QCD at complex coupling, large order in perturbation theory and the gluon condensate

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With A. Denbleyker, D. Du, Y. Liu and A. Velytsky

Content of the talk

1. Limitations of perturbation theory for the standard model
2. Asymptotic series and large field contributions (PRL 88)
3. Lattice gauge theory at imaginary coupling (PRD 71)
4. Large order behavior of the plaquette, complex singularities (PRD 73)
5. Non-perturbative scaling and the gluon condensate (PRD 74)
6. Fisher zeros in pure gauge theory (PRD 76 and in progress)
7. Approximate analytical form of the density of states (SC/WC overlap!)

1. Limitations of Perturbation Theory

- Perturbative series have a zero radius of convergence (Dyson 1952).

- The experimental error bars of precision measurements have been shrinking (LEP, g-2) and a significant effort has been made to calculate higher order terms of perturbative series. Up to now, perturbation theory has been extremely successful but we may be reaching the limit of what can be done with conventional methods.

- QCD series seem to diverge fast. For the hadronic width of the Z^0, the term of order α_s^3 (Larin et al. PLB 320 159 (1994)) is more than 60 percent of the term of order α_s^2 and contributes to one part in 1,000 to the total width (a typical experimental error at LEP).

- For NLO, NNLO, ... QCD corrections in LHC and ILC processes, see

  LoopFest VII

  http://electron.physics.buffalo.edu/loopfest7/

- A common practice to estimate at which order, for a given coupling, we reach the maximal accuracy in perturbation theory is to determine when the ratio of successive contributions reaches one. This method works well for some examples where nonperturbative numerical results are available, but not always. It does not work for the ground state of the double-well (instantons!) but it works for the average of the two lowest energy eigenvalues.

The rule of thumb for divergent series (YM, PRD 74):

Drop the order with the smallest contribution (and all higher orders)

Take a generic asymptotic series: $A \sim \sum_k a_k \lambda^k$

Error at order $k \equiv \Delta_k(\lambda) = A_{numerical}(\lambda) - \sum_{l=0}^{k} a_l \lambda^l$

We assume that $\Delta_k \simeq \lambda^{k+1} a_{k+1}$ (for λ small enough)

Large order behavior: $|a_k| \sim |C_1||C_2|^k \Gamma(k + C_3)$

The error is minimized for $k^\star \simeq (\lambda |C_2|)^{-1} - C_3 - (1/2) + \mathcal{O}(1/k^\star)$

$Min_k \; |\Delta_k| \simeq \sqrt{2\pi} |C_1| (\lambda |C_2|)^{1/2 - C_3} \mathrm{e}^{-\frac{1}{|C_2|\lambda}}$ (order independent)

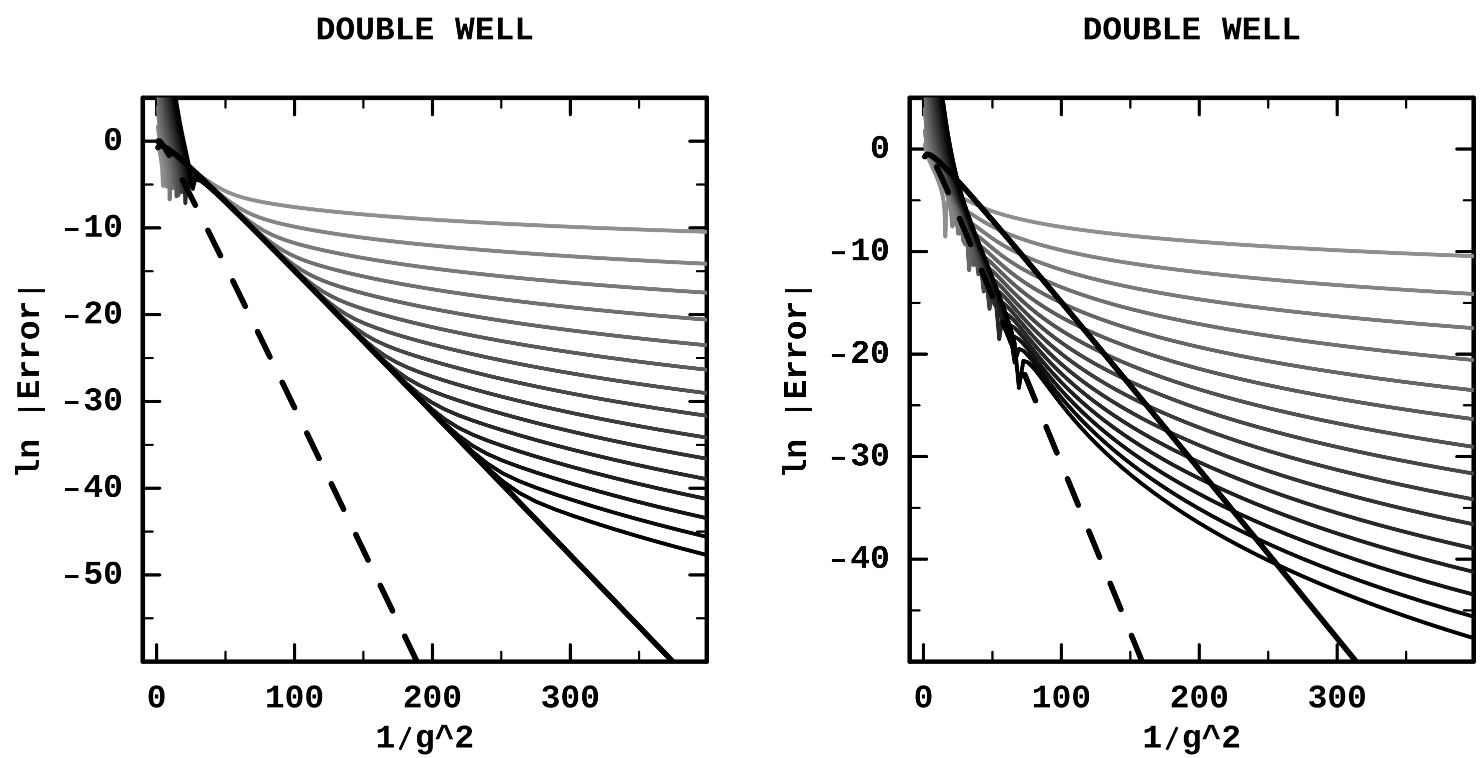


Figure 1: ln($|Error|$) for order 1 to 15 (in g^2) versus $1/g^2$ for the ground state (left) and the average (right) of the two lowest energy states of the double-well potential. As the order increases, the curves get darker. The thicker dark curve is $(g\pi)^{-1/2}e^{-\frac{1}{6g^2}}$ (1 instanton). The dash curve is $\sqrt{6/\pi}g^{-1}e^{-\frac{1}{3g^2}}$ predicted from the large order (Brezin et al., PRD 15).

2. Asymptotic series and large fields (YM, PRL 88)

$$\int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} \neq \sum_0^\infty \frac{(-\lambda)^l}{l!} \int_{-\infty}^{+\infty} d\phi e^{-\frac{1}{2}\phi^2}\phi^{4l} \quad (1)$$

The peak of the integrand of the r.h.s. moves too fast when the order increases. On the other hand, if we introduce a field cutoff, the peak moves outside of the integration range and

$$\int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2-\lambda\phi^4} = \sum_0^\infty \frac{(-\lambda)^l}{l!} \int_{-\phi_{max}}^{+\phi_{max}} d\phi e^{-\frac{1}{2}\phi^2}\phi^{4l} \quad (2)$$

General expectations: for a finite lattice, the partition function Z calculated with a field cutoff is convergent and $\ln(Z)$ has a finite radius of convergence. ϕ_{max} is an optimization parameter fixed using strong coupling, for instance.

Lattice gauge theories with a compact group (e.g., Wilson's lattice QCD) have a build-in large field cutoff: the group elements associated with the links are integrated with dU_l the compact Haar measure.

UV and large field regularization preserve gauge invariance

Definitions (for pure gauge $SU(N)$)

$S = \sum_{plaq.}(1 - (1/N) ReTr(U_p))$

$\beta = 2N/g^2$

$Z = \prod_l \int dU_l \mathrm{e}^{-\beta S} \;= \int_0^\infty n(S)\mathrm{e}^{-\beta S}$; ($n(S)$: density of states)

Number of plaquettes: $\mathcal{N}_p \equiv \; L^D D(D-1)/2$

Average plaquette: $P(\beta) = 1/\mathcal{N}_p) \left\langle \sum_p (1 - (1/N) ReTr(U_p)) \right\rangle$

One plaquette

$Z(\beta) = \int_0^2 dS n(S) \mathrm{e}^{-\beta S} = 2\mathrm{e}^{-\beta} I_1(\beta)/\beta$

$n(S) = \frac{2}{\pi}\sqrt{S(2-S)}$

The large order of the weak coupling expansion $\beta \to \infty$ is determined by the behavior of $n(S)$ near $S = 2$ itself probed when $\beta \to -\infty$

$\sqrt{2-S}$ is easy to approximate near $S = 0$ (finite radius of convergence)

$Z(\beta) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{l=0}^{\infty} (2\beta)^{-l} \frac{\Gamma(l+1/2)}{l!(1/2-l)} \int_0^{2\beta} dt \mathrm{e}^{-t} t^{l+1/2}$ is convergent

$\int_0^{2\beta} dt \mathrm{e}^{-t} t^{l+1/2} \simeq \int_0^{\infty} dt \mathrm{e}^{-t} t^{l+1/2} + O(\mathrm{e}^{-2\beta})$ is the main step to control

3. Gluodynamics at negative β

Continuum expectations: when $g \to ig$, $g^2 \to -g^2$

- The terms $ig\partial AAA$ become non-hermitian
- The terms $g^2 AAAA$ become unbounded from below

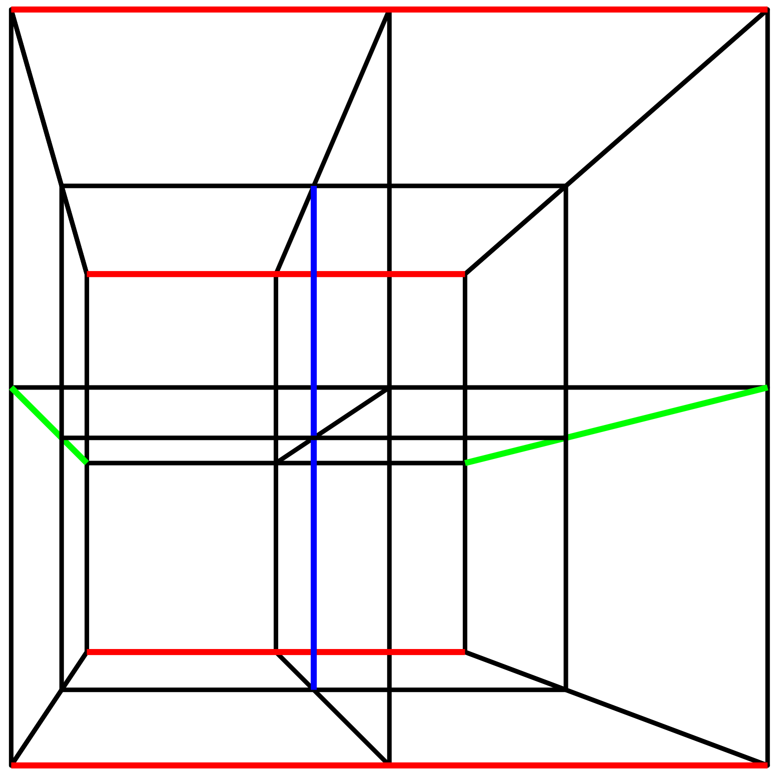
In lattice gauge theory with compact group, the theory is well defined at negative β, but Wilson loops oscillate.

Minimum action configurations at negative β

For $SU(2)$ and $SU(3)$ with $\beta < 0$, an absolute minimum of $\beta \sum_p (1 - (1/N) ReTr(U_p))$ can be obtained if U_p is a non trivial element of the center for each plaquette.

We can construct a set of lines $\mathcal{L}$ such that every plaquette shares one and only one link with this set. These lines cannot intersect. For $D = 2$, parallel lines separated by two lattice spacing do the job. For $SU(2)$, there is only one set of U_p for a configuration of minimal action (all $-\mathbb{1}$). For $SU(3)$, we have two choices:$U_p = \mathrm{e}^{\pm i 2\pi/3} \mathbb{1}$ (Ising like).

For $D = 3$, an example of $\mathcal{L}$ is $\{(A,0,0),(0,A,1),(1,1,A)\}$ with A arbitrary. It is not difficult to show that there are 8 distinct $\mathcal{L}$.



$SU(2)$ duality

Using a *change of variables*, $U_l \to -U_l$ for $l \in \mathcal{L}$ and the invariance of the Haar measure ($-\mathbb{1} \in SU(2)$), we find

$$Z(-\beta) = \mathrm{e}^{2\beta\mathcal{N}_p} Z(\beta) \tag{3}$$

Taking the logarithmic derivative, we obtain

$$P(\beta) + P(-\beta) = 2 \ . \tag{4}$$

This identity can be seen in the symmetry of the curve $P(\beta)$

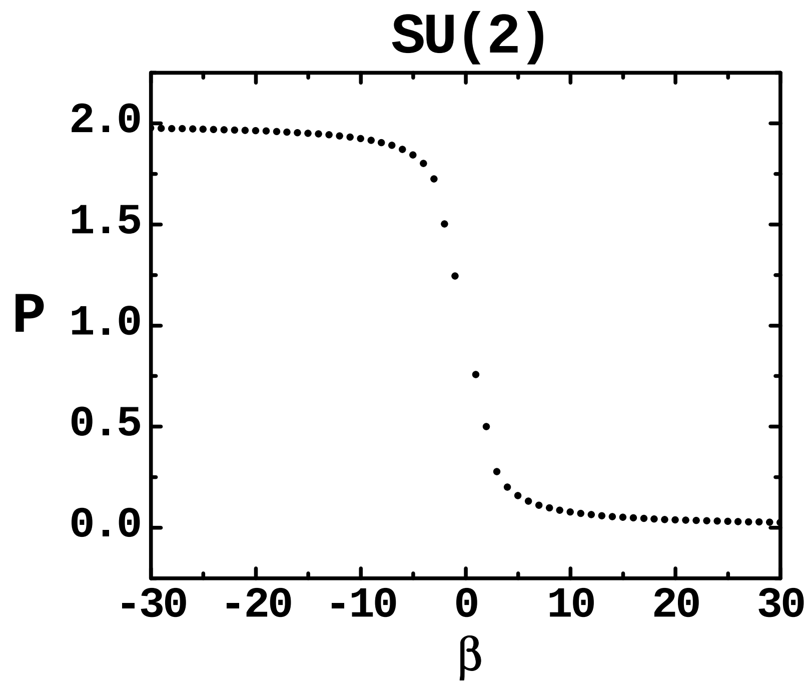


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Figure 2: The average action density $P(\beta)$ for $SU(2)$.

$$\langle W(C)\rangle_{-\beta} = (-1)^{|A|} \langle W(C)\rangle_{\beta} \tag{5}$$

We can now try to interpret the change of the Wilson loop with the area in a term of a potential. We consider a rectangular $R \times T$ contour C and write

$$W(R,T,\beta) \equiv \langle W(C)\rangle_{\beta} \propto \mathrm{e}^{-E(R,\beta)T} \ . \tag{6}$$

From Eq. (5) this implies

$$E(R,-|\beta|) = E(R,|\beta|) + i\pi R \ . \tag{7}$$

This property can be related to the fact that the configurations of minimum action are invariant under translations by two lattice spacings but not under translations by one lattice spacing.

4. Lattice Perturbation Theory ($SU(3)$)

$P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \dots.$

(F. Di Renzo et al. JHEP 10 038, P. Rakow Lat. 05)

Series analysis suggests a singularity: $P \propto (1/5.74 - 1/\beta)^{1.08}$

(Horsley et al, Rakow, Li and YM)

Not expected: zero radius of convergence (the plaquette changes discontinuously at $\beta \rightarrow \pm\infty$

Not seen in 2d derivative of P (would requires massless glueballs!)

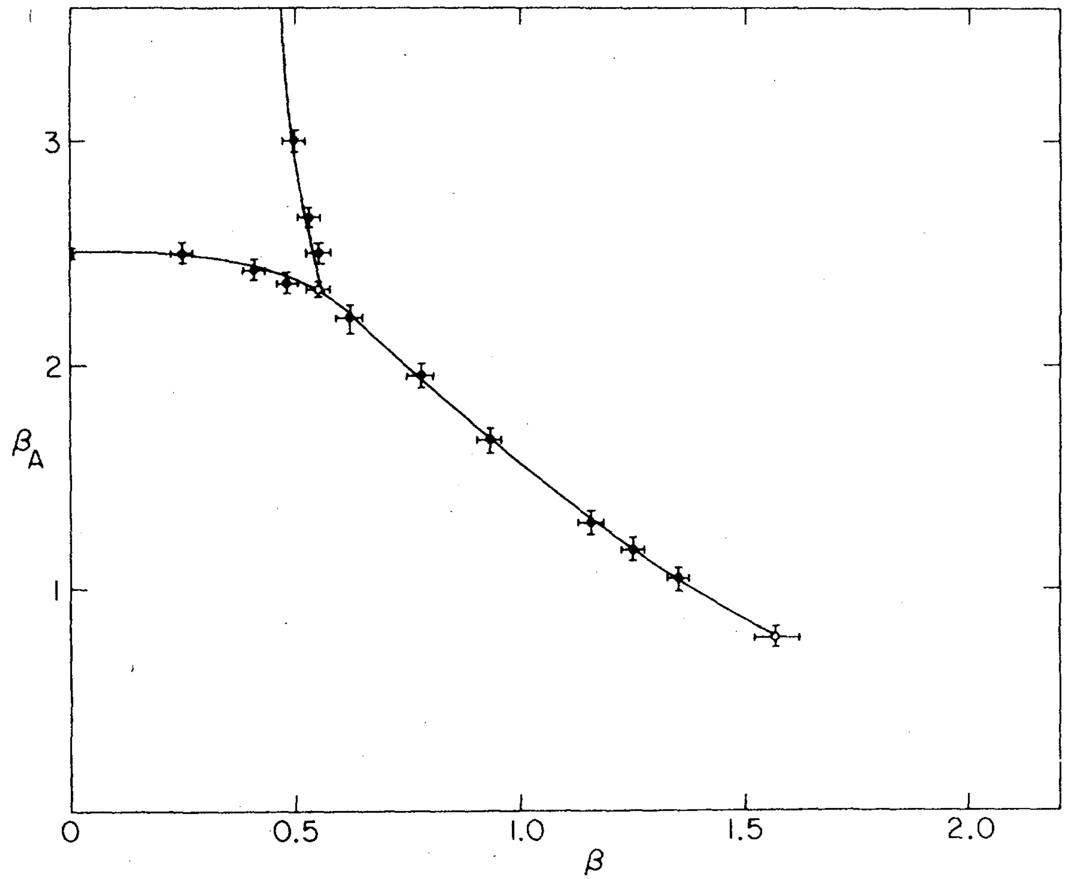


FIG. 3. The full phase diagram. The open circles represent the location of the triple point and the critical point. The solid circles trace out the first-order transition lines. The solid curves are drawn to guide the eye.

Figure 3: From Bhanot and Creutz PRD 24 3212 (for $SU(2)$)

A Small Window for Complex Singularities

A simple alternative: the critical point in the fundamental-adjoint plane has mean field exponents and in particular $\alpha = 0$. On the $\beta_{adj.} = 0$ line, we assume an approximate logarithmic behavior (mean field)

$$-\partial P/\partial\beta \propto \ln((1/\beta_m - 1/\beta)^2 + \Gamma^2) \ , \tag{8}$$

This implies the approximate form (with params. to be fitted from the pert. series)

$$\partial^2 P/\partial\beta^2 \simeq -C \frac{(1/\beta_m - 1/\beta)}{\beta^3((1/\beta_m - 1/\beta)^2 + \Gamma^2)} \tag{9}$$

Typical Fits: $\beta_m \simeq 5.78$, $\Gamma \simeq 0.006$ (i.e $Im\ \beta \simeq 0.2$), and $C \simeq 0.15$

Bounds on Imaginary part (Li and YM PRD 73)

The stability of C and β_m can be used to set a lower bound on Γ. Given that the approximate form of $\partial^2 P/\partial\beta^2$ in Eq. (9) has extrema at $1/\beta = 1/\beta_m \pm \Gamma$. As we do not observe values larger than 0.3 near $\beta = 5.75$ we get the approximate bound $\frac{C}{2\beta_m^3\Gamma} < 0.3$. Large values of Γ would affect the low order coefficients. We never found fitted values of Γ close to 0.01.

$$0.001 < \Gamma < 0.01 \ . \tag{10}$$

This suggests zeros of the partition function in the complex β plane with

$$0.03 \simeq 0.001\beta_m^2 < Im\beta < 0.01\beta_m^2 \simeq 0.33 \tag{11}$$

Large order extrapolations (YM PRD 74 096005)

Model 1:

$\sum_{k=0} b_k \beta^{-k} \simeq C(\mathrm{Li}_2(\beta^{-1}/(\beta_m^{-1} + i\Gamma)) + \mathrm{h.c},$

$\mathrm{Li}_2(x) = \sum_{k=0} x^k/k^2$.

We fixed $\Gamma = 0.003$ and obtained $C = 0.0654$ and β_m=5.787 using of a_9 and a_{10}. The low order coefficients depend very little on Γ (when $\Gamma < 0.01$), larger series are needed!

Very good predictions of the values of $a_8, a_7, \ldots$!

order	predicted	numerical	rel.error
1	0.7567	2	-0.62
2	1.094	1.2208	-0.10
3	2.811	2.961	-0.05
4	9.138	9.417	-0.03
5	33.79	34.39	-0.017
6	135.5	136.8	-0.009
7	575.1	577.4	-0.004
8	2541	2545	-0.0016
9	*exact*	11590	
10	*exact*	54160	

Also $a_{16} = 7.7\ 10^8$ while from Fig. 1 of P. Rakow Lattice 2006 $a_{16} = 0.00027 \times 6^{16} = 7.6\ 10^8$;

Feynman diagram interpretation ???

Model 2 (Mueller 93, di Renzo 95):

$$\sum_{k=0} b_k \bar{\beta}^{-k} \simeq K \int_{t_1}^{t_2} dt e^{-\bar{\beta}t} \, (1 - t\ 33/16\pi^2)^{-1-204/121} \tag{12}$$

$$\bar{\beta} = \beta(1 + d_1/\beta + \dots) \tag{13}$$

$t_1 = 0$ corresponds to the UV cutoff

$t_2 = 16\pi^2/33$: Landau pole; $t_2 = \infty$: usual perturbative series

If we want to study complex zeros, we need to regularize the Borel singularity; connection with the other model or density of states are not well understood.

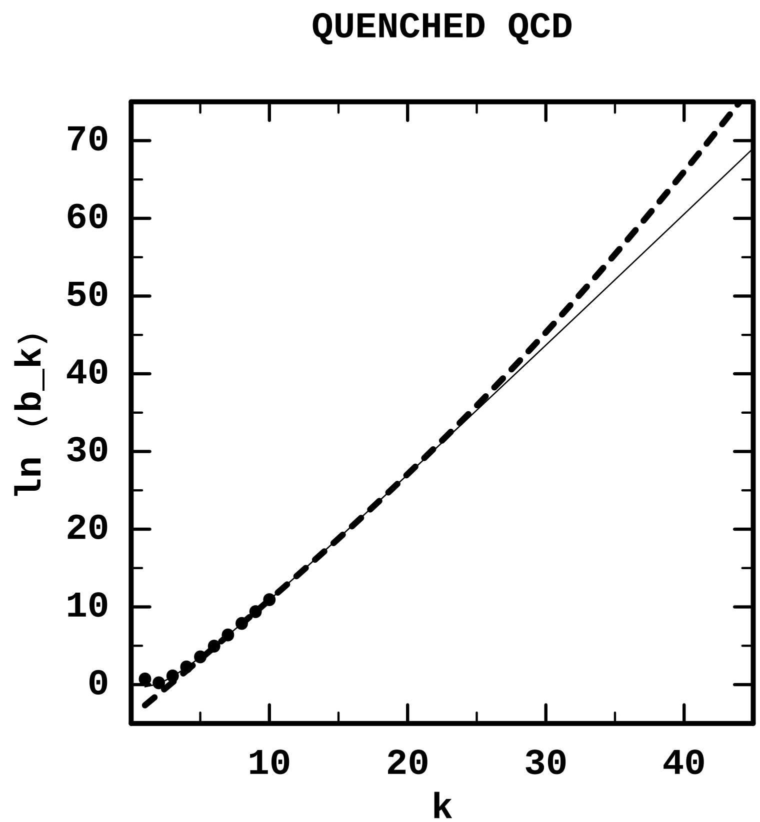


Figure 4: ln(b_k) for the dilogarithm model (solid line) and the integral model (dashes). The dots up to order 10 are the known values. The two models yields similar coefficients up to order 20. After that, the integral model has the logarithm of its coefficients growing faster than linear.

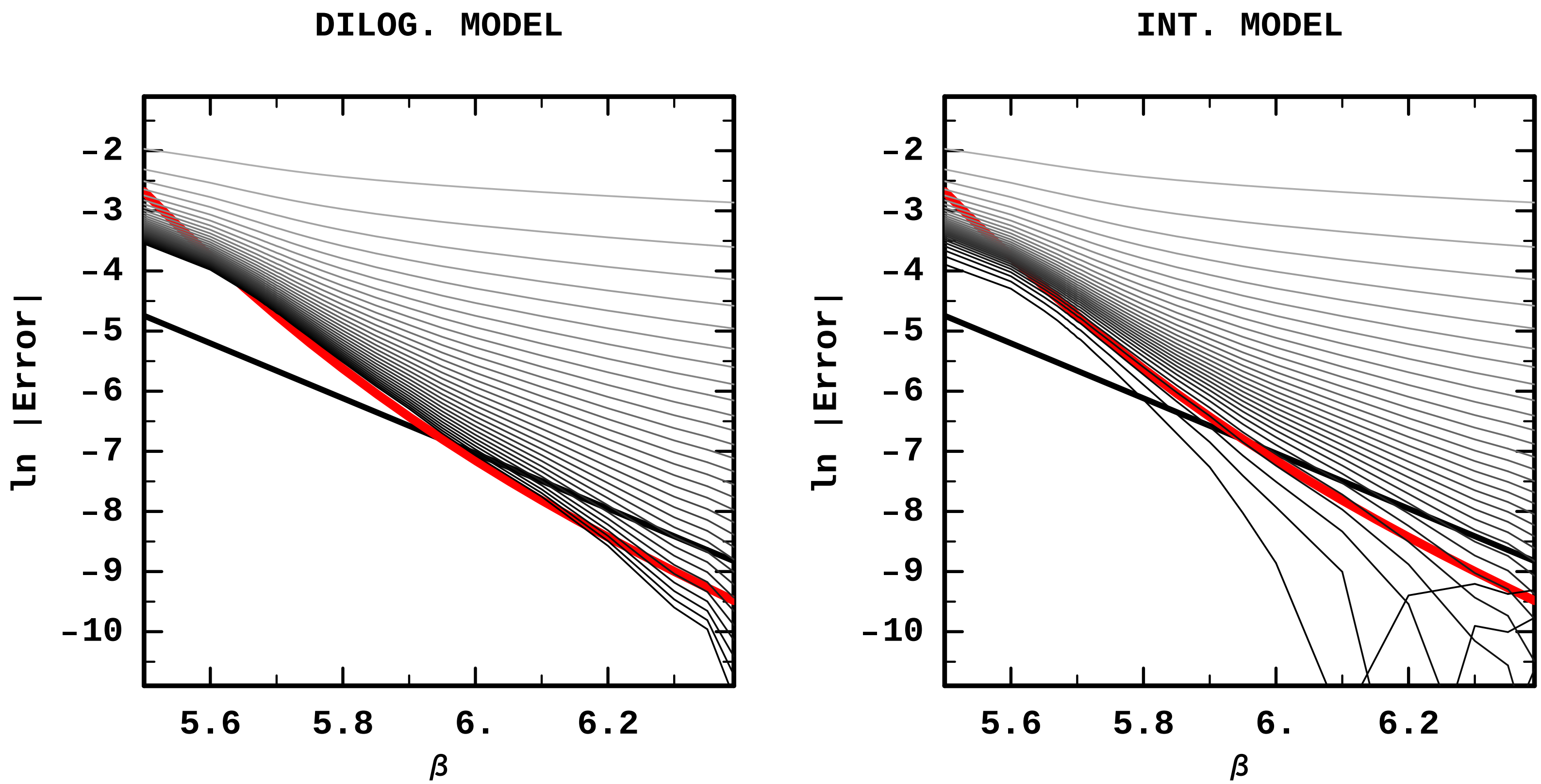


Figure 5: Accuracy curves for the dilogarithm model (left) and the integral model (right) at successive orders. As The red curve is ln$(0.65\ (a/r_0)^4)$. The solid curve is ln $(3.1 \times 10^8 \times (\beta)^{204/121-1/2} \mathrm{e}^{-(16\pi^2/33)\beta})$

5. Gluon Condensate ????

The gluon condensate is not an order parameter, there is no absolute way to decide when it should be zero. (G. C. Rossi)

$P(\beta) - P_{pert.}(\beta) \simeq C(a/r_0)^4$

with $a(\beta)$ defined with Sommer's scale, and P_{pert} appropriately truncated.

C is sensitive to resummation. $C \simeq 0.6$ with the bare series (YM PRD D74 096005) and 0.4 with the tadpole improved series (P. Rakow, Lattice 05). This gives values 2-3 times larger than $0.012\ GeV^4$.

6. Zeros of the partition function

Reweighting (Falcioni et al. 82):

$$Z(\beta_0 + \Delta\beta) = Z(\beta_0) < \exp(-\Delta\beta S) >_{\beta_0} \quad . \tag{14}$$

$$\begin{aligned} < \quad & \exp \quad [-\Delta\beta(S - < S >_{\beta_0})] >_{\beta_0} \\ = \quad & \exp \quad [\Delta\beta < S >_{\beta_0}] Z(\beta_0 + \Delta\beta)/Z(\beta_0) \ , \end{aligned} \tag{15}$$

has the same complex zeros as $Z(\beta_0 + \Delta\beta)$.

$Z(\beta)$ is the Laplace transform of density of states $n(S)$:

$$Z(\beta) = \int_0^\infty dS \ n(S) \exp(-\beta S) \tag{16}$$

A few facts about the density of state $n(S)$

Depends on $L_1, L_2, \ldots$ only.

Can be obtained from $< \mathrm{e}^{-(\beta_1+iu)S} >_{\beta_0}$ (inverse Laplace transform)

$S \sim 0$ probed at weak coupling

$S \sim \mathcal{N}_p$ (number of plaquettes) probed at strong coupling

$n(S) \propto \mathrm{e}^{-(a_1 S + a_2 S^2 + a_3 S^4 + a_4 S^4)}$ in the crossover ?

For $SU(2)$ with L_i even $Z(-\beta) = \mathrm{e}^{2\beta \mathcal{N}_p} Z(\beta)$ and $n(S) = n(2\mathcal{N}_p - S)$

Searches for $SU(2)$ and $SU(3)$ zeros;

see arXiv:0710.5771

$SU(2): \; 4^4, \; 6^4, \; 8^4 \; (T = 0)$

$SU(2): \; 4 \times 6^3, \; 4 \times 8^3 \; (T \neq 0)$

$SU(3): \; 4^4, \; 6^4, \; 8^4 \; (T = 0)$

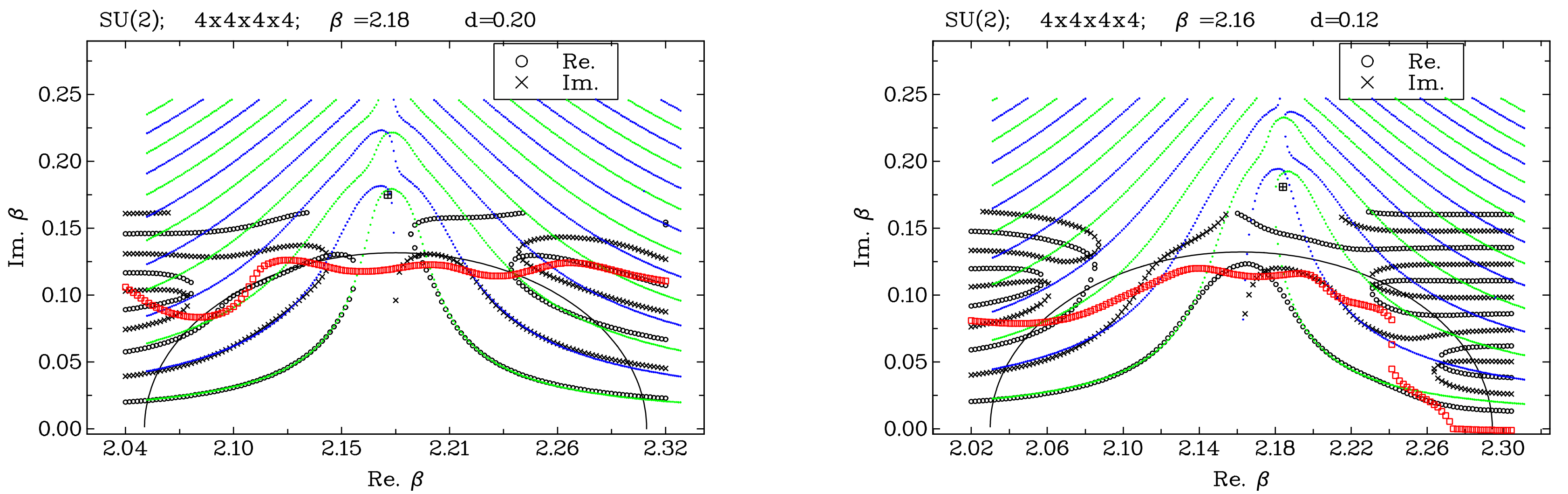
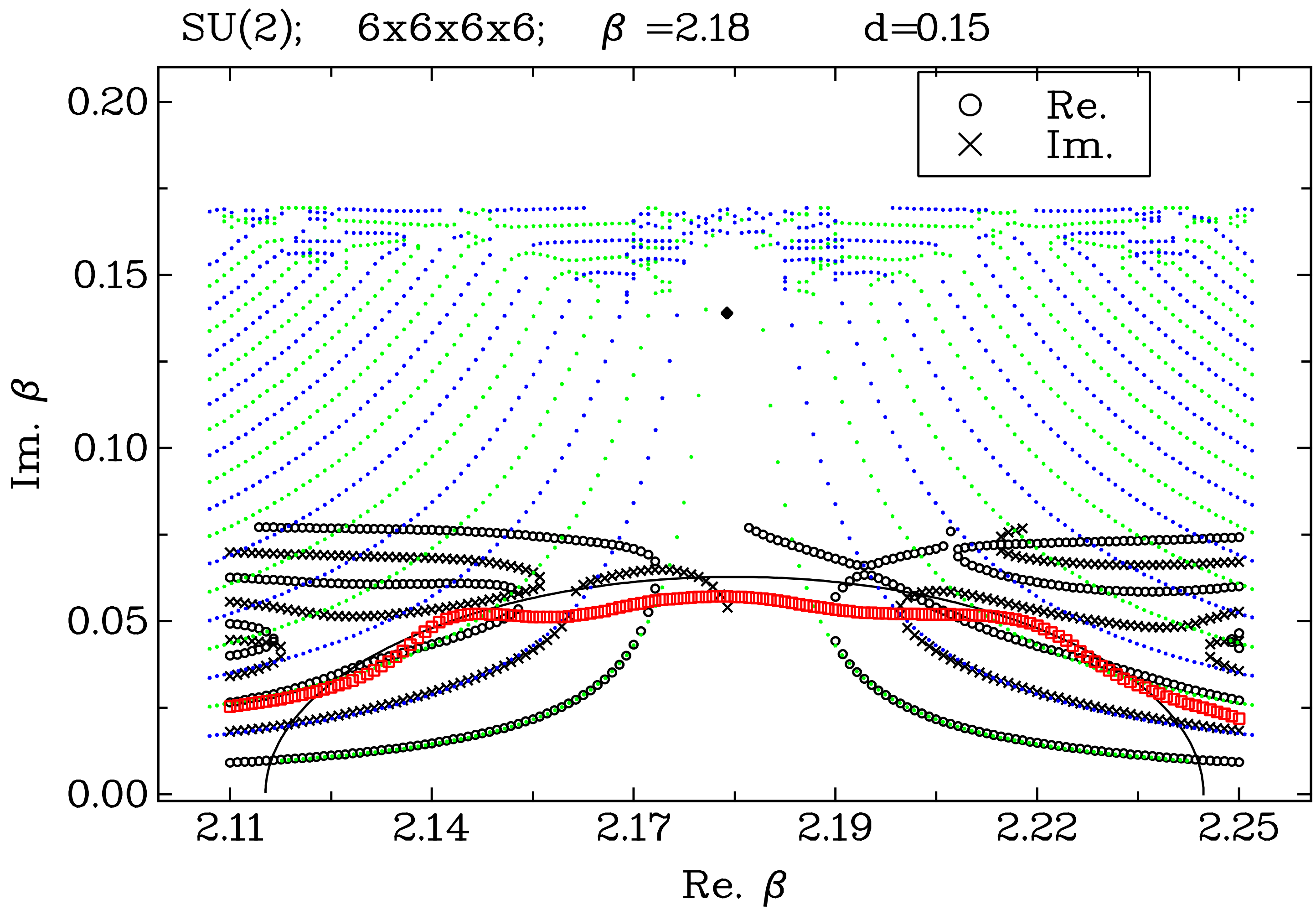


Figure 6: Zeros of the real (crosses) and imaginary (circles) for $SU(2)$ on a 4^4 lattice using reweighting of MC data at $\beta_0 = 2.18$ and 2.16. The small dots are the values for the real (green) and imaginary (blue) parts obtained from the 4 parameter model. Red boxes: boundary of the MC confidence region. The locations of the complex zeros are consistent.


SU(2); 6x6x6x6; β =2.18 d=0.15
Re.
Im.
0.20
0.15
0.10
0.05
0.00
Im. β
2.11
2.14
2.17
2.19
2.22
2.25
Re. β

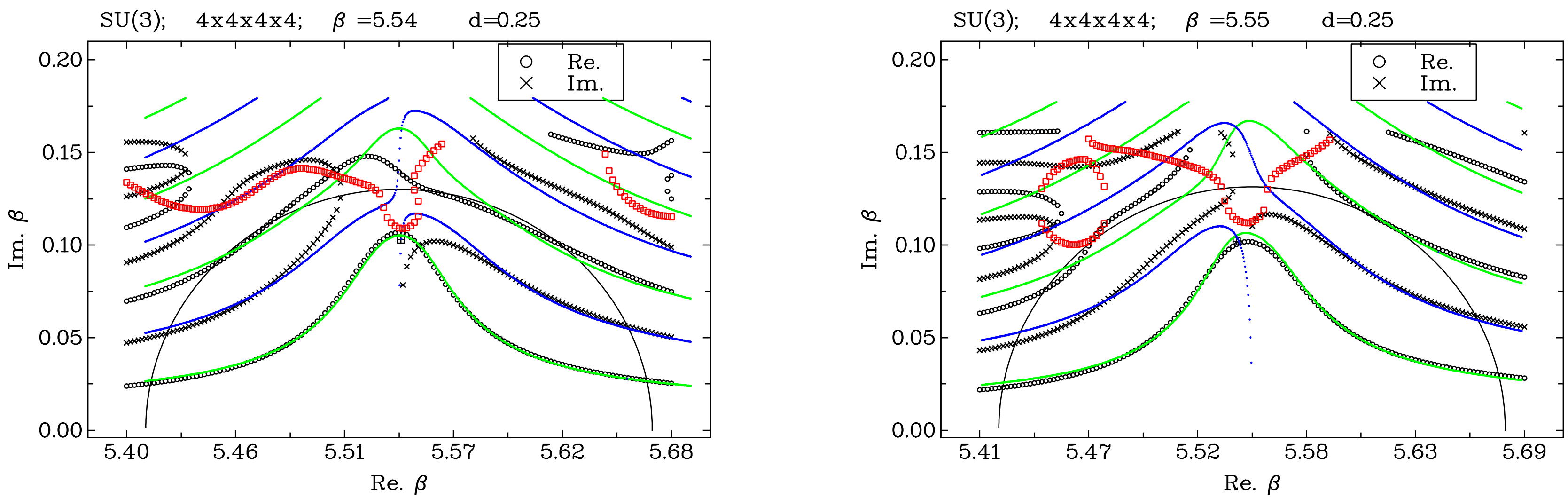


Figure 7: Zeros of the real (crosses) and imaginary (circles) for $SU(3)$ on a 4^4 lattice using reweighting of MC data at $\beta_0 = 5.54$ and 5.55. The small dots are the values for the real (green) and imaginary (blue) parts obtained from the 4 parameter model. Red boxes: boundary of the MC confidence region. The locations of the complex zeros are consistent.

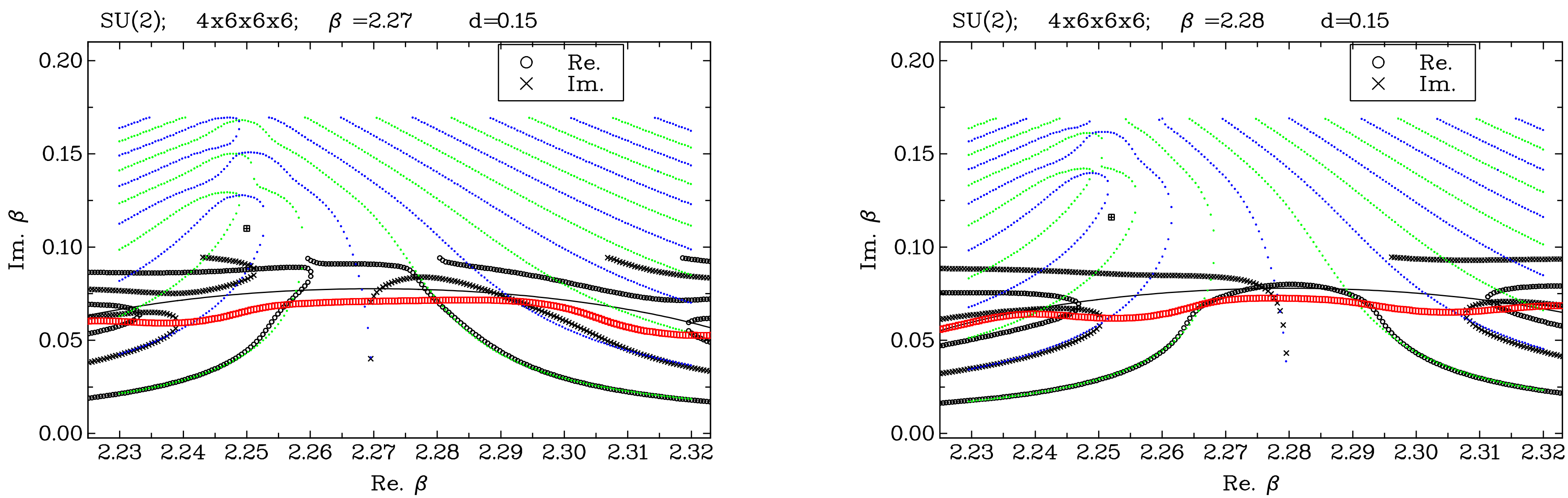


Figure 8: Zeros of the real (crosses) and imaginary (circles) for $SU(2)$ on a 4×6^3 lattice using reweighting of MC data at $\beta_0 = 2.27$ and 2.28. The small dots are the values for the real (green) and imaginary (blue) parts obtained from the 4 parameter model. Red boxes: boundary of the MC confidence region. The locations of the complex zeros are consistent.

- We have build a "ladder" of methods that can be applied for increasing values of the imaginary part of β

- We found a way to distinguish fake and true MC zeros that works well with non-Gaussian examples.

- Fitting methods based on cubic and quartic perturbations work for larger values of the imaginary part. Perturbative and saddle point methods are being developed.

- Im/Re larger for $SU(2)$, effects visible at lower order?

- Finite size scaling study of the zeros is under progress. Nonlinear effects are important.

7. Approximate form of density of state $n(S)$ for $SU(2)$

Ansatz: $n(S) = \mathrm{e}^{\mathcal{N}_p f(S/\mathcal{N}_p)}$

with $f(x)$ volume independent (entropy density)

saddle point: $f'(x) = \beta$ (just like $dS/dE = 1/T$)

Knowing $f(x)$ amounts to solve the theory (in a thermodynamical sense).

For $SU(2)$, $f(x) = f(2-x)$ (symmetric about 1)

We have constructed weak coupling and strong coupling expansions for $SU(2)$ and compared with numerical data on a 4^4 lattice

Weak coupling

We use the large beta expansion near $x = 0$,

$x =< S/\mathcal{N}_p >= \frac{3}{4}\frac{1}{\beta} + 0.156\frac{1}{\beta^2} + \ldots$

($\ldots$ with the dilog form with $\Gamma = 0.18/(2.2)^2$ and $\beta_m = 2.2$, the overall factor was fixed to get the second coefficient 0.156). The coefficients have in principle a small volume dependence that we have not resolved. (Large order numerical calculations are only available for $SU(3)$).

We assume $f(x) \simeq A\ ln(x) + B + Cx + \ldots$, plug the expansion in the saddle point equation and solve for A etc.

$f(x) = \frac{3}{4}ln(x) + 0.208x + 0.0804x^2 + \ldots$

Strong coupling

We use the low beta expansion (Balian, Drouffe and Itzykson) to solve near $x = 1$ for $SU(2)$,

$$x - 1 =< S/\mathcal{N}_p - 1 >= -\frac{\beta}{4} + \frac{\beta^3}{96} - \frac{7\beta^5}{1536} + \frac{31\beta^7}{23040} + \cdots + \frac{1826017873\beta^{15}}{68491306598400}$$

With periodic boundary conditions, the coefficients have no volume dependence for graphs with trivial topology (volume dependence should appear at order β^{2L}). Solving the saddle point equation for an expansion about 1, we get

$$f(x) = -2(x-1)^2 - \frac{2}{3}(x-1)^4 + \cdots + \frac{163150033(x-1)^{16}}{255150} + \ldots$$

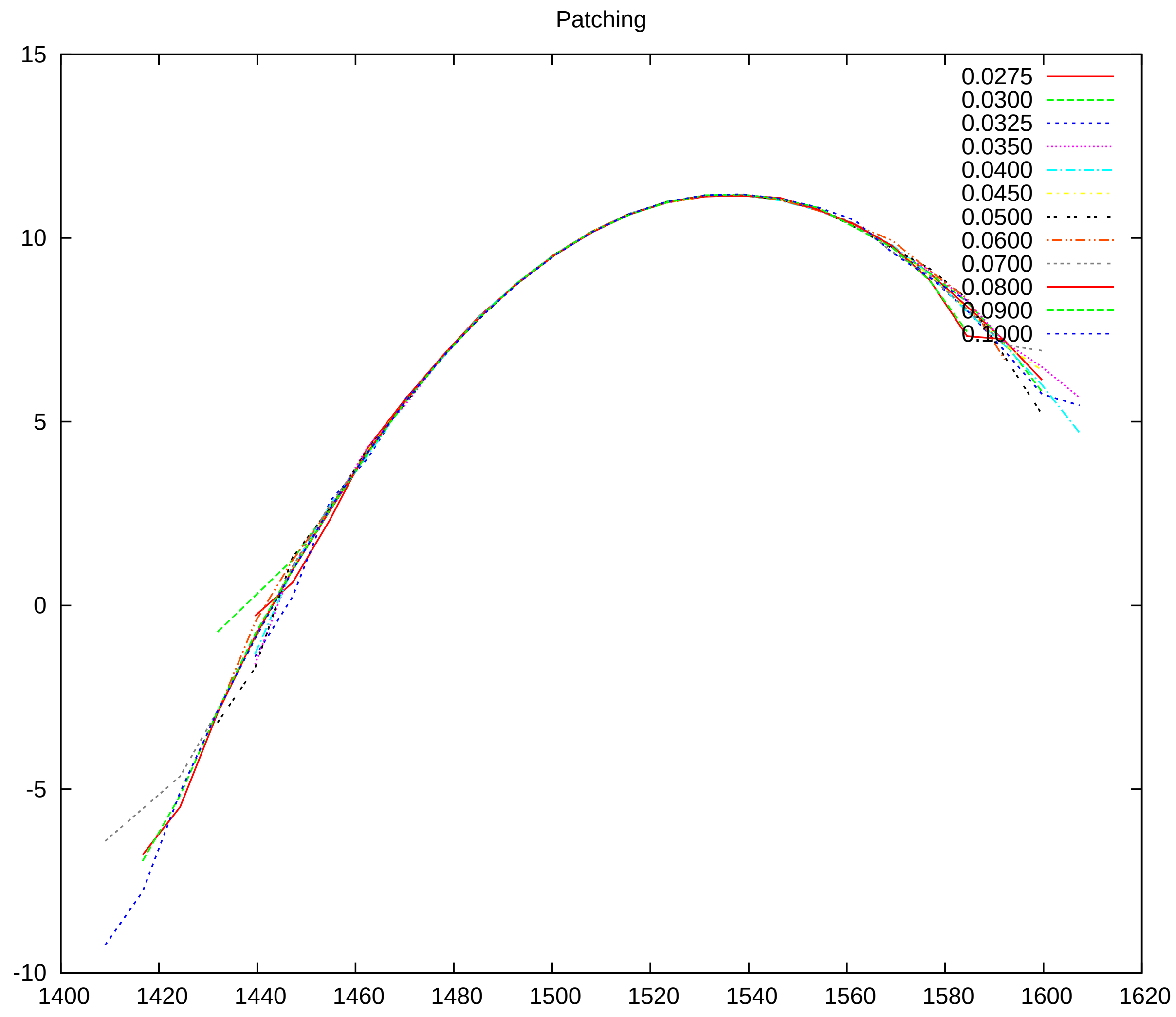


Figure 9: $n(S) = P_{\beta_0}(S)\mathrm{e}^{\beta_0 S}$ for different β_0 (patching).

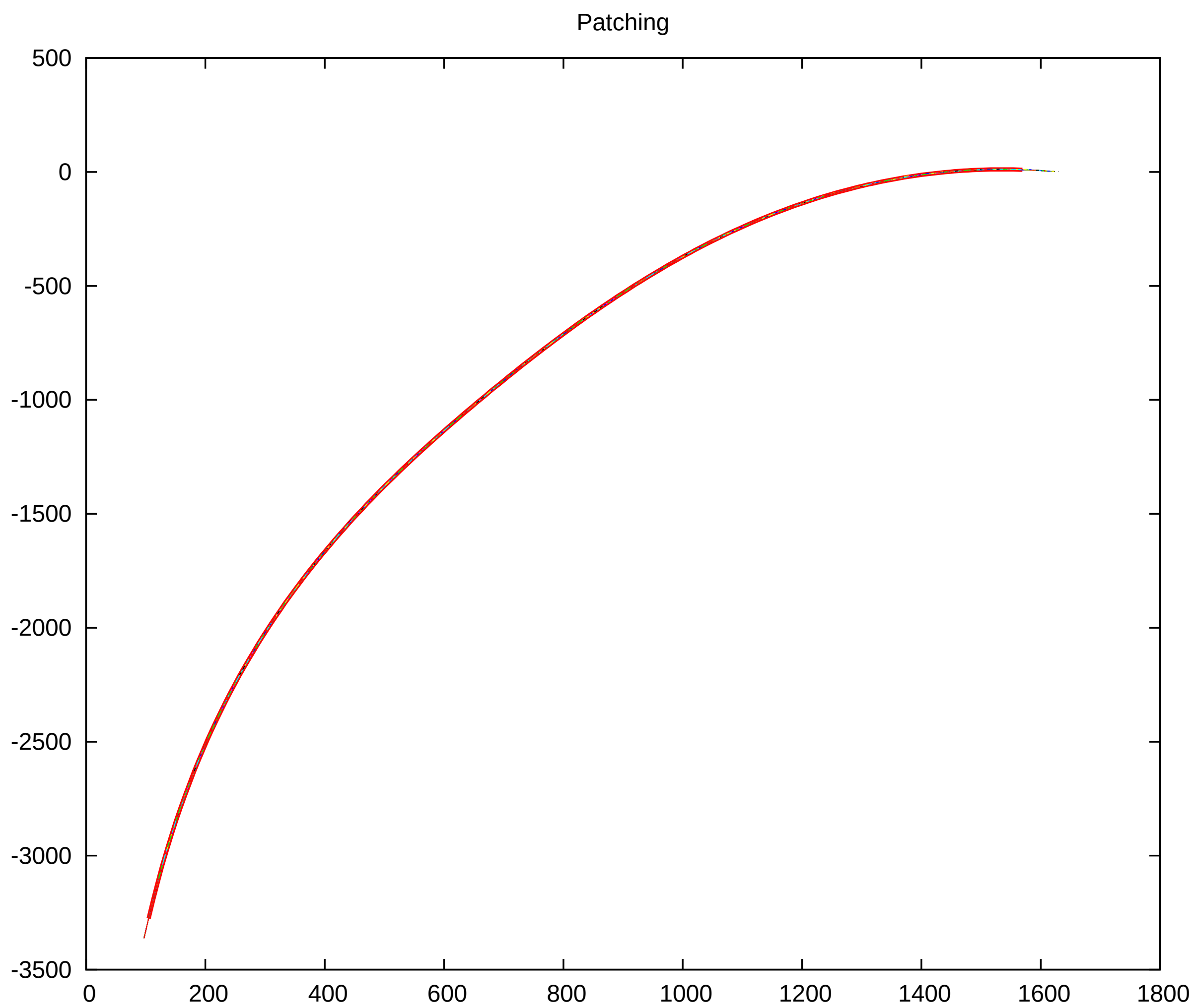


Figure 10: Collection of overlapping data (A. Denbleyker).

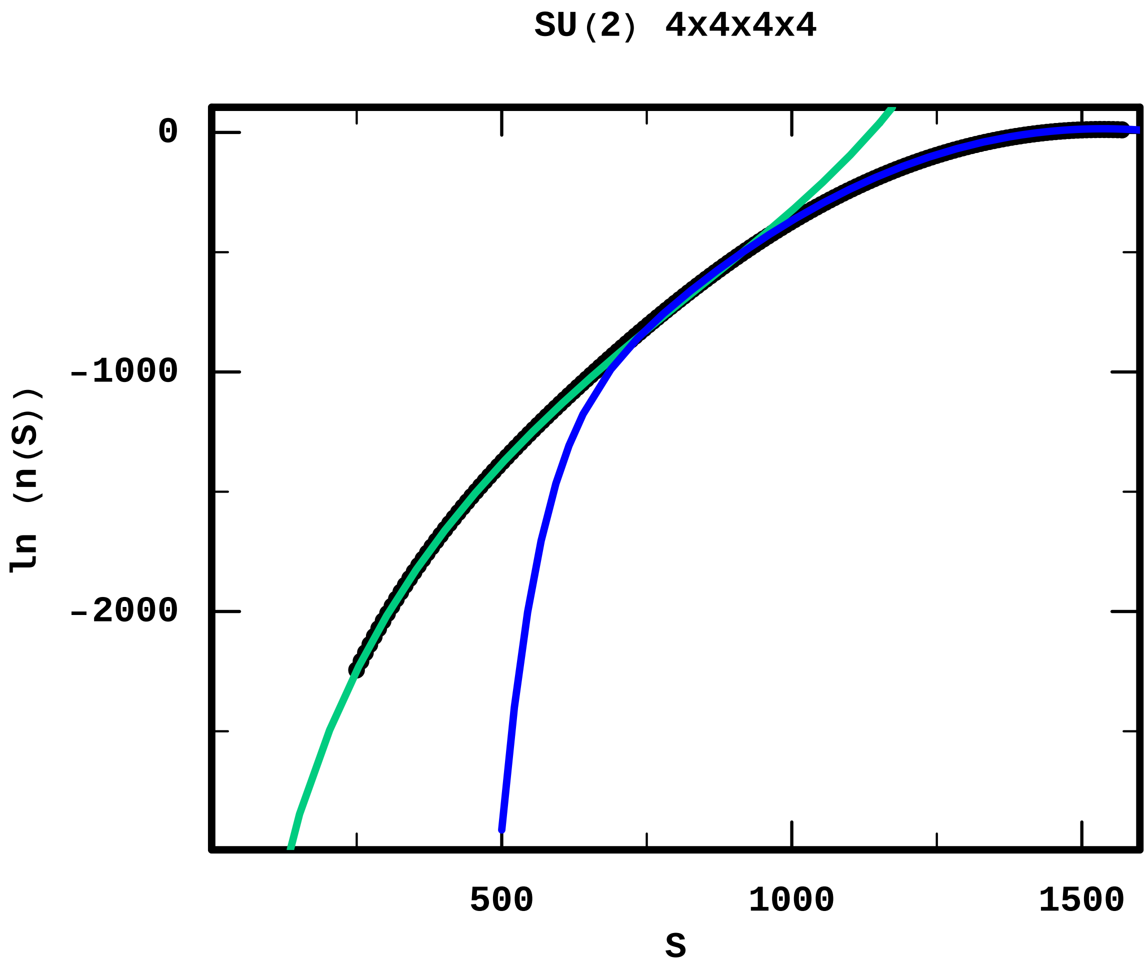


Figure 11: $ln(n(S))$ numerical (black), strong coupling (blue) and weak coupling with dilog model (green) for $SU(2)$ on a 4^4 lattice.

Conclusions

- The density of state has overlapping strong/weak coupling expansions
- Fisher's zeros for this approximate form are being studied (D. Du)
- We need numerical confirmation of guesses made for the weak coupling expansion for $SU(2)$ where $Im\beta_F/Re\beta_F$ is 5 times larger than for $SU(3)$
- We need some agreement between the lattice and the continuum communities regarding the gluon condensate
- We need a better understanding of the large order behavior of QCD series in terms of the behavior at small complex coupling (a picture analog to metastability at $\lambda < 0$ for the anharmonic oscillator)

- Thanks to the organizers!