A "FATOU EQUATION" FOR RANDOMLY STOPPED VARIABLES

by

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Let (Ω, F, P) be a probability space, $\{F_n\}_{n\geq 1}$ an increasing sequence of σ-fields contained in F, and $\{X_n\}_{n\geq 1}$ a sequence of random variables with X_n being F_n-measurable for every n. A random variable t is a <u>stopping variable</u> (sv) if its range is contained in $\{1, 2, \ldots, +\infty\}$, $P[t < +\infty] = 1$, and, for every positive integer n, $[t \leq n] \in F_n$. The stopping variables with their natural partial ordering form a directed set.

The result of this note is the following

<u>Theorem: Suppose there is an integrable random variable</u> Z <u>such that either</u> $X_n \leq Z$ <u>for all</u> n <u>or</u> $X_n \geq Z$ <u>for all</u> n. <u>Then</u>

$$(1) \qquad E(\limsup_{n\to\infty} X_n) = \limsup_{t\to\infty} EX_t,$$

<u>where the first limit is over the positive integers</u> n <u>and the second over the directed set of stopping variables</u> t.

<u>Proof</u>: The proof is in two parts. Case I. $X_n \leq Z$ for all n.

In this case, a minor modification of the usual proof of Fatou's Lemma establishes the easier half of the desired equation. To see this let $X^* = \limsup_{n\to\infty} X_n$ and set $W_n = \sup_{k\geq n} X_k$. Then $W_n \downarrow X^*$ as $n \to \infty$ and $X_t \leq W_n$ a.s. for $t \geq n$. Hence,

$$\begin{aligned}\limsup_{t\to\infty} EX_t &\leq \lim_{n\to\infty} \{\sup_{t\geq n} EX_t\} \\ &\leq \lim_{n\to\infty} EW_n \\ &= EX^*.\end{aligned}$$

In proving the opposite inequality, we can assume that $EX^* > -\infty$. Since $X^* \leq Z$ and Z is integrable, it follows that X^* is integrable.

Now consider random variables Y which satisfy

(a) $Y(\omega) < X^*(\omega)$ a.s.

(b) $Y = \sum_{i=1}^{\infty} a_i 1_{A_i}$ where the A_i

form a partition of Ω and $a_1 > a_2 > \ldots$,

(c) Y is integrable.

One can easily construct a sequence of such random variables converging a.s. up to X^* and hence in expectation. (For example, let

$$Y_n(\omega) = n \text{ if } X^*(\omega) > n \text{ or } X^*(\omega) = -\infty$$

$$= \frac{k-1}{2^n} \text{ if } \frac{k-1}{2^n} < X^*(\omega) \leq \frac{k}{2^n},$$

where k varies over the integers less than or equal to $n\, 2^n$.) Thus it suffices to show, for an arbitrary Y with the above properties, that $\limsup_{t\to\infty} EX_t \geq EY$. So let Y satisfy (a), (b), and (c). Let $\epsilon > 0$ and s be a sv. We must find a sv t with $t \geq s$ and

(2) $$EX_t \geq EY - \epsilon .$$

To construct t, we first define recursively a sequence of variables $t_1, t_2, \ldots$ and a sequence of positive integers $N_0, N_1, \ldots$. Set $N_0 = 1$ and for $k = 1, 2, \ldots$ and $\omega \in \Omega$, let

$$t_k(\omega) = \min\{n: n \geq \max\,(s(\omega), N_{k-1}) \text{ and } X_n(\omega) \geq a_k\},$$

$$= +\infty \text{ if } X_n(\omega) < a_k \text{ for all } n \geq \max\,(s(\omega), N_{k-1}).$$

Choose N_k such that $N_k > N_{k-1}$ and $P[t_k < +\infty] \le P[t_k < N_k] + \epsilon_k$, where $\epsilon_k = \epsilon/[2^k (a_1 - a_{k+1})]$.

Now define

$$t = \min\{t_k : k = 1, 2, \ldots\}.$$

(The intuition behind the definition of X_t is that one first waits to exceed a_1, the largest value of Y, with close to the maximum probability. This can be done before time N_1. One then waits to exceed a_2 and so forth.) Then $t \ge s$ and t is a sv. To check the latter fact, first notice that $[t \le n] = \bigcap_k [t_k \le n] \in F_n$. Also, for every k, the following inclusions hold almost surely:

$$\begin{aligned} A_k &\subseteq [X^* > a_k] \quad \text{(by (a) and (b))} \\ &\subseteq [X_n \ge a_k \text{ for infinitely many } n] \\ &\subseteq [t_k < +\infty]. \end{aligned}$$

Since $\bigcup A_k = \Omega$, we have $t(\omega) < +\infty$ a.s.

It remains to prove (2).

Let $B_k = [t = t_k \text{ and } t \neq t_j \text{ for } j < k]$, for $k = 1, 2, \ldots$. The B_k form a partition of Ω and

(3) $$P\left(\bigcup_{i=1}^{k} A_i - \bigcup_{i=1}^{k} B_i\right) \le \epsilon_k \quad \text{for all } k.$$

To see this, first notice that

$$\begin{aligned} [t_k < N_k] &\subseteq [t < N_k] \\ &\subseteq \bigcup_{i=1}^{k} [t = t_i] \\ &= \bigcup_{i=1}^{k} B_i. \end{aligned}$$

Also,

$$\bigcup_{i=1}^{k} A_i = [Y \geq a_k]$$

$$\subseteq [X^* > a_k] \quad \text{(almost surely)}$$

$$\subseteq [X_n \geq a_k \quad \text{for infinitely many} \quad n]$$

$$\subseteq [t_k < +\infty] \quad \text{(almost surely)}.$$

But, by definition of N_k,

$$P([t_k < +\infty] - [t_k < N_k]) \leq \epsilon_k,$$

which proves (3).

Now let $Y' = \sum_{k=1}^{\infty} a_k 1_{B_k}$. Then

$$\int_{B_1} (Y' - Y)dP \geq 0$$

and, for $k > 1$,

$$\int_{B_k} (Y' - Y)dP \geq \int_{B_k \cap \bigcup_{i=1}^{k-1} A_i} (Y' - Y)dP$$

$$\geq (a_k - a_1)P(B_k \cap \bigcup_{i=1}^{k-1} A_i)$$

$$\geq (a_k - a_1)P(\bigcup_{i=1}^{k-1} A_i - \bigcup_{i=1}^{k-1} B_i)$$

$$\geq (a_k - a_1)\epsilon_{k-1}$$

$$= -\frac{\epsilon}{2^{k-1}} .$$

Summing over k yields $EY' \geq EY - \epsilon$. Since $X_t \geq Y'$ a.s., (2) is established and the theorem proved for Case I.

Case II. $X_n \geq Z$ for all n.

Let c be a real constant. Then $\limsup_{t\to\infty} EX_t \geq \limsup_{t\to\infty} E(X_t \wedge c) = E(X^* \wedge c)$. The last equation follows from Case I. Since $E(X^* \wedge c) \to EX^*$ as $c \to +\infty$, we have $\limsup_{t\to\infty} EX_t \geq EX^*$.

To prove the other inequality, assume, as we may, that $EX^* < +\infty$. Let $\epsilon > 0$ and s be any sv. Choose c such that $E(X^* \wedge c) \geq EX^* - \epsilon/2$. Then choose a sv t such that $t \geq s$ and $E(X_t \wedge c) \geq E(X^* \wedge c) - \epsilon/2$. This is possible by Case I. Since $X_t \geq X_t \wedge c$, we have $EX_t \geq EX^* - \epsilon$. This establishes the desired inequality and the theorem. □

After finding the proof given here, I discovered that a considerably shorter proof is possible if use is made of a result of Siegmund (Theorem 4,[2]). The direct proof given here is more elementary. Also, it may be of interest that the theorem here together with a result of Dubins and Savage (Theorem 3.9.5, [1]) can be used to prove a result parallel to that of Siegmund (cf. Theorem 5, [4]).

For uniformly bounded X_n, a version of the present theorem has been proved when the sv's are not assumed to be measurable (Theorem 3.2, [3]) and there is an interesting interpretation for the Dubins and Savage utility of a measurable strategy. Further applications to measurable gambling problems are anticipated.

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