7}(T; \mathbb{Z}/2),$$

where T is the left-handed trefoil knot. Then, in Section 5.2, we present the construction of and

$$\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T; \mathbb{Z}/2) \to H^{-4,-14}(T \sqcup T; \mathbb{Z}/2),$$

where $T \sqcup T$ is the disjoint union of two left-handed trefoil knots.

4.2.1 ∇_{r-i}

The map $\nabla_{r-i} : C_* \to C_* \otimes C_*$ will be defined as follows

$$\nabla_{r-i} = \mathcal{A}_{\mathbb{Z}/2} \left(\sum_k \nabla^{\binom{k}{k-(r-i)}} \circ \Delta_k \right).$$

We will describe the correspondence Δ_k in Section 4.2.2. Then, in Section 4.2.3, we will define $\nabla^{\binom{k}{k-(r-i)}} \circ \Delta_k$, which is a correspondence from X_k to $X_* \times X_*$. In Section 4.2.4, we will see that the functor $\mathcal{A}_{\mathbb{Z}/2}$ turns this sum of correspondences into the homomorphism ∇_{r-i} from C_* to $C_* \otimes C_*$.

Because we want to use ∇_{r-i} to define $\mathfrak{sq}^i$, whose domain is C^r and whose range is C^{r+i}, we are interested only in the portion of ∇_{r-i} whose domain lies in C_{r+i}. Thus, we will restrict the domain of ∇_{r-i} to simplify the construction. That is, we only consider

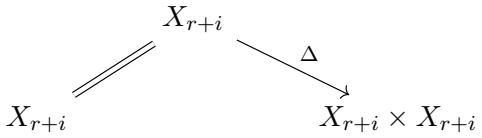
$$\begin{aligned}\nabla_{r-i}|_{C_{r+i}} &= \mathcal{A}_{\mathbb{Z}/2}\left(\nabla^{\binom{r+i}{r+i-(r-i)}} \circ \Delta_{r+i}\right) \\ &= \mathcal{A}_{\mathbb{Z}/2}\left(\nabla^{\binom{r+i}{2i}} \circ \Delta_{r+i}\right)\end{aligned}$$

In the following sections, we describe the morphisms involved in this definition.

4.2.2 Δ_{r+i}

Recall that ∇_{r-i} is a map $C_* \to C_* \otimes C_*$. However, $\nabla^{\binom{r+i}{2i}}$ will be a correspondence from $X_* \times X_*$ to $X_* \times X_*$. Thus, we use the correspondence Δ_{r+i} to "duplicate" the input before applying $\nabla^{\binom{r+i}{2i}}$.

We define the *diagonal correspondence* $\Delta_{r+i} : X_{r+i} \to X_{r+i} \times X_{r+i}$ by:



where the source map is the identity, and the target map is the standard diagonal map Δ that sends $a \in X_{r+i}$ to $(a, a) \in X_{r+i} \times X_{r+i}$.

4.2.3 $\nabla^{\binom{r+i}{2i}}$

In this section, we begin by defining the correspondence

$$\nabla^{\binom{r+i}{2i}} : X_{r+i} \times X_{r+i} \longrightarrow \bigcup_{a+b=2i} X_{(r+i)-a} \times X_{(r+i)-b} \tag{4.1}$$

symbolically as follows

$$\nabla^{\binom{r+i}{2i}} = \sum_{U \in \mathcal{P}_{2i}(r+i)} \bar{\delta}^{r+i}_{U^-} \wedge \bar{\delta}^{r+i}_{U^+}. \tag{4.2}$$

Recall that the individual morphisms $\bar{\delta}^{r+i}_{U^-}$ and $\bar{\delta}^{r+i}_{U^+}$ live in $\mathscr{B}$ and are the images of $X(\tilde{\delta}^{r+i}_{U^-})$ and $X(\tilde{\delta}^{r+i}_{U^+})$, respectively, for the appropriate F^q. However, we still need to define what their wedge product is.

Below, we describe each of the pieces necessary to define $\nabla^{\binom{r+i}{2i}}$, after which, we will run through an example to see how all of these components fit together.

4.2.3.1 The Set $\mathcal{P}_{2i}(r+i)$ and the Subsets U^- and U^+

In this section we describe the set $\mathcal{P}_{2i}(r+i)$ of sequences U. Then, we will explain how we partition U into its odd-indexed elements U^- and its even-indexed elements U^+.

As we saw in Equation 4.2, the index on the summation in $\nabla^{\binom{r+i}{2i}}$ runs over $U \in \mathcal{P}_{2i}(r+i)$. The set $\mathcal{P}_{2i}(r+i)$ is the collection of non-decreasing sequences of length $2i$ from the integers $\{0, 1, \ldots, r+i\}$, where each integer can appear at most twice. The augmented semi-simplicial degree $r+i$ corresponds to the cohomological degree h, where we have

$$r + i = h + n_- - 1,$$

which we can rearrange to get

$$h = r + i - n_- + 1.$$

The cochain groups are only potentially nonzero for:

$$-n_- \leq h \leq n - n_-$$

Using the relation between h and $r+i$ above, we can rewrite this inequality to see that the augmented semi-simplicial object is only potentially nonzero for:

$$-1 \leq r + i \leq n - 1$$

In particular, $r+i$ will be at most one less than the number of crossings in our original link, and so the set $\mathcal{P}_{2i}(r+i)$ will contain sequences of at most n numbers in the case where we have $r + i = n - 1$. Thus, we can view the integers in a sequence $U \in \mathcal{P}_{2i}(r+i)$ as corresponding to coordinates of the vertices of the n-cube diagram or to crossings in our link diagram.

The other extreme case where the augmented semi-simplicial degree is $r + i = -1$ corresponds to the cohomological degree h being $-n_-$.Recall that for our purposes, we

would be considering $\mathcal{P}_{2i}(-1)$ if C_{-n_-} were the domain of ∇_{r-i}, meaning that C^{-n_-} is the range of $\mathfrak{sq}^i$. Since C^{-n_-} has the smallest cohomological degree for which the Khovanov homology group is potentially nonzero, the only potentially nonzero Steenrod square with this as its range would be $\mathfrak{sq}^0$, that is, with $i = 0$. Thus, we are interested here in $\mathcal{P}_0(-1)$, which is the collection of non-decreasing sequences of length 0 of the empty set. That is, $\mathcal{P}_0(-1)$ contains the integers that are greater than or equal to 0 and also less than or equal to -1. In other words, we have $\mathcal{P}_0(-1) = \{\emptyset\}$. This is the set containing a single element, namely the empty set.

Example 4.3. Let $i = 2$ and $r = 0$, then we have $\mathcal{P}_{2i}(r + i) = \mathcal{P}_4(2)$, which contains non-decreasing sequences of length 4 of the integers $\{0, 1, 2\}$, where each integer can appear at most twice. The elements of $\mathcal{P}_4(2)$ are the following sequences:

$$(0, 0, 1, 1)$$
$$(0, 0, 1, 2)$$
$$(0, 0, 2, 2)$$
$$(0, 1, 1, 2)$$
$$(0, 1, 2, 2)$$
$$(1, 1, 2, 2).$$

Now, for a given sequence $U = (u_1, u_2, \ldots, u_{2i}) \in \mathcal{P}_{2i}(r + i)$, we will define it subset of odd-indexed elements, which we call U^-, and its subset of even-indexed elements, which we call U^+. To define the index of the elements in U, we first define the index of the elements in $\bar{U}$, which is the strictly increasing sequence containing one copy of each element of U. Let us write the sequence $\bar{U}$ as $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)$. The index of $\bar{u}_i$ for $i \in [1, m]$ is given by

$$ind_{\bar{U}}(\bar{u}_i) = \bar{u}_i + i.$$

Then, for any non-repeated element in U, its index in U is the same as its index in $\bar{U}$. For any element that appears twice in U, its index in U is the set $\{v, v + 1\}$, where v that element's index in $\bar{U}$. That is, the index of the elements that appear once in U will be a single number, while the elements that appear twice in U will be a set

containing two elements. This corresponds to giving the two copies of those elements a single index.

Then, we say that an element u_j has even index in U if $ind_U(u_j)$ is either an even number or is a set that contains an even number. Similarly, we say that an element u_j has odd index in U if $ind_U(u_j)$ is either an odd number or is a set that contains an odd number. Notice that for elements that appear twice in U, their index will always be a set containing exactly one even number and one odd number. Thus, one copy of that element will appear in each of U^- and U^+.

To help make this concept of index more concrete, consider the following example.

Example 4.4. Let $U = (0, 1, 1, 4, 4, 5, 6)$. Then, the corresponding sub-sequence $\bar{U}$ will be $(0, 1, 4, 5, 6)$, which contains one copy of each element of U in the same relative order. The index of each element of $\bar{U}$, which we define to be that sum of that element and its order in $\bar{U}$ is as follows:

$$
\begin{aligned}
ind_{\bar{U}}(0) &= 0 + 1 = 1 \\
ind_{\bar{U}}(1) &= 1 + 2 = 3 \\
ind_{\bar{U}}(4) &= 4 + 3 = 7 \\
ind_{\bar{U}}(5) &= 5 + 4 = 9 \\
ind_{\bar{U}}(6) &= 6 + 5 = 11
\end{aligned}
$$

Using this, we say that the index of each element with respect in U is then:

$$
\begin{aligned}
ind_U(0) &= ind_{\bar{U}}(0) = 1 \\
ind_U(1) &= \{ind_{\bar{U}}(1), ind_{\bar{U}}(1) + 1\} = \{3, 4\} \\
ind_U(4) &= \{ind_{\bar{U}}(4), ind_{\bar{U}}(4) + 1\} = \{7, 8\} \\
ind_U(5) &= ind_{\bar{U}}(5) = 9 \\
ind_U(6) &= ind_{\bar{U}}(6) = 11
\end{aligned}
$$

Then, we define the collection of odd-indexed elements U^- and the collection of even

indexed elements U^+ as follows:

$$U^- = \{0, 1, 4, 5, 6\},$$
$$U^+ = \{1, 4\}.$$

These subsets U^- and U^+ will be used as the subscripts on face maps in our Steenrod square construction. These subscripts tell us which crossings in our link diagram to leave resolved as they are and which to swap from 0-smoothings to 1-smoothings. This process of splitting up the crossings of a link with each crossing appearing at most twice has a similar flavor to how McClure and Smith define the cup-i product by splitting up a chain via overlapping partitions in their paper "Multivariable Cochain Operations and Little n-Cubes" [MS03].

Now, recall that we will use $\nabla_{r-i} : C_* \to C_* \otimes C_*$ to define $\mathfrak{sq}^i : C^r \to C^{r+i}$. Thus, we will restrict our view of $\nabla^{\binom{r+i}{2i}} : X_{r+i} \times X_{r+i} \to \bigcup_{a+b=2i} X_{(r+i)-a} \times X_{(r+i)-b}$, the correspondence in Equation 4.1, to the case where a and b are both equal to i. Thus, the degree of the range of $\nabla^{\binom{r+i}{2i}}$ will match the degree of the domain of $\mathfrak{sq}^i$. This corresponds to requiring that U^- and U^+ are both sets containing i elements. Note that this is always possible because we have

$$\begin{aligned} |U| &= |U^- \sqcup U^+| \\ &= |U^-| + |U^+| \end{aligned}$$

since U^- and U^+ partition U, and because we defined U to have cardinality $2i$.

4.2.3.2 (s,t)-Good Pairs, (s,t)-Positive Pairs, and Maximal Chains

In this section, we will define when a pair (W'', W°) of subsets of $U^- \cap U^+$ is (s,t)-good for some $(s,t) \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}$. Then, we will describe what makes the subsets (s,t)-positive. Finally, we will describe how to arrange (s,t)-positive pairs into maximal chains. To simplify notation in this and the next few sections, we will let $U^\pm \coloneqq U^- \cap U^+$, that is, the collection of elements that are repeated in U.

To begin, we present the definition of (s,t)-goodness.

Definition 4.5. Consider the pair $(s,t) \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}$. Let (W'', W°) be a pair of

disjoint subsets of $U^{\pm}$. We say that (W'', W°) is an (s,t)-good pair if both of the following conditions are satisfied:

- The images of s and t are equal under the following map

$$\lambda^{r+i}_{W''} : \bar{\delta}^{r+i}_{U^{\pm}} \xrightarrow{\sim} \bar{\delta}^{r+i-|W''|}_{\psi_{W''}(U^{\pm} \setminus W'')} \circ \bar{\delta}^{r+i}_{W''} \xrightarrow{proj} \bar{\delta}^{r+i}_{W''}.$$

  Above, the first arrow is a bijection, and the second arrow is a projection.

- The images of s and t are not equal under the map

$$\lambda^{r+i}_{W'',W^{\circ}} : \bar{\delta}^{r+i}_{U^{\pm}} \xrightarrow{\sim} \bar{\delta}^{r+i-|W'' \cup W^{\circ}|}_{\psi_{W'' \cup W^{\circ}}(U^{\pm} \setminus (W'' \cup W^{\circ}))} \circ \bar{\delta}^{r+i-|W''|}_{\psi_{W''}(W^{\circ})} \circ \bar{\delta}^{r+i}_{W''} \xrightarrow{proj} \bar{\delta}^{r+i-|W''|}_{\psi_{W''}(W^{\circ})}.$$

  Again, the first arrow is a bijection and the second arrow is a projection.

For this first condition, we recall from Section 3.1 that the map $\psi_U(V)$ updates the numbers in V after removing the numbers in U from the universe of numbers we are considering. The map $\lambda^{r+i}_{W''}$ projects s and t onto the first half of their domains. Since the correspondences here go in the opposite direction of the correspondences associated to the n-cube, then s and t are being projected onto the last few vertices that they pass through in the n-cube. For this second condition, the $\lambda^{r+i}_{W'',W^{\circ}}$ map sends s and t to in the middle of their domains.

Notice that this definition is for an individual pair (s,t). It is possible for (W'', W°) to be (s,t)-good for some pair $(s,t) \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}$ but not (s',t')-good for some other pair $(s',t') \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}$.

We also note that pairs of the form $(W, \emptyset)$ are not (s,t)-good for any $(s,t) \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}$. To see this, we let $V = U^{\pm} \setminus W$. Then we define λ^{r+i}_{W} and $\lambda^{r+i}_{W,\emptyset}$ as follows

$$\lambda^{r+i}_{W} : \bar{\delta}^{r+i}_{U^{\pm}} \to \bar{\delta}^{r+i-|W|}_{\psi_W(V)} \circ \bar{\delta}^{r+i}_{W} \to \bar{\delta}^{r+i}_{W}$$
$$\lambda^{r}_{W,\emptyset} : \bar{\delta}^{r+i}_{U^{\pm}} \to \bar{\delta}^{r+i-|W|}_{\psi_W(V)} \circ \bar{\delta}^{r+i-|W|}_{\emptyset} \circ \bar{\delta}^{r+i}_{W} \to \bar{\delta}^{r+i-|W|}_{\emptyset}.$$

Note that the correspondence $\bar{\delta}^{r+i}_{W}$ maps from X_{r+i} to $X_{r+i-|W|}$, and the correspondence

$\bar{\delta}_{\emptyset}^{r+i-|W|}$ is the identity correspondence on $X_{r+i-|W|}$. Recall that for $(W, \emptyset)$ to be (s,t)-good for some pair (s,t), we need

$$\lambda_W^{r+i}(s) = \lambda_W^{r+i}(t) \in \bar{\delta}_W^{r+i},$$

and we also need

$$\lambda_{W,\emptyset}^{r+i}(s) \neq \lambda_{W,\emptyset}^{r+i}(t) \in \bar{\delta}_{\emptyset}^{(r+i)-|W|}.$$

The first (s,t)-goodness condition means that s and t must be projected onto the same element in $\bar{\delta}_W^{r+i}$, which is made up in part by the target of this element in $X_{r+i-|W|}$. However, the second (s,t)-goodness condition means that s and t must be projected onto different target by $\bar{\delta}_W^{r+i}$. These elements contain the target of s and t in $X_{r+i-|W|}$, and to meet this condition, these targets must be different. Therefore, it is not possible for a pair (s,t) to meet both of these conditions necessary for $(W, \emptyset)$ to be (s,t)-good.

In the next definition, we describe what makes an (s,t)-good pair (W'', W°) into an (s,t)-positive pair, where the rough idea of (s,t)-positivity is to ensure that we do not double count pairs (s,t).

Definition 4.6. Let a pair (W'', W°) of disjoint subsets of $U^\pm$ be an (s,t)-good pair for some $(s,t) \in \bar{\delta}_{U^-}^{r+i} \times \bar{\delta}_{U^+}^{r+i}$. We say that the (s,t)-good pair (W'', W°) is (s,t)-*positive* if it satisfies one of the following pairs of requirements:

- We have that $(r+i)+|U^-|+|U^+| = r+3i$ is even, and that the following inequality holds:
$$\lambda_{W'',W^\circ}^{r+i}(s) < \lambda_{W'',W^\circ}^{r+i}(t).$$
- We have that $(r+i)+|U^-|+|U^+| = r+3i$ is odd, and that the following inequality holds:
$$\lambda_{W'',W^\circ}^{r+i}(s) > \lambda_{W'',W^\circ}^{r+i}(t).$$

Notice that in the definition of (s,t)-positive, we assume that the pair of subsets (W'', W°) is (s,t)-good.

Finally, we define the maximal chains of (s,t)-positive pairs. First, we say that two pairs of sets (W_0'', W_0°) and (W_1'', W_1°) are *consecutive* if W_1° is obtained from W_0° by adding a single, new element and W_1'' is the result of removing that element from W_0'',

if it exists there. That is, if that element we added to W_1° is not in W_0'', then we require that $W_1'' = W_0''$. We denote that two pairs are consecutive with the notation

$$(W_0'', W_0^\circ) \prec (W_1'', W_1^\circ).$$

Now, we present the following lemma about consecutive pairs.

Lemma 4.7. [Mor20] Consider $U \in \mathcal{P}_{2i}(r+i)$, consider $(s,t) \in \bar{\delta}_{U^-}^{r+i} \times \bar{\delta}_{U^+}^{r+i}$, and consider (W'', W°) to be an (s,t)-good pair. Suppose that $0 \leq w \leq n$ with $w \notin W^\circ$.

1. The pair $(W'', W^\circ \cup \{w\})$ is (s,t)-good if and only if $w \notin W''$, and
2. the pair $(W'' \setminus \{w\}, W^\circ \cup \{w\})$ is (s,t)-good if and only if $w \in W''$.

Given the definition of consecutive pairs and this property that for consecutive pairs, either both are (s,t)-good or neither is, we now define a maximal chain of (s,t)-good pairs

Definition 4.8. A *maximal chain* of (s,t)-good pairs is a collection of consecutive (s,t)-good pairs such that no other pair can be added into the chain. A maximal chain has the form:

$$(W_1'', W_1^\circ) \prec (W_2'', W_2^\circ) \prec \cdots \prec (W_k'', W_k^\circ), \quad k = |U \cap V|,$$

with $(W_k'', W_k^\circ) = (\emptyset, U \cap V)$.

Definition 4.9. A maximal chain of (s,t)-good pairs is *positive* if all of the pairs in that chain are (s,t)-positive.

4.2.3.3 The Correspondence $\bar{\delta}_{U^-}^{r+i} \wedge \bar{\delta}_{U^+}^{r+i}$

Putting this all together, we define the set $\mathcal{O}_{s,t}(U^\pm)^+$, which contains all the positive maximal chains for the pair $(s,t) \in \bar{\delta}_{U^-}^{r+i} \times \bar{\delta}_{U^+}^{r+i}$. Note that if $U^\pm$ is empty, then we define $\mathcal{O}_{s,t}(U^\pm)^+$ to be $\bar{\delta}_{U^-}^{r+i} \times \bar{\delta}_{U^+}^{r+i}$, where we say that each pair (s,t) has a single, empty positive maximal chain.

Now, suppose we have a set $U \in \mathcal{P}_{2i}(r+i)$. Recall that U^- contains the elements of U with odd index, U^+ contains the elements of U with even index, and we have

$|U^-| = |U^+| = i$. We define the correspondence $\bar{\delta}^{r+i}_{U^-} \wedge \bar{\delta}^{r+i}_{U^+}$ to be

$$X_{r+i} \times X_{r+i} \longleftarrow \sum_{(s,t) \in \bar{\delta}^{r+i}_{U^-} \times \bar{\delta}^{r+i}_{U^+}} \mathcal{O}_{s,t}(U^- \cap U^+)^+ \longrightarrow X_{(r+i)-|U^-|} \times X_{(r+i)-|U^+|} = X_r \times X_r,$$

where the value of the source map and target map for $\mathbb{W} \in \mathcal{O}_{s,t}(U^- \cap U^+)^+$ are given by

$$\text{source}(\mathbb{W}) = (\text{source}(s), \text{source}(t)), \quad \text{target}(\mathbb{W}) = (\text{target}(s), \text{target}(t)).$$

4.2.4 $\mathcal{A}_{\mathbb{Z}/2}$

The final component of the map ∇_{r-i} is the $(\mathbb{Z}/2)$-*linearization functor* $\mathcal{A}_{\mathbb{Z}/2} : \mathscr{B} \to (\mathbb{Z}/2)$-Mod. This linearization functor turns the correspondence

$$X_{r+i} \leftarrow \nabla^{\binom{r+i}{2i}} \circ \Delta^* \to X_r \times X_r$$

into the homomorphism

$$\nabla_{r-i} : C_{r+i} \to C_r \otimes C_r.$$

We define the functor $\mathcal{A}_{\mathbb{Z}/2}$ on objects of X_{r+i} to take a set in $\mathscr{B}$ to the $(\mathbb{Z}/2)$-module generated by that set, that is,

$$X \longmapsto (\mathbb{Z}/2)\langle X \rangle.$$

On morphisms, this functor sends a correspondence $X \xleftarrow{s} Q \xrightarrow{t} Y$ to the homomorphism $(\mathbb{Z}/2)\langle X \rangle \to (\mathbb{Z}/2)\langle Y \rangle$, which sends $x \in X$ to

$$\sum_{y \in Y} |s^{-1}(x) \cap t^{-1}(y)|\, y.$$

That is, this homomorphism sends x to $\sum c_y y$, where c_y counts the number of ways to get from x to y via the correspondence Q.

4.3 Turning ∇_{r-i} into $\mathfrak{sq}^i$

Recall that our goal in constructing $\nabla_{r-i}|_{C_{r+i}} : C_{r+i} \to C_r \otimes C_r$ is to use it to define $\mathfrak{sq}^i : H^{p,q} \to H^{p+i,q}$, where we have $p = r - n_- + 1$. Thus, in this section, we explain

the process of defining $\mathfrak{sq}^i$ given the construction of ∇_{r-i}.

Our first step is to take the dual of ∇_{r-i} to get $\smile_{r-i}\colon C^r \otimes C^r \to C^{r+i}$. To do this, if we let f_l be a basis for C_{r+i} and e_j be a basis for C_r, then we have

$$\nabla_{r-i}(f_l) = \sum_{j,k} a_l^{j,k} e_j \otimes e_k,$$

where $a_l^{j,k}$ is either 1 or 0 depending on whether $e_j \otimes e_k$ is a term in $\nabla_{r-i}(f_l)$ or not. Then, when we take the dual of this map, we get

$$e^j \smile_{r-i} e^k = \sum_l a_l^{j,k} f^l,$$

where we have $e^j, e^k \in C^r$ and $f^l \in C^{r+i}$, which are dual to e_j, e_k, and f_l respectively, and the coefficient $a_l^{j,k}$ is the same coefficient from the definition for $\nabla_{r-i}(f_l)$.

For the second step in determining a formula $\mathfrak{sq}^i$, we note that C^r the cochain group in semi-simplicial degree r is equal to the cochain group is cohomological degree p and quantum degree q, $C^{p,q}$. Thus, it remains to pass the output of $\smile_{r-i}$ to the quotient. That is, for $[\alpha^*] \in H^{p,q}$, we set

$$\mathfrak{sq}^i([\alpha^*]) = [\alpha^* \smile_{r-i} \alpha^*].$$

Again, in terms of basis vectors $e^j \in C^r$, suppose that we have

$$\alpha^* = \sum_j b_j e^j,$$

where b_j is either 1 or 0 depending on whether the value of α^* on e^j is 1 or 0. Then we define $\mathfrak{sq}^i([\alpha^*])$ as follows

$$\begin{aligned} \mathfrak{sq}^i([\alpha^*]) &= [\alpha^* \smile_{r-i} \alpha^*] \\ &= \left[\sum_j b_j e^j \smile_{r-i} \sum_j b_j e^j \right] \end{aligned}$$

$$= \left[\sum_{j,k} b_j b_k (e^j \smile_{r-i} e^k) \right]$$
$$= \left[\sum_{j,k,l} b_j b_k a_l^{j,k} f^l \right].$$

Again here, $a_l^{j,k}$ counts the number of times, modulo 2, that $e_j \otimes e_k$ appears in the image of each basis element $f_l \in C_{r+i}$ under ∇_{r-i}. Then for this computation of $\mathfrak{sq}^i([\alpha^*])$, we are only interested in the terms $e_j \otimes e_k$ where e^j and e^k are both in α^*.

Chapter 5

Examples

In this section, we go through two examples of computing $\mathfrak{sq}^i$ in detail. First, we construct $\mathfrak{sq}^1$ for the left-handed trefoil. Then, we construct $\mathfrak{sq}^2$ for the disjoint union of two left-handed trefoils. This latter example in particular is quite long, but both examples should give the interested reader a clear walk-through of these Steenrod square computations.

5.1 $\mathfrak{sq}^1 : H^{-3,-7}(T;\mathbb{Z}/2) \to H^{-2,-7}(T;\mathbb{Z}/2)$

First, we will go through the computation of

$$\mathfrak{sq}^1 : H^{-3,-7}(T;\mathbb{Z}/2) \to H^{-2,-7}(T;\mathbb{Z}/2)$$

for T the left-handed trefoil. We note that the cohomological degree $p = -3$ corresponds to semi-simplicial degree

$$\begin{aligned} r &= p + n_- - 1 \\ &= (-3) + 3 - 1 \\ &= -1, \end{aligned}$$

where $n_- = 3$ is the number of negative crossings in our link. Thus, here X_{-1} is the set of generators of $C^{-3,-7}(T;\mathbb{Z}/2)$. This is the set in $\mathscr{B}$ containing all labelings by v_+ and

v_- in quantum degree -7 of vertices at height 0, which corresponds to cohomological degree -3. Similarly, cohomological degree $p = -2$ corresponds to semi-simplicial degree 0. Thus, X_0 is the set of generators of $C^{-2,-7}$. In particular, X_0 contains all labelings in quantum degree -7 of vertices at height $p + n_- = 1$, where here we have $p = -2$.

Recall that when we refer to X_i, it is for a specific quantum grading. However, we do not include the quantum grading in the notation. Also, we denote the $(\mathbb{Z}/2)$-linearization of X_i by $C_i \coloneqq C_i(X_\bullet; \mathbb{Z}/2) = \mathbb{Z}/2\langle X_i \rangle$. Notice that C_{p+n_--1} is dual to $C^{p,q}$.

Finally, to simplify notation, we will write vertices of the 3-cube as strings of their coordinates. For example, we write vertex $(0, 1, 0)$ as vertex 010.

5.1.1 Setting up the link

In the top left corner of Figure 5.1, we present a link diagram for the left-handed trefoil. We make a choice of orientation as shown although this will not affect the Khovanov homology or the resulting Steenrod square computation.

We have also imposed an arbitrary but fixed order on the crossings of this link, as shown by the non-underlined number next to each crossing in the trefoil diagram in the top left corner of Figure 5.1. This ordering corresponds to the coordinates of the vertices of the 3-cube diagram for this link. For example, vertex 100 corresponds to the complete smoothing of this link where we resolve crossing 0 via a 1-smoothing, and where we resolve the remaining crossings via 0-smoothings.

Next, given a complete smoothing of this link, we describe our process for ordering its cycles. To do this, we start by numbering each strand of the link, where a strand is a piece of the link between two crossings, regardless of whether that strand goes over in the cross or goes under. Our labelings of the strands are the underlined numbers shown in Figure 5.1. Once we have fixed a labeling of the strands, we number a cycle by the minimum strand labeling involved in that cycle.

For example, in the complete smoothing corresponding to the vertex 100 shown in Figure 5.1, we see that the larger cycle contains strands 1, 2, 4, and 5. Thus, we label this cycle by the minimum number in this list, which here is 1, written inside the cycle. The smaller cycle contains strands 3 and 6, and so we label it as cycle 3, written inside the cycle. Then, if we refer to the enhanced state $(100, v_- \otimes v_+)$, this means that we

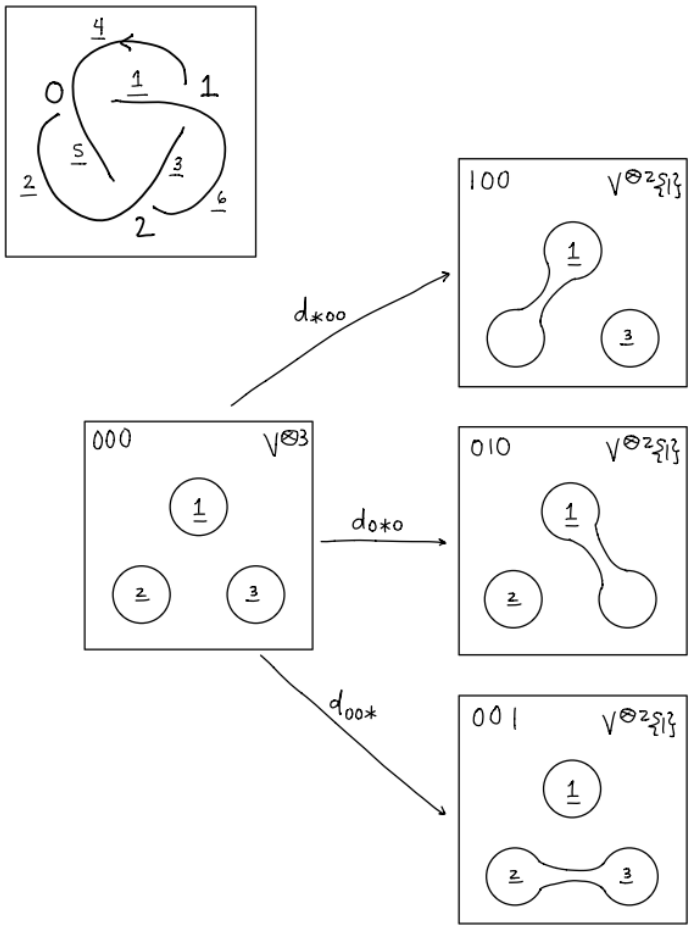


Figure 5.1: Oriented link Diagram of the Left-Handed Trefoil with Crossings and Strands Labeled with Part of its 3-Cube Diagram

have labeled cycle 1 by v_- and cycle 3 by v_+.

5.1.2 Defining the Cochains Groups, Augmented Semi-Simplicial Objects, and Correspondences

In this section, we describe the elements of X_{-1}, which are the enhanced states that generate $C^{-3,-7}$, as well as the elements of X_0, which are the enhanced states that generate $C^{-2,-7}$. To simplify notation throughout the computation, we will also give a

shorted label to each element of X_{-1} and X_0.

The enhanced states of X_{-1} correspond to vertices at height 0 since we have

$$\begin{aligned} height &= r + 1 \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

There is only one such vertex, namely the vertex 000. The complete smoothing of the trefoil corresponding to this vertex contains three cycles. For a labeling of this resolution to be in quantum degree -7, it must have one cycle labeled by v_+ and the remaining two cycles labeled by v_-. We can label the cycle 1 in this resolution as a, cycle 2 as b, and cycle 3 as c. Then, we can refer to one of these enhanced states simply by the letter of the cycle labeled by v_+. The elements of X_{-1} along with their associated one-letter label are as follows:

$$\begin{aligned} X_{-1} = \bigsqcup_{|A|=0} F^{-7}(A) \\ =& F^q(000) \\ =& \{a = (000, v_+ \otimes v_- \otimes v_-) \\ & b = (000, v_- \otimes v_+ \otimes v_-) \\ & c = (000, v_- \otimes v_- \otimes v_+)\} \end{aligned}$$

The enhanced states of X_0 correspond to vertices at height 1. There are three vertices at this height, and each corresponds to a smoothing with two cycles. In order for a labeling of one of these complete smoothings to be in quantum degree -7, both of its cycles must be labeled by v_-. Thus, each vertex gives us exactly one enhanced state. Our simplified notation here comes from ordering the vertices of the 3-cube with height 1 and referring to the enhanced state for each vertex as β_j, where j is the place of that vertex in the overall ordering. The elements of X_0 along with their associated label β_j are as follows:

$$X_0 = \bigsqcup_{|A|=1} F^{-7}(A)$$

$$
\begin{aligned}
&= F^{-7}(100) \sqcup F^{-7}(010) \sqcup F^{-7}(001) \\
&= \{\beta_1 = (100, v_- \otimes v_-) \\
&\quad \beta_2 = (010, v_- \otimes v_-) \\
&\quad \beta_3 = (001, v_- \otimes v_-)\}.
\end{aligned}
$$

Now that we have defined X_0 and X_{-1}, the two objects needed for this $\mathfrak{sq}^1$ construction, we define the correspondences δ^0_U between them. Here, the superscript refers to the degree of the domain group X_0. The subscript U is a sequence in $\mathcal{P}_1(0)$, where the 0 again refers to the degree of the domain and the 1 comes from the fact that we are decreasing semi-simplicial degree by 1 from X_0 to X_{-1}. There is only one sequence in $\mathcal{P}_1(0)$, namely the sequence (0), and so there is only one correspondence from X_0 to X_{-1}. We call this correspondence δ^0_0, where the subscript is simplified notation for the sequence (0). The subscript also refers to the fact that we are replacing the 0-th 1 coordinate with a 0 to go from a height 1 vertex to the height 0 vertex.

In order to define the correspondence δ^0_0, we first write down the images under F^{-7} of the edge maps of the 3-cube. In particular, the correspondences associated to the maps from from the height 0 vertex to the height 1 vertices are:

$$
\begin{aligned}
d_{*,0,0} &= \{(a, \beta_1), (b, \beta_1)\} \\
d_{0,*,0} &= \{(a, \beta_2), (c, \beta_2)\} \\
d_{0,0,*} &= \{(b, \beta_3), (c, \beta_3)\}.
\end{aligned}
$$

Note that each height 1 vertex has only one 1-coordinate, and so all of these correspondences above from A' to A will have the 0-th 1 coordinate in A being a 0 in A'. That is, we will include all of them when constructing δ^0_0. This gives us:

$$
\begin{aligned}
\delta^0_0 &= X(\bar{\delta}^0_0) \\
&= \bigsqcup_{|A|=1} F^{-7}(d_{U'} : (0,0,0) \to A)^* \\
&= F^{-7}(d_{*,0,0} : (0,0,0) \to (1,0,0))^* \sqcup F^{-7}(d_{0,*,0} : (0,0,0) \to (0,1,0))^* \\
&\quad \sqcup F^{-7}(d_{0,0,*} : (0,0,0) \to (0,0,1))^*
\end{aligned}
$$

$$=\{(\beta_1, a), (\beta_1, b), (\beta_2, a), (\beta_2, c), (\beta_3, b), (\beta_3, c)\}.$$

For an element $(x, y) \in \delta_0^0$ with $x \in X_0$ and $y \in X_{-1}$ the source and target maps are defined as follows:

$$s(x, y) = x$$
$$t(x, y) = y.$$

We fix the ordering of the elements of δ_0^0 as shown above, where the first element in the list is the minimal element, and the last element in the list is the maximal element.

5.1.3 Setting up the Steenrod Square Computation

To define $\mathfrak{sq}^1 : H^{-3,-7}(T; \mathbb{Z}/2) \to H^{-2,-7}(T; \mathbb{Z}/2)$, we use the map $\smile_{-2}: C^{-1} \otimes C^{-1} \to C^0$. We get the degree -2 from $((-1) - 1)$, where the -1 term is the semi-simplicial degree of the domain of $\mathfrak{sq}^1$ and the 1 is the degree of $\mathfrak{sq}^1$ itself. To define the map $\smile_{-2}$, we use its dual $\nabla_{-2} : C_* \otimes C_* \to C_*$, where we will have:

$$\begin{aligned}\nabla_{-2} &= \mathcal{A}_{\mathbb{Z}/2}\left(\sum_k \nabla^{\binom{k}{k-(-2)}} \circ \Delta_k\right) \\ &= \mathcal{A}_{\mathbb{Z}/2}\left(\sum_k \nabla^{\binom{k}{k+2}} \circ \Delta_k\right).\end{aligned}$$

Because we want to use this map ∇_{-2} to define $\smile_{-2}: C^{-1} \otimes C^{-1} \to C^0$, we restrict the domain of ∇_{-2} to C_0, which gives us:

$$\nabla_{-2}|_{C_0} = \mathcal{A}_{\mathbb{Z}/2}\left(\nabla^{\binom{0}{2}} \circ \Delta_0\right)$$

Recall that the diagonal correspondence $\Delta_0 : X_0 \to X_0 \times X_0$ is equal to X_0. Here, the source of an element $a \in \Delta_0$ is $a \in X_0$, and the target of a is $(a, a) \in X_0 \times X_0$.

Next, we want to define the correspondence

$$\nabla^{\binom{0}{2}} : X_0 \times X_0 \to \bigcup_{a+b=2} X_{0-a} \times X_{0-b}.$$

Here, the 2 in $a + b = 2$ in the subscript on the union comes from $2i = 2 \cdot 1$, where i is the index of the Steenrod square we are computing. The 0 in the subscript on X_0, X_{0-a}, and X_{0-b} is the semi-simplicial degree of the domain we restricted ∇_{-2} to. We define $\nabla^{\binom{0}{2}}$ as follows:

$$\nabla^{\binom{0}{2}} = \sum_{U \in \mathcal{P}_2(0)} \delta^0_{U^-} \wedge \delta^0_{U^+}.$$

In the line above, $\mathcal{P}_2(0)$ is the set of non-decreasing sequences of length 2 from the non-negative integers less than or equal to 0, namely just $\{0\}$, where each integer can be repeated at most twice. There is exactly one such sequence $U = (0, 0)$, which has:

$$U^- = U^+ = \{0\}.$$

Thus, the correspondence $\nabla^{\binom{0}{2}}$ can be simplified to

$$\nabla^{\binom{0}{2}} = \delta^0_0 \wedge \delta^0_0.$$

We define the correspondence $\delta^0_0 \wedge \delta^0_0$ by:

$$\delta^0_0 \wedge \delta^0_0 : \; X_0 \times X_0 \xleftarrow{source} \sum_{(s,t) \in \delta^0_0 \times \delta^0_0} \mathcal{O}_{s,t}(\{0\})^+ \xrightarrow{target} X_{-1} \times X_{-1}.$$

The subscripts on the term $X_{-1} \times X_{-1}$ come from:

$$0 - |U^-| = 0 - |\{0\}| = -1,$$
$$0 - |U^+| = 0 - |\{0\}| = -1.$$

Normally we would need to restrict $\nabla^{\binom{0}{2}}$ to only terms whose target map has image equal to $X_{-1} \times X_{-1}$ because we are using this map to define about the cup-i product $\smile_{-2}$ whose domain is $C^{-1} \otimes C^{-1}$. However, in this example, the only summand in $\nabla^{\binom{0}{2}}$ already meets this criterion.

5.1.3.1 Determining the Positive Maximal Chains

In this section, we will define $\mathcal{O}_{s,t}(\{0\})^+$ for all $(s,t) \in \delta_0^0 \times \delta_0^0$. The set $\mathcal{O}_{s,t}(\{0\})^+$ contains the maximal chains of (s,t)-good and (s,t)-positive pairs (W'', W°) of disjoint subsets of $\{0\}$. See Section 4.2.3.2 for the definitions of these terms.

The only pair of disjoint subsets of $\{0\}$ that could potentially be (s,t)-positive for any (s,t) is $(W'', W^\circ) = (\emptyset, \{0\})$ since $(\{0\}, \emptyset)$ cannot be (s,t)-good. Then, recall that for the pair $(\emptyset, \{0\})$ to be (s,t)-good for $(s,t) \in \delta_0^0 \times \delta_0^0$, it must satisfy the following conditions.

$$\lambda_\emptyset^0(s) = \lambda_\emptyset^0(t) \tag{5.1}$$

$$\lambda_{\emptyset,0}^0(s) \neq \lambda_{\emptyset,0}^0(t), \tag{5.2}$$

where we will define $\lambda_\emptyset^0$ and $\lambda_{\emptyset,0}^0$ shortly. Further, for $(\emptyset, \{0\})$ to be (s,t)-positive, since $r + |U^-| + |U^+| = 0 + 1 + 1 = 2$ is even, where r is the semi-simplicial degree of the restricted domain of ∇_{-2}, we require:

$$\lambda_{\emptyset,0}^0(s) < \lambda_{\emptyset,0}^0(t).$$

To determine if $(\emptyset, \{0\})$ is (s,t)-positive for any (s,t), we first consider $\lambda_\emptyset^0$:

$$\lambda_\emptyset^0 : \ \delta_0^0 \xrightarrow{\sim} \delta_0^0 \circ \delta_\emptyset^0 \xrightarrow{proj} \delta_\emptyset^0.$$

Because the correspondence $\delta_\emptyset^0$ is the identity on X_0, the map $\lambda_\emptyset^0$ takes each element of δ_0^0 to its image under the source map for this correspondence δ_0^0. Thus, the map $\lambda_\emptyset^0$ makes the following assignments:

$$(\beta_1, a) \mapsto \beta_1$$
$$(\beta_1, b) \mapsto \beta_1$$
$$(\beta_2, a) \mapsto \beta_2$$
$$(\beta_2, c) \mapsto \beta_2$$
$$(\beta_3, b) \mapsto \beta_3$$
$$(\beta_3, c) \mapsto \beta_3$$

The first condition of (s,t)-goodness requires that the image of s and t be equal under $\lambda^0_\emptyset$. The pair $(\emptyset,\{0\})$ satisfies this first condition for the following pairs:

$$\begin{aligned}
&\{((\beta_1,a),(\beta_1,b)), \quad ((\beta_1,a),(\beta_1,a)),\\
&((\beta_1,b),(\beta_1,a)), \quad ((\beta_1,b),(\beta_1,b)),\\
&((\beta_2,a),(\beta_2,c)), \quad ((\beta_2,a),(\beta_2,a)),\\
&((\beta_2,c),(\beta_2,a)), \quad ((\beta_2,c),(\beta_2,c)),\\
&((\beta_3,b),(\beta_3,c)), \quad ((\beta_3,b),(\beta_3,b)),\\
&((\beta_3,c),(\beta_3,b)), \quad ((\beta_3,c),(\beta_3,c))\}.
\end{aligned}$$

Notice that if $(\emptyset,\{0\})$ satisfies this criterion for the pair (s,t), then it also does for the pair (t,s). We also get all pairs of the form (s,s).

Next, we consider $\lambda^0_{\emptyset,0}$:

$$\lambda^0_{\emptyset,0} : \delta^0_0 \xrightarrow{\sim} \delta^0_\emptyset \circ \delta^0_0 \circ \delta^{-1}_\emptyset \xrightarrow{proj} \delta^0_0$$

This map $\lambda^0_{\emptyset,0}$ is the identity. Thus, the second condition of (s,t)-goodness, which says $\lambda^0_{\emptyset,0}(s) \neq \lambda^0_{\emptyset,0}(t)$, is equivalent to the condition $s \neq t$. Therefore, $(\emptyset,\{0\})$ satisfies the second (s,t)-goodness criterion for all of the pairs (s,t) with $s \neq t$. Combining this with the (s,t) pairs from the first condition, we get that $(\emptyset,\{0\})$ is (s,t)-good for the following pairs:

$$\begin{aligned}
&\{((\beta_1,a),(\beta_1,b)),\\
&((\beta_1,b),(\beta_1,a)),\\
&((\beta_2,a),(\beta_2,c)),\\
&((\beta_2,c),(\beta_2,a)),\\
&((\beta_3,b),(\beta_3,c)),\\
&((\beta_3,c),(\beta_3,b))\}.
\end{aligned}$$

We then restrict our view to (s,t)-positivity, where, we require $\lambda^0_{\emptyset,0}(s) < \lambda^0_{\emptyset,0}(t)$.

Again, since $\lambda^0_{\emptyset,0}$ is the identity map, this reduces to $s < t$. Thus, $(\emptyset, \{0\})$ is (s,t)-positive only for the following pairs:

$$\{((\beta_1, a), (\beta_1, b)),$$
$$((\beta_2, a), (\beta_2, c)),$$
$$((\beta_3, b), (\beta_3, c))\}$$

Now, we want to determine the maximal chains of (s,t)-positive pairs. Recall that the length of a maximal chain is $|U^- \cap U^+| = 1$ and the top element is $(\emptyset, U^- \cap U^+)$, which here is $(\emptyset, \{0\})$. Thus, the only maximal chain is the singleton $(\emptyset, \{0\})$, and this chain is positive for the three (s,t) pairs above. That is, $\mathcal{O}_{s,t}(\{0\})^+$ is only nontrivial for (s,t) in the list of three pairs above, and for each of these (s,t) pairs, the set $\mathcal{O}_{s,t}(\{0\})^+$ is $(\emptyset, \{0\})$.

5.1.3.2 Defining $\nabla^{\binom{0}{2}}$

Recall that the correspondence $\nabla^{\binom{0}{2}}$ simplifies as follows:

$$\nabla^{\binom{0}{2}} = \delta^0_0 \wedge \delta^0_0.$$

Thus, we need to define the correspondence

$$\delta^0_0 \wedge \delta^0_0 : \; X_0 \times X_0 \xleftarrow{source} \sum_{(s,t) \in \delta^0_0 \times \delta^0_0} \mathcal{O}_{s,t}(\{0\})^+ \xrightarrow{target} X_{-1} \times X_{-1}.$$

Using the values for $\mathcal{O}_{s,t}(\{0\})^+$ that we determined in the last section, we have that the correspondence $\delta^0_0 \wedge \delta^0_0$ is given by:

$$\begin{aligned}\delta^0_0 \wedge \delta^0_0 &= \mathcal{O}_{((\beta_1,a),(\beta_1,b))}(\{0\})^+ \sqcup \mathcal{O}_{((\beta_2,a),(\beta_2,c))}(\{0\})^+ \sqcup \mathcal{O}_{((\beta_3,b),(\beta_3,c))}(\{0\})^+ \\ &= (\emptyset, \{0\})_{((\beta_1,a),(\beta_1,b))} \sqcup (\emptyset, \{0\})_{((\beta_2,a),(\beta_2,c))} \sqcup (\emptyset, \{0\})_{((\beta_3,b),(\beta_3,c))}.\end{aligned}$$

The subscript on each element in the last line tells us what the source and target of that element is, since they are all copies of the same element. We define the source and

target of these elements as follows

$$\begin{aligned}
source((\emptyset,\{0\})_{((\beta_1,a),(\beta_1,b))}) &= source((\beta_1,a),(\beta_1,b)) = (\beta_1,\beta_1)\\
target((\emptyset,\{0\})_{((\beta_1,a),(\beta_1,b))}) &= target((\beta_1,a),(\beta_1,b)) = (a,b)\\
source((\emptyset,\{0\})_{((\beta_2,a),(\beta_2,c))}) &= source((\beta_2,a),(\beta_2,c)) = (\beta_2,\beta_2)\\
target((\emptyset,\{0\})_{((\beta_2,a),(\beta_2,c))}) &= target((\beta_2,a),(\beta_2,c)) = (a,c)\\
source((\emptyset,\{0\})_{((\beta_3,b),(\beta_3,c))}) &= source((\beta_3,b),(\beta_3,c)) = (\beta_3,\beta_3)\\
target((\emptyset,\{0\})_{((\beta_3,b),(\beta_3,c))}) &= target((\beta_3,b),(\beta_3,c)) = (b,c).
\end{aligned}$$

Finally, note that pre-composing this correspondence with Δ_0 leaves everything the same, except for the source map. Instead of the source map sending an element to (β_i,β_i), it gets mapped simply to β_i.

Putting this all together, we get:

$$\begin{aligned}
\nabla_{(2)}^{(0)} \circ \Delta_0 &= \sum_{U\in\mathcal{P}_2(0)} \left(\delta^0_{U^-} \wedge \delta^0_{U^+}\right) \circ \Delta_0\\
&= \left(\delta^0_0 \wedge \delta^0_0\right) \circ \Delta_0\\
&= \left[\mathcal{O}_{((\beta_1,a),(\beta_1,b))}(\{0\})^+ \circ \Delta_0\right] \sqcup \left[\mathcal{O}_{((\beta_2,a),(\beta_2,c))}(\{0\})^+ \circ \Delta_0\right] \sqcup \left[\mathcal{O}_{((\beta_3,b),(\beta_3,c))}(\{0\})^+ \circ \Delta_0\right]\\
&= \Delta_0((\emptyset,\{0\})_{((\beta_1,a),(\beta_1,b))}) \sqcup \Delta_0((\emptyset,\{0\})_{((\beta_2,a),(\beta_2,c))}) \sqcup \Delta_0((\emptyset,\{0\})_{((\beta_3,b),(\beta_3,c))}).
\end{aligned}$$

The source and target of each element in this composition of correspondences is given in the following table:

element	source $\circ\Delta_0$ in X_0	target in $X_{-1}\times X_{-1}$
$(\emptyset,\{0\})_{((\beta_1,a),(\beta_1,b))}$	β_1	(a,b)
$(\emptyset,\{0\})_{((\beta_2,a),(\beta_2,c))}$	β_2	(a,c)
$(\emptyset,\{0\})_{((\beta_3,b),(\beta_3,c))})$	β_3	(b,c)

Table 5.1: The Elements of $\nabla_{(2)}^{(0)} \circ \Delta_0$ and their Sources and Targets

5.1.4 Taking the $(\mathbb{Z}/2)$-linearization to get ∇_{-2}

Now that we have constructed the correspondence $\nabla^{\binom{0}{2}} \circ \Delta_0$, we will take its $(\mathbb{Z}/2)$-linearization to get the homomorphism ∇_{-2} restricted to the domain C_0. Recall that the $(\mathbb{Z}/2)$-linearization of a correspondence counts the number of ways to get from an element in the image of the source map to an element in the image of the target map.

As we can see in Table 5.1, each element of X_0 appears exactly once as the source of an element of $\nabla^{\binom{0}{2}} \circ \Delta_0$, and each pair from $X_{-1} \times X_{-1}$ appears at most once as the target of an element. Thus, we can define ∇_{-2} on the generators of C_0 as follows:

$$\nabla_{-2}(\beta_1) = (a, b)$$
$$\nabla_{-2}(\beta_2) = (a, c)$$
$$\nabla_{-2}(\beta_3) = (b, c).$$

5.1.5 The Steenrod Square $\mathfrak{sq}^1$

Finally, we can define $\mathfrak{sq}^1 : H^{-3,-7}(T; \mathbb{Z}/2) \to H^{-2,-7}(T; \mathbb{Z}/2)$. Recall from Example 2.12 that we have

$$H^{-3,-7}(T; \mathbb{Z}/2) = \mathbb{Z}/2\langle [a + b + c] \rangle$$
$$H^{-2,-7}(T; \mathbb{Z}/2) = \mathbb{Z}/2\langle [\beta_1] \rangle, \text{ where } [\beta_1] = [\beta_2] = [\beta_3].$$

We will start by taking the dual of

$$\nabla_{-2} : C_0 \to C_{-1} \otimes C_{-1}$$

to get

$$\smile_{-2} \colon C^{-1} \otimes C^{-1} \to C^0.$$

Note that since the image of ∇_{-2} only hits each of (a, b), (a, c), and (b, c) exactly once, when we take the dual of ∇_{-2}, we get the following assignments for $\smile_{-2}$:

$$a \smile_{-2} b = \beta_1$$
$$a \smile_{-2} c = \beta_2$$

$$b \smile_{-2} c = \beta_3.$$

The $\smile_{-2}$ products of any other two basis elements is zero.

Now, to determine the value of $\mathfrak{sq}^1$ on the cohomology class $[a+b+c] \in H^{-3,-7}$, we start with the following computation:

$$\begin{aligned}(a+b+c) \smile_{-2} (a+b+c) =&(a \smile_{-2} a) + (a \smile_{-2} b) + (a \smile_{-2} c)\\ &+ (b \smile_{-2} a) + (b \smile_{-2} b) + (b \smile_{-2} c)\\ &+ (c \smile_{-2} a) + (c \smile_{-2} b) + (c \smile_{-2} c)\\ =&(a \smile_{-2} b) + (a \smile_{-2} c) + (b \smile_{-2} c)\\ =&\beta_1 + \beta_2 + \beta_3\end{aligned}$$

Note that all three of these terms in the last line above are in the same cohomology class, namely $[\beta_1]$ of $H^{-2,-7}(T;\mathbb{Z}/2)$. Thus, when we pass from the cochain group $C^{-2,-7}$ to the cohomology group $H^{-2,-7}$, we get the value for $\mathfrak{sq}^1([a+b+c])$ to be:

$$\begin{aligned}\mathfrak{sq}^1([a+b+c]) &= [(a+b+c) \smile_{-2} (a+b+c)]\\ &= [\beta_1 + \beta_2 + \beta_3]\\ &= [\beta_1] + [\beta_2] + [\beta_3]\\ &= 3 \cdot [\beta_1]\\ &\equiv [\beta_1] \mod 2\end{aligned}$$

That is, $\mathfrak{sq}^1$ takes the generator of $H^{-3,-7}(T;\mathbb{Z}/2)$ to the generator of $H^{-2,-7}(T;\mathbb{Z}/2)$ and thus is nonzero.

5.2 $\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T;\mathbb{Z}/2) \to H^{-4,-14}(T \sqcup T;\mathbb{Z}/2)$

In this section, we will determine the Steenrod square

$$\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T;\mathbb{Z}/2) \to H^{-4,-14}(T \sqcup T;\mathbb{Z}/2)$$

following the method described above. Here, by $T \sqcup T$, we mean the disjoint union of two left-handed trefoils. We note that cohomological degree $p = -6$ corresponds to semi-simplicial degree $r = -1$, and so the set X_{-1} contains the generators of $C^{-6,-14}$. Cohomological degree -5 corresponds to semi-simplicial degree 0, and so the set X_0 contains the generators of $C^{-5,-14}$. Similarly, cohomological degree -4 corresponds to semi-simplicial degree 1, and so the set X_1 contains the generators of $C^{-4,-14}$. Also as in the previous example, we denote the $(\mathbb{Z}/2)$-linearization of X_j by $C_j = \mathbb{Z}/2\langle X_j \rangle$.

Note that we could compute this Steenrod square using a Cartan formula. This is because the link in question is the disjoint union of two trefoils, and in the previous section, we computed $\mathfrak{sq}^1$ for a single trefoil. We present this example to help clarify the definitions and constructions described earlier in this paper.

5.2.1 Setting up the Link

We start by describing the link in this example, which is the disjoint union of two left-handed trefoils. Then we will explain how we order its cycles. To simplify notation in this again, we will again write vertices without parentheses or commas. For example, we will write vertex $(1, 0, 0, 1, 0, 1)$ as vertex 100101.

In Figure 5.2, we present a link diagram for the disjoint union of two trefoils. We do not show the relevant portion of the 6-cube here since it is quite large. Since the two components of this link are each left-handed trefoils, and they do not interact, the reader can reference Figure 5.1 for each component, where the other component will be left as is along edges of the cube diagram.

Since the two components of this link do not interact and are each a single component link, the choice of orientation of each component will not interact with the choice of orientation of the other component and will have no effect on the final Steenrod square. Thus, we arbitrarily impose some choice of orientation, which is as shown in Figure 5.2. This gives us $n_+ = 0$ and $n_- = 3$.

We have also imposed an arbitrary but fixed order on the crossings of this link, as shown by the number next to each crossing in Figure 5.2. This ordering corresponds to the coordinates of the vertices of the 6-cube diagram for this link. For example, vertex 100101 shown in Figure 5.3 corresponds to the complete smoothing of this link where we resolve crossings 0, 3, and 5 via 1-smoothings, and where we resolve the remaining

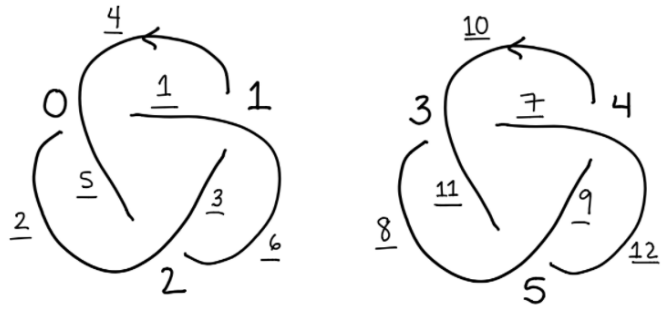


Figure 5.2: Oriented Link Diagram of the Disjoint Union of Two Trefoils with Crossings and Strands Labeled

crossings via 0-smoothings.

Next, we describe our process for ordering the cycles of a complete smoothing of this link. To do this, we start by numbering each strand of the original link, where a strand is a piece of the link between two crossings, regardless of whether that strand goes over in the cross or goes under. See Figure 5.2 for our labeling of the strands, which are the underlined numbers shown. From this, we number a cycle as the minimum strand labeling involved in that cycle.

For example, in the complete smoothing corresponding to vertex 100101 shown in Figure 5.3, we see that the larger cycle from the left trefoil contains strands 1, 2, 4, and 5. We label this cycle by the minimum number in this list, which, here, is 1, which is written inside the cycle. The smaller cycle from this left trefoil contains strands 3 and 6, and so we label it as cycle 3, written inside the cycle. Since there is only one cycle resulting from the right trefoil, it contains all of the strands 7 through 12. The minimum number here is 7, so we label this as cycle 7. Rather than for specific cycles numbers though, we use this system to order the cycles in a complete smoothing. Thus, if we refer to the enhanced state $(100101, v_- \otimes v_+ \otimes v_-)$, this means that we have labeled cycle 1 by v_-, cycle 3 by v_+, and cycle 7 by v_-. That is, we order the cycles in increasing order so that cycle 1 is first, cycle 3 is second, and cycle 7 is third.

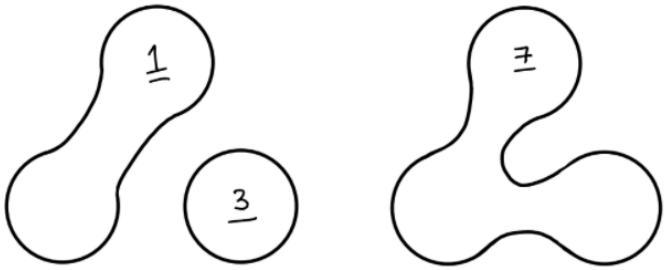


Figure 5.3: Complete Smoothing Corresponding to Vertex 100101 for the Disjoint Union of Two Trefoils with Cycles Labeled

5.2.2 Defining the Cochains Groups, Augmented Semi-Simplicial Objects, and Correspondences

In this section, we present the elements of X_{-1}, which are the enhanced states that generate $C^{-6,-14}$, the elements of X_0, which are the enhanced states that generate $C^{-5,-14}$, and the elements of X_1, which are the enhanced states that generate $C^{-4,-14}$. To simplify notation throughout the computation, we will also give a shortened label to each element of X_{-1}, X_0 and X_1.

The enhanced states of X_{-1} correspond to vertices at height 0 since we have

$$\begin{aligned} height &= r + 1 \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

There is only one vertex at height 0, namely the vertex 000000. The complete smoothing of the trefoil corresponding to this vertex contains six cycles. For a labeling of this resolution to be in quantum degree -14, it must have two cycles labeled by v_+ and the remaining four cycles labeled by v_-. Following Morán's method for labeling these enhanced states from his example 8.5, we label the cycles in this resolution as a, b, c, d, e, and f, as shown in Figure 5.4 [Mor20]. Then, we refer to one of these enhanced states in $C^{-6,-14}$ simply by the two letters of the two cycles labeled by v_+. The elements of X_{-1}, which are the generators of $C^{-6,-14}$, along with their associated

two letter label are as follows:

$$\begin{aligned}
X_{-1} &= \bigsqcup_{|A|=0} F^{-14}(A) \\
&= \{ab = (000000, v_+ \otimes v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-), \\
&\quad ac = (000000, v_+ \otimes v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-), \\
&\quad ad = (000000, v_+ \otimes v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad ae = (000000, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad af = (000000, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+), \\
&\quad bc = (000000, v_- \otimes v_+ \otimes v_+ \otimes v_- \otimes v_- \otimes v_-), \\
&\quad bd = (000000, v_- \otimes v_+ \otimes v_- \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad be = (000000, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad bf = (000000, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_+), \\
&\quad cd = (000000, v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad ce = (000000, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad cf = (000000, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_+), \\
&\quad de = (000000, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_-), \\
&\quad df = (000000, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_+), \\
&\quad ef = (000000, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_+)\}
\end{aligned}$$

The enhanced states of X_0 correspond to vertices at height 1. There are 6 vertices at height 1, and each corresponds to a smoothing with five cycles. In order for a labeling of one of these complete smoothings to be in quantum degree -14, exactly one of its cycles must be labeled by v_+ and the rest must be labeled by v_-. Thus, each vertex gives us five enhanced states. We will refer to one of these enhanced states as $\alpha_{i,j}$, where i refers to the index of the vertex in the fixed, but arbitrary order we specify below, and j refers to the cycle in that complete resolution that is labeled by v_+. Note here that the index j will be between 1 and 5, where we first label the cycles via the the method we described above, and then we order them in increasing order. The elements of X_0

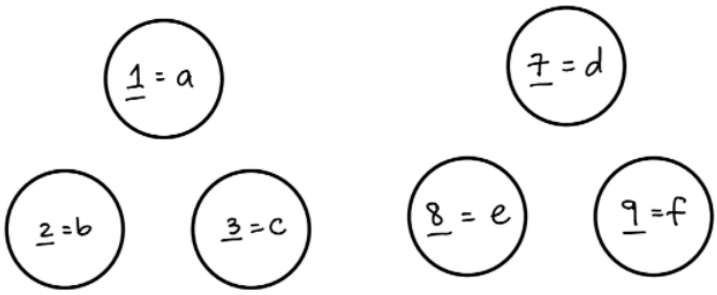


Figure 5.4: Complete Smoothing of the Disjoint Union of Two Trefoils Corresponding to Vertex 000000 with Cycles Labeled

along with their associated label $\alpha_{i,j}$ are as follows:

$$
\begin{aligned}
X_0 &= \bigsqcup_{|A|=1} F^{-14}(A) \\
&= \{\alpha_{1,1} = (100000, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{1,2} = (100000, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{1,3} = (100000, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad \alpha_{1,4} = (100000, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad \alpha_{1,5} = (100000, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+), \\
&\quad \alpha_{2,1} = (010000, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{2,2} = (010000, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{2,3} = (010000, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad \alpha_{2,4} = (010000, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad \alpha_{2,5} = (010000, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+), \\
&\quad \alpha_{3,1} = (001000, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{3,2} = (001000, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-), \\
&\quad \alpha_{3,3} = (001000, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-), \\
&\quad \alpha_{3,4} = (001000, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-), \\
&\quad \alpha_{3,5} = (001000, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+),
\end{aligned}
$$

$$\alpha_{4,1} = (000100, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{4,2} = (000100, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{4,3} = (000100, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-),$$
$$\alpha_{4,4} = (000100, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-),$$
$$\alpha_{4,5} = (000100, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+),$$
$$\alpha_{5,1} = (000010, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{5,2} = (000010, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{5,3} = (000010, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-),$$
$$\alpha_{5,4} = (000010, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-),$$
$$\alpha_{5,5} = (000010, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+),$$
$$\alpha_{6,1} = (000001, v_+ \otimes v_- \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{6,2} = (000001, v_- \otimes v_+ \otimes v_- \otimes v_- \otimes v_-),$$
$$\alpha_{6,3} = (000001, v_- \otimes v_- \otimes v_+ \otimes v_- \otimes v_-),$$
$$\alpha_{6,4} = (000001, v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_-),$$
$$\alpha_{6,5} = (000001, v_- \otimes v_- \otimes v_- \otimes v_- \otimes v_+)\}.$$

The enhanced states of X_1 correspond to vertices at height 2. There are 15 vertices at height 2, and each corresponds to a smoothing with four cycles. In order for a labeling of one of these complete smoothings to be in quantum degree -14, all four of its cycles must be labeled by v_-. Thus, each vertex gives us exactly one enhanced state. Then, our simplified notation here comes from ordering the vertices of the 6-cube with height 2 and referring to the one enhanced state for this vertex as β_j, where j is the place in the overall order of that vertex. The elements of X_1 are their associated label β_j are as follows:

$$X_1 = \bigsqcup_{|A|=2} F^{-14}(A)$$
$$= \{\beta_1 = (110000, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_9 = (010001, v_- \otimes v_- \otimes v_- \otimes v_-)$$
$$\beta_2 = (101000, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{10} = (001100, v_- \otimes v_- \otimes v_- \otimes v_-)$$
$$\beta_3 = (100100, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{11} = (001010, v_- \otimes v_- \otimes v_- \otimes v_-)$$

$$\beta_4 = (100010, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{12} = (001001, v_- \otimes v_- \otimes v_- \otimes v_-)$$
$$\beta_5 = (100001, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{13} = (000110, v_- \otimes v_- \otimes v_- \otimes v_-)$$
$$\beta_6 = (011000, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{14} = (000101, v_- \otimes v_- \otimes v_- \otimes v_-)$$
$$\beta_7 = (010100, v_- \otimes v_- \otimes v_- \otimes v_-), \qquad \beta_{15} = (000011, v_- \otimes v_- \otimes v_- \otimes v_-)\}.$$
$$\beta_8 = (010010, v_- \otimes v_- \otimes v_- \otimes v_-),$$

Now, we will present the correspondences between these augmented semi-simplicial objects that we will use in constructing $\mathfrak{sq}^2$. We start with the correspondences $\delta_0^0 : X_0 \to X_{-1}$, $\delta_0^1 : X_1 \to X_0$, and $\delta_1^1 : X_1 \to X_0$:

$$\begin{aligned}
\delta_0^0 =& X(\bar{\delta}_0^0) \\
=& \bigsqcup_{|A|=1} F^{-14}(d_{U'} : 000000 \to A)^* \\
=& F^{-14}(d_{*,0,0,0,0,0} : 000000 \to 100000)^* \sqcup\ F^{-14}(d_{0,*,0,0,0,0} : 000000 \to 010000)^* \\
& \sqcup\ F^{-14}(d_{0,0,*,0,0,0} : 000000 \to 001000)^* \sqcup\ F^{-14}(d_{0,0,0,*,0,0} : 000000 \to 000100)^* \\
& \sqcup\ F^{-14}(d_{0,0,0,0,*,0} : 000000 \to 000010)^* \sqcup\ F^{-14}(d_{0,0,0,0,0,*} : 000000 \to 000001)^* \\
=& \{(\alpha_{1,1}, ab), \quad (\alpha_{2,1}, ac), \quad (\alpha_{3,1}, ab), \quad (\alpha_{4,1}, ad), \quad (\alpha_{5,1}, ad), \quad (\alpha_{6,1}, ae), \\
& (\alpha_{1,2}, ac), \quad (\alpha_{2,2}, ab), \quad (\alpha_{3,1}, ac), \quad (\alpha_{4,1}, ae), \quad (\alpha_{5,1}, af), \quad (\alpha_{6,1}, af), \\
& (\alpha_{1,2}, bc), \quad (\alpha_{2,2}, bc), \quad (\alpha_{3,2}, bc), \quad (\alpha_{4,2}, bd), \quad (\alpha_{5,2}, bd), \quad (\alpha_{6,2}, be), \\
& (\alpha_{1,3}, ad), \quad (\alpha_{2,3}, ad), \quad (\alpha_{3,3}, bd), \quad (\alpha_{4,2}, be), \quad (\alpha_{5,2}, bf), \quad (\alpha_{6,2}, bf), \\
& (\alpha_{1,3}, bd), \quad (\alpha_{2,3}, cd), \quad (\alpha_{3,3}, cd), \quad (\alpha_{4,3}, cd), \quad (\alpha_{5,3}, cd), \quad (\alpha_{6,3}, ce), \\
& (\alpha_{1,4}, ae), \quad (\alpha_{2,4}, ae), \quad (\alpha_{3,4}, be), \quad (\alpha_{4,3}, ce), \quad (\alpha_{5,3}, cf), \quad (\alpha_{6,3}, cf), \\
& (\alpha_{1,4}, be), \quad (\alpha_{2,4}, ce), \quad (\alpha_{3,4}, ce), \quad (\alpha_{4,4}, de), \quad (\alpha_{5,4}, df), \quad (\alpha_{6,4}, de), \\
& (\alpha_{1,5}, af), \quad (\alpha_{2,5}, af), \quad (\alpha_{3,5}, bf), \quad (\alpha_{4,5}, df), \quad (\alpha_{5,5}, de), \quad (\alpha_{6,4}, df), \\
& (\alpha_{1,5}, bf), \quad (\alpha_{2,5}, cf), \quad (\alpha_{3,5}, cf), \quad (\alpha_{4,5}, ef), \quad (\alpha_{5,5}, ef), \quad (\alpha_{6,5}, ef)\}
\end{aligned}$$

$$\delta_0^1 = X(\bar{\delta}_0^1)$$

$$
\begin{aligned}
&= \bigsqcup_{|A|=2} F^{-14}(d_{U'} : A' \to A)^* \\
&= F^{-14}(d_{*,1,0,0,0,0} : 010000 \to 110000)^* \sqcup\ F^{-14}(d_{*,0,1,0,0,0} : 001000 \to 101000)^* \\
&\quad \sqcup\ F^{-14}(d_{0,*,1,0,0,0} : 001000 \to 011000)^* \sqcup\ F^{-14}(d_{*,0,0,1,0,0} : 000100 \to 100100)^* \\
&\quad \sqcup\ F^{-14}(d_{0,*,0,1,0,0} : 000100 \to 010100)^* \sqcup\ F^{-14}(d_{0,0,*,1,0,0} : 000100 \to 001100)^* \\
&\quad \sqcup\ F^{-14}(d_{*,0,0,0,1,0} : 000010 \to 100010)^* \sqcup\ F^{-14}(d_{0,*,0,0,1,0} : 000010 \to 010010)^* \\
&\quad \sqcup\ F^{-14}(d_{0,0,*,0,1,0} : 000010 \to 001010)^* \sqcup\ F^{-14}(d_{0,0,0,*,1,0} : 000010 \to 000110)^* \\
&\quad \sqcup\ F^{-14}(d_{*,0,0,0,0,1} : 000001 \to 100001)^* \sqcup\ F^{-14}(d_{0,*,0,0,0,1} : 000001 \to 010001)^* \\
&\quad \sqcup\ F^{-14}(d_{0,0,*,0,0,1} : 000001 \to 001001)^* \sqcup\ F^{-14}(d_{0,0,0,*,0,1} : 000001 \to 000101)^* \\
&\quad \sqcup\ F^{-14}(d_{0,0,0,0,*,1} : 000001 \to 000011)^* \\
&= \{(\beta_1, \alpha_{2,1}), \quad (\beta_4, \alpha_{5,1}), \quad (\beta_7, \alpha_{4,1}), \quad (\beta_{10}, \alpha_{4,2}), \quad (\beta_{13}, \alpha_{5,4}), \\
&\quad (\beta_1, \alpha_{2,2}), \quad (\beta_4, \alpha_{5,2}), \quad (\beta_7, \alpha_{4,3}), \quad (\beta_{10}, \alpha_{4,3}), \quad (\beta_{13}, \alpha_{5,5}), \\
&\quad (\beta_2, \alpha_{3,1}), \quad (\beta_5, \alpha_{6,1}), \quad (\beta_8, \alpha_{5,1}), \quad (\beta_{11}, \alpha_{5,2}), \quad (\beta_{14}, \alpha_{6,4}), \\
&\quad (\beta_2, \alpha_{3,2}), \quad (\beta_5, \alpha_{6,2}), \quad (\beta_8, \alpha_{5,3}), \quad (\beta_{11}, \alpha_{5,3}), \quad (\beta_{14}, \alpha_{6,5}), \\
&\quad (\beta_3, \alpha_{4,1}), \quad (\beta_6, \alpha_{3,1}), \quad (\beta_9, \alpha_{6,1}), \quad (\beta_{12}, \alpha_{6,2}), \quad (\beta_{15}, \alpha_{6,4}), \\
&\quad (\beta_3, \alpha_{4,2}), \quad (\beta_6, \alpha_{3,2}), \quad (\beta_9, \alpha_{6,3}), \quad (\beta_{12}, \alpha_{6,3}), \quad (\beta_{15}, \alpha_{6,5})\}
\end{aligned}
$$

$$
\begin{aligned}
\delta_1^1 &= X(\bar{\delta}_1^1) \\
&= \bigsqcup_{|A|=2} F^{-14}(d_{U'} : A' \to A)^* \\
&= \sqcup\ F^{-14}(d_{1,*,0,0,0,0} : 100000 \to 110000)^* \sqcup\ F^{-14}(d_{1,0,*,0,0,0} : 100000 \to 101000)^* \\
&\quad \sqcup\ F^{-14}(d_{1,0,0,*,0,0} : 100000 \to 100100)^* \sqcup\ F^{-14}(d_{1,0,0,0,*,0} : 100000 \to 100010)^* \\
&\quad \sqcup\ F^{-14}(d_{1,0,0,0,0,*} : 100000 \to 100001)^* \sqcup\ F^{-14}(d_{0,1,*,0,0,0} : 010000 \to 011000)^* \\
&\quad \sqcup\ F^{-14}(d_{0,1,0,*,0,0} : 010000 \to 010100)^* \sqcup\ F^{-14}(d_{0,1,0,0,*,0} : 010000 \to 010010)^* \\
&\quad \sqcup\ F^{-14}(d_{0,1,0,0,0,*} : 010000 \to 010001)^* \sqcup\ F^{-14}(d_{0,0,1,*,0,0} : 001000 \to 001100)^* \\
&\quad \sqcup\ F^{-14}(d_{0,0,1,0,*,0} : 001000 \to 001010)^* \sqcup\ F^{-14}(d_{0,0,1,0,0,*} : 001000 \to 001001)^* \\
&\quad \sqcup\ F^{-14}(d_{0,0,0,1,*,0} : 000100 \to 000110)^* \sqcup\ F^{-14}(d_{0,0,0,1,0,*} : 000100 \to 000101)^*
\end{aligned}
$$

$$
\begin{aligned}
&\sqcup\, F^{-14}(d_{0,0,0,0,0,1,*} : 000010 \to 000011)^* \\
=\{&(\beta_1,\alpha_{1,1}), \quad (\beta_4,\alpha_{1,3}), \quad (\beta_7,\alpha_{2,3}), \quad (\beta_{10},\alpha_{3,3}), \quad (\beta_{13},\alpha_{4,4}),\\
&(\beta_1,\alpha_{1,2}), \quad (\beta_4,\alpha_{1,5}), \quad (\beta_7,\alpha_{2,4}), \quad (\beta_{10},\alpha_{3,4}), \quad (\beta_{13},\alpha_{4,5}),\\
&(\beta_2,\alpha_{1,1}), \quad (\beta_5,\alpha_{1,4}), \quad (\beta_8,\alpha_{2,3}), \quad (\beta_{11},\alpha_{3,3}), \quad (\beta_{14},\alpha_{4,4}),\\
&(\beta_2,\alpha_{1,2}), \quad (\beta_5,\alpha_{1,5}), \quad (\beta_8,\alpha_{2,5}), \quad (\beta_{11},\alpha_{3,5}), \quad (\beta_{14},\alpha_{4,5}),\\
&(\beta_3,\alpha_{1,3}), \quad (\beta_6,\alpha_{2,1}), \quad (\beta_9,\alpha_{2,4}), \quad (\beta_{12},\alpha_{3,4}), \quad (\beta_{15},\alpha_{5,4}),\\
&(\beta_3,\alpha_{1,4}), \quad (\beta_6,\alpha_{2,2}), \quad (\beta_9,\alpha_{2,5}), \quad (\beta_{12},\alpha_{3,5}), \quad (\beta_{15},\alpha_{5,5})\}
\end{aligned}
$$

Next, we define $\delta^1_{0,1} : X_1 \to X_{-1}$ in its two equivalent forms, as the composition $\delta^0_{\psi_0(1)} \circ \delta^1_0 = \delta^0_0 \circ \delta^1_0$ and as the composition $\delta^0_0 \circ \delta^1_1$. We write the elements of these compositions as triples since this will be more useful in the $\mathfrak{sq}^2$ computation than just considering the elements as pairs in $\delta^1_{0,1}$. The compositions are defined as follows:

$$
\begin{aligned}
\delta^0_{\psi_0(1)} \circ \delta^1_0 &= \delta^0_0 \circ \delta^1_0 \\
&\cong \delta^1_{0,1} \\
&\cong\{(\beta_1,\alpha_{2,1},ac), \quad (\beta_6,\alpha_{3,1},ab), \quad (\beta_{10},\alpha_{4,3},ce),\\
&\quad (\beta_1,\alpha_{2,2},ab), \quad (\beta_6,\alpha_{3,1},ac), \quad (\beta_{11},\alpha_{5,2},bd),\\
&\quad (\beta_1,\alpha_{2,2},bc), \quad (\beta_6,\alpha_{3,2},bc), \quad (\beta_{11},\alpha_{5,2},bf),\\
&\quad (\beta_2,\alpha_{3,1},ab), \quad (\beta_7,\alpha_{4,1},ad), \quad (\beta_{11},\alpha_{5,3},cd),\\
&\quad (\beta_2,\alpha_{3,1},ac), \quad (\beta_7,\alpha_{4,1},ae), \quad (\beta_{11},\alpha_{5,3},cf),\\
&\quad (\beta_2,\alpha_{3,2},bc), \quad (\beta_7,\alpha_{4,3},cd), \quad (\beta_{12},\alpha_{6,2},bd),\\
&\quad (\beta_3,\alpha_{4,1},ad), \quad (\beta_7,\alpha_{4,3},ce), \quad (\beta_{12},\alpha_{6,2},bf),\\
&\quad (\beta_3,\alpha_{4,1},ae), \quad (\beta_8,\alpha_{5,1},ad), \quad (\beta_{12},\alpha_{6,3},cd),\\
&\quad (\beta_3,\alpha_{4,2},bd), \quad (\beta_8,\alpha_{5,1},af), \quad (\beta_{12},\alpha_{6,3},cf),\\
&\quad (\beta_3,\alpha_{4,2},be), \quad (\beta_8,\alpha_{5,3},cd), \quad (\beta_{13},\alpha_{5,4},df),\\
&\quad (\beta_4,\alpha_{5,1},ad), \quad (\beta_8,\alpha_{5,3},cf), \quad (\beta_{13},\alpha_{5,5},de),\\
&\quad (\beta_4,\alpha_{5,1},af), \quad (\beta_9,\alpha_{6,1},ae), \quad (\beta_{13},\alpha_{5,5},ef),\\
&\quad (\beta_4,\alpha_{5,2},bd), \quad (\beta_9,\alpha_{6,1},af), \quad (\beta_{14},\alpha_{6,4},de),\\
&\quad (\beta_4,\alpha_{5,2},bf), \quad (\beta_9,\alpha_{6,3},ce), \quad (\beta_{14},\alpha_{6,4},df),
\end{aligned}
$$

$$
\begin{array}{lll}
(\beta_5, \alpha_{6,1}, ae), & (\beta_9, \alpha_{6,3}, cf), & (\beta_{14}, \alpha_{6,5}, ef), \\
(\beta_5, \alpha_{6,1}, af), & (\beta_{10}, \alpha_{4,2}, bd), & (\beta_{15}, \alpha_{6,4}, de), \\
(\beta_5, \alpha_{6,2}, be), & (\beta_{10}, \alpha_{4,2}, be), & (\beta_{15}, \alpha_{6,4}, df), \\
(\beta_5, \alpha_{6,2}, bf), & (\beta_{10}, \alpha_{4,3}, cd), & (\beta_{15}, \alpha_{6,5}, ef)\}
\end{array}
$$

$$
\begin{aligned}
\delta_0^0 \circ \delta_1^1 &\cong \delta_{0,1}^1 \\
&\cong \{(\beta_1, \alpha_{1,1}, ab), \quad (\beta_6, \alpha_{2,1}, ac), \quad (\beta_{10}, \alpha_{3,5}, cf), \\
&\quad (\beta_1, \alpha_{1,2}, ac), \quad (\beta_6, \alpha_{2,2}, ab), \quad (\beta_{11}, \alpha_{3,4}, be), \\
&\quad (\beta_1, \alpha_{1,2}, bc), \quad (\beta_6, \alpha_{2,2}, bc), \quad (\beta_{11}, \alpha_{3,4}, ce), \\
&\quad (\beta_2, \alpha_{1,1}, ab), \quad (\beta_7, \alpha_{2,3}, ad), \quad (\beta_{11}, \alpha_{3,5}, bf), \\
&\quad (\beta_2, \alpha_{1,2}, ac), \quad (\beta_7, \alpha_{2,3}, cd), \quad (\beta_{11}, \alpha_{3,5}, cf), \\
&\quad (\beta_2, \alpha_{1,2}, bc), \quad (\beta_7, \alpha_{2,5}, af), \quad (\beta_{12}, \alpha_{3,4}, be), \\
&\quad (\beta_3, \alpha_{1,3}, ad), \quad (\beta_7, \alpha_{2,5}, cf), \quad (\beta_{12}, \alpha_{3,4}, ce), \\
&\quad (\beta_3, \alpha_{1,3}, bd), \quad (\beta_8, \alpha_{2,3}, ad), \quad (\beta_{12}, \alpha_{3,5}, bf), \\
&\quad (\beta_3, \alpha_{1,4}, ae), \quad (\beta_8, \alpha_{2,3}, cd), \quad (\beta_{12}, \alpha_{3,5}, cf), \\
&\quad (\beta_3, \alpha_{1,4}, be), \quad (\beta_8, \alpha_{2,5}, af), \quad (\beta_{13}, \alpha_{4,4}, de), \\
&\quad (\beta_4, \alpha_{1,3}, ad), \quad (\beta_8, \alpha_{2,5}, cf), \quad (\beta_{13}, \alpha_{4,5}, df), \\
&\quad (\beta_4, \alpha_{1,3}, bd), \quad (\beta_9, \alpha_{2,3}, ad), \quad (\beta_{13}, \alpha_{4,5}, ef), \\
&\quad (\beta_4, \alpha_{1,4}, ae), \quad (\beta_9, \alpha_{2,3}, cd), \quad (\beta_{14}, \alpha_{4,4}, de), \\
&\quad (\beta_4, \alpha_{1,4}, be), \quad (\beta_9, \alpha_{2,5}, af), \quad (\beta_{14}, \alpha_{4,5}, df), \\
&\quad (\beta_5, \alpha_{1,3}, ad), \quad (\beta_9, \alpha_{2,5}, cf), \quad (\beta_{14}, \alpha_{4,5}, ef), \\
&\quad (\beta_5, \alpha_{1,3}, bd), \quad (\beta_{10}, \alpha_{3,4}, be), \quad (\beta_{15}, \alpha_{4,4}, df), \\
&\quad (\beta_5, \alpha_{1,4}, ae), \quad (\beta_{10}, \alpha_{3,4}, ce), \quad (\beta_{15}, \alpha_{4,5}, de), \\
&\quad (\beta_5, \alpha_{1,4}, be), \quad (\beta_{10}, \alpha_{3,5}, bf), \quad (\beta_{15}, \alpha_{4,5}, ef)\}
\end{aligned}
$$

The equivalence between $\delta^0_{\psi_0(1)} \circ \delta^1_0$ and $\delta^0_0 \circ \delta^1_1$ matches up two triples where the first and third coordinate are the same. For example $(\beta_1, \alpha_{2,1}, ac)$ is equivalent to

$(\beta_1, \alpha_{1,2}, ac)$. Also, we define the source map of all of these correspondences maps to send each pair or triple to its first coordinate, and we define the target map to send each pair or triple to its last coordinate.

Note that it will be necessary to fix an ordering on the elements within each correspondence. As with the other choices of ordering that we have had to make, the particular choice will not affect the resulting $\mathfrak{sq}^2$ map though. For the correspondences δ_0^0, δ_0^1, and δ_1^1, we order the elements increasing down the column and then increasing from left to right. That is, the first element in the first column is the smallest. The second smallest element is the one in the second row of the first column. The last element in the right-most column is the largest element.

Since $\delta^0_{\psi_0(1)} \circ \delta^1_0$ and $\delta^0_0 \circ \delta^1_1$ are equivalent to $\delta^1_{0,1}$, we want all of these correspondences to have the same order. Here, we will order the elements of $\delta^0_{\psi_0(1)} \circ \delta^1_0$ as described for the other correspondences. This means that the elements of $\delta^0_0 \circ \delta^1_1$ do not quite appear in order.

5.2.3 Setting up the Steenrod Square Computation

To define $\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T; \mathbb{Z}/2) \to H^{-6,-14}(T \sqcup T; \mathbb{Z}/2)$, we use the map $\smile_{-3}$: $C^{-1} \otimes C^{-1} \to C^1$. We get the degree -3 from $((-1) - 2)$, where the -1 term is the semi-simplicial degree of the domain of $\mathfrak{sq}^2$ and the 2 term is the degree of $\mathfrak{sq}^2$ itself. To define the map $\smile_{-3}$, we use its dual $\nabla_{-3} =: C_* \otimes C_* \to C_*$, where we will have:

$$\begin{aligned}\nabla_{-3} &= \mathcal{A}_{\mathbb{Z}/2}\left(\sum_k \nabla^{\binom{k}{k-(-3)}} \circ \Delta_k\right) \\ &= \mathcal{A}_{\mathbb{Z}/2}\left(\sum_k \nabla^{\binom{k}{k+3}} \circ \Delta_k\right).\end{aligned}$$

Because we want to use this map ∇_{-3} to define $\smile_{-3}$: $C^{-1} \otimes C^{-1} \to C^1$, we restrict the domain of ∇_{-3} to C_1, which gives us:

$$\nabla_{-3}|_{C_1} = \mathcal{A}_{\mathbb{Z}/2}\left(\nabla^{\binom{1}{4}} \circ \Delta_1\right)$$

Recall that the diagonal correspondence $\Delta_1 : X_1 \to X_1 \times X_1$ is equal to X_1. Here, the source of an element $a \in \Delta_1$ is $a \in X_1$, and the target of a is $(a, a) \in X_1 \times X_1$.

Next, we want to define the correspondence

$$\nabla^{\binom{1}{4}} : X_1 \times X_1 \to \bigcup_{a+b=4} X_{1-a} \times X_{1-b}.$$

The 4 in $a + b = 4$ in the subscript on the union comes from $2i = 2 \cdot 2$, where i is the index of the Steenrod square we are computing. The 1 in the subscript on X_1, X_{1-a}, and X_{1-b} is the semi-simplicial degree of the domain we restricted ∇_{-3} to. We define $\nabla^{\binom{1}{4}}$ as follows:

$$\nabla^{\binom{1}{4}} = \sum_{U \in \mathcal{P}_4(1)} \delta^1_{U^-} \wedge \delta^1_{U^+}.$$

Recall that $\mathcal{P}_4(1)$ is the collection of non-decreasing sequences of length 4 from the integers $\{0, 1\}$, where each integer can be repeated at most twice. There is exactly one such sequence, namely $U = (0, 0, 1, 1)$, which has both U^- and U^+ equal to $\{0, 1\}$. Thus, the correspondence $\nabla^{\binom{1}{4}}$ can be simplified to

$$\nabla^{\binom{1}{4}} = \delta^1_{0,1} \wedge \delta^1_{0,1}.$$

We define the correspondence $\delta^1_{0,1} \wedge \delta^1_{0,1}$ as follows:

$$\delta^1_{0,1} \wedge \delta^1_{0,1} : \; X_1 \times X_1 \xleftarrow{source} \sum_{(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}} \mathcal{O}_{s,t}(\{0,1\})^+ \xrightarrow{target} X_{-1} \times X_{-1}.$$

The set $\{0, 1\}$ inside the terms of the summation is $U^- \cap U^+$, and the degree of the augmented semi-simplicial objects in the domain is $1 - |U^-|$ and $1 - |U^+|$, where $|U^-| = |U^+| = 2$.

The main work of this Steenrod square construction comes from defining $\mathcal{O}_{s,t}(\{0, 1\})^+$ for each $(s, t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, which contains the positive, maximal chains of $\{0, 1\}$. We will do this in the next section.

5.2.3.1 Determining the Positive Maximal Chains

As we saw at the end of the previous section, the correspondence $\nabla^{\binom{1}{4}}$ simplifies as follows:

$$\nabla^{\binom{1}{4}} = \delta^1_{0,1} \wedge \delta^1_{0,1}.$$

Thus, in this section, we will define the correspondence

$$\delta^1_{0,1} \wedge \delta^1_{0,1} : \ X_1 \times X_1 \xleftarrow{source} \sum_{(s,t)\in\delta^1_{0,1}\times\delta^1_{0,1}} \mathcal{O}_{s,t}(\{0,1\})^+ \xrightarrow{target} X_{-1} \times X_{-1}.$$

That is, for each pair of disjoint subsets (W'', W°) of $U^- \cap U^+ = \{0,1\}$, we will find all pairs $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$ such that (W'', W°) is (s,t)-positive. Then, we will organize these pairs of subsets into maximal chains for each (s,t).

Given our sequence $U = (0,0,1,1)$ with $U^- = \{0,1\}$ and $U^+ = \{0,1\}$, there are five disjoint subsets (W'', W°) of $U^- \cap U^+ = \{0,1\}$ that might be (s,t)-positive for some pair $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$. These pairs of subsets are:

$$(\emptyset, 0)$$
$$(\emptyset, 1)$$
$$(0, 1)$$
$$(1, 0)$$
$$(\emptyset, \{0,1\}).$$

Note that we drop the curly braces around the singleton sets to reduce notation.

Let us now consider each of these pairs (W'', W°) and determine for which (s,t) pairs, (W'', W°) is (s,t)-good and (s,t)-positive. Recall that for a pair (W'', W°) to be (s,t)-good for $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, it must satisfy the following conditions:

$$\lambda^1_{W''}(s) = \lambda^1_{W''}(t) \tag{5.3}$$
$$\lambda^1_{W'',W^\circ}(s) \neq \lambda^1_{W'',W^\circ}(t). \tag{5.4}$$

Then, since $r + |U^-| + |U^+| = 1 + 2 + 2 = 5$ is odd, where r is the semi-simplicial degree of the restricted domain of ∇_{-3}, in order for a pair (W'', W°) to be (s,t)-positive, we further require:

$$\lambda^1_{W'',W^\circ}(s) > \lambda^1_{W'',W^\circ}(t).$$

We will combine the concepts of (s,t)-good and (s,t)-positive since (s,t)-positive simply

further restricts from the set of (s,t)-good pairs. Thus, here we will check the first (s,t)-goodness criterion and the (s,t)-positive criterion.

Now, let us consider each of the (W'', W°) pairs in turn.

- Consider $(W'', W^\circ) = (\emptyset, 0)$. The first criterion for (s,t)-goodness requires that s and t have the same image under the map $\lambda^1_{W''} = \lambda^1_\emptyset$. We define $\lambda^1_\emptyset$ as follows:

$$\lambda^1_\emptyset : \delta^1_{0,1} \xrightarrow{\sim} \delta^1_{0,1} \circ \delta^1_\emptyset \xrightarrow{proj} \delta^1_\emptyset.$$

  Recall that the correspondence $\delta^j_\emptyset$ is the identity correspondence on X_j for all j. Thus, this map $\lambda^1_\emptyset$ projects onto the source of each element of $\delta^1_{0,1}$. That is, if we write an element of $\delta^1_{0,1}$ as a triple (a,b,c), its image under $\lambda^1_\emptyset$ is a. Thus, for the first (s,t)-goodness criterion, we are looking for pairs $(s,t) = ((a,b,c),(d,e,f))$ such that

$$\begin{aligned} \lambda^1_\emptyset(a,b,c) &= \lambda^1_\emptyset(d,e,f) \\ a &= d. \end{aligned}$$

  The criterion for (s,t)-positivity involves the map $\lambda^1_{W'',W^\circ} = \lambda^1_{\emptyset,0}$, which we define as follows:

$$\lambda^1_{\emptyset,0} : \delta^1_{0,1} \xrightarrow{\sim} \delta^0_{\psi_0(1)} \circ \delta^1_0 \circ \delta^1_\emptyset \xrightarrow{proj} \delta^1_0.$$

  Recall that by definition we have $\psi_0(1) = 0$. Note that we defined the middle composition of correspondences in $\lambda^1_{\emptyset,0}$ in the previous section as $\delta^0_0 \circ \delta^1_0$ whose elements we wrote as triples (a,b,c). Since the first arrow in $\lambda^1_{\emptyset,0}$ is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_0$. Then, the map $\lambda^1_{\emptyset,0}$ sends an element $(a,b,c) \in \delta^0_0 \circ \delta^1_0$ to $(a,b) \in \delta^1_0$. Thus, for the (s,t)-positivity criterion, we are looking for pairs $(s,t) = ((a,b,c),(d,e,f))$ such that

$$\begin{aligned} \lambda^1_{\emptyset,0}(a,b,c) &> \lambda^1_{\emptyset,0}(d,e,f) \\ (a,b) &> (d,e). \end{aligned}$$

  Putting these two criteria together for (s,t)-goodness and (s,t)-positivity, we have that $(\emptyset, 0)$ is (s,t)-positive for the following $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, written as elements

of $\delta_0^0 \circ \delta_0^1$:

$$\{((\beta_1,\alpha_{2,2},ab),(\beta_1,\alpha_{2,1},ac)),\quad ((\beta_6,\alpha_{3,2},bc),(\beta_6,\alpha_{3,1},ab)),\quad ((\beta_{10},\alpha_{4,3},ce),(\beta_{10},\alpha_{4,2},bd)),$$
$$((\beta_1,\alpha_{2,2},bc),(\beta_1,\alpha_{2,1},ac)),\quad ((\beta_6,\alpha_{3,2},bc),(\beta_6,\alpha_{3,1},ac)),\quad ((\beta_{10},\alpha_{4,3},ce),(\beta_{10},\alpha_{4,2},be)),$$
$$((\beta_2,\alpha_{3,2},bc),(\beta_2,\alpha_{3,1},ab)),\quad ((\beta_7,\alpha_{4,3},cd),(\beta_7,\alpha_{4,1},ad)),\quad ((\beta_{11},\alpha_{5,3},cd),(\beta_{11},\alpha_{5,2},bd)),$$
$$((\beta_2,\alpha_{3,2},bc),(\beta_2,\alpha_{3,1},ac)),\quad ((\beta_7,\alpha_{4,3},cd),(\beta_7,\alpha_{4,1},ae)),\quad ((\beta_{11},\alpha_{5,3},cd),(\beta_{11},\alpha_{5,2},bf)),$$
$$((\beta_3,\alpha_{4,2},bd),(\beta_3,\alpha_{4,1},ad)),\quad ((\beta_7,\alpha_{4,3},ce),(\beta_7,\alpha_{4,1},ad)),\quad ((\beta_{11},\alpha_{5,3},cf),(\beta_{11},\alpha_{5,2},bd)),$$
$$((\beta_3,\alpha_{4,2},bd),(\beta_3,\alpha_{4,1},ae)),\quad ((\beta_7,\alpha_{4,3},ce),(\beta_7,\alpha_{4,1},ae)),\quad ((\beta_{11},\alpha_{5,3},cf),(\beta_{11},\alpha_{5,2},bf)),$$
$$((\beta_3,\alpha_{4,2},be),(\beta_3,\alpha_{4,1},ad)),\quad ((\beta_8,\alpha_{5,3},cd),(\beta_8,\alpha_{5,1},ad)),\quad ((\beta_{12},\alpha_{6,3},ce),(\beta_{12},\alpha_{6,2},be)),$$
$$((\beta_3,\alpha_{4,2},be),(\beta_3,\alpha_{4,1},ae)),\quad ((\beta_8,\alpha_{5,3},cd),(\beta_8,\alpha_{5,1},af)),\quad ((\beta_{12},\alpha_{6,3},cf),(\beta_{12},\alpha_{6,2},bf)),$$
$$((\beta_4,\alpha_{5,2},bd),(\beta_4,\alpha_{5,1},ad)),\quad ((\beta_8,\alpha_{5,3},cf),(\beta_8,\alpha_{5,1},ad)),\quad ((\beta_{12},\alpha_{6,3},ce),(\beta_{12},\alpha_{6,2},be)),$$
$$((\beta_4,\alpha_{5,2},bd),(\beta_4,\alpha_{5,1},af)),\quad ((\beta_8,\alpha_{5,3},cf),(\beta_8,\alpha_{5,1},af)),\quad ((\beta_{12},\alpha_{6,3},cf),(\beta_{12},\alpha_{6,2},bf)),$$
$$((\beta_4,\alpha_{5,2},bf),(\beta_4,\alpha_{5,1},ad)),\quad ((\beta_9,\alpha_{6,3},ce),(\beta_9,\alpha_{6,1},ae)),\quad ((\beta_{13},\alpha_{5,2},de),(\beta_{13},\alpha_{5,1},df)),$$
$$((\beta_4,\alpha_{5,2},bf),(\beta_4,\alpha_{5,1},af)),\quad ((\beta_9,\alpha_{6,3},ce),(\beta_9,\alpha_{6,1},af)),\quad ((\beta_{13},\alpha_{5,2},ef),(\beta_{13},\alpha_{5,1},df)),$$
$$((\beta_5,\alpha_{6,2},be),(\beta_5,\alpha_{6,1},ae)),\quad ((\beta_9,\alpha_{6,3},cf),(\beta_9,\alpha_{6,1},ae)),\quad ((\beta_{14},\alpha_{6,2},ef),(\beta_{14},\alpha_{6,1},de)),$$
$$((\beta_5,\alpha_{6,2},be),(\beta_5,\alpha_{6,1},af)),\quad ((\beta_9,\alpha_{6,3},cf),(\beta_9,\alpha_{6,1},af)),\quad ((\beta_{14},\alpha_{6,2},ef),(\beta_{14},\alpha_{6,1},df)),$$
$$((\beta_5,\alpha_{6,2},bf),(\beta_5,\alpha_{6,1},ae)),\quad ((\beta_{10},\alpha_{4,3},cd),(\beta_{10},\alpha_{4,2},bd)),\quad ((\beta_{15},\alpha_{6,2},ef),(\beta_{15},\alpha_{6,1},de)),$$
$$((\beta_5,\alpha_{6,2},bf),(\beta_5,\alpha_{6,1},af)),\quad ((\beta_{10},\alpha_{4,3},cd),(\beta_{10},\alpha_{4,2},be)),\quad ((\beta_{15},\alpha_{6,2},ef),(\beta_{15},\alpha_{6,1},df))\}.$$

- Consider $(W'',W^\circ) = (\emptyset,1)$. The first criterion for (s,t)-goodness is that s and t have the same image under the map $\lambda^1_{W''} = \lambda^1_\emptyset$. We define $\lambda^1_\emptyset$ as follows:

$$\lambda^1_\emptyset : \delta^1_{0,1} \xrightarrow{\sim} \delta^1_{0,1} \circ \delta^1_\emptyset \xrightarrow{proj} \delta^1_\emptyset.$$

  Recall that the correspondence $\delta^j_\emptyset$ is the identity correspondence on X_j for all j. Thus, this map $\lambda^1_\emptyset$ projects onto the source of each element in $\delta^1_{0,1}$. That is, if we write an element of $\delta^1_{0,1}$ as a triple (a,b,c), its image under $\lambda^1_\emptyset$ is a. Thus, for the first (s,t)-goodness criterion, we are looking for pairs $(s,t) = ((a,b,c),(d,e,f))$

such that

$$\lambda^1_{\emptyset}(a,b,c) = \lambda^1_{\emptyset}(d,e,f)$$
$$a = d.$$

The criterion for (s,t)-positivity involves the map $\lambda^1_{W'',W^\circ} = \lambda^1_{\emptyset,1}$, which we define as follows:

$$\lambda^1_{\emptyset,1} : \delta^1_{0,1} \xrightarrow{\sim} \delta^0_0 \circ \delta^1_1 \circ \delta^1_{\emptyset} \xrightarrow{proj} \delta^1_1.$$

Note that we defined the middle composition of correspondences in $\lambda^1_{\emptyset,1}$ in the previous section as $\delta^0_0 \circ \delta^1_1$ whose elements we wrote as triples (a,b,c). Since the first map in $\lambda^1_{\emptyset,1}$ is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_1$. Then, the map $\lambda^1_{\emptyset,1}$ sends an element $(a,b,c) \in \delta^0_0 \circ \delta^1_0$ to $(a,b) \in \delta^1_0$. Thus, for the (s,t)-positivity criterion, we are looking for pairs $(s,t) = ((a,b,c),(d,e,f))$ such that

$$\lambda^1_{\emptyset,1}(a,b,c) > \lambda^1_{\emptyset,1}(d,e,f)$$
$$(a,b) > (d,e).$$

Putting these two criteria together for (s,t)-goodness and (s,t)-positivity, we have that $(\emptyset,1)$ is (s,t)-positive for the following $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, written as elements of $\delta^0_0 \circ \delta^1_1$:

$$\begin{array}{lll}
\{((\beta_1,\alpha_{1,2},ac),(\beta_1,\alpha_{1,1},ab)), & ((\beta_6,\alpha_{2,2},ab),(\beta_6,\alpha_{2,1},ac)), & ((\beta_{10},\alpha_{3,5},cf),(\beta_{10},\alpha_{3,4},be)),\\
((\beta_1,\alpha_{1,2},bc),(\beta_1,\alpha_{1,1},ab)), & ((\beta_6,\alpha_{2,2},bc),(\beta_6,\alpha_{2,1},ac)), & ((\beta_{10},\alpha_{3,5},cf),(\beta_{10},\alpha_{3,4},cd)),\\
((\beta_2,\alpha_{1,2},ac),(\beta_2,\alpha_{1,1},ab)), & ((\beta_7,\alpha_{2,5},af),(\beta_7,\alpha_{2,3},ad)), & ((\beta_{11},\alpha_{3,5},bf),(\beta_{11},\alpha_{3,4},be)),\\
((\beta_2,\alpha_{1,2},bc),(\beta_2,\alpha_{1,1},ab)), & ((\beta_7,\alpha_{2,5},af),(\beta_7,\alpha_{2,3},cd)), & ((\beta_{11},\alpha_{3,5},bf),(\beta_{11},\alpha_{3,4},ce)),\\
((\beta_3,\alpha_{1,4},ae),(\beta_3,\alpha_{1,3},ad)), & ((\beta_7,\alpha_{2,5},cf),(\beta_7,\alpha_{2,3},ad)), & ((\beta_{11},\alpha_{3,5},cf),(\beta_{11},\alpha_{3,4},be)),\\
((\beta_3,\alpha_{1,4},ae),(\beta_3,\alpha_{1,3},bd)), & ((\beta_7,\alpha_{2,5},cf),(\beta_7,\alpha_{2,3},cd)), & ((\beta_{11},\alpha_{3,5},cf),(\beta_{11},\alpha_{3,4},ce)),\\
((\beta_3,\alpha_{1,4},be),(\beta_3,\alpha_{1,3},ad)), & ((\beta_8,\alpha_{2,5},af),(\beta_8,\alpha_{2,3},ad)), & ((\beta_{12},\alpha_{3,5},bf),(\beta_{12},\alpha_{3,4},be)),\\
((\beta_3,\alpha_{1,4},be),(\beta_3,\alpha_{1,3},bd)), & ((\beta_8,\alpha_{2,5},af),(\beta_8,\alpha_{2,3},cd)), & ((\beta_{12},\alpha_{3,5},bf),(\beta_{12},\alpha_{3,4},ce)),
\end{array}$$

$$((\beta_4,\alpha_{1,4},ae),(\beta_4,\alpha_{1,3},ad)),\quad ((\beta_8,\alpha_{2,5},cf),(\beta_8,\alpha_{2,3},ad)),\quad ((\beta_{12},\alpha_{3,5},cf),(\beta_{12},\alpha_{3,4},be)),$$
$$((\beta_4,\alpha_{1,4},ae),(\beta_4,\alpha_{1,3},bd)),\quad ((\beta_8,\alpha_{2,5},cf),(\beta_8,\alpha_{2,3},cd)),\quad ((\beta_{12},\alpha_{3,5},cf),(\beta_{12},\alpha_{3,4},ce)),$$
$$((\beta_4,\alpha_{1,4},be),(\beta_4,\alpha_{1,3},ad)),\quad ((\beta_9,\alpha_{2,5},af),(\beta_9,\alpha_{2,3},ad)),\quad ((\beta_{13},\alpha_{4,5},df),(\beta_{13},\alpha_{4,4},de)),$$
$$((\beta_4,\alpha_{1,4},be),(\beta_4,\alpha_{1,3},bd)),\quad ((\beta_9,\alpha_{2,5},af),(\beta_9,\alpha_{2,3},cd)),\quad ((\beta_{13},\alpha_{4,5},ef),(\beta_{13},\alpha_{4,4},de)),$$
$$((\beta_5,\alpha_{1,4},af),(\beta_5,\alpha_{1,3},ad)),\quad ((\beta_9,\alpha_{2,5},cf),(\beta_9,\alpha_{2,3},ad)),\quad ((\beta_{14},\alpha_{4,5},df),(\beta_{14},\alpha_{4,4},de)),$$
$$((\beta_5,\alpha_{1,4},af),(\beta_5,\alpha_{1,3},bd)),\quad ((\beta_9,\alpha_{2,5},cf),(\beta_9,\alpha_{2,3},cd)),\quad ((\beta_{14},\alpha_{4,5},ef),(\beta_{14},\alpha_{4,4},de)),$$
$$((\beta_5,\alpha_{1,4},cf),(\beta_5,\alpha_{1,3},ad)),\quad ((\beta_{10},\alpha_{3,5},bf),(\beta_{10},\alpha_{3,4},be)),\quad ((\beta_{15},\alpha_{5,5},de),(\beta_{15},\alpha_{5,4},df)),$$
$$((\beta_5,\alpha_{1,4},cf),(\beta_5,\alpha_{1,3},bd)),\quad ((\beta_{10},\alpha_{3,5},bf),(\beta_{10},\alpha_{3,4},ce)),\quad ((\beta_{15},\alpha_{5,5},ef),(\beta_{15},\alpha_{5,4},df))\}.$$

- Consider $(W'',W^\circ)=(\emptyset,\{0,1\})$. The first criterion for (s,t)-goodness is that s and t have the same image under the map $\lambda^1_{W''}=\lambda^1_\emptyset$. We define $\lambda^1_\emptyset$ as follows:

$$\lambda^1_\emptyset:\delta^1_{0,1}\xrightarrow{\sim}\delta^1_{0,1}\circ\delta^1_\emptyset\xrightarrow{proj}\delta^1_\emptyset.$$

  Recall that the correspondence $\delta^j_\emptyset$ is the identity correspondence on X_j for all j. Thus, this map $\lambda^1_\emptyset$ projects onto the source of each element in $\delta^1_{0,1}$. That is, if we write an element of $\delta^1_{0,1}$ as a triple (a,b,c), its image under $\lambda^1_\emptyset$ is a. Thus, for the first (s,t)-goodness criterion, we are looking for pairs $(s,t)=((a,b,c),(d,e,f))$ such that

$$\lambda^1_\emptyset(a,b,c)=\lambda^1_\emptyset(d,e,f)$$
$$a=d.$$

  The criterion for (s,t)-positivity invovles the map $\lambda^1_{W'',W^\circ}=\lambda^1_{\emptyset,\{0,1\}}$, which we define as follows:

$$\lambda^1_{\emptyset,\{0,1\}}:\delta^1_{0,1}\xrightarrow{\sim}\delta^0_\emptyset\circ\delta^1_{0,1}\circ\delta^1_\emptyset\xrightarrow{proj}\delta^1_{0,1}.$$

  Note that this map $\lambda^1_{\emptyset,\{0,1\}}$ is the identity map on $\delta^1_{0,1}$. Thus, for the (s,t)-positivity criterion, we are looking for pairs $(s,t)=((a,b,c),(d,e,f))$ such that

$$\lambda^1_{\emptyset,\{0,1\}}(a,b,c)>\lambda^1_{\emptyset,\{0,1\}}(d,e,f)$$
$$(a,b,c)>(d,e,f).$$

Putting these two criteria together for (s,t)-goodness and (s,t)-positivity, $(\emptyset, 0)$ is (s,t)-positive for the following $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, written as elements of $\delta^0_0 \circ \delta^1_0$:

$$
\begin{array}{lll}
\{((\beta_1,\alpha_{2,1},ac),(\beta_1,\alpha_{2,2},ab)), & ((\beta_6,\alpha_{3,1},ac),(\beta_6,\alpha_{3,1},ab)), & ((\beta_{10},\alpha_{4,3},ce),(\beta_{10},\alpha_{4,3},cd)),\\
((\beta_1,\alpha_{2,2},bc),(\beta_1,\alpha_{2,1},ac)), & ((\beta_6,\alpha_{3,2},bc),(\beta_6,\alpha_{3,1},ac)), & ((\beta_{10},\alpha_{4,3},ce),(\beta_{10},\alpha_{4,2},be)),\\
((\beta_1,\alpha_{2,2},bc),(\beta_1,\alpha_{2,2},ab)), & ((\beta_6,\alpha_{3,2},bc),(\beta_6,\alpha_{3,1},ab)), & ((\beta_{10},\alpha_{4,3},ce),(\beta_{10},\alpha_{4,2},bd)),\\
((\beta_2,\alpha_{3,1},ac),(\beta_2,\alpha_{3,1},ab)), & ((\beta_7,\alpha_{4,1},ae),(\beta_7,\alpha_{4,1},ad)), & ((\beta_{11},\alpha_{5,2},bf),(\beta_{11},\alpha_{5,2},bd)),\\
((\beta_2,\alpha_{3,2},bc),(\beta_2,\alpha_{3,1},ac)), & ((\beta_7,\alpha_{4,2},cd),(\beta_7,\alpha_{4,1},ae)), & ((\beta_{11},\alpha_{5,3},cd),(\beta_{11},\alpha_{5,2},bf)),\\
((\beta_2,\alpha_{3,2},bc),(\beta_2,\alpha_{3,1},ab)), & ((\beta_7,\alpha_{4,2},cd),(\beta_7,\alpha_{4,1},ad)), & ((\beta_{11},\alpha_{5,3},cd),(\beta_{11},\alpha_{5,2},bd)),\\
((\beta_3,\alpha_{4,1},ae),(\beta_3,\alpha_{4,1},ad)), & ((\beta_7,\alpha_{4,2},ce),(\beta_7,\alpha_{4,2},cd)), & ((\beta_{11},\alpha_{5,3},cf),(\beta_{11},\alpha_{5,3},cd)),\\
((\beta_3,\alpha_{4,2},bd),(\beta_3,\alpha_{4,1},ae)), & ((\beta_7,\alpha_{4,2},ce),(\beta_7,\alpha_{4,1},ae)), & (\beta_{11},\alpha_{5,3},cf),(\beta_{11},\alpha_{5,2},bf)),\\
((\beta_3,\alpha_{4,2},bd),(\beta_3,\alpha_{4,1},ad)), & ((\beta_7,\alpha_{4,2},ce),(\beta_7,\alpha_{4,1},ad)), & ((\beta_{11},\alpha_{5,3},cf),(\beta_{11},\alpha_{5,2},bd)),\\
((\beta_3,\alpha_{4,2},be),(\beta_3,\alpha_{4,2},bd)), & ((\beta_8,\alpha_{5,1},af),(\beta_8,\alpha_{5,1},ad)), & ((\beta_{12},\alpha_{6,2},bf),(\beta_{12},\alpha_{6,2},be)),\\
((\beta_3,\alpha_{4,2},be),(\beta_3,\alpha_{4,1},ae)), & ((\beta_8,\alpha_{5,3},cd),(\beta_8,\alpha_{5,1},af)), & ((\beta_{12},\alpha_{6,3},ce),(\beta_{12},\alpha_{6,2},bf)),\\
((\beta_3,\alpha_{4,2},be),(\beta_3,\alpha_{4,1},ad)), & ((\beta_8,\alpha_{5,3},cd),(\beta_8,\alpha_{5,1},ad)), & ((\beta_{12},\alpha_{6,3},ce),(\beta_{12},\alpha_{6,2},be)),\\
((\beta_4,\alpha_{5,1},af),(\beta_4,\alpha_{5,1},ad)), & ((\beta_8,\alpha_{5,3},cf),(\beta_8,\alpha_{5,3},cd)), & ((\beta_{12},\alpha_{6,3},cf),(\beta_{12},\alpha_{6,3},ce)),\\
((\beta_4,\alpha_{5,2},bd),(\beta_4,\alpha_{5,1},af)), & ((\beta_8,\alpha_{5,3},cf),(\beta_8,\alpha_{5,1},af)), & ((\beta_{12},\alpha_{6,3},cf),(\beta_{12},\alpha_{6,2},bf)),\\
((\beta_4,\alpha_{5,2},bd),(\beta_4,\alpha_{5,1},ad)), & ((\beta_8,\alpha_{5,3},cf),(\beta_8,\alpha_{5,1},ad)), & ((\beta_{12},\alpha_{6,3},cf),(\beta_{12},\alpha_{6,2},be)),\\
((\beta_4,\alpha_{5,2},bf),(\beta_4,\alpha_{5,2},bd)), & ((\beta_9,\alpha_{6,1},af),(\beta_9,\alpha_{6,1},ae)), & ((\beta_{13},\alpha_{5,4},df),(\beta_{13},\alpha_{5,5},de)),\\
((\beta_4,\alpha_{5,2},bf),(\beta_4,\alpha_{5,1},af)), & ((\beta_9,\alpha_{6,3},ce),(\beta_9,\alpha_{6,1},af)), & ((\beta_{13},\alpha_{5,5},ef),(\beta_{13},\alpha_{5,4},df)),\\
((\beta_4,\alpha_{5,2},bf),(\beta_4,\alpha_{5,1},ad)), & ((\beta_9,\alpha_{6,3},ce),(\beta_9,\alpha_{6,1},ae)), & ((\beta_{13},\alpha_{5,5},ef),(\beta_{13},\alpha_{5,5},de)),\\
((\beta_5,\alpha_{1,4},be),(\beta_5,\alpha_{1,3},ae)), & ((\beta_9,\alpha_{6,3},cf),(\beta_9,\alpha_{6,3},ce)), & ((\beta_{14},\alpha_{6,4},df),(\beta_{14},\alpha_{6,4},de)),\\
((\beta_5,\alpha_{1,4},be),(\beta_5,\alpha_{1,3},af)), & ((\beta_9,\alpha_{6,3},cf),(\beta_9,\alpha_{6,1},af)), & ((\beta_{14},\alpha_{6,5},ef),(\beta_{14},\alpha_{6,4},df)),\\
((\beta_5,\alpha_{1,4},bf),(\beta_5,\alpha_{1,3},ae)), & ((\beta_9,\alpha_{6,3},cf),(\beta_9,\alpha_{6,1},ae)), & ((\beta_{14},\alpha_{6,5},ef),(\beta_{14},\alpha_{6,4},de)),\\
((\beta_5,\alpha_{1,4},bf),(\beta_5,\alpha_{1,3},af)), & ((\beta_{10},\alpha_{4,2},be),(\beta_{10},\alpha_{4,2},bd)), & ((\beta_{15},\alpha_{6,4},df),(\beta_{15},\alpha_{6,4},de)),\\
((\beta_5,\alpha_{2,2},bc),(\beta_5,\alpha_{2,1},ab)), & ((\beta_{10},\alpha_{4,3},cd),(\beta_{10},\alpha_{4,2},be)), & ((\beta_{15},\alpha_{6,5},ef),(\beta_{15},\alpha_{6,4},df)),\\
((\beta_5,\alpha_{2,2},bc),(\beta_5,\alpha_{2,1},ac)), & ((\beta_{10},\alpha_{4,3},cd),(\beta_{10},\alpha_{4,2},bd)), & ((\beta_{15},\alpha_{6,5},ef),(\beta_{15},\alpha_{6,4},de))\}.
\end{array}
$$

- Consider $(W'', W^\circ) = (0, 1)$. The first criterion for (s, t)-goodness is that s and t have the same image under the map $\lambda^1_{W''} = \lambda^1_0$. We define λ^1_0 as follows:

$$\lambda^1_0 : \delta^1_{0,1} \xrightarrow{\sim} \delta^0_{\psi_0(1)} \circ \delta^1_0 \xrightarrow{proj} \delta^1_0.$$

  Recall that by definition we have $\psi_0(1) = 0$. Note that we defined the middle composition of correspondences in λ^1_0 in the previous section as $\delta^0_0 \circ \delta^1_0$ whose elements we wrote as triples (a, b, c). Since the first map in λ^1_0 is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_0$. Then, the map λ^1_0 sends an element $(a, b, c) \in \delta^0_0 \circ \delta^1_0$ to $(a, b) \in \delta^1_0$. Thus, for the first (s, t)-goodness criterion, we are looking for pairs $(s, t) = ((a, b, c), (d, e, f))$ such that

$$\begin{aligned} \lambda^1_{\emptyset,0}(a, b, c) &= \lambda^1_{\emptyset,0}(d, e, f) \\ (a, b) &= (d, e). \end{aligned}$$

  The criterion for (s, t)-positivity involves the map $\lambda^1_{W'',W^\circ} = \lambda^1_{0,1}$, which we define as follows:

$$\lambda^1_{0,1} : \delta^1_{0,1} \xrightarrow{\sim} \delta^{-1}_\emptyset \circ \delta^0_{\psi_0(1)} \circ \delta^1_0 \xrightarrow{proj} \delta^0_{\psi_0(1)}.$$

  Again, we recall that we have $\psi_0(1) = 0$. Note that the middle composition $\delta^{-1}_\emptyset \circ \delta^0_{\psi_0(1)} \circ \delta^1_0$ in $\lambda^1_{0,1}$ is equivalent to $\delta^0_{\psi_0(1)} \circ \delta^1_0$, whose elements we wrote as triples (a, b, c). Since the first map in $\lambda^1_{0,1}$ is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_0$. Then, we say that the map $\lambda^1_{0,1}$ sends an element $(a, b, c) \in \delta^0_0 \circ \delta^1_0$ to $(b, c) \in \delta^0_0$. Thus, for the (s, t)-positivity criterion, we are looking for pairs $(s, t) = ((a, b, c), (d, e, f))$ such that

$$\begin{aligned} \lambda^1_{0,1}(a, b, c) &> \lambda^1_{0,1}(d, e, f) \\ (b, c) &> (e, f). \end{aligned}$$

  Putting these two criteria together for (s, t)-goodness and (s, t)-positivity, $(0, 1)$ is (s, t)-positive for the following $(s, t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, written as elements of $\delta^0_0 \circ \delta^1_0$:

$$\{((\beta_1, \alpha_{2,2}, bc), (\beta_1, \alpha_{2,2}, ab)), \quad ((\beta_6, \alpha_{3,1}, ac), (\beta_6, \alpha_{3,1}, ab)), \ ((\beta_{10}, \alpha_{4,3}, ce), (\beta_{10}, \alpha_{4,3}, cd)),$$

$((\beta_2, \alpha_{3,1}, ac), (\beta_2, \alpha_{3,1}, ab))$, $((\beta_7, \alpha_{4,1}, ae), (\beta_7, \alpha_{4,1}, ad))$, $((\beta_{11}, \alpha_{5,2}, bf), (\beta_{11}, \alpha_{5,2}, bd))$,
$((\beta_3, \alpha_{4,1}, ae), (\beta_3, \alpha_{4,1}, ad))$, $((\beta_7, \alpha_{4,3}, ce), (\beta_7, \alpha_{4,3}, cd))$, $((\beta_{11}, \alpha_{5,3}, cf), (\beta_{11}, \alpha_{5,3}, cd))$,
$((\beta_3, \alpha_{4,2}, bd), (\beta_3, \alpha_{4.2}, be))$, $((\beta_8, \alpha_{5,1}, af), (\beta_8, \alpha_{5,1}, ad))$, $((\beta_{12}, \alpha_{6,2}, bf), (\beta_{12}, \alpha_{6,2}, be))$,
$((\beta_4, \alpha_{5,1}, af), (\beta_4, \alpha_{5,1}, ad))$, $((\beta_8, \alpha_{5,3}, cf), (\beta_8, \alpha_{5,3}, cd))$, $((\beta_{12}, \alpha_{6,3}, cf), (\beta_{12}, \alpha_{6,3}, ce))$,
$((\beta_4, \alpha_{5,2}, bf), (\beta_4, \alpha_{5,2}, bd))$, $((\beta_9, \alpha_{6,1}, af), (\beta_9, \alpha_{6,1}, ae))$, $((\beta_{13}, \alpha_{5,5}, ef), (\beta_{13}, \alpha_{5,5}, de))$,
$((\beta_5, \alpha_{6,1}, af), (\beta_5, \alpha_{6,1}, ae))$, $((\beta_9, \alpha_{6,3}, cf), (\beta_9, \alpha_{6,3}, ce))$, $((\beta_{14}, \alpha_{6,4}, df), (\beta_{14}, \alpha_{6,4}, de))$,
$((\beta_5, \alpha_{6,2}, bf), (\beta_5, \alpha_{6,2}, be))$, $((\beta_{10}, \alpha_{4,2}, be), (\beta_{10}, \alpha_{4,2}, bd))$, $((\beta_{15}, \alpha_{6,4}, df), (\beta_{15}, \alpha_{6,4}, de))\}$.

- Finally, consider $(W'', W^\circ) = (1, 0)$. The first criterion for (s, t)-goodness is that s and t have the same image under the map $\lambda^1_{W''} = \lambda^1_1$. We define λ^1_1 as follows:

$$\lambda^1_1 : \delta^1_{0,1} \xrightarrow{\sim} \delta^0_0 \circ \delta^1_1 \xrightarrow{proj} \delta^1_1.$$

  Recall that we defined the middle composition of correspondences in λ^1_1 in the previous section as $\delta^0_0 \circ \delta^1_1$ whose elements we wrote as triples (a, b, c). Since the first map in λ^1_1 is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_1$. Then, the map λ^1_1 sends an element $(a, b, c) \in \delta^0_0 \circ \delta^1_1$ to $(a, b) \in \delta^1_1$. Thus, for the first (s, t)-goodness criterion, we are looking for pairs $(s, t) = ((a, b, c), (d, e, f))$ such that

$$\lambda^1_{\emptyset,0}(a, b, c) = \lambda^1_{\emptyset,0}(d, e, f)$$
$$(a, b) = (d, e).$$

  The criterion for (s, t)-positivity involves the map $\lambda^1_{W'',W^\circ} = \lambda^1_{1,0}$, which we define as follows:

$$\lambda^1_{1,0} : \delta^1_{0,1} \xrightarrow{\sim} \delta^{-1}_\emptyset \circ \delta^0_0 \circ \delta^1_1 \xrightarrow{proj} \delta^0_0.$$

  Note that the middle composition $\delta^{-1}_\emptyset \circ \delta^0_0 \circ \delta^1_1$ in $\lambda^1_{1,0}$ is equivalent to $\delta^0_0 \circ \delta^1_1$, whose elements we wrote as triples (a, b, c). Since the first map in $\lambda^1_{1,0}$ is a bijection, we can think of $\delta^1_{0,1}$ as simply being $\delta^0_0 \circ \delta^1_1$. Then, we say that the map $\lambda^1_{1,0}$ sends an element $(a, b, c) \in \delta^0_0 \circ \delta^1_1$ to $(b, c) \in \delta^0_0$. Thus, for the (s, t)-positivity criterion, we

are looking for pairs $(s,t) = ((a,b,c),(d,e,f))$ such that

$$\lambda^1_{0,1}(a,b,c) > \lambda^1_{0,1}(d,e,f)$$
$$(b,c) > (e,f).$$

Putting these two criteria together for (s,t)-goodness and (s,t)-positivity, $(1,0)$ is (s,t)-positive for the following $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, written as elements of $\delta^0_0 \circ \delta^1_1$:

$$\begin{array}{lll}
\{((\beta_1,\alpha_{1,2},bc),(\beta_1,\alpha_{1,2},ac)), & ((\beta_6,\alpha_{2,2},bc),(\beta_6,\alpha_{2,2},ab)), & ((\beta_{10},\alpha_{3,5},cf),(\beta_{10},\alpha_{3,5},bf)),\\
((\beta_2,\alpha_{1,2},bc),(\beta_2,\alpha_{1,2},ac)), & ((\beta_7,\alpha_{2,3},cd),(\beta_7,\alpha_{2,3},ad)), & ((\beta_{11},\alpha_{3,4},ce),(\beta_{11},\alpha_{3,4},be)),\\
((\beta_3,\alpha_{1,3},bd),(\beta_3,\alpha_{1,3},ad)), & ((\beta_7,\alpha_{2,5},cf),(\beta_7,\alpha_{2,5},af)), & ((\beta_{11},\alpha_{3,5},cf),(\beta_{11},\alpha_{3,5},bf)),\\
((\beta_3,\alpha_{1,4},be),(\beta_3,\alpha_{1,4},ae)), & ((\beta_8,\alpha_{2,3},cd),(\beta_8,\alpha_{2,3},ad)), & ((\beta_{12},\alpha_{3,4},ce),(\beta_{12},\alpha_{3,4},be)),\\
((\beta_4,\alpha_{1,3},bd),(\beta_4,\alpha_{1,3},ad)), & ((\beta_8,\alpha_{2,5},cf),(\beta_8,\alpha_{2,5},af)), & ((\beta_{12},\alpha_{3,5},cf),(\beta_{12},\alpha_{3,5},bf)),\\
((\beta_4,\alpha_{1,4},be),(\beta_4,\alpha_{1,4},ae)), & ((\beta_9,\alpha_{2,3},cd),(\beta_9,\alpha_{2,3},ad)), & ((\beta_{13},\alpha_{4,5},ef),(\beta_{13},\alpha_{4,5},df)),\\
((\beta_5,\alpha_{1,3},bd),(\beta_5,\alpha_{1,3},ad)), & ((\beta_9,\alpha_{2,5},cf),(\beta_9,\alpha_{2,5},af)), & ((\beta_{14},\alpha_{4,5},ef),(\beta_{14},\alpha_{4,5},df)),\\
((\beta_5,\alpha_{1,4},be),(\beta_5,\alpha_{1,4},ae)), & ((\beta_{10},\alpha_{3,4},ce),(\beta_{10},\alpha_{3,4},be)), & ((\beta_{15},\alpha_{5,5},ef),(\beta_{15},\alpha_{5,5},de))\}.
\end{array}$$

For each pair of subsets (W'',W°), we now know whether they are (s,t)-positive for each $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$. Now, for each (s,t), we arrange the (W'',W°) that are (s,t)-positive into maximal chains. Recall that each maximal chain has length $|U^- \cap U^+| = 2$, and the top element of each maximal chain is always $(\emptyset,\{0,1\})$. To do this, for each (s,t), we find the (W'',W°) that are (s,t)-positive and, if possible, arrange these pairs into length 2 maximal chains with maximal element as just mentioned.

Based on our work above, for each pair $(s,t) \in \delta^1_{0,1} \times \delta^1_{0,1}$, we can describe the set $\mathcal{O}_{s,t}(\{0,1\})^+$, which contains the positive, maximal chains. We present this information in Table 5.2, which lists each relevant (s,t) pair, its associated set $\mathcal{O}_{s,t}(\{0,1\})^+$, the composition $source \circ \Delta_1$, which gives the source of each positive maximal chain pre-composed with the diagonal correspondence, and the target of each positive maximal chain. Recall that the source of an element in $\mathcal{O}_{s,t}(\{0,1\})^+$ is the tuple containing the sources of s and t, and the target of an element in $\mathcal{O}_{s,t}(\{0,1\})^+$ is the tuple containing the targets of s and t. We use this table of elements to define $\binom{1}{4} \circ \Delta_1$.

(s,t)	$\mathcal{O}_{s,t}(\{0,1\})^+$	$source \circ \Delta_3$	$target$
$((\beta_1, ac), (\beta_1, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\})$	β_1	(ac, ab)
$((\beta_1, bc), (\beta_1, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{1,0\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_1	(bc, ac)
$((\beta_1, bc), (\beta_1, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_1	(bc, ab)
$((\beta_2, ac), (\beta_2, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_2	(ac, ab)
$((\beta_2, bc), (\beta_2, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_2	(bc, ac)
$((\beta_2, bc), (\beta_2, ab))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_2	(bc, ab)
$((\beta_3, ae), (\beta_3, ad))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_3	(ae, ad)
$((\beta_3, bd), (\beta_3, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\})\}$	β_3	(bd, ae)
$((\beta_3, bd), (\beta_3, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_3	(bd, ad)
$((\beta_3, be), (\beta_3, bd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_3	(be, bd)
$((\beta_3, be), (\beta_3, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_3	(be, ae)
$((\beta_3, be), (\beta_3, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_3	(be, ad)
$((\beta_4, af), (\beta_4, ad))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_4	(af, ad)
$((\beta_4, bd), (\beta_4, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\})\}$	β_4	(bd, af)
$((\beta_4, bd), (\beta_4, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_4	(bd, ad)
$((\beta_4, bf), (\beta_4, bd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_4	(bf, bd)
$((\beta_4, bf), (\beta_4, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_4	(bf, af)
$((\beta_4, bf), (\beta_4, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_4	(bf, ad)
$((\beta_5, af), (\beta_5, ae))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_5	(af, ae)
$((\beta_5, be), (\beta_5, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\})\}$	β_5	(be, af)
$((\beta_5, be), (\beta_5, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_5	(be, ae)
$((\beta_5, bf), (\beta_5, be))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_5	(bf, be)
$((\beta_5, bf), (\beta_5, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_5	(bf, af)
$((\beta_5, bf), (\beta_5, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_5	(bf, ae)
$((\beta_6, ac), (\beta_6, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_6	(ac, ab)
$((\beta_6, bc), (\beta_6, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_6	(bc, ac)
$((\beta_6, bc), (\beta_6, ab))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_6	(bc, ab)
$((\beta_7, ae), (\beta_7, ad))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_7	(ae, ad)
$((\beta_7, cd), (\beta_7, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\})\}$	β_7	(cd, ae)
$((\beta_7, cd), (\beta_7, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_7	(cd, ad)

$((\beta_7, ce), (\beta_7, cd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_7	(ce, cd)
$((\beta_7, ce), (\beta_7, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_7	(ce, ae)
$((\beta_7, ce), (\beta_7, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (\emptyset, 1) \prec (\emptyset, \{0, 1\})\}$	β_7	(ce, ad)
$((\beta_8, af), (\beta_8, ad))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_8	(af, ad)
$((\beta_8, cd), (\beta_8, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\})\}$	β_8	(cd, af)
$((\beta_8, cd), (\beta_8, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_8	(cd, ad)
$((\beta_8, cf), (\beta_8, cd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_8	(cf, cd)
$((\beta_8, cf), (\beta_8, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_8	(cf, af)
$((\beta_8, cf), (\beta_8, ad))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (\emptyset, 1) \prec (\emptyset, \{0, 1\})\}$	β_8	(cf, ad)
$((\beta_9, af), (\beta_9, ae))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_9	(af, ae)
$((\beta_9, ce), (\beta_9, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\})\}$	β_9	(ce, af)
$((\beta_9, ce), (\beta_9, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_9	(ce, ae)
$((\beta_9, cf), (\beta_9, ce))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_9	(cf, ce)
$((\beta_9, cf), (\beta_9, af))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_9	(cf, af)
$((\beta_9, cf), (\beta_9, ae))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (\emptyset, 1) \prec (\emptyset, \{0, 1\})\}$	β_9	(cf, ae)
$((\beta_{10}, be), (\beta_{10}, bd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(be, bd)
$((\beta_{10}, cd), (\beta_{10}, be))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(cd, be)
$((\beta_{10}, cd), (\beta_{10}, bd))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(cd, bd)
$((\beta_{10}, ce), (\beta_{10}, cd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(ce, cd)
$((\beta_{10}, ce), (\beta_{10}, be))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(ce, be)
$((\beta_{10}, ce), (\beta_{10}, bd))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (\emptyset, 1) \prec (\emptyset, \{0, 1\})\}$	β_{10}	(ce, bd)
$((\beta_{11}, bf), (\beta_{11}, bd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(bf, bd)
$((\beta_{11}, cd), (\beta_{11}, bf))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(cd, bf)
$((\beta_{11}, cd), (\beta_{11}, bd))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(cd, bd)
$((\beta_{11}, cf), (\beta_{11}, cd))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(cf, cd)
$((\beta_{11}, cf), (\beta_{11}, bf))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(cf, bf)
$((\beta_{11}, cf), (\beta_{11}, bd))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (\emptyset, 1) \prec (\emptyset, \{0, 1\})\}$	β_{11}	(cf, bd)
$((\beta_{12}, bf), (\beta_{12}, be))$	$\{(\emptyset, 1) \prec (\emptyset, \{0, 1\}), (0, 1) \prec (\emptyset, \{0, 1\})\}$	β_{12}	(bf, be)
$((\beta_{12}, ce), (\beta_{12}, bf))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\})\}$	β_{12}	(ce, bf)
$((\beta_{12}, ce), (\beta_{12}, be))$	$\{(\emptyset, 0) \prec (\emptyset, \{0, 1\}), (1, 0) \prec (\emptyset, \{0, 1\})\}$	β_{12}	(ce, be)

$((\beta_{12}, cf), (\beta_{12}, ce))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_{12}	(cf, ce)
$((\beta_{12}, cf), (\beta_{12}, bf))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_{12}	(cf, bf)
$((\beta_{12}, cf), (\beta_{12}, be))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_{12}	(cf, be)
$((\beta_{13}, ac), (\beta_{13}, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_{13}	(ac, ab)
$((\beta_{13}, bc), (\beta_{13}, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_{13}	(bc, ac)
$((\beta_{13}, bc), (\beta_{13}, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_{13}	(bc, ab)
$((\beta_{14}, ac), (\beta_{14}, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_{14}	(ac, ab)
$((\beta_{14}, bc), (\beta_{14}, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_{14}	(bc, ac)
$((\beta_{14}, bc), (\beta_{14}, ab))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_{14}	(bc, ab)
$((\beta_{15}, ac), (\beta_{15}, ab))$	$\{(\emptyset, 1) \prec (\emptyset, \{0,1\}), (0,1) \prec (\emptyset, \{0,1\})\}$	β_{15}	(ac, ab)
$((\beta_{15}, bc), (\beta_{15}, ac))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (1,0) \prec (\emptyset, \{0,1\})\}$	β_{15}	(bc, ac)
$((\beta_{15}, bc), (\beta_{15}, ab))$	$\{(\emptyset, 0) \prec (\emptyset, \{0,1\}), (\emptyset, 1) \prec (\emptyset, \{0,1\})\}$	β_{15}	(bc, ab)

Table 5.2: The Correspondence $\left(\delta^1_{0,1} \wedge \delta^1_{0,1}\right) \circ \Delta_1$

5.2.4 Taking the $(\mathbb{Z}/2)$-Linearization to get ∇_1

In the previous section, we defined the correspondence

$$\begin{aligned} \nabla^{\binom{1}{4}} \circ \Delta_1 &= \left(\delta^1_{0,1} \wedge \delta^1_{0,1}\right) \circ \Delta_1 \\ &= \left(\delta^1_{0,1} \wedge \delta^1_{0,1}\right) \circ \Delta_1. \end{aligned}$$

Now, in this section, we take the $(\mathbb{Z}/2)$-linearization to get the homomorphism ∇_{-3} restricted to the domain C_1, which we will use to define $\mathfrak{sq}^2$. Recall that $\mathcal{A}_{\mathbb{Z}/2}$ counts the number of ways, modulo 2, to get from an element in the image of the source map to an element in the image of the target map. In particular, for the map

$$\nabla_{-3}|_{C_1} = \mathcal{A}_{\mathbb{Z}/2}\left(\nabla^{\binom{1}{4}} \circ \Delta_1\right),$$

we define ∇_{-3} applied to an element $(\alpha) \in C_1$ as:

$$\nabla_{-3}(\alpha) = \sum_{\beta \in C_{-1}} |source^{-1}(\alpha) \cap target^{-1}(\beta)|\, \beta,$$

where *source* and *target* are the source and target maps, respectively, of the correspondence $\nabla_{(4)}^{(1)} \circ \Delta_1$.

Based on the work in the previous section summarized in Table 5.2, we define ∇_{-3} as making the following assignments on the generators of C_1:

$$\nabla_{-3}(\beta_1) = (ac, ab) + 2(bc, ac) + 2(bc, ab) = (ac, ab)$$
$$\nabla_{-3}(\beta_2) = 2(ac, ab) + 2(bc, ac) + 2(bc, ab) = 0$$
$$\nabla_{-3}(\beta_3) = 2(ae, ad) + (bd, ae) + 2(bd, ad) + 2(be, bd) + 2(be, ae) + 2(be, ad) = (bd, ae)$$
$$\nabla_{-3}(\beta_4) = 2(ae, ad) + (bd, ae) + 2(bd, ad) + 2(be, bd) + 2(be, ae) + 2(be, ad) = (bd, ae)$$
$$\nabla_{-3}(\beta_5) = 2(ae, ad) + (bd, ae) + 2(bd, ad) + 2(be, bd) + 2(be, ae) + 2(be, ad) = (bd, ae)$$
$$\nabla_{-3}(\beta_6) = 2(ac, ab) + 2(bc, ac) + 2(bc, ab) = 0$$
$$\nabla_{-3}(\beta_7) = 2(af, ad) + (cd, af) + 2(cd, ad) + 2(cf, cd) + 2(cf, af) + 2(cf, ad) = (cd, af)$$
$$\nabla_{-3}(\beta_8) = 2(af, ad) + (cd, af) + 2(cd, ad) + 2(cf, cd) + 2(cf, af) + 2(cf, ad) = (cd, af)$$
$$\nabla_{-3}(\beta_9) = 2(af, ad) + (cd, af) + 2(cd, ad) + 2(cf, cd) + 2(cf, af) + 2(cf, ad) = (cd, af)$$
$$\nabla_{-3}(\beta_{10}) = 2(bf, be) + (ce, bf) + 2(ce, be) + 2(cf, 8e) + 2(cf, bf) + 2(cf, be) = (ce, bf)$$
$$\nabla_{-3}(\beta_{11}) = 2(bf, be) + (ce, bf) + 2(ce, be) + 2(cf, 8e) + 2(cf, bf) + 2(cf, be) = (ce, bf)$$
$$\nabla_{-3}(\beta_{12}) = 2(bf, be) + (ce, bf) + 2(ce, be) + 2(cf, 8e) + 2(cf, bf) + 2(cf, be) = (ce, bf)$$
$$\nabla_{-3}(\beta_{13}) = (df, de) + 2(ef, df) + 2(ef, de) = (df, de)$$
$$\nabla_{-3}(\beta_{14}) = 2(df, de) + 2(ef, df) + 2(ef, de) = 0$$
$$\nabla_{-3}(\beta_{15}) = 2(df, de) + 2(ef, df) + 2(ef, de) = 0$$

5.2.5 The Steenrod Square $\mathfrak{sq}^2$

With all of the work from the previous sections, we can now finally define the Steenrod square

$$\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T; \mathbb{Z}/2) \to H^{-4,-14}(T \sqcup T; \mathbb{Z}/2).$$

Recall that these cohomology groups are given as follows:

$$H^{-6,-14}(T \sqcup T; \mathbb{Z}/2) = \mathbb{Z}/2\langle [ad + ae + af + bd + be + bf + cd + ce + cf] \rangle$$
$$H^{-4,-14}(T \sqcup T; \mathbb{Z}/2) = \mathbb{Z}/2\langle [\beta_3] \rangle,$$

where $[\beta_3] = [\beta_4] = [\beta_5] = [\beta_7] = [\beta_8] = [\beta_9] = [\beta_{10}] = [\beta_{11}] = [\beta_{12}]$.

We will start by taking the dual of

$$\nabla_{-3} : C_1 \to C_{-1} \otimes C_{-1}$$

to get

$$\smile_{-3}\colon C^{-1} \otimes C^{-1} \to C^1.$$

Note that the image of ∇_{-3} hits each of $(bd, ae), (cd, af)$, and (ce, bf) three times and hits each of (ac, ab) and (df, de) exactly once. This means that when we take the dual of ∇_{-3}, we get the following assignments:

$$\begin{aligned}
bd \smile_{-3} ae &= \beta_3 + \beta_4 + \beta_5 \\
cd \smile_{-3} af &= \beta_7 + \beta_8 + \beta_9 \\
cf \smile_{-3} be &= \beta_{10} + \beta_{11} + \beta_{12} \\
ac \smile_{-3} ab &= \beta_1 \\
df \smile_{-3} de &= \beta_{13}.
\end{aligned}$$

Now, to determine the value of $\mathfrak{sq}^2$ on the cohomology class $[ad + ae + af + bd + be + bf + cd + ce + cf] \in H^{-6,-14}$, we start with the following computation:

$$\begin{aligned}
&(ad+ae+af+bd+be+bf+cd+ce+cf) \smile_{-3} (ad+ae+af+bd+be+bf+cd+ce+cf) \\
&\quad = (ad \smile_{-3} ad) + (ad \smile_{-3} ae) + (ad \smile_{-3} af) + (ad \smile_{-3} bd) + (ad \smile_{-3} be) + (ad \smile_{-3} bf) + (ad \smile_{-3} cd) \\
&\qquad + (ad \smile_{-3} ce) + (ad \smile_{-3} cf) + (ae \smile_{-3} ad) + (ae \smile_{-3} ae) + (ae \smile_{-3} af) + (ae \smile_{-3} bd) + (ae \smile_{-3} be) \\
&\qquad + (ae \smile_{-3} bf) + (ae \smile_{-3} cd) + (ae \smile_{-3} ce) + (ae \smile_{-3} cf) + (af \smile_{-3} ad) + (af \smile_{-3} ae) + (af \smile_{-3} af) \\
&\qquad + (af \smile_{-3} bd) + (af \smile_{-3} be) + (af \smile_{-3} bf) + (af \smile_{-3} cd) + (af \smile_{-3} ce) + (af \smile_{-3} cf) + (bd \smile_{-3} ad) \\
&\qquad + (bd \smile_{-3} ae) + (bd \smile_{-3} af) + (bd \smile_{-3} bd) + (bd \smile_{-3} be) + (bd \smile_{-3} bf) + (bd \smile_{-3} cd) + (bd \smile_{-3} ce) \\
&\qquad + (bd \smile_{-3} cf) + (be \smile_{-3} ad) + (be \smile_{-3} ae) + (be \smile_{-3} af) + (be \smile_{-3} bd) + (be \smile_{-3} be) + (be \smile_{-3} bf) \\
&\qquad + (be \smile_{-3} cd) + (be \smile_{-3} ce) + (be \smile_{-3} cf) + (bf \smile_{-3} ad) + (bf \smile_{-3} ae) + (bf \smile_{-3} af) + (bf \smile_{-3} bd) \\
&\qquad + (bf \smile_{-3} be) + (bf \smile_{-3} bf) + (bf \smile_{-3} cd) + (bf \smile_{-3} ce) + (bf \smile_{-3} cf) + (cd \smile_{-3} ad) + (cd \smile_{-3} ae) \\
&\qquad + (cd \smile_{-3} af) + (cd \smile_{-3} bd) + (cd \smile_{-3} be) + (cd \smile_{-3} bf) + (cd \smile_{-3} cd) + (cd \smile_{-3} ce) + (cd \smile_{-3} cf) \\
&\qquad + (ce \smile_{-3} ad) + (ce \smile_{-3} ae) + (ce \smile_{-3} af) + (ce \smile_{-3} bd) + (ce \smile_{-3} be) + (ce \smile_{-3} bf) + (ce \smile_{-3} cd) \\
&\qquad + (ce \smile_{-3} ce) + (ce \smile_{-3} cf) + (cf \smile_{-3} ad) + (cf \smile_{-3} ae) + (cf \smile_{-3} af) + (cf \smile_{-3} bd) + (cf \smile_{-3} be)
\end{aligned}$$

$$
\begin{aligned}
&+ (cf \smile_{-3} bf) + (cf \smile_{-3} cd) + (cf \smile_{-3} ce) + (cf \smile_{-3} cf) \\
=&(bd \smile_{-3} ae) + (cd \smile_{-3} af) + (cf \smile_{-3} be) \\
=&(\beta_3 + \beta_4 + \beta_5) + (\beta_7 + \beta_8 + \beta_9) + (\beta_{10} + \beta_{11} + \beta_{12}).
\end{aligned}
$$

Note that all of the terms in the last line above are in the same cohomology class, namely $[\beta_3]$ of $H^{-4,-14}(T \sqcup T; \mathbb{Z}/2)$. Thus, when we pass from the cochain group $C^{-4,-14}$ to the cohomology group $H^{-4,-14}$, we get the value for $\mathfrak{sq}^2([ad + ae + af + bd + be + bf + cd + ce + cf])$ to be:

$$
\begin{aligned}
&\mathfrak{sq}^2([ad + ae + af + bd + be + bf + cd + ce + cf]) \\
&= [(ad + ae + af + bd + be + bf + cd + ce + cf) \smile_{-3} \\
&\qquad\qquad (ad + ae + af + bd + be + bf + cd + ce + cf)] \\
&= [\beta_3 + \beta_4 + \beta_5 + \beta_7 + \beta_8 + \beta_9 + \beta_{10} + \beta_{11} + \beta_{12}] \\
&= [\beta_3] + [\beta_4] + [\beta_5] + [\beta_7] + [\beta_8] + [\beta_9] + [\beta_{10}] + [\beta_{11}] + [\beta_{12}] \\
&= 9 \cdot [\beta_3] \\
&\equiv [\beta_3]
\end{aligned}
$$

That is, $\mathfrak{sq}^2$ takes the generator of $H^{-6,-14}(T \sqcup T; \mathbb{Z}/2)$ to the generator of $H^{-4,-14}(T \sqcup T; \mathbb{Z}/2)$ and thus is nonzero.

Chapter 6

The Implementation of $\mathfrak{sq}^i$ and Some Initial Results

For this project, we implemented code in SageMath to compute the Steenrod square $\mathfrak{sq}^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2)$ as constructed by Morán. This code forms a package called `KhovanovSteenrodSquare` and is available at `https://github.com/samilstein/KhovanovSteenrodSquare`.

6.1 Overview of `KhovanovSteenrodSqare`

To construct this code, we relied on some of SageMath's already-implemented link capabilities [SD22b]. We also pulled some functions directly from Lipshitz and Sarkar's `KhovanovSteenrod` package, which can be found at `https://github.com/sucharit/KhovanovSteenrod`. The functions in `KhovanovSteenrodSquare` that come from or are based on functions from these other sources are marked as such in the code. Note that Lipshitz and Sarkar's `KhovanovSteenrod` does not work for a link if any of its components are the unknot or the Hopf link. Because our code relies on theirs, then our code `KhovanovSteenrodSquare` also does not work for any such links. However, our code does work for any other oriented link.

Now, to use the code in `KhovanovSteenrodSquare` to compute a Steenrod square

$$\mathfrak{sq}^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2),$$

the user specifies their oriented link L via its planar diagram code or Dowker-Thistlewaite code, which are standardized formats for describing links, the desired Steenrod map index i, and the cohomology degree p and quantum degree q of the domain homology group. The code will then compute the corresponding map $\mathfrak{sq}^i$, following Morán's construction described above. It will return the rank of this map, and for each cohomology class in $H^{p,q}$, it will return the image of that element as a linear combination of the cohomology classes in $H^{p+i,q}$.

6.1.1 Description of the Code

The code in `KhovanovSteenrodSquare` defines two Sagemath classes. The first is the `Link` class. Each instance of this class represents an oriented link. When we instantiate a `Link` object, we determine its n-cube diagram and its enhanced states. The code that computes these is based on functions from Lipshitz and Sarkar's `KhovanovSteenrod`.

The other class in `KhovanovSteenrodSquare` is the `KhovanovComplex` class. Each instance of this class represents the Khovanov complex for an oriented link. One of the data of this class is the `Link` object for the oriented link we are considering. This class has a method called `compute_sqi()` for computing the Steenrod square

$$\mathfrak{sq}^i : H^{p,q}(L;\mathbb{Z}/2) \to H^{p+i,q}(L;\mathbb{Z}/2)$$

on the Khovanov homology of the given link.

There are three main functions, which are all methods on the `KhovanovComplex` class, that do the work of computing the Steenrod square in `compute_sqi()`. The first is called `define_khovanov_complex()`, which takes a quantum degree q as input, and, as the name suggests, it defines the Khovanov complex in that quantum degree. The content of this function is pulled directly from the function `_khovanov_homology_cached()` in SageMath, except that we removed the last two lines that take the Khovanov complex that this function has defined so far and compute its homology [SD22a].

The second main function in `compute_sqi()` is called `nabla_choose()`. This function takes as input the Steenrod square index of interest i, the cohomological degree p of $\mathfrak{sq}^i$'s domain group, and the quantum degree q of the map's domain and range groups. Its output is the dual mp to Morán's map $\sum_k \left(\nabla^{\binom{k}{n}} \circ \Delta_k\right)$. Since this is a method on

the `KhovanovComplex` class it already knows the link of interest, as well as its Khovanov complex in the given quantum degree q.

The work of this function `nabla_choose()` encompasses the construction of

$$\sum_k \nabla^{\binom{k}{n}} \circ \Delta_k.$$

This correspondence is part of the map ∇_r that is dual to $\mathfrak{sq}^i$, in Morán's construction. The name `nabla_choose` comes from having referred to the map $\nabla^{\binom{k}{n}}$ as the "nabla choose" map while working on this project since the symbol ∇ is called "nabla" and the superscript is "k choose n".

As part of the function `nabla_choose()`, we define correspondences on the augmented semi-simplicial object as constructed by Morán. The work for defining all of these correspondences is one of the more complicated sections of the code. Since face maps can be compositions of multiple smaller face maps, this means we can represent elements of a correspondence in different ways. Determining which representations are equivalent ended up requiring significant effort. To do this, we also relied on Lipshitz and Sarkar's code for defining the ladybug matching in `KhovanovSteenrod`. This gave us the matchings across an individual 2-dimensional face. However, in order to be able to compute all higher Steenrod squares $\mathfrak{sq}^i$ for $i > 2$, we needed to be able to match properly across arbitrarily long stretches.

In another part of the function `nabla_choose()`, we define the maps $\lambda_{W''}$ and λ_{W'',W° for a pair (W'', W°) as constructed by Morán and as described above. We use these maps to determine whether each (W'', W°) is (s,t)-positive for each (s,t). This part of the code is the slowest since there are so many different pairs (s,t) and so many different (W'', W°) to consider. For example, for the disjoint union of four trefoils, we had about 81,000 (s,t)-pairs to consider across the entire computation and 65 pairs (W'', W°). We had to consider each pair of pairs (W'', W°) and (s,t).

The last main function in `compute_sqi()` is called `nabla_on_kh()`. This function takes as input the output of `nabla_choose()`, which is the dual map to Morán's map $\sum_k \nabla^{\binom{k}{n}} \circ \Delta_k$. From that, this function take the $\mathbb{Z}/2$-linearization and then passes this result to the quotient. This gives us the Steenrod square $\mathfrak{sq}^i$. Then, for each generator of $H^{p,q}$, the function `nabla_on_kh()` returns the image of that generator as a linear

combination of the generators of $H^{p+i,q}$. Note that these linear combinations may not be unique. This function also computes and returns the rank of this map modulo 2.

6.2 Results

In this section, we briefly describe some initial results from running our code on different links. As we will see in this section, while the Steenrod squares that we computed via `KhovanovSteenrodSquare` took between seconds and hours to compute, this is still significantly more efficient than doing these computations by hand. In particular, the larger computation that took hours to compute using `KhovanovSteenrodSquare` was unreasonable to do by hand given the number of (s,t) and (W'', W°) pairs to consider and the amount of bookkeeping needed.

6.2.1 Results for Trefoils

To test this code, we started by considering the Steenrod square

$$\mathfrak{sq}^1 : H^{-3,-7}(T;\mathbb{Z}/2) \to H^{-2,-7}(T;\mathbb{Z}/2),$$

where T is the left-handed trefoil. We computed this map both by hand and using our code, and both results matched. We similarly computed both by hand and using our code the Steenrod square

$$\mathfrak{sq}^2 : H^{-6,-14}(T \sqcup T;\mathbb{Z}/2) \to H^{-4,-14}(T \sqcup T;\mathbb{Z}/2),$$

where $T \sqcup T$ is the disjoint union of two left-handed trefoils. The computations by hand of these $\mathfrak{sq}^1$ and $\mathfrak{sq}^2$ maps are presented in the previous section.

Using a Cartan formula for the Khovanov homology of disjoint links combined with the fact that the above two Steenrod squares are nontrivial, we expect the following maps to be nontrivial:

$$\mathfrak{sq}^3 : H^{-9,-21}(T \sqcup T \sqcup T;\mathbb{Z}/2) \to H^{-6,-21}(T \sqcup T \sqcup T;\mathbb{Z}/2)$$
$$\mathfrak{sq}^4 : H^{-12,-28}(T \sqcup T \sqcup T \sqcup T;\mathbb{Z}/2) \to H^{-8,-28}(T \sqcup T \sqcup T \sqcup T;\mathbb{Z}/2).$$

Our code correctly returned this results for both $\mathfrak{sq}^3$ and $\mathfrak{sq}^4$.

In Table 6.1, we present the timing of our code on the four above mentions Steenrod square computations.

link	Steenrod square	cohomological and quantum degrees	rank modulo 2	code timing
T	$\mathfrak{sq}^1$	$(-3,-7) \to (-2,-7)$	1	0.0946 s
$T \sqcup T$	$\mathfrak{sq}^2$	$(-6,-14) \to (-4,-14)$	1	0.2217 s
$T \sqcup T \sqcup T$	$\mathfrak{sq}^3$	$(-9,-21) \to (-6,-21)$	1	22.3647 s
$T \sqcup T \sqcup T \sqcup T$	$\mathfrak{sq}^4$	$(-12,-28) \to (-8,-28)$	1	32770.0201 s (9h, 6m, 10s)

Table 6.1: Timing of Steenrod Squares on Disjoint Unions of Left-Handed Trefoils

6.2.2 The Steenrod Square $\mathfrak{sq}^4$

Recall from the brief discussion above that Lipshitz and Sarkar, as well as Seed, have written code to compute Sq^1 and Sq^2 on the Khovanov homology of a link [LS12, See15]. Again, because of the Adem relations, we can use these computations to define Sq^3. However, we cannot use these Steenrod squares to define Sq^4. Thus, we are interested in any nontrivial Sq^4 computations we can find.

As a proof of concept, we computed $\mathfrak{sq}^4$ on the disjoint union of 4 trefoils, as shown in the results above. However, because of a Cartan formula for the Khovanov homology of disjoint links, we did expect that result.

In search of a nontrivial $\mathfrak{sq}^4$ that we were not necessarily expecting, we turned to the links in the Knot Atlas [BNM15]. From this source, we were able to download the PD code or DT code for the knots in the Rolfsen Knot Table, which contains all the prime knots with 10 crossings or fewer. We also downloaded information about knots with 11, 12, 13, 14, and 15 crossings, available at `http://katlas.org/Data`. For each knot, we first looked for quantum degrees where the Khovanov homology groups were 4 cohomological degrees apart. This would mean that we might possibly have a nontrivial $\mathfrak{sq}^4$. Some of these knot tables included the rank of the knot's integral Khovanov homology groups. For other knots, these homology groups were not already computed. Instead, the knot tables included their rational Khovanov polynomials. From this information, we were able to restrict which Khovanov homology groups we computed. We checked

the Khovanov homology groups for all knots up to 15 crossings and for many knots with 15 crossings from the sources mentioned above. None of these knots had pairs of Khovanov homology groups that were 4 cohomological degrees apart. Thus, since we already knew that none of these knots had a nontrivial $\mathfrak{sq}^4$, we did not try using `KhovanovSteenrodSquare` to compute their Steenrod squares.

6.3 Future Directions

As mentioned above, the only nontrivial $\mathfrak{sq}^4$ that we found was defined on the disjoint union of four trefoils. However, also as mentioned, this was not a surprising result, but was more so a proof of concept. Thus, one future direction of this work is to continue running the code in `KhovanovSteenrodSquare` on the 15-crossing knots and other knots in order to find an unexpected, nontrivial $\mathfrak{sq}^4$.

Another future direction would be to consider a family of knots or links, like torus knots. Then we would like to use the code in `KhovanovSteenrodSquare` to look for patterns in the Khovanov homology groups and Steenrod squares for the link in one of these families.

In order to make these searches more tractable, another next step is to find or write a faster Khovanov homology computation than the current method that we are using from Sagemath. This step is one of the slowest parts of the computation. In addition to speeding up functions from elsewhere that we are using, there is certainly work to be done to speed up our own code in `KhovanovSteenrodSquare`.

References

[BN02] Dror Bar-Natan, *On Khovanov's Categorification of the Jones Polynomial*, Algebraic & Geometric Topology **2** (2002), no. 1, 337 – 370.

[BNM15] Dror Bar-Natan and Scott Morrison, *The Knot Atlas*, `http://katlas.org/wiki/Main_Page`, 2015.

[Kho00] Mikhail Khovanov, *A Categorification of the Jones Polynomial*, Duke Mathematical Journal **3** (2000), 359 – 426.

[KM11] Peter Kronheimer and Tomasz Mrowka, *Khovanov homology is an unknot-detector*, Publications Mathématiques de l'IHÉS **113** (2011), no. 1, 97 – 208.

[LLS20] Tyler Lawson, Robert Lipshitz, and Sucharit Sarkar, *Khovanov homotopy type, Burnside category and products*, Geometry & Topology **24** (2020), no. 2, 623 – 745.

[LS12] Robert Lipshitz and Sucharit Sarkar, *KhovanovSteenrod*, `https://github.com/sucharit/KhovanovSteenrod`, 2012, commit 56b965b.

[LS14a] ———, *A Khovanov stable homotopy type*, Journal of the American Mathematical Society **27** (2014), 983 – 1042.

[LS14b] ———, *A Steenrod Square on Khovanov Homology*, Journal of Topology **7** (2014), no. 3, 817 – 848.

[Mor20] Federico Cantero Morán, *Higher Steenrod squares for Khovanov homology*, Advances in Mathematics **369** (2020).

[MS03] James E. McClure and Jeffrey H. Smith, *Multivariable Cochain Operations and Little n-Cubes*, Journal of the American Mathematical Society **16** (2003), no. 3, 681 – 704.

[SD22a] The SageMath Developers, *link.py*, `https://github.com/sagemath/sage/blob/f8df80820dc7321dc9b18c9644c3b8315999670b/src/sage/knots/link.py`, 2022, version 9.6.

[SD22b] ———, *Links*, `https://doc.sagemath.org/html/en/reference/knots/sage/knots/link.html`, 2022, version 9.7.

[See12] Cotton Seed, *Computations of the Lipshitz-Sarkar Steenrod Square on Khovanov Homology*, `https://arxiv.org/abs/1210.1882`, 2012.

[See15] ———, *knotkit*, `https://github.com/cseed/knotkit`, 2015, commit 4fdd08f.

[Thi01] Morwen Thistlethwaite, *Links with Trivial Jones Polynomial*, Journal of Knot Theory and its Ramifications **10** (2001), no. 4, 641 – 643.

Appendix A

Morphism Notation

Throughout this paper, we have referenced several different but similar morphisms. Here, we summarize these morphisms for quick reference to help keep them straight.

- Within the cube category $\underline{2}^n = \{0,1\}^n$, the morphisms are written $d_U : A' \to A$, where we have $A', A \in \{0,1\}^n$ and the sequence $U \in \{0,1,*\}^n$. The elements of the sequence U in $\{0,1\}$ tell us which coordinates A' and A share. The elements of U that are $*$ tell us for which coordinates A' and A differ. In the positions that U has a $*$, A' has a 0, and A has a 1. We define the cube category in Section 3.2.2.

- For a given quantum degree q, the functor $F^q : \underline{2}^n \to \mathscr{B}$, which we present in Section 3.2.2.1, sends vertices of the n-cube to sets of enhanced states in the Burnside category. This functor sends morphisms to correspondences, as we saw in the previous bullet. These correspondences do not have special notation. We simply denote a morphism by $F^q(d_U)$ where we have $d_U \in Mor(\underline{2}^n)$.

- Within the semi-simplicial category Δ_{inj}, the morphisms are maps between ordinals. We denote these maps by $\partial^i_U : [i-|U|] \to [i]$, where we have $U \in \{0,1,\ldots,i\}$.

- Within the augmented semi-simplicial category Δ_{inj*}, the morphisms are the same maps between ordinals as in semi-simplicial category Δ_{inj}. We now also have a map from $[-1]$ to $[0]$, the identity morphism on $[-1]$, and a morphism $[-1] \to [n]$ for each $n > 0$. To distinguish the maps in the semi-simplicial category from

those in the augmented semi-simplicial category, even though, when they both exist, they are the same, we denote the maps in Δ_{inj*} by $\bar{\partial}^i_U : [i - |U|] \to [i]$. Again, we note that we have the equivalence $\partial^i_U = \bar{\partial}^i_U$ for all $i \geq 0$ and for all $U \in \{0, 1, \ldots, i\}$. We define Δ_{inj*} in Section 3.2.3.2.

- Within the opposite of the augmented semi-simplicial category Δ^{op}_{inj*}, which we define in Section 3.2.3.3, we write the maps as $\bar{\delta}^i_U$. Here, a morphism $\bar{\delta}^i_U$ is dual to the morphism $\bar{\partial}^i_U \in Mor(\Delta_{inj*})$.

- The functor $X : \Delta^{op}_{inj*} \to \mathscr{B}$, which we describe in Section 4.1, takes ordinals in the opposite of the augmented semi-simplicial category to sets of enhanced states in the Burnside category. In particular, it maps to disjoint unions of $F^q(A)$ for some $A \in \underline{2}^n$. This functor X takes morphisms $\bar{\delta}^i_U$ to correspondences δ^i_U.