Adaptive Control of Linear Discrete Time Systems with External Disturbances under Inaccurate Modelling: a case study

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Introduction

This paper deals with adaptive control techniques implemented to control linear systems with unknown parameters.

In many practical cases the hypothesis that the system parameters values are known it is not realistic, instead, it may be possible to estimate them by observing the system response under appropiated experimental conditions. Based on this idea, adaptive control techniques consist of an on-line parameter estimation combined with an on-line control law (see *Goodwin-Sin* [1984]).

An essential requirement of a practical adaptive system is to preserve its perfomance when faced with disturbances. For a linear, discrete time system, the disturbances can be mainly of two types: inaccurate modelling and external noise.

When dealing with a linear, discrete time system which is suppose to track a reference y_k^* with bounded external disturbance, the bound being known, the projection algorithm with dead zone, introduced in *Edgart [1978]*, can be implemented to cope with the disturbance. This algorithm briefly switches off the adaptation when the estimated error is within a selected threshold. For input-output stable systems of the form:

$$y_{k+1} = \theta_0^T \phi(k)$$

where Θ_0 is the vector of parameters and $\Phi(k)$ is a vector containing past inputs and outputs, it can be proved that the tracking error, $e_k = |y_{k+1} - y^*_k|$ verifies(see *Goodwin-Sin [1984]*):

$$\limsup_{k\to\infty} e_k^2 \leq 4\Delta^2$$

where Δ is the bound of the external disturbance.

Recently, different algorithms based on the idea of dead zone were developed, (*Ortega and Lozano-Leal [87], Ortega and Lozano-Leal [87], Canudas de Witt-Carrillo [90]*). Their results are concerned with the control or parameter identification of linear discrete time systems with external bounded disturbances, where the upper bound is known.

In the case of inaccurate modelling different situations, from stable to chaotic behaviour, can arise. In *Mareels and Bitmead [86-88]* the authors studied a particular case of a second order linear discrete time system with no external perturbation, undermodelled by a first order one. The objective was to regulate the system output to zero. They observed chaotic behaviour on the succesion of estimated parameters for some values of the real parameters of the system although the convergence of the output was obtained. The same example was picked up by *González- Troparevsky- D'Attellis [93]*, and with a little modification on the adaptation algorithm, no chaos appeared and the objective was fulfilled for a range of parameter values taking part in the system.

In this work we obtained results about stabilization of a linear discrete time stable system, with bounded external disturbances, under inaccurate modelling. Conditions between the value of the "unmodelled parameters" and an upper bound of the disturbance (Δ) that ensure that the tracking error satisfies:

$$\limsup |e_k| \leq \beta\Delta$$

for some $\beta > 1$, are stablished.

In this way the result for the stabilization of linear, discrete time systems with external perturbation is generalized to the case of inaccurate modelling.

Description of the system

We will consider the linear, second order, discrete time system presented in *Mareels-Bitmead [86-88]* and in *González- Troparevsky-D'Attellis [93]* added with an external perturbation:

$$(1) \qquad y_{k+1} = ay_k + by_{k-1} + u_k + d_k \ ,$$

where a,b are real unknown constants, y_k,u_k are respectively the output and the control at time k, and d_k denotes an external disturbance that verifies:

$$|d_k| \leq \Delta \ .$$

The objective is to regulate the output to zero.

The proposed model to represent the system is a first order one:

$$y_{k+1} = \hat{a}_k y_k + u_k \ .$$

The control is calculated directly from the proposed model to achieve the control objective, that is:

$$u_k = -\hat{a}_k y_k \ .$$

The parameter is updated via the projection algorithm with dead zone:

$$\hat{a}_{k+1} = \hat{a}_k + \alpha_k y_{k+1} \frac{y_k}{\gamma + y_k^2} \ .$$

where $a_k = a - \hat{a}_k$, $\gamma > 0$ and α_k depends on the output value:

$$(2) \qquad \alpha_k = \begin{cases} 0 & y_{k+1}^2 \leq \beta^2 \Delta^2 \\ \alpha & y_{k+1}^2 > \beta^2 \Delta^2 \end{cases}$$

The closed-loop system can be described by:

$$(3) \qquad \begin{cases} y_{k+1} = a_k y_k + b y_{k-1} + d_k \\ a_{k+1} = a_k - \alpha_k y_{k+1} \dfrac{y_k}{\gamma + y_k^2} \end{cases}$$

We stated that in presence of a stable system with $|y_k| \leq M$, $\forall k$, and if $M|b| < \varepsilon\beta/2\ \Delta$, $\beta > 2/(2-\varepsilon)$, $2 > \varepsilon > 0$ there exists $\alpha > 0$ such that the algorithm (3) generates succesions $\{y_{k\}}$ and $\{a_k\}$ that verifies:

$$a_{k+1}^2 \leq a_k^2$$

$$\limsup |y_k| \leq \beta \Delta$$

Main result

From now on we will work with the system (1) and the closed loop (3). All the constants that will appear correspond to those of the preceeding section.

Proposition 1: If $|b|M \leq \varepsilon(\beta/2)\Delta$ with $0 < \varepsilon < 2$, there exists $\alpha > 0$ such that:

$$a_{k+1}^2 \le a_k^2$$

$$\limsup |y_k| \le \beta\Delta$$

Proof:

From (2) and (3)

$$a_{k+1}^2 - a_k^2 \le \frac{\alpha_k^2 y_{k+1}^2 - 2\alpha_k y_{k+1}^2 + 2\alpha_k |b| \, |y_{k+1}| \, |y_{k-1}| + 2\alpha_k y_{k+1} d_k}{\gamma + y_k^2} ,$$

and as

$$2y_{k+1} d_k \le \frac{y_{k+1}^2}{\xi} + \xi d_k^2 \qquad \forall \xi > 0 ,$$

we obtained

$$a_{k+1}^2 - a_k^2 \le \frac{y_{k+1}^2 (\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi}) + 2\alpha_k |b| \, |y_{k+1}| \, |y_{k-1}| + \alpha_k \xi \Delta^2}{\gamma + y_k^2} , \tag{4}$$

Suposse that there exists R<0 such that :

$$a_{k+1}^2 - a_k^2 \le \frac{\alpha_k \, R (y_{k+1}^2 - \beta^2 \Delta^2)}{\gamma + y_k^2} . \tag{5}$$

We will demonstrate the proposition under the hypothesis (5); finally, we will demonstrate (5) to complete the proof.

In that case, as the right side of the last inequality is always less than or equal to 0 (see(2)) we can conclude that a_k^2 is a positive decreasing sequence. In fact, calling

$$S_N \triangleq \sum_0^N \frac{\alpha_k (-R)(y_{k+1}^2 - \beta^2 \Delta^2)}{\gamma + y_k^2} ,$$

from (5) it results:

$$S_N \leq a_0^2 - a_{N+1}^2 \leq a_0^2 \ .$$

We can conclude

$$\lim_{k\to\infty} \frac{\alpha_k(y_{k+1}^2 - \beta^2\Delta^2)}{\gamma + y_k^2} = 0 \ .$$

In view of the hypothesis $|y_k| \leq M \ \forall k$, and from the fact

$$\alpha_k(y_{k+1}^2 - \beta^2\Delta^2) \geq 0 \ ,$$

we have the following result:

$$\lim_{k\to\infty} \alpha_k(y_{k+1}^2 - \beta^2\Delta^2) = 0 \ .$$

As

$$\alpha_k(y_{k+1}^2 - \beta^2\Delta^2) \geq \alpha(y_{k+1}^2 - \beta^2\Delta^2) \qquad \forall k \ ,$$

we have that

$$0 = \lim_{k\to\infty} \alpha_k(y_{k+1}^2 - \beta^2\Delta^2) \geq \limsup_{k\to\infty} \alpha(y_{k+1}^2 - \beta^2\Delta^2) \ ,$$

and from there we obtained:

$$\limsup_{k\to\infty} |y_k| \leq \beta\Delta \ .$$

To complete the proof of the Proposition we have to demonstrate (5) .

From the fact that

$$\alpha_k > 0 \Rightarrow y_{k+1}^2 > \beta^2\Delta^2 \ ,$$

and under the hypothesis $|b|M \leq (\beta/2)\varepsilon\Delta$, it results:

$$\varepsilon |y_{k+1}| > \beta\varepsilon\Delta \geq 2|b|M \geq 2|b|\,|y_{k-1}| \ ,$$

that is:

$$-\alpha_k \varepsilon y_{k+1}^2 + 2\alpha_k |b|\,|y_{-1}|\,|y_{k+1}| \leq 0 \ .$$

We can now return to (4) to state:

$$a_{k+1}^2 - a_k^2 \leq \frac{y_{k+1}^2(\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k\varepsilon) + \Delta^2\alpha_k\xi}{\gamma + y_k^2} \ .$$

Choosing α, ξ and R verifing: $-2 < \omega = \alpha + \varepsilon - 2 \leq -2/\beta$, $\zeta \in (0,[-\omega\beta^2 + (\omega^2\beta^4 - 4\beta^2)^{1/2})]1/2)$ and $R \in (\omega + 1/\zeta, -\zeta/\beta^2)$ we have:

$$(6) \qquad y_{k+1}^2(\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k\varepsilon) + \Delta^2\alpha_k\xi \leq \alpha_k R(y_{k+1}^2 - \beta^2\Delta^2) \ .$$

In that case equation (5) is verified and the proof ends. ■

The width of the band where the output of the system asymptotically remains is $2\beta\Delta$, $\beta > 1$. Note that if $\varepsilon < 1$ we can choose $\beta < 2$, while in the ideal case when there is no undermodelling (see *Goodwin-Sin [84]*) the band width is 4Δ.

The following Proposition shows that the result of Proposition 1 cannot be improved in the sense of obtaining a narrower band, that is $\beta \leq 1$.

Proposition 2 : Under the hypothesis of the previous sections, we cannot choose $\beta \leq 1$ so that equation (6) holds.

Proof:

Let us suposse that equation (6) holds for $\beta = 1$. In that case we will have:

$$(7) \qquad \alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k \varepsilon - \alpha_k R + \frac{\Delta^2}{y_{k+1}^2}(\alpha_k \xi + \alpha_k R \beta^2) \le 0 \ .$$

As $A_k \triangleq \Delta^2/y_{k+1}^2$ can be as near to 0 as wanted, we obtained:

$$\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k \varepsilon - \alpha_k R \le 0 \ .$$

We have two possibilities:
i)$\xi + R \le 0$
ii)$\xi + R > 0$.
If i) holds we will have:

$$0 \ge \omega + \frac{1}{\xi} - R \ge \omega + \frac{1}{\xi} + \xi \ge 2 + \omega > 0 \ ,$$

so the only possibility which last is ii). Remembering that (7) must hold for every A_k, we take A_k converging to 1 and always less than 1. The expression appearing in (7) converges to $a^2 + \varepsilon a - 2a + a/\xi - Ra + \xi a + Ra = a(\omega + 1/\xi + \xi) > 0$. On the other hand as the expression in (7) is always negative, its limit must also be less than or equal to 0. With this contradiction we prove that β cannot be 1.

If β is less than 1 we will have:

$$\alpha_k(-R)(y_{k+1}^2 - \beta^2 \Delta^2) \ge \alpha_k(-R)(y_{k+1}^2 - \Delta^2) \ ,$$

and this contradicts what we have just proved for $\beta = 1$. ■

Conclusions

Proposition 1 demonstrates that under appropiated hypothesis a slight modification to the standard dead zone algorithm enables the system to tolerate both: external bounded perturbation and undermodelling. The hypothesis relate the magnitude of $|b|$ (the undermodelled parameter) with the upper bounds of the disturbance and that of the output of the system. The result of the Proposition 1 can be resumed by saying that if $|b|M < \varepsilon(\beta/2)\Delta$ we can choose a_k so that the output y_k of the closed loop system (3) asymptotically remains between $-\beta\Delta$ and $\beta\Delta$ while a_k^2 is a decreasing sequence. In this way, the result appearing in Proposition 1 generalizes that of the dead-zone algorithm for linear, discrete-time systems with external bounded perturbation to the case where we also have inaccurate modelling. It is worth noting that if $\varepsilon < 1$ we can choose $\beta < 2$. With this choice we ensure that the output of the system will asymptotically remain within a band of width $2\beta\Delta$, a narrower one than that of the general case of complete model(see *Goodwin-Sin [84]*).

Proposition 2 shows that the band width obtained in Proposition 1 is the best one in this scheme, in the sense that $\beta > 1$.

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$$\limsup |y_k| \leq \beta\Delta$$

Main result

From now on we will work with the system (1) and the closed loop (3). All the constants that will appear correspond to those of the preceeding section.

Proposition 1: If $|b|M \leq \varepsilon(\beta/2)\Delta$ with $0 < \varepsilon < 2$, there exists $\alpha > 0$ such that:

$$a_{k+1}^2 \le a_k^2$$

$$\limsup |y_k| \le \beta\Delta$$

Proof:

From (2) and (3)

$$a_{k+1}^2 - a_k^2 \le \frac{\alpha_k^2 y_{k+1}^2 - 2\alpha_k y_{k+1}^2 + 2\alpha_k |b|\,|y_{k+1}|\,|y_{k-1}| + 2\alpha_k y_{k+1} d_k}{\gamma + y_k^2},$$

and as

$$2y_{k+1}d_k \le \frac{y_{k+1}^2}{\xi} + \xi d_k^2 \qquad \forall \xi > 0,$$

we obtained

$$(4) \qquad a_{k+1}^2 - a_k^2 \le \frac{y_{k+1}^2(\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi}) + 2\alpha_k |b|\,|y_{k+1}|\,|y_{k-1}| + \alpha_k \xi \Delta^2}{\gamma + y_k^2},$$

Suposse that there exists R<0 such that :

$$(5) \qquad a_{k+1}^2 - a_k^2 \le \frac{\alpha_k\, R(y_{k+1}^2 - \beta^2\Delta^2)}{\gamma + y_k^2}.$$

We will demonstrate the proposition under the hypothesis (5); finally, we will demonstrate (5) to complete the proof.

In that case, as the right side of the last inequality is always less than or equal to 0 (see(2)) we can conclude that a_k^2 is a positive decreasing sequence. In fact, calling

$$S_N \triangleq \sum_0^N \frac{\alpha_k(-R)(y_{k+1}^2 - \beta^2\Delta^2)}{\gamma + y_k^2},$$

from (5) it results:

$$S_N \le a_0^2 - a_{N+1}^2 \le a_0^2 \ .$$

We can conclude

$$\lim_{k\to\infty} \frac{\alpha_k (y_{k+1}^2 - \beta^2 \Delta^2)}{\gamma + y_k^2} = 0 \ .$$

In view of the hypothesis $|y_k| \le M \ \forall k$, and from the fact

$$\alpha_k (y_{k+1}^2 - \beta^2 \Delta^2) \ge 0 \ ,$$

we have the following result:

$$\lim_{k\to\infty} \alpha_k (y_{k+1}^2 - \beta^2 \Delta^2) = 0 \ .$$

As

$$\alpha_k (y_{k+1}^2 - \beta^2 \Delta^2) \ge \alpha (y_{k+1}^2 - \beta^2 \Delta^2) \qquad \forall k \ ,$$

we have that

$$0 = \lim_{k\to\infty} \alpha_k (y_{k+1}^2 - \beta^2 \Delta^2) \ge \limsup_{k\to\infty} \alpha (y_{k+1}^2 - \beta^2 \Delta^2) \ ,$$

and from there we obtained:

$$\limsup_{k\to\infty} |y_k| \le \beta \Delta \ .$$

To complete the proof of the Proposition we have to demonstrate (5) .

From the fact that

$$\alpha_k > 0 \Rightarrow y_{k+1}^2 > \beta^2 \Delta^2 \ ,$$

and under the hypothesis $|b| M \le (\beta/2)\varepsilon\Delta$, it results:

$$\varepsilon |y_{k+1}| > \beta\varepsilon\Delta \geq 2|b|M \geq 2|b||y_{k-1}| ,$$

that is:

$$-\alpha_k\varepsilon y_{k+1}^2 + 2\alpha_k|b||y_{-1}||y_{k+1}| \leq 0 .$$

We can now return to (4) to state:

$$a_{k+1}^2 - a_k^2 \leq \frac{y_{k+1}^2(\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k\varepsilon) + \Delta^2\alpha_k\xi}{\gamma + y_k^2} .$$

Choosing α, ξ and R verifing: $-2<\omega=\alpha+\varepsilon-2\leq -2/\beta$, $\zeta\in(0,[-\omega\beta^2+(\omega^2\beta^4-4\beta^2)^{1/2})]1/2)$ and $R\in(\omega+1/\zeta,-\zeta/\beta^2)$ we have:

$$(6) \qquad y_{k+1}^2(\alpha_k^2 - 2\alpha_k + \frac{\alpha_k}{\xi} + \alpha_k\varepsilon) + \Delta^2\alpha_k\xi \leq \alpha_k R(y_{k+1}^2 - \beta^2\Delta^2) .$$

In that case equation (5) is verified and the proof ends. ■

The width of the band where the output of the system asymptotically remains is $2\beta\Delta$, $\beta>1$. Note that if $\varepsilon<1$ we can choose $\beta<2$, while in the ideal case when there is no undermodelling (see *Goodwin-Sin [84]*) the band width is 4Δ.

The following Proposition shows that the result of Proposition 1 cannot be improved in the sense of obtaining a narrower band, that is $\beta\leq 1$.

<u>Proposition 2</u> : Under the hypothesis of the previous sections, we cannot choose $\beta\leq 1$ so that equation (6) holds.

Proof:

Let us suposse that equation (6) holds for $\beta=1$. In that case we will have:

$$(7) \qquad \alpha_k^2-2\alpha_k+\frac{\alpha_k}{\xi}+\alpha_k\varepsilon-\alpha_k R+\frac{\Delta^2}{y_{k+1}^2}(\alpha_k\xi+\alpha_k R\beta^2) \le 0 .$$

As $A_k \triangleq \Delta^2/y_{k+1}^2$ can be as near to 0 as wanted, we obtained:

$$\alpha_k^2-2\alpha_k+\frac{\alpha_k}{\xi}+\alpha_k\varepsilon-\alpha_k R \le 0 .$$

We have two possibilities:
i)$\xi+R \le 0$
ii)$\xi+R > 0$.
If i) holds we will have:

$$0 \ge \omega+\frac{1}{\xi}-R \ge \omega+\frac{1}{\xi}+\xi \ge 2+\omega > 0 ,$$

so the only possibility which last is ii). Remembering that (7) must hold for every A_k, we take A_k converging to 1 and always less than 1. The expression appearing in (7) converges to $a^2+\varepsilon a-2a+a/\xi-Ra+\xi a+Ra=a(\omega+1/\xi+\xi)>0$. On the other hand as the expression in (7) is always negative, its limit must also be less than or equal to 0. With this contradiction we prove that β cannot be 1.

If β is less than 1 we will have:

$$\alpha_k(-R)(y_{k+1}^2-\beta^2\Delta^2) \ge \alpha_k(-R)(y_{k+1}^2-\Delta^2) ,$$

and this contradicts what we have just proved for $\beta=1$. ■

Conclusions

Proposition 1 demonstrates that under appropiated hypothesis a slight modification to the standard dead zone algorithm enables the system to tolerate both: external bounded perturbation and undermodelling. The hypothesis relate the magnitude of $|b|$ (the undermodelled parameter) with the upper bounds of the disturbance and that of the output of the system. The result of the Proposition 1 can be resumed by saying that if $|b|M<\varepsilon(\beta/2)\Delta$ we can choose a_k so that the output y_k of the closed loop system (3) asymptotically remains between $-\beta\Delta$ and $\beta\Delta$ while a_k^2 is a decreasing sequence. In this way, the result appearing in Proposition 1 generalizes that of the dead-zone algorithm for linear, discrete-time systems with external bounded perturbation to the case where we also have inaccurate modelling. It is worth noting that if $\varepsilon<1$ we can choose $\beta<2$. With this choice we ensure that the output of the system will asymptotically remain within a band of width $2\beta\Delta$, a narrower one than that of the general case of complete model(see *Goodwin-Sin [84]*).

Proposition 2 shows that the band width obtained in Proposition 1 is the best one in this scheme, in the sense that $\beta > 1$.

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