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Abstract

We consider the convergence theory of the Successive Overrelaxation (SOR) iterative method for the solution of linear systems $Ax = b$, when the matrix A has block a $p \times p$ partitioned p–cyclic form. Our purpose is to extend much of the p–cyclic SOR theory for nonsingular A to consistent singular systems and to apply the results to the solution of large scale systems arising, *e.g.,* in queueing network problems in Markov analysis. Markov chains and queueing models lead to structured singular linear systems and are playing an increasing role in the understanding of complex phenomena arising in computer, communication and transportation systems.

For certain important classes of singular problems, we develop a convergence theory for p–cyclic SOR, and show how to repartition for optimal convergence. Recent results by Kontovasilis, Plemmons and Stewart on the new concept of convergence of SOR in an *extended* sense are further analyzed and applied to the solution of periodic Markov chains.

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1 Introduction

Markov chains sometimes possess the block partitioning property that the minimum number of transitions that must be made on leaving any state to return to that state, is a multiple of some integer $p > 1$. These models are said to be periodic of period p, or p–cyclic of index p. Bonhoure, Dallery and Stewart [2] have shown that Markov chains that arise from queueing network models frequently have this important property. These models usually possess extremely large numbers of states and iterative methods are normally used in their analysis. The problem of computing the stationary probabilities of such chains leads to linear systems of equations where one should take advantage of the resulting block cyclic structure of the coefficient matrix A in the iterations. With this in mind, we proceed to examine the application of block p–cyclic SOR iterative methods to the analysis of Markov chains.

More generally, block iterative methods are suitable for the solution of large and sparse systems of linear equations having matrices that possess a special structure. First we consider block p–cyclic SOR for arbitrary consistent systems of linear equations. We are given

$$Ax = b, \quad A \in \Re^{n\times n}, \ x, b \in \Re^n \tag{1}$$

and the usual block decomposition

$$A = D - L - U \tag{2}$$

where D, L and U are block diagonal, lower and upper triangular matrices respectively, and D is non-singular. The block SOR method for any $\omega \neq 0$ is defined as:

$$Dx^{(m)} = Dx^{(m-1)} + \omega(Lx^{(m)} - Dx^{(m-1)} + Ux^{(m-1)} + b), \quad m = 1, 2, \ldots \tag{3}$$

The method can be equivalently described as

$$x^{(m)} = \mathcal{L}_\omega x^{(m-1)} + c, \quad m = 1, 2, \ldots \tag{4}$$

where

$$\mathcal{L}_\omega = (D - \omega L)^{-1}\left[(1-\omega)D + \omega U)\right], \quad c = \omega(D - \omega L)^{-1}. \tag{5}$$

It is well known that, for nonsingular systems (1), SOR converges iff $\rho(\mathcal{L}_\omega) < 1$. The associated spectral convergence factor is then $\rho(\mathcal{L}_\omega)$.

For arbitrary systems (1), very little is known about the optimal relaxation parameter ω which minimizes $\rho(\mathcal{L}_\omega)$ as a function of ω. However, considerable information is known for the situations where the matrix A has a special block cyclic structure. For the important case of matrices with "Property A", Young [13] (see also [14]) discovered his famous result on the optimum ω. Here A in (1) has a special two–by–two cyclic block form. Young's result was generalized by Varga [11] (see also [12]) to consistently ordered p–cyclic matrices. In this case it is assumed (without loss of generality) that A has the partitioned block form

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & B_1 \\ B_2 & A_2 & 0 & \cdots & 0 \\ 0 & B_3 & A_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_p & A_p \end{pmatrix} \tag{6}$$

where each diagonal submatrix A_i is square and nonsingular. With D in (2) defined by $D \equiv \operatorname{diag}(A_1, A_2, \ldots, A_p)$, the associated block Jacobi matrix J_p defined by $J_p \equiv I - D^{-1}A$, has the form

$$J_p = \begin{pmatrix} 0 & 0 & 0 & \cdots & C_1 \\ C_2 & 0 & 0 & \cdots & 0 \\ 0 & C_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & C_p & 0 \end{pmatrix} \tag{7}$$

where $C_i \equiv -A_i^{-1}B_i,\ 1 \le i \le p$.

The matrix J_p is weakly cyclic of index p if there exists a permutation matrix P such that PJ_pP^T has the form (7), where the null diagonal blocks are square. When J_p is already in the form (7) it is called *consistently ordered.* For such matrices Varga [11] proved the important relationship

$$(\lambda + \omega - 1)^p = \lambda^{p-1}\omega^p\mu^p \tag{8}$$

between the eigenvalues μ of J_p and λ of $\mathcal{L}_\omega$. Assuming further that all eigenvalues of J_p^p satisfy

$$0 \leq \mu^p \leq \rho(J_p^p) < 1,$$

he showed that the optimum ω value ω_{opt} is the unique positive solution of the equation

$$[\rho(J_p)\omega]^p = p^p(p-1)^{1-p}(\omega - 1) \tag{9}$$

in the interval $(1, \frac{p}{p-1})$. This ω_{opt} yields a convergence factor equal to

$$\rho(\mathcal{L}_\omega) = (p-1)(\omega_{opt} - 1). \tag{10}$$

Similar results have been obtained (see,e.g., [5]) for the case where the eigenvalues of J_p^p are non-positive, that is,

$$-(\frac{p}{p-2})^p < -\rho(J_p^p) \leq \mu^p \leq 0.$$

Few applications of p–cyclic SOR have been found for $p > 2$ (see, e.g., Berman and Plemmons [1]), but it turns out that, for example, least squares computations lead naturally to a 3–cyclic SOR iterative scheme. Markham, Neumann and Plemmons [9] have described a formulation of the problem leading to a 2–cyclic SOR method, obtained by repartitioning the 3–cyclic coefficient matrix into a 2–cyclic form. They showed that this method always converges for sufficiently small SOR parameter ω, in contrast to the 3–cyclic formulation, and that the 2–cyclic approach is asymptotically faster. Recently, Pierce, Hadjidimos and Plemmons [10] have generalized and extended the technology of p–cyclic SOR by showing that if the spectrum of the p-th power, J_p^p, of the block Jacobi matrix given in (7) is either nonpositive or nonnegative, then repartitioning a block p–cyclic matrix into a block q–cyclic form, $q < p$, results in asymptotically faster SOR convergence for the same amount of work per iteration. As a consequence, 2–cyclic SOR is asymptotically optimal for SOR under these conditions. In particular, it follows that 2–cyclic is optimal for SOR applied to linear equality constrained least squares in the Kuhn–Tucker formulation since here the spectrum of J_3^3 is nonpositive.

In general, the requirement that the spectrum of J_p^p be either nonpositive or nonnegative is critical. Eiermann, Niethammer and Ruttan [3] have shown by considering experimental and theoretical counterexamples that, without this requirement, 2–cyclic SOR is not always superior to p–cyclic SOR, $p > 2$. Galanis and Hadjidimos [4] have now generalized all of this work for nonsingular systems by showing how to repartition a block p–cyclic consistently ordered matrix for optimal SOR convergence for the general real case of the eigenvalues of J_p^p. One must keep in mind that all these convergence results hold only in an asymptotic sense. Golub and de Pillis [6] have pointed out that short–term convergence, i.e., error reduction in the early iterations, may only be controlled by reducing the spectral norm of the iteration matrix, while long–term or asymptotic convergence is generally improved by minimizing the spectral radius, as described above.

All results mentioned thus far consider only *nonsingular* systems of equations of the form (1). For many applications, for example to Markov chains, the coefficient matrix A will be singular. Hadjidimos [7] has examined the singular case ($\det(A) = 0$ and $b \in \mathcal{R}(A)$), under the assumptions that: (1) the Jacobi matrix $J = J_p$ is weakly cyclic of index p, (2) the eigenvalues of J_p^p are nonnegative with $\rho(J) = 1$, and (3), that J has either a simple unit eigenvalue or a multiple one associated with 1×1 Jordan blocks. Hadjidimos proved, among other results, that ω_{opt} is the unique root of (9) (in the same interval as in the nonsingular case), where $\rho(J_p)$ has to be replaced by $\gamma(J_p)$, the maximum of the moduli of the eigenvalues of J_p, excluding those that have modulus 1, viz.,

$$\gamma(J_p) \equiv \max\left\{|\lambda|,\ \lambda \in \sigma(J),\ |\lambda| \neq 1\right\}.$$

Recall now that, if A is a singular irreducible M–matrix then $1 \in \sigma(\mathcal{L}_\omega)$ for all ω and, the conditions for *semiconvergence* (see, e.g., [1]) become:

- $\rho(\mathcal{L}_\omega) = 1$.
- Elementary divisors associated with 1 are linear, i.e., $\text{rank}(I - \mathcal{L}_\omega)^2 = \text{rank}(I - \mathcal{L}_\omega)$.
- If $\lambda \in \sigma(\mathcal{L}_\omega)$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\gamma(\mathcal{L}_\omega) < 1$.

For consistency, will use the term convergence to mean semiconvergence in the singular case.

The results we will obtain here on optimal p–cyclic SOR for consistent linear systems $Ax = b$ have applications to discrete ergodic Markov Chain problems with a transition probability matrix P being cyclic with period p, as discussed in [8]. In particular, Markov chains sometimes possess the property that the minimum number of transitions that must be made on leaving any state to return to that state, is a multiple of some integer $p > 1$. These models are said to be periodic of period p, or p–cyclic of index p. In a recent interesting paper, Bonhoure, Dallery and Stewart [2] have shown that Markov chains that arise from queueing network models frequently possess this property.

Indeed, in the discrete case, the problem to be solved is

$$\pi^T P = \pi^T, \quad \|\pi\|_1 = 1 \tag{11}$$

or, equivalently,

$$(I - P^T)\pi = 0, \quad \|\pi\|_1 = 1,$$

where the element π_i is the probability of being in state i when the system reaches statistical equilibrium. It is immediate that, setting $A \equiv I - P^T$, and noting that if P is a cyclic stochastic matrix with transpose of the form (7), the corresponding homogeneous problem has a matrix that is of the form (6) and the associated Jacobi matrix is $J_p = P^T$. Therefore, all the results of this paper carry over to p–cyclic Markov Chains, simply by replacing J_p with P^T. In particular, the matrix A is a singular M–matrix and is irreducible when the chain is ergodic. Thus the conditions for semiconvergence described earlier apply to this Markov chain application.

For homogeneous continuous time p–cyclic Markov chains with infinitesimal generator Q, considered in [2,8], we are interested in solving

$$\pi^T Q = 0, \quad \|\pi\|_1 = 1. \tag{12}$$

Equation (12) may also be written in the form (11), where

$$P = Q\Delta t + I,$$

if Δt is sufficiently small. In the p–cyclic case, the infinitesimal matrix Q in (12) is such that Q^T has the block form (6).

Markov chains and queueing models thus lead to structured singular, irreducible linear systems of the type considered in this paper. Queueing models are playing an increasing role in the understanding of complex phenomena arising in computer, communication and transportation systems.

Our purpose in this paper is to extend the p–cyclic SOR theory to the singular case and to apply the results to the solution of large scale singular systems arising, *e.g.,* in queueing network problems in Markov analysis. For certain important classes of singular problems, we provide a convergence theory for p–cyclic SOR in §2, and show in §3 how to repartition for optimal convergence. Recent results by Kontovasilis, Plemmons and Stewart [8] on the new concept of convergence of SOR in an *extended* sense are rigorously analyzed in §4, for the important case where J_2^2 (or J_q^q for some $q \leq p$) has all real eigenvalues with the same sign.

2 The General p-Cyclic Case

For the determination of ω_{opt} in the p-cyclic singular case arising in Markov chains, we begin our analysis by giving the corresponding result for the nonsingular one. This is stated in [3] or, in a more compact form, in Theorem 2.2 of [4]. More specifically:

Lemma 2.1 *Suppose we are given a nonsingular system of the form (1), and let ω_p and ρ_p denote the relaxation factor and the convergence factor, respectively, of the optimal p–cyclic SOR for which*

$$\begin{array}{c} \sigma(J_p^p) \subset [-\alpha^p, \beta^p], \quad -\alpha^p,\ \beta^p\ \in\ \sigma(J_p^p), \\ 0 \leq \alpha < \frac{p}{p-2}, \quad 0 \leq \beta < 1. \end{array} \tag{13}$$

Then ω_p and ρ_p are determined from the equations

$$\left(\frac{(\alpha_p + \beta_p)}{2}\ \omega \right)^p - \frac{(\alpha_p + \beta_p)}{(\beta_p - \alpha_p)}\ (\omega - 1) = 0 \tag{14}$$

and

$$\rho_p = \left(\frac{\alpha_p + \beta_p}{2} \right) (\omega_p - 1) = \left(\frac{(\alpha_p + \beta_p)}{2}\ \omega_p \right)^p \tag{15}$$

where ω_p is the unique positive root of (14) in

$$\left(\min\{1, 1 + \frac{\beta_p - \alpha_p}{\alpha_p + \beta_p}\}, \quad \max\{1, 1 + \frac{\beta_p - \alpha_p}{\alpha_p + \beta_p}\} \right) \tag{16}$$

and where

$$
\begin{array}{llll}
i) & \alpha_p = \frac{p-2}{p}\beta_p, & \beta_p = \beta & \text{iff } \frac{\alpha}{\beta} \le \frac{p-2}{p} \\
ii) & \alpha_p = \alpha, & \beta_p = \beta & \text{iff } \frac{p-2}{p} \le \frac{\alpha}{\beta} \le \frac{p}{p-2} \\
iii) & \alpha_p = \alpha, & \beta_p = \frac{p-2}{p}\alpha_p & \text{iff } \frac{p}{p-2} \le \frac{\alpha}{\beta}.
\end{array} \tag{17}
$$

Note: The limiting cases $\alpha = \beta (= 0 \text{ or } \neq 0)$ lead to $\omega_p = 1$ and $\rho_p = \alpha^p = \beta^p$; while for $\alpha \neq 0$, $\beta = 0$ it is assumed that $\frac{\alpha}{\beta} = \infty$ and also for $p = 2$, $\frac{p}{p-2} = \infty$.

For the singular case we are studying here we recall a result from Theorem 3.1 of [7] which will be used in the sequel.

Lemma 2.2 *If the block Jacobi matrix J_p in (7) satisfies the assumption* index $(I - J_p) = 1$, *then for all $\omega \in (0, 2) \setminus \{p/(p-1)\}$ it follows that* index$(I - \mathcal{L}_\omega) =$ index$(I - J_p) = 1$.

For the general singular case of interest in in this paper we assume that

$$
\sigma(J_p^p) \subset [-\alpha^p, \beta^p] \cup \{1\}, \tag{18}
$$

with α and β being defined as in (13). Thus, under the assumption that index$(I - J_p) = 1$ and with (18) replacing the first part of (13), the main result of this section is identically the same as that of Lemma 2.1. Evidently, in (15) the optimal semiconvergence factor $\gamma_p = \gamma(\mathcal{L}_{\omega_p})$ must replace the optimal spectral radius $\rho_p = \rho(\mathcal{L}_{\omega_p})$. This results extends that obtained in Theorem 3.3 of [7], where the nonnegative case ($\alpha = 0 \le \beta < 1$) was treated, to that of the general real case of $\sigma(J_p^p)$. The resulting theorem can thus be stated.

Theorem 2.1 *Suppose we are given a possibly singular system of the form (1), and let ω_p and γ_p denote the relaxation factor and the convergence factor, respectively, of the optimal p-cyclic SOR for which*

$$
\begin{array}{c}
\sigma(J_p^p) \subset [-\alpha^p, \beta^p] \cup \{1\}, \ -\alpha^p, \ \beta^p \ \in \ \sigma(J_p^p), \\
0 \le \alpha < \frac{p}{p-2}, \quad 0 \le \beta < 1,
\end{array} \tag{19}
$$

and, moreover, assume that index$(I - J_p) = 1$. *Then ω_p and γ_p are determined by equations (13) through (17), with γ_p replacing ρ_p.*

3 Best Cyclic Repartitioning

As was mentioned in the Introduction, Markhan, Neumann and Plemmons [9] were the first who considered the problem of repartitioning a block 3-cyclic consistently ordered matrix into a 2-cyclic form for optimal SOR convergence. The most recent result on the general problem of the best cyclic repartitioning seems to be that obtained by Galanis and Hadjidimos [4]. It covers the case where the spectrum $\sigma(J_p^p)$ is real, under the assumptions (13), where the conditions on α are relaxed to $0 \leq \alpha < \infty$. The result is given in Theorem 2.1 of [4]. In the following lemma we give the main part of Theorem 2.1 of [4] and provide its accompanying Table R at the end of the paper.

Lemma 3.1 *Let J_p be the block Jacobi matrix (7) associated with the linear system (1), where A has the p-cyclic consistently ordered form (6), $p \geq 3$, and let $\sigma(J_p^p)$ satisfy (13), where the bound $\frac{p}{p-2}$ on α is replaced by ∞. Assume that A is repartitioned into a block q-cyclic consistently ordered form ($2 \leq q < p$) and denote by ω_q and ρ_q the relaxation factor and the spectral radius of the optimal q-cyclic SOR. Let r be the value of q that gives the best cyclic repartitioning; i.e., the smallest optimal spectral radius ρ_q. Then the value of r is given in Table R, where the quantities $\alpha_{\ell,\ell+1}$ and $\beta_{\ell,\ell+1}$ in the table are found from the expressions*

$$\begin{aligned} \alpha_{\ell,\ell+1} &= \left(\frac{2\rho^{1/\ell} - (1+\rho)\beta^{p/\ell}}{1-\rho} \right)^{\ell/p}, \\ \beta_{\ell,\ell+1} &= \left(\frac{2\rho^{1/\ell} - (1-\rho)\alpha^{p/\ell}}{1+\rho} \right)^{\ell/p}. \end{aligned} \tag{20}$$

In (20), ρ is the unique root, in $(0,1)$, of the equation

$$\beta^p(\ell+\rho)^{\ell+1} - (\ell+1)^{\ell+1}\rho = 0,$$

for $\alpha_{\ell,\ell+1}$, and of the equation

$$\alpha^p(\ell-\rho)^{\ell+1} - (\ell+1)^{\ell+1}\rho = 0,$$

for $\beta_{\ell,\ell+1}$. The values of ω_r and ρ_r are determined via (14)–(17), where in all these formulas, r replaces p and then $\alpha^{p/r}$ and $\beta^{p/r}$ replace α and β, respectively.

Note: As was pointed out in Kontovasilis, Plemmons and Stewart [8], Lemma 3.1 is of a more general value since it also covers the case of complex eigenvalues $\mu \in \sigma(J_p)$, with $\mu \in \mathbb{C}$, provided $|\mu| \leq \min\{\alpha, \beta\}$.

Suppose now that in the singular case of the p–cyclic consistently ordered matrix A $\operatorname{index}(I - J_p) = 1$. Suppose also that $\sigma(J_p^p)$ is given by (19), in which $0 \leq \alpha < \frac{p}{p-2}$ has been replaced by $0 \leq \alpha < \infty$. Suppose also that A is repartitioned into a block q–cyclic consistently ordered form. From the analysis in §2 it is obvious that Lemma 2.2 and therefore Theorem 2.1 apply to the singular case for $q = p$ provided $\alpha < \frac{p}{p-2}$. For Lemma 2.2 to apply for any $2 \leq q \leq p-1$ one must have $\omega \in (0,2) \setminus \{\frac{q}{q-1}\}$. This, however, assumes that the relationship

$$\operatorname{index}(I - J_q) = 1, \quad q = 2, \ldots, p-1, \tag{21}$$

is valid. There are certainly cases of vital practical importance where the implication (21) is a straightforward consequence of some further property of the matrix A. For example, (21) follows directly when A is a singular *irreducible* M-matrix (see, e.g, [1]), as in the case of the matrix coefficient in the Markov Chain problem with $A = I - P^T$, where P is the transition probability matrix and the chain is ergodic. However, (21) is always true under the assumption $\operatorname{index}(I - J_p) = 1$. This is stated in the following theorem.

Theorem 3.1 *Let J_p be the block Jacobi matrix (7) associated with the linear system (1), where A has the p-cyclic consistently ordered form (6), $p \geq 3$, and let* $\operatorname{index}(I - J_p) = 1$. *Assume that A is repartitioned into a block q–cyclic consistently ordered form* $(2 \leq q < p)$ *and denote by J_q be the block Jacobi matrix corresponding to the new repartinioning. Then (21) holds.*

So, under no further assumption, Theorem 2.1 holds for any q, when $\alpha^{p/q} < \frac{q}{q-2}$. Hence, a statement let us call it Theorem 3.2 (which will not be formally stated here), completely analogous to Lemma 3.1 holds true. We note that since Theorem 3.2 refers to the singular case instead of the optimal spectral radius ρ_q, one must use the optimal semiconvergence factor $\gamma_q = \gamma(\mathcal{L}_{\omega_q})$ in its place. Thus we obtain formulas on how to repatition A for optimal SOR convergence for computing the stationary distribution vectors of Markov chains. (See Table R at the end of the paper.)

4 Optimal Extended SOR

In this section we provide a convergence analysis for extended SOR convergence for the important 2-cyclic consistently ordered case. Here we assume that J_2^2 has all real eigenvalues with the same sign. It is shown that small perturbations around the optimal ω in the extended SOR method affect the convergence factor much less than for the usual SOR method. This formally confirms the validity of observations about numerical tests showing this phenomenom reported in [8].

We first introduce the following notation from [8]:

$$\begin{array}{rcl} \alpha(\omega) & := & \max\{|\lambda| := f_1(\lambda,\omega) = 0\} \geq 1, \\ \vartheta(\omega) & := & \max\{|\lambda| := f_\mu(\lambda,\omega) = 0,\ \mu \in \sigma(J_p), |\mu| < 1\}, \\ r(\omega) & := & \vartheta(\omega)/\alpha(\omega), \end{array} \tag{22}$$

where

$$f_\mu(\lambda,\omega) := \lambda^p - \omega\mu\lambda^{p-1} + \omega - 1. \tag{23}$$

In [8] a detailed analysis led to the determination of the optimal parameter(s) ω_p and therefore of the optimal convergence factor $r^p(\omega_p)$ of the extended SOR in the case of nonnegative spectra $\sigma(J_p^p)$. In this section we study in more detail the behavior of the asymptotic convergence factor of the extended SOR in both cases of the nonnegative ($\alpha = 0$) and the nonpositive ($\beta = 0$) spectra $\sigma(J_2^2)$ (see (19)). Also, the detailed study of the behavior of $r^2(\omega)$ around ω_p will subsequently allow us to explain the phenomenon observed in [8]; namely, that small perturbations around ω_p affect the convergence factor in the extended SOR much less than small perturbations around the corresponding ω_p in the usual SOR. In both cases to be studied the interval for ω will be considered as being $(-\infty,\infty) \setminus (0,2)$, i.e.,

$$\omega \in (-\infty, 0] \cup [2, \infty) \tag{24}$$

while from (22)

$$f_1(\lambda,\omega) = \lambda^2 - \omega\lambda + \omega - 1 = 0 \tag{25}$$

and

$$f_\mu(\lambda,\omega) = \lambda^2 - \omega\mu\lambda + \omega - 1 = 0, \tag{26}$$

with $\mu \in \sigma(J_2)$ and $|\mu| \neq 1$. From (22), (24) and (25) we immediately obtain that

$$\alpha(\omega) = |\omega - 1| \geq 1 \tag{27}$$

while from (22) and (26)

$$\vartheta(\omega) := \frac{1}{2} \max |\omega\mu \pm (\mu^2\omega^2 - 4\omega + 4)^{1/2}|. \tag{28}$$

4.1 The Nonnegative Case

In this case it is assumed that $\sigma(J_2)$ satisfies the following conditions

$$\sigma(J_2^2) \subset [0, \beta^2] \cup \{1\}, \quad 0 \leq \beta < 1. \tag{29}$$

Let then $0 \leq \mu^2 \leq \beta^2$, $\mu \in \sigma(J_2)$, where without loss of generality, we may consider μ such that $0 \leq \mu \leq \beta$. Denoting the discriminant in (28) by

$$D(\mu) := \mu^2\omega^2 - 4\omega + 4 \tag{30}$$

we readily have $D(\beta) \geq D(\mu) \geq D(0)$. So, we distinguish the three subcases a) $D(\beta) \leq 0$, b) $D(0) \geq 0$, and c) $D(\beta) \geq 0 \geq D(0)$ which are studied separately.

In subcase (a), (26) has two complex conjugate roots of modulus $|\omega - 1|^{1/2}$ each and this is the case when

$$\frac{2}{1 + (1 - \beta^2)^{1/2}} \leq \omega \leq \frac{2}{1 - (1 - \beta^2)^{1/2}}, \tag{31}$$

whence

$$\vartheta(\omega) = |\omega - 1|^{1/2}.$$

Therefore, setting

$$r^2(\omega) = \frac{\vartheta^2(\omega)}{\alpha^2(\omega)} = \frac{1}{|\omega - 1|} < 1, \tag{32}$$

we conclude that the extended SOR converges for all

$$\omega \in (2, \frac{2}{1 - (1 - \beta^2)^{1/2}}].$$

Table 1:

ω	$-\infty$		0	$-$	2		ω_2		∞
$r^2(\omega)$	β^2	$\nearrow$	1	$-$	1	$\searrow$	$r^2(\omega_2)$	$\nearrow$	β^2

From (32) it is also readily seen that $r^2(\omega)$ is a strictly decreasing function of ω. Specifically as ω increases from 2 to

$$2/(1-(1-\beta^2)^{1/2})$$

$r^2(\omega)$ decreases from 1 to

$$(1-(1-\beta^2)^{1/2})/(1+(1-\beta^2)^{1/2}).$$

After the study of subcases (b) and (c) takes place, the main result for the nonnegative case, which is stated below, confirms the corresponding one obtained in [8].

Theorem 4.1 *Let $\sigma(J_2)$ satisfy (29) and let* $\mathrm{index}(I - J_2) = 1$. *Then, the extended SOR converges for all $\omega \in (-\infty, 0) \cup (2, \infty)$. The optimal relaxation factor is given by*

$$\omega_2 = \frac{2}{1-(1-\beta^2)^{1/2}} \tag{33}$$

while for the (optimal) convergence factor there holds for $\omega_2 \neq \omega \in (-\infty, 0) \cup (2, \infty)$,

$$1 > r^2(\omega) > r^2(\omega_2) = \frac{1-(1-\beta^2)^{1/2}}{1+(1-\beta^2)^{1/2}}. \tag{34}$$

Furthermore, the behavior of $r^2(\omega)$ is illustrated in Table 1.

Note: As is seen from Table 1 when $\omega \to \pm\infty$, $r^2(\omega) \to \beta^2$, in other words, the extended SOR converges (in the limiting cases) as fast as the usual Gauss-Seidel method.

4.2 The Nonpositive Case

This time it is assumed that $\sigma(J_2)$ satisfies

$$\sigma(J_2^2) \subset [-\alpha^2, 0] \cup \{1\}, \quad 0 \leq \alpha < 1, \tag{35}$$

so $\tilde{\sigma}(J_2) := \sigma(J_2) \setminus \{-1, 1\}$ is purely imaginary. Let then $i\mu \in \tilde{\sigma}(J_2)$ be any eigenvalue of J_2, where without loss of generality we may assume that $0 \leq \mu \leq \alpha$ whence $-\alpha^2 \leq -\mu^2 \leq 0$. Obviously (25) will remain the same, leading again to the expression (27) for $\alpha(\omega)$, while (26) will become

$$f_{i\mu} = \lambda^2 - i\omega\mu\lambda + \omega - 1 = 0 \tag{36}$$

and hence

$$\vartheta(\omega) = \frac{1}{2} \max |i\omega\mu \pm (-\mu^2\omega^2 - 4\omega + 4)^{1/2}|. \tag{37}$$

The discriminant $D(\mu)$ in (37) is now

$$D(\mu) := -\mu^2\omega^2 - 4\omega + 4 \tag{38}$$

implying that $D(0) \geq D(\mu) \geq D(\alpha)$. Three subcases are considered again. Specifically, a) $D(\alpha) \geq 0$, b) $D(0) \leq 0$, and c) $D(0) \geq 0 \geq D(\alpha)$. As in the nonnegative case the simplest of the three subcases, that is (a), will be examined in the sequel. For this we have $D(\mu) \geq 0$ for all $\mu \in [0, \alpha]$, so $(D(\mu))^{1/2}$ is real. In view of the purely imaginary nature of $i\omega\mu$, (36) has two complex roots having the same imaginary parts and opposite in sign real parts. Hence, the two roots have equal moduli, consequently

$$\vartheta(\omega) = |\omega - 1|^{1/2}.$$

However, from $D(\alpha^2) \geq 0$ it is obtained that

$$\omega \in [\frac{2}{1 - (1 + \alpha^2)^{1/2}}, \frac{2}{1 + (1 + \alpha^2)^{1/2}}]$$

which together with

$$r^2(\omega) = \frac{\vartheta^2(\omega)}{\alpha^2(\omega)} = \frac{1}{|\omega - 1|} < 1, \tag{39}$$

Table 2:

ω	$-\infty$		ω_2		0	—	2		$\frac{2}{1-\alpha}$		∞
$r^2(\omega)$	α^2	$\searrow$	$r^2(\omega_2)$	$\nearrow$	1	—		$\searrow$	1	$\searrow$	α^2

which implies that $\omega \in (-\infty, 0) \cup (2, \infty)$, gives

$$\omega \in [\frac{2}{1-(1+\alpha^2)^{1/2}}, 0).$$

It can be readily found out that in the previous interval $r^2(\omega)$ strictly increases from the value

$$\frac{(1+\alpha^2)^{1/2}-1}{(1+\alpha^2)^{1/2}+1}$$

to the value 1.

After the examination of the subcases (b) and (c), the main result of the present section can be stated as follows:

Theorem 4.2 *Let $\sigma(J_2)$ satisfy (35) and let* $\mathrm{index}(I-J_2) = 1$. *Then, the extended SOR converges for all $\omega \in (-\infty, 0) \cup (\frac{2}{1-\alpha}, \infty)$ The optimal relaxation factor is given by*

$$\omega_2 = \frac{2}{(1+\alpha^2)^{1/2}-1},$$

while for the (optimal) convergence factor there holds for $\omega_2 \neq \omega \in (-\infty, 0) \cup (2, \infty)$,

$$r^2(\omega) > r^2(\omega_2) = \frac{(1+\alpha^2)^{1/2}-1}{(1+\alpha^2)^{1/2}+1}.$$

Moreover, the behavior of $r^2(\omega)$ is illustrated in Table 2.

Note: As in the nonnegative case when $\omega \to \pm\infty$, $r^2(\omega) \to \alpha^2$ and the extended (SOR) converges as fast as the usual Gauss-Seidel method.

5 Final Comments

Our purpose in this paper has been to extend the p–cyclic SOR theory from nonsingular problems to consistent singular systems and to apply the results to the solution of large scale singular linear systems arising in Markov analysis. We have developed a convergence theory for p–cyclic SOR, and shown how to repartition for optimal convergence.

Results by Kontovasilis, Plemmons and Stewart [8] on the new concept of convergence of SOR in an *extended sense* have been analyzed for the case where the spectrum $\sigma(J_p^p)$ is either nonnegative or nonpositive. Further work in this area is needed before the concept of extended SOR convergence can be fully applied. We have analyzed in detail the phenomenon observed in [8]; that small perturbations around ω_p affect the convergence factor in the extended SOR much less than small perturbations around the corresponding ω_p in the usual SOR. The extended SOR method is thus preferable to the usual SOR method for solving singular linear systems of equations associated with Markov chains.

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Table R: **Cyclicity r of the Best Repartitioning.**

<table>
<tr><th>Case</th><th>Value or domain of the ration $\frac{\alpha}{\beta}$</th><th>Values of ℓ and $\alpha_{\ell,\ell+1}$ (or $\beta_{\ell,\ell+1}$) if further subcases have to be considered</th><th colspan="2">Further Subcases</th><th>Cyclicity r of the best repartitioning</th></tr>
<tr><td rowspan="4">I</td><td>0
i) $(0 = \alpha < \beta < 1)$</td><td>—</td><td colspan="2">—</td><td>2</td></tr>
<tr><td rowspan="3">ii) $(0, \frac{p-2}{p})$</td><td rowspan="3">Determine the largest integer
$\ell \in \{2,3,\ldots,p-1\} : (\frac{\ell-2}{\ell})^{\ell} \leq (\frac{\alpha}{\beta})^{p}$
and then $\alpha_{\ell,\ell+1}$ from (20)</td><td colspan="2">a) $(\frac{\ell-2}{\ell})^{\ell/p}\beta \leq \alpha < \alpha_{\ell,\ell+1}$</td><td>$\ell$</td></tr>
<tr><td colspan="2">b) $\alpha = \alpha_{\ell,\ell+1}$</td><td>$\ell, \ell+1$*</td></tr>
<tr><td colspan="2">c) $\alpha_{\ell,\ell+1} < \alpha < (\frac{\ell-1}{\ell+1})^{(\ell+1)/p}\beta$</td><td>$\ell+1$</td></tr>
<tr><td>II</td><td>$[\frac{p-2}{p}, 1)$</td><td>—</td><td colspan="2">—</td><td>p</td></tr>
<tr><td>III</td><td>$(0 \leq \alpha = \beta < 1)$</td><td>—</td><td colspan="2">—</td><td>$2,3,\ldots,p$**</td></tr>
<tr><td>IV</td><td>$(1, \frac{p}{p-2}]$</td><td>—</td><td colspan="2">—</td><td>p</td></tr>
<tr><td rowspan="4">V</td><td rowspan="4">i) $(\frac{p}{p-2}, \infty)$</td><td rowspan="4">Determine the largest integers
$k \in \{2,3,\ldots,p\} : (\frac{k-2}{k})^{k} \leq \frac{1}{\alpha^{p}}$,
$\ell \in \{2,3,\ldots,\min(p-1,k)\} : (\frac{\ell-2}{\ell})^{\ell} \leq (\frac{\beta}{\alpha})^{p}$
and, if $\ell < k$, then $\beta_{\ell,\ell+1}$ from (20)</td><td>A)</td><td>$\ell = k$</td><td>ℓ</td></tr>
<tr><td rowspan="3">B) $\ell < k$</td><td>a) $(\frac{\ell-2}{\ell})^{\ell/p}\alpha \leq \beta < \beta_{\ell,\ell+1}$</td><td>$\ell$</td></tr>
<tr><td>b) $\beta = \beta_{\ell,\ell+1}$</td><td>$\ell, \ell+1$*</td></tr>
<tr><td>c) $\beta_{\ell,\ell+1} < \beta < (\frac{\ell-1}{\ell+1})^{(\ell+1)/p}\alpha$</td><td>$\ell+1$</td></tr>
<tr><td></td><td>∞
ii) $(\alpha > 0,\ \beta = 0)$</td><td>—</td><td colspan="2">—</td><td>2</td></tr>
</table>

* Either will do.

** Any will do. The optimal SOR is the Gauss-Seidel method.

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