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ABSTRACT

Energy functions on knots are continuous and scale-invariant functions defined from knot conformations into non-negative real numbers. The infimum of an energy function is an invariant which defines (not necessarily unique) "canonical conformations" of knots in three space. Many (infinite) hierarchies of energy functions for knots in the mathematical and physical science literature have been studied, each energy function with its own (characteristic) set of properties. In this paper we examine energy functions, and classify them as either *basic*, *strong*, *charge* or *tight*, depending on the properties of the energies for different knot conformations. *Semi-ideal* energy functions are basic, strong and charge, and *ideal* energy functions are tight semi-ideal energy functions. Semi-ideal and ideal energy functions may be expected to be particularly powerful as knot invariants.

Keywords: Knots, Energy of Knots, Properties of Energy Functions, Thickness

1. INTRODUCTION AND DEFINITIONS

The energy of knots was studied as a means of finding canonical conformations of a given knot [2, 4, 11, 14]. There are several energy functions which have received attention, and the number of possibilities for alternative energies are infinite [12, 13]. However, a closer examination reveals that many energies are related in the sense that they have the same properties when analyzed as knot invariants. Thus, in order to carry out a systematic study of knot energies, we propose a classification scheme

for energy functions of knots based on some elementary properties of the energy functions: Two energies functions which are classified similarly may be expected to be close relatives when considered as invariants of knots.

There are several questions one can ask when considering a functional as a knot energy. For example, is it true that only the unknot is at the global infimum of the energy functional, and is the minimal conformation of the unknot a circle? Are infimums of the non-trivial knots attained by nonsingular conformations? How many distinct knot types will have infimums below a given threshold? From a mathematical point of view, the infimum of an energy function will be a knot invariant where we take the infimum of the energy function over all conformations of a given knot within a certain class (for example, differentiable). At the realization of the infimum we hope that there will be a “canonical conformation” of the knot. The definition of an energy function should include finiteness on all non-singular knots, continuity, scale invariance and invariance under rigid motions, and the possible resolution of knot-types from one another. If in addition, the energy function distinguishes the circle from all other knots, and if it is infinite when the knot embedding is singular and assigns the same infimum to only finitely many different knot types, then it is a particularly good energy function which we will call *semi-ideal*. Another desirable property, called *tight*, is that the knot energy becomes infinite if one tries to pull the knot “tight” (meaning that there is a knotted arc ball pair where the ball becomes smaller and smaller.) There are knot energies which are infinite for singular knot embeddings, but not tight, as well shall see in the next section. If one uses a knot energy function which is not tight to find a canonical form of a knot by minimizing the knot energy, then it may happen that the knot energy is minimized by creating a tiny knotted ball pair on an almost perfect circle. In the limit all knotted curves may approach the circle under energy minimization. We will call a tight semi-ideal energy function *ideal*.

Definition 1. Let $\alpha(s)$ and $\beta(s)$ be two simple closed curves of the same length parametrized by arc length. If the curves are k -times differentiable, then the k -distance between them is defined as

$$d^k(\alpha(s), \beta(s)) = \min_{A \in O} \left\{ \max_{0 \leq s \leq L, 0 \leq t \leq L} \sum_{i=0}^k |A\alpha^{(i)}(s+t) - \beta^{(i)}(s)| \right\}, \quad (1)$$

where L is the length of $\alpha(s)$ and $\beta(s)$, A is a rigid motion (the composition of a translation and a rotation), O is the group of all rigid motions in $\mathcal{R}^3$ and $\alpha^{(i)}(s)$, $\beta^{(i)}(s)$ are the i -th derivatives of $\alpha(s)$ and $\beta(s)$.

Note that the k -th distance between two curves can be achieved at certain A and t_0 , i.e., there exist $A \in O$ and $t_0 \in [0, L]$ such that $d^k(\alpha(s), \beta(s)) = d^k(\hat{\alpha}(s), \beta(s)) = \max_{0 \leq s \leq L} \sum_{i=0}^k |\hat{\alpha}^{(i)}(s) - \beta^{(i)}(s)|$ where $\hat{\alpha}(s) = A\alpha(s+t_0)$. We leave it to the reader to verify that d^k defines a metric on the set of all simple closed curves of equal length that are k times differentiable, if we define two such curves α and β to be the same when $d^k(\alpha, \beta) = 0$.

Definition 2. Let $C^k(T)$ be the set of all simple closed curves of unit length that are k times differentiable. Then $f : C^k(T) \rightarrow \mathcal{R}$ is said to be k -continuous

on $C^k(T)$ if $\forall \epsilon > 0, \exists \delta > 0$, such that $|f(\alpha) - f(\beta)| < \epsilon$ whenever the k -th distance between α and β is less than δ .

Remark: If $f : C^k(T) \rightarrow \mathcal{R}$ is k -continuous, then f is invariant under rigid motion, i.e., $f(A\alpha) = f(\alpha)$ for any $\alpha \in C^k(T)$ and $A \in O$. Also, f will be m -continuous on $C^m(T)$ if $m \geq k$ since $C^m(T) \subset C^k(T)$. On the other hand, one can easily find examples in which f is k -continuous but not m -continuous for some $m < k$.

Definition 3. Let $C^k(T')$ be the set of all simple closed curves (of any length) that are k times differentiable. Let $f : C^k(T') \rightarrow \mathcal{R}$. Then f is said to be scale invariant if we have $f(\lambda\alpha(s)) = f(\alpha(s))$ for any real number λ and $\alpha(s) \in C^k(T)$.

Definition 4. Let $f : C^k(T') \rightarrow \mathcal{R}^+$. Then f is called an *energy function* on $C^k(T')$ if it is scale invariant and continuous when restricted to $C^k(T)$.

The essential properties of an energy function are given in definition 4. Let f be an energy function on $C^k(T')$. Then for a given knot type $\mathcal{K}$, the energy of $\mathcal{K}$ denoted $f(\mathcal{K})$ is defined to be the infimum of f over all knots K of knot type $\mathcal{K}$ in $C^k(T')$.

Definition 5. Let $f : C^k(T') \rightarrow \mathcal{R}^+$, where $k > 0$, be an energy function. Then

- (a) f is called a *basic* energy function if $f(\alpha)$ is absolute minimum if and only if α is a circle;
- (b) f is called a *strong* energy function if for any given positive number a , there are only finitely many knot types $\mathcal{K}$ such that $f(\mathcal{K}) \leq a$;
- (c) f is called a *charge* energy if f goes to infinity when α approaches a closed curve with self-intersections;
- (d) f is called a *tight* energy if $f(\alpha_i)$ goes to infinity for any sequence of embeddings α_i of the same knot type K and of equal length, where α_i contains a knotted arc of knot type K in a ball B_i and the diameter of the ball B_i converges to zero.
- (e) f is called a *semi-ideal* energy function if it satisfies conditions (a), (b) and (c).
- (f) f is called a *ideal* energy function if it satisfies conditions (a), (b), (c) and (d).

It is obvious but worthwhile to point out that if f and g are both energy functions and satisfy the condition $f(K) \leq g(K)$ for any K , then if f is strong or charge, then so is g .

2. SOME EXAMPLES

In this section, we give several examples of various energy function of smooth knots without detailed proofs. These are all well known energy functions. A careful reader will find from these examples that the conditions *basic*, *charge*, *strong* and *tight* do not contain each other.

Example 1. Let $C^2(T')$ be the set of all twice differentiable simple closed curves and let $K \in C^2(T')$ be parametrized by arc length s . Let $\alpha(s)$ be the parametric equation of K with arc-length or unit speed parametrization. Then $|\alpha'(s)| = 1$, and the curvature of K is $\kappa(s) = |\alpha''(s)|$. The total curvature and total squared curvature are well understood energy functions [8,9]. These may be generalized as

follows into a hierarchy of energy functions involving $\kappa(s)$: Let L be the length of K and define $E_p^f(K) = \frac{1}{L} \int_0^L (L\kappa(s))^p ds$, where p is a constant and L is the total length of K . This defines a continuous function over $C^2(T')$ and we can prove that $E_p^f(K)$ is a basic energy function for $p > 1$ but not a strong energy function, nor a charge energy function. The proof of the case $p = 2$ can be found in [8]. For the other cases of p , it is rather obvious that $E_p^f(K)$ is not charge. The fact that it is basic can be obtained by using Hölder's inequality with the well known result of Milnor [9] that any knot has total curvature at least 4π . A constructive proof can show that all $(2, k)$ torus knots have the same minimal energy hence $E_p^f(K)$ is not strong.

We consider tightness next by outlining a proof of tightness if $p > 1$, and proving that $E_p^f(K)$ is not tight if $p \leq 1$. Let B_i be a sequence of knotted arc ball-pairs of radii $r_i > 0$ where $r_i \rightarrow 0$ as $i \rightarrow \infty$. Let l_i be the arc in B_i and suppose that it has length $|l_i|$ and that l_i is parametrized by $s \in [0, |l_i|]$. The curvature at a point s on l_i is $\kappa_i(s)$. If the B_i are scaled versions of the same knotted arc ball-pair, then $\kappa_i = \kappa_0 \frac{r_0}{r_i}$. The contribution to the energy from l_i is $\int_0^{|l_i|} [\kappa_i(s)]^p ds = \int_0^{|l_0|} [\kappa_0(s) \frac{r_0}{r_i}]^p r_i ds$. This goes to infinity as $r_i \rightarrow 0$ if $p > 1$ but is finite if $p \leq 1$. Thus $E_p^f(K)$ is not tight if $p \leq 1$.

Now let $p > 1$ and let B be a ball that contains a knotted arc C . Let $P = \alpha(s_0)$ be one end point of C and let $Q = \alpha(s_1)$ ($s_1 > s_0$) be the first point on C such that the angle between $\alpha'(s_0)$ and $\alpha'(s_1)$ becomes $1/2$ radian. Such point must exist otherwise C cannot be knotted. This implies that $|\alpha'(s) - \alpha'(s_0)| \leq \sqrt{2(1 - \cos 1/2)} < 1/2$ for all $s_0 \leq s \leq s_1$. We claim that $s_1 - s_0 < 4r$ where r is the radius of B . If not, we would have

$$\begin{aligned} |\alpha(s_1) - \alpha(s_0)| &= \left| \int_{s_0}^{s_1} \alpha'(s) ds \right| \\ &= \left| \int_{s_0}^{s_1} (\alpha'(s) - \alpha'(s_0)) ds + (s_1 - s_0) \alpha'(s_0) \right| \\ &\geq (s_1 - s_0) - \int_{s_0}^{s_1} |\alpha'(s) - \alpha'(s_0)| ds \\ &\geq (s_1 - s_0) - (s_1 - s_0)/2 = (s_1 - s_0)/2 > 2r. \end{aligned}$$

This means that Q is outside of B which is a contradiction. From the condition on the angle between $\alpha'(s_0)$ and $\alpha'(s_1)$ it follows that $\int_{s_0}^{s_1} \kappa(s) ds \geq \frac{1}{2}$. Let $u = (s - s_0)/(s_1 - s_0)$ in the above integral, then we have $\int_{s_0}^{s_1} \kappa(s) ds = (s_1 - s_0) \int_0^1 \kappa_1(u) du$ where $\kappa_1(u) = \kappa(s_0 + (s_1 - s_0)u)$. Thus $\int_0^1 \kappa_1(u) du = \frac{1}{s_1 - s_0} \int_{s_0}^{s_1} \kappa(s) ds \geq \frac{1}{2(s_1 - s_0)}$. Finally, by Hölder's inequality, we have

$$\begin{aligned} \int_{s_0}^{s_1} \kappa^p(s) ds &= (s_1 - s_0) \int_0^1 \kappa_1^p(u) du \\ &\geq (s_1 - s_0) \left(\int_0^1 \kappa_1(u) du \right)^p \end{aligned}$$

$$\geq \frac{1}{2^p(s_1 - s_0)^{p-1}} \geq \frac{1}{2^{3p-2}r^{p-1}}.$$

This shows that if $r \rightarrow 0$, then $E_p^f(K) \rightarrow \infty$. So $E_p^f(K)$ is tight if $p > 1$.

Example 2. Let K be any C^1 -knot and let $c(K)$ be the crossover number of K . Then $c(K)$ is scale invariant and continuous. $c(K)$ is a strong energy function, but is not basic, not charge and not tight.

Example 3. Let $K \in C^1(T')$ be arclength parametrized and $\alpha(s)$ be such a parametric equation of K . Let L be the length of K so $\alpha(s)$ is of period L . Define

$$g(s, t) = \frac{t}{|\alpha(s) - \alpha(s+t)|},$$

where $0 < t \leq \frac{L}{2}$ and $|\alpha(s) - \alpha(s+t)|$ is the distance between $\alpha(s)$ and $\alpha(s+t)$. Observe that $g(s, t)$ can be extended continuously to $0 \leq t \leq \frac{L}{2}$ if we define $g(s, 0) = 1$ for any s , since $\lim_{t \rightarrow 0} g(s, t) = \lim_{t \rightarrow 0} \frac{t}{|\alpha(s) - \alpha(s+t)|} = 1$ for any s . Define

$$\mathcal{F}(K) = \sup_{0 \leq s \leq L, 0 \leq t \leq \frac{L}{2}} g(s, t).$$

Then $\mathcal{F}(K)$ is the same as the Gromov distorsion $D_s(K)$ of K defined as

$$D_s(K) = \sup_{u, v \in [0, 1], u \neq v} \frac{D(\alpha(u), \alpha(v))}{|\alpha(u) - \alpha(v)|}$$

in [5,6], where $D(\alpha(u), \alpha(v))$ is the distance between $\alpha(u)$ and $\alpha(v)$ along the curve $\alpha(s)$. $\mathcal{F}(K)$ is scale invariant and continuous since $g(s, t)$ has these properties so $\mathcal{F}(K)$ is an energy function. It is obvious that $\mathcal{F}(K)$ is a charge function. We show below that $\mathcal{F}(K)$ (or the Gromov's distorsion of K) is not tight. $\mathcal{F}(K)$ is basic since $\mathcal{F}(K) = \frac{\pi}{2}$ if K is a circle and $\mathcal{F}(K) > \frac{\pi}{2}$ if K is not a circle (See the lemma in section 3). It is known that there exists infinitely many knot types with Gromov's distorsion bounded by a fixed number M [5,12], and so $D_s(K)$ is not a strong energy function. In the next paragraph we show that Gromov's distorsion is neither tight nor strong (it is known that Gromov's distorsion is not strong, see references [5,12], the proof below is an alternative to that of reference [12]).

$\mathcal{F}(K)$ can be defined in a scale invariant manner on any open simple C^1 curve. Thus we can calculate its value on a knotted curve, such as a trefoil. The energy is scale invariant, so the trefoil can be shrunk to be very small without changing its energy. Now let K be a knot with n copies of the trefoil as shown in figure 1 such that each copy is very tiny compare to the length of K such that K takes on the appearance of a circle when viewed macroscopically. Assume furthermore that the copies of trefoils are identical up to rotation and translation. Then the maximum of $g(s, t)$ is attained either in each copy of the small trefoils, or in the macroscopic circle. In both cases we can multiply the copies of the trefoils without bound, and hence $\mathcal{F}(K)$ is not strong. By using only one copy of the trefoils and shrinking it to a point, we see that $\mathcal{F}(K)$ is not tight either. $\mathcal{F}(K)$ can be easily modified into a charge energy function which is not strong nor basic by defining $\mathcal{F}'(K) = \max\{\mathcal{F}(K), k\}$ where $k > 0$ is large enough.

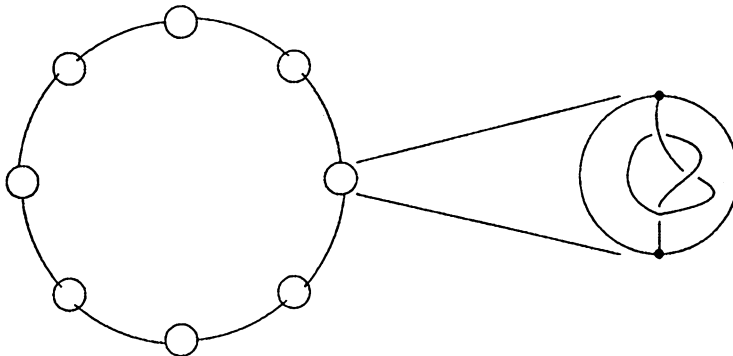


Figure 1: A Knot Containing n Copies of Trefoil.

Example 4. In [3,7] the following energy function E^b is defined. Let K be a C^2 simple closed curve in $\mathcal{R}^3$ with a parametric equation $\alpha(s)$. Then

$$E^b(K) = \int \int \left(\frac{1}{|\alpha(u) - \alpha(v)|^2} - \frac{1}{D^2(\alpha(u), \alpha(v))} \right) |\alpha'(u)| |\alpha'(v)| du dv.$$

Note that this energy function is equivalent to a knot energy function defined by O'Hara [11] which equals $\frac{1}{2}E^b(K) - 2$.

In [3] it is shown that $E^b(K) \geq 4$ where equality holds if and only if K is a circle, so $E^b(K)$ is basic. $E^b(K)$ is a charge energy function by its definition. It is also shown in [3] that $2\pi c(K) + 4 \leq b(K)$ where $c(K)$ is the minimal crossover number of the knot K . So $E^b(K)$ is also a strong energy function. In [12] it is shown that $E^b(K)$ is not tight and hence is semi-ideal. It is shown in [12] that this energy function is not tight.

$E^b(K)$ is a special case in a class of energy functions defined by O'Hara [12,13]. Let K be a C^1 immersion of S^1 in $\mathcal{R}^3$ with unit speed parametrization $\alpha(t)$ (that is, $|\alpha'(t)| = 1$). We define the O'Hara functional of order p and degree j of K by

$$e_j^p(K) = \frac{1}{j} \left[\int_0^1 \int_0^1 \left(\frac{1}{|\alpha(u) - \alpha(v)|^j} - \frac{1}{D^j(\alpha(u), \alpha(v))} \right)^p du dv \right]^{\frac{1}{p}}.$$

Note that $e_2^1(K) = \frac{1}{2}E^b(K) + 2$. It has been shown that $e_j^p(K)$ is strong if $0 < j \leq 2$ and $jp > 2$ [12]. (The case $j = 2, p = 1$ is discussed in the above). It is now also known that e_j^p is charge and tight if $jp > 2$ and $0 < j \leq 2$ [12]. On the other hand, if $jp \leq 2$ and $0 < j \leq 2$, then this energy is not tight. It is not known that $e_j^p(K)$ is basic, except in the case discussed above. Finally, $\lim_{j \rightarrow 0} \lim_{p \rightarrow \infty} e_j^p(K) = \log D_s(K)$ where $D_s(K)$ is the Gromov distortion defined in example 3 [12].

3. IDEAL ENERGY FUNCTIONS.

In this section, two new energy functions are defined and a complete proof will show that they are ideal energy functions.

3.1: The c -neighborhood of a simple closed C^1 -curve K is the set of all points with distances less than or equal to c from K . We denote the c -neighborhood of K by K_c . Also, for any $x \in K$, let Σ_x denote the plane that intersects K perpendicularly at x . If c is small enough, then for any $x \in K$, the component of $K_c \cap \Sigma_x$ containing x is a meridian disk of K_c which only intersects K at x . Of course, in this case K_c is homotopic to K (this is written as $K \sim K_c$). A number c with this property is called a *nice* number and the *thickness* of K [10] is defined by

$$t(K) = \sup\{c : t \text{ is nice } \forall t \in (0, c)\}. \quad (2)$$

Theorem 1. Let K be a C^1 knot of length L , then $\frac{L}{t(K)}$ is an ideal energy function.

Proof. Let $E^h(K) = \frac{L}{t(K)}$. For any C^1 knot, we have $t(K) > 0$ hence $E^h(K)$ is defined. To prove that $E^h(K)$ is continuous, it suffices to prove that $t(K)$ is continuous. Let K_1 and K_2 be two C^1 knots of equal length and t_1, t_2 be their thickness respectively. Let $\epsilon > 0$ be a small positive number. Consider the $(t_1 - \epsilon)$ neighborhood N of K_1 . By definition of the thickness, for any $x \in K_1$, the component of $\Sigma_x \cap N$ containing x is a meridian disk that only intersects K_1 at x . Such a meridian disk also only intersects K_2 at one point if K_1 and K_2 are close enough. A small perturbation in the directions of these Σ_x (with x fixed) will result in a new plane Σ which still intersects K_1 at x and the component of $\Sigma \cap N$ containing x will still be a meridian disk of N that only intersects K_1 at x (and only intersects K_2 at one point if K_1 and K_2 are close enough). Assume that the 1-distance between K_1 and K_2 is δ and δ is small enough to make the above possible and also the following possible:

- (1) The $(t_1 - 2\epsilon)$ neighborhood M of K_2 is contained in N .
- (2) For any $y \in K_2$, Σ_y intersects K_1 at a point x such that $|x - y| + |x^{(1)} - y^{(1)}|$ is small enough such that the directions of Σ_x and Σ_y are close enough so Σ_y will also intersect N at a meridian disk which only contains x from K_1 .
- (3) The component of $\Sigma_y \cap M$ containing y will only intersect M at y .

Therefore, $\Sigma_y \cap M$ contains a meridian disk of M which intersects K_2 only once at y . This shows that $t_2 \geq t_1 - 2\epsilon$. Similarly, we will have $t_1 \geq t_2 - 2\epsilon$. This finishes the proof that $E^h(K)$ is continuous.

It is obvious that $E^h(K)$ is scale invariant hence an energy function.

If K is a circle of radius 1, then the thickness of K is 1 and we have $E^h(K) = 2\pi$. If K is of length 2π but has thickness $c \geq 1$, then let m_ϵ be a meridian of $K_{c-\epsilon}$ (which is a solid torus homotopic to K). Fix a point X on K and let Y be a point on m_ϵ . Let l_Y be a line containing X and Y . As Y moves along m_ϵ , l_Y will meet K at a point Z_ϵ other than X for some Y_ϵ , since m_ϵ and K are linked. For $\epsilon = \frac{1}{n}$, we will then have a sequence of points $(Y_{\frac{1}{n}}, Z_{\frac{1}{n}})$ on $S^1 \times S^1$. This sequence will have a convergent subsequence since $S^1 \times S^1$ is compact. Let (Y_0, Z_0) be the limit point and let $B(Y_0)$ be the ball of radius c centered at Y_0 . Clearly, $B(Y_0)$ does not intersect K at an interior point and the line segment XZ_0 contains a diameter of $B(Y_0)$ hence the length of K is greater than or equal to $2\pi c$. The equality is true

only when X and Z_0 are both on $\partial B(Y_0)$ and K is the great circle on $B(Y_0)$. Since K is of length 2π , $c = 1$ and K is a circle. This shows that $E^h(K)$ is a basic energy function.

It is obvious that $E^h(K)$ is a tight and charge energy function so we will now prove that $E^h(K)$ is also a strong energy function. Let $a > 0$ be a constant. We need to show that there are only finitely many knot types for a knot K such that $E^h(K) \leq a$. Let K be any knot of unit length such that $E^h(K) \leq a$, so the thickness of K is at least $\frac{1}{a}$. Let $c = \frac{1}{2a}$ and consider K_c . Let $n = \lfloor 2a \rfloor + 1$ and divide K in n pieces of arclength $\frac{1}{\lfloor 2a \rfloor + 1}$. This defines n points on K with a natural order. Connect the consecutive ones with line segments and we obtain a polygonal knot. Obviously, this polygonal knot is homeotopic to K so they will have the same crossover number. Since the crossover number of this polygonal knot is bounded by $n^2 \leq (2a+1)^2$, the crossover number of K is also bounded by $(2a+1)^2$. So there are only finitely many knot types for K since there are only finitely many knot types with crossover number less than any given constant. $\square$

Remark: A similar definition for the thickness of knots is given in [1] together with an upper bound of the crossover number of a knot with given thickness.

3.2: Let $K \in C^3(T')$ be arclength parametrized and $\alpha(s)$ be such a parametric equation of K . Let L be the length of K so $\alpha(s)$ is of period L . We define an energy function $E^e(K)$, related to $E^b(K)$, as follows: Let

$$G(s, t) = L^2 \left(\frac{1}{|\alpha(s) - \alpha(s+t)|^2} - \frac{1}{t^2} \right),$$

where $0 < t \leq \frac{L}{2}$. Observe that $G(s, t)$ can be extended continuously to $0 \leq t \leq \frac{L}{2}$ if we define $G(s, 0) = \frac{L^2}{12} |\alpha''(s)|^2 = \frac{L^2}{12} \kappa^2(s)$ for any s , since

$$\lim_{t \rightarrow 0} G(s, t) = -\frac{L^2}{4} |\alpha''(s)|^2 - \frac{L^2}{3} \alpha'(s) \cdot \alpha'''(s)$$

and

$$\alpha'(s) \cdot \alpha'''(s) = -\alpha''(s) \cdot \alpha''(s).$$

Define

$$E^e(K) = \max_{0 \leq s \leq L, 0 \leq t \leq \frac{L}{2}} G(s, t).$$

In order to prove that $E^e(K)$ is a ideal energy function, we need the following result:

Lemma. Let $K \in C^1(K')$ be a non circle of length 2π and $\alpha(s)$ be parametric equation of K . Then there exists $s_0 \in [0, 2\pi]$ such that $|\alpha(s_0) - \alpha(s_0 + \pi)| < 2$.

Proof. If this is not true, then for all s we will have $|\alpha(s) - \alpha(s + \pi)| \geq 2$. Let $t \in [0, 2\pi]$ be such that

$$|\alpha(t) - \alpha(t + \pi)| = \min_{0 \leq s \leq 2\pi} |\alpha(s) - \alpha(s + \pi)| = r \geq 2.$$

Consider the sphere centered at the origin O with radius r . The curve $\beta(s) = \alpha(s) - \alpha(s + \pi)$ does not intersect the interior bounded by this sphere. Furthermore, the points $\beta(t)$ and $\beta(t + \pi)$ define a diameter of this sphere since $\beta(t) = -\beta(t + \pi)$. The shortest curve on or outside the sphere that can join these two points is half of a great circle on the sphere, which is of length πr . Therefore, the length L of the curve $\beta(s)$ is at least $2\pi r$. On the other hand, the length of $\beta(s)$ can be calculated directly as follows.

$$L = \int_0^{2\pi} |\beta'(s)| ds = \int_0^{2\pi} \sqrt{2 - 2\alpha'(s) \cdot \alpha'(s + \pi)} ds. \quad (3)$$

Note that in the above we have used the fact that $|\alpha'(s)| = 1$ for any s . Clearly, L is at most 4π since $2 - 2\alpha'(s) \cdot \alpha'(s + \pi)$ is at most 4 for any s . So we have $4\pi \leq 2\pi r \leq L \leq 4\pi$, hence $r = 2$ and $\beta(s)$ is a great circle on this ball. And this is so if and only if $2 - 2\alpha'(s) \cdot \alpha'(s + \pi) = 4$ for all s . That is, $\alpha'(s) = -\alpha'(s + \pi)$ for all s . But then the point $\frac{\alpha(s) + \alpha(s + \pi)}{2}$ is a constant point and $\alpha(s)$ has constant distance 1 from this point for all s . So $\alpha(s)$ is on a ball of radius 1 with the property $|\alpha(t) - \alpha(t + \pi)| = 2$. The shortest curve joining two diameter points is the half great circle. Since $\alpha(t)$ has length 2π , it has to be a great circle on this ball. This contradicts to our assumption. $\square$

Theorem 2. $E^e(K)$ is an ideal energy function.

Proof. $E^e(K)$ is scale invariant and continuous since $G(s, t)$ is scale invariant and continuous. So $E^e(K)$ is an energy function.

For a circle C , we can easily calculate that

$$G(s, t) = 4\pi^2 \left(\frac{1}{4 \sin^2 \frac{t}{2}} - \frac{1}{t^2} \right) \leq \pi^2 - 4.$$

This function is strictly increasing so the maximum $\pi^2 - 4$ is obtained when $t = \pi$. Let K be a non-circle of length 2π and let $\alpha(s)$ be a parametric equation of K . Then by the above lemma, there exists a point s_0 such that $|\alpha(s_0) - \alpha(s_0 + \pi)| < 2$. Thus, $G(s_0, \pi) = 4\pi^2 \left(\frac{1}{|\alpha(s_0) - \alpha(s_0 + \pi)|^2} - \frac{1}{\pi^2} \right) > \pi^2 - 4$ hence $E^e(K) > \pi^2 - 4$. This shows that $E^e(K)$ is a basic energy function.

It is obvious that $E^e(K)$ is a charge function but it is not obvious that it is tight. To see this, assume that K has a tiny knotted arc on it. Call a (geometric) ball B on K a minimal knotted arc-ball pair if the ball contains a knotted arc of K (not properly embedded), such that any larger ball with the same origin as B forms a proper knotted arc-ball pair, and no smaller ball can contain a knotted arc of K . We leave it to our reader to prove the existence of a minimum ball pair. Let B be such a minimum ball with a knotted arc C of K in it. It is obvious that $\partial B \cap C$ contains at least three disjoint components where two of them contain the end points of C . Let $A = \alpha(s_0)$ be a point in one component of $\partial B \cap C$ that does not contain the end points of C . Then it is clear that the curvature of K at A is at least $1/r$, i.e., $|\alpha''(s)| \geq 1/r$ where r is the radius of B . This shows that $E^e(K) \geq G(s_0, 0) = (L^2/12)|\alpha''(s_0)|^2 \geq L^2/(12r^2)$. Therefore, $E^e(K) \rightarrow \infty$ as $r \rightarrow 0$.

Finally, $E^e(K)$ is strong since we have $E^e(K) \geq E^b(K)$ where $E^b(K)$ is defined in Example 4 of last section, which is strong. $\square$

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