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ABSTRACT

In this paper we introduce the concept of an inertial manifold for nonlinear evolutionary equations, in particular for ordinary and partial differential equations. These manifolds, which are finite dimensional invariant Lipschitz manifolds, seem to be an appropriate tool for the study of questions related to the long-time behavior of solutions of the evolutionary equations. The inertial manifolds contain the global attractor, they attract exponentially all solutions, and they are stable with respect to perturbations. Furthermore, in the infinite dimensional case they allow for the reduction of the dynamics to a finite dimensional ordinary differential equation.

0. INTRODUCTION

The study of the long time behavior of solutions of evolutionary equations is a problem of interest in many areas of mathematical physics and mechanics. This is particularly true when the effects of the nonlinear terms become significant. It is well known that, even for an autonomous equation, the long time behavior can indeed be time dependent, since the solutions do not necessarily converge to stationary states.

The first natural tool for describing the dynamics of non-stationary "permanent" flows is the concept of the global attractor for dissipative evolutionary equations. We are uncertain when the global attractor for a flow on an infinite dimensional space was first studied, but it was known to Billotti-LaSalle (1971). The finite dimensionality of the global attractor was proved by Mallet-Paret (1976) for certain flows on a Hilbert space, and by Mane (1981) in the Banach space setting. More detailed studies of the global attractor, including good estimates of its Hausdorff or fractal dimensions, for the Navier-Stokes equations and related dynamics systems can be found in Foias-Temam (1979), Babin-Vishik (1982), Constantin-Foias-Manley-Temam (1983), Hale (1984), Nicolaenko-Scheurer-Temam (1985), Temam (1985), and Foias-Manley-Temam (1986).

The concept of an inertial manifold, which we introduce in this article, allows one to go deeper into the study of dissipative systems. These manifolds, which are finite dimensional, will contain the universal attractor and, most importantly, they are invariant. Furthermore, the ambient dynamics of the evolutionary equation, when restricted to the inertial manifold, reduces to a finite dimensional ordinary differential equation, which we call an inertial form of the given evolutionary equation. We believe that the inertial manifold and the corresponding inertial forms will prove to be useful tools for the investigation of the finite dimensional behavior of dissipative equations.

We consider here a class of nonlinear evolutionary equations of the type

$$\frac{du}{dt} + Au + R(u) = 0, \tag{0.1}$$

where A is a suitable linear, unbounded, self-adjoint operator on a Hilbert space H, while R is a nonlinear operator which (in a sense to be made precise later) is "dominated" by A. Let $S(t)\colon u(0) \to u(t)$ be the semigroup of operators defining the solutions of (0.1).

For the class of equations (0.1) that we consider, we prove the existence of a finite dimensional Lipschitz manifold $\mathcal{M}$, which is invariant (i.e. $S(t)\mathcal{M} \subseteq \mathcal{M}$, for all $t > 0$). This manifold is constructed as the graph of a Lipschitz function Φ mapping PH into QH, where P is an orthogonal projection with finite dimensional range in H and $Q = I - P$. The function Φ is obtained as the fixed point of an appropriate mapping $\mathcal{J}$, which is introduced in Section 2. The manifold $\mathcal{M}$ contains the universal attractor of (0.1) since it attracts exponentially all the trajectories. The proof of the last phenomenon relies on the Squeezing Property of the semigroup $S(t)$, a property which was introduced and studied in Foias-Temam (1979) and Constantin-Foias-Temam (1985). Another property of $\mathcal{M}$, which is proved here, is its stability with respect to perturbations: For simplicity we restrict ourselves to the perturbations of (0.1) corresponding to its Galerkin approximations, i.e.

$$\frac{du_M}{dt} + A_M u_M + R_M(u_M) = 0. \tag{0.2}$$

It is found that there exists $M_0 > 0$ such that for every $M > M_0$ (0.2) possess an inertial manifold $\mathcal{M}_M$, which is the graph above PH of a Lipschitz function Φ_M. Furthermore Φ_M converges to Φ as $M \to \infty$. In particular one has $\dim \mathcal{M}_M = \dim \mathcal{M}$ for $M > M_0$.

It is reasonably accurate to describe an inertial manifold as a global center (or center-unstable) manifold, cf. Hartman (1964), Pliss (1964), Kelley (1967), Chaffee (1968), Ball (1973), and other references in Carr (1981). The methods which we use to construct the inertial manifolds are adaptation of various theories found in the study of center manifolds and integral manifolds, cf. Sacker (1964), Hale (1968) and the references cited above. Since we wish to study nonlinearities which involve (lower order) spatial derivatives, the term $R(u)$ need not be smooth, nor Lipschitz continuous, nor even continuous in the given norm on H. Consequently the theory we present here has essential differences, which do not appear in the references cited above.

Let us now turn to the organization of the paper. In Section 1 we describe the assumptions which we make on the terms in (0.1). These assumptions will insure that (0.1) is dissipative and that there is a global attractor which is compact and has finite Hausdorff dimension. Since the nonlinear terms in (0.1) may behave badly at infinity, it is appropriate to modify (0.1). The modifications which we make do not change (0.1) in the vicinity of the global attractor. Consequently the global attractor for (0.1) is a compact invariant set for the modified equation. The modification of the nonlinear terms in the infinite dimensional equation (0.1) is similar to other modifications one finds in the theory of dynamical systems in the study of center and center-unstable manifolds, cf. Hartman (1964), Kelley (1967), Sell (1978) and the other references cited above. The statement of the Squeezing Property also appears in Section 1.

In Section 2 we properly define the inertial manifold and we show how the existence of this manifold can be reduced to a fixed point property for a mapping $\mathcal{T}$. We conclude this section by stating our main results, i.e. the existence and uniqueness of the fixed point. In Section 3 we study the properties of $\mathcal{T}$ and prove in particular that the contraction fixed point theorem is applicable. In Section 4 we consider the approximation problem and prove that $\mathcal{M}$ is stable with respect to perturbations. In the next section we study further properties of the inertial manifolds, including some alternate characterizations as well as a comparison between two inertial manifolds with different dimensions. In Section 6 we describe some examples including a modified Navier-Stokes equation with an artificial viscosity term involving a power of the operator $(-\Delta)$ for an arbitrary space dimension. (It should be emphasized that, even in space dimension 2, the Navier-Stokes equation itself does not enter the class of equations considered here.) We conclude the paper with a discussion of some open problems in the theory of inertial manifolds.

In a subsequent paper we will study the application to another equation (i.e. the Kuramoto-Sivashinsky equation) of the general results presented here, Foias-Nicolaenko-Sell-Temam (1986). The results proved here were announced in a previous Note in the Comptes Rendus, Foias-Sell-Temam (1985).

Several persons, while not presenting the general theory of inertial manifolds which we give in this paper, have investigated related questions. Special mention should be made of the work of Kurzweil (1970) for differential-delay equations with small time-delays and the theory of Conway-Hoff-Smoller (1978) for reaction-diffusion equations with high diffusivity. Other references can be found in Hale-Magalhaes-Oliva (1984).

We are grateful to Professor Jack Hale for bringing to our attention some of the references cited above. We express our sincere appreciation to Dr. E. Titi for his careful reading of the manuscript and his helpful suggestions.

1. PRELIMINARY RESULTS

1.1 The Evolution Equation

Assume we are given a Hilbert space H with an inner (or scalar) product $(\cdot,\cdot)$. The nonlinear evolution equation which we will study has the form

$$\frac{du}{dt} + Au + R(u) = 0 \tag{1.1}$$

where

$$R(u) = B(u,u) + Cu - f. \tag{1.2}$$

The term Au is a linear unbounded self-adjoint operator on H with domain $D(A)$ dense in H. We assume that A is positive, i.e.

$$(Av,v) > 0, \qquad \text{for all} \quad v \in D(A), \quad v \neq 0$$

and that A^{-1} is compact. This implies that the mapping $u \to Au$ is an isomorphism from $D(A)$ (equipped with the graph norm) onto H. Recall that under these hypotheses one can define the powers A^s of A for $s \in R$ and that the space $V_{2s} := D(A^s)$ is a Hilbert space for the scalar product

$$(u,v)_{2s} = (A^s u, A^s v), \qquad u,v \in D(A^s).$$

For $u \in V_s$ we set $|u|_s = (u,u)_s^{1/2}$.

Since A^{-1} is compact and self-adjoint there exists an orthonormal basis $\{w_j\}$ of H consisting of eigenvectors of A,

$$Aw_j = \lambda_j w_j \tag{1.3}$$

where the eigenvalues satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \to \infty \quad \text{as} \quad j \to \infty. \tag{1.4}$$

It follows easily from (1.3) and (1.4) that

$$(1.5) \qquad |A^{1/2}u| \geq \lambda_1^{1/2}|u|, \qquad \text{for all} \quad u \in D(A^{1/2})$$

$$(1.6) \qquad |A^{p+1/2}u| \geq \lambda_1^{1/2}|A^p u|, \qquad \text{for all} \quad u \in D(A^{p+1/2}), \text{ and all } p.$$

For $N = 1,2,\ldots$ we let P_N denote the orthogonal projection onto $\mathrm{Span}\{w_1,\ldots,w_N\}$ and let $Q_N = I - P_N$.

The nonlinear term $R(u)$ satisfies (1.2) where $B(u,v)$ is a bilinear operator from $D(A) \times D(A)$ into H, C is a linear operator from $D(A)$ into H and $f \in D(A^{1/2})$. Furthermore we assume that

$$(1.7) \qquad (B(u,v),v) = 0, \qquad \text{for all} \quad u,v \in D(A),$$

$$(1.8) \qquad |B(u,v)| \leq c_1|u|^{1/2}|A^{1/2}u|^{1/2}|A^{1/2}v|^{1/2}|Av|^{1/2}, \qquad \text{for all} \quad u,v \in D(A),$$

$$(1.9) \qquad |Cu| \leq c_2|A^{1/2}u|^{1/2}|Au|^{1/2}, \qquad \text{for all} \quad u \in D(A),$$

where c_1, c_2, and the c_i's appearing in the sequel are positive constants. In addition we assume the following continuity properties for B and C:

$$(1.10) \qquad |A^{1/2}B(u,v)| \leq c_3|Au||Av|, \qquad \text{for all} \quad u,v \in D(A),$$

$$(1.11) \qquad |A^{1/2}Cu| \leq c_4|Au|, \qquad \text{for all} \quad u \in D(A),$$

Finally it is assumed that $A + C$ is positive, i.e.

$$(1.12) \qquad ((A + C)u,u) \geq \alpha|A^{1/2}u|^2, \qquad \text{for all} \quad u \in D(A),$$

where $\alpha > 0$.

<u>Remark</u> 1.1 Since A is self-adjoint (1.8) follows from the stronger condition

$$|B(u,v)| \leq c_1|A^{1/4}u| \; |A^{3/4}v|.$$

By using (1.6) we see that (1.9) is a consequence of

$$|Cu| \leq c_2 \lambda_1^{1/4} |A^{1/2}u|.$$

The latter inequalities imply that B is continuous from $D(A^{1/4}) \times D(A^{3/4})$ into H while C is continuous from $D(A^{1/2})$ into H.

Eventhough B and C may be discontinuous in terms of the given norm $|\cdot|$ on H, the above inequalities mean that that B and C are suitably dominated by A. The assumptions made above insure that the initial value problem for (1.1) with initial condition

$$u(0) = u_0, \quad u_0 \in H \tag{1.13}$$

has a unique solution $S(t)u_0$ defined for all $t > 0$ and $S(t)u_0 \in D(A)$ for all $t > 0$. The mapping $S(t)$ enjoys the usual semigroup property and several continuity properties which will not be recalled here.

The solution $S(t)u_0$ is uniformly bounded in time in H, and if $u_0 \in D(A^{1/2})$ (resp. $u_0 \in D(A)$) then $S(t)u_0$ is uniformly bounded in $D(A^{1/2})$ (resp. $D(A)$). We will not prove these classical results here but instead we refer the reader to Temam (1983). However we will present the a priori estimates which play an essential role in the proof of such results.

These estimates are obtained by taking the scalar product of (1.1) respectively with u, Au and A^2u and making use of (1.3)-(1.12). We will also use the following form of the Hölder inequality:

$$\Sigma|x_i y_i| = \Sigma|\beta x_i||\beta^{-1} y_i| \leq \frac{\beta^p}{p} (\Sigma|x_i|^p) + \frac{\beta^{-q}}{q} (\Sigma|y_i|^q) \tag{1.14}$$

where $\beta > 0$, $p + q = pq$ and $1 < p,q < \infty$. In the first case we obtain

$$\frac{1}{2}\frac{d}{dt} |u|^2 + \alpha|A^{1/2}u|^2 \leq |(f,u)| \leq \lambda_1^{-1/2}|f||A^{1/2}u|$$

$$\leq \frac{\alpha}{2} |A^{1/2}u|^2 + \frac{1}{2\lambda_1 \alpha} |f|^2.$$

where (1.14) is used in the last inequality. Consequently one has

(1.15) $$\frac{d}{dt}|u|^2 + \alpha\lambda_1|u|^2 \leq \frac{d}{dt}|u|^2 + \alpha|A^{1/2}u|^2 \leq \frac{1}{\alpha\lambda_1}|f|^2.$$

When taking the scalar product of (1.1) with Au we use (1.8) and (1.9) to obtain

$$|(B(u,u) + Cu,Au)| \leq c_1|u|^{1/2}|A^{1/2}u||Au|^{3/2} + c_2|A^{1/2}u|^{1/2}|Au|^{3/2}$$
$$\leq 49(c_1^4|u|^2|A^{1/2}u|^4 + c_2^4|A^{1/2}u|^2) + \frac{1}{4}|Au|^2,$$

where the last inequality follows from (1.14) with $p = 4$, $q = 4/3$, $\beta^4 = 196$. Similarly one has

$$|(f,Au)| \leq |f||Au| \leq 2|f|^2 + \frac{1}{4}|Au|^2.$$

Consequently we get

(1.16) $$\frac{d}{dt}|A^{1/2}u|^2 + \lambda_1|A^{1/2}u|^2 \leq \frac{d}{dt}|A^{1/2}u|^2 + |Au|^2$$
$$\leq c_6|u|^2|A^{1/2}u|^4 + c_7|A^{1/2}u|^2 + 4|f|^2$$

(The new constants c_6 and c_7, as well as others which are used below, are derived easily from the other c_i's.) Finally the scalar product with A^2u uses (1.10) and (1.11) to yield

$$\frac{1}{2}\frac{d}{dt}|Au|^2 + |A^{3/2}u|^2 \leq |(B(u,u) + Cu,A^2u)| + |(f,A^2u)|$$
$$\leq |A^{1/2}(B(u,u) + Cu),A^{3/2}u)| + |(A^{1/2}f,A^{3/2}u)|$$
$$\leq c_3|Au|^2|A^{3/2}u| + c_4|Au||A^{3/2}u| + |A^{1/2}f||A^{3/2}u|$$
$$\leq \frac{1}{2}c_8|Au|^4 + \frac{1}{2}c_9|Au|^2 + \frac{3}{2}|A^{1/2}f|^2 + \frac{1}{2}|A^{3/2}u|^2$$

where the last inequality follows from (1.14) with $p = q = 2$, $\beta^2 = 3$.

Consequently one has

$$\text{(1.17)} \qquad \frac{d}{dt}|Au|^2 + \lambda_1|Au|^2 \le \frac{d}{dt}|Au|^2 + |A^{3/2}u|^2$$

$$\le c_8|Au|^4 + c_9|Au|^2 + 3|A^{1/2}f|^2.$$

Inequalities (1.15)-(1.17) provide the a priori estimates which are necessary for proving the existence and regularity of solutions and uniform bounds on the norms. For this purpose we shall use the following "uniform" Gronwall inequality which is proved in Foias-Manley-Temam (1986);

(1.18) *Let* g, h, y *be three positive locally integrable functions for* $t_0 \le t < +\infty$ *which satisfy*

$$\frac{dy}{dt} \le gy + h, \quad \text{for all} \quad t \ge t_0,$$

and $\int_t^{t+1} g\,ds \le \alpha_1$, $\int_t^{t+1} h\,ds \le \alpha_2$, $\int_t^{t+1} y\,ds \le \alpha_3$ *for all* $t \ge t_0$, *where* α_1, α_2, α_3 *are positive constants. Then*

$$y(t+1) \le (\alpha_3 + \alpha_2)\exp(\alpha_1), \quad \text{for all} \quad t \ge t_0.$$

The standard Gronwall inequality applies to (1.15) and we obtain for $u(t) = S(t)u_0$

$$\text{(1.19)} \qquad |u(t)|^2 \le |u(0)|^2\exp(-\alpha\lambda_1 t) + \rho_0^2(1 - \exp(-\alpha\lambda_1 t))$$

where $\rho_0 = \frac{1}{\alpha\lambda_1}|f|$. Hence $|u(t)|$ is uniformly bounded in H and

$$\text{(1.20)} \qquad \limsup_{t\to\infty} |u(t)|^2 \le \rho_0^2.$$

It then follows from (1.15) that $\int_t^{t+1}|A^{1/2}u(s)|^2 ds$ is uniformly bounded. Furthermore (1.16) implies that

$$\frac{d}{dt}|A^{1/2}u|^2 < c_{10}|A^{1/2}u|^4 + (c_7 - \lambda_1)|A^{1/2}u|^2 + 4|f|^2$$

where $c_{10} = c_6 b_0^2$ and $|u(t)|^2 < b_0^2$ for all $t > 0$. It then follows from (1.18) with $g = c_{10}|A^{1/2}u|^2$, $y = |A^{1/2}u|^2$ and $h = (c_7 - \lambda_1)|A^{1/2}u|^2 + 4|f|^2$ that $|A^{1/2}u(t)|^2$ is uniformly bounded in H. Returning to (1.16) we see that $\int_t^{t+1}|Au(s)|^2 ds$ is uniformly bounded. By iterating with (1.17) we see that $|Au(t)|^2$ and $\int_t^{t+1}|A^{3/2}u(S)|^2 ds$ are uniformly bounded in t. In particular we have

(1.21) $$\limsup_{t\to\infty} |A^{1/2}u(t)|^2 < \rho_1^2,$$

(1.22) $$\limsup_{t\to\infty} |Au(t)|^2 < \rho_2^2,$$

however explicit expressions for ρ_1 and ρ_2 are not as simple as that of ρ_0.

It follows from (1.20)-(1.22) that

(1.23) *Any solution* $S(t)u_0$ *of* (1.1) *will, after a certain time, enter a ball of* H *centered at* 0 *of radius* $> \rho_0$, *say* $2\rho_0$. *The same is true in* $D(A^{1/2})$ *and* $D(A)$ *with balls of radii* $2\rho_1$ *and* $2\rho_2$.

We denote these balls by $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2$. Due to (1.23) each ball is uniformly ultimately bounded (or point dissipative, or *absorbing*) in H, $D(A^{1/2})$, $D(A)$, respectively. The ω-limit set of $\mathcal{B}_2$:

$$\mathcal{A} = \omega(\mathcal{B}_2) := \bigcap_{s>0} C\ell\Big(\bigcap_{t>s} S(t)\mathcal{B}_2\Big) \qquad (*)$$

is the *global attractor* for (1.1), cf. Billotti-LaSalle (1971), Mallet-Paret (1976), Foias-Temam (1979), Mane (1981), and Constantin-Foias-Temam (1985). Its basin of attraction is the whole space H.   is the largest attractor for (1.1). Clearly one has

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(*) The closure $C\ell$ is taken in H.

$$\mathcal{A} \subseteq \mathcal{B}_2 \cap \mathcal{B}_1 \cap \mathcal{B}_0.$$

1.2 The Modified or Prepared Equation.

As a result of (1.18) we see that any ball in H centered at 0 with radius $> \rho_0$ is positively invariant under $S(t)$ and attracts all solutions exponentially. In particular $\mathcal{B}_0$, the ball of radius $2\rho_0$, has this feature. Unfortunately a similar property is not known in $D(A^{1/2})$ and $D(A)$. In order to avoid certain technical difficulties at infinity in $D(A)$ resulting from the nonlinear term, we borrow an idea arising in the study of invariant manifolds for ordinary differential equations and truncate this term; thereby considering a modified equation which provides the same asymptotic behavior (as $t \to \infty$) near the global attractor but is different for $|Au|$ large.

Let $\theta: R_+ \to [0,1]$ be a fixed smooth function with $\theta(s) = 1$ for $0 \leq s \leq 1$, $\theta(s) = 0$ for $s \geq 2$, and $|\theta'(s)| \leq 2$ for $s \geq 0$. Fix $\rho = 2\rho_2$ and define $\theta_\rho(s) = \theta(s/\rho)$ for $s \geq 0$. The modified equation of (1.1) is

$$\frac{du}{dt} + Au + \theta_\rho(|Au|)R(u) = 0. \tag{1.24}$$

The proof of the existence and uniqueness of solutions of (1.24) with initial condition $u(0) = u_0 \quad H$ is straightforward. The advantage of (1.24) compared to (1.1) is that (1.24) possess an absorbing invariant ball in $D(A)$, namely any ball centered at 0 of radius $>2\rho$. Indeed, take the scalar product of (1.24) with A^2u. For $|Au| \geq 2\rho$ we have

$$\frac{1}{2}\frac{d}{dt}|Au|^2 + \lambda_1|Au|^2 \leq \frac{1}{2}\frac{d}{dt}|Au|^2 + |A^{3/2}u|^2 = 0 ,$$

since $\theta_\rho(|Au|) = 0$ for $|Au| \geq 2\rho$. (Compare this with (1.17).) Thus if $|Au_0| > \rho_3$, where $\rho_3 \geq 2\rho$, the orbit $u(t)$ will converge exponentially in $D(A)$ to the ball of radius ρ_3, while if $|Au_0| \leq \rho_3$, then $u(t)$ cannot leave this ball.

Since $\theta_\rho(|Au|) = 1$ for $|Au| < \rho$ we see that (1.1) and (1.24) are identical in the $D(A)$-neighborhood of the global attractor $\mathcal{A}$ given by $\{u \in D(A): \ |Au| < \rho\}$.

A perusal of the computations leading to (1.15)-(1.17) shows that the inequalities are still valid for a solution of (1.24) instead of (1.1). Hence the asymptotic bounds (1.20)-(1.22) are still valid for solutions of (1.24) with the *same values* for ρ_0, ρ_1 and ρ_2. More generally it is useful to observe that (1.15)-(1.17), and thus (1.20)-(1.22), are still valid when $u(t)$ is a solution of an equation

$$\frac{du}{dt} + Au + g(t)R(u) = 0$$

where $g(t)$ is any continuous scalar function with $0 < g(t) < 1$ for all $t > 0$.

For the remainder of the paper we shall use $S(t)u_0$ to denote the solution of (1.24) satisfying the initial condition $u(0) = u_0 \in H$. If one has $|AS(t)u_0| < \rho$ for $t > 0$, then $S(t)u_0$ is also a solution of (1.1).

1.3 The Squeezing Property

Let $S(t)u_0$ and $S(t)v_0$ be two solutions of (1.24) where u_0, v_0 H. Let P_N and Q_N be the projections on H described above.

The following Lipschitz property for $S(t)$ is easy to prove, see Foias-Temam (1979), Constantin-Foias-Temam (1985):

(1.25) *For every* $r > 0$ *there is a* K_1, *depending on* r *and the operators, such that if* $u_0, v_0 \in D(A)$ *with* $|Au_0| < r$, *and* $|Av_0| < r$, *then one has*

$$|S(t)u_0 - S(t)v_0| < \exp(K_1 t)|u_0 - v_0|, \quad \textit{for all} \ t > 0.$$

The squeezing property, which we state below, was proved in Foias-Temam (1979), Constantin-Foias-Temam (1985) under assumptions for the operators B and C which are very close to the assumptions made here. Since the argument can be easily adapted to our case, we will not repeat the proof here.

Squeezing Property: *For every* $T > 0$, $\gamma > 0$, $r > 0$ *there exist constants* K_2, K_3 (*depending on* T, γ, r *and the constants* c_1-c_4, *but not on the explicit nature of the operators or on* N) *such that for every* t, $0 < t < T$, *and for every* u_0, v_0 *with* $|Au_0| < r$, $|Av_0| < r$ *one of the following inequalities hold for every* $N > 1$:

$$(1.26) \qquad |Q_N(S(t)u_0 - S(t)v_0)| < \gamma |P_N(S(t)u_0 - S(t)v_0)|$$

$$(1.27) \qquad |S(t)u_0 - S(t)v_0| < K_2 \exp(-K_3 \alpha \lambda_{N+1} t)|u_0 - v_0|.$$

In the sequel we will apply this result with t satisfying $t_0 < t < 2t_0$, where $t_0 := \frac{1}{2K_1} \log 2$ (See (1.25).), $\gamma = \frac{1}{8}$ and $N > N_0$ where N_0 satisfies

$$(1.28) \qquad \lambda_{N_0+1} > (2K_3 \alpha t_0)^{-1} \log(2K_2)$$

In this case, (1.26), (1.27) become

$$(1.29) \qquad |Q_N(S(t)u_0 - S(t)v_0)| < \frac{1}{8} |P_N(S(t)u_0 - S(t)v_0)|$$

$$(1.30) \qquad |S(t)u_0 - S(t)v_0| < \frac{1}{2} |u_0 - v_0|$$

with $u_0, v_0 \in D(A)$, $|Au_0| < r$, $|Av_0| < r$ and $t_0 < t < 2t_0$.

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(1) The coefficient γ does not appear in Constantin-Foias-Temam (1985). It can be easily inserted in (1.26) provided K_2 and K_3 are modified accordingly.

1.4 Locally Lipschitz Nonlinearities

With the assumptions on the nonlinear terms B and C described above, the theory which we describe here does apply when B and C are not continuous on H. For example if A is a differential operator of order $2k$, then the assumptions on B and C would permit these operators to have derivatives of order $\leq k$. We will see an illustration of this in Section 4.

Another situation, which arises in the theory of reaction-diffusion equations, occurs when the nonlinear term $R(u)$ in (1.1) is locally Lipschitz continuous on H. For example, the equation

$$u_t = \nu \Delta u + g(u)$$

with $g(u) = u(1 - u)(u - a)$, for $0 < a < 1$, and with Neumann, Dirichlet or periodic boundary conditions on a suitable bounded region $\Omega_n \subseteqq R^n$ gives rise to an abstract equation of the form

$$\frac{du}{dt} + Au + R(u) = 0 \tag{1.31}$$

on $L_2(\Omega_n)$. Furthermore (1.31) is dissipative and $R(u)$ is locally Lipshitz continuous in u. In this case, the modified equation would have the form

$$\frac{du}{dt} + Au + \theta_\rho(|u|)R(u) = 0 \tag{1.32}$$

for a suitable $\rho > 0$. Furthermore the nonlinear term $F(u) = \theta_\rho(|u|)R(u)$ would be globally Lipshitz continuous, i.e. there is a K such that

$$|F(u) - F(v)| \leq K|u - v|, \qquad \text{for all } u,v \in H. \tag{1.33}$$

We will return to this example later in the paper.

2. FORMULATION OF THE PROBLEM AND STATEMENT OF MAIN RESULTS

2.1 The Inertial Manifold $\mathcal{M}$ and the Inertial Form: The Induced Ordinary Differential Equation on $\mathcal{M}$.

We consider the solution operator $S(t)$ generated by the modified equation (1.24). A subset $\mathcal{M} \subseteq H$ is said to be an <u>inertial manifold</u> for (1.24) if it has the following three properties:

(i) $\mathcal{M}$ is a finite dimensional Lipschitz manifold,

(ii) $\mathcal{M}$ is invariant, i.e. $S(t)\mathcal{M} \subseteq \mathcal{M}$, for all $t > 0$,

(iii) $\mathcal{M}$ attracts exponentially all solutions of (1.24), i.e.

$$\operatorname{dist}(S(t)u_0, \mathcal{M}) \to 0 \quad \text{as} \quad t \to \infty \tag{2.1}$$

for every $u_0 \in H$ and the rate of decay in (2.1) is exponential, uniformly for u_0 in bounded sets in H.

Property (iii) implies that an inertial manifold must contain the universal attractor.

The next step is to show how the search for the existence of an inertial manifold can be reduced to a fixed point problem. Let P_N and Q_N be the projections described above and write $P = P_N$ and $Q = Q_N$. If $u = u(t)$ is a solution of (1.24) we define $p = p(t)$ and $q = q(t)$ by $p = Pu$, $q = Qu$. Then p,q are solutions of the following differential equations on PH and QH:

$$\frac{dp}{dt} + Ap + PF(u) = 0 \tag{2.2}$$

$$\frac{dq}{dt} + Aq + QF(u) = 0 \tag{2.3}$$

where

$$F(u) = \theta_\rho(|Au|)R(u) \tag{2.4}$$

and $u = p + q$. We have used the fact that $PA = AP$ and $QA = AQ$, which implies that $PAu = APu = Ap$ and $QAu = AQu = Aq$.

We will be looking for an inertial manifold $\mathcal{M}$ which is constructed as the graph of a Lipschitz function $\Phi: PD(A) \to QD(A)$. (Note that $PH = PD(A)$.) The function Φ will be sought as the fixed point of an operator $\mathcal{J}$ on a class $\mathcal{F}_{b,\ell}$, where b,ℓ are positive numbers to be determined. $\mathcal{F}_{b,\ell}$ is defined as the class of those functions Φ from $PD(A)$ into $QD(A)$ which satisfy

(2.5) $\quad |A\Phi(p)| \leq b, \quad$ for all $p \in PD(A)$,

(2.6) $\quad |A\Phi(p_1) - A\Phi(p_2)| \leq \ell|Ap_1 - Ap_2|, \quad$ for all $p_1,p_2 \in PD(A)$,

(2.7) $\quad \operatorname{Supp} \Phi \subseteq \{p \in PD(A): |Ap| \leq 4\rho\}$

Now $u(t) = p(t) + \Phi(p(t))$ is a solution of (1.24) if and only if $p(t)$ and $q(t) = \Phi(p(t))$ satisfy (2.2), (2.3) with $u = p + \Phi(p)$. Assume that Φ is given in $\mathcal{F}_{b,\ell}$ and $p_0 \in PD(A)$. Since Φ is Lipschitz continuous, we can integrate (2.2) with $u = p + \Phi(p)$ and $p(0) = p_0$. This defines $p(t)$ uniquely for all $t \in R$ (including $t < 0$). Furthermore if $|Ap_0| \leq 4\rho$, then $|Ap(t)| \leq 4\rho$ for all $t \geq 0$. (The last property follows from the definition of θ_ρ and the fact that $|Ap| \leq |A(p + q)| = |Au|$.) This function $p(t)$ actually depends on Φ and p_0. We will write it as $p = p(t) = p(t;\Phi,p_0)$. Of course, the truncation θ_ρ is essential for the definition of p for all $t \in R$.

The main object of analysis in this paper is the operator $\mathcal{J}$, which is an infinite integral operator on $\mathcal{F}_{b,\ell}$. The use of this infinite integral is quite old (it goes back to Lyapunov in his study of the stable manifold) and is widely used in the theory of ordinary differential equations, see for example, Coppel (1965), Kelley (1967), Hale (1968) and Sell (1978). For the benefit of readers who have not encountered this construction before, we made the following observations to motivate the definition of $\mathcal{J}$.

Given a continuous, bounded function $\sigma: R \to H$ we first note that the equation

(2.8) $$\frac{d\xi}{dt} + A\xi = \sigma$$

possess a unique solution that remains bounded as $t \to -\infty$. Indeed by integrating (2.8) between s and t, $s < t$, yields

$$\xi(t) = e^{-(t-s)A}\xi(s) + \int_s^t e^{-(t-\tau)A}\,\sigma(\tau)d\tau.$$

Now let $s \to -\infty$. From the boundedness assumption we find that, necessarily one has

$$\xi(t) = \int_{-\infty}^{t} e^{-(t-\tau)A}\,\sigma(\tau)d\tau. \tag{2.9}$$

This shows the uniqueness of ξ. It is elementary to check that the function ξ defined by (2.9) is a solution of (2.8) and it is bounded as $t \to -\infty$.

The last paragraph can now be applied to (2.3) with $\xi = q$, $\sigma = -QF(u)$, $u = p + \Phi(p)$, $\Phi \in \mathcal{F}_{b,\ell}$ and $p = p(t)$ given as above. Then $\sigma(t)$ is bounded from R into H, and $\xi(t)$ is given by (2.8) for all $t \in R$. In particular $\xi(0)$ is given by

$$\xi(0) = -\int_{-\infty}^{0} e^{\tau AQ}\,QF(p + \Phi(p))d\tau \tag{2.10}$$

where $p = p(\tau) = p(\tau;\Phi,p_0)$ is given above. Note that $\xi(0)$ depends on $\Phi \in \mathcal{F}_{b,\ell}$ and $p_0 \in PD(A)$. We have thus defined a <u>formal</u> mapping

$$\Phi \to \mathcal{J}\Phi$$

where Φ is the function $p_0 \to \Phi(p_0)$ and $\mathcal{J}\Phi$ is the function which maps p_0 into $\xi(0)$ as given by (2.10). In other words,

$$\mathcal{J}\Phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ}\,QF(u)d\tau \tag{2.11}$$

where $u = u(\tau) = p(\tau;\Phi,p_0) + \Phi(p(\tau;\Phi,p_0))$.

Finally we will require that $(p(0) + q(0))$ belongs to the manifold $\mathcal{M}$ defined by $\mathcal{M} = \mathrm{Graph}(\Phi)$. This means that

$$q(0) = \Phi(p_0) = \mathcal{J}\Phi(p_0), \qquad \text{for all } p_0 \quad PD(A),$$

i.e. Φ is to be a fixed point of the operator $\mathcal{J}$. This leads to the fixed point problem, but first note that the dimension of $\mathcal{M}$ is N, which is the dimension of PH.

The fixed point problem is to find conditions on N, b and ℓ so that

(i) $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into itself, and

(ii) $\mathcal{J}$ has a fixed point Φ in $\mathcal{F}_{b,\ell}$.

After establishing (i) we will prove (ii) by showing that $\mathcal{J}$ is a contraction on $\mathcal{F}_{b,\ell}$.

Before stating our main results let us note that with this construction we can describe the induced ordinary differential equation on $\mathcal{M}$. Indeed if Φ is the fixed point of $\mathcal{J}$ then dynamics of (1.24) on $\mathcal{M}$ is completely determined by the ordinary differential equation

(2.12) $$\frac{dp}{dt} + Ap + \theta_\rho(|A(p + \Phi(p))|)PR(p + \Phi(p)) = 0.$$

Thus (2.12) is an inertial form of the equation (1.1), (1.2). (Strictly speaking, (2.12) describes an ordinary differential equation on PD(A) = PH. The trajectories on $\mathcal{M}$ itself are given by $u(t) = p(t) + \Phi(p(t))$).

2.2 Statement of Results

The main assumption which we make below is concerned with the determination of the dimension N of the inertial manifold. The form of this assumption will involve a comparison between

λ_N (the largest eigenvalue of $A|_{PH}$), and

λ_{N+1} (the smallest eigenvalue of $A|_{QH}$).

What we will show is that if the gap between λ_N and λ_{N+1} is sufficiently large then there exists an inertial manifold. Now for the details.

Theorem 2.1

Assume that hypotheses (1.1)-(1.12) *are satisfied and let* ℓ, $0 < \ell < 1/8$, *be given. Let* N_0 *be given by* (1.28). *Then there exist a constant* K_{10}, K_{11} (*dependent on* ℓ *and the data of the problem*) *such that if one has*

$$N > N_0 , \quad \lambda_{N+1}^{1/2} > K_{11} \tag{2.13}$$

$$\lambda_{N+1}^{1/2} - \lambda_N^{1/2} > K_{10} \tag{2.14}$$

then there is a $b > 0$ *such that:*

(i) $\mathcal{J}$ *maps* $\mathcal{F}_{b,\ell}$ *into itself;*

(ii) $\mathcal{J}$ *has a fixed point* $\Phi \in \mathcal{F}_{b,\ell}$;

(iii) $\mathcal{M} = \text{Graph}(\Phi)$ *is an inertial manifold for* (1.24);

(iv) $\mathcal{M}$ *contains the global attractor* $\mathcal{A}$ *for* (1.1);

(v) *The dynamics of* $\mathcal{M}$ *is completely determined by the ordinary differential equation* (2.12).

Partial Proof:

We will postpone to Section 3 the proof of items (i) and (ii) above. By assuming the validity of (i)-(ii) it is not difficult to prove that $\mathcal{M} = \text{Graph}(\Phi)$ is invariant and that it attracts all trajectories exponentially. For this purpose we will use the Squeezing Property and (1.29), (1.30).

The invariance of $\mathcal{M}$, i.e. $S(t)\mathcal{M} \subseteq \mathcal{M}$ for all t, follows from the fixed point equation $\Phi = \mathcal{J}\Phi$ which becomes:

$$\Phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ}\, QF(u(\tau,p_0))d\tau \tag{2.15}$$

where $u(\tau,p_0) = p(\tau;\Phi,p_0) + \Phi(p(\tau;\Phi,p_0))$. Now replace p_0 by $p(t) = p(t;\Phi,p_0)$ in (2.14) and use the fact that

$$p(\tau;\Phi,p(t,\Phi,p_0)) = p(\tau + t;\Phi,p_0)$$

to conclude that

$$\Phi(p(t)) = -\int_{-\infty}^{0} e^{\tau AQ}\, QF(u(\tau,p(t)))d\tau \tag{2.16}$$

$$= -\int_{-\infty}^{t} e^{-(t-\tau)AQ}\, QF(u(\tau,p_0))d\tau$$

for all $t \in R$. Furthermore by differentiating (2.16) with respect to t, we see that $(p(t),q(t))$ is a solution of (2.2), (2.3) and $u(t) = p(t) + q(t)$ is a solution of (1.24), where $q(t) = \Phi(p(t))$. This shows that $S(t)\mathcal{M} \subseteq \mathcal{M}$ for all $t > 0$.

Let $t_0 = \frac{1}{2K_1}\log 2$ and let N_0 be given by (1.28). Fix $r = 4\rho + b$. According to (1.22) every trajectory of (1.24) eventually enters the ball in $D(A)$ centered at 0 with radius $4\rho = 8\rho_2$. Since we want to show that a given trajectory $S(t)u_0$ is exponentially attracted to $\mathcal{M}$, we can assume that $|Au_0| < 4\rho$ and $|AS(t)u_0| < 4\rho$, for $t > 0$.

We will first show that for any t_1 with $t_0 < t_1 < 2t_0$ one has

$$\mathrm{dist}(S(t_1)u_0,\mathcal{M}) < \frac{1}{2}\,\mathrm{dist}(u_0,\mathcal{M}) \tag{2.17}$$

where $\mathrm{dist}(\phi,\mathcal{M}) = \inf\{|\phi - v|: v \in \mathcal{M}\}$. For that purpose choose v_0 such that $|u_0 - v_0| = \mathrm{dist}(u_0,\mathcal{M})$. Then $v_0 = Pv_0 + \Phi(Pv_0)$. We claim that $|APv_0| < 4\rho$. If, on the contrary, one had $|APv_0| > 4\rho > |APu_0|$, then $\Phi(Pv_0) = 0$ and $v_0 = Pv_0$. In addition there is a β, $0 < \beta < 1$ such that $|APv_\beta| = 4\rho$ where $v_\beta = \beta Pu_0 + (1 - \beta)v_0 \in PD(A)$. One then has $\Phi(v_\beta) = 0$, therefore $v_\beta \in \mathcal{M}$. Finally one has

$$
\begin{aligned}
|v_\beta - u_0|^2 &= |v_\beta - Pu_0|^2 + |Qu_0|^2 \\
&= |(1-\beta)v_0 - Pu_0|^2 + |Qu_0|^2 \\
&< |v_0 - Pu_0|^2 + |Qu_0|^2 = |v_0 - u_0|^2
\end{aligned}
$$

which contradicts the fact that $|v_0 - u_0| = \mathrm{dist}(u_0, \mathcal{M})$. Since $|\Phi(Pv_0)| < b$ one has

$$|Av_0| < |APv_0| + |A\Phi(Pv_0)| < 4\rho + b = r.$$

Next we apply the Squeezing Property (1.29), (1.30) to $S(t_1)u_0$ and $S(t_1)v_0$. If (1.30) holds then

$$
\begin{aligned}
\mathrm{dist}(S(t_1)u_0, \quad) &< |S(t_1)u_0 - S(t_1)v_0| \\
&< \frac{1}{2}\,|u_0 - v_0| < \frac{1}{2}\,\mathrm{dist}(u_0, \quad).
\end{aligned}
$$

On the other hand if (1.29) holds then we write

$$
\begin{aligned}
\mathrm{dist}(S(t_1)u_0, \mathcal{M}) &< |S(t_1)u_0 - (PS(t_1)u_0 + \Phi(PS(t_1)u_0))| \\
&< |QS(t_1)u_0 - \Phi(PS(t_1)u_0)| \\
&< |QS(t_1)u_0 - QS(t_1)v_0| + |\Phi(PS(t_1)v_0) - \Phi(PS(t_1)u_0)|
\end{aligned}
$$

(since $S(t_1)v_0 \in \mathcal{M}$ one has $QS(t_1)v_0 = \Phi(PS(t_1)v_0)$)

$$< \left(\frac{1}{8} + \ell\,\frac{\lambda_N}{\lambda_{N+1}}\right)|PS(t_1)v_0 - PS(t_1)u_0|$$

(by (1.29), (2.6) and since $|q| < \lambda_{N+1}^{-1}|Aq|$ for $q \in QD(A)$ and $|Ap| < \lambda_N|p|$ for $p \in PD(A)$)

$$< \frac{1}{4}\,|S(t_1)v_0 - S(t_1)u_0|$$

$$< \frac{1}{2}\,|v_0 - u_0| = \frac{1}{2}\,\mathrm{dist}(u_0, \mathcal{M})$$

(by (1.25) and the choice of t_0 and K_1). Once (2.17) is proved it is clear that

$$\operatorname{dist}(S(nt_1),\mathcal{M}) < \left(\tfrac{1}{2}\right)^n \operatorname{dist}(u_0,\mathcal{M}) \to 0$$

as $n \to \infty$. Then for an arbitrary $t > t_0$ we write $t = nt_1$ where $t_0 < t_1 < 2t_0$, and thus

$$\begin{aligned}
(2.18) \qquad \operatorname{dist}(S(t)u_0,\mathcal{M}) &< \left(\tfrac{1}{2}\right)^n \operatorname{dist}(u_0,\mathcal{M}) \\
&< \exp\left(-\frac{t}{t_1}\log 2\right)\operatorname{dist}(u_0,\mathcal{M}) \\
&< \exp\left(-\frac{t}{2t_0}\log 2\right)\operatorname{dist}(u_0,\mathcal{M}),
\end{aligned}$$

which gives the exponential convergence with the rate $\frac{1}{2t_0}\log 2$.

The same considerations lead to a direct proof that the global attractor $\mathcal{A}$ lies in $\mathcal{M}$. Indeed if $u \in \mathcal{A}$, then the solution $S(t)u$ is defined for all t R. Furthermore from (1.22-23) one has

$$\operatorname{dist}(S(t)u,\mathcal{M}) < 2\rho_2, \qquad \text{for all } t \text{ in } R.$$

Let $v = S(-t)u$, where $t > t_0$. Then from (2.18) we get

$$\operatorname{dist}(u,\mathcal{M}) = \operatorname{dist}(S(t)v,\mathcal{M}) < \exp\left(-\frac{t}{2t_0}\log 2\right) \cdot 2\rho_2,$$

which implies that $\operatorname{dist}(u,\mathcal{M}) = 0$. $\blacksquare$

In the case where the nonlinear terms $R(u)$ are locally Lipschitz continuous on H one can derive a sharper inertial manifold result. In the next theorem we make reference to Section 1.4.

Theorem 2.2

Let Equations (1.31), (1.32) *be given on* H *where the nonlinear term* $F(u) = \theta_\rho(|u|)R(u)$ *satisfies* (1.33). *Let* ℓ *be given,* $0 < \ell < 1/8$. *Then there exist constants* N_0, K_{12}, K_{13} (*dependent on* ℓ *and the data of the problem*) *such that if one has*

(2.19)
$$N > N_0, \quad \lambda_{N+1} > K_{12}$$

(2.20)
$$\lambda_{N+1} - \lambda_N > K_{13}$$

then the conclusions of Theorem 2.1 are valid.

We will say more about the gap conditions (2.14), (2.20) in Section 4 when we discuss some applications of our theory.

Since the Squeezing Property is valid for (1.31), (1.32) when $F(u)$ satisfies (1.33), the proof of the invariance of $\mathcal{M}$ and the attraction property given above is applicable in this case as well. We will omit the details.

3. PROPERTIES OF THE OPERATOR $\mathcal{J}$.

In this Section we establish the properties of $\mathcal{J}$ which lead to the application of the contraction fixed point theorem. We successively show that $\mathcal{J}$ is well-defined (§3.1) and that it maps $\mathcal{F}_{b,\ell}$ into $\mathcal{F}_{b,L}$ for an appropriate L (§3.3). Then we show that under suitable assumptions it maps $\mathcal{F}_{b,\ell}$ into itself and is a strict contraction in this space (§3.4). This will complete the proof of Theorem 2.1. The modifications of our argument needed to prove Theorem 2.2 are given in §3.5.

3.1 The Operator $\mathcal{J}$ is Well-Defined.

Recall that we are considering functions Φ from $PD(A)$ into $QD(A)$ ($P = P_N$, $Q = Q_N = I - P_N$) which belong to the class $\mathcal{F}_{b,\ell}$, i.e. they satisfy (2.5) - (2.7). Since $QD(A)$ is complete in the norm $|\cdot|_2$, the set $\mathcal{F}_{b,\ell}$ is a complete metric space when endowed with the distance

$$\|\Phi - \Psi\| := \sup_{p \in PD(A)} |A\Phi(p) - A\Psi(p)|. \tag{3.1}$$

The mapping $\mathcal{J}$ associates to each Φ in $\mathcal{F}_{b,\ell}$ a function on $PD(A)$ defined by

$$\mathcal{J}\Phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ}\, QF(u)d\tau, \qquad p_0 \in PD(A) \tag{3.2}$$

where $u = u(\tau) = p(\tau;\Phi,p_0) + \Phi(p(\tau;\Phi,p_0))$ and $p(\tau;\Phi,p_0)$ is the solution of (2.2) satisfying $p(0;\Phi,p_0) = p_0$. We will now show that $\mathcal{J}\Phi(p_0)$ is well-defined for all p_0, that its range lies in an a priori bounded set of $D(A)$ and even $D(A^{5/4})$, and that is has compact support, i.e. that $\mathcal{J}\Phi(p_0) = 0$ for $|Ap_0| > 4\rho$.

Let us start with a technical lemma.

Lemma 3.1

For $\alpha > 0$ and $\tau < 0$, the operator $(AQ)^{\alpha}e^{\tau AQ}$ is linear and continuous on QH. Furthermore, its norm in $\mathcal{L}(QH)$, which we denote by $|(AQ)^{\alpha}e^{\tau AQ}|_{op}$, is bounded by

$$K_3|\tau|^{-\alpha}, \qquad \text{for} \quad -\alpha\lambda_{N+1}^{-1} < \tau < 0 \tag{3.3}$$

and by

$$\lambda_{N+1}^{\alpha}e^{\tau\lambda_{N+1}}, \qquad \text{for} \quad -\infty < \tau < -\alpha\lambda_{N+1}^{-1} \tag{3.4}$$

Proof. Let $v = \sum_{j=N+1}^{\infty} b_j w_j$ be an element of QH. Then $(AQ)^{\alpha} e^{\tau AQ} v = \sum_{j=N+1}^{\infty} \lambda_j^{\alpha} e^{\tau\lambda_j} b_j w_j$ and by the Plancheral formula

$$\begin{aligned} |(AQ)^{\alpha} e^{\tau AQ} v|^2 &= \sum_{j=N+1}^{\infty} (\lambda_j^{\alpha} e^{\tau\lambda_j})^2 b_j^2 \\ &\leq \sup_{\lambda \geq \lambda_{N+1}} (\lambda^{\alpha}e^{\tau\lambda})^2 \Sigma_j b_j^2 \\ &= \sup_{\lambda \geq \lambda_{N+1}} (\lambda^{\alpha}e^{\tau\lambda})^2 |v|^2. \end{aligned}$$

Consequently $|(AQ)^{\alpha}e^{\tau AQ}|_{op}$ is bounded by

$$\sup_{\lambda \geq \lambda_{N+1}} (\lambda^{\alpha}e^{\tau\lambda}).$$

An elementary calculation shows that this supremum is equal to $|\tau|^{-\alpha}(\alpha e^{-1})^{\alpha}$ if $-\alpha\lambda_{N+1}^{-1} \leq \tau < 0$ and is equal to $\lambda_{N+1}^{\alpha}e^{\tau\lambda_{N+1}}$ for $\tau \leq -\alpha\lambda_{N+1}^{-1}$. We thus obtain (3.3), (3.4) with $K_3 = K_3(\alpha) = (\alpha e^{-1})^{\alpha}$. ∎

An immediate consequence of (3.3), (3.4) is that

$$(3.5)\qquad \int_{-\infty}^{0} |(AQ)^{\alpha} e^{\tau AQ}|_{op} d\tau \le (1-\alpha)^{-1} e^{-\alpha} \lambda_{N+1}^{\alpha-1}, \qquad 0 < \alpha < 1.$$

From (1.10), (1.11) we get

$$\begin{aligned} |(AQ)^{1/2}R(u)| &\le |A^{1/2}B(u,u)| + |A^{1/2}Cu| + |A^{1/2}f| \\ &\le c_3|Au|^2 + c_4|Au| + |A^{1/2}f|. \end{aligned}$$

Since $\theta_\rho(|Au|) = 0$ for $|Au| > 2\rho$ one has

$$(3.6)\qquad |(AQ)^{1/2}F(u)| \le K_4$$

where $K_4 = 4c_3\rho^2 + 2c_4\rho + |A^{1/2}f|$ and F is given by (2.4). Next we prove

<u>Lemma 3.2</u>

<u>For every</u> p_0 <u>in</u> $PD(A)$, $\mathcal{J}\Phi(p_0)$ <u>belongs to</u> $QD(A)$ <u>and</u>

$$(3.7)\qquad |A\mathcal{J}\Phi(p_0)| \le K_5\lambda_{N+1}^{-1/2}$$

$$(3.8)\qquad |A^{5/4}\mathcal{J}\Phi(p_0)| \le K_6\lambda_{N+1}^{-1/4}$$

<u>where</u> K_5, K_6 <u>are appropriate constants which are independent of</u> p_0 <u>and</u> Φ.

<u>Proof</u>. Since $Qe^{\tau AQ} = e^{\tau AQ}$ it is easy to see that $\mathcal{J}\Phi(p_0) \in QD(A)$. Next note that

$$|A\mathcal{J}\Phi(p_0)| \le \int_{-\infty}^{0} |AQe^{\tau AQ}F(u)|d\tau \le \int_{-\infty}^{0} |(AQ)^{1/2}e^{\tau AQ}|_{op}|(AQ)^{1/2}F(u)|d\tau$$

$$|A^{5/4}\mathcal{J}\Phi(p_0)| \le \int_{-\infty}^{0} |(AQ)^{5/4}e^{\tau AQ}F(u)|d\tau \le \int_{-\infty}^{0} |(AQ)^{3/4}e^{\tau AQ}|_{op}|(AQ)^{1/2}F(u)|d\tau.$$

Inequalities (3.7), (3.8) now follow from (3.5), (3.6) where

$$K_5 = 2K_4 e^{-1/2}, \quad K_6 = 4K_4 e^{-3/4}. \qquad ∎$$

From now on we fix

$$b = K_5 \lambda_{N+1}^{-1/2}\ . \tag{3.9}$$

It is then clear that for every Φ in $\mathcal{F}_{b,\ell}$, $\mathcal{J}\Phi$ satisfies the analog of (2.5), namely

$$|A\,\mathcal{J}\Phi(p_0)| \leq b, \qquad \text{for all} \quad p_0 \in PD(A). \tag{3.10}$$

Furthermore due to (3.8) the range of $\mathcal{J}\Phi$ is a bounded set in $D(A^{5/4})$. Since $A^{-1/4}$ is compact (like A^{-1}) we have shown that:

(3.11) *The range of* $\mathcal{J}\Phi$ *is in a compact subset of* $QD(A)$, *which does not depend on* Φ.

We next prove the property of the support of $\mathcal{J}\Phi$, (2.7).

Lemma 3.3

For every Φ *in* $\mathcal{F}_{b,\ell}$ *the support of* $\mathcal{J}\Phi$ *is included in the set*

$$\{p \in PD(A) : |Ap| \leq 4\rho\}.$$

<u>Proof</u>. We observe that if $u = p + \Phi(p) = Pp + Q\Phi(p)$ then $|Au|^2 = |Ap|^2 + |A\Phi(p)|^2$, and $|Au| \geq |Ap|$. Therefore if $|Ap| > 2\rho$ then $|Au| > 2\rho$ and $\theta_\rho(|Au|) = 0$.

Now assume that $|Ap_0| > 4\rho$. Then $|Ap(t)| > 2\rho$ on some interval I containing $t = 0$. On I equation (2.2) reduces to

$$\frac{dp}{dt} + Ap = 0.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}|Ap|^2 + \lambda_1|Ap|^2 \leq \frac{1}{2}\frac{d}{dt}|Ap|^2 + |A^{3/2}p|^2 = 0.$$

Hence for $\tau < 0$ one has

$$2\rho < |Ap(0)| \leq |Ap(\tau)|\exp(\lambda_1\tau) \leq |Ap(\tau)|.$$

Consequently $|Ap(\tau)|$ will never reach the value 2ρ for $\tau < 0$, i.e. $|Ap(\tau)| > 2\rho$, for all $\tau < 0$ and $\theta_\rho(|Au(\tau)|) = 0$, for all $\tau \leq 0$. Therefore the integral on the right side of (3.2) vanishes and $\Phi(p_0) = 0$ for all $\Phi \in \mathcal{F}_{b,\ell}$. ▨

3.2 Lipschitz Properties of F.

We return to the nonlinear function $F(u)$ defined by (2.4). Note that F depends on p and Φ since $u = p + \Phi(p)$. Let $p_1, p_2 \in PD(A)$, $\Phi_1, \Phi_2 \in \mathcal{F}_{b,\ell}$ and set $u_i = p_i + \Phi_i(p_i)$, $i = 1,2$. Our object here is to show that

$$|A^{1/2}F(u_1) - A^{1/2}F(u_2)| \leq K_7[(1 + \ell)|Ap_1 - Ap_2| + \|\Phi_1 - \Phi_2\|] \tag{3.12}$$

for some constant K_7 which does not depend on p_i or Φ_i, $i = 1,2$.

First notice that (1.10) and (1.11) imply that

$$\begin{aligned}|A^{1/2}R(u_1) - A^{1/2}R(u_2)| &\leq |A^{1/2}[B(u_1,u_1) - B(u_1,u_1) - B(u_1,u_2) + B(u_1,u_2) - B(u_2,u_2)]| \\ &\quad + |A^{1/2} C(u_1 - u_2)| \\ &\leq c_3(|Au_1| + |Au_2|)||Au_1 - Au_2| + c_4|Au_1 - Au_2|\end{aligned}$$

and

$$|A^{1/2} R(u_1)| \leq c_3|Au_1|^2 + c_4|Au_1| + |A^{1/2}f|.$$

Define G by

$$G = A^{1/2} F(u_1) - A^{1/2}F(u_2) = \theta_\rho(|Au_1|)\, A^{1/2}R(u_1) - \theta_\rho(|Au_2|)\, A^{1/2}R(u_2).$$

We then distinguish between 3 cases:

1) $2\rho \leq |Au_1|, |Au_2|$; 2) $|Au_1| < 2\rho \leq |Au_2|$ (or $|Au_2| < 2\rho \leq |Au_1|$)

3) $|Au_1|, |Au_2| < 2\rho$

By using the facts that $\theta_\rho(|Au|) = 0$ for $|Au| > 2\rho$ and $|\theta'| \le 2\rho^{-1}$ we obtain the following:

In case 1 has $G = 0$. In case 2 we have

$$|G| = |\theta_\rho(|Au_1|)\, A^{1/2}R(u_1)|$$

$$= |\theta_\rho(|Au_1|)\, A^{1/2}R(u_1) - \theta_\rho(|Au_2|)\, A^{1/2}R(u_1)|$$

$$\le |\theta_\rho(|Au_1|) - \theta_\rho(|Au_2|)|\; |A^{1/2}R(u_1)|$$

$$\le 2\rho^{-1}\; ||Au_1| - |Au_2|| \cdot |A^{1/2}R(u_1)|$$

$$\le 2\rho^{-1}\; (c_3 \cdot 4\rho^2 + c_4 \cdot 2\rho + |A^{1/2}f|)\; |Au_1 - Au_2|.$$

The argument for $|Au_2| \le 2\rho \le |Au_1|$ is similar. In case 3 we obtain

$$|G| \le |\theta_\rho(|Au_1|) - \theta_\rho(|Au_2|)|\,|A^{1/2}R(u_1)|$$

$$+ \theta_\rho(|Au_2|)\,|A^{1/2}R(u_1) - A^{1/2}R(u_2)|$$

$$\le 2\rho^{-1}||Au_1| - |Au_2||\; (c_3|Au_1|^2 + c_4|Au_1| + |A^{1/2}F|)$$

$$+ [c_3(|Au_1| + |Au_2|) + c_4]\; |Au_1 - Au_2|.$$

Hence

$$|A^{1/2}F(u_1) - A^{1/2}F(u_2)| \le K_7|Au_1 - Au_2| \tag{3.13}$$

where $K_7 = 2\rho^{-1}(c_3 4\rho^2 + c_4 \cdot 2\rho + |Au^{1/2} f|) + c_3\, 4\rho + c_4$.

Since $u_1 - u_2 = p_1 - p_2 + (\Phi_1(p_1) - \Phi_1(p_2)) + (\Phi_1(p_2) - \Phi_2(p_2))$ one has

$$|Au_1 - Au_2| \le (1 + \ell)|Ap_1 - Ap_2| + \|\Phi_1 - \Phi_2\|.$$

By combining this with (3.13) we obtain (3.12).

3.3 Properties of $\mathcal{J}$ (continued)

We now show that under suitable assumptions $\mathcal{J}$ is a Lipschitz mapping from $\mathcal{F}_{b,\ell}$ into $\mathcal{F}_{b,L}$ and we estimate L. Then we show that as a result of the hypotheses of Theorem 2.1, $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into itself and that it is a strict contraction.

First let Φ be fixed. Let p_{01}, p_{02} belong to $PD(A)$ and let $p_1 = p_1(t)$ and $p_2 = p_2(t)$ be the corresponding solutions of (2.2) satisfying $p_i(0) = p_{0i}$, $i = 1,2$. Define $\Delta = p_1 - p_2$, then Δ is a solution of the evolutionary equation

$$(3.14) \qquad \frac{d\Delta}{dt} + A\Delta + PF(u_1) - PF(u_2) = 0$$

with $u_i = p_i + \Phi(p_i)$, $i = 1,2$. Taking the scalar product of (3.14) with $A^2\Delta$ we obtain

$$(3.15) \qquad \frac{1}{2}\frac{d}{dt}|A\Delta|^2 + |A^{3/2}\Delta|^2 = -(A^{1/2}P(F(u_1) - F(u_2)), A^{3/2}\Delta).$$

Therefore using (3.12) with $\Phi = \Phi_1 = \Phi_2$ we obtain

$$\left|\frac{1}{2}\frac{d}{dt}|A\Delta|^2 + |A^{3/2}\Delta|^2\right| \leq K_7(1 + \ell)|A\Delta||A^{3/2}\Delta|$$

and thus

$$|A\Delta|\frac{d}{dt}|A\Delta| \geq -|A^{3/2}\Delta|^2 - K_7(1 + \ell)|A\Delta||A^{3/2}\Delta|.$$

Since $\Delta \in PD(A)$ one has

$$|A^{3/2}\Delta| = |A^{1/2}A\Delta| \leq \lambda_N^{1/2}|A\Delta|,$$

and consequently

$$|A\Delta| \frac{d}{dt} |A\Delta| \geq -\lambda_N |A\Delta|^2 - K_7(1 + \ell)\lambda_N^{1/2} |A\Delta|^2,$$

or

(3.16) $$\frac{d}{dt} |A\Delta| + (\lambda_N + K_7(1 + \ell)\lambda_N^{1/2}) |A\Delta| \geq 0.$$

We easily infer from (3.16) that

(3.17) $$|A\Delta(\tau)| \leq |A\Delta(0)| \exp(-\tau(\lambda_N + K_7(1 + \ell)\lambda_N^{1/2})), \quad \text{for all} \quad \tau \leq 0.$$

We can now prove

<u>Lemma 3.4</u>

<u>Assume that</u>

(3.18) $$\gamma_N := \lambda_{N+1} - \lambda_N - K_7(1 + \ell)\lambda_N^{1/2} > 0.$$

<u>Then for</u> $\Phi \in \mathcal{F}_{b,\ell}$ <u>and</u> $p_{01}, p_{02} \in PD(A)$ <u>one has</u>

(3.19) $$|A\mathcal{J}\Phi(p_{01}) - A\mathcal{J}\Phi(p_{02})| \leq L |Ap_{01} - Ap_{02}|$$

<u>with</u>

(3.20) $$L = K_7(1 + \ell)\lambda_{N+1}^{-1/2} [1 + (1 - r_N\alpha_N)^{-1}] e^{-1/2} \exp\left(\frac{r_N\alpha_N}{2} \right)$$

(3.21) $$r_N = \lambda_N/\lambda_{N+1}, \quad \alpha_N = (1 + K_7(1 + \ell)\lambda_N^{-1/2}).$$

<u>Consequently</u> $\Phi \in \mathcal{F}_{b,L}$.

<u>Proof</u>. Due to (3.2) and (3.12) one has

$$|A\mathcal{J}\Phi(p_{01}) - A\mathcal{J}\Phi(p_{02})| \leq \int_{-\infty}^{0} |AQ\, e^{\tau AQ} Q(F(u_1) - F(u_2))| d\tau$$

$$\leq K_7(1 + \ell) \int_{-\infty}^{0} |(AQ)^{1/2} e^{\tau AQ}|_{op} |A\Delta(\tau)| d\tau$$

where $\Delta = p_1 - p_2$, as above. Then with Lemma 3.1 and (3.17) the last integral is bounded by

$$\left(\int_{-\infty}^{-a} \lambda_{N+1}^{1/2} \exp[\tau(\lambda_{N+1} - \lambda_N - K_7(1 + \ell)\lambda_N^{1/2})]d\tau \right.$$
$$\left. + \int_{-a}^{0} K_3(\tfrac{1}{2})|\tau|^{-1/2}\exp[-\tau(\lambda_N + K_7(1 + \ell)\lambda_N^{1/2})]d\tau \right) \cdot |Ap_{01} - Ap_{02}|,$$

where $a = \frac{1}{2}\lambda_{N+1}^{-1}$. An elementary calculation shows that the last expression is bounded by

$$\lambda_{N+1}^{-1/2} e^{-1/2}[1 + (1 - r_N\alpha_N)^{-1}]\exp(\frac{r_N\alpha_N}{2})|Ap_{01} - Ap_{02}|.$$

This proves (3.19) with L given by (3.20). The fact that $\mathcal{J}\Phi$ belongs to $\mathcal{F}_{b,L}$ now follows from (3.9), (3.19) and Lemma 3.3. ▨

At this point we have shown that $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into $\mathcal{F}_{b,L}$. Next we want to show that $\mathcal{J}$ is a Lipschitz mapping on these spaces. For that purpose we consider two functions Φ_1 and Φ_2 and a single initial condition $p_0 \in PD(A)$. Let $p_i = p(t,\Phi_i,p_0)$ and set $u_i = p_i + \Phi_i(p_i)$, $i = 1,2$. We will now estimate $|A\mathcal{J}\Phi_1(p_0) - A\mathcal{J}\Phi_2(p_0)|$ by using essentially the same methods as above.

Define $\Delta = p_1 - p_2$, then (3.15) and (3.16) are still valid. However instead of (3.17) one obtains

$$\frac{d}{dt}|A\Delta| + \lambda_N\alpha_N|A\Delta| \geq -K_7\lambda_N^{1/2}\|\Phi_1 - \Phi_2\| \tag{3.22}$$

where α_N is given by (3.21). Here one has $\Delta(0) = 0$ and it follows from (3.22) that

$$(3.23)\qquad |A\Delta(\tau)| \leq \frac{K_7\lambda_N^{1/2}\|\Phi_1 - \Phi_2\|}{\alpha_N\lambda_N}\left([\exp(-\alpha_N\lambda_N\tau)] - 1\right), \quad \tau \leq 0$$

$$\leq K_7\lambda_N^{-1/2}\|\Phi_1 - \Phi_2\|\exp(-\alpha_N\lambda_N\tau), \qquad \tau \leq 0$$

since $\alpha_N \geq 1$.

As in Lemma 3.6, using (3.2), (3.12) and (3.23) we get

$$(3.24)\quad |A\mathcal{J}\Phi_1(p_0) - A\mathcal{J}\Phi_2(p_0)| \leq \int_{-\infty}^{0} |AQe^{\tau AQ}\, Q(F(u_1) - F(u_2))|d\tau$$

$$\leq K_7 \int_{-\infty}^{0} |(AQ)^{1/2}e^{\tau AQ}|_{op}|(1+\ell)|A\Delta| + \|\Phi_1-\Phi_2\||d\tau$$

$$\leq K_7\|\Phi_1-\Phi_2\|\int_{-\infty}^{0} |(AQ)^{1/2}e^{\tau AQ}|_{op}(1+K_7\lambda_N^{-1/2}(1+\ell)\exp(-\alpha_N\lambda_N\tau))d\tau.$$

By Lemma 3.1, the integral in the last term in (3.24) is bounded by

$$2e^{-1/2}\lambda_{N+1}^{-1/2} + K_7(1 + \ell)\lambda_N^{-1/2}\left(\int_{-\infty}^{-a} \lambda_{N+1}^{1/2} \exp[\tau(\lambda_{N+1} - \lambda_N\alpha_N)]d\tau\right.$$

$$\left. + \int_{-a}^{0} K_3(\tfrac{1}{2})|\tau|^{-1/2}\exp(-\lambda_N\alpha_N\tau)d\tau\right)$$

$$\leq 2e^{-1/2}\lambda_{N+1}^{-1/2} + K_7(1 + \ell)\lambda_N^{-1/2}\left(\lambda_{N+1}^{-1/2}\, e^{-1/2}(1 + (1 - r_N\alpha_N)^{-1}))\exp(\frac{r_N\alpha_N}{2})\right)$$

$$\leq 2e^{-1/2}\lambda_{N+1}^{-1/2} + \lambda_N^{-1/2}\, L$$

where L is given by (3.20). We conclude therefore that

$$(3.25)\qquad |A\mathcal{J}\Phi_1(p_0) - A\mathcal{J}\Phi_2(p_0)| \leq L'\|\Phi_1 - \Phi_2\|, \qquad \text{for all } p_0 \in PD(A)$$

where

$$L' = K_7(2e^{-1/2}\lambda_{N+1}^{-1/2} + \lambda_N^{-1/2}L). \tag{3.26}$$

3.4 Conclusion of the Proof of Theorem 2.1.

As indicated above, we seek conditions which insure that $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into itself and that it is a strict contraction on $\mathcal{F}_{b,\ell}$. This amounts to finding sufficient conditions (primarily on λ_N and λ_{N+1}) which insure that

$$L < \ell \text{ and } L' < 1$$

where L and L' are given by (3.20) and (3.26).

First notice that (3.18) is equivalent to

$$1 - r_N\alpha_N > 0, \tag{3.27}$$

or $1 > r_N\alpha_N > 0$. It then follows from (3.27) and (3.20) that

$$L < K_7(1 + \ell)\lambda_{N+1}^{-1/2} [1 + (1 - r_N\alpha_N)^{-1})].$$

In order to achieve $L < \ell$, it suffices to have N chosen so that both of the following two inequalities are satisfied:

$$K_7(1 + \ell)\lambda_{N+1}^{-1/2} < \frac{\ell}{2} \tag{3.28}$$

$$K_7(1 + \ell)\lambda_{N+1}^{-1/2} (1 - r_N\alpha_N)^{-1} < \frac{\ell}{2} . \tag{3.29}$$

Now (3.28) can be rewritten as

(3.30) $$K_{10} < \lambda_{N+1}^{1/2}$$

where $K_{10} = 2K_7(1 + \ell)\ell^{-1}$. We assume now that N is chosen so that (3.30) is valid. Inequality (3.29) can be rewritten as

(3.31) $$K_{10}\lambda_{N+1}^{-1/2} < (1 - r_N\alpha_N)$$

or equivalently as

(3.32) $$K_{10}\lambda_{N+1}^{-1/2} - 1 + r_N + K_7(1 + \ell)\lambda_{N+1}^{-1/2} r_N^{1/2} < 0$$

where $r_N = \lambda_N/\lambda_{N+1}$. Let us assume for the moment that

(3.33) $$r_N^{1/2} + K_{10}\lambda_{N+1}^{-1/2} = (\lambda_N\lambda_{N+1}^{-1})^{1/2} + K_{10}\lambda_{N+1}^{-1/2} < 1$$

By applying (3.33) twice one has

(3.34) $$K_{10}\lambda_{N+1}^{-1/2} - 1 + r_N + K_{10}\lambda_{N+1}^{-1/2} r_N^{1/2} < K_{10}\lambda_{N+1}^{-1/2} - 1 + r_N^{1/2} < 0$$

Since $\ell < 1/8$ one has $K_7(1 + \ell) < K_{10}$, and consequently (3.33) implies (3.32) which, in turn, implies (3.31).

Let us summarize the situation to this point. In order to conclude that $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into $\mathcal{F}_{b,L}$ we need to assume that $\gamma_N > 0$, or equivalently that $(1 - r_N\alpha_N) > 0$. This assumption is guaranteed by (3.31). A sufficient condition for $\mathcal{J}$ to map $\mathcal{F}_{b,\ell}$ into itself is that both (3.28) and (3.31) be satisfied, however this in turn is guaranteed by (3.30) and (3.33). Both of the latter inequalities are consequences of a single condition:

(3.35) $$K_{10} < \lambda_{N+1}^{1/2} - \lambda_N^{1/2}.$$

In order for $\mathcal{J}$ to be a contraction mapping on $\mathcal{F}_{b,\ell}$ we also want $L' < 1$, say $L' \le 1/2$. However this follows immediately from (3.36) when one has

$$K_{11} \le \lambda_{N+1}^{1/2} \tag{3.36}$$

where $K_{11} = 2K_7(2e^{-1/2} + \ell)$.

Under the conditions (3.35), (3.36) $\mathcal{J}$ maps $\mathcal{F}_{b,\ell}$ into itself and it is strictly contracting and compact. The existence of a fixed point for $\mathcal{J}$ follows in fact either by the contraction fixed point theorem or by Schauder's theorem. This completes the proof of Theorem 2.1.

3.5 Proof of Theorem 2.2

In Theorem 2.2 we assume that modified nonlinear term $F(u) = \theta_\rho(|u|)R(u)$ satisfies a global Lipschitz condition

$$|F(u) - F(v)| \le K|u - v|, \qquad \text{for all } u,v \in H.$$

While the proof follows the same outline as that of Theorem 2.1, there is a significiant simplification of some of the technicalities. We shall only indicate the major changes in the argument here.

First the parameter ρ is defined by $\rho = 2\rho_0$ since one will be limiting the analysis to the Hilbert space H. Next the space $\mathcal{F}_{b,\ell}$ consists of those functions Φ: $PH \to QH$ that satisfy:

$$\begin{aligned} &|\Phi(p)| \le b, && \text{for all } p \in PD(A)\\ &|\Phi(p_1) - \Phi(p_2)| \le \ell|p_1 - p_2|, && \text{for all } p_1,p_2 \in PD(A)\\ &\operatorname{Supp} \Phi \subseteq \{p \in PD(A): |p| \le 4\rho\}. \end{aligned}$$

The operator $\mathcal{J}$ is defined by

$$\mathcal{J}\Phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ}\, QF(u)d\tau$$

where $u = u(\tau) = p(\tau;\Phi,p_0) + \Phi(p(\tau;\Phi,p_0))$. Inequality (3.6) is replaced by

$$|F(u)| \leq K_4'$$

where K_4' depends on $R(u)$, θ and ρ. Lemma 3.1 remains valid for $\alpha = 0$, and it is this version which will be used in the proof. Likewise, inequality (3.5) remains valid for $\alpha = 0$. In place of (3.7) one shows that

$$|\mathcal{J}\Phi(p_0)| \leq K_5' \lambda_{N+1}^{-1}$$

where $K_5' = K_4'$. Therefore one can take $b = K_5'\lambda_{N+1}^{-1}$. Lemma 3.3 is changed in that the norm $|Av|$ is replaced by $|v|$ throughout. Inequality (3.12) reduces to

$$(3.37) \qquad |F(u_1) - F(u_2)| \leq K_7'[(1 + \ell)|p_1 - p_2| + \|\Phi_1 - \Phi_2\|].$$

where $\|\Phi\| = \sup\{|\Phi(p)|: p \in PD(A)\}$.

A more significant change occurs in the arguments in Section 3.3. First of all, if Φ is fixed and $\Delta = p_1 - p_2$ then (3.14) is the same. However we now take the scalar product of (3.14) with Δ to obtain

$$\frac{1}{2}\frac{d}{dt}|\Delta|^2 + |A^{1/2}\Delta|^2 = -(P(F(u_1) - F(u_2)),\Delta).$$

From (3.37) one obtains

$$\left|\frac{1}{2}\frac{d}{dt}|\Delta|^2 + |A^{1/2}\Delta|^2\right| \leq K_7'(1 + \ell)|\Delta|^2,$$

which implies that

$$|\Delta|\frac{d}{dt}|\Delta| \geq -|A^{1/2}\Delta|^2 - K_7'(1 + \ell)|\Delta|^2$$

$$\geq -\lambda_N|\Delta|^2 - K_7'(1 + \ell)|\Delta|^2.$$

Consequently one has

$$|\Delta(\tau)| \le |\Delta(0)| \exp(-\tau(\lambda_N + K_7'(1+\ell))), \quad \tau < 0$$

in place of (3.17).

Hypothesis (3.18) in Lemma 3.4 is then replaced by

$$\Gamma_N := \lambda_{N+1} - \lambda_N - K_7'(1+\ell) > 0.$$

The Lipschitz condition (3.19) is replaced by

$$|\mathcal{J}\Phi(p_{01}) - \mathcal{J}\Phi(p_{02})| \le L|p_{01} - p_{02}|$$

where

$$L = K_7'(1+\ell)\Gamma_N^{-1}$$

Indeed the argument in Lemma 3.4 reduces to

$$\begin{aligned}|\mathcal{J}\Phi(p_{01}) - \mathcal{J}\Phi(p_{02})| &\le \int_{-\infty}^{0} |e^{\tau AQ}|_{op} |F(u_1) - F(u_2)| d\tau \\ &\le K_7'(1+\ell)\int_{-\infty}^{0} e^{\tau\lambda_{N+1}} |\Delta(\tau)| d\tau \\ &\le K_7'(1+\ell)|p_{01}-p_{02}| \int_{-\infty}^{0} \exp(\tau(\lambda_{N+1} - \lambda_N - K_7'(1+\ell)))d\tau \\ &\le K'(1+\ell)(\lambda_{N+1} - \lambda_N - K_7'(1+\ell))^{-1}|p_{01} - p_{02}|\end{aligned}$$

Similarly one has

$$|\mathcal{J}\Phi_1(p_0) - \mathcal{J}\Phi_2(p_0)| \le L'\|\Phi_1 - \Phi_2\|$$

where

$$L' = K_7'\lambda_{N+1}^{-1} + K_7'[\lambda_N + K_7'(1+\ell)]^{-1}L$$

Finally the two conditions

$$L < \ell, \quad L' < 1/2$$

are satisfied when $\ell < 1/8$ and

(3.42) $$K_{12} < \lambda_N, \qquad K_{13} < \lambda_{N+1} - \lambda_N$$

where $K_{12} = 4K_7'$ and $K_{13} = K_7'(1 + \ell)(1 + \ell^{-1})$. This completes the proof of Theorem 2.2.

Remark 3.1 The arguments used to prove the existence of inertial manifolds in Theorem 2.1 and 2.2 also imply that these manifolds are normally hyperbolic, cf. Hirsch-Pugh-Shub (1977) and Sacker-Sell (1980).

4. APPROXIMATION AND STABILITY WITH RESPECT TO PERTURBATIONS

The inertial manifold which we have constructed in Sections 2 and 3 is stable with respect to certain perturbations of the evolution equation. In order to give the main ideas without concern for some technical difficulties, we will restrict ourselves to the perturbations of (1.1), or better (1.24), corresponding to a Galerkin approximation associated with the eigenfunctions $\{w_j\}$. Thus for any integer $M > 1$ the perturbed equation is

$$\frac{du_M}{dt} + Au_M + P_M F(u_M) = 0 \tag{4.1}$$

where $F(u) = \theta_\rho(|Au|)R(u)$ and u_M takes its values in $P_M D(A)$. (We have used the fact that $P_M Au_M = AP_M u_M = Au_M$.) Equation (4.1) is an ordinary differential equation on the finite dimensional space $P_M D(A)$.

It is easy to see that (4.1) satisfies the same properties as (1.1) and (1.24). Consequently for all M large enough, Theorem 2.1 is applicable[1] and provides an inertial manifold $\mathcal{M}_M$ for (4.1). The inertial manifold for (4.1) is the graph of a Lipschitz function

$$\Phi_M : PD(A) \to QP_M D(A) \subseteq QD(A)$$

We will now investigate this point in more detail and study the convergence of Φ_M to Φ as $M \to \infty$. We will prove the following result:

<u>Theorem 4.1</u>

The assumptions are those of Theorem 2.1. Let $\ell > 0$ and N be given satisfying the conditions in Theorem 2.1.

Then for every $M > N$ equation (4.1) has an inertial manifold $\mathcal{M}_M$ which is constructed as the graph of a Lipschitz function

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(1) We will still write $P = P_N$, $Q = Q_N$ where N is given by Theorem 2.1. Of course we must assume that $M > N$.

$$\Phi_M : PD(A) \to QP_M D(A) \subseteq QD(A).$$

Moreover the Lipschitz constant ℓ for Φ_M is the same as that for the function $\Phi : PD(A) \to QD(A)$ constructed in Theorem 2.1. Finally one has

$$\|\Phi_M - \Phi\| \le 2K_6 \, \lambda_{N+1}^{-1/4} \, \lambda_{M+1}^{-1/4} \tag{4.2}$$

where

$$\|\Phi_M - \Phi\| = \sup_{p \in PD(A)} |A\Phi_M(p) - A\Phi(p)|.$$

Proof. It is straightforward to verify that (4.1) satisfies the same assumptions as (1.1) and (1.24). Consequently Theorem 2.1 is applicable to (4.1), and if N satisfies (2.13) for (1.24), the same conditions[(1)] are satisfied for (4.1) provided $M > N$.

It follows that (4.1) has an inertial manifold $\mathcal{M}_M$ for every $M > N$. Let us now look at the construction of this manifold with some care. First we define $\mathcal{F}_{b,\ell,M}$ to be the set of those functions

$$\Phi_M: PD(A) \to QP_M D(A) \subseteq QD(A)$$

which satisfy:

$$|A\Phi_M(p)| \le b \quad , \qquad \text{for all} \quad p \in PD(A)$$

$$|A\Phi_M(p_1) - A\Phi_M(p_2)| \le \ell |Ap_1 - Ap_2|, \quad \text{for all} \quad p_1, p_2 \in PD(A)$$

$$\operatorname{Supp} \Phi_M \subseteq \{p \in PD(A): |Ap| \le 4\rho\}.$$

Notice that one has $\mathcal{F}_{b,\ell,M} \subseteq \mathcal{F}_{b,\ell}$ for for all $M > N$. One then obtains $\mathcal{M}_M = \text{Graph}(\Phi_M)$ where Φ_M is a fixed point of the operator

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(1) This follows from the fact that $\|P_M\| \le 1$, $\|Q_M\| \le 1$. Therefore all the estimates for (4.1) are uniformly valid in M.

$\mathcal{J}_M : \mathcal{F}_{b,\ell,M} \to \mathcal{F}_{b,\ell,M}$ given by

$$\mathcal{J}_M \Phi_M(p_0) = -\int_{-\infty}^{0} e^{\tau AQP_M} \theta_\rho(|Au_M|)QP_M R(u_M)d\tau. \tag{4.3}$$

In (4.3) we use $u_M = p_M + \Phi_M(p_M)$ where $p_M = p_M(\tau;\Phi_M,p_0)$ is a solution (2.2). Becuase the P_M's and Q_M's are orthogonal commuting projections one has $QP_M = P_M - P$.

Let us also rewrite equations (2.2), (2.3) as they apply to (4.1). Let $u_M = p_M + q_M$ where $p_M = Pu_M \in PD(A)$ and $q_M = Qu_M = QP_M u_M \in QP_M D(A)$. Then (4.1) becomes

$$\frac{dp_M}{dt} + Ap_M + \theta_\rho(|Au_M|)PR(u_M) = 0 \tag{4.4}$$

$$\frac{dq_M}{dt} + Aq_M + \theta_\rho(|Au_M|)QP_M R(u_M) = 0. \tag{4.5}$$

Since $P_M e^{\tau AQ} = e^{\tau AQP_M} = e^{\tau AQP_M}P_M$, it follows from (2.11) and (4.3) that $\mathcal{J}_M$ is the restriction of $P_M \mathcal{J}$ to $\mathcal{F}_{b,\ell,M}$. This fact allows us to both derive properties of $\mathcal{J}_M$ from those of $\mathcal{J}$ and to compare the fixed points Φ_M and Φ. We will study next the convergence of $\mathcal{J}_M$ to $\mathcal{J}$ and the convergence of Φ_M to Φ.

Let $\Phi_M \in \mathcal{F}_{b,\ell,M} \subseteq \mathcal{F}_{b,\ell}$ and $p_0 \in P_M D(A)$. Let $u = u_M = p_M + \Phi_M(p_M)$ where $p_M = p_M(t;\Phi_M,p_0)$ is the solution of (4.4) (<u>and</u> also (2.2)). (These two equations are identical on $PD(A)$ for $\Phi \in \mathcal{F}_{b,\ell,M}$.) From (4.5) one has

$$\mathcal{J}\Phi_M(p_0) - \mathcal{J}_M\Phi_M(p_0) = Q_M \mathcal{J}\Phi_M(p_0).$$

In addition for all $M > N$, all $\Phi_M \in \mathcal{F}_{b,\ell,M}$ and all $p_0 \in PD(A)$ one has

$$|A\mathcal{J}\Phi_M(p_0) - A\mathcal{J}_M\Phi_M(p_0)| = |Q_M A\mathcal{J}\Phi_M(p_0)|$$

$$= |Q_M A^{-1/4}A^{5/4}\mathcal{J}\Phi_M(p_0)| < \lambda_{M+1}^{-1/4}|A^{5/4}\mathcal{J}\Phi_M(p_0)| < K_6\lambda_{N+1}^{-1/4}\lambda_{M+1}^{-1/4}$$

by (3.8). In other words one has

(4.6) $$\|\mathcal{J}\Phi_M - \mathcal{J}_M\Phi_M\| \le K_6\lambda_{N+1}^{-1/4}\ \lambda_{M+1}^{-1/4}, \quad \text{for all } \Phi_M \in \mathcal{F}_{b,\ell,M}.$$

Let us now compare Φ_M and Φ. By adding and subtracting $\mathcal{J}\Phi_M$ in

$$\Phi - \Phi_M = \mathcal{J}\Phi - \mathcal{J}_M\Phi_M$$

we obtain

$$\|\Phi - \Phi_M\| \le \|\mathcal{J}\Phi - \mathcal{J}\Phi_M\| + \|\mathcal{J}\Phi_M - \mathcal{J}_M\Phi_M\|.$$

From (3.25) and (4.6) we then get

$$\|\Phi - \Phi_M\| \le L'\|\Phi - \Phi_M\| + K_6\lambda_{N+1}^{-1/4}\lambda_{M+1}^{-1/4}$$

Since $L' < 1/2$ this yields (4.2). ▨

Remark 4.1. Inequality (4.2) should be useful not only in approximating the inertial manifold $\mathcal{M} = \text{Graph}(\Phi)$ but also in approximating the dynamical properties of $\mathcal{M}$. Recall that the dynamics on $\mathcal{M}$ is completely determined by the ordinary differential equation (see (2.12))

(4.7) $$\frac{dp}{dt} + Ap + PF(p + \Phi(p)) = 0,$$

where $\mathcal{J}\Phi = \Phi$. An approximation to (4.7) is given by

$$\frac{dp}{dt} + Ap + PF(p + \Phi_M(p)) = 0$$

where $\mathcal{J}_M\Phi_M = \Phi_M$. In order to compare these two equations we rewrite (4.7) as

$$\frac{dp}{dt} + Ap + PF(p + \Phi_M(p)) + E(p) = 0$$

where the "error" term E is defined by

$$E(p) = P(F(p + \Phi(p)) - F(p + \Phi_M(p))).$$

It follows from (3.12) and (4.2) that

$$|A^{1/2}E(p)| < K_7\|\Phi - \Phi_M\| < 2K_6K_7\lambda_{N+1}^{-1/4}\lambda_{M+1}^{-1/4}, \qquad \text{for all } p \in PD(A).$$

Remark 4.2. The contraction fixed point theorem is robust. What this means in our situation is that if the coefficients in (1.1) depend continuously on a parameter μ, and if the estimates (1.4)-(1.12), or (1.33) are valid uniformly in μ, then the inertial manifolds $\mathcal{M}(\mu)$ and the fixed point Φ_μ: $PD(A) \to QD(A)$ vary continuously in μ. In the same way Φ_μ will be Lipschitz continuous in μ if the coefficients have this property and the estimates are valid uniformly in μ.

Remark 4.3. The mapping $\mathcal{J}_M$ above appears as an approximation of $\mathcal{J}$. Of course one can consider other approximations of $\mathcal{J}$ and study, by similar methods, their convergence to $\mathcal{J}$.

5. FURTHER PROPERTIES OF INERTIAL MANIFOLDS

In this section we want to examine some additional properties of inertial manifolds. First we present several alternate characterizations of the inertial manifold constructed in Theorem 2.1 (or Theorem 2.2). Next we reformulate the operator $\mathcal{J}$ as a "vector" mapping. This reformulation will lead to a calculation of $\Phi(p)$ in terms of the eigenstates of A. Finally we derive an important relation between $\mathcal{M}_1$ and $\mathcal{M}_2$ when both $\mathcal{M}_1$ and $\mathcal{M}_2$ are inertial manifolds for (1.24). While our comments in this section apply equally to the inertial manifold theories described in Theorem 2.1 and Theorem 2.2, we will restrict them to the former in order to simplify the notation.

5.1 Characterization of the Inertial Manifold

Let N satisfy the condition of Theorem 2.1 and let Φ be the unique fixed point of the operator $\mathcal{J}$. Recall that from (2.11) one has

$$\Phi(p(t)) = -\int_{-\infty}^{t} e^{-(t-\tau)AQ}\, QF(u(\tau,p_0))d\tau$$

where $u(\tau,p_0) = p(\tau) + \Phi(p(\tau))$ and $p(\tau) = p(\tau;\Phi,p_0)$. Let $q(t) = \Phi(p(t))$, then $(p(t),q(t))$ is a solution of

$$\frac{dp}{dt} + Ap + PF(u) = 0 \tag{5.1}$$

$$\frac{dq}{dt} + Aq + QF(u) = 0. \tag{5.2}$$

From (3.7), (3.9) we have

$$|Aq(t)| \leq b, \quad \text{for all} \quad t \in R, \tag{5.3}$$

i.e. $q(t)$ has a "bounded" A-norm. The next theorem shows that this line of reasoning can be reversed.

Theorem 5.1

Let $p:R \to PH$ and $q:R \to QH$ be given continuous functions and set $u(t) = p(t) + q(t)$. Assume that $p(t)$ is a solution of (5.1). Then the following statements are equivalent:

(A) $u(t) \in \mathcal{M}$ for all $t \in R$.

(B) $q(t) = \Phi(p(t))$ for all $t \in R$.

(C) $q(t)$ is a solution of (5.2) and $q(0) = \Phi(p(0))$.

(D) $q(t)$ is a solution of (5.2) and (5.3) is valid.

(E) $q(t)$ is a solution of (5.2) and there is a $b_0 < +\infty$ with

$$|Aq(t)| \leq b_0, \text{ for all } t \in R. \tag{5.4}$$

(F) $q(t)$ is a solution of

$$q(t) = -\int_{-\infty}^{t} e^{-(t-\tau)AQ} QF(p(\tau) + q(\tau))d\tau. \tag{5.5}$$

Proof. (A) <==> (B). This follows from the fact that $\mathcal{M} = \text{Graph}(\Phi)$.

(B) <==> (C). This follows from the invariance of $\mathcal{M}$.

(D) ==> (E). Trivial.

(E) ==> (F). We will use here the ideas outlined in Section 2.1. By integrating (5.2) between s and t, $s < t$, we get

$$q(t) = e^{-(t-s)AQ} q(s) - \int_{s}^{t} e^{-(t-\tau)AQ} QF(p(\tau) + q(\tau))d\tau.$$

However from Lemma 3.1 and (5.4) we get

$$\begin{aligned} |e^{-(t-s)AQ} q(s)| &\leq |A^{-2}| \; |(AQ)e^{-(t-s)AQ}|_{op} \; |AQ(s)| \\ &\leq |A^{-2}| \; b_0 \; \lambda_{N+1} \; e^{-(t-s)\lambda_{N+1}}, \end{aligned}$$

for $t - s \geq \lambda_{N+1}^{-1}$. Let $s \to -\infty$ to conclude that $|e^{-(t-s)AQ}q(s)| \to 0$, which yields (5.5).

(F) ==> (C). By differentiating (5.5) with respect to t we see that $q(t)$ is a solution of (5.2). Equation (5.5) is, of course, another fixed point equation. Let $\mathcal{Q}$ denote the collection of all continuous functons $q:R \to QH$ for which

$$\sup\{|Aq(t)|: t \in R\} < +\infty.$$

We define a mapping $\hat{\mathcal{J}}$ on $\mathcal{Q}$ as follows: Fix $q(t) \in \mathcal{Q}$ and $p_0 \in PH$. Let $p(t)$ denote the solution of

$$\frac{dp}{dt} + Ap + PF(p + q(t)) = 0$$

that satisfies $p(0) = p_0$. Define

$$\hat{\mathcal{J}}q(t) = -\int_{-\infty}^{t} e^{-(t-\tau)AQ}\, QF(p(\tau) + q(\tau))d\tau.$$

The methods from Section 3 show that $\hat{\mathcal{J}}$ maps $\mathcal{Q}$ into itself and that $\hat{\mathcal{J}}$ is a strict contraction in the norm

$$\|q\| = \sup\{|Aq(t)|: t \in R\}.$$

By the contraction fixed point theorem, $\hat{\mathcal{J}}$ has a unique fixed point in $\mathcal{Q}$. Since $\hat{q}(t) := \Phi(p(t);\Phi,p_0))$ is a fixed point of $\hat{\mathcal{J}}$, we see that the $q(t)$ in (5.6) and $\hat{q}(t)$ are the same and that $p(t) = p(t;\Phi,p_0)$. Consequently one has $\hat{q}(0) = q(0) = \Phi(p(0))$. ▨

<u>Remark</u> 5.1 Other characterizations of the inertial manifold $\mathcal{M}$ are possible. For example, let $u(t)$ be any solution of (1.24). Then the following statements are equivalent:

(A) $u(t) \in \mathcal{M}$, for all $t \in R$.

(B) $Qu(0) = \Phi(Pu(0))$.

(C) There is a $b_0 < +\infty$ such that

$$|AQu(t)| \leq b_0, \quad \text{for all} \quad t \in R.$$

5.2 A Vector Formulation of the Operator $\mathcal{J}$.

Let $N = N_1$ satisfy the conditions of Theorem 2.1 and let Φ be the unique fixed point of the operator $\mathcal{J}$. Let N_2 be another integer with $N_1 < N_2$. Let P_i be the orthogonal projection onto $\mathrm{Span}\{w_j: 1 \leq j \leq N_i\}$ and set $Q_i = I - P_i$, $i = 1,2$. Let $R = P_2 - P_1$ and introduce two local coordinate systems on H:

$$u = p_1 + q_1 = p_2 + q_2$$

where $p_i = P_i u$, $q_i = Q_i u$, $i = 1,2$. Also define $p,r,s \in H$ by $p = P_1 u$, $r = Ru$, $s = Q_2 u$. Then one has

$$u = p + r + s, \quad q_1 = r + s, \quad p_2 = p + r.$$

Recall that $\mathcal{J}$ is given by

$$\mathcal{J}\Phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ_1} Q_1 F(u(\tau,p_0))d\tau$$

where $u(\tau,p_0) = p(\tau) + \Phi(p(\tau))$ and $p(\tau) = p(\tau;\Phi,p_0)$ for $p_0 \in P_1 D(A)$. Also $\mathcal{J}$ is a strict contraction on $\mathcal{F}_{b,\ell}$, as guaranteed by Theorem 2.1. For $\Phi \in \mathcal{F}_{b,\ell}$ we define

$$\Phi_r := R\Phi, \quad \Phi_s := Q_2\Phi.$$

Since $|R| \leq 1$, $|Q_2| \leq 1$ we see that Φ_r, $\Phi_s \in \mathcal{F}_{b,\ell}$ whenever $\Phi \in \mathcal{F}_{b,\ell}$ and that $\Phi = \Phi_r + \Phi_s$.

The fixed point problem $\mathcal{J}\Phi = \Phi$ can now be written as a vector fixed point problem

$$
(5.6) \qquad \begin{cases} \mathcal{J}_r\Phi = \Phi_r \\ \mathcal{J}_s\Phi = \Phi_s \\ \quad \Phi = \Phi_r + \Phi_s \end{cases}
$$

where $\mathcal{J}_r$ and $\mathcal{J}_s$ are defined by $\mathcal{J}_r = R\mathcal{J}$, $\mathcal{J}_s = Q_2\mathcal{J}$, or equivalently

$$
\begin{aligned}
\mathcal{J}_r\Phi(p_0) &= -\int_{-\infty}^{0} Re^{\tau AQ_1} Q_1F(u(\tau,p_0))d\tau \\
&= -\int_{-\infty}^{0} e^{\tau AR} RF(u(\tau,p_0))d\tau \\
\mathcal{J}_s\Phi(p_0) &= -\int_{-\infty}^{0} Q_2e^{\tau AQ_1} Q_1F(u(\tau,p_0))d\tau \\
&= \int_{-\infty}^{0} e^{\tau AQ_2} Q_2F(u(\tau,p_0))d\tau
\end{aligned}
$$

The fixed point problem (5.6) can be rewritten as

$$
(5.7) \qquad \begin{cases} \Phi_r(p_0) = -\int_{-\infty}^{0} e^{\tau AR} RF(p + \Phi_r(p) + \Phi_s(p))d\tau \\ \Phi_s(p_0) = -\int_{-\infty}^{0} e^{\tau AQ_2} Q_2F(p + \Phi_r(p) + \Phi_s(p))d\tau \end{cases}
$$

where $p = p(\tau) = p(\tau;\Phi,p_0)$. By replacing p_0 by $p(t)$ in (5.7) and using the group property of the solution $p(\tau;\Phi,p_0)$, we get

$$
\Phi_r(p(t)) = -\int_{-\infty}^{t} e^{-(t-\tau)AR} RF(p(\tau) + \Phi_r(p(\tau)) + \Phi_s(p(\tau)))d\tau
$$

$$
\Phi_s(p(t)) = -\int_{-\infty}^{t} e^{-(t-\tau)AQ_2} Q_2F(p(\tau) + \Phi_r(p(\tau)) + \Phi_s(p(\tau)))d\tau.
$$

Remark 5.2. By using the argument of Lemma 3.2 one can easily verify that

$$
(5.8) \qquad |A\Phi_r(p(t))| < b, \quad |A\Phi_s(p(t))| < (\lambda_{N_1+1}\cdot\lambda_{N_2+1}^{-1})^{1/2}\, b
$$

for all $t \in R$.

<u>Remark 5.3</u>. The reformulation of $\mathcal{J}$ given above can be extended to get a complete expansion of Φ in terms of the eigenvectors of A. For $i = 1,2,...$ let R_i denote the orthogonal projection of H onto $\text{Span}\{w_i\}$. For $\Phi \in \mathcal{F}_{b,\ell}$ define $\Phi_i = R_i \Phi$ and $\mathcal{J}_i = R_i \mathcal{J}$, $N + 1 \leq i$. The fixed point problem $\mathcal{J}\Phi = \Phi$ then becomes

$$(5.9) \qquad \begin{cases} \mathcal{J}_i \Phi = \Phi_i, \quad N + 1 \leq i \\ \Phi = \sum_{i=N+1}^{\infty} \Phi_i. \end{cases}$$

The solution of (5.9) is

$$\Phi_i(p_0) = -\int_{-\infty}^{0} e^{\tau AR_i} R_i F(p + \Phi(p))d\tau$$

$$= -\int_{-\infty}^{0} e^{\tau\lambda_i} R_i F(p + \Phi(p))d\tau,$$

$N + 1 \leq i$, which can be rewritten as

$$\Phi_i(p(t)) = -\int_{-\infty}^{t} e^{-(t-\tau)\lambda_i} R_i F(p + \Phi(p))d\tau, \quad N + 1 \leq i.$$

Furthermore one has

$$|A\Phi_i(p(t))| \leq (\lambda_{N_1+1} \lambda_i^{-1})^{1/2} b.$$

for all $t \in R$ and $i \geq N + 1$.

5.3 The Nesting of Inertial Manifolds.

Let $N_1 < N_2$ be two values of N that satisfy the conditions of Theorem 2.1. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be the associated operators defined by (2.12) and let Φ_1 and Φ_2 be the fixed points of $\mathcal{J}_1$ and $\mathcal{J}_2$. As in Section 5.2 we define

P_i, Q_i, $i = 1,2$ and set $R = P_2 - P_1$. Let

$$u = p_1 + q_1 = p_2 + q_2 = p + r + s$$

be coordinates on H where $p_i = P_i u$, $q_i = Q_i u$, $i = 1,2$, and $p = P_1 u$, $r = Ru$, $s = Q_2 u$. Also define

$$\Phi_r = R\Phi_1, \quad \Phi_s = Q_2\Phi_1.$$

Theorem 5.2

With the above notation one has

$$\Phi_2(p_0 + \Phi_r(p_0)) = \Phi_s(p_0), \quad \textit{for all} \quad p_0 \in P_1 D(A). \tag{5.10}$$

In particular one has $\mathcal{M}_1 \subseteq \mathcal{M}_2$, (*see Figure* 1).

Proof. Fix $p_0 \in P_1 D(A)$ and let $p(t) = p(t;\Phi_1,p_0)$, $r(t) = \Phi_r(p(t))$ and $s(t) = \Phi_s(p(t))$. Next define $u(t) = p_2(t) + q_2(t)$ where $p_2(t) = p(t) + r(t)$ and $q_2(t) = s(t)$. Then $(p_2(t); q_2(t))$ is a solution of the system

$$\frac{dp_2}{dt} + Ap_2 + P_2F(u) = 0$$

$$\frac{dq_2}{dt} + Aq_2 + Q_2F(u) = 0.$$

From (5.18) we have $|Aq_2(t)| < b_2$ for all $t \in R$. Therefore we can apply Theorem 5.1 to N_2, $\mathcal{J}_2$ and Φ_2 to obtain

$$q_2(0) = \Phi_2(p_2(0)),$$

which can be rewritten as (5.10).

Remark 5.4 Since an inertial manifold attracts all solutions at an exponential rate, we see that $\mathcal{M}_1$ is exponentially asymptotically stable in $\mathcal{M}_2$.

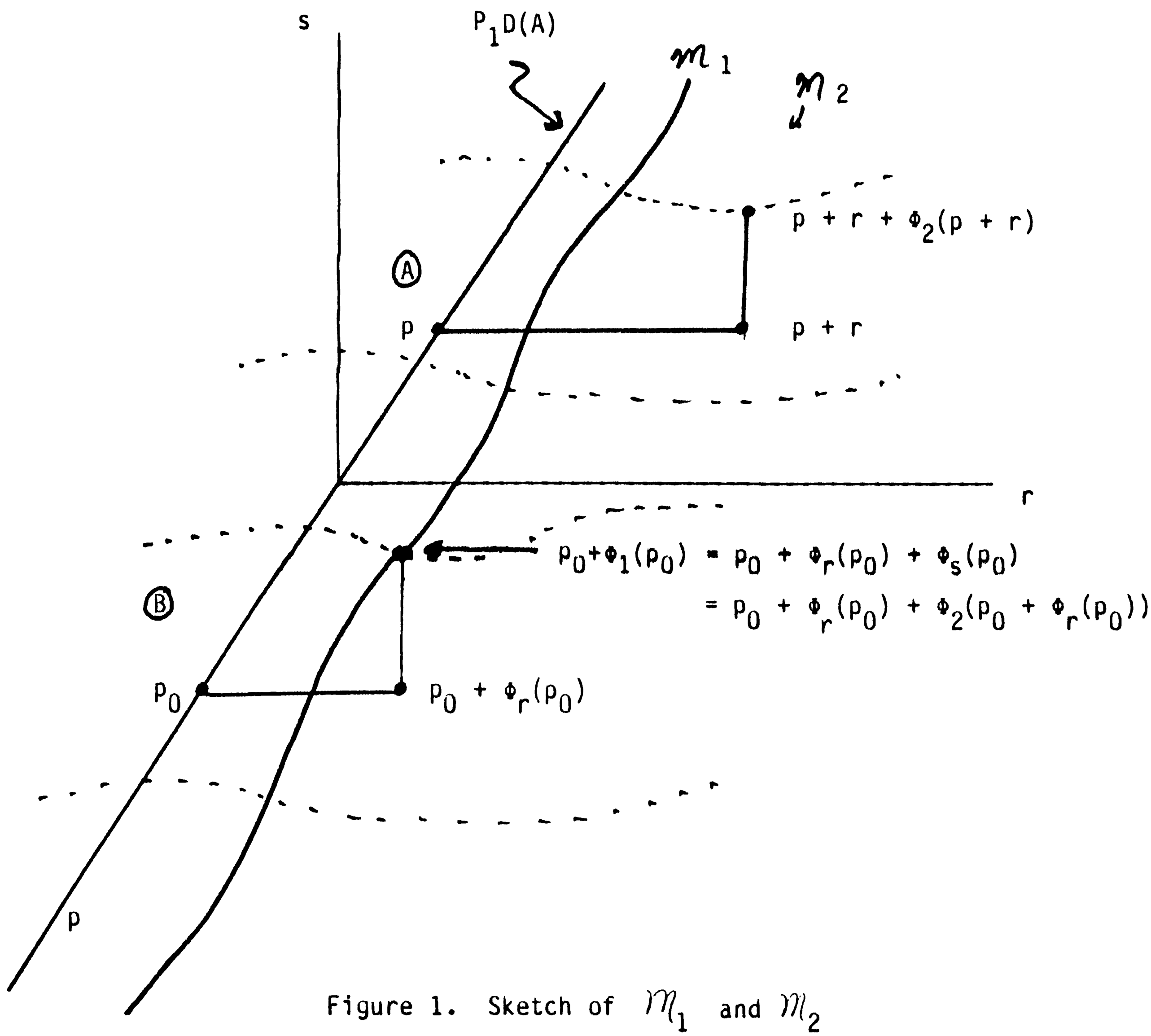

Figure 1. Sketch of $\mathcal{M}_1$ and $\mathcal{M}_2$

$$\mathcal{M}_1 = \{p + \Phi_1(p): p \in P_1D(A)\}$$

$$\mathcal{M}_2 = \{p + r + \Phi_2(p + r): p \in P_1D(A),\ r \in RD(A)\}.$$

Ⓐ typical point on $\mathcal{M}_2$

Ⓑ typical point on $\mathcal{M}_1$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$

6. SOME EXAMPLES

We present here a brief description of two examples to which the previous results apply. We refer the reader to Foias-Nicolaenko-Sell-Temam (1985,1986) for another example and to Constantin-Foias-Nicolaenko-Temam (1986) for a more geometrical construction of the inertial manifold with several examples. Also see Mallet-Paret and Sell (1986) for a different construction of the inertial manifold for a reaction-diffusion equation.

6.1 A Modified Navier-Stokes Equation

The first example is a modified Navier-Stokes equation in space dimension n with a higher order viscosity term, cf. Lions (1969):

$$\frac{\partial u}{\partial t} + \alpha\Delta^{2n} \cdot u - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \tag{6.1}$$

$$\nabla \cdot u = 0. \tag{6.2}$$

The functions $u = u(x,t)$ and $p = p(x,t)$ are defined on $R^n \times R_+$ taking values in R^n and R, respectively. (Note that $u = (u_1,\ldots,u_n)$.) Also α and ν are strictly positive numbers[1].

We assume that u and p are periodic in each direction $x_1,\ldots,x_n$ with period $L > 0$:

$$\left.\begin{aligned} u(x + Le_i,t) &= u(x,t) \\ p(x + Le_i,t) &= p(x,t) \end{aligned}\right\} \quad i = 1,\ldots,n, \tag{6.3}$$

where $\{e_1,\ldots,e_n\}$ is the natural basis of R^n. Furthermore we assume that

$$\int_\Omega u(x,t)dx = 0, \quad \int_\Omega p(x,t)dx = 0 \tag{6.4}$$

---

[1] (6.1) reduces to the usual Navier-Stokes equation when $\alpha = 0$.

where $\Omega = [0,L]^n$ is the n-cube, see Temam (1983).

By classical results (6.1)-(6.4) reduces to an evolutionary equation for u of the form (1.1) where the assumptions (1.3)-(1.12) are satisfied. Let $H^m_{per}(\Omega)$ denote the restriction to Ω of the Ω-periodic function v from R^n to R which are locally in $H^m(R^n)$ $(m > 1)$. Let $\dot{H}^m_{per}(\Omega)$ denote the subspace consisting of the functions u that satisfy $\int_\Omega v(x)dx = 0$. The spaces $H^m_{per}(\Omega)$ and $\dot{H}^m_{per}(\Omega)$ are both Hilbert subspaces of $H^m(\Omega)$.

For the application of Theorem 2.1 we define the space H to be the subspace of $L_2(\Omega)^n$ consisting of the restrictions to Ω of the locally L^2 vector functions with a free divergence and a null average on Ω. We set $D(A) = \dot{H}^2_{per}(\Omega)^n \cap H$ and $D(A^{1/2}) = \dot{H}^1_{per}(\Omega)^n \cap H$. Then

$$Au = \alpha\Delta^2 u \qquad , \text{ for all } u \in D(A)$$

$$Cu = -\nu\Delta u \qquad , \text{ for all } u \in D(A^{1/2})$$

and $B(u,v)$ is defined by

$$(B(u,v),w) = \int_\Omega ((u \cdot \nabla)v)w \, dx, \quad \text{for all } u,v,w \in D(A).$$

All the assumptions of Section 1 are satisfied. The reader is referred to Temam (1983) for the details.

It remains to verify the assumptions of Theorem 2.1, especially (2.15). It follows from the methods of Temam (1983), Metivier (1978) that there is a $c > 0$ such that

$$\lambda_N \sim c\lambda_1 N^4 \qquad , \text{ as } N \to \infty.$$

Clearly (2.15) is satisfied for N sufficiently large and consequently there is an inertial manifold for (6.1)-(6.4)

6.2 A Reaction-Diffusion Equation

The next example is the vector-valued reaction-diffusion equation in space dimension n:

$$u_t = \nu\Delta u + g(u) \tag{6.5}$$

where $u = (u_1,\ldots,u_m)$ and $\nu > 0$. We assume that (6.5) is given on the n-cube $\Omega_n = [0,2\pi]^n$ with periodic boundary conditions[(1)]. This gives rise to an abstract equation

$$\frac{du}{dt} + Au + G(u) = 0 \tag{6.6}$$

on $H = L_2(\Omega_n)^m$ where $A = -\nu\Delta$ and $G(u)(x) = g(u(x))$. We assume that g is chosen so that (i) the solutions of (6.5) are well-defined and regular for $t > 0$, (ii) $G(u)$ is locally Lipschitz continuous on H and (iii) equation (6.6) is dissipative, cf. Henry (1981). An example of this occurs when (6.5) is a scalar equation $(m = 1)$ and $g(u)$ is an odd degree polynomial with $ug(u) < 0$ for $|u|$ large.

With periodic boundary conditions the operator A is not positive (it is nonnegative) since $\lambda = 0$ is an eigenvalue. We then modify the equation by replacing Au by $(Au + \alpha u)$ and replace $G(u)$ by $G(u) - \alpha u$, where $\alpha > 0$. With this change the eigenvalues of A now have the form

$$\nu(m_1^2 + \ldots + m_n^2) + \alpha$$

where $m_1,\ldots,m_n$ are integers.

The global attractor $\mathcal{A}$ for (6.6) lies in a bounded set in H. Choose $\rho > 0$ so that

---

(1) The theory described here also applies with Dirichlet or Neuman boundary conditions on Ω_n. Also the nonlinear term can depend on $x \in \Omega_n$.

$$\mathcal{A} \subseteq \{u \in H\colon |u| < \rho/2\},$$

the modified equation is

$$\frac{du}{dt} + Au + \theta_\rho(|u|)G(u) = 0$$

and $F(u) = \theta_\rho(|u|)G(u)$ satisfies the global Lipschitz condition (1.33).

In order to apply Theorem 2.2 we need to verify the spectral gap condition (2.20), that is we want to choose N so that

$$\lambda_{N+1} - \lambda_N > K_{13}.$$

However the last inequality is valid (for an arbitrary $\nu > 0$) only in space dimensions $n = 1,2$, cf. Richards (1982) and Hardy-Wright (1962). We conclude therefore that (6.5) has an inertial manifold for every $\nu > 0$ when $n = 1,2$.

Remark 6.1 The existence of an inertial manifold for reaction-diffusion equations (with Neumann boundary conditions) was shown implicitly by Conway-Hoff-Smoller (1978) under the assumption that the diffusion coefficient (i.e. viscosity) is very large.

7. CONCLUSIONS AND OPEN PROBLEMS

In the Introductory Section we had said that our theory of inertial manifolds is not applicable to the Navier-Stokes equation in any space dimension $n \geq 2$. We want to explain the reason for this. However before doing that, it is informative to point out some features of the two inertial manifold theorems which may not be apparent upon first reading. For this purpose it is convenient to introduce a viscosity factor into (1.1) and (1.32) and write these equations as

$$\frac{du}{dt} + \nu Au + R(u) = 0 \tag{7.1}$$

where ν is a positive constant. If the eigenvalues of A satisfy (1.4) then the eigenvalues of νA are

$$0 < \nu\lambda_1 \leq \nu\lambda_2 \leq \dots .$$

We assume that the conditions (1.5)-(1.12), or condition (1.34), is also satisfied.

The main point which must be checked in order to determine whether (7.1) has an inertial manifold (as a consequence of Theorem 2.1 or 2.2) is the spectral gap condition in (2.14) or (2.20). Our hope is to show that for <u>each</u> $\nu > 0$, equation (7.1) has an inertial manifold. This means that we must seek an N such that

$$\lambda_{N+1} - \lambda_N \geq \nu^{-1} K_{13} \tag{7.2}$$

in the case of Theorem 2.2, or

$$\lambda_{N+1}^{1/2} - \lambda_N^{1/2} \geq \nu^{-1/2} K_{10}$$

in the case of Theorem 2.1. The last inequality can be rewritten as

$$\lambda_{N+1} - \lambda_N > \nu^{-1/2} K_{10}(\lambda_{N+1}^{1/2} + \lambda_N^{1/2}). \tag{7.3}$$

We see that both Theorems 2.1 and 2.2 require arbitrarily large spectral gaps in the spectrum of A in order to guarantee the existence of an inertial manifold for every $\nu > 0$. The two conditions (7.2) and (7.3) can be reformulated heuristically as follows:

(7.2 new) "The spectrum of A should have arbitrarily large gaps".

(7.3 new) "The spectrum of A should have arbitrarily large gaps <u>and</u> these gaps should occur soon enough."

In the case of the Navier-Stokes equation (with periodic boundary conditions) one has that A is the restriction of the Laplacian (in space dimension $n = 2,3$) to divergence free, periodic vector fields. As noted in the last section, A satisfies (7.2) on the 2-cube $\Omega_2 = [0,2\pi]^2$. However Theorem 2.2 is not applicable because the nonlinear terms for the Navier-Stokes equation do not appear to be locally Lipschitz in the sense required for Theorem 2.2 Instead these nonlinearities do satisfy conditions similar to (1.7)-(1.10). While we do not have a rigorous proof, it appears that the spectrum of A does not satisfy (7.3), cf. Richards (1982). Therefore neither is Theorem 2.1 applicable for the Navier-Stokes equation.

The question of whether the Navier-Stokes equation in space dimension 2 admits an inertial manifold for every value of $\nu > 0$ remains one of the very interesting outstanding problems about inertial manifolds.

Another question which arises is whether the spectral gap conditions (7.2) or (7.3) are necessary for the existence of inertial manifolds. Since the Squeezing Property does not depend on large spectral gaps, there is some hope for a theory of inertial manifolds which uses weaker conditions than (7.2) or (7.3). (See Mallet-Paret and Sell (1986) for example.)

If Theorem 2.1 (or 2.2) is applicable to (7.1) for every $\nu > 0$, then for a fixed $\nu > 0$ there exist infinitely many choices of N that satisfy the spectral gap condition (7.2) or (7.3). Denote these N's by

$$N_1 < N_2 < N_3 < \dots$$

Then for each N_i there is an inertial manifold $\mathcal{M}(N_i)$ with $\dim \mathcal{M}(N_i) = N_i$. Obviously the manifold with lowest dimension is the object of greatest dynamical interest. In particular good estimates of this dimension, along with the basic question of the existence of inertial manifolds, are central issues in the study of nonlinear evolutionary equations, see for instance Foias-Nicolaenko-Sell-Temam (1985,1986).

As seen in Section 5 one has

$$\mathcal{M}(N_1) \subseteq \mathcal{M}(N_2) \subseteq \mathcal{M}(N_3) \subseteq \dots$$

It would be of great interest to study the global bifurcation of these manifolds as the viscosity crosses certain critical values.

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