The Super Mumford Form and the Sato Grassmannian

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Abstract

We describe a supersymmetric generalization of the construction of Kontsevich and Arbarello, De Concini, Kac, and Procesi, which utilizes a relation between the moduli space of curves with the infinite-dimensional Sato Grassmannian. Our main result is the existence of a flat holomorphic connection on the line bundle $\lambda_{3/2} \otimes \lambda_{1/2}^{-5}$ on the moduli space of triples: a super Riemann surface, a Neveu-Schwarz puncture, and a formal coordinate system.

Contents

1 Introduction

The constant $c_j = 6j^2 - 6j + 1$, proportional to the 2^{nd} Bernoulli polynomial, arises independently in studying representations of the Virasoro algebra and by studying the geometry of the moduli space of algebraic curves. Manin [28] conjectured based on this numerical coincidence that there exists a direct connection between these seemingly independent mathematical areas. This conjecture was shown to be true in the simultaneous publications [2, 5, 22, 25], which was important to the quantum theory of (bosonic) strings, as it unified path integral quantization and operator quantization.

The numerical coincidence goes as follows. The Witt algebra has natural representations ϱ_j defined by Lie derivative action on $\mathbb{C}((z))dz^{\otimes j}$. The Japanese cocycle η on $\mathfrak{gl}(\mathbb{C}((z)))$ by restriction to the Witt algebra induces the unique central extension of the Witt algebra, known as the Virasoro algebra. Then we find

$$\varrho_j^*(\eta) = c_j \varrho_1^*(\eta), \qquad c_j = 6j^2 - 6j + 1.$$

When the Virasoro algebra is realized as operators on the state space of a string, the string must propagate in in $2c_j$ (real) spacetime dimensions for both unitarity and Lorentz invariance of the string theory to hold. On the other hand, for any smooth proper family of curves $\pi\colon X \to S$, the Mumford isomorphism is the isomorphism of line bundles

$$\lambda_j \cong \lambda_1^{c_j},$$

where $\lambda_j := \det R\pi_*(\omega_X^{\otimes j})$ and $\omega_X := \Omega_X^1$. If S is the moduli space of algebraic curves and $\pi\colon X \to S$ is the universal curve, then we find the dualizing sheaf $\omega_S := \det \Omega_S^1 \cong \lambda_2$, and therefore $\omega_S \cong \lambda_1^{c_j} \cong (\pi_* \omega_X)^{c_j}$. This is important to Polyakov path integration on the moduli space.

An alternative description of the Polyakov measure comes through the connection between representations of the Virasoro algebra and the moduli space; this connection is roughly the following. The Virasoro algebra acts on the moduli space $\mathcal{M}_{g,1^\infty}$ of triples (C, p, z), a Riemann surface, an point, and a parameter. This action can be used to show that there exists a flat holomorphic connection on the line bundle $\lambda_j \otimes \lambda_1^{-c_j}$, which can be used to write differential equations for the Polyakov measure. For another summary, see the introduction of [30].

In fact, Manin in [28] hypothesized this relationship existed in the superized case as well. The numerical coincidence is analogous: The super Witt algebra has representations ϱ_j defined by Lie derivative action on $\mathbb{C}((z))[\zeta][dz|d\zeta]^{\otimes j}$ such that pulling back the super Japanese cocycle η (defining the unique central extension, the Neveu-Schwarz algebra) gives

$$\varrho_j^*(\eta) = c_j \varrho_1^*(\eta), \qquad c_j = -(-1)^j(2j-1).$$

And for any smooth proper family of supercurves $\pi\colon X \to S$, the super Mumford isomorphism is the isomorphism of line bundles

$$\lambda_{j/2} \cong \lambda_{1/2}^{c_j},$$

where $\lambda_{j/2} := \operatorname{Ber} R\pi_*(\omega_X^{\otimes j})$ and $\omega_X = \operatorname{Ber} \Omega_X^1$.

The goal of the paper is to describe a supersymmetric generalization of the construction of Kontsevich [25] and Arbarello, De Concini, Kac, and Procesi [2], which utilizes a relation between the moduli space of curves with the infinite-dimensional Grassmannian. The consequences of their construction shows that the Chern classes of the line bundles of the Mumford isomorphism are equal. Manin proved a supersymmetric generalization of this result in Theorem 3.3 of [30] by generalizing the methods of Beilinson and Schechtman in [5], which prove a version of the Riemann-Roch theorem for the Atiyah algebras of vector bundles. Instead, our paper extends the results of Ueno and Yamada [40] of the representations of the Neveu-Schwarz algebra and the results of Mulase and Rabin [33] of the super Sato Grassmannian and the super Krichever map. Our method depends on showing that the Lie superalgebra of superconformal vector fields on an affine super Riemann surface is perfect, which we currently state as a conjecture, but expect to have a proof soon.

Our use of the super Grassmannian $\operatorname{Gr}\left(\mathbb{C}((z))[\zeta]\right)$ appears to provide an alternative approach for integrating over the moduli space of SRSs $\mathfrak{M}_g$. The Torelli map sending a super Riemann surface to its Jacobian $J\colon \mathfrak{M}_g \to \mathfrak{A}_g$, where $\mathfrak{A}_g$ is the moduli space of principally polarized abelian supervarieties, plays a prominent role in the papers [12, 13, 14, 15] of D'Hoker and Phong and [19] of Grushevsky, who compute explicitly or propose an ansatz for chiral superstring measure in low genus. In higher genus, these methods face the problem that the locus of moduli space inside $\mathfrak{A}_g$, known as the super Jacobian locus, is very hard to describe. The known characterizations of the super Jacobian locus, i.e. solutions to the super Schottky problem, are very implicit. Since the known solution of the super Schottky problem, as in [32] of Mulase, goes through the super Krichever map $\mathfrak{M}_g \to \operatorname{Gr}(\mathbb{C}((z))[\zeta])$, and since a more explicit description of the moduli space locus is an orbit of the super Witt algebra under the super Krichever map, it appears that the computation of physically meaningful integrals, such as scattering amplitudes, can be carried out in the super Grassmannian, rather than on the moduli space $\mathfrak{A}_g$. This idea does not seem to have been utilized in the classical (i.e. non-super) case.

2 Superalgebra

In this section we review the fundamentals of superalgebras starting with super vector spaces and ending with super Lie algebroids. In this section, we consider k to be a field of characteristic not equal to 2.

2.1 Super vector spaces

Super linear algebra is the study of the category of super vector spaces. Some references are part 1 chapter 1 of [11], chapter 3 of [41], and chapter 2 of [43], chapter 3 of [29], chapter 2 of [16]. chapter 1 of [8]

A *supervector space* is a $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$-graded vector space, that is a vector space $V = V_0 \oplus V_1$.[1] The elements of V_i are called *homogeneous*, specifically nonzero elements of V_0 are *even* and nonzero elements of V_1 are *odd*. We denote the parity of a homogeneous element $v \in V_i$ as $|v| \in \mathbb{Z}_2$. For an arbitrary element $v \in V$, we may decompose it as $v = v_0 + v_1$ into even and odd components.

A super vector space $V = V_0 \oplus V_1$ with $\dim(V_0) = m$ and $\dim(V_1) = n$ is said to have dimension $\dim(V) = m|n$. A super vector space V is said to be finite dimensional if $m < \infty$ and $n < \infty$. A classical vector space is a super vector space of dimension $m|0$ for some m. We define the *super dimension*

$$\operatorname{sdim}(V) = \dim(V_0) - \dim(V_1) = m - n.$$

A *super subspace* $U \subset V$ is defined as a subspace of V which is $\mathbb{Z}_2$-graded: $U = U_0 \oplus U_1$, in which case $U_i = U \cap V_i$. Clearly, a super subspace is a super vector space itself.

A *morphism of super vector spaces* is a partity perserving k-linear map. Let $\operatorname{Hom}(V, W)$ denote the vector space of all morphisms from V to W. We denote the category of super vector spaces with this defintion of morphisms as `sVec`. The category `sVec` is abelian. Direct sum, direct product, and tensor product functors are defined as one would expect.

[1] An alternate notation is $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

The parity of these constructions is given by

$$(V \oplus W)_i = V_i \oplus W_i \qquad (V \times W)_i = V_i \times W_i \qquad (V \otimes W)_i = \bigoplus_{j+k=i} V_j \otimes W_k$$

for $i, j, k \in \mathbb{Z}_2$. An important difference from classical vector spaces is the commutativity isomorphism for tensor products:

$$c_{V,W} \colon V \otimes W \xrightarrow{\sim} W \otimes V \qquad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \tag{1}$$

This is an example of writing a formula implicitly assuming that v and w are homogeneous elements, and the formula is understood as extended linearly to the full super vector space. The associativity isomorphism for super vector spaces is same as for classical vector spaces. This isomorphism is the source of the rule of signs used by phsyicists when working in supermathematics. The *rule of signs* states that two terms commute if at least one term is even, and two terms anticommute if both terms are odd.

A functor unique to super vector spaces is the *parity reversing functor* $\Pi \colon \texttt{sVec} \to \texttt{sVec}$ defined by

$$(\Pi V)_0 := V_1 \qquad (\Pi V)_1 := V_0$$

and a morphism $f \colon v \in V \mapsto w \in W$ becomes $\Pi f \colon v \in \Pi V \mapsto w \in \Pi W$. With the partity reversing functor we can write explicit formulas for the internal hom in the category of vector spaces, denoted $\underline{\mathrm{Hom}}(V, W)$.

$$\left(\underline{\mathrm{Hom}}(V, W)\right)_0 = \mathrm{Hom}(V, W) \qquad \left(\underline{\mathrm{Hom}}(V, W)\right)_1 = \mathrm{Hom}(\Pi V, W) = \mathrm{Hom}(V, \Pi W)$$

The internal hom $\underline{\mathrm{Hom}}(V, W)$ is the super vector space of all k-linear maps from V to W which has the parity preserving maps as even component and partity reversing maps as odd component. We denote $\underline{\mathrm{End}}(V) := \underline{\mathrm{Hom}}(V, V)$. The *dual* of a super vector space is defined as $V^* := \underline{\mathrm{Hom}}(V, k)$. In other words, $(V^*)_i = (V_i)^*$. With this definition we have that $\underline{\mathrm{Hom}}(V, W) \cong V^* \otimes W$. Any element $T \in \underline{\mathrm{Hom}}(V, W)$ can be written as a 2×2 matrix of linear maps $T_{j,i} \colon V_i \to W_j$:

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}.$$

Example 2.1. *The super vector space is $k^{m|n}$ is defined as*

$$k^{m|n} := \bigoplus_m k \oplus \bigoplus_n (\Pi k)$$

Any super vector space of dimension $m|n$ over k is isomorphic as a super vector space to $k^{m|n}$.

Later, the notation $k^{m|n}$ will also denote a geometrical space instead of a vector space, which should be clear from context.

2.2 Superalgebras

A *superalgebra* over k is a super vector space A over k and a morphism $A \otimes A \to A$. The parity of a product of elements xy is given by the sum of each elements' parities: $|xy| = |x| + |y|$. We also assume that our superalagebras are associative and have a unit. It is easy to check that the even component of a superalgebra is a classical associative algebra with unit.

If A and B are superalgebras, we define the tensor product superalgebra $A \otimes B$ to have the product morphism

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2 \tag{2}$$

The *supercommutator* or just *commutator* of a superalgebra A is

$$[a, b] := ab - (-1)^{|a||b|} ba.$$

We denote by $[A, A]$ the subsuperalgebra which is generated by elements of the form $[a, a]$ for $a \in A$.

A superalgebra is said to be *supercommutative* or just *commutative* if for all (homogeneous) $a, b \in A$ we have $[a, b] = 0$. In particular, this means that any odd element is nilpotent, and specifically it must square to 0. An equivalent (more abstract) definition of a commutative superalgebra is a superalgebra A with product morphism $\mu\colon A \otimes A \to A$ such that

$$\mu = \mu \circ c_{A,A}.$$

In fact, this abstract definition is the same statement as for commutativity of classical algebras. Supercommutativity may also be described as graded commutative with $\mathbb{Z}_2$ grading.

Example 2.2. *The superalgebra $k[\theta]/(\theta^2 - 1)$ of dimnesion $1|1$, with even generator 1 and odd generator θ, is a noncommutative superealgebra. The commutativity as a superalagebra fails since $\theta^2 \neq 0$ and θ is odd. However, considered as a classical algebra of dimension 2, $k[\theta]/(\theta^2 - 1)$ is commutative. This example highlights the differences in the commutativity property in the classical and super settings.*

Example 2.3. *For any super vector space V, $\underline{\mathrm{End}}(V)$ with the product morphism given by composition is a noncommutative superalgebra. The noncommutativity as a superalgebra is simply due to the noncommutivity as a classical algebra.*

Example 2.4. *Let V be a classical algebra of finite dimension m. The exterior algebra $\bigwedge^\bullet V$ considered with $\mathbb{Z}_2$-grading is a commutative superalgebra. In particular, $\bigwedge^\bullet V \cong k[\theta_1, \dots, \theta_m]/(\theta_i\theta_j + \theta_j\theta_i = 0)$ as superalgebras. It is customary to write $k[\theta_1, \dots, \theta_m]$ and state that the θ_i are odd instead of explicitly writing the quotient.*

This isomorphism is clear since for any basis $v_1, \dots, v_m$ for V, we have that $v_i \wedge v_j = -v_j \wedge v_i$. The isomorphism can then be written as $v_i \mapsto \theta_i$.

Clearly, this example generalizes to the sheaf of differential forms $\Omega^\bullet$ over a classical manifold, so that $\Omega^\bullet$ may be viewed as a commutative superalgebra over $\mathbb{R}$ or $\mathbb{C}$.

2.3 Graded superalgebras

For a super vector space V, define the tensor algebra of V as $T(V) := \bigoplus_{n\geq 0}(V^{\otimes n})$ with the natural product morphism $\otimes$. Then define the symmetric algebra of V as

$$S(V) := T(V)\Big/\left(v \otimes w - (-1)^{|v||w|} w \otimes v\right)$$

where the quotient is by the ideal generated by all expressions of the form given with $v, w \in V$. Clearly then $S(V)$ is a commutative superalegbra. The above definition is in analogy with the classical definition of the symmetric algebra, and we see that this gives $S(V) = S(V_0) \otimes \bigwedge(V_1)$, where the righthand side uses the classical definitions of symmetric and exterior algebras.

Carefully considering the definition of $T(V)$, we see that is has both a $\mathbb{Z}$-grading and a $\mathbb{Z}_2$-grading. If we consider $T(V)$ as bigraded we will denote it $T^\bullet(V)$ to emphasize the $\mathbb{Z}$-grading, and similarly for $S^\bullet(V)$. When considering these as bigraded algebras, the Z-graidng may be called (co)homological grading or *degree*, and the Z_2-grading may be called super grading or *parity*. In this perspective, the degree 1 component of $T^\bullet(V)$ and $S^\bullet(V)$ is V. However, we can choose to ignore the $\mathbb{Z}$-grading, which we will denote with $T(V)$ and $S(V)$. In this perspective, V is contained in degree 0, and therefore so are the entire algebras $T(V)$ and $S(V)$.

We define the exterior algebra $\bigwedge^\bullet(V) := S(V[-1])$, where the grading shift $[-1]$ denotes shifting V from degree 0 to degree 1. As in the previous paragraph, $\bigwedge^\bullet(V)$ is considered a bigraded algebra.

We adopt the perspective that graded algebras are internal to the category of superalgebras. In brief, the degree and parity are kept separate. For $\bigwedge^\bullet(V)$, this is known as

Deligne's sign convention:

$$a \wedge b = (-1)^{\deg(a)\deg(b)+|a||b|} b \wedge a. \tag{3}$$

The alternative perspective is to consider parity as degree modulo 2. This is done in Example 2.4. This is known as Bernstein's convention and results in the sign rule $a \wedge b = (-1)^{(\deg(a)+|a|)(\deg(b)+|b|)} b \wedge a$. Using this convention, the appropriate definition of the exterior algebra is $\bigwedge^\bullet(V) = S(\Pi V)$. These two sign conventions are incompatible, but essentially equivalent. A comparison of the two approaches and the precise statement of equivalence can be found in part 1 chapter 1 appendix of [11].

Example 2.5. *Consider $A[z_1, \ldots, z_m|\theta_1, \ldots, \theta_n]$ where A is a superalgebra. The z_i are even, and the θ_i are odd. It is freely generated as an A-algebra by the z_i and θ_i. It is isomorphic to $A \otimes S(k^{m|n}) \cong S(A^{m|n})$, where $A^{m|n}$ is defined in the next section.*

2.4 Modules over superalgebras

For a superalgebra A, a *left A-supermodule* or just *left A-module* is a super vector space M with a morphism $A \otimes M \to M$ which satisfies $|am| = |a| + |m|$ for all (homogeneous) $a \in A$ and $m \in M$, and such that M is a classical A-module where A is considered as a classical vector space. Similarly defined is a *right A-module*. An *A-module* will refer to a left and right A-module under where the action of A is related by $am = (-1)^{|a||m|} ma$ for $a \in A$ and $m \in M$.

If M is a right A-module and N is a left A-module, we define the tensor product $M \otimes_A N$ as the quotient of $M \otimes_k N$ by $ma \otimes n = m \otimes an$ for $a \in A$, $m \in M$, and $n \in N$. If M and N are A-modules, then $M \otimes_A N$ is a supermodule with product morphism $a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an)$, where the last equality is due to the commutativity isomorphism (1). This tensor product functor has the usual properties of associativity and commutativity, and has a unit, namely A-module A.

The internal hom in the category of A-modules $\underline{\mathrm{Hom}}_A(M, N)$ is defined as those T in the internal hom between M and N as super vector space such that $aT(m) = (-1)^{|a||T|} T(am)$. The internal hom $\underline{\mathrm{Hom}}_A(M, N)$ is an A-module under the multiplication $(aT)(m) := aT(m)$. We denote $\underline{\mathrm{End}}_A(M) := \underline{\mathrm{Hom}}_A(M, M)$ as for super vector spaces. Accordingly, morphisms in the category of A-modules will be k-linear, parity perserving, and A-commutative maps[2]. The dual of a A-module is defined as $M^* := \underline{\mathrm{Hom}}_A(M, A)$.

[2] That is maps such that $aT(m) = T(am)$ for all $a \in A$.

We define the A-module

$$A^{m|n} := \bigoplus_m A \oplus \bigoplus_n (\Pi A).$$

An A-module is *free* if it is isomorphic to $A^{m|n}$ for some m, n. The *rank* of such a module is $m|n$. If M and N are free A-modules, any element $T \in \underline{\mathrm{Hom}}_A(M, N)$ can be written as a 2×2 matrix of linear maps $T_{j,i}\colon\ M_i \to N_j$:

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix}.$$

Care must be taken with the commutation relations of this matrix when written in coordinates. If M has a basis $(e_i|\varepsilon_j)_{1\le i\le m_0, 1\le j\le m_1}$ with e_i even and ε_j odd, then we write an element $m \in M$ as $m = \sum_i e_i x^i + \varepsilon_j \xi^j$ where $x^i, \xi^j \in A$ of any parity. We may do similarly for N with a basis $(f_k|\varphi_l)_{1\le k\le n_0, 1\le l\le n_1}$ with f_k even and φ_l odd. Then we may write a map $T \in \underline{\mathrm{Hom}}_A(M, N)$ as

$$Te_i = \sum_k f_k a^k_i + \sum_l \varphi_l c^l_i \qquad\qquad T\varepsilon_j = \sum_k f_k b^k_j + \sum_l \varphi_l d^l_j$$

so that we may identify T with the matrix

$$\begin{pmatrix} (a^k_i) & (b^k_j) \\ (c^l_i) & (d^l_j) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{4}$$

with $1 \le i \le m_0, 1 \le j \le m_1, 1 \le k \le n_0, 1 \le l \le n_1$. Where clearly, A is a the map T_{00} written in the chosen bases, etc. We note that if T is a morphism (parity preserving) then A and D are even and B and C are odd.

Using these definitions, we may identify $T(m)$ with $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)$, and composition of maps by the usual product of matrices.

2.5 The Berezinian and related operations

Let $T \in \underline{\mathrm{Hom}}_A(M, N)$ in the last section.

The supertranspose may be defined using the block decomposition of a supermatrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{sT}} := \begin{pmatrix} A_0^{\mathrm{T}} + A_1^{\mathrm{T}} & -C_0^{\mathrm{T}} + C_1^{\mathrm{T}} \\ B_0^{\mathrm{T}} - B_1^{\mathrm{T}} & D_0^{\mathrm{T}} + D_1^{\mathrm{T}} \end{pmatrix} \tag{5}$$

This is the appropriate definition so that the matrix representation of the dual $T^* := \underline{\mathrm{Hom}}_A(T, A)$ is given by the supertranspose of the matrix representation of T.

The dual $T^* \in \underline{\mathrm{Hom}}_A(N^*, M^*)$ satisfies the equations

$$T^* f_k^* = \sum_i e_i^* {a^*}^i_k + \sum_j \varepsilon_j^* {c^*}^j_k \qquad T^* \varphi_l^* = \sum_i e_i^* {b^*}^i_l + \sum_j \varepsilon_j^* {d^*}^j_l$$

Then consider $(T^* {f^*}_k)(e_i)$, $(T^* {f^*}_k)(\varepsilon_j)$, $(T^* {\varphi^*}_l)(e_i)$, and $(T^* {\varphi^*}_l)(\varepsilon_j)$. Each may be evaluated in two ways. We show this for $(T^* {\varphi^*}_l)(e_i)$.

$$(T^* {\varphi^*}_l)(e_i) = \left(\sum_i e_i^* {b^*}^i_l + \sum_{j'} \varepsilon_{j'}^* {d^*}^{j'}_l \right)(e_i) = {b^*}^i_j$$
$$(T^* {\varphi^*}_l)(e_i) = (Te_i)\varphi_l^* = \left(\sum_k f_k a_i^k + \sum_{l'} \varphi_{l'} c_i^{l'} \right) \varphi_l^* = -(-1)^{|c_i^l|} c_i^l$$

Note that the last equality is found using $\varphi_l \varphi_l^* = -1$ since $\varphi_l^* \varphi_l = 1$. In summary, we find

$$\begin{aligned} {a^*}^i_k &= a_i^k & {b^*}^i_l &= -(-1)^{|c_i^l|} c_i^l \\ {c^*}^j_k &= (-1)^{|b_j^k|} b_j^k & {d^*}^j_l &= d_j^l \end{aligned}$$

As appearing in this calculation, there is difference between the evaluation maps $M^* \otimes M \to A$ and $M \otimes M^* \to A$. Arguably, the more natural evaluation map is

$$M^* \otimes M \to A \qquad \omega \otimes m \mapsto \omega(m) \tag{6}$$

This however will generate a sign when considering the isomorphism

$$M^* \otimes M \xrightarrow{\sim} \underline{\mathrm{End}}(M) \qquad \omega \otimes m \mapsto (m' \mapsto (-1)^{|m||\omega|} m\omega(m'))$$

Therefore, the supertrace will be

$$\begin{aligned} &\underline{\mathrm{End}}(M) \xleftarrow{\sim} M^* \otimes M \to A \\ &T \leftarrow\!\shortmid \left(\sum_i e_i^* \otimes T(e_i) + (-1)^{|T|} \sum_j \varepsilon_j^* \otimes T(\varepsilon_j) \right) \mapsto \left(\sum_i e_i^* f(e_i) - (-1)^{|T|} \sum_j \varepsilon_j^* f(\varepsilon_j) \right) \end{aligned}$$

or given in terms of the block matrices of T:

$$\operatorname{str} T := \operatorname{tr} A - (-1)^{|T|} \operatorname{tr} D. \tag{7}$$

One can easily check that the desired properties of the supertranspose and supertrace are satisfied:

$$\begin{aligned} \operatorname{str}(X+Y) &= \operatorname{str} X + \operatorname{str} Y & (X+Y)^{\mathrm{sT}} &= X^{\mathrm{sT}} + Y^{\mathrm{sT}} \\ \operatorname{str}(XY) &= (-1)^{|X||Y|} \operatorname{str}(YX) & (XY)^{\mathrm{sT}} &= (-1)^{|X||Y|} Y^{\mathrm{sT}} X^{\mathrm{sT}} \\ \operatorname{str}(X^{\mathrm{sT}}) &= \operatorname{str} X \end{aligned}$$

One property that differs from the classical operations is that $\mathrm{sT} \circ \mathrm{sT} \neq \mathrm{Id}$. Instead we have $\mathrm{sT} \circ \mathrm{sT} \circ \mathrm{sT} \circ \mathrm{sT} = \mathrm{Id}$.

Before describing the superdeterminant, we must describe the inverse of a morphism. The inverse of $T \in \underline{\mathrm{End}}_A(M)$ is the usual definition, that there must exist a T^{-1} such that $TT^{-1} = T^{-1}T = \mathrm{Id}_M$. However, if T is odd, this violates the rule of signs since $TT^{-1} = -T^{-1}T$. Therefore, only morphisms $T \in \mathrm{End}(M)$ may be invertible. The invertible morphisms are called automorphisms $\mathrm{Aut}(M)$ as usual.

For $T \in \mathrm{End}_A(M)$ with block decomposition as before, Lemma 3.6.1 of [41] states that T is invertible iff both A and D are invertible over the even algebra A_0.

The notion of a superdeterminant was a breakthrough discovery due to Berezin, and therefore the superdeterminant is called the Berezinian and denoted Ber. It satisfies the usual properties

$$\exp(\mathrm{str}\, X) = \mathrm{Ber}(\exp X) \tag{8}$$

$$\mathrm{Ber}(XY) = \mathrm{Ber}(X)\,\mathrm{Ber}(Y) \tag{9}$$

Consider $T \in \mathrm{End}(M)$ with block decomposition $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$. Using the property 8 and the definition of the supertrace 7, we must have that $\mathrm{Ber}\, T = \det A \cdot \det^{-1} D$. Importantly, notice that we are forced to define the Berezinian such that it is undefined if D is not invertible.

Now consider $T \in \mathrm{End}(M)$ such that D is invertible. We have the UDL-decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}.$$

If we then extend our limited definition of the Berezinian and enforce the property 9, then we see that

$$\mathrm{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det{}^{-1} D. \tag{10}$$

Alternatively, if we had assumed A was invertible, we could used the LDU-decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}.$$

to arrive at

$$\mathrm{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det{}^{-1}(D - CA^{-1}B). \tag{11}$$

These two definitions of the Berezinian are equivalent, which can be seen after showing the property 9 holds. A proof of this multiplicative property is Theorem 3.6.2 in [41].

We comment that the classical determinants in the formulas above are well-defined since T is partity-preserving that gives that all the arguments of det above are even and are over the classical algebra A_0. In particular, we may consider $\mathrm{Ber}\colon\ \mathrm{GL}(m_0|m_1, A) \to \mathrm{GL}(1|0, A_0)$, where we define $\mathrm{GL}(m_0|m_1, A) := \mathrm{Aut}(A^{m_0|m_1})$. Put simply, $\mathrm{Ber}(T) \in A_0$.

We add to our list of properties with the following where $Z \in \underline{\mathrm{End}}(M)$ and $X, Y \in \mathrm{Aut}(M)$.

$$\mathrm{str}(XZX^{-1}) = \mathrm{str}\, Z \qquad \mathrm{Ber}(X^{\mathrm{sT}}) = \mathrm{Ber}\, X$$
$$\mathrm{Ber}(X^{-1}) = (\mathrm{Ber}\, X)^{-1} \qquad \mathrm{Ber}(X + Y) = \mathrm{Ber}\, X\, \mathrm{Ber}\, Y$$

2.6 Berezinian of a free A-module

Classically, the determinant of a free module L of rank l is $\det L := \bigwedge^l L$. Naively applying this definition to a supermodule will not work since a supermodule of rank $p|q$ with $q > 0$ has an exterior algebra which is nonzero in every degree.

Consider M a free A-module of rank $m_0|m_1$. We define the A-module $\mathrm{Ber}\, M$ of rank $1|0$ if m_1 is even or rank $0|1$ if m_1 is odd as generators and relations among them as follows. Let e_i and ε_j for $1 \le i \le m_0$ and $1 \le j \le m_1$ be a basis for M where e_i is even and ε_j is odd. Then we denote $[e_1, \dots, e_{m_0}|\varepsilon_1, \dots, \varepsilon_{m_1}]$ a generator of $\mathrm{Ber}\, M$ as an A-module. Elements of $\mathrm{Ber}\, M$ are subject to the relations

$$[Te_1, \dots, Te_{m_0}|T\varepsilon_1, \dots, T\varepsilon_{m_1}] = \mathrm{Ber}\, T \cdot [e_1, \dots, e_{m_0}|\varepsilon_1, \dots, \varepsilon_{m_1}]$$

where $T \in \mathrm{Aut}\, M$. This definition is analogous to the property determinant satisfies when acting on the top exterior power.

There does exist a alternative basis-independent definition of the Berezinian:

$$\mathrm{Ber}\, M = \underline{\mathrm{Ext}}^{m_0}_{S^\bullet(M^*)}(A, S^\bullet(M^*))$$

This matches the definition of determinant if M is a classical module, which can be shown using a Koszul complex. More details are in [11].

Lastly, we note that a short exact sequence of A-modules

$$0 \to L' \to L \to L'' \to 0$$

gives an isomorphism

$$\mathrm{Ber}\, L' \otimes \mathrm{Ber}\, L'' \xrightarrow{\sim} \mathrm{Ber}\, L.$$

3 Supergeometry

First, we describe superspaces and their properties which differentiate them from classical geometrical spaces. We denote the category of superspaces as `sSpa`.

Next we introduce the two particular types of superspaces, supermanifolds and superschemes. We denote the category of supermanifolds by `sMan`, and the category of superschemes by `sSch`.

We then discuss super vector bundles, in particular the tangent, cotangent, and Berezinian sheaves. We discuss in some detail the construction of the finite dimensional super Grassmannian as an example.

3.1 Superspaces

Unlike classical geometry, supergeometry requires the use of sheaves in order to define a superspace. The presence of odd nilpotent functions can only be expressed via sheaves.

Definition 3.1. *A* locally ringed superspace *is a pair* $(|X|, \mathcal{O}_X)$ *where* $|X|$ *is a topological space and* $\mathcal{O}_X$ *is a a sheaf of supercommutative rings such that for every point* $x \in |X|$ *the stalk* $\mathcal{O}_{X,x}$ *is a local superring.*

A morphism *of locally ringed superspaces is a pair* $(f, f^\sharp)$ *where* $f\colon X \to Y$ *is a continuous map of the topological spaces and* $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ *is a homogeneous morphism of sheaves of supercommutative rings, such that for every* $x \in |X|$*, the induced morphism* $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ *is local.*

We will frequently omit the structure sheaf when referring to a superspace, and simply write X. We denote the even and odd components of the structure sheaf as $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$.

From a superspace there are a few ways to create an associated classical locally ringed space. A classical method is to take the reduction. Define $\mathcal{N}_X \subset \mathcal{O}_X$ to be the subsheaf generated by all nilpotent elements. The usual reduction is then $X_{\mathrm{red}} := (|X|, \mathcal{O}_X/\mathcal{N}_X)$.

The most useful classical space associated to a superspace is the *bosonic reduction* or *odd reduction*, denoted X_{rd} or X_{bos}.[3]

$$X_{\text{bos}} := (|X|, \mathcal{O}_X/\mathcal{J}_X)$$

We may write this sheaf as $\mathcal{O}_{|X|} := \mathcal{O}_X/\mathcal{J}_X$. The subsheaf $\mathcal{J}_X \subset \mathcal{O}_X$ is the sheaf generated by all odd elements of the structure sheaf. Explicitly, that is

$$\mathcal{J}_X := (\mathcal{O}_{X,1})^2 \oplus \mathcal{O}_{X,1}.$$

Further, the *bosonic quotient* or *even quotient* is defined as

$$X/\Gamma := (|X|, \mathcal{O}_{X,0})$$

where the group $\Gamma = \{\pm 1\}$ acts by $\pm 1 \mapsto (f \mapsto f_0 \pm f_1)$.

In fact, the bosonic reduction is right adjoint and the bosonic quotient is left adjoint to the inclusion from locally ringed spaces to locally ringed superspaces [7]. This is stated below for Y a classical locally ringed space and X a locally ringed superspace.

$$\mathrm{Mor}_{\texttt{sSpa}}(Y, X) \cong \mathrm{Mor}_{\texttt{Spa}}(Y, X_{\text{bos}}) \qquad \mathrm{Mor}_{\texttt{sSpa}}(X, Y) \cong \mathrm{Mor}_{\texttt{Spa}}(X/\Gamma, Y) \tag{12}$$

There is a natural closed immersion of superspaces $i\colon X_{\text{bos}} \hookrightarrow X$ corresponding to the projection $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X/\mathcal{J}_X$. For this reason, we refer to X_{bos} as the *underlying classical local ringed space* of X.

We can associate a nicer superspace to every superspace like this:

$$\begin{aligned} \mathrm{gr}(X) &:= (X, \mathrm{gr}(\mathcal{O}_X)) \\ \mathrm{gr}(\mathcal{O}_X) &:= \bigoplus_{i\geq 0} \mathcal{J}_X^i/\mathcal{J}_X^{i+1} = \mathcal{O}_X/\mathcal{J}_X \oplus \mathcal{J}_X/\mathcal{J}_X^2 \oplus \mathcal{J}_X^2/\mathcal{J}_X^3 \oplus \cdots \end{aligned}$$

Note that each sheaf $\mathcal{J}_X^i/\mathcal{J}_X^{i+1}$ is a $\mathcal{O}_{|X|}$-module, so that $\mathrm{gr}(\mathcal{O}_X)$ is also a $\mathcal{O}_{|X|}$-module, that is a locally free sheaf on X_{bos}. The superspace $\mathrm{gr}(X)$ is called the *retract* of X or the *associated split superspace* to X, which makes sense given the definitions below.

Definition 3.2. *Let $X = (|X|, \mathcal{O}_X)$ be a superspce and let $\mathcal{J}_X$ denote the subsheaf generated by all odd elements. Then we define*

1. *X is* locally split *if $\mathcal{E} := \mathcal{J}_X/\mathcal{J}_X^2$ is a locally free $\mathcal{O}_{|X|}$-module such that locally $\mathrm{gr}(\mathcal{O}_X|_U) \cong \bigwedge_{\mathcal{O}_U} \mathcal{E}|_U$.*
2. *X is* split *if $\mathcal{O}_X \cong \bigwedge_{\mathcal{O}_{|X|}} \mathcal{E}$ for some locally free sheaf $\mathcal{E}$ on X_{bos} which generates the odd structure, that is $\mathcal{E} \cong \mathcal{J}_X/\mathcal{J}_X^2$.*

[3]The usual reduction and bosonic reduction are equivalent for nice enough superspaces including supermanifolds. [29]

3. X is projected *if there exists a map $p\colon X \to X_{\text{bos}}$ such that $p \circ i = \text{Id}$.*

We can see that a split superspace is both locally split and projected. As well, a superspace X is split iff $X \cong \text{gr}(X)$. Note that a projected superspace has structure sheaf $\mathcal{O}_X$ which has a $\mathcal{O}_{|X|}$-module structure, that is a map $\mathcal{O}_{|X|} \to \mathcal{O}_X$.

The terms 'locally split' and 'split' supermanifold is equivalent to the the local or global splitting respectively of the exact sequence of sheaves on X below.

$$0 \to \mathcal{J}_X \to \mathcal{O}_X \to \mathcal{O}_{|X|} \to 0$$

Remark 3.3. *We may view a superspace $X = (|X|, \mathcal{O}_X)$ as the classical space $X/\Gamma = (|X|, \mathcal{O}_{X,0})$ with the coherent sheaf $\mathcal{O}_{X,1}$ of $\mathcal{O}_{X,0}$-modules. Even if X is a reduced superspace, X/Γ may be nonreduced due to the nilpotents $\mathcal{O}_{X,1}^2 \subset \mathcal{O}_{X,0}$. ([29] §1 Def 6)*

On the other hand, we may view a superspace $(X, \mathcal{O}_X)$ as the classical space $X_{\text{bos}} = (|X|, \mathcal{O}_{|X|})$ with instead the sheaf $\mathcal{O}_X$, which is not an $\mathcal{O}_{|X|}$-module unless X is projected.

In the next section we will see examples of superspaces and define the concept of dimension of a superspace.

3.2 Superschemes and supermanifolds

Various types of superspaces can be defined based on the local structure of the superspace. We will most often work in the analytic category with complex supermanifolds or in the algebraic category with superschemes.

Definition 3.4. *A* supermanifold *X is a locally ringed superspace which is locally split and such that $X_{\text{bos}} = (|X|, \mathcal{O}_{|X|})$ is a manifold.*

While the underlying manifold may can be any classical notion of a manifold, we will be most concerned with the following.[4]

A smooth supermanifold *is a supermanifold with structure sheaf $\mathcal{O}_X$ locally isomorphic to $\mathcal{C}^\infty_{|X|} \otimes_{\mathbb{R}} \mathbb{R}[\xi_1, \ldots, \xi_n]$ where the ξ_i are odd and therefore anticommuting.*

A complex supermanifold *is a supermanifold with structure sheaf $\mathcal{O}_X$ locally isomorphic to $\mathcal{O}_{|X|} \otimes_{\mathbb{C}} \mathbb{C}[\xi_1, \ldots, \xi_n]$ where the ξ_i are odd and therefore anticommuting and where $\mathcal{O}_{|X|}$ consists of holomorphic functions over $|X|$.*

The literature on supermanifolds is bountiful, for intsance see part 1 ch2,3 of [11], [27], and chapter 4 of [29]. The details of superschemes are less common in the early literature. Recent accounts include [7] and [43].

[4] Another interesting option is that of a cs manifold as explained in part 1 §4.8 of [11].

Definition 3.5. *[7][43] The* superspectrum *of a supercommutative ring A, where we require the ideal J generated by odd elements to be finitely generated, is defined as follows.*

$$\mathbb{S}\mathrm{pec}(A) := (\mathrm{Spec}(A/J), \mathcal{O}) \qquad \text{where} \quad \mathcal{O}(D(f)) := A_f.$$

The element $f \in A$ must be nonnilpotent for the basic open set $D(f)$ to be defined. Note that A_f denotes the localization of A at the multiplicative set generated by f. A superspace defined this way is an affine superscheme*.*

A superschme *is a locally ringed superspace which is locally isomorphic an affine superscheme.*

The definition of superspectrum is analogous to the usual defintion of a spectrum. These notions do not actually differ in considering prime superideals vs prime ideals, since the ideal J is contained in every prime ideal, then every prime ideal is in fact a prime superideal of A. One difference is the implicit commutation relations among the odd variables, that is as classical schemes $\mathbb{S}\mathrm{pec}(A) \cong \mathrm{Spec}(A/(\xi_i\xi_j + \xi_j\xi_i))$ for any odd $\xi_i, \xi_j \in A$.

Another difference is the definition of a reduced superscheme vs reduced classical scheme. We define the superscheme $\mathbb{S}\mathrm{pec}(A)$ to be reduced when $\mathrm{Spec}(A/J)$ is reduced as a classical scheme. Whereas $\mathbb{S}\mathrm{pec}(A)$ considered as a classical scheme is nonreduced whenever A has nonzero odd component.

Yet another difference is the existence of a group law on the superscheme $\mathbb{A}^{m|n}$ given by

$$\begin{aligned} \mathbb{A}^{m|n} \times \mathbb{A}^{m|n} &\to \mathbb{A}^{m|n} \\ (x_1, \ldots, x_m|\xi_1, \ldots, \xi_n) \times (x'_1, \ldots, x'_m|\xi'_1, \ldots, \xi'_n) & \\ &\mapsto (x_1 + x'_1, \ldots, x_m + x'_m|\xi_1 + \xi'_1, \ldots, \xi_n + \xi'_n). \end{aligned}$$

No such group law exists on the classical scheme $\mathrm{Spec}(A/(\xi_i\xi_j + \xi_j\xi_i))$. For instance, $\mathrm{Spec}(k[\xi]/(\xi^2))$ would need a map $k[\xi'']/(\xi''^2) \to k[\xi]/(\xi^2) \otimes k[\xi']/(\xi'^2)$, but if $\xi'' \mapsto \xi + \xi'$ then $(\xi + \xi')^2 = \xi\xi' + \xi'\xi \neq 0$, so the map is not a ring homomorphism. [11]

Example 3.6. [5] *We define affine superspace in various categories.*

In the smooth category, $(m|n)$-dimensional affine space is

$$\mathbb{R}^{m|n} := (\mathbb{R}^m, \mathcal{C}^\infty_{\mathbb{R}^m}[\xi_1, \ldots, \xi_n])$$

where the ξ_i are odd and therefore anticommuting.

[5] In some liturature, the process of associating a superspace to a super vector space may be called 'supermanifoldification'.

In the complex analytic category, $(m|n)$-dimensional affine space is

$$\mathbb{C}^{m|n} := (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m}[\xi_1, \dots, \xi_n])$$

where the ξ_i are odd and therefore anticommuting and where $\mathcal{O}_{\mathbb{C}^m}$ consists of holomorphic functions over $\mathbb{C}^m$.

In the algebraic category, affine space associated to a $(m|n)$-dimensional supervector space V over a ground field k is

$$\mathbb{A}_V := \mathbb{S}\mathrm{pec}(S(V^*)) \cong \mathbb{S}\mathrm{pec}\,(k[x_1, \dots, x_m|\xi_1, \dots, \xi_n]) =: \mathbb{A}_k^{m|n}$$

where $(x_1, \dots, x_m|\xi_1, \dots, \xi_n)$ is a basis for V^*.

Considering in particular a complex supermanifold X, we see that local coordinates on X are of the form $(x_1, \dots, x_m|\xi_1, \dots, \xi_n)$, where the x_i are even functions and the ξ_i are odd functions on some small enough open subsupermaniold U of X. Note that the x_i may not be considered as coordinates on X_{bos} unless X is projected, that is the x_i are not pullbacks of coordinates on X_{bos} unless X is projected 3.2.

The definition of dimension of a superspace should not come as a surprise.

Definition 3.7. *The odd dimension of a supermanifold X is the rank of the locally free $\mathcal{O}_{|X|}$-module $\mathcal{J}_X/\mathcal{J}_X^2$. The even dimension is the dimension of $|X|$ as a real or complex manifold depending on the type of supermanifold.*

The odd dimension of a superscheme X is the supremum of the odd dimensions of the stalks $\mathcal{O}_{X,x}$ for all $x \in |X|$. The even dimension is the dimension of the scheme X_{bos}. Both may be infinite.

The super GAGA principle has been studied in particular in [39]. Here it is shown that there is a analyticifcation function taking subsupervarieties $X \in \mathbb{P}^{m|n}$ to a complex supermanifold X^h such that the maps $H^q(X, \mathcal{F}) \to H^q(X^h, \mathcal{F}^h)$ are isomorphisms when $\mathcal{F}$ is coherent. We use the super GAGA principle to pass freely between the algebraic and analytic categories, which is done mostly for convienience

Supermanifolds may be constructed similarly to classical manifolds using methods such as gluing charts and submanifolds defined by equations. See [17] for more examples.

Example 3.8. *Consider the nondegenerate conic in $\mathbb{P}^{2|2}$ given by the equation*

$$x_0^2 + x_1^2 + x_2^2 + \theta_1\theta_2 = 0.$$

As shown in [29], this defines a non-split supermanifold of dimension $1|2$.

Example 3.9. *[44] Consider the complex supermanifold* $\mathbb{P}^{2|1}$ *along with a homogeneous equation* $f(z_0, z_1, z_2|\zeta) = 0$*. The submanifold defined by this equation will be a* $1|1$ *complex supermanifold. However, it will not in general be a super Riemann surface.*

Remark 3.10. *In certain cases, the property of being split is easily characterized. [17]*

Every smooth supermanifold is split. This follows from the fact that every locally free C^∞ *sheaf is fine.*

Every $(m|1)$*-dimensional supermanifold is split.*

A $(m|2)$*-dimensional complex supermanifold is split if and only if it is projected.*

3.3 Functor of points

Alternatively to viewing superspaces as ringed spaces, we may adopt the notion common in algebraic geometry and physics of the functor of points. The functor of points has the advantage of being closer to geometric intuition.

Definition 3.11. *Let* X *be a superspace. The* functor of points of X *is the contravariant functor*

$$\mathrm{Hom}(\ \ , X)\colon \textit{sSpa} \to \textit{Sets} \qquad\qquad S \mapsto \mathrm{Hom}(S, X).$$

The key utility of the functor of points in supergeometry comes from the fact that superspaces cannot be described via their classical points. The concept of an S-point provides the analogous geometric intuition while fully capturing the super structure.

Definition 3.12. *An* S-point *of a superspace* X *is an element of* $\mathrm{Hom}(S, X)$*.*

Example 3.13.

- *Consider* $S = \mathbb{C}^{0|0}$*, a point. Then an* S*-point of* X *is simply a point of the topological space* $|X|$*. That is* $\mathrm{Hom}(S, X) \cong |X|$*.*
- *Consider* $X = \mathbb{C}^{m|n}$*. Then an* S*-point of* X *is a choice of* m *even functions on* S *and* n *odd functions on* S*. That is* $\mathrm{Hom}(S, X) \cong \mathcal{O}_{S,0}(S)^m \times \mathcal{O}_{S,1}(S)^n$*. We can rewrite this more concisely as* $\mathrm{Hom}(S, X) \cong (\mathcal{O}_S(S) \otimes X)_0$*, where* X *on the right hand side is considered as a super vector space.*
- *Consider* $S = \mathbb{C}^{0|s}$*, a superpoint. For* X *a complex supermanifold of dimension* $m|n$*, then an* S*-point of* X *is a point of* $|X|$ *and* n *odd functions on* S*.*

Sometimes it is best to construct a superspace first by defining its functor of points and then showing it is representable. The Yoneda embedding is the functor

$$よ\colon \texttt{sSpa} \to \texttt{PSh}(\texttt{sSpa}) \qquad\qquad X \mapsto \mathrm{Hom}(\quad, X)$$

where PSh denotes the category of presheaves. We say a functor $\texttt{sSpa} \to \texttt{Sets}$ is representable if it is in the image of the functor よ. We also commonly check if a functor is representable as a supermanifold or superscheme specifically, which is done by modifying the Yoneda embedding above for the appropriate category sMan or sSch. As well, a version of the Yoneda embedding exists for families of superspaces.

3.4 Super vector bundles

As in classical differential geometry, super vector bundles are viewed interchangeably with locally free $\mathcal{O}$-modules of finite constant rank. While in most settings the sheaf viewpoint will be used, we will need the fiber bundle definition when considering Lie algebroids in 4.3.

Definition 3.14 (Balduzzi, Carmeli, Cassinelli [3])**.** *A* super vector bundle *is* (E, π, X, V) *such that*

- *E and X are superspaces and $\pi\colon E \to X$ is a surjective submersion*
- *there exists an cover $\{U_i\}$ of X where $|U_i|$ is open with isomorphisms $t_i\colon \pi^{-1}(U_i) \to U_i \times V$ such that $\mathrm{pr}_{U_i} \circ t_i = \pi$ and V is a super vector space*
- *if $U_{i,j} := U_i \cap U_j \neq \varnothing$, then there exists a map $\varphi_{i,j}\colon U_{i,j} \to \mathrm{GL}(V)$ such that the automorphism $t_i \circ t_j^{-1}$ of $U_{i,j} \times V$ factors through $U_{i,j} \times \mathrm{GL}(V) \times V$ as:*

$$\begin{array}{ccccc} & & \pi^{-1}(U_{i,j}) & & \\ & \swarrow^{t_j} & & \searrow^{t_i} & \\ U_{i,j}\times V & \xrightarrow{(\mathrm{Id}_{U_{i,j}},\varphi_{i,j})\times \mathrm{Id}_V} & U_{i,j}\times \mathrm{GL}(V)\times V & \xrightarrow{\mathrm{Id}_{U_{i,j}}\times \mathrm{ev}} & U_{i,j}\times V\end{array}$$

  where ev *is the evaluation of* $\mathrm{GL}(V)$ *on* V*.*

We call a pair $(U, t\colon \pi^{-1}(U) \to U \times V)$ a local trivialization of the supervector bundle.

A morphism of super vector bundles *from (E, π, X, V) to (F, ϖ, Y, W) is a morphism $f\colon E \to F$ such that*

- *there exists a morphism $f_\flat\colon X \to Y$ such that $f \circ \varpi \cong \pi \circ f_\flat$*
- *for all $x \in |X|$ there exists $x \in U \subset X$ and local trivializations (U, u) on E and (T, t) on F such that $f_\flat(|U|) \subseteq |T|$.*

- *there exists a map* $\varphi_U \colon U \to \underline{\mathrm{Hom}}(V, W)$ *such that*

$$\begin{array}{ccccc} \pi^{-1}(U) & & \xrightarrow{\quad f|_{\pi^{-1}(U)} \quad} & & \pi^{-1}(T) \\ \downarrow u & & & & \downarrow t \\ U \times V & \xrightarrow{(\mathrm{Id}_U, \varphi_U) \times \mathrm{Id}_V} & U \times \underline{\mathrm{Hom}}(V, W) \times V & \xrightarrow{f_b \times \mathrm{ev}} & T \times W \end{array}$$

commutes, where ev *is the evaluation of* $\underline{\mathrm{Hom}}(V, W)$ *on* V*.*

In [3] Proposition 4.22, it is proved that the category of supervector bundles described above and the category of locally free $\mathcal{O}$-modules of finite constant rank are equivalent. This can be seen by describing the functors between the categories. We describe these functors over a small enough superdomain that the vector bundle trivializes. For a super vector bundle E trivial over a superdomain U, such that $E|_U \cong U \times V$, then the associated sheaf of sections is $\mathcal{O}_U \otimes V$. In the opposite direction, for a free sheaf $\mathcal{E}$ over U, the S-points are defined as pairs (f, s) where $f \colon S \to U$ is a morphism and $s \colon S \to f^*\mathcal{E}$ is an even section. The superspace representing these S-points is $U \times V$ where $\dim(V) = \mathrm{rank}(\mathcal{E})$. [11]

Note that we refrain from calling a super vector bundle even or odd. For example, a rank $m|0$ vector bundle is locally isomorphic to $\mathcal{O}_X^m$ which certainly has nonzero even and nonzero odd parts (when X has nonzero odd dimension).

As in classical geometry, a super vector bundle morphism naturally gives a morphism of $\mathcal{O}$-modules, but the converse is not true. In particular, not every subsheaf of a locally free $\mathcal{O}$-modules of finite constant rank is subbundle. Let $\mathcal{E}$ be a locally free $\mathcal{O}$-modules of finite constant rank. Then the subsheaf $\mathcal{F}$ is a subbundle of $\mathcal{E}$ if $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are locally free.

3.5 Super Grassmannians

We follow Ch. 4 §3 of [29] in this section.

The (relative) finite dimensional super Grassmannian can be defined by its functor of points.

Definition 3.15. *Let* E *be a supervector bundle of on a superspace* X*. For* $c|d \leq \mathrm{rank}(E)$*, let* $\mathrm{Gr}_X(c|d, E) = \mathrm{Gr}_X(c|d, \mathcal{E})$ *be the functor that takes a superspace* S *over* X *to the family of rank* $c|d$ *subbundles of* $f^*\mathcal{E} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_S$ *where* $f \colon S \to X$*.*

For X *a one point space, we have* E *reduced to a supervector space* V*, and therefore denote the (absolute) super Grassmannian as* $\mathrm{Gr}(c|d, V)$*.*

The bosonization of $\mathrm{Gr}(c|d, V)$ is given by $\mathrm{Gr}(c, V_0) \times \mathrm{Gr}(d, V_1)$, the product of two classical Grassmannians. We verify this below.

Define an embedding $\mathrm{Gr}(c, V_0) \times \mathrm{Gr}(d, V_1) \hookrightarrow \mathrm{Gr}(c|d, V)$ as follows. An S-point of $\mathrm{Gr}(c, V_0) \times \mathrm{Gr}(d, V_1)$ is a pair

$$\mathcal{L}_0 \to \mathcal{O}_S \otimes V_0 \to \mathcal{Q}_0 \to 0 \qquad\qquad \mathcal{L}_1 \to \mathcal{O}_S \otimes V_1 \to \mathcal{Q}_1 \to 0 \tag{13}$$

where $\mathrm{rank}(\mathcal{L}_0) = c$, $\mathrm{rank}(\mathcal{L}_1) = d$, and $\mathcal{Q}_i$ are locally free. This maps to the S-point of $\mathrm{Gr}(c|d, V)$ given by the direct sum

$$\mathcal{L}_0 \oplus L_1 \to \mathcal{O}_S \otimes V \to \mathcal{Q}_0 \oplus \mathcal{Q}_1 \to 0$$

where $\mathcal{L}_1$ and $\mathcal{Q}_1$ are now considered odd.

We wish to show this embedding satisfies the universal property (12), that is that any morphism from a classical space to $\mathrm{Gr}(c|d, V)$ factors through this embedding. This fact is easily seen since for S a classical space, any S-point of $\mathrm{Gr}(c|d, V)$

$$\mathcal{L} \to \mathcal{O}_S \otimes V \to \mathcal{Q} \to 0$$

with $\mathcal{L}$ locally free of rank $c|d$ and $\mathcal{Q}$ locally free, can be canonically decomposed into its even and odd parts given precisely as (13).

It is known that super Grassmannians may not to superprojected. This is an interesting feature of supregeometry. Super Grassmannians seem to be important since most superspaces with projected bosonization embedd into a super Grassmannian. [7]

Example 3.16. *Projective superspace may be defined as the super Grassmannian of rank* $1|0$ *subbundles.*

Let V *be a supervector space. Then we define* $\mathbb{P}(V) := \mathrm{Gr}(1|0, V)$*. Further, denote* $\mathbb{P}^{m|n} := \mathrm{Gr}(1|0, \mathbb{C}^{m+1|n})$*.*

It is known that $\mathbb{P}^{m|n}$ *is split [29]. Explicitly*

$$\mathbb{P}^{m|n} \cong \mathrm{gr}\mathbb{P}^{m|n} \cong \left(\mathbb{P}^m, S\left(\left(\mathbb{C}^{0|n}\right)^* \otimes \mathcal{O}_{\mathbb{P}^m}(-1)\right)\right).$$

We now show that $\mathrm{Gr}_X(c|d, E)$ is representable. Consider a covering of X by affine superspaces or Stein open superspaces depending on the category such that E trivializes over the elements of the cover. Then we may reduce to the case of $\mathrm{Gr}_A(c|d, M)$ for M an A-module of rank $m|n$, where $X = \mathrm{Spec}(A)$ or similarly in the complex analytic or smooth category.

Let $L \in \mathrm{Gr}_A(c|d, M)$. Then to each L we can associate a superdomain U_L of the Grassmannian. Consider all possible morphisms

$$\begin{pmatrix} \mathrm{Id}_L \\ Z \end{pmatrix} : \; L \to M$$

where $Z\colon L \to M/L$ may be written as a $(m-c|n-d) \times c|d$ matrix with entries from A. We define $U_L \subseteq \mathrm{Gr}_A(c|d, M)$ to be the superdomain with Z representing its coordinates. We notice that $U_L \cong X \times \mathbb{C}^{(m-c)c+(n-d)d|(m-c)d+(n-d)c}$ in the complex analytic category and $U_L \cong \mathrm{Spec}(A[z_i^k, z_j^l|\zeta_j^k, \zeta_i^l])$ in the algebraic category where z are the even entries of Z and ζ are the odd entries of Z using the indexing notation of (4).

We describe the gluing of these superdomains. Let U_L and $U_{L'}$ be two superdomains as above. We write the identity map on M as the block matrix

$$\mathrm{Id}_M = \begin{pmatrix} T^{L'L} & T^{L'K} \\ T^{K'L} & T^{K'K} \end{pmatrix}$$

where we denote $K := M/L$ and $K' := M/L'$, and $T^{L'K}\colon K \to L'$, etc. Then we define the change of coordinates from Z_L to $Z_{L'}$ as

$$Z_{L'} = \left(T^{K'L} + T^{K'K} Z_L\right)\left(T^{L'L} + T^{L'K} Z_L\right)^{-1}. \tag{14}$$

Clearly this is only defined on the subdomain of U_L where $T^{L'L} + T^{L'K} Z_L$ is invertible, which is an open condition. As well, this definition clearly gives transition functions which are rational functions, which are a part of the functions in any of the categories we are considering. One can check that this satisfies the cocycle condition on the intersection on three superdomains.

To elucidate the change of coordinates definition and to define a group action, consider any automorphism T on M instead of the identity. A nautral left action of the supergroup $\mathrm{GL}(M, A) := \mathrm{Aut}(M)$ is defined by the matrix multiplication below.

$$\begin{pmatrix} T^{L'L} & T^{L'K} \\ T^{K'L} & T^{K'K} \end{pmatrix} \begin{pmatrix} I \\ Z_L \end{pmatrix} = \begin{pmatrix} I \\ Z_{L'} \end{pmatrix} Q$$

where $Q\colon L \to L'$ is some isomorphism.

As for the classical Grassmannian, there is a tautological sheaf over the super Grassmannian. Over the superdomain U_L, we define the tautological bundle to be the trivial bundle over U_L with supervector space the column space of $\left(\begin{smallmatrix} \mathrm{Id}_L \\ Z_L \end{smallmatrix}\right)$. This free bundle of rank $c|d$ is then glued across superdomains U_L using the same change of coordinates as in (14).

The proof that this construction represents the functor $\mathrm{Gr}_X(c|d, E)$ is analogous to the proof in classical geometry. See Chapter 4 §3 theorem 10 of [29]. We note that the S-points of our constructed superspace G above are the morphisms $\mathrm{Hom}_X(S, G)$ to which we can associate the pullback to S of the tautological sheaf of G. Representability is equivalent to showing that any rank $c|d$ subbundle of the pullback of E corresponds uniquely to a morphism in $\mathrm{Hom}_X(S, G)$.

3.6 Differential structure

In this section we define the tangent and cotangent sheaf of a superspace and discuss their properties. We follow §3.3 of [11]. Then we discuss the Berezinian bundle following §3.12 of [11].

Definition 3.17. *Let $D \in \underline{\mathrm{Hom}}(A, A)$, where A is a superalgebra. Then D is a* derivation *of A if it satisfies the sign rule*

$$D(ab) = D(a)b + (-1)^{|a||D|}aD(b)$$

for all $a, b \in A$.

We define the tangent sheaf of a superspace X to be the sheaf $\mathrm{Der}(\mathcal{O}_X)$, the sheaf of derivations of the structure sheaf. We denote the tangent sheaf by $\mathcal{T}_X$ and its sections are called vector fields as usual. The space of all vector fields $\mathrm{Vect}(M)$ is a super vector space and a super Lie algebra as discussed in section 4.1.

On $\mathbb{C}^{m|n}$ with coordinates z^i and ζ^i, we have the derivations $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \zeta^i}$ defined by

$$\frac{\partial}{\partial z^i}\left(\sum_I f_I \zeta^I\right) := \sum_I \frac{\partial}{\partial z^i}(f_I)\zeta^I$$

$$\frac{\partial}{\partial \zeta^i}\left(\sum_{i \notin I}(f_I \zeta^I + f_{i,I}\zeta^i\zeta^I)\right) := \sum_I \frac{\partial}{\partial \zeta^i}(f_{i,I})\zeta^I.$$

Accordingly, $\frac{\partial}{\partial z^i}$ is even and $\frac{\partial}{\partial \zeta^i}$ is odd. This definition suffices to define the tangent sheaf for any complex supermanifold as locally the space is isomorphic to $\mathbb{C}^{m|n}$.

Proposition 3.18 (Leites [27])**.** *For $X = \mathbb{C}^{m|n}$, the tangent sheaf $\mathcal{T}_X$ is free of rank $m|n$ with basis $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \zeta^i}$.*

Then for any complex supermanifold X, $\mathcal{T}_X$ is locally free of rank $m|n$, with local basis $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \zeta^i}$ in the local coordinates $(z_1, \ldots, z_m|\zeta_1, \ldots, \zeta_n)$.

The cotangent sheaf to a superspace X is defined as the dual to the tangent sheaf, that is $\Omega_X := \underline{\mathrm{Hom}}(\mathcal{T}_X, \mathcal{O}_X)$. Therefore, the cotangent sheaf is locally free of rank $m|n$. The local basis consists of the one-forms dx^i and $d\zeta^i$, which satisfy the duality pairing

$$\left\langle \frac{\partial}{\partial z^j}, dz^i \right\rangle = \delta_{i,j} \qquad \left\langle \frac{\partial}{\partial \zeta^j}, dz^i \right\rangle = 0$$

$$\left\langle \frac{\partial}{\partial z^j}, d\zeta^i \right\rangle = 0 \qquad \left\langle \frac{\partial}{\partial \zeta^j}, d\zeta^i \right\rangle = \delta_{i,j}.$$

Accordingly, dz^i is even and $d\zeta^i$ is odd. We adopt this sign convention, which means

$$\omega(D) := (-1)^{|D||\omega|}\langle D, \omega\rangle. \tag{15}$$

The exterior derivative can be defined as in classical geometry with care taken for the sign convention. Define $d\colon \mathcal{O}_X \to \Omega_X$ by to satisfy

$$\langle D, df\rangle = Df. \tag{16}$$

As in classical geometry, d is a derivation of Ω which extends to a degree 1 derivation of the graded superalgebra $\wedge^*\Omega$.

$$d(\alpha\beta) = d\alpha \cdot \beta(-1)^{\deg(\alpha)}\alpha \cdot d\beta \qquad\qquad d^2 = 0$$

Since indeed this derivation squares to zero, this defines the differential graded superalgebra of differential forms on a supermanifold. The sign convention for graded superalgebras was discussed in section 2.3 and we follow Deligne's sign convention (3). Accordingly

$$dz \wedge dz = -dz \wedge dz = 0 \qquad\qquad d\zeta \wedge d\zeta \neq 0$$

and thus $d\zeta \wedge \cdots \wedge d\zeta \neq 0$ for any number of wedge products.

Analogously to the classical case, de Rham cohomology holds:

Lemma 3.19 (Super Poincaré Lemma, 3.3.4 of [11])**.** *The complex $\Omega^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}$ on $|X|$.*

However, in contrast to the classical dga of differential forms, the super dga of differential forms does not have top forms. That is, $\Omega^p_X \neq 0$ for all $p \geq 0$ when X has dimension $m|n$ with $n > 0$. This indicates that differential forms are not the correct objects to integrate over a supermanifold.

Unique to supergeometry is the Berezinian, the super determinant. From the Berezinian we can construct integral forms, which are the correct objects for integration.

Let $\mathcal{F}$ be a vector bundle of rank $m|n$ with associated transition functions $\{f_{ij}\}$. Then we define $\operatorname{Ber}\mathcal{F}$ to be the bundle with transition functions $\{\operatorname{Ber} f_{ij}\}$. We enforce that $\operatorname{Ber}\mathcal{F}$ is rank $1|0$ if n is even and rank $0|1$ if n is odd. We denote a trivializing section of $\operatorname{Ber}\mathcal{F}$ as $[e_1, \ldots, e_m|f_1, \ldots, f_n]$, where $e_1, \ldots, e_m|f_1 \ldots f_n$ are trivializing local coordinates for $\mathcal{F}$.

Analogously to classical determinants, for any exact sequence of supervector bundles on a supermanifold X

$$\cdots \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+2} \to \cdots$$

results in an isomorphism $\otimes(\operatorname{Ber}\mathcal{F}_i)^{(-1)^i} = \mathcal{O}_X$.

For a supermanifold X, we define the Berezinian of the superspace X to be $\operatorname{Ber} X := \operatorname{Ber}\Omega^1_X$, the Berezinian of the cotangent sheaf. Explicitly the change of coordinates can be written as

$$[dz_1, \ldots, dz_m | d\zeta_1, \ldots, d\zeta_n] = \operatorname{Ber}\left(\frac{\partial(z|\zeta)}{\partial(x|\xi)}\right)[dx_1, \ldots, dx_m | d\xi_1, \ldots, d\xi_n].$$

A section of $\operatorname{Ber} X$ is called a density.

Consider the real supermanifold $X = \mathbb{R}^{m|n}$. For $f \in \mathcal{O}_X$ we define the integral of f to be

$$\int_{\mathbb{R}^{m|n}} f(x|\xi)[dx_1, \ldots, dx_m | d\xi_1, \ldots, d\xi_n] := \int_{\mathbb{R}^m} \frac{\partial^n}{\partial\xi_n \cdots \partial\xi_1} f(x|\xi) dx_1 \ldots dx_m.$$

This formula reduces the integral to an ordinary integral over the reduced space, of the coefficient of the highest term expanded in the ξs. A similar definition holds for any other split real supermanifold. Extending this to nonsplit supermanifolds only requires a partition of unity argument.

Consider now a complex supermanifold X. It is customary to write $\omega := \operatorname{Ber} X$. As in classical complex analysis, we must consider smooth densities instead of holomorphic densities for integration. We denote by $\mathcal{E}$ the sheaf of smooth superfunctions on X. Then the smooth Berezinian sheaf is $\omega \otimes \overline{\omega} \otimes \mathcal{E}$. The sections of the smooth Berezinian sheaf are of Hodge type $(m|n, m|n)$ and are the correct objects for integration over X, where as the sections of ω are of Hodge type $(m|n, 0)$.

Alternatively, we may write the change of coordinates $\operatorname{Ber}(d\varphi)$ where $d\varphi$ is the differential map on the tangent spaces of supermanifolds with morphism $\varphi\colon (x_i|\xi_i) \to (z_i|\zeta_i)$ between local coordinate systems. We may call this map the Jacobian matrix

$$d\varphi = \frac{\partial(z|\zeta)}{\partial(x|\xi)} := \begin{pmatrix} \frac{\partial z}{\partial x} & -\frac{\partial z}{\partial \xi} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial \xi} \end{pmatrix}$$

The block matrix above is written in even-odd decomposition to show its structure as a supermatrix. The negative appearing in the block decomposition arises due to the definition of the chain rule below.

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial z_j}{\partial x_i}\frac{\partial}{\partial z_j} + \sum_k \frac{\partial \zeta_k}{\partial x_i}\frac{\partial}{\partial \zeta_k} \qquad \frac{\partial}{\partial \xi_i} = \sum_j \frac{\partial z_j}{\partial \xi_i}\frac{\partial}{\partial z_j} + \sum_k \frac{\partial \zeta_k}{\partial \xi_i}\frac{\partial}{\partial \zeta_k}$$

In order to write the chain rule using matrix multiplication, the sign rule generates the negative. Details may be found in [41].

The definitions of a super diffeomorphism, immersion, and submersion may be defined similarly to the classical setting. In brief, an immersion is a morphism $\varphi\colon X \to Z$ of supermanifolds for which the tangent map $d\varphi_x\colon TX_x \to TZ_{\varphi(x)}$ is injective map of super vector spaces for all $x \in X$. Similarly, a submersion is when $d\varphi_x$ is surjective, and a diffeomorphism is when $d\varphi_x$ is bijective.

4 Super Lie theory

In this section we review the basic construction of Lie theoretic objects in super algebra and super geometry. Then Neveu-Schwarz Lie superalgebra is discussed in detail which is key to the main argument in this paper.

4.1 Lie superalgebras

We follow [41] and [21] in this section.

Definition 4.1. *A* Lie superalgebra *is a super vector space* $\mathfrak{g}$ *with a morphism* $[\ ,\ ]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ *satisfying*

- $[x,y]+(-1)^{|x||y|}[y,x]=0$
- *the super Jacobi identity* $(-1)^{|x||z|}[x,[y,z]]+(-1)^{|y||x|}[y,[z,x]]+(-1)^{|z||y|}[z,[x,y]]=0$

The differences in the definition of a Lie superalgebra and classical Lie algebra are precisely the rule of signs. Therefore one can easily see a Lie superalgebra is simply a Lie object in the category of super vector spaces.

As well, one can see that the even component $\mathfrak{g}_0$ of a Lie superalgebra has the structure of a classical Lie algebra. If we then describe the structure on top of this classical Lie algebra, we see that a super Lie algebra $\mathfrak{g}$ with $[\ ,\ ]$ is precisely:

- a classical Lie algebra $\mathfrak{g}_0$ under $[\ ,\ ]$
- with $\mathfrak{g}_1$ a $\mathfrak{g}_0$-module under the adjoint map $a\mapsto \mathrm{ad}_a=[a,\ ]$
- such that $[\ ,\ ]\colon \mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{g}_0$ is a asymmetric $\mathfrak{g}_0$-modules map
- and such that $[a,[a,a]]=0$ for all $a\in\mathfrak{g}_1$.

For any superalgebra, the definition $[a,b]:=ab-(-1)^{|a||b|}ba$ endows it with the structure of a Lie superalgebra. Clearly, if the superalgebra is supercommutative, then the bracket is identically zero and the Lie superalgebra is abelian.

Definition 4.2. *The superalgebra* $\underline{\mathrm{End}}(V)$ *when considered with a Lie structure is denoted* $\mathfrak{gl}(V)$*, the* general linear *Lie superalgebra. We denote* $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$*.*

In other literature, $\mathfrak{gl}(m|n)$ may refer to $\mathfrak{gl}(\mathbb{R}^{m|n})$ instead.

Definition 4.3. *The kernel of* $\mathrm{str}\colon \mathfrak{gl}(V) \to \mathfrak{gl}(1|0) = \mathbb{C}$ *is the* special linear *Lie superalgebra* $\mathfrak{sl}(V)$*. We denote* $\mathfrak{sl}(m|n) := \mathfrak{sl}(\mathbb{C}^{m|n})$*.*

The even component of $\mathfrak{gl}(V)$ is $\mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1)$. The even component of $\mathfrak{sl}(V)$ contains $\mathfrak{sl}(V_0) \oplus \mathfrak{sl}(V_1)$, but for V of nonzero even and nonzero odd dimension, it is not isomorphic to it.

Definition 4.4. *The* orthosymplectic *Lie superalgebra is denoted* $\mathfrak{osp}(\Phi)$ *where* $\Phi\colon V \times V \to \mathbb{C}$ *is a symmetric nondegenerate even bilinear form. In particular, this means* Φ *restricted to* $V_0 \times V_0$ *is symmetric nondegenerate, restricted to* $V_0 \times V_1$ *is identitcally zero, and restricted to* $V_1 \times V_1$ *is a symplectic form. Thus* n *must be even. The orthosymplectic Lie superalgebra is defined*

$$\mathfrak{osp}(\Phi) := \{X \in \mathfrak{gl}(V)\colon\ \Phi(Xv, w) + (-1)^{|X||v|}\Phi(v, Xw) = 0 \text{ for all } v, w, \in V\}.$$

For $V = \mathbb{C}^{m|n}$*, we denote* $\mathfrak{osp}(m|n) := \mathfrak{osp}(\Phi)$*.*

We can see that the even component of $\mathfrak{osp}(m|n)$ is $\mathfrak{so}(m) \oplus \mathfrak{sp}(n)$. And the odd component of $\mathfrak{osp}(m|n)$ is $\mathfrak{so}(m) \otimes \mathfrak{sp}(n)$.

A complete classification of simple finite dimensional Lie superalgebras over $\mathbb{C}$ was found in Theorem 5 of [21]. We will not discuss the other simple Lie superalgebras since they will not come up later in this paper.

4.2 Lie supergroups

We discuss Lie supergroups rather minimally. Further info can be found in Ch 7 of [8] or Chapter VII of [4]. We have not discussed the Lie superalgebra associated to a Lie supergroup since we do not make use of this, but the expected relationship holds.

Definition 4.5. *A* Lie supergroup *is a supermanifold* $G = (|G|, \mathcal{O}_G)$ *with the additional structure:*

- *a product morphism* $\mu\colon G \times G \to G$ *which is associative:* $\mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_3)) \cong \mu \circ (\mu \circ \pi_1 \times \pi_2) \times \pi_3)\colon G \times G \times G \to G$
- *an identity point* $e \in |G|$ *which defines a map* $e\colon G \to G$ *which factors through the one-point space such that* $\mu \circ (\mathrm{Id} \times e) = \mathrm{Id} = \mu \circ (e \times \mathrm{Id})$

- *an inverse morphism which is an involutive superdiffeomorphism* $\sigma\colon G \to G$ *such that* $\mu \circ (\mathrm{Id} \times \sigma) \cong \varepsilon \cong \mu \circ (\sigma \times \mathrm{Id})$.

Example 4.6 ([6]). • *In classical Lie theory, we may consider* $\mathbb{R}^1$ *and* $\mathbb{C}^1$ *as Lie groups with their natural additive group structure.*

We may do the same with the supermanifolds $\mathbb{R}^{1|1}$ *and* $\mathbb{C}^{1|1}$. *Let* $(z|\zeta)$ *denote coordinates on either space. The product morphism* μ *corresponds to the sheaf map* $\mu^\sharp$ *which is* $\mu^\sharp(z) = z_1 + z_2$ *and* $\mu^\sharp(\zeta) = \zeta_1 + \zeta_2$. *The distinguished point is clearly the origin* $e = 0$ *and the inverse morphism is* $\sigma^\sharp(z) = -z$ *and* $\sigma^\sharp(\zeta) = -\zeta$.

The same construction works to endow $\mathbb{R}^{m|n}$ *and* $\mathbb{C}^{m|n}$ *with a group structure.*

- *Analogously to the classical Lie groups* $\mathbb{R}^*$ *and* $\mathbb{C}^*$, *we may consider supergroup counterparts.*

  Define $\mathbb{R}^{1*|1} := \mathbb{R}^{1|1}|_{\mathbb{R}^*}$ *and* $\mathbb{C}^{1*|1} := \mathbb{C}^{1|1}|_{\mathbb{C}^*}$. *In local coordinates, that is those* $(z|\zeta)$ *such that* $z \neq 0$. *The product morphism* μ *corresponds to the sheaf map* $\mu^\sharp$ *which is* $\mu^\sharp(z) = z_1 \cdot z_2 + \zeta_1 \cdot \zeta_2$ *and* $\mu^\sharp(\zeta) = z_1 \cdot \zeta_2 + \zeta_1 \cdot z_2$. *The distinguished point is* $e = 1$ *and the inverse morphism is* $\sigma^\sharp(z) = \frac{1}{z}$ *and* $\sigma^\sharp(\zeta) = -\left(\frac{1}{z}\right)^2 \zeta$.

Definition 4.7. *For* V *a super vector space, the notation* $\mathrm{GL}(V)$ *may denote the super vector space* $\mathrm{Aut}(V)$ *or the super Lie group associated to it.*

The Lie supergroup $\mathrm{GL}(V)$ *is the supermanifold whose bosonization is the classical manifold* $\mathrm{GL}(V_0) \times \mathrm{GL}(V_1)$. *Then* $\mathrm{GL}(V)$ *is then then corresponding split supermanifold of odd dimension* $2\dim(V_0) \cdot \dim(V_1)$.

Alternatively we may define the supergroup by the functor of points

$$S \mapsto \mathrm{Aut}(\mathcal{O}_S \otimes V). \tag{17}$$

More explicitly, for V *of dimension* $m|n$, *an* S*-point of* $\mathrm{GL}(V)$ *is* $m^2 + n^2$ *even functions on* S *and* $2mn$ *odd functions on* S *which can be written as a matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad A, D \text{ even and } B, C \text{ odd}$$

where $\det A \cdot \det D \in \mathcal{O}(S)^* = \mathcal{O}(S)_0^*$, *an invertible function on* S.

The product map, identity element, and inverse map are given by the usual matrix operations. We may denote $\mathrm{GL}(m|n) := \mathrm{GL}(V)$ *when* $\dim(V) = m|n$.

4.3 Lie superalgebroids

We summarize the basics of Lie superalgebroids and specifically Atiyah superalgebras. A Lie (super)algebroid is simply the many-object generalization of a Lie (super)algebra. A good reference is [26].

Definition 4.8. *A* Lie algebroid *on a $\mathbb{C}$-manifold M is a $\mathbb{C}$ vector bundle $E \to M$ with a vector bundle map $a\colon E \to TM$, called the anchor map, where TM is the tangent bundle of M, and with a $\mathbb{C}$-bilinear alternating bracket $[\ ,\ ] : \Gamma(E, M) \wedge \Gamma(E, M) \to \Gamma(E, M)$, which satisfies the Jacobi identity, $a([X, Y]) = [a(X), a(Y)]$, and $[X, fY] = f[X, Y] + a(X)(f)Y$ for all $X, Y \in \Gamma(E, M)$ and $f \in \mathcal{O}_M(M)$.*

Given Lie algebroids E and F on the same base M, a Lie algebroid morphism *is a $\mathbb{C}$ vector bundle morphism $\varphi\colon E \to F$ over M such that $b \circ \varphi = a$ and $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$, for all $X, Y \in \Gamma(E, M)$.*

Definition 4.9. *Let L be a line bundle on a manifold X. We take as definition the Atiyah algebra $\mathcal{A}_L$ on X is the Lie algebroid of order 1 operators on the line bundle L. It is a Lie algebra extension with a compatible left $\mathcal{O}_X$-module structure:*

$$0 \to \mathcal{O}_X \to \mathcal{A}_L \xrightarrow{\text{sym}_1} \mathcal{T}_X \to 0, \tag{18}$$

where the anchor map is the symbol map defined as $\text{sym}_1(D)(f) = [D, f]$.

Definition 4.10. *The action Lie algebroid associated to a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \text{Vect}(X)$ is given by the trivial vector bundle $X \times \mathfrak{g}$ with anchor map $a\colon X \times \mathfrak{g} \to TX$ such that $a(x, X) = \varphi(X)(x)$, and bracket $[V, W] = \widehat{\varphi}(V)(W) - \widehat{\varphi}(W)(V) + [V, W]^\bullet$, where $V, W \in \Gamma(X \times \mathfrak{g}, X)$ and $\widehat{\varphi}(V)(x) = \varphi(V(x))(x)$.*

To generalize the above definitions to Lie superalgebroids and Atiyah superalgebras, we simply require that any vector bundle map is a super vector bundle map (parity preserving) and that any commutator is the super commutator [30].

The Atiyah (super)algebra $\mathcal{A}_L$ of a line bundle L on a (super)manifold X may be regarded as the Lie (super)algebroid of infinitesimal symmetries of the pair (X, L).

We describe the pushforward of an Atiyah algebra. For $\lambda \in \mathbb{C}$, we define $\lambda\mathcal{A}_L$ as the semi-direct product $\mathcal{O}_X \otimes \mathcal{A}_L$ subject to the relations $(\lambda f, 0) = (0, f)$ for $f \in \mathcal{O}_X$. There is a canonical map of Lie algebroids $\mathcal{A}_L \to \lambda\mathcal{A}_L$ which commutes as below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_L & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda\,\text{Id}} & & \downarrow & & \downarrow{\scriptstyle \text{Id}} & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \lambda\mathcal{A}_L & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \end{array} \tag{19}$$

Further we define the Atyiah algebra $\mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2}$ to be the semi-direct product $\mathcal{O}_X \rtimes (\mathcal{A}_{L_1} \times_{\mathcal{T}_X} \mathcal{A}_{L_2})$ subject to the relations $(f+g, 0) = (0, (f, g))$. This definition gives the commutative diagram of the first two rows below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X \times \mathcal{O}_X & \longrightarrow & \mathcal{A}_{L_1} \times_{\mathcal{T}_X} \mathcal{A}_{L_2} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle (f,g)\mapsto f+g} & & \Big\downarrow \quad \searrow^{\varphi} & & \Big\downarrow {\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \mathrm{Id}} & & \Big\downarrow \wr & & \Big\downarrow {\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{A}_{L_1 \otimes L_2} & \longrightarrow & \mathcal{T}_X & \longrightarrow & 0
\end{array}$$

Now consider $\varphi\colon \mathcal{A}_{L_1} \times_{\mathcal{T}_X} \mathcal{A}_{L_2} \to \mathcal{A}_{L_1 \otimes L_2}$ given by the Leibniz rule

$$\varphi(\delta_1, \delta_2)(s_1 \otimes s_2) = \delta_1(s_1) \otimes s_2 + s_1 \otimes \delta_2(s_2),$$

where $\delta_i \in \mathcal{A}_{L_i}$ and $s_i \in \Gamma(L_i)$. From this it follows that there is a morphism of Lie algebroids $\mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2} \xrightarrow{\sim} \mathcal{A}_{L_1 \otimes L_2}$ which is a canonical isomorphism.

Comparing the definitions of the Atiyah algebras $\mathcal{A}_L \otimes \mathcal{A}_L$ and $2\mathcal{A}_L$, you find a canonical isomorphism $\mathcal{A}_L \otimes \mathcal{A}_L \xrightarrow{\sim} 2\mathcal{A}_L\colon (f, (\delta_1, \delta_2)) \mapsto (f, \frac{\delta_1 + \delta_2}{2})$, and the inverse given by the natural double inclusion. Let $L^\vee$ be the inverse line bundle of L. The Leibniz rule gives a map $\psi\colon \mathcal{A}_L \to \mathcal{A}_{L^\vee}$ given by $\psi(\delta)(s^\vee) = -s^\vee \otimes \delta(s) \otimes s^\vee$, where $\delta \in \mathcal{A}_L$ and $s \in \Gamma(L)$. By composing ψ^{-1} and the canonical map in (19), we find a canonical isomorphism of Atiyah algebras $\mathcal{A}_{L^\vee} \xrightarrow{\sim} -\mathcal{A}_L$. Therefore for $n \in \mathbb{Z}$, canonically $n\mathcal{A}_L \cong \mathcal{A}_{L^{\otimes n}}$.

These operations generalize straightforwardly to super Atiyah algebras with the important replacement of the Leibniz rule with the super Leibniz rule:

$$\begin{aligned}
&\varphi\colon \mathcal{A}_{L_1} \times_{\mathcal{T}_X} \mathcal{A}_{L_2} \to \mathcal{A}_{L_1 \otimes L_2} \quad && \varphi(\delta_1, \delta_2)(s_1 \otimes s_2) = \delta_1(s_1) \otimes s_2 + (-1)^{|\delta_2||s_1|} s_1 \otimes \delta_2(s_2) \\
&\psi\colon \mathcal{A}_L \to \mathcal{A}_{L^\vee} && \psi(\delta)(s^\vee) = -(-1)^{|\delta||s|} s^\vee \otimes \delta(s) \otimes s^\vee
\end{aligned}$$

where $\delta_i \in \mathcal{A}_{L_i}$ and $s_i \in \Gamma(L_i)$, and $\delta \in \mathcal{A}_L$ and $s \in \Gamma(L)$.

4.4 Connections and the Lie derivative

We mostly follow §3.6 and §3.7 of [11] in this section.

The classical notion of a connection on a vector bundle extends to supergeometry as expected, with some care needed to keep track of signs.

For a super vector bundle $\mathcal{E}$ considered as a locally free sheaf, a connection on $\mathcal{E}$ is defined as a map

$$\nabla\colon \mathcal{E} \to \Omega^1 \otimes \mathcal{E}$$

which obeys the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s \qquad\qquad f \in \mathcal{O}, \quad s \in \Gamma(\mathcal{E}).$$

A connection allows sections of $\mathcal{E}$ to be differentiated along vector fields. The notation used is

$$\nabla_X s := \langle X, \nabla s\rangle$$

where X is a vector field and the sign convention as in (15) is used.

Alternatively, in either the classical or super setting, a connection on a line bundle L is a splitting of the Atiyah sequence (18) as a sequence of $\mathcal{O}_X$-modules, i.e. an $\mathcal{O}_X$-linear map $\nabla\colon \mathcal{T}_X \to \mathcal{A}$ such that $\mathrm{sym}_1 \circ \nabla = \mathrm{Id}_{\mathcal{T}_X}$. The curvature $c_\nabla \in \Omega^2$ is defined by

$$c_\nabla(v \wedge w) = \Big[\nabla(v), \nabla(w)\Big] - \nabla\Big([v,w]\Big).$$

When $c_\nabla = 0$, we say the connection ∇ is flat or integrable. A connection is flat exactly when it is a morphism of Lie algebroids.

A connection is called a holomorphic connection when the superspace has structure sheaf consisting of holomorphic functions.

Keeping consistent with the sign rules of the differential structure, the Lie derivative is given by

$$\mathcal{L}_X(f) = X(f) \qquad\qquad \mathcal{L}_X(Y) = [X, Y]$$

for $f \in \mathcal{O}$ and X, Y vector fields. As well, the super Cartan formula on the super deRham complex Ω^* holds.

$$\mathcal{L}_X = d\iota_X + \iota_X d \tag{20}$$

where the exterior derivative is defined in (16) and insertion operator ι_X satisfies the sign rule

$$\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{|\alpha||X| + d(\alpha)} \alpha \wedge \iota_X(\beta).$$

These statements follow from defining the Lie derivative when X an even vector fields as in the classical setting and then extending the definition to odd vector fields X so that the definition is compatible under base change.

4.5 The Neveu-Schwarz Lie superalgebra

The Neveu-Schwarz Lie superalgebra is an infinite dimensional Lie superalgebra central to the result in this paper. We first discuss the infinite dimensional general linear superalgebra, and the witt superalgebra. The relationship between these infinite dimensional Lie superalgebras is also important to the main result.

Define $\mathfrak{gl}(H) = \mathrm{Hom}(H, H)$ to be the space of continuous linear maps on H. For any endomorphism of H we may write it as

$$F = \begin{pmatrix} F^{--} & F^{-+} \\ F^{+-} & F^{++} \end{pmatrix}$$

where $F^{+-} : H^- \to H^+$ etc.

It is known that $H^2(\mathfrak{gl}(H))$ is one-dimensional. We choose a distinguished 2-cocycle:

$$\eta(F, G) := \mathrm{str}(F^{-+}G^{+-} - F^{-+}G^{+-}),$$

which is known as the Japanese cocycle [24]. The unique Lie superalgebra central extension defined by the super Japanese cocycle will be denoted $\widetilde{\mathfrak{gl}}(H)$.[6] We denote the bracket on $\widetilde{\mathfrak{gl}}(H)$ as $[F, G]^{\sim} = [F, G] + \eta(F, G)$.

The super Witt algebra $\mathfrak{switt}$ is the Lie superalgebra of superconformal vector fields on a punctured formal neighborhood in $\mathbb{C}^{1|1}$. Explicitly, the elements of $\mathfrak{switt}$ are of the form (25) with f a formal super Laurent series:

$$\mathfrak{switt} = \left\{ f(z,\zeta)\frac{\partial}{\partial z} + \frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2} D_\zeta : f(z,\zeta) \in \mathbb{C}((z))[\zeta] \right\}$$

Expressed as a basis of even and odd vector fields:

$$f = z^{-n+1} \qquad L_n = z^{-n+1}\frac{\partial}{\partial z} + \frac{-n+1}{2} z^{-n}\zeta\frac{\partial}{\partial \zeta} \qquad n \in \mathbb{Z}$$

$$f = 2i\zeta z^{-r+\frac{1}{2}} \qquad G_r = i\, z^{-r+\frac{1}{2}}\left(\zeta\frac{\partial}{\partial z} - \frac{\partial}{\partial \zeta}\right) \qquad r \in \mathbb{Z} + \frac{1}{2}$$

$$[L_m, L_n] = (m-n)L_{m+n} \qquad [L_n, G_r] = \left(\frac{n}{2} - r\right) G_{n+r} \qquad [G_r, G_s] = 2L_{r+s}$$

Just as in classical Lie theory, infinite dimensional Lie supergroups and their Lie superalgebras do not behave as nicely as in finite dimensions. Lie's third theorem states that every finite dimensional (abstract) Lie algebra is the Lie algebra of some Lie group. However, there exist infinite dimensional Lie algebras which are not the Lie algebra of some Lie group.

[6] The group central extension $\widetilde{\mathrm{GL}}^0_F(H)$ corresponding to the Japanese cocycle is described in [36].

For example, the classical witt Lie algebra $\mathfrak{witt}$ over the complex numbers does not correspond to any Lie group.[24] This is a violation of Lie's third theorem. Since $\mathfrak{witt} \subset \mathfrak{switt}$, then similarly there does not exist a Witt supergroup.

As described by Ueno and Yamada in [40], the super Witt algebra $\mathfrak{switt}$ acts on $\omega_\Sigma^{\otimes j}$ by Lie derivative[7]

$$\mathcal{L}_{\partial_{f(z,\zeta)}}\big(g(z,\zeta)\ [dz|d\zeta]^{\otimes j}\big) = \Big(\partial_f(g)(z,\zeta) + \frac{j}{2}\frac{\partial f}{\partial z}(z,\zeta)g(z,\zeta)\Big)\ [dz|d\zeta]^{\otimes j}$$

where $\partial_{f(z,\zeta)}$ is as in (25). Since $\omega_\Sigma^{\otimes j} \cong \Pi^j H$ as super vector spaces, where Π is the parity reversing operator, we define the map $\rho_j : \mathfrak{switt} \to \mathfrak{gl}(H)$ as

$$\rho_j\big(\partial_{f(z,\zeta)}\big)(g(z,\zeta)) = \partial_f(g)(z,\zeta) + \frac{j}{2}\frac{\partial f}{\partial z}(z,\zeta)g(z,\zeta).$$

The map ρ_j does not take into account the parity of the section $[dz|d\zeta]^{\otimes j}$ upon which the map was based. Essentially, the map ρ_j treats $\mathrm{Ber}(\omega^{\otimes j})$ as rank $1|0$ independent of j. In order to keep with the convention that $\mathrm{Ber}(\omega^{\otimes j})$ is of rank $1|0$ when j is even and $0|1$ when j is odd, we define the Lie superalgebra homomorphism $\varrho_j : \mathfrak{switt} \to \mathfrak{gl}(H)$ as

$$\varrho_j = \Pi^j \circ \rho_j \circ \Pi^j.$$

Proposition 4.11. *The pullbacks of the super Japanese cocycle along the representations ϱ_j satisfies:*

$$\varrho_j^*(\eta) = -(-1)^j(2j-1)\varrho_1^*(\eta) \qquad\qquad \varrho_j^*(\eta) = \varrho_{1-j}^*(\eta).$$

Proof. The proof is a direct computation. We note that $\rho_j^*(\eta) = (2j-1)\rho_1^*(\eta)$ as in Proposition 4 of [40]. Since the supertrace is defined as the trace of the even-even matrix minus the trace of the odd-odd matrix, the effect of the parity operators in the definition of ϱ_j is to negate the pullback of the cocycle η exactly when j is odd. □

It is known that $H^2(\mathfrak{switt})$ is one dimensional. Let $c_j = -(-1)^j(2j-1)$ and define $\mathfrak{ns}_j$ to be the central extension of $\mathfrak{switt}$ by $\varrho_j^*(\eta) = c_j\varrho_0^*(\eta)$. When $j = 0$ or $j = 1$, we recover the standard definition of the Neuev-Schwarz algebra, denoted $\mathfrak{ns}$. On the basis of $\mathfrak{switt}$ the standard cocycle is given by

$$\varrho_0^*(\eta)(L_m, L_n) = \frac{m^3-m}{4}\delta_{m+n,0} \quad \varrho_0^*(\eta)(L_n, G_r) = 0 \quad \varrho_0^*(\eta)(G_r, G_s) = \frac{4r^2-1}{4}\delta_{r+s,0}$$

[7]For reference, the Lie derivative of sections of the Berezinian is described in §3.11 of [11].

This is summarized in the following commutative diagram of Lie superalgebras.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathfrak{ns} & \longrightarrow & \mathfrak{switt} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \cdot c_j} & & \big\downarrow{\scriptstyle \wr} & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathfrak{ns}_j & \longrightarrow & \mathfrak{switt} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \big\downarrow & \ulcorner & \big\downarrow{\scriptstyle \varrho_j} & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{\mathfrak{gl}}(H) & \longrightarrow & \mathfrak{gl}(H) & \longrightarrow & 0
\end{array}
\tag{21}
$$

5 Super Reimann surfaces and their moduli space

Super Riemann surfaces (SRS), also called SUSY curves, are a particular type of superspace with extra structure. SRSs have been widely studied and have been found to have many properties similar to classical Riemann surfaces. Two references which cover much of the background on SRS are [44] and [18].

5.1 Super Riemann surfaces and their moduli space

Definition 5.1. *A super Riemann surface (SRS, or SUSY curve) is a complex supermanifold of dimension* $1|1$ *with a maximally nonintegrable distribution* $\mathcal{D}_\Sigma$ *of rank* $0|1$*. Precisely, a super Riemann surface is the data* $(\Sigma, \mathcal{O}_\Sigma, \mathcal{D}_\Sigma)$ *such that*

- Σ *is a complex manifold of dimension* 1
- $\mathcal{O}_\Sigma$ *is a sheaf on* Σ *of supercommutative* $\mathbb{C}$ *algebras*
- $\mathcal{O}_\Sigma$ *is locally isomorphic to* $\mathcal{O}_\mathbb{C}[\xi]$
- $\mathcal{D}_\Sigma$ *is an odd subbundle of the tangent bundle* $\mathcal{T}_\Sigma$
- *the induced map by Lie bracket* $[\ ,\ ] : \mathcal{D}_\Sigma^{\otimes 2} \to \mathcal{T}_\Sigma/\mathcal{D}_\Sigma$ *is an isomorphism.*

Define a family of SUSY curves in the algebro-geometric sense: a smooth proper morphism $\pi : X \to S$ *of super Riemann surfaces (i.e. relative dimension* $1|1$*) parameterized by the superscheme* S *with a relative distribution* $\mathcal{D}_{X/S} \subset \mathcal{T}_{X/S}$ *of rank* $0|1$ *such that the map* $[\ ,\ ] : \mathcal{D}_{X/S}^{\otimes 2} \to \mathcal{T}_{X/S}/\mathcal{D}_{X/S}$ *is an isomorphism.*

Lemma 5.2. *Locally, we can always find coordinates (called superconformal coordinates)* $z|\zeta$ *such that*

$$D_\zeta = \frac{\partial}{\partial \zeta} + \zeta \frac{\partial}{\partial z}$$

generates the distribution $\mathcal{D}_\Sigma$*.*

In more generality, we can find relative superconformal coordinates $z|\zeta$ *such that* D_ξ *generates* $\mathcal{D}_{X/S}$*.*

Proof. We summarize the proof given in [44].

Since $\mathcal{D}$ is locally free of rank $0|1$, we may trivialize it in a local coordinate chart $z|\zeta$. Consider an nonzero odd vector field of $\mathcal{D}$ in this chart

$$D = a\frac{\partial}{\partial \zeta} + b\frac{\partial}{\partial z},$$

and this generates $\mathcal{D}$ in the chart. We must have then that a is an odd function and b is an even function. And further, the nonzero property means that the reduction $D_{bos} = b_0\frac{\partial}{\partial z}$ is nonzero, where $b = b_0 + b_1\zeta$.

We can therefore consider $D = \frac{\partial}{\partial \zeta} + b\frac{\partial}{\partial z}$, since performing a multiplication by a nonzero function $D \mapsto b_0^{-1}(1 - \frac{b_1}{b_0}\zeta)D$ maps a section of $\mathcal{D}$ to another section of $\mathcal{D}$, i.e. preserves the property that D^2 is not a multiple of D. A straightforward calculation shows that $(D^2) = (b_0 b_0' + b_1 + \zeta(b_0' b_1 - b_0 b_1'))\frac{\partial}{\partial z}$. Thus, b_1 must be nonzero so that D^2 is not a multiple of D.

Then we can preform the change of coordinates $\xi = b/b_1^{1/2}$ and $x = z$, where $b_1^{1/2}$ makes sense as a holomorphic sqaureroot of the nonzero even function b_1. We then find that $D = \left(b_1^{1/2}\xi\frac{\partial \xi}{\partial z} + b_1^{1/2}\right)D_\xi$.

□

Naturally, we call a change of coordinates $z|\zeta \mapsto x|\xi$ superconformal if D_ζ and D_ξ are multiples of one another.

Consider $z|\zeta$ and $x|\xi$ two superconformal coordinate systems. By definition, we see that $D_\zeta = fD_\xi$ for some nonzero function f. Further, we find that $f = D_\zeta \xi$ by asking on ξ with both sides. On the other hand, chain rule gives

$$D_\zeta = (D_\zeta \xi)D_\xi + (D_\zeta x - \xi D_\zeta \xi)\frac{\partial}{\partial z}.$$

Comparing the two expressions, we must have that

$$D_\zeta x = \xi D_\zeta \xi.$$

This condition is enough to ensure the change of coordinates is superconformal.

In general, it is difficult to construct super Riemann surfaces since the existence of a maximally nonintegrable distribution is not easily checked. See for instance, example 3.9. But clearly, gluing coordinate systems using superconformal changes of coordinates is a reliable way to construct SUSY curves. A change of coordinates given by

$$x = a + \zeta\alpha \qquad\qquad \xi = \beta + \zeta b \tag{22}$$

must satisfy $\alpha = b\beta$ and $b^2 = a' + \beta\beta'$ where the prime denotes derivative with respect to z.

The change of coordinates (22) must include the odd functions α and β for full generality. This odd functions may be nonzero only in the presence of odd moduli, that is when $\pi\colon X \to S$ is a family with S having nonzero odd dimension. It is these odd moduli that cause much of the difficulty in studying super Riemann surfaces, while also providing much of the interesting structure. If the base S is a classical space, it is well-known that the family of SRSs is in fact a family of spin curves, which can be studied without the use of supergeometry.

From the perspective of the change of coordinates (22), the absence of odd moduli reduces the gluing to

$$x = a \qquad \xi = \zeta b \tag{23}$$

where $b^2 = a'$ if the change is superconformal. Clearly, the change of coordinates restricted to $z \mapsto x$ is constructs a classical Riemann surface. A SRS Σ glued from charts via change of coordinates of the form (23) will have a bosonization Σ_{bos} such that $\Sigma \to \Sigma_{bos}$ is a line bundle with fibers of rank $0|1$. Essentially the bosonization and the odd coordinates are independent. Recalling definition 3.2, we see that Σ is split. It is therefore natural to call (23) a split change of coordinates.

The line bundle $\Sigma \to \Sigma_{bos}$ for a split SRS is isomorphic to a squareroot of the canonical bundle of Σ_{bos}, which is known as a spin structure. For Σ_{bos} of genus g, there are 2^{2g} nonisomorphic spin structures.

It is useful to consider the moduli space of spin curves, that is a classical Riemann surface with choice of spin structure, since topologically this moduli space and the moduli space of SRSs are the same. The difference being the absence of presence of odd moduli. The moduli space of spin curves has two topological components: even spin structures and odd spin structures. An even (odd) spin structure is a spin structure with an even (odd) integer dimensional space of global holomorphic sections.

Analogously to considering classical Riemann surfaces with a marked point, we may consider punctured SRSs. The natural super analog of a marked point is called a Neveu-Schwarz puncture. However, there exists another more exotic type of puncture without analogy ot the classical case. These are Ramond (R) punctures.

Consider a family of SUSY curves $\pi\colon X \to S$. A section $\sigma : S \to X$ of the morphism π is a coherent way to choose a Neuev-Schwarz (NS) puncture in each fiber. Since the section σ is locally $\operatorname{Spec} A \to \operatorname{Spec} A[z, \zeta]$, these NS punctures may be described by $z = z_0$ and $\zeta = \zeta_0$ for some even function z_0 and some odd function ζ_0 in $\mathcal{O}_S$. Note that the image of

σ is a codimension $1|1$ subsuperscheme of X. However, the orbit generated by the $\mathcal{D}_{X/S}$ canonically associates a divisor of X to the section σ. Expressed with an odd parameter α, the divisor, denoted $P = \operatorname{div}(s)$, is the codimension $1|0$ subsuperscheme [44]

$$z = z_0 + \alpha\zeta$$
$$\zeta = \zeta_0 + \alpha.$$

Ramond (R) punctures must be studied as they occur when compactifying the moduli space of SUSY curves. However, for the purposes of this paper, we will not need to discuss R punctures. In brief, a R puncture is a singularity in the SUSY structure. That is a $0|1$ submanifold along which $\mathcal{D}^2 = 0$. In local coordinates, a generator of $\mathcal{D}$ is then $\frac{\partial}{\partial\zeta} + z\zeta\frac{\partial}{\partial z}$. Note that a super Riemann surface with Ramond puncture is not a true super Riemann surface due to the singularity in the SUSY structure.

Definition 5.3. *The moduli space of super Riemann surfaces $\mathfrak{M}_g$ is the superspace representing the functor*

$$S \mapsto \{\textit{families of SUSY curves } \pi\colon\ X \to S\}.$$

Similar to the moduli space of classical Riemann surfaces $\mathcal{M}_g$, the moduli space of SRSs is representable as a stack, specifically an orbifold or Deligne-Mumford stack.

The stack structure is needed since even a classical Riemann surface may have automorphisms. For genus at least 3, a Reimann surface has no nontrivial automorphisms. For low genus, the extra automorphisms occur along a superspace of lower dimension, and so are well-behaved.

However, the moduli space of super Riemann surfaces has a dense locus of singularity. This is due to the extra $\mathbb{Z}_2$ symmetry of split SRSs. Every split SRS has nontrivial automorphism which is trivial on the reduced space and acts by -1 along the odd direction. A generic SRS will not have this $\mathbb{Z}_2$ symmetry, but will be infinitesimally close to a split SRS. We note also that continuous symmetries are absent for genus at least 2, analogously to the classical case. We will not address the singularities in this paper, and so will not go into more detail here.

One method for avoiding singularities in the moduli stack to to consider (super) Riemann surfaces with enough additional structure so that there are no remaining nontrivial automorphisms. The moduli space we are most interested in, $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$ takes this to an extreme, such that no nontrivial automorphisms of these marked SRSs remain.

Definition 5.4. *The moduli stack $\mathfrak{M}_{g,1^k_{NS}}$ is the moduli space of triples $(\Sigma, p, z|\zeta)$, where Σ is a genus g SUSY curve, p is the divisor associated to a NS puncture on Σ, both z and ζ are a k-jet equivalence class of an even (resp. odd) formal parameter vanishing at p.*

Specifically, we have the parameters $z, \zeta \in \widehat{\mathcal{O}}_p$, the formal neighborhood at p, and z and ζ each have a zero of order one at p. And since $z|\zeta$ is a k-jet, the coordinate system is equivalent to any $y|\gamma$ such that $y|\gamma$ equals $z|\zeta$ modulo $\mathfrak{m}_p^{k+1}|\mathfrak{m}_p^{k+1}$. Note that $\mathfrak{M}_{g,1^k_{\mathrm{NS}}}$ has dimension $3g-3+1+k\,|\,2g-2+1+k$.

Definition 5.5. *By taking the projective limit of* $\mathfrak{M}_{g,1^k_{NS}}$ *as* $k \to \infty$*, we construct the pro-Deligne-Mumford stack*

$$\mathfrak{M}_{g,1^\infty_{NS}} = \varprojlim \mathfrak{M}_{g,1^k_{NS}}.$$

Simply by forgetting the coordinate system $z|\zeta$ and the NS puncture p, we have the projection

$$\begin{aligned} \mathfrak{M}_{g,1^\infty_{\mathrm{NS}}} &\to \mathfrak{M}_g \\ (\Sigma, p, z|\zeta) &\mapsto \Sigma \end{aligned}$$

where $\mathfrak{M}_g$ is the supermoduli space of SUSY curves of genus g.

5.2 Superconformal vector fields and the Berezinian

Let $\pi\colon X \to S$ be a family of SUSY curves. The exact sequence below reveals much of the structure of SUSY curves.

$$0 \to \mathcal{D}_{X/S} \to \mathcal{T}_{X/S} \to \mathcal{D}^2_{X/S} \to 0 \tag{24}$$

To be precise, the projection above is the composition $\mathcal{T}_{X/S} \to \mathcal{T}_{X/S}/\mathcal{D}_{X/S} \xleftarrow{\sim} \mathcal{D}^2_{X/S}$ using the isomorphism induced by the Lie bracket.

While the distribution $\mathcal{D}$ may be called the supersymmetric structure of a SRS, we are naturally interested in transformations which preserve this structure. The vector fields which preserve $\mathcal{D}$ are called supreconformal.

The sheaf of superconformal vector fields $\mathcal{T}^s_{X/S}$ on a family of super Riemann surfaces is the largest subsheaf of $\mathcal{T}_{X/S}$ such that $[\mathcal{T}^s_{X/S}, \mathcal{D}_{X/S}] = \mathcal{D}_{X/S}$. In local superconformal coordinates $z|\zeta$, a vector field may be expressed as

$$\partial = f(z,\zeta)\frac{\partial}{\partial z} + g(z,\zeta)D_\zeta.$$

And further, we can calculate

$$[\partial, D_\zeta] = -(-1)^{|f|}(D_\zeta f)\frac{\partial}{\partial z} + 2g\frac{\partial}{\partial z} + (-1)^{|g|}(D_\zeta g)D_\zeta.$$

Therefore if $\partial \in \mathcal{T}^s_\Sigma$, then

$$\partial = [fD_\zeta, D_\zeta] = f(z,\zeta)\frac{\partial}{\partial z} + \frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2}D_\zeta. \tag{25}$$

Proposition 5.6. *The sheaf of superconformal vector fields $\mathcal{T}^s$ is not a $\mathcal{O}$-submodule of the tangent sheaf $\mathcal{T}$. However, the sheaf $\mathcal{S}$ has an $\mathcal{O}$-module structure via the map $[\ ,\ ]\colon \mathcal{D}\otimes\mathcal{D} \to \mathcal{T}^s$.*

Proof. We consider the following commutative diagram of $\mathbb{C}$-modules.

$$\begin{array}{ccc} \mathcal{D}\otimes\mathcal{D} & \xhookrightarrow{i\otimes i} & \mathcal{T}\otimes\mathcal{T} \\ {\scriptstyle [\,,\,]}\big\downarrow & & \big\downarrow{\scriptstyle [\,,\,]} \\ \mathcal{T}^s & \underset{i}{\hookrightarrow} & \mathcal{T} \end{array}$$

The vertical maps are derived from the Lie bracket, and the horizontal maps are the natural inclusion into the tangent sheaf. All these maps are well-defined except perhaps the left map.

Any superalgebra must have $[[\alpha,\alpha],\alpha] = 0$ for any odd element α. (pg 89 of [41]) If we consider a odd generator D of $\mathcal{D}$, by this $[[D,D],D] = 0$. From this, it can be shown that $[[\mathcal{D},\mathcal{D}],\mathcal{D}] \subset \mathcal{D}$. Then by definition of a superconformal vector field, $[\mathcal{D},\mathcal{D}] \subset \mathcal{T}^s$. This shows the left map is well-defined.

We consider the map $\mathcal{D}\otimes\mathcal{D} \to \mathcal{T}^s$ as a map of $\mathcal{O}$-modules. We notice that $\mathcal{D}\otimes\mathcal{D}$ is of rank $1|0$ and $\mathcal{T}^s$ can be at most rank $1|0$. Since the map is not trivial, then it is locally an isomorphism.

Finally, we remark on the diagram. We cannot consider the right map derived from $[\ ,\ ]$ as an $\mathcal{O}$-module map with the canonical $\mathcal{O}$-module structure on each of the three copies of $\mathcal{T}$. Therefore, the lower map $i\colon \mathcal{T}^s \hookrightarrow \mathcal{T}$ which commutes in the diagram cannot be a $\mathcal{O}$-module map. Alternatively, this fact that $\mathcal{T}^s$ is not a subbundle of $\mathcal{T}$ can be seen by oberserving the local coordinate expression (25). □

Consider a superconformal chart $z|\zeta$. It is common to consider the tangent space with local basis $\frac{\partial}{\partial z}$ and D_ζ. This is precisely how the local coordinate expression (25) is expressed. Using this expression, we can see that locally the map

$$\mathcal{T}^s \hookrightarrow \mathcal{T} \twoheadrightarrow \mathcal{T}/\mathcal{D} \qquad\qquad [fD_\zeta, D_\zeta] \mapsto f\frac{\partial}{\partial z}$$

is an isomorphism. Given this, it is sometimes convenient to consider the decomposition of the tangent sheaf as $\mathcal{T} \cong \mathcal{T}^s \oplus \mathcal{D}$ instead.

Further, via the Lie bracket map, we then have that $\mathcal{T}^s \cong \mathcal{D}^2$ is an isomorphism of sheaves of $\mathbb{C}$ vector spaces. [37] This matches with what is explained in the proof of proposition 5.6 and in the local form (25).

Returning to (24), and dualizing the sequence gives

$$0 \to \mathcal{D}^{\otimes -2}_{X/S} \to \Omega^1_{X/S} \to \mathcal{D}^{-1}_{X/S} \to 0. \tag{26}$$

Since $\mathcal{D}_{X/S}^{\otimes -2}$ is the dual of the quotient $\mathcal{T}_{X/S}/\mathcal{D}_{X/S}$, then we find $\mathcal{D}_{X/S}^{\otimes -2}$ is generated in local superconformal coordinates by $dx - \xi d\xi$.

The Berezinian of the exact sequence (26) induces an isomorphism

$$\operatorname{Ber}\Omega^1_{X/S} \cong \operatorname{Ber}\mathcal{D}_{X/S}^{\otimes -2} \otimes \operatorname{Ber}\mathcal{D}_{X/S}^{-1} \cong \mathcal{D}_{X/S}^{\otimes -2} \otimes \mathcal{D}_{X/S} \cong \mathcal{D}_{X/S}^{-1}.$$

We define $\omega_{X/S} := \operatorname{Ber}\Omega^1_{X/S}$, which we call the relative Berezinian of X over S. Therefore in local superconformal coordinates, we may describe the relative Berezinian as relative one-forms modulo $dz - \zeta d\zeta$.

5.3 The super Mumford isomorphism

The super Mumford isomorphism is the analogous statement for the moduli space of super Riemann surfaces $\mathfrak{M}_g$ as exists for the classical moduli space of Riemann surfaces $\mathcal{M}_g$. The classical Mumford isomorphism was proved in [34]. It states that all line bundles on $\mathcal{M}_g$ are tensor multiples of the Hodge line bundle. The statement is $\lambda_j \cong \lambda_1^{6j^2-6j+1}$, where λ_1 is the Hodge line bundle.

Note that the Berezinian of cohomology used below is the natural generalization of the determinant of cohomology defined in [9].

Definition 5.7. *Let $\pi : X \to S$ be a smooth proper family of complex supermanifolds of dimension $1|1$. Let $\mathcal{F}$ be a locally free sheaf on X. Then the Berezinian of cohomology of $\mathcal{F}$ is a sheaf on S of dimension $1|0$ or $0|1$ given by*

$$B(\mathcal{F}) := \otimes_i \left(\operatorname{Ber} R^i\pi_*\mathcal{F}\right)^{(-1)^i}$$

We define the Berezinian line bundles $\lambda_{j/2}$ for the universal family $\pi : X \to \mathfrak{M}_{g,1^\infty_{NS}}$ as

$$\lambda_{j/2} := B(\omega_{X/\mathfrak{M}}^{\otimes j})$$

where $\omega_{X/\mathfrak{M}} := \operatorname{Ber}\Omega^1_{X/\mathfrak{M}}$ is the relative Berezinian.

Theorem 5.8 (Deligne [10], and Voronov [42]). *The super Mumford isomorphism is the collection of canonical isomorphisms*

$$\lambda_{j/2} \cong \lambda_{1/2}^{-(-1)^j(2j-1)}, \qquad \textit{in particular} \qquad \lambda_{3/2} \cong \lambda_{1/2}^5.$$

The proof of the super Mumford isomorphism is in fact easier than the proof of the classical Mumford isomorphism, and results in a stronger statement. The super Mumford form μ is a trivializing section of $\lambda_{3/2} \otimes \lambda_{1/2}^{-5}$, specifically th cannonical section which corresponds to 1 via the super Mumford isomorphism. In the classical Mumford isomorphism the isomorphism is non-canonical, and therefore the classical Mumford form is defined up to a scaling factor.

6 The super Sato Grassmannian

The construction of the finite dimensional super Grassmannians was done in section 3.5. In this section, we first review the construction of the classical Sato Grassmannian. Then the super Sato Grassmannian and its properties are discussed. All of these Grassmannians are established in the literature, however our definition of the super Sato Grassmannian differs somewhat from existing definitions.

6.1 The classical Sato Grassmannian

The semi-infinite Grassmannian, defined originally by M. Sato in [38], originated via the study of soliton equations and KP equations. Following the spirit of Kontsevich, we define the discrete subspaces which form the infinite-dimensional Grassmannian. We state here the important findings highlighting any differences with the superized case.

Consider $\mathbb{C}((z))$, the infinite-dimensional topological space of formal Laurent series with the z-adic topology. Define the vector spaces $H = \mathbb{C}((z))$, $H^+ = \mathbb{C}[[z]]$, and $H^- = z^{-1}\mathbb{C}[z^{-1}]$.

Definition 6.1. *Define a subspace of H to be* compact *if it is commensurable with H^+. Explicitly, subspaces K and H^+ are commensurable when $K/(K \cap H^+)$ and $H^+/(K \cap H^+)$ are both finite-dimensional. Define a subspace D of H to be* discrete *if there exists a compact subspace K such that the natural map $D \oplus K \to H$ is an isomorphism.*

Lemma 6.2. *Let K be any compact subspace. Then D is discrete if and only if the natural map $D \oplus K \to H$ is Fredholm.*

Definition 6.3. *The Grassmannian $\mathrm{Gr}(H)$ is the set of all discrete subspaces $D \subset H$.*

Proposition 6.4. *$\mathrm{Gr}(H)$ is locally modeled on vector spaces $\mathrm{Hom}_{\mathbb{C}}(H^-, H^+)$. And thus $\mathrm{Gr}(H)$ is an infinite-dimensional manifold.*

Proof. This proof is essentially follows the check of Def. 6.10. See also Proposition 7.1.2 of [36]. □

Define $\mathfrak{gl}(H) = \mathrm{Hom}(H, H)$ to be the space of continuous linear maps on H. For any endomorphism of H we may write it as

$$F = \begin{pmatrix} F^{--} & F^{-+} \\ F^{+-} & F^{++} \end{pmatrix}$$

where $F^{+-} : H^- \to H^+$ etc. Define the group $\mathrm{GL}_F(H) \subset \mathrm{Hom}(H, H)$ to be the subspace of invertible maps with F^{--} and F^{++} Fredholm.

Define an endomorphism of a discrete subspace to be trace class if it factors through some compact subspace.

Proposition 6.5 (cf. Pressley and Segal Proposition 7.1.3 [36])**.** *The group* $\mathrm{GL}_F(H)$ *acts transitively on* $\mathrm{Gr}(H)$*, and the stabilizer of* D *is* $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ *expressed in a decomposition* $D \oplus K$*.*

Proof. The proof reads the same as the proof of Proposition 7.2 with the condition $v(z^i\theta) = z^{i-m}\theta$ for some m omitted. □

Therefore we may describe $\mathrm{Gr}(H)$ as the homogeneous space

$$\mathrm{Gr}(H) \cong \mathrm{GL}_F(H)/P \tag{27}$$

where $P = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ in the $H^- \oplus H^+$ decomposition.

Proposition 6.6 (Kontsevich [25])**.** *The Lie algebra* $\mathfrak{gl}(H)$ *acts by vector fields on* $\mathrm{Gr}(H)$*. Explicitly,* $F \mapsto L_F$ *is a Lie algebra homomorphism* $\mathfrak{gl}(H) \to \mathcal{T}_{\mathrm{Gr}(H)}$*. In the chart* $U_{D,K}$*, this action is given by the formula*

$$L_F(A) = F^{KD} + F^{KK}A - AF^{DD} - AF^{DK}A$$

where $L_F \in \mathrm{Hom}_{\mathbb{C}}(S^\bullet(\mathrm{Hom}_{\mathbb{C}}(D, K)), \mathrm{Hom}_{\mathbb{C}}(D, K))$*.*

Proof. The proof is the same as the proof of Proposition 7.3. □

We can take as definition

$$\det_K(D) := \frac{\det(D \cap K)}{\det(H/(D + K))},$$

which is well-defined for discrete D and compact K.

Definition 6.7. *As in [2], we define the determinant line bundle on* $\mathrm{Gr}(H)$ *as* $\det_{\mathrm{Gr}(H)} := \det_{H^+}$*, in other words,* $\det_K$ *with* $K = H^+$*, the distinguished compact subspace.*

6.2 Definition

Consider $\mathbb{C}((z))$, the infinite-dimensional topological space of formal Laurent series with the z-adic topology. Define the super vector space $H = \mathbb{C}((z))[\zeta] = \mathbb{C}((z))\zeta + \mathbb{C}((z))$ where ζ is a Grassmann variable, i.e. $\zeta^2 = 0$. We define two classes of subspaces: compact and discrete.

Firstly, we note that two super subspaces $V, W \subseteq H$ are *commensurable* if both $V/(V \cap W)$ and $W/(V \cap W)$ have both finite even dimension and finite odd dimension. We define the notation $V_S := V \otimes_{\mathbb{C}} \mathcal{O}_S$ and $\widehat{V}_S := V \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S = \varprojlim V/(K + V) \otimes_{\mathbb{C}} \mathcal{O}_S$, where S is a superscheme with structure sheaf $\mathcal{O}_S$. The definition below is a generalization of the classical Sato Grassmannian as defined by Álvarez Vázquez, Muñoz Porras, and Plaza Martín in [1].

Definition 6.8. *Define a super subspace of H to be* compact *if it is commensurable with $H^+ = \mathbb{C}[[z]][\zeta]$. Define a super subspace D of H to be* discrete *if there exists a compact subspace K such that the natural map $D \oplus K \to H$ is an isomorphism of super vector spaces.*[8]

More generally for a superscheme S with structure sheaf $\mathcal{O}_S$, a super $\mathcal{O}_S$-submodule $L \subset \widehat{H}_S$ is discrete *if for every $s \in S$ there exists a neighborhood U of s and a compact K such that the natural map $L_U \oplus \widehat{K}_U \to \widehat{H}_U$ is an isomorphism, where $L_U := L \otimes_{\mathcal{O}_S} \mathcal{O}_U$.*

Lemma 6.9. *Let K be any compact subspace. Then $L \subset \widehat{H}_S$ is discrete if and only if for every $s \in S$ there exists a neighborhood U of s such that the kernel and cokernel are free of finite type in the exact sequence below*

$$0 \to L_U \cap \widehat{K}_U \to L_U \oplus \widehat{K}_U \to \widehat{H}_U \to \widehat{H}_U/(L_U + \widehat{K}_U) \to 0 \tag{28}$$

where $L_U := L \otimes_{\mathcal{O}_S} \mathcal{O}_U$.

Proof. The proof is a straight forward linear algebraic argument.

Firstly we show that commensurablity is transitive, i.e. any two compact subspaces are commensurable to each other. For K and K' compact, using the exact sequence

$$0 \to \frac{(K \cap H^+) + (K \cap K')}{K \cap K'} \to \frac{K}{K \cap K'} \to \frac{K}{(K \cap H^+) + (K \cap K')} \to 0,$$

where the first term is isomorphic to $(K \cap H^+)/(K \cap K' \cap H^+)$, so we see $\dim(K/(K \cap K')) \leq \dim(H^+/(K' \cap H^+)) + \dim(K/(K \cap H^+)) < \infty$. And $\dim(K/(K \cap K')) < \infty$ similarly.

In the below we work in $\widehat{H}_U$, but we have omitted the subscript $_U$.

[8] We have defined discrete and compact subspaces purely algebraically. The topological names are motivated by the z-adic topology on H.

Next we show that if $L\cap K$ and $H/(L+K)$ are finite-dimensional for some compact K, then $L \cap K'$ and $H/(L+K')$ are finite-dimensional for any other compact K'. Using the exact sequence

$$0 \to L\cap K\cap K' \to L\cap K' \to L\cap K'/(L\cap K\cap K') \to 0,$$

we see $\dim(L\cap K') \le \dim(L\cap K) + \dim(K'/(K\cap K')) < \infty$. Using the exact sequence

$$0 \to \frac{K}{(L+K')\cap K} \cong \frac{L+K'+K}{L+K'} \to \frac{H}{L+K'} \to \frac{H}{L+K+K'} \to 0,$$

we see $\dim(H/(L+K')) \le \dim(K/(K'\cap K)) + \dim(H/(L+K)) < \infty$.

Assume L is discrete. Then $L\oplus K' = H$ for some compact K'. Thus $L\cap K' = 0$ and $H/(L+K') = 0$. Then $D\cap K$ and $H/(L+K)$ are finite-dimensional by the previous paragraph.

Assume $L\cap K$ and $H/(L+K)$ are finite-dimensional. We show L is discrete by showing H/L is compact. It suffices to notice

$$\frac{K}{K\cap(H/L)} \cong L\cap K, \qquad \frac{H/L}{K\cap(H/L)} \cong \frac{H}{L+K}.$$

□

Just as H^+ is a natural choice for a compact subspace, the discrete subspace $H^- = z^{-1}\mathbb{C}[z^{-1}][\zeta]$ is a natural choice for a discrete subspace since $H = H^+ \oplus H^-$. We call H^+ the distinguished compact subspace, and H^- the distinguished discrete subspace. Equivalent to (28), we may check that $p_-|_D$ is Fredholm, where p_- is the projection $H \to H^-$.

$$0 \to D\cap H^+ \to D \xrightarrow{p_-} H^- \to H/(D+H^+) \to 0 \tag{29}$$

Definition 6.10. *We define the (semi-infinite or Sato) super Grassmannian $Gr(H)$ as the complex superscheme representing the functor of the discrete super subspaces:*

$$S \mapsto \left\{ \textit{discrete super } \mathcal{O}_S\textit{-submodules } L \subseteq \widehat{H}_S \right\}$$

The fact that this functor is representable by a complex superscheme may be proven just following the style of [1] and [35], which prove the analogous nonsuper result for the infinite-dimensional Grassmannian of discrete subspaces D in $\mathbb{C}((z))$. We explain a modified version of this argument in the remainder of this section.

Observe that the underlying scheme of $\mathrm{Gr}(H)$ is $\mathrm{Gr}_{\mathrm{rd}}(H)$ which is the set of points

$$\mathrm{Gr}_{\mathrm{rd}}(H) := \{\text{discrete super subspaces } D \subset H\}.$$

We may identify this with the product of two classical Sato Grassmannians. One copy which corresponds to even discrete subspaces $\mathrm{Gr}(\mathbb{C}(\!(z)\!))$, and the other which corresponds to odd discrete subspaces $\mathrm{Gr}(\zeta\mathbb{C}(\!(z)\!))$.

Definition 6.11. *Given discrete D and compact K such that the natural map $D \oplus K \to H$ is an isomorphism, define an* open chart $U_{D,K}$ *as the subsuperscheme of the super Grassmannian representing the planes L which project isomorphically onto D along K. To make this precise using the functor of points, the superscheme $U_{D,K}$ is*

$$S \mapsto \left\{ \text{discrete super } \mathcal{O}_S\text{-submodules } L \subseteq \widehat{H}_S \text{ such that } L \cap \widehat{K}_S = \widehat{H}_S/(L + \widehat{K}_S) = 0 \right\}$$

Equivalently, $U_{D,K}$ is those L such that $L \xrightarrow{p_D} D_S$ is an isomorphism. Taking the inverse $D_S \to L$ of the isomorphism and composing it with the inclusion $L \hookrightarrow \widehat{H}_S$, we get a map $A : D_S \to \widehat{H}_S$. On the other hand, for any $\mathcal{O}_S$-module map $B\colon D_S \to \widehat{K}_S$, we have that the natural map $\mathrm{graph}(B) \oplus \widehat{K}_S \to \widehat{H}_S$ is an isomorphism, so $L' = \mathrm{graph}(B)$ is an S-point of $U_{D,K}$. We say that L' is represented by the coordinates $B\colon D_S \to \widehat{K}_S$. We therefore identify the affine coordinate charts $U_{D,K}$ with the superspaces $\underline{\mathrm{Hom}}_{\mathbb{C}}(D, K)$. Where we use the notation $\underline{\mathrm{Hom}}_{\mathbb{C}}(D, K)$ for the corresponding (infinite dimensional) affine superscheme in addition to the super vector space, by a slight abuse of notation.

We check that the functor $U_{D,K}$ does define an open chart of $\mathrm{Gr}(H)$. Consider an S-point L of $\mathrm{Gr}(H)$. Taking the pullback, whave a morphism $S \times_{\mathrm{Gr}(H)} U_{D,K} \to U_{D,K}$. We need to show that this $S \times_{\mathrm{Gr}(H)} U_{D,K}$-point is represented by an open subspace of S.

Let $s \in S$ be a superpoint such that L_s satisfies $L_s \cap \widehat{K}_s = \widehat{H}_s/(L_s + \widehat{K}_s) = 0$. Then since $\widehat{H}_S/(L + \widehat{K}_S)$ is locally finitely generated, we may apply the super Nakayama lemma (Lemma 4.7.1 ii [41]) to conclude that there exists a neighboorhood $U \ni s$ such that $\widehat{H}_U/(L_U + \widehat{K}_U) = 0$. Similarly, since $L \cap \widehat{K}_S$ is finitely generated, there exists $U' \ni s$ such that $L_{U'} \cap \widehat{K}_{U'} = 0$. Therefore the $S \times_{\mathrm{Gr}(H)} U_{D,K}$-point must be open, because we have shown that any s contained in $U_{D,K}$ is contained in an open neighborhood of $U_{D,K}$.

The fact that the charts $U_{D,K}$ cover $\mathrm{Gr}(H)$ follows directly from their definition. Therefore, since each chart UD, K is representable as an infinite dimensional superscheme, then $\mathrm{Gr}(H)$ is representable as well.

Lastly we describe the gluing of the charts $U_{D,K}$ following [36]. Consider two such charts: $U_{D,K}$ and $U_{D',K'}$. Let $I \subseteq \underline{\mathrm{Hom}}_{\mathbb{C}}(D, K)$ and $I' \subseteq \underline{\mathrm{Hom}}_{\mathbb{C}}(D', K')$ each correspond to $U_{D,K} \cap U_{D',K'}$. We wish to show I and I' are open and that the change of coordinates $I \to I'$ is algebraic.

Let

$$\begin{pmatrix} T^{D'D} & T^{D'K} \\ T^{K'D} & T^{K'K} \end{pmatrix}$$

be the identity map expressed as $D \oplus K \to D' \oplus K'$. Since K' and K are commensurable, then $T^{K'K}$ is Fredholm. Suppose $W = \operatorname{graph}(A\colon D \to K) = \operatorname{graph}(B\colon D' \to K')$, i.e. W corresponds to a point in $U_{D,K} \cap U_{D',K'}$. Then we must have

$$\begin{pmatrix} T^{D'D} & T^{D'K} \\ T^{K'D} & T^{K'K} \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} = \begin{pmatrix} I \\ B \end{pmatrix} Q$$

where $Q\colon D \to D'$ is some isomorphism. We find

$$B = (T^{K'D} + T^{K'K}A)(T^{D'D} + T^{D'K}A)^{-1}.$$

So B is a algebraic function of A and $I = \{A \in \underline{\operatorname{Hom}}_{\mathbb{C}}(D, K)\colon\ T^{D'D} + T^{D'K}A \text{ is invertible}\}$ and so I is indeed open; and similarly for I'.

6.3 Berezinian line bundle

The Berezinian line bundle on the super Grassmannian is a generalization to the super and infinite dimensional setting of the determinant line bundle.

For a discrete $L \in \widehat{H}_S$, we define the $\mathcal{O}_S$-module

$$\operatorname{Ber}_K(L) := \frac{\operatorname{Ber}(L \cap \widehat{K}_S)}{\operatorname{Ber}(\widehat{H}_S/(L + \widehat{K}_S))}, \tag{30}$$

which is well defined for compact K. Considering the exact sequence (28) and taking its Berezinian, locally we have that $\operatorname{Ber}_K(L_U) \cong \frac{\operatorname{Ber}(L_U)\operatorname{Ber}(\widehat{K}_U)}{\operatorname{Ber}(\widehat{H}_U)}$ if L, K, H were finite-dimensional. Since this finite-dimensional expression is $\operatorname{Ber}(L_U)$ multiplied by other constant[9], though actually infinite, factors, it is an appropriate choice of the Berezinian line bundle over the U-point corresponding to L.

To define the Berezinian line bundle as a sheaf over $\operatorname{Gr}(H)$, we consider $\operatorname{Ber}_K(I)$ where I is the tautological bundle, a discrete subbundle of $\widehat{H}_{\operatorname{Gr}(H)}$.

The price we paid with this construction is that now our line bundle depends on a choice of compact subspace K. However, any two Berezinian line bundles are isomorphic by

$$\operatorname{Ber}_K(I) \cong \frac{\operatorname{Ber}(K/(K \cap K'))}{\operatorname{Ber}(K'/(K \cap K'))} \otimes \operatorname{Ber}_{K'}(I). \tag{31}$$

[9]If we fix any compact K, for example $K = H^+$.

Definition 6.12. *In the spirit of [2] and [1], we define the Berezinian line bundle on* $\mathrm{Gr}(H)$ *as the sheaf given by*

$$\mathrm{Ber}_{\mathrm{Gr}(H)} := \mathrm{Ber}_{H^+}(I),$$

in other words, (30) *with the distinguished compact subspace* $K = H^+$ *and with the tautological discrete* $L = I \subset \widehat{H}_{\mathrm{Gr}(H)}$.

We can check this construction on the geometrical points $D \subset H$. We have the fiber over D given by

$$\mathrm{Ber}_{\mathrm{Gr}(H)}(D) = \frac{\mathrm{Ber}(D \cap H^+)}{\mathrm{Ber}(H/(D + H^+))}$$

which can be seen as a natural generalization of the determinant line bundle defined over the classical Sato Grassmannian in defintion 6.7.

7 A flat holomorphic connection

The main result of this paper is Theorem 7.10. In this section, firstly the action of the Witt superalgebra and general linear superlagebra are described, and then the compatibility with the geometric Krichever map.

7.1 Witt action on $\mathfrak{M}_{g,1^{\infty}_{\mathrm{NS}}}$

We use the sheaf of superconformal vector fields to analyze the tangent spaces of the moduli spaces. For $\pi : X \to \mathfrak{M}_g$ the universal family of super Riemann surfaces, we have the short exact sequence

$$0 \to \mathcal{T}^s_{X/\mathfrak{M}_g} \to \mathcal{T}^s_X \to \pi^*(\mathcal{T}_{\mathfrak{M}_g}) \to 0.$$

In the long exact sequence of higher direct images, we find the Kodaira-Spencer map [20] is

$$\delta : \mathcal{T}_{\mathfrak{M}_g} \xrightarrow{\sim} R^1\pi_*\mathcal{T}^s_{X/\mathfrak{M}_g},$$

where the source $\mathcal{T}_{\mathfrak{M}_g}$ follows from the projection formula $\pi_*(\pi^*(\mathcal{T}_{\mathfrak{M}_g})) = \mathcal{T}_{\mathfrak{M}_g} \otimes \pi_*(\mathcal{O}_X) = \mathcal{T}_{\mathfrak{M}_g}$. The restriction to a SRS Σ is

$$\delta_\Sigma \colon\ T_\Sigma\mathfrak{M}_g \xrightarrow{\sim} H^1(\Sigma, \mathcal{T}^s_\Sigma).$$

This map is a natural bijection between the isomorphism classes of infinitesimal deformations $v \in T_{(\Sigma,p,z|\zeta)}\mathfrak{M}_g$ and elements of $H^1(\Sigma, \mathcal{T}^s_\Sigma)$. From this we find the dimension of $\mathfrak{M}_g$ is $3g-3\,|\,2g-2$ for $g \geq 2$.

Further, for $\pi : X \to \mathfrak{M}_{g,1^k_{\mathrm{NS}}}$ the family of super Riemann surfaces with NS punctures represented by the divisor P and a k-jet coordinate system near the punctures, the Kondaira-Spencer map preserving this extra structure is

$$\delta : \mathcal{T}_{\mathfrak{M}_{g,1^k_{\mathrm{NS}}}} \xrightarrow{\sim} R^1\pi_*\mathcal{T}^s_{X/S}(-(k+1)P),$$

which locally for $(\Sigma, p, z|\zeta)$ is

$$\delta_{(\Sigma,p,z|\zeta)} \colon\ T_{(\Sigma,p,z|\zeta)}\mathfrak{M}_{g,1^k_{\mathrm{NS}}} \xrightarrow{\sim} H^1(\Sigma, \mathcal{T}^s_\Sigma(-(k+1)p))$$

where $P|_\Sigma = p$ is the NS puncture on Σ. From this we find the dimension of $\mathfrak{M}_{g,1^k_{\mathrm{NS}}}$ is $3g-2+k\,|\,2g-1+k$ for $g \geq 2$.

Let U be a formal neighborhood of a NS puncture p in Σ. Choosing the superconformal formal parameters z, ζ on U such that the divisor p is given by $z=0$ gives a trivialization of $\mathcal{T}^s_\Sigma$ on U. Using Čech cohomology, then

$$\begin{aligned} T_{(\Sigma,p,z|\zeta)}(\mathfrak{M}_{g,1^k_{\mathrm{NS}}}) &\cong \Gamma(U \setminus p, \mathcal{T}^s_\Sigma) \Big/ \Big(\Gamma(U, \mathcal{T}^s_\Sigma(-(k+1)p)) + \Gamma(\Sigma \setminus p, \mathcal{T}^s_\Sigma)\Big) \\ &\cong \mathfrak{switt} \Big/ \Big((\mathfrak{switt}/z^k \mathbb{C}[z^{-1}]\mathfrak{switt}) + D_{\Sigma,p,z|\zeta} \Big), \end{aligned}$$

where we denote by $D_{\Sigma,p,z|\zeta}$ the image of $\Gamma(\Sigma \setminus p, \mathcal{T}^s_\Sigma)$ in $\mathfrak{switt}$ using the local coordinate system $z|\zeta$. Further, taking the projective limit gives a description of the tangent space of $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$:

$$T_{(\Sigma,p,z|\zeta)}(\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}) \cong \mathfrak{switt}/D_{\Sigma,p,z|\zeta}. \tag{32}$$

Proposition 7.1. *The super Witt algebra acts on the moduli space $\mathfrak{M}_{g,1^\infty_{NS}}$ by vector fields.*

Proof. The isomorphism given in (32) can be used to define the global map

$$P : \mathfrak{switt} \to \mathrm{Vect}(\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}).$$

This map P is a Lie superalgebra homomorphism by the naturality of the Kodaira-Spencer map.

□

7.2 General linear action on the Sato Grassmannian

Consider $\mathfrak{gl}(H)$ as in definition 4.2. For any endomorphism of H, we may write it as

$$F = \begin{pmatrix} F^{--} & F^{-+} \\ F^{+-} & F^{++} \end{pmatrix}$$

where $F^{+-} : H^- \to H^+$ etc.

Define the restricted linear supergroup $\mathrm{GL}_F(H)$ to be the supergroup of continuous linear isomorphisms $G\colon H \to H$, where $G = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$, with G^{--} and G^{++} Fredholm. It is defined by the functor of points assigning a supercommutative algebra R the group $\mathrm{GL}_F(R \otimes H)$ of R-linear automorphisms $G\colon R \otimes H \to R \otimes H$ with G^{--} and G^{++} having finitely generated locally free kernels and cokernels.

Define an endomorphism of a discrete subspace to be supertrace class if it factors through some compact subspace.

Proposition 7.2. *The group* $\mathrm{GL}_F(H)$ *acts transitively on* $\mathrm{Gr}(H)$*, and the stabilizer of* D *is* $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ *expressed in a decomposition* $D \oplus K$*.*

Proof. Consider $G \in \mathrm{GL}_F(H)$ and discrete D. Every D is the image of an operator $w = \left(\begin{smallmatrix} w_- \\ w_+ \end{smallmatrix}\right) : H^- \to H$ such that $p_- \circ w = w_-$ is Fredholm. The action of G on D can be easily computed by matrix multiplication resulting in the new subspace given by the operator

$$Gw = \begin{pmatrix} G^{--}w_- + G^{-+}w_+ \\ G^{+-}w_- + G^{++}w_+ \end{pmatrix}$$

where the top map is clearly Fredholm.

To show the transitivity of the action, consider the map w as above and a map $v = \left(\begin{smallmatrix} v_- \\ v_+ \end{smallmatrix}\right) : H^+ \to H$ where the columns of v are a basis for K such that the natural map $D \oplus K \to H$ is an isomorphism, and where eventually $v(z^i) = z^{i-n}$ for some n and $v(z^i\theta) = z^{i-m}\theta$ for some m. We claim that

$$G = \begin{pmatrix} w_- & v_- \\ w_+ & v_+ \end{pmatrix}$$

is in $\mathrm{GL}_F(H)$ and $G(H^-) = D$. We need only to check that v is continuous and v_+ is Fredholm, both of which are satisfied by the condition that eventually $v(z^i) = z^{i-n}$ for some n and $v(z^i\theta) = z^{i-m}\theta$ for some m.

The stabilizer is obvious. □

Therefore we may describe $\mathrm{Gr}(H)$ as the homogeneous superspace

$$\mathrm{Gr}(H) \cong \mathrm{GL}_F(H)/P$$

where $P = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ in the $H^- \oplus H^+$ decomposition.

Proposition 7.3. *The Lie superalgebra* $\mathfrak{gl}(H)$ *acts by vector fields on* $\mathrm{Gr}(H)$*. Explicitly,* $F \mapsto L_F$ *is a Lie superalgebra homomorphism* $\mathfrak{gl}(H) \to \mathcal{T}_{\mathrm{Gr}(H)}$*. In the chart* $U_{D,K}$*, this action is given by the formula*

$$L_F(A) = F^{KD} + F^{KK}A - AF^{DD} - AF^{DK}A$$

where $L_F \in \underline{\mathrm{Hom}}_{\mathbb{C}}(S^\bullet(\underline{\mathrm{Hom}}_{\mathbb{C}}(D,K)), \underline{\mathrm{Hom}}_{\mathbb{C}}(D,K))$*.*

Proof. Consider the action of $G \in \mathrm{GL}_F$ on the point in $U_{D,K}$ represented by coordinates $A \colon D \to K$.

$$\begin{bmatrix} G^{DD} & G^{DK} \\ G^{KD} & G^{KK} \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} = \begin{bmatrix} G^{DD} + G^{DK}A & G^{DK} \\ G^{KD} + G^{KK}A & G^{KK} \end{bmatrix}$$
$$\sim \begin{bmatrix} I & 0 \\ (G^{KD} + G^{KK}A)(G^{DD} + G^{DK}A)^{-1} & I \end{bmatrix}$$

The equivalence relation is given by multiplication on the right by the stabilizer of D, which in the $D \oplus K$ decomposition is $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. We calculate $L_F(A)$ as the derivative at $t = 0$ of the lower left block with $G = I + tF$. □

It is known that $H^2(\mathfrak{gl}(H))$ is one-dimensional. For any choice of discrete D and compact K such that the natural map $D \oplus K \xrightarrow{\sim} H$ is an isomorphism, we define a 2-cocycle on $\mathfrak{gl}(H)$ as

$$\eta_{D,K}(F, G) := \operatorname{str}(F^{DK}G^{KD} - (-1)^{|F||G|}F^{DK}G^{KD}).$$

We choose a distinguished 2-cocycle to correspond our distinguished decomposition $H^- \oplus H^+ \cong H$:[10]

$$\eta(F, G) := \operatorname{str}(F^{-+}G^{+-} - (-1)^{|F||G|}F^{-+}G^{+-}),$$

which we call the super Japanese cocycle. The unique Lie superalgebra central extension defined by the super Japanese cocycle will be denoted $\widetilde{\mathfrak{gl}}(H)$. We denote the bracket on $\widetilde{\mathfrak{gl}}(H)$ as $[F, G]^{\sim} = [F, G] + \eta(F, G)$.

Proposition 7.4. *The Lie superalgebra $\widetilde{\mathfrak{gl}}(H)$ acts by first order differential operators on $\operatorname{Ber}_{\operatorname{Gr}(H)}$. Explicitly, $F + c \mapsto \widetilde{L}_F + c$ is a Lie superalgebra homomorphism $\widetilde{\mathfrak{gl}}(H) \to \mathcal{A}_{\operatorname{Ber}}(\operatorname{Gr}(H))$ sending $[F, G]^{\sim}$ to $[\widetilde{L}_F, \widetilde{L}_G]$. In the chart $U_{D,K}$, this action is given by the formula*

$$\widetilde{L}_F(A) = L_F(A) + \operatorname{str}(F^{DK}A) + \alpha(F)$$

where $\alpha \in C^1(\mathfrak{gl})$ is the unique 1-cochain such that

$$d\alpha(F, G) = \alpha([F, G]) = \eta_{D,K}(F, G) - \eta(F, G).$$

Before providing the proof, we remark about the case that D, K, H were finite-dimensional as motivation for the definition of the Lie superalgebra action. Consider the natural action of $G \in \operatorname{GL}_F$ on $\operatorname{Ber}_K(D_A)$ where $D_A \in U_{D,K}$ is represented by coordinates $A \colon D \to K$.

$$\begin{array}{ccc}
\dfrac{\operatorname{Ber}(D)\operatorname{Ber}(K)}{\operatorname{Ber}(H)} & \xrightarrow{\frac{\operatorname{Ber}(G^{DD}+G^{DK}A)\operatorname{Ber}(G^{KK})}{\operatorname{Ber}(G)}} & \dfrac{\operatorname{Ber}(D)\operatorname{Ber}(K)}{\operatorname{Ber}(H)} \\
{\scriptstyle \operatorname{graph}(A)}\big\downarrow & & \big\uparrow{\scriptstyle \pi_D, \pi_K} \\
\dfrac{\operatorname{Ber}(D_A)\operatorname{Ber}(K)}{\operatorname{Ber}(H)} & \xrightarrow{\quad G \quad} & \dfrac{\operatorname{Ber}(D_A)\operatorname{Ber}(K)}{\operatorname{Ber}(H)}
\end{array}$$

[10] The cocycle defined by Ueno and Yamada in [40] is $\eta_{D,K}$ with choice $D = \mathbb{C}[z^{-1}, \zeta]$ and $K = z\mathbb{C}[[z]][\zeta]$.

Using the canonical isomorphism of the fiber $\mathrm{Ber}_K(D_A)$ with $\mathrm{Ber}_K(D)$, we find the multiplicative factor $\frac{\mathrm{Ber}(G^{DD}+G^{DK}A)\,\mathrm{Ber}(G^{KK})}{\mathrm{Ber}(G)}$. Then for $G = I + tF$, we derive the Lie superalgebra action of $\mathrm{str}(F^{DK}A)$.

Proof. Direct computation shows that $\mathrm{str}([F_1, F_2]^{DK}A) = \eta_{D_A,K}(F_1, F_2) - \eta_{D,K}(F_1, F_2)$. Similarly, if $D_A \in U_{D,K}$ and $D_A \in U_{D',K'}$ then the change of coordinates is given by the unique 1-cochain whose exterior derivative is $\eta_{D_A,K'}(F_1, F_2) - \eta_{D_A,K}(F_1, F_2)$. Thus, this Lie superalgebra action is glues between charts.

The Lie superalgebra homomorphism follows from:

$$
\begin{aligned}
[\widetilde{L}_{F_1}, \widetilde{L}_{F_2}]A &= [L_{F_1}, L_{F_2}] + \mathrm{str}(F_1^{DK}L_{F_2}A) - \mathrm{str}(F_2^{DK}L_{F_1}A) \\
&= L_{[F_1,F_2]}A + \eta_{D,K}(F_1, F_2) + \mathrm{str}([F_1, F_2]^{DK}A) \\
&= L_{[F_1,F_2]}A + \eta_{D,K}(F_1, F_2) - \eta(F_1, F_2) + \eta(F_1, F_2) + \mathrm{str}([F_1, F_2]^{DK}A) \\
&= L_{[F_1,F_2]}A + \alpha([F_1, F_2]) + \eta(F_1, F_2) + \mathrm{str}([F_1, F_2]^{DK}A) \\
&= \widetilde{L}_{[F_1,F_2]^{\sim}}A
\end{aligned}
$$

□

We now summarize the above Lie superalgebra action using the concept of an action Lie superalgebroid as defined in section 4.3. The global map $\widetilde{\mathfrak{gl}}(H) \to \mathcal{A}_{\mathrm{Ber}}(\mathrm{Gr}(H))$ in Proposition 7.4 may be used to define a morphism of Lie algebroids $\widetilde{\mathcal{G}} \to \mathcal{A}_{\mathrm{Ber}}$, where $\widetilde{\mathcal{G}}$ is denotes the action Lie superalgebroid over $\mathrm{Gr}(H)$ of the Lie superalgebra $\widetilde{\mathfrak{gl}}(H)$. By restriction to the action Lie superalgebroid $\mathcal{G}$ corresponding to $\mathfrak{gl}(H)$, we have the Lie superalgebroid morphism $\mathcal{G} \to \mathcal{T}_{\mathrm{Gr}(H)}$ corresponding to the global map in Proposition 7.3. In summary, we have the commutative diagram of Lie superalgebroids below.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathrm{Gr}(H)} & \longrightarrow & \widetilde{\mathcal{G}} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \big\downarrow \mathrm{Id} & & \big\downarrow \widetilde{L} & & \big\downarrow L & & \\
0 & \longrightarrow & \mathcal{O}_{\mathrm{Gr}(H)} & \longrightarrow & \mathcal{A}_{\mathrm{Ber}} & \longrightarrow & \mathcal{T}_{\mathrm{Gr}(H)} & \longrightarrow & 0
\end{array} \tag{33}
$$

7.3 The super Krichever map

Consider the super vector subspace of sections $\Gamma(\Sigma \setminus p, \omega_\Sigma^{\otimes j}) \subset \mathbb{C}((z))[\zeta]\ [dz|d\zeta]^{\otimes j}$. Using the super vector space isomorphism $H \cong \mathbb{C}((z))[\zeta]\ [dz|d\zeta]^{\otimes j}$, we may regard $\Gamma(\Sigma \setminus p, \omega_\Sigma^{\otimes j}) \subset H$.

Definition 7.5 (Mulase and Rabin [33]). *The super Krichever map $\mathfrak{M}_{g,1^\infty_{NS}} \to \mathrm{Gr}(H)$ is an injective analytic map given by*

$$\kappa_j(\Sigma, p, z|\zeta) = \Gamma(\Sigma \setminus p, \omega_\Sigma^{\otimes j}) \subset \mathbb{C}((z))[\zeta]\ [dz|d\zeta]^{\otimes j}.$$

where $\omega_\Sigma := \mathrm{Ber}(\Omega^1_\Sigma)$ *is a rank* $0|1$ *invertible sheaf, the Berezinian of* Σ.

Proof. We check this definition is well-defined. Notice $\Gamma(\Sigma\setminus p, \omega_\Sigma^{\otimes j}) \in \mathrm{Gr}(H)$ since the map p_- restricted to $\Gamma(\Sigma\setminus p, \omega_\Sigma^{\otimes j})$ is Fredholm, see (29).

$$0 \to \Gamma(\Sigma, \omega_\Sigma^{\otimes j}) \to \Gamma(\Sigma\setminus p, \omega_\Sigma^{\otimes j}) \xrightarrow{p_-} H^- \to H^1(\Sigma, \omega_\Sigma^{\otimes j}) \to 0.$$

See Theorem 4.2 of [33] for a proof of that the map is injective analytic. □

Proposition 7.6. *For* $m = (\Sigma, p, z|\zeta) \in \mathfrak{M}_{g,1^\infty_{NS}}$*, we have the following commutative diagram of Lie superalgebras with exact horizontal sequences.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{gl}(H)_{\kappa_j(m)} & \longrightarrow & \mathfrak{gl}(H) & \xrightarrow{L|_{\kappa_j(m)}} & T_{\kappa_j(m)}\,\mathrm{Gr}(H) & \longrightarrow & 0 \\ & & \varrho_j\big\uparrow & & \varrho_j\big\uparrow & & d\kappa_j\big\uparrow & & \\ 0 & \longrightarrow & \Gamma(\Sigma\setminus p, \mathcal{T}^s_\Sigma) & \longrightarrow & \mathfrak{switt} & \xrightarrow{P|_m} & T_m\mathfrak{M}_{g,1^\infty_{NS}} & \longrightarrow & 0 \end{array} \tag{34}$$

Proof. The top exact sequence is the action of $\mathfrak{gl}(H)$ along a fiber. This gives $T_D\,\mathrm{Gr}(H) \cong \underline{\mathrm{Hom}}_{\mathbb{C}}(D, H/D)$, where in the diagram above we have $D = \kappa_j(m)$. The bottom exact sequence is given by the action of $\mathfrak{switt}$ along a fiber. The left map is well-defined since the Lie derivative of an element of $\kappa_j(m) = \kappa_j(\Sigma, p, z|\zeta) = \Gamma(\Sigma\setminus p, \omega_\Sigma^{\otimes j})$ by an element of $\Gamma(\Sigma\setminus p, \mathcal{T}^s_\Sigma)$ is again an element of $\kappa_j(\Sigma, p, z|\zeta)$. Then we must check that the right map induced by ϱ_j is isomorphic to the map $d\kappa_j$. Let $\partial_{f(z,\zeta)} \in \mathfrak{switt}$ be as in (25), and $v_f \in T_m\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$ be its image. Then for $g(z,\zeta)[dz|d\zeta]^{\otimes j} \in \kappa_j(m)$ and ε such that $|\varepsilon| = |f|$

$$\begin{aligned} &d\kappa_j(v_f)\Big(g(z,\zeta)[dz|d\zeta]^{\otimes j}\Big) \\ &\quad= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\, g\left(z + \varepsilon f(z) + \varepsilon\zeta\frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2}\,\middle|\, \zeta + \varepsilon\frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2}\right) \\ &\qquad\qquad \left[d(z + \varepsilon f(z) + \varepsilon\zeta\frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2})\,\middle|\, d(\zeta + \varepsilon\frac{(-1)^{|f|}D_\zeta f(z,\zeta)}{2})\right]^{\otimes j} \mod \kappa_j(m) \\ &\quad= \mathcal{L}_{\partial_f}\, g(z,\zeta)[dz|d\zeta]^{\otimes j} \mod \kappa_j(m) \end{aligned}$$

which matches the definition of the Lie derivative. □

We easily upgrade the diagram (34) into a diagram of Lie superalgebroids on $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {\kappa_j}^*\mathcal{J} & \longrightarrow & {\kappa_j}^*\mathcal{G} & \xrightarrow{{\kappa_j}^*L} & {\kappa_j}^*\mathcal{T}_{\mathrm{Gr}(H)} & \longrightarrow & 0 \\ & & \varrho_j\big\uparrow & & \varrho_j\big\uparrow & & d\kappa_j\big\uparrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{W} & \xrightarrow{P} & \mathcal{T}_{\mathfrak{M}} & \longrightarrow & 0 \end{array} \tag{35}$$

7.4 Proof of a flat holomorphic connection

Lemma 7.7. *In an abelian category, if the following commutative diagram has exact short exact rows and the left map is an isomorphism, then the right square is a pullback square, i.e.* $B = B' \times_{C'} C$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & {\scriptstyle \wr}\big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{\iota'} & B' & \xrightarrow{\pi'} & C' & \longrightarrow & 0 \end{array}$$

Proof. Suppose we have X such that the following square commutes.

$$\begin{array}{ccc} X & \xrightarrow{g} & C \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle c} \\ B' & \xrightarrow{\pi'} & C' \end{array}$$

Let $x \in X$. Choose $y \in B$ such that $\pi(y) = g(x)$. Then $b(y) - f(x) \in B'$, and further $\pi'(b(y) - f(x)) = \pi' \circ b(y) - \pi' \circ f(x) = c \circ \pi(y) - c \circ g(x) = c(\pi(y) - g(x)) = 0$. Therefore we have $b(y) - f(x) \in A'$. And using the isomorphism a and inclusion into B, we have $a^{-1}(b(y) - f(x)) \in B$.

Define $\varphi\colon X \to B$ as $\varphi(x) = y - a^{-1}(b(y) - f(x))$. We first check this map commutes, then check it does not depend on the choice of y.

$$\pi \circ \varphi(x) = \pi(y - a^{-1}(b(y) - f(x))) = \pi(y) - \pi(a^{-1}(b(y) - f(x))) = \pi(y) = g(x)$$
$$b \circ \varphi(x) = b(y - a^{-1}(b(y) - f(x))) = b(y) - b(a^{-1}(b(y) - f(x))) = b(y) - (b(y) - f(x)) = f(x)$$

Now consider two lifts $y, y' \in B$ such that $\pi(y) = \pi(y') = g(x)$. Consider the subtraction of the resulting maps

$$\begin{aligned} \varphi(x) - \varphi'(x) &= [y - a^{-1}(b(y) - f(x))] - [y' - a^{-1}(b(y') - f(x))] \\ &= (y - y') - a^{-1}(b(y - y')) = 0 \end{aligned}$$

where the last equality results from noticing that $y - y' \in A$ and $b(y - y') \in A'$. $\square$

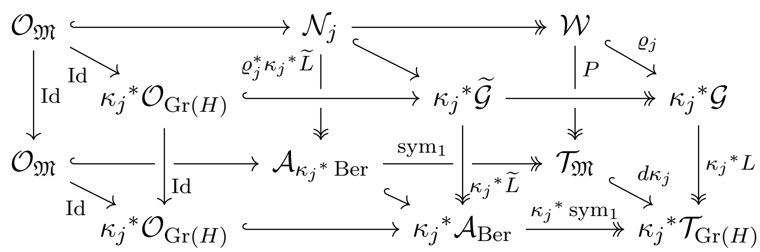
Corollary 7.8. *The Atiyah algebra of the pullback of the Berezinian line bundle is the pullback in the following diagram.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathfrak{M}} & \longrightarrow & \mathcal{A}_{\kappa_j{}^* \mathrm{Ber}} & \xrightarrow{\mathrm{sym}_1} & \mathcal{T}_{\mathfrak{M}} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_{j*}} & \ulcorner & \big\downarrow{\scriptstyle d\kappa_j} & & \\ 0 & \longrightarrow & \kappa_j{}^* \mathcal{O}_{\mathrm{Gr}(H)} & \longrightarrow & \kappa_j{}^* \mathcal{A}_{\mathrm{Ber}} & \xrightarrow{\kappa_j{}^* \mathrm{sym}_1} & \kappa_j^* \mathcal{T}_{\mathrm{Gr}(H)} & \longrightarrow & 0 \end{array} \tag{36}$$

Proof. By Lemma 7.7, it suffices to notice that κ_{j*} restricts to an isomorphism $\mathcal{O}_{\mathfrak{M}} \to {\kappa_j}^*\mathcal{O}_{\mathrm{Gr}(H)}$, and that the right square (with the similarly defined map $d\kappa_j$ on the tangent sheaves) commutes. □

We now combine several commuting squares to pullback the diagram (33) from the Grassmannian to the moduli space $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$. The transfer of this diagram from the Grassmannian to the moduli space originates from compatibility of the representation of the super Witt algebra with the derivative of the Krichever map, as given in (35).

Beginning with diagram (33) as the front face and the square of (35) as the right face, we build the following commuting cube of Lie superalgebroids on $\mathfrak{M}_{g,1^\infty_{\mathrm{NS}}}$.



The top face and bottom face are pullback diagrams. The top face is the Lie superalgebroid version of (21), and the bottom face is given by (36). Since $\mathcal{N}_j$ and $\mathcal{A}_{{\kappa_j}^*\mathrm{Ber}}$ are both pullbacks in the top and bottom squares respectively, there is a unique action map $\varrho_j^*{\kappa_j}^*\widetilde{L} : \mathcal{N}_j \to \mathcal{A}_{{\kappa_j}^*\mathrm{Ber}}$.

Further, using the relationship given in (21) between the standard $\mathfrak{ns}$ and the non-standard $\mathfrak{ns}_j$, we arrive at the diagram below. The crucial property of the action of the Neveu-Schwarz Lie superalgebroid $\mathcal{N}$ is that the central charge acts by c_j.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathfrak{M}} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_j\,\mathrm{Id}} & & \downarrow{\scriptstyle \widetilde{P}_j} & & \downarrow{\scriptstyle P} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathfrak{M}} & \longrightarrow & \mathcal{A}_{{\kappa_j}^*\mathrm{Ber}} & \longrightarrow & \mathcal{T}_{\mathfrak{M}} & \longrightarrow & 0 \end{array} \tag{37}$$

Proposition 7.9. *The pullback of the Berezinian line bundle from the super Grassmannian along the j^{th} super Krichever map is canonically isomorphic to $\lambda_{j/2}$.*

$${\kappa_j}^*\mathrm{Ber}_{\mathrm{Gr}(H)} \cong \lambda_{j/2}$$

Proof. We see $\kappa_j(\Sigma, p, z|\zeta) = \Gamma(\Sigma \setminus p, \omega_\Sigma^{\otimes j})$. Using the definition in definition 6.12 of the Berezinian line bundle on the super Grassmannian, we have the fiber

$$\mathrm{Ber}_{\mathrm{Gr}(H)}\left(\Gamma(\Sigma \setminus p, \omega_\Sigma^j)\right) = \frac{\mathrm{Ber}(H^0(\Sigma, \omega_\Sigma^j))}{\mathrm{Ber}(H^1(\Sigma, \omega_\Sigma^j))}$$

which can be seen using Čech cohomology. Further, we may consider the discrete $\mathcal{O}_{\mathfrak{M}}$-module $L \in \widehat{H}_{\mathfrak{M}}$ corresponding to the Krichever map. By a similar argument,

$$\mathrm{Ber}_{H^+}(L) = \frac{\mathrm{Ber}(R^0\pi_*\omega^j_{X/\mathfrak{M}})}{\mathrm{Ber}(R^1\pi_*\omega^j_{X/\mathfrak{M}})}$$

which is exactly definition 5.7 of $\lambda_{j/2}$. □

The standard operations of Atiyah algebras have straightforward generalizations to super Atiyah algebras, which are described in the section 4.3. Using these standard properties of Atiyah algebras, we describe the action of the Neveu-Schwarz Lie superalgebroid on the line bundles of the super Mumford isomorphism. Denote $\mathcal{A}_j := \mathcal{A}_{\lambda_{j/2}\otimes\lambda_{1/2}^{-c_j}}$. Firstly, as in (37), $\mathcal{N}$ acts on $-c_j\mathcal{A}_{\lambda_{1/2}}$ with central charge $-c_j$. Then the action of $\mathcal{N}$ on $\mathcal{A}_j$ is defined via the action on $\mathcal{A}_{\lambda_{j/2}} \times_{\mathcal{T}_X} -c_j\mathcal{A}_{\lambda_{1/2}}$. Importantly, we see that $\mathcal{N}$ acts with central charge $c_j - c_j = 0$, as in the diagram of Lie superalgebroids below.

$$\begin{array}{ccccccccc} & & & & & & \mathcal{K} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathfrak{M}} & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & \downarrow & \swarrow_{\alpha_j} & \downarrow{\scriptstyle P} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathfrak{M}} & \longrightarrow & \mathcal{A}_j & \longrightarrow & \mathcal{T}_{\mathfrak{M}} & \longrightarrow & 0 \end{array} \tag{38}$$

A straightforward diagram chase gives the unique morphism of Lie superalgebroids $\alpha_j \colon \mathcal{W} \to \mathcal{A}_j$.

The final lemma needed to show our main result is the perfectness of the Lie superalgebra of superconformal vector fields on an affine super Riemann surface. The analogous result in classical geometry was proven by [2] and [23] and is expected to generalize to the super setting. The proof may use a superconformal Noether normalization based on the results of [31].

Conjecture 1. *Let Σ be an* affine *super Riemann surface. Denote the Lie superalgebra of global superconformal vector fields as $\mathfrak{k} := \Gamma(\Sigma, \mathcal{T}^s_\Sigma)$. Then the Lie superalgebra $\mathfrak{k}$ is perfect, that is to say, $H_1(\mathfrak{k};\mathbb{C}) = \mathfrak{k}/[\mathfrak{k},\mathfrak{k}] = 0$.*

Locally, this conjecture is true. For an open superdomain in $\mathbb{C}^{1|1}$ we can use the standard superconformal coordinates $z|\zeta$ such that $\mathcal{D}$ is generated by $D_\zeta = \frac{\partial}{\partial\zeta} + \zeta\frac{\partial}{\partial z}$ and elements of $\mathfrak{k}$ are of the form $[fD_\zeta, D_\zeta]$where $f = f(z,\zeta)$. The supercommutator of superconformal vector fields gives

$$\Big[[fD_\zeta, D_\zeta], [gD_\zeta, D_\zeta]\Big] = \left[\left(f(\partial_z g) + \frac{(-1)^{|f|}}{2}(D_\zeta f)(D_\zeta g) - (\partial_z f)g\right) D_\zeta, D_\zeta\right].$$

For an arbitrary function h, we find $[hD_\zeta, D_\zeta] \in [\mathfrak{k}, \mathfrak{k}]$ since

$$[hD_\zeta, D_\zeta] = \frac{2}{3}\Big[[hD_\zeta, D_\zeta], [zD_\zeta, D_\zeta]\Big] + \frac{2}{3}\Big[[D_\zeta, D_\zeta], [hzD_\zeta, D_\zeta]\Big] + \frac{2}{3}\Big[[h\zeta D_\zeta, D_\zeta], [\zeta D_\zeta, D_\zeta]\Big].$$

The definition of a holomorphic connection on a line bundle is recalled in section 4.4.

Theorem 7.10. *There exists a flat holomorphic connection on the line bundle* $\lambda_1^{-c_j} \otimes \lambda_j$.

Proof. Consider the kernel Lie algebroid $\mathcal{K}$ as in (38). In fact, $\mathcal{K}$ is a bundle of Lie algebras with fiber $\mathfrak{k} = \Gamma(\Sigma \setminus p, \mathcal{T}_\Sigma^s)$, meaning that the bracket on $\mathcal{K}$ is simply the pointwise bracket of its fibers. By conjecture 1, we know every fiber is equal to its commutant, therefore $[\mathcal{K}, \mathcal{K}] = \mathcal{K}$. So wlog, let $k = [k_1, k_2] \in \mathcal{K}$. Then notice $\alpha_j(k_1), \alpha_j(k_2) \in \ker(\mathcal{A}_{p^*j} \to \mathcal{T}_X) = \mathcal{O}_\mathfrak{M}$. Which means $\alpha_j(k) = [\alpha_j(k_1), \alpha_j(k_2)] = 0$. Then by the universal property of the cokernel of $\mathcal{T}_\mathfrak{M}$, we have a unique morphism of Lie algebroids $\nabla\colon \mathcal{T}_\mathfrak{M} \to \mathcal{A}_j$. □

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