GROUND STATES AND DIRICHLET PROBLEMS FOR $-\Delta = F(U)$ IN R^2

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GROUND STATES AND DIRICHLET PROBLEMS FOR

$-\Delta u = f(u)$ in R^2

F.V. Atkinson and L.A. Peletier

Dedicated to Jim Serrin on his sixtieth birthday

1. INTRODUCTION

In an earlier paper [1] we considered the existence of solutions of the problem

$$\text{(I)} \qquad \begin{aligned} &-\Delta u = f(u), \quad u > 0 \quad \text{in} \quad R^N \\ &u(x) \to 0 \qquad \text{as} \quad |x| \to \infty \end{aligned}$$

in which $f(u)$ is to be positive for large u, but not for all $u > 0$. Such solutions are sometimes called "ground states", a term borrowed from the physical context (nonlinear field equations) in which Problem I arises. Following in part the approach of [5], we used a "shooting method" (in place of variational arguments) to prove under suitable conditions the existence of such a ground state. The principal difficulty lay in showing that if $u(0)$ were chosen sufficiently large, then the associated radially symmetric solution had a zero, i.e. that the Dirichlet problem on some finite ball had a solution.

The solubility of Dirichlet problems for

$$-\Delta u = f(u),$$

where $f(u)$ is positive for large u, or perhaps for all $u > 0$ depends very critically on the rate of growth of $f(u)$ as $u \to \infty$. Here the cases $N \geq 3$ and $N = 2$ are strikingly different. In the case $N \geq 3$ one finds the condition (see for instance [1,4,5,9,10,11])

$$f(u) = O(u^p) \quad \text{as} \quad u \to \infty, \qquad 1 < p < \frac{N+2}{N-2}, \tag{1.1}$$

and for $N = 2$ conditions such as (see [10,12])

$$\log f(u) = o(u^2) \quad \text{as} \quad u \to \infty. \tag{1.2}$$

There has been much interest recently in borderline cases for $N > 3$, involving the "critical exponent" $(N + 2)/(N - 2)$. In particular the case

$$f(u) = u^q + u^{(N + 2)/(N - 2)}, \quad 1 \leq q < (N + 2)/(N - 2) \tag{1.3}$$

has been studied in [6] and in [2], where the ODE approach used in [1] was shown to be effective for this case.

In the present paper we use the same approach to fill in a gap in our earlier treatment [1] of the case $N = 2$, and to deal with some borderline cases corresponding to (1.2). In [1] we did not cover the whole range (1.2); writing

$$F(u) = \int_0^u f(s)ds \tag{1.4}$$

we asked that

$$\liminf_{u \to \infty} \frac{1}{u} \log F(u) = 0. \tag{1.5}$$

This condition allows less rapid growth than (1.2) when the growth is regular, though the "lim inf" and integral features do permit quite erratic growth. Here we confine attention to the case of regular growth, and we shall ask in particular that $\log f(u)$ be <u>convex</u> for large u.

Coming to specifics, we shall make the following general hypotheses about f.

<u>H1</u>. $f(u)$ is locally Lipschitz continuous on $[0,\infty)$;

H2. $f(0) = 0$ and there exists a number $\zeta > 0$ such that

$$F(u) < 0 \quad \text{for} \quad 0 < u < \zeta \quad \text{and} \quad F(\zeta) = 0 \tag{1.6}$$

$$f(u) > 0 \quad \text{for} \quad u \geq \zeta. \tag{1.7}$$

H3. There exists a number $y_0 \geq 0$ such that

(i)
$$f(u) > 0 \quad \text{for} \quad u \geq y_0, \tag{1.8}$$

and the function

$$g(u) \overset{\text{def}}{=} \log f(u) \qquad u \geq y_0 \tag{1.9}$$

satisfies

(ii)
$$g \in C^2([y_0,\infty));$$

(iii)
$$g'(u) > 0,\ g''(u) \geq 0 \qquad \text{for} \quad u \geq y_0. \tag{1.10}$$

We shall always assume that f satisfies H1 and H3. When considering the existence of ground states we shall assume H2 as well. In that case y_0 will have to be a positive number.

We formulate our main result for ground states first in its most general form in which the growth condition on f is replaced by an inequality involving the function

$$h(u) = \{g(u) - \tfrac{1}{2}ug'(u)\} - \tfrac{1}{2}y_0g'(u)[e^{1/2\{g(u) - g(y_0)\}} - 1]^{-1} \tag{1.11}$$

and the lower bound of f

$$\inf\{f(u) : u > 0\} = -M. \tag{1.12}$$

Theorem 1. Suppose f *satisfies H1, H2 and H3. If there exists a* $\gamma >$ $\max\{y_0, \zeta\}$ *such that*

$$h(\gamma) > \log M + 1, \tag{1.13}$$

then Problem I, with $N = 2$, *has a solution* u *in* R^2 *with* $u < \gamma$.

The behaviour of $f(u)$ or $g(u)$ for $u > \gamma$ is, of course, irrelevant. All that is needed is that there should be one γ satisfying (1.13).

Less precise, but simpler criteria can be obtained by ensuring that (1.13) is asymptotically satisfied, and in particular, by making $h(u)$ take suitably large values for large u. Thus we have the following corollary to Theorem 1.

Theorem 2. Suppose H1, H2 and H3 are satisfied. If

$$\limsup_{u \to \infty} \{g(u) - \tfrac{1}{2} ug'(u)\} > \log M + 1, \tag{1.14}$$

then Problem I has a solution in R^2.

To indicate the scope of these results we conclude with some examples. If

$$g(u) = u^q + u^r \qquad 0 < r < q$$

then (1.14) is satisfied if $q < 2$ but not if $q > 2$, in conformity with (1.2).

The borderline case, in which $g(u)$ behaves like u^2 as $u \to \infty$ can also be investigated. Thus

$$g(u) = u^2 + \log u$$

satisfies (1.14), and so does

$$g(u) = u^2 + b$$

provided $b > \log M + 1$.

An example of a function f, which satisfies all the requirements is

$$f(u) = (u^2 - u)e^{u^2}. \tag{1.15}$$

Another example, in a similar vein, is given by

$$f(u) = (u^2 - au)e^{u^q},$$

where $q > 1$ and $a > 0$. Here $\log M$ will become large and negative as $a \to 0$ so that for any given $\gamma > 0$, (1.13) will be satisfied if a is chosen small enough.

The proofs of these results are based on a study of the asymptotically constant solutions of a class of highly nonlinear, generalized Emden-Fowler equations

$$y'' + e^{g(y)-t} = 0 \qquad t \in R$$
$$y(t) \to \infty \qquad \text{as } t \to \infty.$$

For g increasing and convex and γ large, the graphs of such solutions turn out to have a very characteristic shape (see Fig. 1)

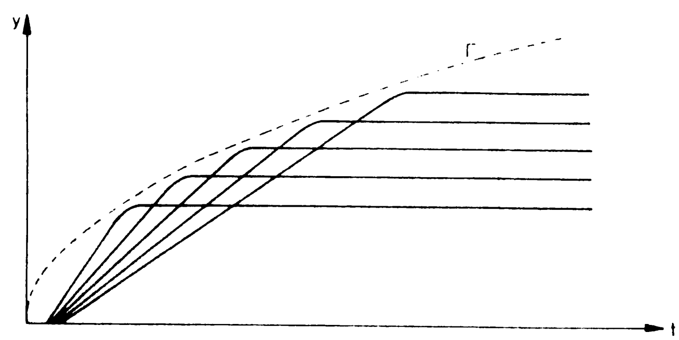

Fig. 1 Solution graphs

This property is of some intrinsic interest. The final section is therefore devoted to obtaining some first quantitative results about it.

2. PRELIMINARY TRANSFORMATION AND SHOOTING ARGUMENT

We are concerned with proving the existence of spherically symmetric solution of Problem I when $N = 2$, that is with solutions $u(r)$ of the equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + f(u) = 0 \qquad \text{on} \quad (0,\infty). \tag{2.1}$$

Here we transform the singularity at $r = 0$ to one at $t = +\infty$ by taking

$$t = -2\log(\tfrac{1}{2}r), \qquad y(t) = u(r). \tag{2.2}$$

We thus replace Problem I by Problem II, namely

(II)

$$\frac{d^2y}{dt^2} + e^{-t}f(y) = 0 \qquad t \in R \tag{2.3}$$

$$y(t) > 0 \qquad t \in R \tag{2.4}$$

$$\sup\{y(t):\ t \in R\} < \infty \tag{2.5}$$

$$y(t) \to 0 \qquad \text{as} \quad t \to -\infty \tag{2.6}$$

This does not, of course, change the nature of the problem, but brings it into line with the classical theory of the generalized Emden-Fowler equation [7,8,14] and the arguments of [1,2].

In particular, it follows from (2.3)-(2.5) that as $t \to \infty$, $y'(t)$ tends to a limit, which can only be zero and, using (2.3) again, that $y(t)$ also tends to a limit which must be finite. We shall generally write

$$\lim_{t\to\infty} y(t) = \gamma. \tag{2.7}$$

On the other hand, for any $\gamma \in (0,\infty)$ there exists for sufficiently large t a solution $y(t) = y(t,\gamma)$ of (2.3) with the property (2.7); this solution will be unique, in view of the Lipschitz condition on f.

The subsequent argument centres on the behavior of $y(t,\gamma)$ as t decreases from $+\infty$. If $y(t,\gamma)$ reaches the value 0 for some t we denote this value of t by $T = T(\gamma)$:

$$T(\gamma) = \inf\{t \in R: \ y(\cdot,\gamma) > 0 \quad \text{as} \quad (t,\infty)\}. \tag{2.8}$$

We then have a solution of a certain Dirichlet problem which we term Problem III, namely

$$\text{(III)} \qquad \begin{aligned} &\frac{d^2y}{dt^2} + e^{-t}f(y) = 0 && t > T(\gamma) \\ &y(t,\gamma) > 0, && t > T(\gamma) \\ &y(T(\gamma),\gamma) = 0. \end{aligned}$$

It may of course happen that $T(\gamma)$ does not exist, i.e. that $y(t,\gamma) > 0$ for all $t \in R$ and $T(\gamma) = -\infty$. We denote by S the set $\gamma \in R^+$ for which $T(\gamma)$ does exist.

From [1] (see also [5]) we need

Lemma 0. *Suppose H1 and H2 are satisfied. If S is nonempty, Problem II has a solution $y(t)$ with*

$$\lim_{t\to\infty} y(t) < \inf S.$$

We thus have to show that Problem III has a solution for some $\gamma > 0$ and $T(\gamma) > -\infty$, to ensure the existence of a ground state.

3. ESTIMATES FOR THE DIRICHLET PROBLEM

Our approach to Problem III is based on viewing $\gamma = y(\infty)$ as a parameter. We first ascertain the existence of $T(\gamma)$, the "first zero" of y as t decreases from $+\infty$, and then estimate $T(\gamma)$ in terms of γ.

For the standard Dirichlet problem

$$\text{(IV)}\qquad \begin{aligned} -\Delta u &= f(u) && \text{in } B_R && (3.1)\\ u &> 0 && \text{in } B_R && (3.2)\\ u &= 0 && \text{on } \partial B_R, && (3.3)\end{aligned}$$

where $B_R = \{x \in \mathbb{R}^2\colon |x| < R\}$, the existence of $T(\gamma)$ means that there is a solution $u(r)$ with

$$u(0) = \gamma \quad\text{and}\quad R = R(\gamma) \overset{\text{def}}{=} 2e^{-T(\gamma)/2}, \tag{3.4}$$

in view of (2.2).

Our main result concerning the Dirichlet problem is the following.

<u>Theorem 3.</u> <u>Suppose</u> f <u>satisfies H1 and H3 and</u> γ <u>satisfies (1.13).</u> <u>Then</u> $T(\gamma)$ <u>exists and</u>

$$T(\gamma) > h(\gamma) + \log\{\tfrac{1}{2}\, g'(\gamma)\} - 1. \tag{3.5}$$

<u>If</u> $M = 0$ <u>in (1.12), the term</u> -1 <u>may be omitted.</u>

Theorem 3 can be used to determine when $T(\gamma) \to \infty$ (and therefore $R(\gamma) \to 0$) as $\gamma \to \infty$. We given a few examples.

<u>Example 1</u> $f(u) = \pi(u)e^{u^q}$, $u \geq y_0$, $q > 1$. (3.6)

Here $\pi(u)$ is a polynomial which is positive for $u \geq y_0$. The requirement $q > 1$ ensures the convexity property (1.10) for suitable y_0. Theorem 3 yields

$$T(\gamma) > (1 - \frac{1}{2} q)\gamma^q + O(\log \gamma) \quad \text{as} \quad \gamma \to \infty \tag{3.7}$$

provided either $q < 2$, in which case (1.13) is certainly satisfied for large γ, or else $M = 0$ that is to say $\pi(n) \geqslant 0$ for all $u > 0$.

It follows from (3.7) that

$$\lim_{\gamma\to\infty} T(\gamma) = \infty \quad \text{if} \quad 1 < q < 2.$$

The possibility that $T(\gamma) \to -\infty$ as $\gamma \to \infty$ if $q > 2$ is left open by (3.7) but does not seem supported by numerical evidence.

In the following examples we examine the borderline cases $q = 2$ and $q = 1$.

Example 2: $f(u) = u^m e^{u^2}$, $m \in R$. (3.8)

Here $M = 0$ and the existence of $T(\gamma)$ for $\gamma > 0$ is automatic. We now conclude from Theorem 3 that

$$T(\gamma) > (m + 1)\log \gamma + O(1) \quad \text{as} \quad \gamma \to \infty \tag{3.9}$$

so that $T(\gamma) \to \infty$ as $\gamma \to \infty$ logarithmically if $m > -1$ (The departure from Lipschitz continuity at $u = 0$ does not affect this.)

The case $q = 1$ is noteworthy for the presence of an exact solution.

Example 3. $f(u) = e^{a+bu}$, $a \in R$, $b \in R^+$. (3.10)

Here the solution $y(t)$ which converges to γ as $t \to \infty$ is given by

$$y(t,\gamma) = \gamma - \frac{2}{b} \log(1 + \frac{1}{2} be^{a+b\gamma-t}). \tag{3.11}$$

Hence in this case

$$T(\gamma) = a + \frac{1}{2} b\gamma + \log(\frac{1}{2} b) - \log(1 - e^{-b\gamma/2}). \tag{3.12}$$

Theorem 3 gives, if we take $y_0 = 0$:

$$T(\gamma) > a + \frac{1}{2} b\gamma + \log(\frac{1}{2} b) - 1. \tag{3.13}$$

Thus (3.13) is correct within a bounded error as $\gamma \to \infty$, though it fails to show that $T(\gamma) \to \infty$ as $\gamma \to 0$; the latter will be a general feature of the situation when $f(0) > 0$.

These remarks can be interpreted in terms of the nonlinear eigenvalue problem

(V)

$$-\Delta u = \lambda f(u) \quad \text{if} \quad |x| < 1 \tag{3.14}$$

$$u > 0 \quad \text{if} \quad |x| < 1 \tag{3.15}$$

$$u = 0 \quad \text{if} \quad |x| = 1. \tag{3.16}$$

If Problem III has a solution $y(t,\gamma)$ with first zero $T(\gamma)$, then Problem V has a solution u with $u(0) = \gamma$ if $\lambda = R^2(\gamma)$ is related to $T(\gamma)$ by (3.4). Thus, in the above situations in which $T(\gamma) \to \infty$ as $\gamma \to \infty$, we shall have

$$R(\gamma) \to 0 \quad \text{and hence} \quad \lambda \to 0$$

when $\gamma = \max_{B_1} u \to \infty$.

If $f(0) > 0$, we shall have the same behaviour as $\gamma \to 0$, and so we shall have $\lambda \to 0$ both as $\gamma \to \infty$ and as $\gamma \to 0$. This leads to the situation that there exists a $\lambda_0 > 0$ such that Problem V has no solution if $\lambda > \lambda_0$ and at least two if $0 < \lambda < \lambda_0$.

The case $f(u) = e^u$ in general deomains is discussed by Bandle [3] and Weston [13].

We pass now to the proof of our theorems, which will be accomplished by proving Theorem 3.

4. ASYMPTOTICALLY CONSTANT SOLUTIONS OF A HIGHLY NONLINEAR EQUATION

For the proof of Theorem 3 we need detailed estimates concerning solutions of the problem

VI
$$y'' + e^{g(y)-t} = 0 \qquad y > y_0 \tag{4.1}$$
$$\lim_{t\to\infty} y(t) = \gamma \tag{4.2}$$

in which $y_0 > 0$, $\gamma \in (y_0,\infty)$ and g satisfies

<u>A1</u>. $g \in C^2([y_0,\gamma])$

<u>A2</u>. $g' > 0$ and $g'' \geq 0$ on $[y_0,\gamma]$

Here $g(u) = \log f(u)$, as in (1.9); the behavior of $g(u)$ for $u > \gamma$ will be irrelevant.

Thanks to the rapid decay of the coefficient e^{-t} in (4.1) as $t \to \infty$, a solution $y = y(t,\gamma)$ of Problem VI will certainly exist for large t; if it is continued backwards for decreasing t, it will necessariliy reach the value y_0 for some $t = T_0 = T_0(\gamma)$, with slope $y'(T_0)$. We need mainly to estimate T_0 and $y'(T_0)$ in order to ascertain whether $y(t)$ will reach the value zero at some $T(\gamma) < T_0(\gamma)$.

We recall that if $g''(u) \equiv 0$, so that $g(u) = a + bu$ with constants a and b, an exact solution can be given (cf. (3.11)). In the sequel we obtain upper and lower bounds for y which coincide with the exact solution when the latter is valid.

In what follows it will be convenient to write

$$g = g(\gamma) \quad \text{and} \quad g' = g(\gamma).$$

We then have

Lemma 1. For $T_0 < t < \infty$

(i) $$y(t,\gamma) \leq \gamma - \frac{2}{g'} \log(1 + \frac{1}{2} g' e^{g-t}), \tag{4.4}$$

(ii) $$g(y(t,\gamma)) \geq g - 2 \log(1 + \frac{1}{2} g' e^{g-t}). \tag{4.5}$$

As we just observed, these bounds are exact if g is linear.

Proof. Writing $y(t)$ (or y) for $y(t,\gamma)$ with fixed γ, we note that $y' > 0$ and $y'' < 0$ for $t \geq T_0$, and that $y' \to 0$ as $t \to \infty$. Thus, from (4.1) we obtain on integration that

$$y'(t) = \int_t^\infty e^{g(y(s))-s} ds \tag{4.6}$$

and so, using the monotonicity of $y(t)$ and $g(y)$, that

$$e^{g(y(t))-t} < y'(t) < e^{g-t}. \tag{4.7}$$

This implies in particular that

$$\log y'(t) + t - g \to 0 \quad \text{as} \quad t \to \infty. \tag{4.8}$$

We now multiply (4.1) by $\{t - g(y(t))\}'$. This yields

$$E'(t) = -\frac{1}{2} y'^3 g''(y), \tag{4.9}$$

where

$$E(t) = y' - \frac{1}{2} y'^2 g'(y) - e^{g(y)-t}. \tag{4.10}$$

Thus, since $g''(u) \geq 0$, $E(t)$ is a non-negative, non-increasing function which tends to zero as $t \to \infty$.

We next divide (4.10) by y' and get, using (4.1):

$$0 < 1 - \frac{1}{2} y'g'(y) - \frac{y''}{y'} < \frac{1}{2y'(t)} \int_t^\infty y'^3 g''(y)dt$$

$$< \frac{1}{2} y'(t) \int_t^\infty y'g''(y)dt$$

$$= \frac{1}{2} y'\{g' - g'(y)\}.$$

These inequalities yield upon integration

$$0 < [t - \frac{1}{2} g(y) - \log y']_t^\infty < \frac{1}{2} (\gamma - y)g' - \frac{1}{2} \{g - g(y)\}.$$

Computing the limit as $t \to \infty$ in the middle term by means of (4.8) we arrive at the two inequalities:

$$\frac{1}{2} \{g + g(y(t))\} - t - \log y'(t) > 0 \tag{4.11}$$

and

$$g - t - \log y'(t) < \frac{1}{2} \{\gamma - y(t)\}g'. \tag{4.12}$$

The first of these - (4.11) - implies that

$$y'e^{-g(y)/2} < e^{(g/2)-t}. \tag{4.13}$$

Because $g'(y) > 0$ and $g''(y) > 0$ we have $0 < g'(y) < g'$ so that (4.13) yields

$$y'g'(y)e^{-g(y)/2} < g'e^{(g/2)-t}.$$

Integration over (t,∞) now gives

$$2\{e^{-g(y)/2} - e^{-g/2}\} < g'e^{-(g/2)-t},$$

which leads after some manipulation to the desired lower bound (4.5).

The second inequality - (4.12) - may be rewritten as

$$y'e^{(\gamma-y)(g'/2)} > e^{g-t} \qquad (4.14)$$

from which we may derive (4.4) by an integration over (t,∞).

We note a number of further results, all valid for $T_0 < t < \infty$. In the first one we rewrite (4.4) as a lower bound for t in terms of $y(t)$.

Lemma 2. We have

$$t > g - \frac{1}{2}(\gamma - y)g' + \log(\frac{1}{2}g') - \log\{1 - e^{(y-\gamma)(g'/2)}\} \qquad (4.15)$$

$$> g - \frac{1}{2}(\gamma - y)g' + \log(\frac{1}{2}g'). \qquad (4.16)$$

As a simple but significant illustration we have the following result.

Corollary 1. If $g(u) = u^m$, $m < 2$ and $y_0 = 0$, then

$$T_0(\gamma) \to \infty \text{ as } \gamma \to \infty. \qquad (4.17)$$

We discuss such cases in more detail in Section 7.

We can of course use (4.5) similarly to obtain an upper bound for t in terms of y, namely

$$t < \frac{1}{2}\{g + g(y)\} + \log(\frac{1}{2}g') - \log[1 - e^{\{g(y)-g\}/2}] \qquad (4.18)$$

but this does not seem adequate in the situation of Corollary 1.

A number of additional inequalities, of interest in their own right, should be noted at this point.

Lemma 3. For $T_0 < t < \infty$

(i) $$y'g'(y) < 2 \qquad (4.19)$$

(ii) $$2g'(y)e^{g(y)-t} < 1 \qquad (4.20)$$

(iii) $$y'(t) < e^{(g/2)+(g(y)/2)-t} \tag{4.21}$$

(iv) $$y'(t) > e^{g-(\gamma-y)(g'/2)-t}. \tag{4.22}$$

Proof. (i) Since $E(t) > 0$, we have

$$y' - \frac{1}{2} y'^2 g'(y) > e^{g(y)-t} > 0$$

and the result follows at once since $y' > 0$.

(ii) We write E as

$$E = -\frac{1}{2} g'(y)\{y' + \frac{1}{g'(y)}\}^2 + \frac{1}{2g'(y)} - e^{g(y)-t}.$$

Since $E > 0$, the last two terms must together be positive; this gives (ii). (iii) and (iv) follow respectively from (4.13) and (4.14).

We note that the upper bound for y' given in (4.21) is intermediate between the bounds given in (4.7). The lower bound given by (4.22) is intermediate when y is close to γ.

In later discussions (see Section 7) the curve

$$\Gamma = \{(t,u): t = g(u)\}, \tag{4.23}$$

will play a critical role. Choosing u to be the ordinate and t the abscissa, the solution graph $\{(t,y(t)): t > T_0\}$ will lie to the right of Γ for large t (see Fig. 1). Considering this graph as t decreases from ∞, we see from (4.20) that it can only cross Γ, if it does so at all, at points (t,y) where $g'(y) < \frac{1}{2}$ and so, in cases of main interest when $g'(u) \to \infty$ as $u \to \infty$, in a bounded range of y-values.

5. FURTHER BOUNDS FOR $y'(t)$

We continue assembling inequalities for the proof of Theorem 3 and need mainly a good lower bound for $y'(t)$, $t > T_0$. The lower bound (4.22) has the disadvantage of involving both y and t on the right, compounded by the lack of a good upper bound for t in terms of y, or lower bound for y in terms of t.

Lemma 4. For $t > T_0$,

$$y'(t) > \frac{2}{g'}\left(1 - \frac{1}{1 + \frac{1}{2} g' e^{g-t}}\right) = \left(e^{t-g} + \frac{1}{2} g'\right)^{-1}. \tag{5.1}$$

We remark first that this lower bound for y' exhibits what appears to be true behaviour in a certain range of cases, in that the lower bound is exponentially small for large t, and tends to $2/g'$ for small t, with a transition zone centered on

$$T_c = g + \log\left(\frac{1}{2} g'\right). \tag{5.2}$$

For the proof of Lemma 4 we insert in (4.6) the bound (4.5) for $g(y)$. This gives

$$y'(t) > \int_t^\infty \exp\{g - 2\log(1 + \tfrac{1}{2} g' e^{g-s}) - s\} ds$$

or, with the substitution $s - g = r$,

$$y'(t) > \int_{t-g}^\infty e^{-r}(1 + \tfrac{1}{2} g' e^{-r})^{-2} dr$$

which yields (5.1).

For the more detailed discussion of the graph of $y(t)$ in Section 7 we need a corresponding upper bound for y', valid in some suitable asymptotic sense as t decreases through the transition zone around T_c, that is to say for $T_c - t$ possibly large. For this we need an upper bound for $g(y)$ and note that, by (4.4),

$$g(y) = g - 2\log(1 + \tfrac{1}{2} g' e^{g-t}) + $$

$$+ 2g''(\xi)(g')^{-2}\log^2(1 + \tfrac{1}{2} g' e^{g-t}) \qquad (5.3)$$

for some $\xi \in (y,\gamma)$. Inserting this in (4.6) again we now obtain the desired upper bound for y'.

<u>Lemma 5.</u> <u>For</u> $t > T_0$

$$y'(t) < \frac{2L}{g'} \left(1 - \frac{1}{1 + \frac{1}{2} g' e^{g-t}}\right) \qquad (5.4)$$

<u>where</u>

$$L = \exp\{ \frac{2}{|g'|^2} \log^2(1 + \tfrac{1}{2} g' e^{g-t}) \sup_{I} g''(\xi)\} \qquad (5.5)$$

<u>and</u> I <u>is the set of values</u> ξ <u>such that</u>

$$g - 2\log(1 + \tfrac{1}{2} g' e^{g-t}) < g(\xi) < g. \qquad (5.6)$$

Note that in (5.6) we have extended the range for ξ on the basis of the lower bound of $g(y)$ given by (4.5).

6. PROOF OF THEOREM 3

We first take the case $y_0 = 0$, for which the existence of $T(\gamma) = T_0(\gamma)$ is ensured, and (1.13) is not needed, it is formally satisfied with $M = 0$.

By Lemma 2 we have for $y = 0$

$$T(\gamma) \geq g - \frac{1}{2}\gamma g' + \log(\frac{1}{2}g') - \log(1 - e^{-\gamma g'/2}). \tag{6.1}$$

Setting $y_0 = 0$ in the definition (1.11) of h we can write (6.1) as

$$T(\gamma) \geq h(\gamma) + \log(\frac{1}{2}g'(\gamma)). \tag{6.2}$$

This completes the proof for $y_0 = 0$.

We proceed on the basis that $y_0 > 0$ and that for some $M \geq 0$

$$f(u) \geq -M \quad \text{for} \quad 0 \leq u \leq y_0. \tag{6.3}$$

We continue to denote $g(\gamma)$ and $g'(\gamma)$ by respectively g and g' and by T_0 the value of t such that $y(t) = y_0$. We also write g_0 for $g(y_0)$.

It is intuitively evident that if T_0 and $y'(T_0)$ are suitably large, then, as t decreases from T_0 there will come a T such that $y(T) = 0$. This thought, slightly extended, is embodied in the next lemma.

<u>Lemma 6</u>. <u>Suppose that for some</u> $s \geq T_0$ <u>and</u> $M > 0$, (6.3) <u>holds and</u>

$$s - \frac{y(s)}{y'(s)} + \log y'(s) > \log M + 1. \tag{6.4}$$

<u>Then</u> $T(\gamma) > -\infty$.

Proof. It follows from the differential equation that

$$y''(t) \leqslant Me^{-t} \qquad t \leqslant s$$

and so, by two integrations,

$$y'(t) \geqslant y'(s) - Me^{-t}$$
$$y(s) - y(t) \geqslant (s - t)y'(s) - Me^{-t},$$

these being valid in any interval $[T',s]$ in which $y \geqslant 0$. It follows that if there exists a $t^* < s$ such that

$$(s - t^*)y'(s) - Me^{-t^*} > y(s) \tag{6.5}$$

then $T(\gamma) > -\infty$ and, moreover,

$$t^* < T(\gamma) < T_0. \tag{6.6}$$

We take

$$t^* = \log M - \log y'(s). \tag{6.7}$$

Then, by (6.4), $t^* < s$ as required and substitution in (6.5) yields

$$s + \log y'(s) - \log M - 1 > \frac{y(s)}{y'(s)}$$

which is equivalent to (6.4). This proves Lemma 6.

In the following Lemma we derive a lower bound for $T(\gamma)$.

Lemma 7. Let s *be as in Lemma 6. Then*

$$T(\gamma) > s - \frac{y(s)}{y'(s)} - 1. \tag{6.8}$$

If $M = 0$ *in (6.3), the term* -1 *may be omitted.*

Proof. The last remark in Lemma 7 is obvious because if $M = 0$, $y'' < 0$ for $t < s$. Thus we proceed on the basis that $M > 0$.

Our criterion (6.5) for the existence of $T(\gamma)$ may be written as

$$s - \frac{y(s)}{y'(s)} - t - M\frac{e^{-t}}{y'(s)} > 0, \qquad (6.9)$$

to be satisfied for some $t < s$. The left hand side of (6.9) attains its maximum value at $t = t^*$; it is positive by virtue of assumption (6.4). For $t > t^*$ the left of (6.9) is decreasing, but we assert that for

$$t^{**} = s - \frac{y(s)}{y'(s)} - 1$$

it is still positive. Note that by (6.4) and (6.7), $t^{**} > t^*$. To prove the assertion, observe that

$$s - \frac{y(s)}{y'(s)} - t^{**} - M\frac{e^{-t^{**}}}{y'(s)}$$

$$= 1 - M\frac{e^{-t^{**}}}{y'(s)}$$

$$> 1 - M\frac{e^{-t^{*}}}{y'(s)} = 0.$$

Thus $T(\gamma) > t^{**}$, as was to be proved.

Having established these two preliminary lemmas, we now turn to the proof of Theorem 3. Since the case $y_0 = 0$ has been dealt with we shall take $y_0 > 0$.

We begin by using Lemma 6 to show that $T(\gamma) > -\infty$. Setting $s = T_0$ we require an estimate from below for the expression

$$J = T_0 + \log y'(T_0) - \frac{y_0}{y'(T_0)}\ . \qquad (6.10)$$

We have from (5.1) that

$$e^{t}y'(t) \geq e^{g}(1 + \frac{1}{2}\, g'e^{g-t})^{-1}$$

and hence, that

$$T_0 + \log y'(T_0) \geq g - \log(1 + \frac{1}{2}\, g'e^{g-T_0}). \tag{6.11}$$

In the remaining term on the left of (6.10) we use once more the lower bound (5.1) for y' and so get

$$\begin{aligned} - \frac{y_0}{y'(T_0)} &\geq -y_0 e^{T_0-g} - \frac{1}{2}\, g'y_0 \\ &\geq -y_0 e^{T_0-g} - \frac{1}{2}\, g'\gamma + \log(1 + \frac{1}{2}\, g'e^{g-T_0}) \end{aligned} \tag{6.12}$$

by (4.4). Combining (6.11) and (6.12) we obtain

$$J \geq g - \frac{1}{2}\, \gamma g' - y_0 e^{T_0-g}.$$

Finally, we observe that by (4.18)

$$e^{T_0-g} \leq \frac{1}{2}\, g'e^{(g_0-g)/2}\, \{1 - e^{(g_0-g)/2}\}^{-1}. \tag{6.13}$$

This shows, in the notation of (1.11), that $J \geq h(\gamma)$.

This completes the proof of Theorem 3 so far as the existence of $T(\gamma)$ is concerned, if $M = 0$ the existence is automatic, and if $M > 0$ the condition $h(\gamma) \geq \log M + 1$ ensures that $J > \log M + 1$, and hence that (6.4) is satisfied with $s = T_0$. This also completes the proof of Theorem 1 on the existence of a ground state when hypothesis H2 holds as well.

To complete the proof of Theorem 3 we need to discuss the lower bound (3.5) for $T(\gamma)$ in the case $y_0 > 0$. By Lemma 7

$$T(\gamma) > T_0 - \frac{y_0}{y'(T_0)} - 1, \tag{6.14}$$

where the term -1 should be omitted if $M = 0$. From (6.12) we obtain

$$T_0 - \frac{y_0}{y'(T_0)} > T_0 + \log(1 + \tfrac{1}{2} g' e^{g-T_0}) - y_0 e^{T_0-g} - \tfrac{1}{2}\gamma g'$$
$$= g - \tfrac{1}{2}\gamma g' - y_0 e^{T_0-g} + \log(\tfrac{1}{2} g') + \log(1 + \tfrac{2}{g'} e^{T_0-g}). \tag{6.15}$$

Thus, using (6.13), and dropping the last term of (6.15) we arrive at

$$T_0 - \frac{y_0}{y'(T_0)} > h(\gamma) + \log(\tfrac{1}{2} g').$$

Substitution in (6.14) yields the desired lower bound, whence the proof is complete.

7. THE GRAPH OF $y(t)$ WHEN γ IS LARGE

If the limit value $\gamma = \lim_{t\to\infty} y(t)$ becomes large the solution of Problem VI:

$$\text{(VI)} \qquad y'' + e^{g(y)-t} = 0, \qquad y(t) \to \gamma \quad \text{as} \quad t \to \infty$$

acquires some striking features, which first became apparent on the perusal of computer-generated graphics; we are indebted to S. Angenent in this connection. We describe these features briefly in this section and show how they can be used to obtain improved bounds for T_0. A more detailed discussion is left to a subsequent paper.

For simplicity we begin with the explicitly soluble case

$$g(y) = ay \qquad a \in R^+ \tag{7.1}$$

with solution (see (3.10)):

$$y(t,\gamma) = \gamma - \frac{2}{a}\log(1 + \frac{a}{2}e^{a\gamma - t}). \tag{7.2}$$

Setting $T_c = a\gamma + \log(a/2)$ (see (5.2)) we readily see that

$$y(t,\gamma) \approx \begin{cases} \gamma & \text{if } t > T_c \quad (7.3) \\ \gamma + \frac{2}{a}(t - T_c) & \text{if } t < T_c \quad (7.4) \end{cases}$$

provided $|t - T_c|$ is not too small. The graph of $y(t,\gamma)$ is given in Fig. 2.

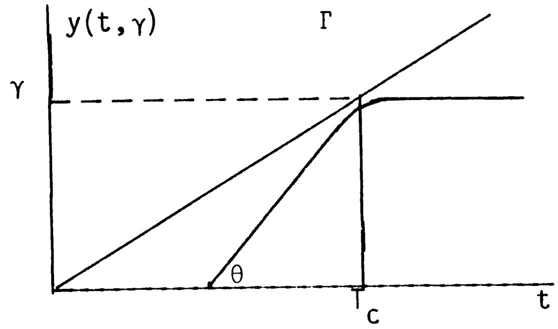

Fig. 2 The graph of y when $a = 2$, $\gamma = 4$.

One has

$$y'(t) = (e^{t-a\gamma} + \frac{a}{2})^{-1}$$

so that $y'(1) = o(1)$ or $2a^{-1} + o(1)$ if $|t - a\gamma|$ is large, according to whether $t > a\gamma$ or $t < a\gamma$.

Thus, shooting from $t = \infty$, the solution comes in nearly horizontally and is "reflected" by the curve Γ (see (4.23)), in this case the line $y = t/a$. After reflection, and passing through a transition zone, it proceeds again in nearly a straight line, but now at an angle θ to the t-axis, where

$$\tan\theta = \frac{2}{a}, \tag{7.5}$$

so that the slope of the solution graph after reflection is twice that of the curve Γ. This illustrates the general pattern according to which the slopes of incident and reflected rays have as their mean the slope of the tangent to Γ at the point of incidence, roughly speaking.

The special case $g(u) = u^2$ deserves comment. If the reflection pattern sketched above were followed exactly, the solutions which come in horizontally from $t = \infty$ would all be reflected through the vertex $(0,0)$ of the parabola $y^2 = t$ (rather than through the focus, as in optical reflection). Actually, as may be seen from (4.16), or from Theorem 3, $T(\gamma)$ in this case grows logarithmically as $\gamma \to \infty$. Cases such as $g(u) = u^m$, $m > 2$ appear to lead to multiple reflections of the solution before it reaches the t-axis.

We now sketch a strategy for justifying these observations. For simplicity, we confine attention to the case $y_0 = 0$ and to cases similar to $g(u) = u^m$ $1 < m < 2$. Specifically we shall assume about g, in addition to A1 and A2 (see section 4):

A3. $g(u) - \frac{1}{2} ug'(u) > 0$ for all $u > 0$.

A4. $g \in C^3([0,\infty))$ and there exist nonzero constants L_p such that

$$\lim_{u\to\infty} \frac{ug^{(p+1)}(u)}{g^{(p)}(u)} = L_p \qquad \text{for } p = 0,1,2.$$

Here $g^{(p)}$ denotes the p^{th} derivative of g.

The strategy consists of choosing a point T_1 below the transition point (see (5.2))

$$T_c = g + \log(\tfrac{1}{2} g') \tag{7.6}$$

at which (i) y' can be shown - by means of Lemma 4 and 5 - to be close to $2/g'$ and (ii) $t - g(y)$ can be shown - by means of Lemma 1 - to be large. The second requirement will ensure that y'' is small and y' does not charge much for values of t below T_1. Together with an estimate for y at $t = T_1$, this strategy enables us to obtain an estimate for T_0.

Thus set

$$T_1(\gamma) = g + \log(\tfrac{1}{2} g') - \delta(\gamma), \tag{7.7}$$

where $\delta > 0$. We shall assume throughout that

(i) $\quad \delta(\gamma) \to \infty \quad$ as $\quad \gamma \to \infty$

(ii) $\quad \delta(\gamma) = o(\gamma) \quad$ as $\quad \gamma \to \infty$.

We begin by estimating y and $g(y)$ at $t = T_1$, then we estimate y' at $t = T_1$ and finally we estimate y' on $[T_0, T_1]$.

<u>Lemma 8</u>. <u>Suppose</u> g <u>satisfies A1-4</u>. <u>Then</u>

(a)
$$y(T_1) = \gamma - \frac{2}{g'} + O\left(\frac{\delta^2}{g}\right) \quad \underline{\text{as}} \quad \gamma \to \infty, \tag{7.8}$$

(b)
$$g(y(T_1)) = g - 2\delta + O\left(\frac{\delta^2}{g}\right) \quad \underline{\text{as}} \quad \gamma \to \infty. \tag{7.9}$$

<u>Proof</u>. By Lemma 4.1, as $\gamma \to \infty$

$$\begin{aligned} y(T_1) &\leq \gamma - \frac{2}{g'} \log(1 + e^{\delta}) \\ &\leq \gamma - \frac{2}{g'} \delta + O\left(\frac{2}{g'} e^{-\delta}\right) \end{aligned} \tag{7.10}$$

and

$$g(y(T_1)) > g - 2\delta + O(e^{-\delta}). \tag{7.11}$$

Since g is increasing we obtain

$$g(y(T_1)) \leq g(\gamma - \frac{2}{g'}\delta + O(\frac{2}{g'}e^{-\delta}))$$
$$= g - 2\delta + O(\frac{\delta^2}{g})$$

and similarly

$$y(T_1) > g^{-1}(g - 2\delta + O(e^{-\delta}))$$
$$= \gamma - \frac{2}{g'}\delta + O(\frac{\delta^2}{g}).$$

In view of the regularity assumption A4.

<u>Lemma 9</u>. <u>If</u> g <u>satisfies A1-4, then</u>

$$y'(T_1) = \frac{2}{g'}\frac{1}{1 + e^{-\delta}}[1 + O(\frac{\delta^2}{g})] \quad \underline{\text{as}} \quad \gamma \to \infty. \tag{7.12}$$

<u>Proof</u>. By Lemma 4

$$y'(T_1) > \frac{2}{g'}\frac{1}{1 + e^{-\delta}}$$

and by Lemma 5

$$y'(T_1) \leq \frac{2}{g'}\frac{L}{1 + e^{-\delta}},$$

where

$$L = \exp[\frac{1}{2}\{\frac{2}{g'}\log(1 + e^{\delta})\}^2 \sup_{\xi} s''(\xi)]$$

and ξ satisfies, in view of Lemma 8,

$$\gamma - \frac{4}{g'}\delta < \xi < \gamma.$$

Hence, by assumption A4,

$$L = \exp[O(\delta^2/g)] = 1 + O(\frac{\delta^2}{g}),$$

from which the assertion follows.

<u>Lemma 10</u>. <u>For</u> $T_0 < t < T_1$:

$$y'(t) = \frac{2}{g'}[1 + O(\frac{\delta^2}{g}) + O(e^{-\delta})] +$$

$$+ O(\gamma e^{\gamma(g'/2)-g}) + O(\gamma e^{-\delta}) \quad \underline{\text{as}} \quad \gamma \to \infty \qquad (7.13)$$

<u>uniformly with respect to</u> t.

<u>Proof</u>. Integrating equation (4.1) over (t,T_1) we obtain

$$y'(t) = y'(T_1) + \int_t^{T_1} e^{g(y(s))-s} ds. \qquad (7.14)$$

By (4.10)

$$g(y) - t < \{g(y) - \frac{1}{2} yg'\} - (g - \frac{1}{2}\gamma g') - \log(\frac{1}{2}g').$$

$$:= \psi(y).$$

Hence, because $\psi'' > 0$,

$$g(y) - t \leq \max\{\psi(y(T_0)), \psi(y(T_1))\}.$$

Now

$$\psi(t(T_0)) = \psi(0) = g(0) - (g - \tfrac{1}{2}\gamma g') - \log(\tfrac{1}{2}g')$$

and, by Lemma 8,

$$\psi(y(T_1)) = -2\delta + \{\gamma - y(T_1)\}\tfrac{1}{2}g' - \log(\tfrac{1}{2}g') + O(\frac{\delta^2}{g})$$

$$= -\delta - \log(\tfrac{1}{2}g') + O(\frac{\delta^2}{g}) + O(\frac{\delta^2}{\gamma}).$$

Thus, for g sufficiently large,

$$g(y) - t < -\log(\tfrac{1}{2}g') + \max\{-(g - \tfrac{1}{2}\gamma g'), -\delta\}.$$

It follows that for $t \in [T_0, T_1]$

$$\int_t^T e^{g(y(s))-s}ds < \frac{2}{s'}\max\{e^{-(g-\gamma(g'/2))}, e^{-\delta}\}(T_1 - T_0).$$

By Theorem 3, since $g - \frac{1}{2}\gamma g' > 0$, $T_0 > \log(\frac{1}{2}g')$ and hence $T_1 - T_0 < g$. Therefore, when $T_0 < t < T_1$

$$y'(t) - y'(T_1) = O(\gamma e^{-(g-\gamma(g'/2))}) + O(\gamma e^{-\delta}).$$

The desired estimate now follows from Lemma 9.

It remains to make a choice for δ. If we set

$$\delta(\gamma) = k \log \gamma, \quad k > 0$$

and choose k sufficiently large the error terms involving $e^{-\delta}$ become small compared to $O(\delta^2/g)$, whence (7.7), (7.8) and (7.13) become

$$T_1 = g + \log(\tfrac{1}{2}g') - k \log \gamma, \tag{7.15}$$

$$y(T_1) = \gamma - \frac{2k \log \gamma}{g'} + O(\frac{\log^2\gamma}{g}), \tag{7.16}$$

$$y'(t) = \frac{2}{g} [1 + O(\frac{\log^2\gamma}{g})] + O(\gamma e^{\gamma(g'/2)-g}), \qquad (7.17)$$

for $T_0 < t < T_1$. By an elementary computation they yield an estimate for T_0. We make this the content of the next theorem.

Theorem 4. *Let* $y(t,\gamma)$ *be the solution of Problem VI in which* g *satisfies A1-4 with* $y_0 = 0$. *Then the first zero* $T(\gamma)$ *of* y *satisfies*

$$T(\gamma) = g(\gamma) - \frac{1}{2}\gamma g'(\gamma) + \log(\frac{1}{2}g'(\gamma)) + O(\frac{\log^2\gamma}{\gamma}) \quad \text{as} \quad \gamma \to \infty.$$

Remark. When

$$g(u) = u^m \quad 1 < m < 2,$$

Theorem 4 yields

$$T(\gamma) = (1 - \frac{1}{2}m)\gamma^m + (m - 1)\log\gamma - \log(2/m) + O(\gamma^{-1}\log^2\gamma) \quad \text{as} \quad \gamma \to \infty.$$

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