NONSMOOTH ANALYSIS AND SHAPE OPTIMIZATION IN FLOW PROBLEM

By

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Abstract

New approach to some Shape optimization (i.e., Free boundary) problems is introduced. The variable domain problem is relaxed so that it becomes a nonsmooth optimization problem on the fixed domain for the somewhat singular state equation. State equation is considered, and the multivalued generalized gradient of the variational functional is studied. The general approach is discussed here in the case of a perfect fluid flow. The method is constructive.

1 Statement of the Problem

The general problem in Shape optimization consists in searching for the shape of the domain where certain pde holds so that the obtained solution exhibits desired properties. In that broad sense it is closely related to the theory of Free boundary problems, where, indeed, solution of pde should satisfy additional condition on the designated (free) part of the boundary of the domain. Nevertheless, aims and means of the two subjects are significantly different. In some sense the free boundary problems are the most difficult shape optimization problems, since, broadly speaking, the functional to be minimized is concentrated on the variable boundary of the domain.

In this paper we introduce a new approach to some Shape optimization (i.e., Free boundary) problems. The approach is discussed in one very particular situation, but we note that the method extends to other more interesting problems,

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like for example the analogous problem for the Stokes equation, which will be discussed elsewhere.

Consider the set of admissible shapes, i.e., the control set[1]

$$U = \left\{ u \in H_0^3(-1,1); 0 \le u(x) \le 1, -1 < x < 1 \right\}. \tag{1.1}$$

Denote

$$\Gamma_u = \{(x, u(x)); -1 < x < 1\}, \tag{1.2}$$

and extend $u \in U$ as zero outside of $(-1, 1)$. Define the domain

$$\Omega_u = \{(x, y);\ |x| < a,\ u(x) < y < 2\}. \tag{1.3}$$

Also, let $\Omega \supset \Omega_u$ be a domain defined by

$$\Omega = \{(x, y);\ |x| < a,\ 0 < y < 2\}. \tag{1.4}$$

Now, consider inviscid, incompressible, irrotational flow in a finite channel Ω with an immersed obstacle Γ_u with shape $u \in U$. So, the flow actually takes place in Ω_u.

It is well known that such a flow can be described by the stream function $w = w^u$, which is a solution of (to fix ideas, we take flux to be equal to one)

$$\begin{aligned} \Delta w &= 0 \text{ in } \Omega_u \\ w &= 0 \text{ in } \{(x,0); -a < x < -1 \text{ or } 1 < x < a\} \\ w &= 0 \text{ in } \Gamma_u \\ w &= 2 \text{ in } \{(x,2); -a < x < a\} \\ w_x &= 0 \text{ in } \{(\pm a, y); 0 < y < 1\}. \end{aligned} \tag{1.5}$$

We could also take $a = \infty$ in (1.5), i.e., consider a flow in an infinite channel. Then the last condition in (1.5) is substituted by the requirement that w is bounded.

If the stream function w is known then, of course, the velocity vector field $\mathbf{v}$ can be computed easily as $\mathbf{v} =< w_y, -w_x >$.

The problem we propose is the following:

For given $g = g(x, y) \in W^{2,q}(\Omega)$, $q > 2$, (occasionally we will not have to assume that much) and such that

$$g \ge 0, \tag{1.6}$$

$$g = 0 \text{ in } \Omega \cap \{|x| > 1\}, \tag{1.7}$$

[1]We note that the regularity requirement $U \subset H^3(-1, 1)$ is for some technical reasons. As a matter of fact, it is unfortunate to assume ad hoc so much, and we will see later that it is important to require as little as possible, ideally only $H^1(-1, 1)$. We shall get back to this issue when we make some comments about numerical solution.

find (if possible) $u \in U$ (the shape of the immersed obstacle) such that, if w^u is the corresponding solution of (1.5), then also

$$|\mathbf{v}| = |\nabla w^u| = g \text{ in } \Gamma_u. \tag{1.8}$$

By Bernoulli's law

$$P + \frac{1}{2}|\nabla w^u|^2 = const. \tag{1.9}$$

throughout the fluid (here P denotes the pressure). Hence, we see that the requesting specific velocity profile on the immersed obstacle is equivalent to requesting the specific pressure profile. Obviously, this is a problem with wide possibilities for applications.

We note that this problem is closely related to the following, by now, well known variational problem (see [2, 3]; see also [11] for numerical considerations):

Find $w \in H^1(\Omega)$ satisfying the boundary conditions

$$\begin{aligned} w &= 0 \text{ in } \{(x,0); -a < x < a\} \\ w &= 2 \text{ in } \{(x,2); -a < x < a\} \end{aligned} \tag{1.10}$$

and such that the variational functional

$$\mathbf{J}(w) = \int_\Omega \left[|\nabla w|^2 + g^2 \mathbf{I}_{\{w>0\}}\right] \tag{1.11}$$

is minimized. Here $\mathbf{I}_D$ is a characteristic function of the set D, i.e.,

$$\mathbf{I}_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}. \tag{1.12}$$

The reason to develop the method introduced here is that it extends to other equations for which there is no known analog of $\mathbf{J}$ (for example, Stokes problem, to be discussed in the forthcoming paper).

2 A Relaxation of the Problem

In this section we assume that

$$g \in H^2(\Omega). \tag{2.1}$$

Suppose that there exists an $u \in U$ such that corresponding w^u solves (1.5,1.8). We shall say then that u is an *exact control*. Now, extend w^u from Ω_u to Ω as z^u:

$$z^u = \begin{cases} w^u & \text{on } \Omega_u \\ 0 & \text{on } \Omega \backslash \Omega_u \end{cases}. \tag{2.2}$$

It follows

Lemma 2.1 *If $u \in U$ is an exact control, then $z^u \in H^1(\Omega)$, and it is a solution of the following elliptic boundary value problem (with singular right hand side)*

$$\begin{aligned} \Delta z^u &= \xi_u \text{ in } \Omega \\ z^u &= 0 \text{ in } \{(x,0); -a < x < a\} \\ z^u &= 2 \text{ in } \{(x,2); -a < x < a\} \\ (z^u)_x &= 0 \text{ in } \{(\pm a, y); 0 < y < 1\} \end{aligned} \tag{2.3}$$

where $\xi_u \in H^{-1}(\Omega)$ is a measure given by

$$\xi_u(\varphi) = \int_{\Gamma_u} g\varphi d\sigma. \tag{2.4}$$

Proof: Obviously, since by elliptic estimates w^u is regular in Ω_u, $z^u \in C^{0,1}(\bar{\Omega})$ (regarding regularity near corners see the begining of the proof of the Theorem 3.1), and in particular $z^u \in H^1(\Omega)$.

By the Trace Theorem,

$$\begin{aligned} |\xi_u(\varphi)| = \left| \int_{\Gamma_u} g\varphi d\sigma \right| &\leq \|g\|_{L^2(\Gamma_u)} \|\varphi\|_{L^2(\Gamma_u)} \\ &\leq c_u \|g\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}. \end{aligned} \tag{2.5}$$

So, in particular, $\xi_u \in H^{-1}(\Omega)$. Also, since $g \geq 0$, ξ_u is a measure.

Now, more explicitly, (2.3) can be written as: Find $z \in H^1(\Omega)$ such that

$$\begin{aligned} z^u &= 0 \text{ in } \{(x,0); -a < x < a\} \\ z^u &= 2 \text{ in } \{(x,2); -a < x < a\} \end{aligned} \tag{2.6}$$

and

$$-\int_{\Omega} \nabla z^u \cdot \nabla \varphi = \int_{\Gamma_u} g\varphi d\sigma \tag{2.7}$$

for all $\varphi \in H^1(\Omega)$ such that

$$\varphi = 0 \text{ in } \{(x,0); -a < x < a\} \cup \{(x,2); -a < x < a\}. \tag{2.8}$$

To check (2.7), we note that by the maximum principle, a solution of (1.5) is positive. Hence (1.8) and the boundary condition in (1.5) imply that

$$\frac{\partial w^u}{\partial \nu_u} = -g \text{ in } \Gamma_u, \tag{2.9}$$

where ν_u is the exterior unit normal to $\partial\Omega_u$. Hence,

$$\begin{aligned} -\int_{\Omega} \nabla z^u \cdot \nabla \varphi &= -\int_{\Omega_u} \nabla w^u \cdot \nabla \varphi = \\ = \int_{\Omega_u} (\Delta w^u)\varphi - \int_{\partial\Omega_u} \frac{\partial w^u}{\partial \nu_u} \varphi d\sigma &= \int_{\Gamma_u} g\varphi d\sigma, \end{aligned} \tag{2.10}$$

which completes the proof of the Lemma. □

Lemma 2.2 *Let z^u be a solution of (2.6-2.8). If it happens that $z^u|_{\Gamma_u} = 0$, then $z^u|_{\Omega_u}$ is a solution of (1.5,1.8), i.e., u is an exact control.*

Proof: In the next section we shall prove that z^u is regular enough so that calculations performed here are legitimate. More precisely, by (3.6) below, it suffices to assume that $\varphi \in C_0^1(\Omega)$. We have

$$\int_{\Gamma_u} g\varphi d\sigma = -\int_{\Omega_u} \nabla z^u \cdot \nabla\varphi - \int_{\Omega\backslash\Omega_u} \nabla z^u \cdot \nabla\varphi$$
$$= \int_{\Omega_u} (\Delta z^u)\varphi + \int_{\Omega\backslash\Omega_u} (\Delta z^u)\varphi$$
$$-\int_{\partial\Omega_u} \frac{\partial z^u}{\partial\nu}\varphi d\sigma - \int_{\partial(\Omega\backslash\Omega_u)} \frac{\partial z^u}{\partial\nu}\varphi d\sigma. \tag{2.11}$$

Let ν be exterior to Ω_u, and let

$$z^{u,int} \stackrel{\text{def}}{=} z^u|_{\Omega\backslash\Omega_u}$$
$$z^{u,ext} \stackrel{\text{def}}{=} z^u|_{\Omega_u}. \tag{2.12}$$

Then (2.11) implies that

$$\int_{\Gamma_u} g\varphi d\sigma = \int_{\Gamma_u} \left(\frac{\partial z^{u,int}}{\partial\nu} - \frac{\partial z^{u,ext}}{\partial\nu}\right)\varphi d\sigma,\ \forall\varphi \in C_0^1(\Omega). \tag{2.13}$$

So,

$$g = \frac{\partial z^{u,int}}{\partial\nu} - \frac{\partial z^{u,ext}}{\partial\nu} \text{ on } \Gamma_u. \tag{2.14}$$

We observe that (2.14) always holds for the solution of (2.3).

Now, if $z^u|_{\Gamma_u} = 0$, then $z^u|_{\Omega\backslash\Omega_u} = 0$, so that $\frac{\partial z^{u,int}}{\partial\nu} = 0$, and then

$$g = -\frac{\partial z^{u,ext}}{\partial\nu} \text{ on } \Gamma_u, \tag{2.15}$$

i.e.,

$$g = |\nabla (z^u|_{\Omega_u})| \text{ on } \Gamma_u, \tag{2.16}$$

i.e., (1.8) holds. □

Lemma 2.2 motivates the following

Definition 2.1 *$u^* \in U$ is said to solve the* relaxed shape optimization problem *if the corresponding z^u defined by (2.3), is such that*

$$\Phi(u) = \frac{1}{2}\int_{\Gamma_u} (z^u)^2 d\sigma \tag{2.17}$$

is minimized, i.e., that there exists an $u^* \in U$ such that

$$\Phi(u^*) = \min_{u\in U} \Phi(u). \tag{2.18}$$

Of course, an exact control is a minimizer, i.e., a solution of (2.18). On the other hand, a solution of (2.18) is an exact control provided an exact control exists.

We do not consider the exact controllability. Rather, we shall study the relaxed problem introduced in Definition 2.1.

3 The State Equation

It will be convenient to state the regularity theorem for the general boundary value. So let ψ be a given function on Ω such that $z^u = \psi$ on $\partial_0\Omega \subset \partial\Omega$. We assume that the boundary and ψ are sufficiently regular (see [12] for details; also we shall give some details in the case of the boundary and boundary values in our case). For any $z \in H^1(\Omega)$, we define $\|z\|_{L^\infty(\partial\Omega)}$ as

$$\|z\|_{L^\infty(\partial\Omega)} \stackrel{\text{def}}{=} \inf\left\{m \geq 0; -m \leq z \leq m \text{ on } \partial\Omega \text{ in } H^1(\Omega)\right\}, \tag{3.1}$$

where inequalities in $H^1(\Omega)$ are defined in e.g. [9]. Also, we define

$$W^{2,q}_{\{y>0\}-loc}(\Omega) \stackrel{\text{def}}{=} \cap_{\epsilon>0} W^{2,q}(\Omega \cap \{y > \epsilon\}). \tag{3.2}$$

We have

Theorem 3.1 *For any $u \in U$ the state equation (2.3) has a unique weak solution. Let q be such that $2 \leq q < \infty$. If $g \in W^{1,q}(\Omega)$ then*

$$z^u \in W^{1,q}(\Omega) \cap C^\infty(\bar{\Omega} \setminus \Gamma_u), \tag{3.3}$$

and the apriori estimate

$$\|z^u\|_{W^{1,q}(\Omega)} \leq c\left(1 + \|u\|_{C^{0,1}(-1,1)}\right)\left(\|g\|_{W^{1,q}(\Omega)} + \|\psi\|_{W^{1,q}(\Omega)}\right). \tag{3.4}$$

holds. If in addition $q > 2$ then also

$$\|z^u\|_{L^\infty(\Omega)} \leq c\left(1 + \|u\|_{C^{0,1}(-1,1)}\right)\left(\|g\|_{W^{1,q}(\Omega)} + \|\psi\|_{L^\infty(\partial\Omega)}\right). \tag{3.5}$$

Moreover, if $g \in W^{2,q}(\Omega)$, and (1.7) holds, then (see (2.12))

$$z^{u,ext} \in W^{2,q}(\Omega_u),\ z^{u,int} \in W^{2,q}_{\{y>0\}-loc}(\Omega \setminus \bar{\Omega_u}). \tag{3.6}$$

and the apriori estimates

$$\|z^{u,ext}\|_{W^{2,q}(\Omega_u)} \leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \tag{3.7}$$

and

$$\begin{aligned} &\|z^{u,int}\|_{W^{2,q}((\Omega\setminus\Omega_u)\cap\{y>\epsilon\})} \leq \\ &\leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right). \end{aligned} \tag{3.8}$$

hold.

Proof: Since $\xi_u \in H^{-1}(\Omega)$ existence and uniqueness of a weak solution z^u of (2.3) is trivial. Also, since z^u is harmonic in $\Omega \setminus \Gamma_u$, it follows that $z^u \in C^\infty(\bar{\Omega} \setminus \Gamma_u)$. Few words are needed here due to the presence of corners in Ω. To prove regularity of z^u in the neighborhood of corners, say in the neighborhood of $(-a, 0)$, one can extend z^u in $\{x < -a, 0 < y < 2\}$ as $\widetilde{z^u}$ by the formula

$$\widetilde{z^u}(x,y) \stackrel{\text{def}}{=} \begin{cases} z^u(-2a - x, y) & \text{if } x < -a \\ z^u(x,y) & \text{if } x \geq -a. \end{cases} \tag{3.9}$$

Then since $\widetilde{z^u}$ is continuous on $\{x = -a\}$ and $\widetilde{z^u}_x = 0$ on $\{x = -a\}$, it is elementary to show that $\widetilde{z^u}$ is harmonic across $\{x = -a\}$. Indeed, let $B_\rho(A) = B_1 \cup (B_\rho(A) \cap \{x = -a\}) \cup B_2 \subset \{0 < y < 2\}$ be a ball centered at $A \in \{x = -a\}$ with radius ρ. Here, $B_1 = B_\rho(A) \cap \{x > -a\}$ and $B_2 = B_\rho(A) \cap \{x < -a\}$. Then,

$$\begin{gathered} \int_{B_\rho(A)} \widetilde{z^u} \Delta \varphi = \int_{B_1} \widetilde{z^u} \Delta \varphi + \int_{B_2} \widetilde{z^u} \Delta \varphi = \\ \int_{\{x=-a\} \cap B_\rho(A)} \left[-\varphi_x \widetilde{z^u} + \varphi \widetilde{z^u}_x + \varphi_x \widetilde{z^u} - \varphi \widetilde{z^u}_x \right] dy = 0, \end{gathered} \tag{3.10}$$

for all $\varphi \in C_0^\infty(B_\rho(A))$, so that $\widetilde{z^u}$ is harmonic across $\{x = -a\}$ as clamed. Henceforth $\widetilde{z^u}$ is as regular in the neighborhood of $(-a, 0)$ as the (extended) boundary data is. In particular, in our case $\psi = 0$ there, so that (3.3) follows.

Set

$$\varphi = \psi - z^u \tag{3.11}$$

in (2.7). It easily follows

$$\begin{aligned} \int_\Omega |\nabla z^u|^2 &= \int_{\Gamma_u} g(\psi - z^u) d\sigma - \int_\Omega \nabla z^u \cdot \nabla \psi \\ &\leq \left(\int_{\Gamma_u} g^2 d\sigma \right)^{\frac{1}{2}} \left[\left(\int_{\Gamma_u} \psi^2 d\sigma \right)^{\frac{1}{2}} + \left(\int_{\Gamma_u} (z^u)^2 d\sigma \right)^{\frac{1}{2}} \right] \\ &\qquad + \left| \int_\Omega \nabla z^u \cdot \nabla \psi \right|. \end{aligned} \tag{3.12}$$

Now since $z^u = (z^u - \psi) + \psi$, using Poincaré inequality, we have

$$\begin{aligned} \|z^u\|_{H^1(\Omega)} &\leq c \left(\|\nabla (z^u - \psi)\|_{L^2(\Omega)} + \|\psi\|_{H^1(\Omega)} \right) \\ &\leq c \left(\|\nabla z^u\|_{L^2(\Omega)} + \|\psi\|_{H^1(\Omega)} \right) \end{aligned} \tag{3.13}$$

Combining (3.12,3.13) we get

$$\begin{aligned} \|z^u\|^2_{H^1(\Omega)} \leq c \left(1 + \|u\|_{C^{0,1}(-1,1)}\right) &\left[\|g\|_{H^1(\Omega)} \left(\|\psi\|_{H^1(\Omega)} + \|z^u\|_{H^1(\Omega)} \right) \right] \\ &+ c \|z^u\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)} + c \|\psi\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.14}$$

In (3.14) the inequality follows from the proof of the Trace Theorem (see e.g. [10], or [5]). Indeed, one can see ([5], p. 132) that for $1 \leq q < \infty$ one has

$$\|z^u\|^q_{L^q(\Gamma_u)} \leq c\left(1+\|u\|^2_{C^{0,1}(-1,1)}\right)^{\frac{1}{2}} \|z^u\|^q_{W^{1,q}(\Omega)}, \tag{3.15}$$

which implies

$$\|z^u\|_{L^q(\Gamma_u)} \leq c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q}} \|z^u\|_{W^{1,q}(\Omega)}, \tag{3.16}$$

From (3.14) we easily conclude that (3.4) holds for $q = 2$.

Proceeding, (we assume $\frac{1}{q} + \frac{1}{q'} = 1$)

$$\begin{aligned} |\xi_u(\varphi)| &\leq \|g\|_{L^q(\Gamma_u)}\|\varphi\|_{L^{q'}(\Gamma_u)} \\ &\leq c\left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q}} \|g\|_{W^{1,q}(\Omega)} \left(1+\|u\|_{C^{0,1}(-1,1)}\right)^{\frac{1}{q'}} \|\varphi\|_{W^{1,q'}(\Omega)} \\ &= c\left(1+\|u\|_{C^{0,1}(-1,1)}\right) \|g\|_{W^{1,q}(\Omega)}\|\varphi\|_{W^{1,q'}(\Omega)}. \end{aligned} \tag{3.17}$$

So, $\xi_u \in \left(W^{1,q'}(\Omega)\right)^*$ (here X^* represents the dual space of the space X) and

$$\|\xi_u\|_{\left(W^{1,q'}(\Omega)\right)^*} \leq c\left(1+\|u\|_{C^{0,1}(-1,1)}\right) \|g\|_{W^{1,q}(\Omega)}. \tag{3.18}$$

We know (see e.g. [1]) that ξ has a representation

$$\xi(\varphi) = \int_\Omega \left[f_0\varphi + f_1\varphi_x + f_2\varphi_y\right], \tag{3.19}$$

for some $f_i \in L^q(\Omega), i = 0, 1, 2$, and

$$\|\xi_u\|_{\left(W^{1,q'}(\Omega)\right)^*} = \sum_{i=0}^{2} \|f_i\|_{L^q(\Omega)}. \tag{3.20}$$

Now from elliptic regularity (see [12], p. 179) we have

$$\|z^u\|_{W^{1,q}(\Omega)} \leq c\left(\sum_{i=0}^{2} \|f_i\|_{L^q(\Omega)} + \|\psi\|_{W^{1,q}(\Omega)} + \|z^u\|_{H^1(\Omega)}\right). \tag{3.21}$$

From (3.18,3.20,3.21) and since we already proved (3.4) in the case $q = 2$, we conclude that (3.4) holds.

To prove (3.5), we recall (see e.g. [12] p. 103) that if $q > 2$ and if $z^u \leq 0$ on $\partial_0\Omega$ in the sense of $H^1(\Omega)$, then

$$\operatorname*{ess\,sup}_\Omega z^u \leq c\left(\sum_{i=0}^{2} \|f_i\|_{L^q(\Omega)} + \|z^u\|_{L^2(\Omega)}\right). \tag{3.22}$$

Hence,

$$\operatorname*{ess\,sup}_{\Omega}\left(z^u - \|z^u\|_{L^\infty(\partial\Omega)}\right) \leq$$

$$\leq c\left(\sum_{i=0}^{2}\|f_i\|_{L^q(\Omega)} + \|z^u\|_{L^2(\Omega)} + \|z^u\|_{L^\infty(\partial\Omega)}\right), \tag{3.23}$$

and similarly for $-z^u + \|z^u\|_{L^\infty(\partial\Omega)}$. This easily implies (3.5).

Now, we shall consider further regularity of $z^u|_{\Omega_u}$ and $z^u|_{\Omega\setminus\Omega_u}$. Since the singular set is on Γ_u, we expect higher regularity in the tangential direction. To prove that this is the case we flatten the Γ_u first, since then it is easier to differentiate.

Define v, $\tilde{g}$ and $\tilde{\varphi}$ by

$$v(x,y) = z^u(x, y + u(x)), \tag{3.24}$$

$$\tilde{g}(x,y) = g(x, y + u(x))\sqrt{1 + u'^2(x)}, \tag{3.25}$$

$$\tilde{\varphi}(x,y) = \varphi(x, y + u(x)) \tag{3.26}$$

and operator L by

$$Lv = \Delta v + v_{yy}(u_x)^2 - 2v_{xy}u_x - v_y u_{xx}. \tag{3.27}$$

Of course, L is uniformly elliptic, since the matrix

$$[l_{ij}] = \begin{bmatrix} 1 & -u_x \\ -u_x & 1 + u_x^2 \end{bmatrix} \tag{3.28}$$

is positive definite. Indeed,

$$l_{ij}\xi_i\xi_j = (\xi_1 - u_x\xi_2)^2 + \xi_2^2. \tag{3.29}$$

So, if c is such that $|u_x| \leq c$, then if $|\xi_2| < \frac{1}{2c}|\xi_1|$ then

$$(\xi_1 - u_x\xi_2)^2 > \frac{1}{4}\xi_1^2. \tag{3.30}$$

On the other hand if $|\xi_2| \geq \frac{1}{2c}|\xi_1|$ then

$$\xi_2^2 \geq \frac{1}{4c^2}\xi_1^2. \tag{3.31}$$

So, it is easy to see that if we take $\alpha = \min\left(\frac{1}{4}, \frac{1}{8c^2}\right)$ then

$$l_{ij}\xi_i\xi_j \geq \alpha|\xi|^2. \tag{3.32}$$

Let, also, Ξ_u be the map with the image Ω given by the formula

$$\Xi_u(x,y) = (x, y + u(x)). \tag{3.33}$$

Then,

$$\Delta z^u \circ \Xi_u = Lv, \tag{3.34}$$

and since $|detD\Xi_u| = 1$ (here $D\Xi_u$ is the gradient matrix of the map Ξ_u so that $|detD\Xi_u|$ is the Jacobian)

$$(Lv)(\tilde{\varphi}) = (\Delta z^u)(\varphi). \tag{3.35}$$

Hence

$$\begin{aligned}(Lv)(\tilde{\varphi}) &= \int_{\Gamma_u} g\varphi d\sigma \\ &= \int g(x,u(x))\varphi(x,u(x))\sqrt{1+u'^2(x)}dx \\ &= \int_{\{y=0\}} \tilde{g}\tilde{\varphi}dx \stackrel{\text{def}}{=} \tilde{\xi}(\tilde{\varphi}).\end{aligned} \tag{3.36}$$

So,

$$Lv = \tilde{\xi} \tag{3.37}$$

in the sense of distributions. Since the singular set is now on $\{y = 0\}$, we expect higher regularity in x-direction. To prove that, we want to differentiate (or more precisely, difference) equation (3.37) with respect to x. Somewhat more precisely, define the standard difference operator (in the x-direction) δ_h^1 as

$$\left(\delta_h^1 u\right)(x) = \frac{1}{h}\left(u(x+h,y) - u(x,y)\right),\ h \neq 0. \tag{3.38}$$

Then from (3.37) we get

$$(Lv)\left(\delta_{-h}^1\tilde{\varphi}\right) = \tilde{\xi}\left(\delta_{-h}^1\tilde{\varphi}\right). \tag{3.39}$$

We shall discuss in some details only the right-hand side. We have

$$\begin{aligned}\tilde{\xi}\left(\delta_{-h}^1\tilde{\varphi}\right) &= \int_{\{y=0\}} \tilde{g}\delta_{-h}^1\tilde{\varphi}dx \\ &= -\int_{\{y=0\}} \left(\delta_h^1\tilde{g}\right)\tilde{\varphi}dx \longrightarrow -\int_{\{y=0\}} \tilde{g}_x\tilde{\varphi}dx,\end{aligned} \tag{3.40}$$

as $h \to 0$. We conclude that

$$(Lv)_x(\tilde{\varphi}) = \tilde{\xi}_x(\tilde{\varphi}), \tag{3.41}$$

and hence

$$L_1 v_x = \tilde{\xi}_x - v_{yy}2u_x u_{xx} + v_y u_{xxx}, \tag{3.42}$$

where

$$L_1 w = \Delta w + (u_x)^2 w_{yy} - 2u_x w_{xy} - 3u_{xx}w_y, \tag{3.43}$$

and where

$$\tilde{\xi}_x(\tilde{\varphi}) \stackrel{\text{def}}{=} \int_{\{y=0\}} \tilde{g}_x \tilde{\varphi} dx. \tag{3.44}$$

We observe that the differencing performed above is legitimate, since

$$\tilde{\xi}_x - v_{yy} 2u_x u_{xx} + v_y u_{xxx} \in \left(W^{1,q'}\right)^*. \tag{3.45}$$

Indeed, $g \in W^{2,q}(\Omega)$, and also observe that $u_{xxx} \in L^2$, and that $L^2 \hookrightarrow \left(W^{1,q'}\right)^*$. Also, since L_1 has the same principal part as L, L_1 is uniformly elliptic, as well.

Now we can conclude from (3.42,3.45) that

$$v_x \in W^{1,q}. \tag{3.46}$$

This implies, by the Trace Theorem, that $v_x|_{\{y=0\}} \in W^{1-\frac{1}{q},q}$, so that

$$v|_{\{y=0\}} \in W^{2-\frac{1}{q},q}. \tag{3.47}$$

We observe that because of (1.1,1.7), the preceding analysis is true also in the $\{y > 0\}$-neighborhood of (the pre-image of) $(\pm 1, 0)$, so that (3.47) holds up to the initial and terminal points of (the pre-image of) Γ_u. Elliptic regularity then yields

$$v|_{\{y \geq 0\}} \in W^{2,q}. \tag{3.48}$$

Unfortunately, we can not claim the same global result for $v|_{\{y \leq 0\}}$ because of the nonsmoothness of $\partial(\Omega \setminus \Omega_u)$, i.e., we have to localize in $\{y > 0\}$. This concludes the proof of (3.6). Now, regarding estimates (3.7,3.8) we have

$$\begin{aligned} \|z^{u,ext}\|_{W^{2,q}(\Omega_u)} &\leq c\left(\|u\|_{H^3(-1,1)}\right) \|v|_{\{y\geq 0\}}\|_{W^{2,q}\left(\Xi_u^{-1}(\Omega_u)\right)} \\ &\leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \end{aligned} \tag{3.49}$$

and similarly (after localization in $\{y > 0\}$) for $z^{u,int}$, which completes the proof of the Theorem. □

Corollary 3.1 *If $g \in W^{2,q}(\Omega)$ for some $q > 2$, and if (1.7) holds, then $z^u \in C^{0,1}_{\{y>0\}-loc}(\bar{\Omega})$ and the following apriori estimate holds*

$$\|z^u\|_{C^{0,1}(\bar{\Omega}\cap\{y\geq\epsilon\})} \leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \tag{3.50}$$

for any $\epsilon > 0$.

Proof: From (3.7,3.8) and by the Imbedding theorem (see e.g. [7]), we have

$$\begin{aligned} &\|z^{u,ext}\|_{C^1(\bar{\Omega}_u)} + \|z^{u,int}\|_{C^1(\overline{\Omega\setminus\Omega_u}\cap\{y\geq 0\})} \leq \\ &\leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right). \end{aligned} \tag{3.51}$$

This implies (3.50). □

Corollary 3.2 *If $g \in W^{2,q}(\Omega)$ for some $q > 2$, and if (1.7) holds, then*

$$z_y^{u,int} \in C^{0,1-\frac{2}{q}}\left(\overline{\Gamma_u}\right), \tag{3.52}$$

and the following apriori estimate

$$\|z_y^{u,int}\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} \leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right) \tag{3.53}$$

holds.

The interest in this Corollary is due to the lack of $(\Omega \setminus \Omega_u)$-global regularity of z^u.

Proof: Let τ and ν be unit tangent and unit normal to Γ_u. More precisely, set

$$\tau = \frac{1}{\sqrt{1+u'^2}}\langle 1, u'\rangle, \tag{3.54}$$

and

$$\nu = \frac{1}{\sqrt{1+u'^2}}\langle u', -1\rangle. \tag{3.55}$$

It is elementary to compute that then

$$z_y^{u,int} = \frac{u'}{\sqrt{1+u'^2}} z_\tau^{u,int} - \frac{1}{\sqrt{1+u'^2}} z_\nu^{u,int}. \tag{3.56}$$

But since, by Theorem 3.1 $z_\tau^{u,int} = z_\tau^{u,ext}$ and (also, by Lemma 2.2) $z_\nu^{u,int} = g + z_\nu^{u,ext}$ on Γ_u, we have

$$z_y^{u,int}\big|_{\Gamma_u} = \left(\frac{u'}{\sqrt{1+u'^2}} z_\tau^{u,ext} - \frac{1}{\sqrt{1+u'^2}}\left[g + z_\nu^{u,ext}\right]\right)\Bigg|_{\Gamma_u}. \tag{3.57}$$

The Corollary follows due to the Ω_u-*global* regularity of $z^{u,ext}$, and by the Imbedding theorem. Indeed,

$$\begin{aligned}
\|z_y^{u,int}\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} &= \left\|\frac{u'}{\sqrt{1+u'^2}} z_\tau^{u,ext} - \frac{1}{\sqrt{1+u'^2}}\left[g + z_\nu^{u,ext}\right]\right\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})} \\
&\leq c\|u\|_{H^3(-1,1)}\left[\|z^{u,ext}\|_{C^{1,1-\frac{2}{q}}(\overline{\Gamma_u})} + \|g\|_{C^{0,1-\frac{2}{q}}(\overline{\Gamma_u})}\right] \\
&\leq c\|u\|_{H^3(-1,1)}\left[\|z^{u,ext}\|_{W^{2,q}(\Omega_u)} + \|g\|_{W^{1,q}(\Omega_u)}\right] \\
&\leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right).
\end{aligned} \tag{3.58}$$

□

4 Existence of a Minimizer

In order to claim existence of a minimizer, i.e., existence of a solution of the relaxed problem, one needs compactness. One way of introducing compactness would be to bound the set of admissible controls to

$$U_b = \{u \in U; \|u\|_{H^3(-1,1)} \leq b\} \tag{4.1}$$

where b is some prescribed (large) positive constant.

Proposition 4.1 *Let $g \in W^{1,q}(\Omega)$, for some $q > 2$. Then, there exists an $u^* \in U_b$ such that*

$$\Phi(u^*) = \min_{u \in U_b} \Phi(u). \tag{4.2}$$

Proof: Let $(u_n)_{n=1,2,\ldots} \subset U_b$ be a minimizing sequence. By Theorem 1. we know that

$$\|z^{u_n}\|_{H^1(\Omega)} + \|z^{u_n}\|_{C^0(\bar{\Omega})} \leq c. \tag{4.3}$$

By taking subsequences, if necessary, we can assume without loss of generality that there exist $u^* \in U_b$ and $z^* \in H^1(\Omega)$ such that

$$u^n \to u^* \text{ in } H^2(-1,1) \tag{4.4}$$

$$z^{u_n} \rightharpoonup z^* \text{ weakly in } H^1(\Omega) \tag{4.5}$$

$$z^{u_n} \to z^* \text{ in } C^0(\bar{\Omega}). \tag{4.6}$$

Recall that

$$-\int_\Omega \nabla z^{u_n} \cdot \nabla \varphi = \int_{\Gamma_{u_n}} g \varphi d\sigma \tag{4.7}$$

for all $\varphi \in H^1(\Omega)$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. If, in addition, $\varphi \in C^1(\bar{\Omega})$ then it is easy to see that

$$\lim_{n \to \infty} \int_{\Gamma_{u_n}} g \varphi d\sigma = \int_{\Gamma_{u^*}} g \varphi d\sigma. \tag{4.8}$$

Hence, for such φ we can pass $n \to \infty$ in (4.7) to conclude

$$-\int_\Omega \nabla z^* \cdot \nabla \varphi = \int_{\Gamma_{u^*}} g \varphi d\sigma \tag{4.9}$$

for any $\varphi \in C^1(\bar{\Omega})$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. But then, by the density, (4.9) holds for all $\varphi \in H^1(\Omega)$ such that $\varphi|_{\{y=0\}} = \varphi|_{\{y=2\}} = 0$. We conclude, by uniqueness, that

$$z^* = z^{u^*}. \tag{4.10}$$

Now since

$$\Phi(u_n) = \frac{1}{2} \int_{\Gamma_{u_n}} (z^{u_n})^2 d\sigma \tag{4.11}$$

(4.4,4.6) imply that

$$\lim_{n\to\infty} \Phi(u_n) = \Phi(u^*). \tag{4.12}$$

This completes the proof of the Proposition. □

5 Differentiability Properties of the Variational Functional Φ

Our goal is to derive information about the *multivalued* generalized gradient of Φ. To make our results more precise we shall introduce several definitions.

Let Φ be a real-valued function on the subset U of the Banach space X.

Definition 5.1 *Φ is said to be* directionally differentiable *at $u \in U$ if the limit*

$$\lim_{\lambda\downarrow 0} \frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda} \tag{5.1}$$

exists for any $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. If that is the case, then the limit in (5.1) is called directional derivative *and it is denoted by $\Phi'(u; v)$.*

Definition 5.2 *Φ is said to be* subdifferentiable *at u, if there exists an $f \in X^*$ such that*

$$\Phi'(u; v) \geq f(v) \tag{5.2}$$

for every $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. Set of all such f's is called subdifferential, *and it is denoted by $\partial_*\Phi(u)$.*

Definition 5.3 *Φ is said to be* superdifferentiable *at u, if there exists an $f \in X^*$ such that*

$$\Phi'(u; v) \leq f(v) \tag{5.3}$$

for every $v \in X$ such that $u + \lambda v \in U$, for small enough $\lambda > 0$. Set of all such f's is called superdifferential, *and it is denoted by $\partial^*\Phi(u)$.*

If Φ is both sub- and superdifferentiable at $u \in int(U)$, and moreover $\partial_*\Phi(u) \cap \partial^*\Phi(u) \neq \emptyset$, then $\partial_*\Phi(u) \cap \partial^*\Phi(u)$ is a singleton and Φ is Gâteaux differentiable.

We go back now to our problem. Of course, $X = H_0^3(-1,1)$, U is defined in (1.1).

Proceeding, define the adjoint variable p^u, as a solution of the (adjoint) equation

$$\begin{aligned} \Delta p^u &= \eta_u \text{ in } \Omega \\ p^u &= 0 \text{ in } \{(x,0); -a < x < a\} \cup \{(x,2); -a < x < a\} \\ p^u_x &= 0 \text{ in } \{(\pm a, y); 0 < y < 2\} \end{aligned} \tag{5.4}$$

where $\eta_u \in H^{-1}(\Omega)$ is a (signed) measure given by

$$\eta_u(\varphi) = \int_{\Gamma_u} z^u \varphi d\sigma. \tag{5.5}$$

Obviously, (5.4) is the same type of equation as (2.3).

In this section, as before, $z^{u,ext} = z^u|_{\Omega_u}$ and $z^{u,int} = z^u|_{\Omega\setminus\Omega_u}$; also, below we shall use the notation $p^{u,ext} = p^u|_{\Omega_u}$ and $p^{u,int} = p^u|_{\Omega\setminus\Omega_u}$. That is essential in this calculation, since z^u and p^u are *not* differentiable *across* the Γ_u.

Lemma 5.1 *Let $g \in W^{2,q}(\Omega)$, for some $q \geq 2$. Then*

$$p^{u,ext} \in W^{2,q}(\Omega_u),\ p^{u,int} \in W^{2,q}_{\{y>0\}-loc}(\Omega \setminus \bar{\Omega_u}). \tag{5.6}$$

and the apriori estimates

$$\|p^{u,ext}\|_{W^{2,q}(\Omega_u)} \leq c\left(\|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right), \tag{5.7}$$

and

$$\begin{aligned} &\|p^{u,int}\|_{W^{2,q}((\Omega\setminus\Omega_u)\cap\{y\geq\epsilon\})} \leq \\ &\leq c\left(\epsilon, \|u\|_{H^3(-1,1)}, \|g\|_{W^{2,q}(\Omega)}, \|\psi\|_{W^{2,q}(\Omega)}\right). \end{aligned} \tag{5.8}$$

hold.

Proof: Comparing (2.3) and (5.4) we see that the only difference is in right-hand sides. Namely, in (5.5), $z^u \notin W^{2,q}(\Omega)$. Nevertheless, for example, $z^{u,ext} \in W^{2,q}(\Omega_u)$, and since η_u depends on z^u only through the trace on Γ_u, and since z^u and $z^{u,ext}$ have same traces on Γ_u we easily conclude the proof of the Lemma. □

We shall use the usual notation: $v^+ \stackrel{\text{def}}{=} v\mathbf{I}_{\{v>0\}}$, and $v^- \stackrel{\text{def}}{=} -v\mathbf{I}_{\{v<0\}}$. So, $v = v^+ - v^-$.

Now we are ready to state the following

Theorem 5.1 *Let $g \in W^{2,q}(\Omega)$, for some $q > 2$. Then Φ is directionally differentiable at any $u \in U$ such that $u(x) > 0$ for $-1 < x < 1$, and*

$$\begin{aligned} &\Phi'(u;v) = \\ &= \int_{-1}^{1} \left(z^u (z_y^{u,ext} v^+ - z_y^{u,int} v^-)\sqrt{1+u'^2} + (z^u)^2 \frac{u'v'}{\sqrt{1+u'^2}} \right) dx \\ &+ \int_{\Gamma_u} \left((gp^{u,ext})_y v^+ - (gp^{u,int})_y v^- \right) d\sigma + \int_{-1}^{1} gp^u \frac{u'v'}{\sqrt{1+u'^2}} dx. \end{aligned} \tag{5.9}$$

Moreover, if

$$z^u z_y^{u,int} + (gp^{u,int})_y \leq z^u z_y^{u,ext} + (gp^{u,ext})_y \quad a.e.\ in\ (-1,1), \tag{5.10}$$

then Φ *is subdifferentiable at* u *and*

$$\begin{aligned}
&\partial_*\Phi(u) = \\
&= \left[\left(z^u z_y^{u,int} + (gp^{u,int})_y\right)\sqrt{1+u'^2}, \left(z^u z_y^{u,ext} + (gp^{u,ext})_y\right)\sqrt{1+u'^2}\right] \\
&\qquad - \left(\frac{u'}{\sqrt{1+u'^2}}\left((z^u)^2 + gp^u\right)\right)' \\
&\stackrel{\text{def}}{=} [l\partial_*\Phi(u), r\partial_*\Phi(u)] \subset L^\infty(-1,1).
\end{aligned} \tag{5.11}$$

On the other hand, if

$$z^u z_y^{u,int} + (gp^{u,int})_y \geq z^u z_y^{u,ext} + (gp^{u,ext})_y \quad a.e. \ in\ (-1,1), \tag{5.12}$$

then Φ *is superdifferentiable at* u *and*

$$\begin{aligned}
&\partial^*\Phi(u) = \\
&= \left[\left(z^u (z_y^{u,ext} + (gp^{u,ext})_y\right)\sqrt{1+u'^2}, \left(z^u (z_y^{u,int} + (gp^{u,int})_y\right)\sqrt{1+u'^2}\right] \\
&\qquad - \left(\frac{u'}{\sqrt{1+u'^2}}\left((z^u)^2 + gp^u\right)\right)' \\
&\stackrel{\text{def}}{=} [l\partial^*\Phi(u), r\partial^*\Phi(u)] \subset L^\infty(-1,1)
\end{aligned} \tag{5.13}$$

Proof: We attempt to differentiate Φ. To this end, for given $u \in U$ and a suitable direction $v \in H_0^3(-1,1)$ (suitable in a sense that $u + \lambda v \in U$ for small enough $\lambda > 0$) we try to compute the (one sided) directional derivative $\Phi'(u;v)$. Using the regularity result (Theorem 3.1, and Corollary 3.2), we compute

$$\begin{aligned}
\Phi'(u;v) &= \lim_{\lambda\downarrow 0} \frac{\Phi(u+\lambda v) - \Phi(u)}{\lambda} \\
&= \lim_{\lambda\downarrow 0} \frac{1}{2\lambda}\left(\int_{\Gamma_{u+\lambda v}} (z^{u+\lambda v})^2 d\sigma - \int_{\Gamma_u} (z^u)^2 d\sigma\right) \\
&= \int_{-1}^{1}\left(z^u (z_y^{u,ext} v^+ - z_y^{u,int} v^-)\sqrt{1+u'^2} + (z^u)^2 \frac{u'v'}{\sqrt{1+u'^2}}\right) dx + \\
&\qquad + \lim_{\lambda\downarrow 0} \frac{1}{2\lambda}\int_{\Gamma_u} \left((z^{u+\lambda v})^2 - (z^u)^2\right) d\sigma,
\end{aligned} \tag{5.14}$$

Before proceeding with the proof, we shall need the following Lemma (more precisely, its Corollary).

Lemma 5.2 *Under previous assumptions on* u, *and* v, *and for any* $\alpha < 1$ *the following estimate holds*

$$\|z^{u+\lambda v} - z^u\|_{C^0(\bar\Omega)} \leq c\lambda^\alpha. \tag{5.15}$$

Proof: We need to compare $z^{u+\lambda v}$ and z^u. This is difficult to do in the original domain Ω since (singular) right hand sides of the equations that they satisfy act on disjoint sets, so that there is no obvious cancelation. So, the idea of the proof is to map the original domain into different domains in such a way that the cancelation does take place.

Let, as before, Ξ_u be the map with the image Ω given by the formula

$$\Xi_u(x,y) = (x, y + u(x)). \tag{5.16}$$

Then

$$\Xi_u^{-1}(x,y) = (x, y - u(x)), \tag{5.17}$$

and (set $A = (x,y)$)

$$dist\left(\Xi_{u+\lambda v}^{-1}(A) - \Xi_u^{-1}(A)\right) \le c\lambda. \tag{5.18}$$

Now consider $\tilde{z}^{u+\lambda v}$ and $\tilde{z}^u$ defined as

$$\begin{aligned} \tilde{z}^{u+\lambda v} &= z^{u+\lambda v} \circ \Xi_{u+\lambda v}, \\ \tilde{z}^u &= z^u \circ \Xi_u. \end{aligned} \tag{5.19}$$

Then $\tilde{z}^{u+\lambda v} - \tilde{z}^u$ satisfies the equation

$$\Delta\left(\tilde{z}^{u+\lambda v} - \tilde{z}^u\right) = \gamma, \text{ in } \Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega), \tag{5.20}$$

where

$$\gamma(\varphi) \stackrel{\text{def}}{=} \int_{\{y=0\}} (G_1 - G_2)\, \varphi dx \tag{5.21}$$

and where

$$\begin{aligned} G_1(x,y) &\stackrel{\text{def}}{=} g(x, y + u(x) + \lambda v(x))\sqrt{1 + (u'(x) + \lambda v'(x))^2}, \\ G_2(x,y) &\stackrel{\text{def}}{=} g(x, y + u(x))\sqrt{1 + (u'(x))^2}. \end{aligned} \tag{5.22}$$

Observe that

$$\|G_1 - G_2\|_{W^{1,q}\left(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)\right)} \le c\lambda. \tag{5.23}$$

Now since

$$dist\left(\partial\left(\Xi_{u+\lambda v}^{-1}(\Omega)\right), \partial\left(\Xi_u^{-1}(\Omega)\right)\right) \le c\lambda \tag{5.24}$$

and because of the Hölder continuity of $z^{u+\lambda v}$ and z^u, we conclude that

$$\|\tilde{z}^{u+\lambda v} - \tilde{z}^u\|_{C^0\left(\partial\left(\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)\right)\right)} \le c\lambda^\alpha. \tag{5.25}$$

Then (3.5,5.23,5.25) imply that

$$\|\tilde{z}^{u+\lambda v} - \tilde{z}^u\|_{C^0\left(\overline{\Xi_{u+\lambda v}^{-1}(\Omega) \cap \Xi_u^{-1}(\Omega)}\right)} \le c\lambda^\alpha. \tag{5.26}$$

Then we have (set $A = (x, y)$)

$$\begin{aligned}
&\left|z^{u+\lambda v}(A) - z^u(A)\right| = \\
&= \left|\tilde{z}^{u+\lambda v}\left(\Xi^{-1}_{u+\lambda v}(A)\right) - \tilde{z}^u\left(\Xi^{-1}_u(A)\right)\right| \leq \\
&\leq \left|\tilde{z}^{u+\lambda v}\left(\Xi^{-1}_{u+\lambda v}(A)\right) - \tilde{z}^u\left(\Xi^{-1}_{u+\lambda v}(A)\right)\right| + \\
&+ \left|\tilde{z}^u\left(\Xi^{-1}_{u+\lambda v}(A)\right) - \tilde{z}^u\left(\Xi^{-1}_u(A)\right)\right| \leq \\
&c\lambda^\alpha + c\lambda^\alpha = c\lambda^\alpha.
\end{aligned} \tag{5.27}$$

In (5.27) we also used Hölder continuity of $\tilde{z}^u$. This completes the proof of the Lemma. □

Corollary 5.1

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\Gamma_u} \left(z^{u+\lambda v} - z^u\right)^2 d\sigma = 0. \tag{5.28}$$

Proof: Take $\alpha > \frac{1}{2}$ in the Lemma. Then

$$\frac{\left\|z^{u+\lambda v} - z^u\right\|^2_{C^0(\bar{\Omega})}}{\lambda} \leq c\lambda^\beta, \ \beta = 2\alpha - 1 > 0. \tag{5.29}$$

□

Now, we can proceed with the proof of the Theorem. We compute the last term in (5.14).

$$\begin{aligned}
&\lim_{\lambda \downarrow 0} \frac{1}{2\lambda} \int_{\Gamma_u} \left((z^{u+\lambda v})^2 - (z^u)^2\right) d\sigma \\
&= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\Gamma_u} (z^{u+\lambda v} - z^u) z^u d\sigma + \lim_{\lambda \downarrow 0} \frac{1}{2\lambda} \int_{\Gamma_u} \left(z^{u+\lambda v} - z^u\right)^2 d\sigma \\
&= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_{\Gamma_u} (z^{u+\lambda v} - z^u) z^u d\sigma \\
&= -\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_\Omega \nabla p^u \cdot \nabla (z^{u+\lambda v} - z^u) \\
&= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left(\int_{\Gamma_{u+\lambda v}} g p^u d\sigma - \int_{\Gamma_u} g p^u d\sigma\right)
\end{aligned}$$

$$(5.30) = \int_{\Gamma_u} \left((gp^{u,ext})_y v^+ - (gp^{u,int})_y v^-\right) d\sigma + \int_{-1}^1 g p^u \frac{u'v'}{\sqrt{1 + u'^2}} dx.$$

Now from (5.14,5.30) we conclude that Φ is directionally differentiable, and that (5.9) holds. Furthermore, if (5.10) holds, then

$$\begin{aligned}
&\Phi'(u; v) = \\
&= \int_{-1}^1 \left(z^u (z_y^{u,ext} v^+ - z_y^{u,int} v^-)\sqrt{1 + u'^2} + (z^u)^2 \frac{u'v'}{\sqrt{1 + u'^2}}\right) dx
\end{aligned}$$

$$+\int_{\Gamma_u}\left((gp^{u,ext})_y v^+ - (gp^{u,int})_y v^-\right) d\sigma + \int_{-1}^{1} gp^u \frac{u'v'}{\sqrt{1+u'^2}} dx$$

$$\geq \int_{-1}^{1}\left(\tau - \left(\frac{u'}{\sqrt{1+u'^2}}\left((z^u)^2 + gp^u\right)\right)'\right) v dx \tag{5.31}$$

for all

$$\tau \in \left[\left(z^u z_y^{u,int} + (gp^{u,int})_y\right)\sqrt{1+u'^2},\right.$$

$$\left.\left(z^u z_y^{u,ext} + (gp^{u,ext})_y\right)\sqrt{1+u'^2}\right]. \tag{5.32}$$

This proves that Φ is subdifferentiable at u and that (5.11) holds. Similarly, one can consider superdifferentiability of Φ. So, the Theorem follows. □

6 Concluding Remarks

The above suggests the numerical algorithm (*the steepest descent method*) for minimization of Φ, i.e., for the numerical solution of the relaxed shape optimization problem:

Choose $u_0 \in U$. If $u_n \in U$ is already known, then u_{n+1} is determined by:

- compute z^{u_n} as a solution of (2.3);
- compute p^{u_n} as a solution of (5.4);
- if (5.10) holds, compute an u_{n+1} such that

$$u_{n+1} \in \left(u_n - \rho_n A^{-1}\left(\partial_*\Phi(u_n)\right)\right) \cap U,\ \rho_n > 0, \tag{6.1}$$

and if (5.12) holds, compute an u_{n+1} such that

$$u_{n+1} \in \left(u_n - \rho_n A^{-1}\left(\partial^*\Phi(u_n)\right)\right) \cap U,\ \rho_n > 0. \tag{6.2}$$

Here, A is the isomorphism between $H_0^3(-1,1)$ and its dual. So we see that it would be much better to work on $H_0^1(-1,1)$ instead, since then A would be a second order operator $-\frac{d^2}{dx^2}$ instead of the sixth order operator.

If neither (5.10) nor (5.12) holds, i.e., if Φ is neither convex nor concave at the point u_n, then it is more delicate to determine the steep(est) descent direction.

The actual *choice* of u_n in (6.1) or (6.2) is an interesting question. Somewhat formal considerations suggest that the following rules should be adopted:

- if (5.10) holds and $\partial_*\Phi(u_n) \geq 0$ a.e. in $(-1,1)$, then

$$u_{n+1} = u_n - \rho_n A^{-1}\left(l\partial_*\Phi(u_n)\right),\ \rho_n > 0; \tag{6.3}$$

- if (5.10) holds and $\partial_*\Phi(u_n) \leq 0$ a.e. in $(-1,1)$, then

$$u_{n+1} = u_n - \rho_n A^{-1}\left(r\partial_*\Phi(u_n)\right),\ \rho_n > 0; \tag{6.4}$$

• if (5.12) holds and $\partial^*\Phi(u_n) \geq 0$ a.e. in $(-1,1)$, then

$$u_{n+1} = u_n - \rho_n A^{-1}\left(r\partial^*\Phi(u_n)\right), \ \rho_n > 0; \tag{6.5}$$

• if (5.12) holds and $\partial^*\Phi(u_n) \leq 0$ a.e. in $(-1,1)$, then

$$u_{n+1} = u_n - \rho_n A^{-1}\left(l\partial^*\Phi(u_n)\right), \ \rho_n > 0. \tag{6.6}$$

One can show that if u is a local minimizer for Φ then (5.10) does hold. Also, we observe that in terms of Clarke's nonsmooth analysis (5.32) implies that (if (5.10) holds)

$$\partial\Phi(u) \supset \partial_*\Phi(u) \tag{6.7}$$

where $\partial\Phi$ is the generalized gradient of Φ (observe that Φ is nonsmooth, i.e., $\partial\Phi$ is *multivalued*).

Finally, we note that the method introduced here is an unexpected follow-up of the research in the completely different context (electrophotography, see [4]; see also [8]). The difference is that in [4], instead of (2.3), the state equation is (up to nonessential details)

$$\begin{aligned} \Delta z^u &= \mathbf{I}_{D_u} \text{ in } \Omega \\ z^u &= 0 \text{ in } \{(x,0); -a < x < a\} \\ z^u &= 1 \text{ in } \{(x,1); -a < x < a\} \\ z^u_x &= 0 \text{ in } \{(\pm a, y); 0 < y < 1\} \end{aligned} \tag{6.8}$$

where D_u is the set enclosed by Γ_u, and the functional to minimize is, instead of (2.17),

$$\Psi(u) = \frac{1}{2}\int_{\Gamma_u}\left(\frac{\partial z^u}{\partial\nu}\right)^2 d\sigma. \tag{6.9}$$

Observe that in (2.3) the right hand side, i.e., the measure ξ_u is, *essentially*, "derivative" of $\mathbf{I}_{D_u}$, the right hand side in (6.8). On the other hand in (6.9), z^u is under derivative. So, in the final balance those two problems have the same level of smoothness (which happens to be a kind of Lipschitz continuity), and hence, the analogous general ideas apply.

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