ON THE COMPUTATION OF THE BELTRAMI EQUATION IN THE COMPLEX PLANE

By

Prabir Daripa

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Prabir Daripa
Division of Applied Mathematics
Department of Mathematics
Texas A&M University
College Station, TX 77843

Abstract

In this paper we derive some results that can be used to improve the computational methods of solving the Beltrami equation. This equation arises in the study of partial differential equations and many physical problems of interest may be reduced to this form. A substantial part of the computational effort is spent in the evaluation of the integral solution of this equation. The ideas and results presented here can be used to speed up this computation. Details of its implememtation and applications to various practical problems will appear elsewhere.

1. Introduction.

In many branches of computational physics, one encounters the following problem [1,2]: solve

$$u_{\bar{\sigma}} = f(\sigma, \bar{\sigma}); \quad |\sigma| \le 1, \tag{1}$$

subject to the data,

$$\text{Real}[u(\sigma = e^{i\alpha})] = u_0(\alpha); \qquad 0 \le \alpha \le 2\pi$$
$$\text{Imag}[u(\sigma = 0)] = v_0.$$

Here the non-analytic function f is assumed to satisfy a Hölder condition with exponent α in a circle Ω: $|\sigma| \le 1$. This equation is known as the Beltrami equation.

In [3], an iterative algorithm has been proposed to solve this problem which is based on splitting the solution of (1) as

$$u(\sigma) = u_p(\sigma) + u_a(\sigma) \tag{2}$$

where $u_a(\sigma)$ is the analytic part of the solution and $u_p(\sigma)$ is the non-analytic part of the solution. The non-analytic part $u_p(\sigma)$ is then given by

$$u_p(\sigma) = -\frac{1}{\pi}\int\int_{\Omega} \frac{f(\zeta)}{\zeta - \sigma} d\xi d\eta \tag{3}$$

where $\zeta = \xi + i\eta$.

To implement the iterative algorithm one needs to compute u_p at all N^2 grid points for σ, where N is the number of grid point in each of the radial and the angular directions. The straight forward computation of $u_p(\sigma)$ at each point requires performing an integral with an operation count of the order of $O(N^2)$, giving a net operation count of $0(N^4)$ for N^2 points. Because this method is computationally intensive, the process of evaluation has been optimized in this paper giving an operation count of the order of $0(N^2 \ln N)$, a reduction of over two thousand times for N=128. Besides, our algorithm has the advantage of working in place, meaning that no additional memory storage is required beyond that of the initial data.

Our method is basically a recursive routine in Fourier space that divides the the entire domain (the interior of the unit circle) into a collection of annular regions and expands the integral in Fourier series in angular direction with radius dependent Fourier coefficients. Appropriate one dimensional integrals are evaluated only once in each annular region which are then scaled and combined to produce the desired radius dependent Fourier coefficients. The desired integrals at all N^2 gridpoints are then easily obtained by the FFT (Fast Fourier Transform).

2. Mathematical foundation of the algorithm.

In the following we use the notations Ω_r: $|\sigma| \leq r < 1$, $\Omega_{\bar{r}}$: $\Omega \backslash \Omega_r$, and Ω_{ij}: $r_i \leq |\sigma| \leq r_j$. The following theorem is crucial for the later development of the algorithm.

Theorem 2.1. The particular solution of $u_{\bar{\sigma}} = f(\sigma)$ with $\sigma = re^{i\alpha}$, can be written as,

$$u_p(\sigma) = \sum_{n=-\infty}^{\infty} c_n(r)e^{in\alpha} \tag{4}$$

where

$$c_n(r) = \begin{cases} \frac{1}{\pi}\int\int_{\Omega_r} f(\zeta)(\frac{r}{\zeta})^n(\frac{1}{\zeta})d\xi d\eta; & n < 0 \\ -\frac{1}{\pi}\int\int_{\Omega_{\bar{r}}} f(\zeta)(\frac{r}{\zeta})^n(\frac{1}{\zeta})d\xi d\eta; & n \geq 0. \end{cases} \tag{5}$$

Proof. We expand the particular solution (3) in the fourier series, i.e.

$$-\frac{1}{\pi}\int\int_{\Omega} \frac{f(\zeta)}{\zeta - re^{i\alpha}} d\xi d\eta = \sum_{n=-\infty}^{\infty} c_n(r)e^{in\alpha}. \tag{6}$$

It follows that

$$\begin{aligned} -\pi c_n(r) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\alpha}\left(\int\int_{\Omega} \frac{f(\zeta)}{\zeta - re^{i\alpha}} d\xi d\eta\right) d\alpha \\ &= \frac{1}{2\pi}\int\int_{\Omega} f(\zeta)P_n(r,\zeta)d\xi d\eta \end{aligned} \tag{7}$$

where

$$P_n(r,\zeta) = \int_0^{2\pi} \frac{e^{-in\alpha}}{\zeta - re^{i\alpha}} d\alpha. \tag{8}$$

The integral in (8) can be evaluated using complex variable. It is an elementary exercise in complex variable to show that,

$$P_n(r,\zeta) = -2\pi r^n S_n(\zeta) \tag{9}$$

where

$$S_n(\zeta) = -\delta(n)\zeta^{-(n+1)} + \begin{cases} \zeta^{-(n+1)}; & |\zeta| < |\sigma| \\ 0.5\zeta^{-(n+1)}; & |\zeta| = |\sigma| \\ 0; & |\zeta| > |\sigma|. \end{cases} \tag{10}$$

In (10), $\delta(n) = 0$ for $n < 0$ and $\delta(n) = 1$ for $n \geq 0$. Substitution of (9) in (7) gives,

$$c_n(r) = \frac{r^n}{\pi} \int\int_{\Omega} f(\zeta) S_n(\zeta) d\xi d\eta. \tag{11}$$

Substitution of (10) in (11) yields the desired result , i.e. (5).

Remark: The above theorem can also be derived by an alternative approach (probably easier): expand the $\frac{1}{\zeta-\sigma}$ of the integrand in (3) in an appropriate manner.

Corollary 2.1. Suppose that $\zeta = \rho e^{i\theta}$ and

$$f(\zeta) = \sum_{n=-\infty}^{\infty} f_n(\rho) e^{in\theta}. \tag{12}$$

Then the coeffiecients $c_n(r)$ in (5) can be written as

$$c_n(r) = \begin{cases} \frac{1}{\pi} \int_0^r f_{n+1}(\rho)(\frac{r}{\rho})^n d\rho; & n < 0 \\ -\frac{1}{\pi} \int_r^1 f_{n+1}(\rho)(\frac{r}{\rho})^n d\rho; & n \geq 0. \end{cases} \tag{13}$$

Proof. Note first that $d\xi d\eta = \rho d\rho d\theta$ and $\zeta = \rho e^{i\theta}$. Substituting these in (5) we obtain,

$$c_n(r) = \begin{cases} \frac{1}{\pi} \int\int_{\Omega_r} f(\zeta)(\frac{r}{\rho})^n e^{-i(n+1)\theta} d\theta d\rho; & n < 0 \\ -\frac{1}{\pi} \int\int_{\Omega_{\bar{r}}} f(\zeta)(\frac{r}{\rho})^n e^{-i(n+1)\theta} d\theta d\rho; & n \geq 0. \end{cases} \tag{14}$$

The above corollary (13) now follows from (12) and (14).

Corollary 2.2. It follows directly from (13) that $c_n(1) = 0$ for $n \geq 0$, $\quad c_n(0) = 0 \; \forall \, n \neq 0$. Similarly it follows directly from (12) that $f_n(0) = 0$ for $n \neq 0$, and $f_0(0) = f(0)$.

Corollary 2.3. Let $r_j > r_i$. Define

$$c_n^{ij} = \frac{1}{\pi}\int_{r_i}^{r_j} f_{n+1}(\rho)(\frac{r_i}{\rho})^n d\rho \tag{15}$$

and

$$c_n^{ji} = \frac{1}{\pi}\int_{r_i}^{r_j} f_{n+1}(\rho)(\frac{r_j}{\rho})^n d\rho. \tag{16}$$

Then

$$c_n(r_j) = (\frac{r_j}{r_i})^n c_n(r_i) + c_n^{ji}; \quad n < 0 \tag{17}$$

and

$$c_n(r_i) = (\frac{r_i}{r_j})^n c_n(r_j) - c_n^{ij}; \quad n \geq 0. \tag{18}$$

Proof. For $n < 0$, we have from (13)

$$\begin{aligned} c_n(r_j) &= \frac{1}{\pi}\int_0^{r_j} f_{n+1}(\rho)(\frac{r_j}{\rho})^n d\rho \\ &= \frac{1}{\pi}\int_0^{r_i} f_{n+1}(\rho)(\frac{r_i}{\rho})^n(\frac{r_j}{r_i})^n d\rho + \frac{1}{\pi}\int_{r_i}^{r_j} f_{n+1}(\rho)(\frac{r_j}{\rho})^n d\rho \\ &= (\frac{r_j}{r_i})^n c_n(r_i) + c_n^{ji}. \end{aligned} \tag{19}$$

For $n \geq 0$, we have from (13)

$$\begin{aligned} c_n(r_i) &= -\frac{1}{\pi}\int_{r_i}^{1} f_{n+1}(\rho)(\frac{r_i}{\rho})^n d\rho \\ &= -\frac{1}{\pi}\int_{r_i}^{r_j} f_{n+1}(\rho)(\frac{r_i}{\rho})^n d\rho - \frac{1}{\pi}\int_{r_j}^{1} f_{n+1}(\rho)(\frac{r_i}{\rho})^n d\rho \\ &= (\frac{r_i}{r_j})^n c_n(r_j) - c_n^{ij}. \end{aligned} \tag{20}$$

Corollary 2.4. Let $0 = r_1 < r_2 < r_3 .... < r_m = 1$, $c_n^{i,i-1} = 0$ if $i \leq 1$ and $c_n^{i,i+1} = 0$ if $i \geq m$. Then $c_n(r_l)$ can be written as

$$c_n(r_l) = \begin{cases} \sum_{i=1}^{l}(\frac{r_l}{r_i})^n c_n^{i,i-1}; & n < 0 \\ -\sum_{i=m}^{l}(\frac{r_l}{r_i})^n c_n^{i,i+1}; & n \geq 0. \end{cases} \tag{21}$$

Proof. From (17) we have for $n < 0$,

$$
\begin{aligned}
c_n(r_l) &= (\frac{r_l}{r_{l-1}})^n c_n(r_{l-1}) + c_n^{l,l-1} \\
&= (\frac{r_l}{r_{l-1}})^n [(\frac{r_{l-1}}{r_{l-2}})^n c_n(r_{l-2}) + c_n^{l-1,l-2}] + c_n^{l,l-1} \\
&= (\frac{r_l}{r_{l-2}})^n c_n(r_{l-2}) + (\frac{r_l}{r_{l-1}})^n c_n^{l-1,l-2} + c_n^{l,l-1} \\
&= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= (\frac{r_l}{r_1})^n c_n(r_1) + (\frac{r_l}{r_2})^n c_n^{2,1} + \dots + (\frac{r_l}{r_{l-1}})^n c_n^{l-1,l-2} + c_n^{l,l-1} \\
&= \sum_{i=2}^{l} (\frac{r_l}{r_i})^n c_n^{i,i-1}; \quad l = 2, 3, \dots, m.
\end{aligned}
\tag{22}
$$

We have used $c_n(r_1 = 0) = 0$ from Corollary 2.2 to obtain (22). From (18) we have for $n \geq 0$,

$$
\begin{aligned}
c_n(r_l) &= (\frac{r_l}{r_{l+1}})^n c_n(r_{l+1}) - c_n^{l,l+1} \\
&= (\frac{r_l}{r_{l+1}})^n [(\frac{r_{l+1}}{r_{l+2}})^n c_n(r_{l+2}) - c_n^{l+1,l+2}] - c_n^{l,l+1} \\
&= (\frac{r_l}{r_{l+2}})^n c_n(r_{l+2}) - (\frac{r_l}{r_{l+1}})^n c_n^{l+1,l+2} - c_n^{l,l+1} \\
&= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= (\frac{r_l}{r_m})^n c_n(r_m) - (\frac{r_l}{r_{m-1}})^n c_n^{m-1,m} - \dots - (\frac{r_l}{r_{l+1}})^n c_n^{l+1,l+2} - c_n^{l,l+1} \\
&= - \sum_{i=m-1}^{l} (\frac{r_l}{r_i})^n c_n^{i,i+1}; \quad l = m-1, m-2, \dots, 1.
\end{aligned}
\tag{23}
$$

We have used $c_n(r_m = 1) = 0$ from Corollary 2.2 to obtain (23).

3. The Fast Algorithm.

Initialization.

Choose M and N. Define $K = \frac{N}{2}$.

Step 1

For $l \in [1, M]$ and $n \in [-K+1, K]$, compute the Fourier coefficients $f_n(r_l)$ of $f(\zeta)$ from the known values of $f(\zeta = r_j e^{\frac{2\pi i k}{N}}), j = 1, .., M; k = 1, ..., N$.

Step 2

Compute $c_n^{i,i+1}, i \in [1, M-1]$ for $n \in [0, K-1]$ using (15) and $c_n^{i,i-1}, i \in [2, M]$ for $n \in [-K, -1]$ using (16).

Step 3

Compute the Fourier coefficients $c_n(r_l); n \in [-K, K-1], l \in [1, M]$ using the relations (17) and (18).

set $c_n(r_M) = 0 \forall n \in [0, K]$

do n = 0,.....,K

  do l = M-1,.....,1

  Use (18) of Corollary 2.3 to compute $c_n(r_l)$.

  $c_n(r_l) = (\frac{r_l}{r_{l+1}})^n c_n(r_{l+1}) - c_n^{l,l+1}$

  enddo

enddo

set $c_n(r_1) = 0 \ \forall \ n \in [-K, -1]$

do n =$-K$,....,1

  do l = 2,.....,M

  Use (17) of Corollary 2.3 to compute $c_n(r_l)$.

  $c_n(r_l) = (\frac{r_l}{r_{l-1}})^n c_n(r_{l-1}) + c_n^{l,l-1}$

  enddo

enddo

Step 4

Finally compute $u_p(\sigma = r_j e^{\frac{2\pi i k}{N}}); j \in [1, M], k \in [1, N]$ using (4).

The Algorithmic Complexity.

Here we consider the computational complexity of the above algorithm. We discuss the asymptotic operation count, the asymptotic time complexity and asymptotic storage requirement in order. A brief analysis of the algorithmic complexity follows:

Step	Operation Count	Explanation
1	0(MNlnN)	Each set of discrete Fourier transform of N data sets contribute NlnN operations. There are M such discrete Fourier transforms.
2	0(MN)	For each n, computations of $c_n^{i,i+1}, i \in [1, M-1]$ and $c_n^{i,i-1}, i \in [2, M]$ contribute 2M operations. There are n such computations for $n \in [-K+1, K]$.
3	0(MN)	Computation of each $c_n(r_l)$ takes one operation. There are 2MN such computations.
4	0(MNlnN)	Computation of $u_p(\sigma = r_j e^{i\alpha_k}), k \in [1, N]$ for each fixed 'j' by FFT contributes $N \ln N$ operations. There are M such FFTs to be performed.

From the above table we see that the asymptotic operation count and hence the asymptotic time complexity is 0(MNlnN).

The algorithm requires to store the MN Fourier coefficients $f_n(r_l)$ in step 1, the MN Fourier coefficients $c_n(r_l)$ in step 3 and the MN values of the desired u_p at MN grid points in step 4. Therefore the asymptotic storage requirement is 0(MN).

Remark: The computation c_n^{ij} in step 2 can be embedded within the inner do-loops of step 3, thus avoiding the storage requirement for these. Note, we present the algorithm in the form as shown above, for the sake of clarity and without any sacrifice in the asymptotic time complexity.

The implementaion and its applications to various applied problems will be reported elsewhere.

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