ON THE GEOMETRY OF SEMI-LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THE PLANE INTEGRABLE BY THE METHOD OF DARBOUX

By

P.J. Vassiliou

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Permanent address: School of Information Science and Engineering, Canberra College of Advanced Education, Belconnen, A.C.T., Australia.

§2 Darboux's method for solving semilinear hyperbolic equations in the plane. In this section a brief description is given of the application of Darboux's method to the construction of explicit general solutions of semilinear hyperbolic equations. The reader should note that the method is applicable to more general equations but this is the case of interest in this paper. To keep this report within bounds, we will not give any proofs in this section. These will be given in [3].

Since we will only be considering local problems, all manifolds will be regarded as subsets of Euclidean spaces. Let $T\mathbf{R}^n$ and $T^*\mathbf{R}^n$ denote the tangent and cotangent bundles on R^n respectively. Let $C^\infty(\mathbf{R}^n)$ denote the ring of real valued, C^∞ functions on $\mathbf{R}^n$. Let $\Gamma^*(T^*\mathbf{R}^n)$ and $\Gamma(T\mathbf{R}^n)$ denote the $C^\infty(\mathbf{R}^n)$-modules of smooth sections of $T^*\mathbf{R}^n$ respectively. For simplicity, we will assume that all mappings and vector field systems, etc., are smooth even when some statements don't need that assumption. By a *distribution* of vector fields or *vector field system* we mean a $C^\infty(\mathbf{R}^n)$-submodule of $\Gamma(T\mathbf{R}^n)$ of constant dimension on $\mathbf{R}^n$ or some open subset of $\mathbf{R}^n$.

Consider the hyperbolic semilinear partial differential equation

$$\frac{\partial^2 \mathbf{u}}{\partial x_1 \partial x_2} = \mathbf{f}(x_1, x_2, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \frac{\partial \mathbf{u}}{\partial x_2}) \tag{2.1}$$

where $\mathbf{f}$ is a smooth $\mathbf{R}^m$-valued function of all its arguments and $\mathbf{u}$ is an $\mathbf{R}^m$-valued function of x_1 and x_2. Let $J^k(\mathbf{R}^2, \mathbf{R}^m)$ denote the k^{th} order jet bundle of map $\mathbf{R}^2 \longrightarrow \mathbf{R}^m$ and let $x_1, x_2, z^A, z^A_{i_1}, z^A_{i_1 i_2}, \ldots z^A_{i_1 i_2} \cdots {}_{i_k}$ be local coordinates on $J^k(\mathbf{R}^2, \mathbf{R}^m), i_1, \ldots i_k$ take values 1 or 2 and $1 \le A \le m$. Let $\mathcal{E}^2_m$ be the submanifold in $J^2(\mathbf{R}^2, \mathbf{R}^m)$ defined by the equation (2.1). Throughout, summation over repeated indices is assumed.

DEFINITION 2.1. The *Vessiot distribution* $\tilde{\Omega}_{\mathcal{E}^2_m}$ of equation (2.1) is the $2m$+2-generated distribution (over $C^\infty(\mathcal{E}^2_m)$)

$$\Big\{ D^{(2)}_1 = \partial_{x_1} + z^A_1 \frac{\partial}{\partial x^A} + z^A_{11} \frac{\partial}{\partial z^A_1} + f^A \frac{\partial}{\partial z^A_2}$$

$$D^{(2)}_2 = \frac{\partial}{\partial x_2} + z^A_2 \frac{\partial}{\partial z^A} + f^A \frac{\partial}{\partial z^A_1} + z^A_{22} \frac{\partial}{\partial z^A_2}$$

$$\frac{\partial}{\partial z^A_{11}}, \frac{\partial}{\partial z^A_{22}} \Big\}.$$

We note that $\tilde{\Omega}_{\mathcal{E}^2_m}$ is dual to the contact structure on $J^2(\mathbf{R}^2, \mathbf{R}^m)$ restricted to $\mathcal{E}^2_m$ and hence the solutions of the equation (2.1) correspond to the integral submanifolds of $\tilde{\Omega}_{\mathcal{E}^2_m}$. Clearly we can define the Vessiot distribution of a k^{th} order p.d.e. by writing out the dual of the appropriate contact structure suitably restricted to the submanifold of the corresponding jet bundle. Mostly in this paper we will stick to $k = 2$ and reserve only some remarks for general k.

Now from the point of view of the integration of hyperbolic partial differential equations, the most important construction is that of *Monge characteristics.* We have the following.

THEOREM 2.2. *The Monge characteristics for equation (2.1) are defined by a pair of vector field systems*

$$_1\tilde{\Omega}_{\mathcal{E}^2_m} : \left\{ D_1^{(2)} + (D_2^{(2)} f^A)\frac{\partial}{\partial z^A_{22}}, \frac{\partial}{\partial z^A_{11}} \right\} = \{\mathcal{D}_1,\ \mathcal{D}_{2A+1}\}$$
$$_2\tilde{\Omega}_{\mathcal{E}^2_m} : \left\{ D_2^{(2)} + (D_1^{(2)} f^A)\frac{\partial}{\partial z^A_{11}}, \frac{\partial}{\partial z^A_{22}} \right\} = \{\mathcal{D}_2,\ \mathcal{D}_{2A}\}$$

each of which is a $C^\infty(\mathcal{E}^2_m)$-submodule of $\tilde{\Omega}_{\mathcal{E}^2_m}$.

We note that each of $_1\tilde{\Omega}_{\mathcal{E}^2_m}$ and $_2\tilde{\Omega}_{\mathcal{E}^2_m}$ have dimension $m+1$ on $\mathcal{E}^2_m$ and that $_1\tilde{\Omega}_{\mathcal{E}^2_m} \oplus_2 \tilde{\Omega}_{\mathcal{E}^2_m} = \tilde{\Omega}_{\mathcal{E}^2_m}$. Also, as we shall see, each of $_1\tilde{\Omega}_{\mathcal{E}^2_m}$ and $_2\tilde{\Omega}_{\mathcal{E}^2_m}$ have one dimensional integral submanifold in $\mathcal{E}^2_m$ which depend on arbitrary functions which is a well known fact about Monge characteristics. The next theorem is basic to Darboux's algorithm for solving partial differential equation

THEOREM 2.3 (*Darboux, second order*). *Suppose* $\mathbf{u} : (x_1, x_2) \longrightarrow u^A(x_1, x_2)$ *satisfies equation (2.1) and that*

$$\pi : \mathcal{E}^2_m \longmapsto \mathbf{R}$$

is a nontrivial (this means apart from x_1 or x_2) invariant of either system $_1\tilde{\Omega}_{\mathcal{E}^2_m}$ or $_2\tilde{\Omega}_{\mathcal{E}^2_m}$ then

$$\pi \circ j^2\mathbf{u} = \lambda_1(x_2) \tag{2.2}$$

if π is an invariant of $_1\tilde{\Omega}_{\mathcal{E}^2_m}$ or

$$\pi \circ j^2\mathbf{u} = \lambda_2(x_1) \tag{2.3}$$

if π is an invariant of $_2\tilde{\Omega}_{\mathcal{E}^2_m}$, where λ_1 and λ_2 are real valued functions which depend upon the solution $\mathbf{u}$ chosen.

In fact, classically, Darboux's theorem is stated only for the scalar case $m = 1$, however, the extension to the vector case is straightforward. We note that since x_1 and x_2 are characteristic variables, the differential equations (2.2) and (2.3) are really ordinary differential equations with each of x_1 and x_2 appearing parametrically.

Now the point of Darboux's method is as follows. Suppose a collection of functionally independent invariants $\pi_1^1, \dots \pi_s^1$ for ${}_1\tilde{\Omega}_{\mathcal{E}^2_m}$ and $\pi_1^2, \dots, \pi_r^2$ for ${}_2\tilde{\Omega}_{\mathcal{E}^2_m}$ have been found. Then

$$\pi_i^1 \circ j^2\psi = \lambda_i^1(x_2), \qquad \pi_j^2 \circ j^2\psi = \lambda_j^2(x_1), \tag{2.4}$$

$1 \le i \le s, 1 \le j \le r$ is a collection of partial differential equations with solution space $\mathcal{S}$ (say). (Note that even though (2.4) may be overdetermined, $\mathcal{S}$ will not be empty by Darboux's theorem). By Darboux's theorem, there exists some subset $\mathcal{S}' \subseteq \mathcal{S}$ which solves (2.1). The problem is to derive $\mathcal{S}'$ from a knowledge of $\mathcal{S}$. This is possible when $r + s \ge m$ and in this case we will call the equation *Darboux integrable.* We will conclude our brief discussion of Darboux's method with an example and some remarks.

Example. (Goursat [2]) the equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{2}{x_1 + x_2}\sqrt{\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}} \tag{2.5}$$

is one of Goursat's canonical equations [5]. The second order Monge characteristics are given by theorem 2.2:

$${}_1\tilde{\Omega}_{\mathcal{E}^2_1} : \left\{ \frac{\partial}{\partial x_1} + z_1\frac{\partial}{\partial x} + z_{11}\frac{\partial}{\partial z_1} + \frac{2\sqrt{z_1 z_2}}{x_1 + x_2}\frac{\partial}{\partial z_2} + \frac{2}{x_1 + x_2}\left(\frac{z_{22}}{2}\sqrt{\frac{z_1}{z_2}} + \frac{z_2}{x_1 + x_2} - \frac{\sqrt{z_1 z_2}}{x_1 + x_2}\right) \times \frac{\partial}{\partial z_{22}}, \frac{\partial}{\partial z_{11}} \right\},$$

$${}_2\tilde{\Omega}_{\mathcal{E}^2_1} : \left\{ \frac{\partial}{\partial x_2} + z_2\frac{\partial}{\partial z} + \frac{2\sqrt{z_1 z_2}}{x_1 + x_2}\frac{\partial}{\partial z_1} + z_{22}\frac{\partial}{\partial z_2} + \frac{2}{x_1 + x_2}\left(\frac{z_{11}}{2}\sqrt{\frac{z_2}{z_1}} + \frac{z_1}{x_1 + x_2} - \frac{\sqrt{z_1 z_2}}{x_1 + x_2}\right) \times \frac{\partial}{\partial z_{11}}, \frac{\partial}{\partial z_{22}} \right\},$$

The (Riemann) invariants for equation (2.5) may now be computed.

$${}_1\tilde{\Omega}_{\mathcal{E}^2_1} : \begin{cases} \pi_1^1 = x_2 \\ \pi_2^1 = \frac{z_{22}}{2\sqrt{z_2}} + \frac{\sqrt{z_2}}{x_1+x_2} \end{cases}, \qquad {}_2\tilde{\Omega}_{\mathcal{E}^2_1} : \begin{cases} \pi_1^2 = x_1 \\ \pi_2^2 = \frac{z_{11}}{2\sqrt{z_1}} + \frac{\sqrt{z_1}}{x_1+x_2} \end{cases} \tag{2.6}$$

Equation (2.5) is now seen to be Darboux integrable having a nontrivial invariant for each of the characteristic systems. We now consider the equations (2.4) in this case. These are

$$\left(2\sqrt{\frac{\partial\psi}{\partial x_1}}\right)^{-1}\frac{\partial^2\psi}{\partial {x_1}^2} + \frac{1}{x_1 + x_2}\sqrt{\frac{\partial\psi}{\partial x_1}} = \lambda_1(x_1) \tag{2.7}$$

$$\left(2\sqrt{\frac{\partial\psi}{\partial x_2}}\right)^{-1}\frac{\partial^2\psi}{\partial {x_2}^2} + \frac{1}{x_1 + x_2}\sqrt{\frac{\partial\psi}{\partial x_2}} = \lambda_2(x_2) \tag{2.8}$$

where λ_1 and λ_2 are arbitrary real-valued functions. Setting $w^2 = \frac{\partial\psi}{\partial x_1}$, $v^2 = \frac{\partial\psi}{\partial x_2}$, $\lambda_1 = f''(x_1)$ and $\lambda_2 = g''(x_2)$ where f and g are arbitrary and substituting into (2.7) and (2.8) we obtain

$$\left.\begin{aligned} \frac{\partial w}{\partial x_1} + \frac{w}{x_1+x_2} &= f''(x_1), \\ \frac{\partial v}{\partial x_2} + \frac{v}{x_1+x_2} &= g''(x_2), \\ \frac{\partial}{\partial x_2}(w^2) - \frac{\partial}{\partial x_1}(v^2) &= 0. \end{aligned}\right\} \tag{2.9}$$

The system (2.9) has general solution

$$u = f' + \frac{g-f}{x_1+x_2}, \quad v = g' + \frac{f-g}{x_1+x_2}$$

and hence

$$\psi = \int (f' + \frac{g-f}{x_1+x_2})^2 dx_1 \quad + \quad \int (g' + \frac{f-g}{x_1+x_2})^2 dx_2 \tag{2.10}$$

is the most general solution of the overdetermined system (2.7), (2.8). Since the solution (2.10) contains two arbitrary functions we conclude that $\mathcal{S}' = \mathcal{S}$ and (2.10) is the general solution of (2.5).

There are a number of questions raised by this discussion of Darboux's method which we will not attempt to answer in this paper. For example, what is the relationship between $\mathcal{S}'$ and $\mathcal{S}$? We have not in this section even given a precise definition of the idea of "general solution" and the reader must be referred to [3] for a discussion of these matters. We simply wish to emphasize that the method depends upon the existence of invariants carried by the characteristics and these are in reality nothing more than the Riemann invariants familiar from another context. Now if no invariants exist on $J^2(\mathbf{R}^2, \mathbf{R}^m)$ then the next prolongation $J^3(\mathbf{R}^2, \mathbf{R}^m)$ must be considered. In this case one would investigate the third order characteristics. In this way we proceed to $J^k(\mathbf{R}^2, \mathbf{R}^m)$ for some k for which nontrivial invariants exist. Of course it isn't true that for any equation (2.1) there exists a k for which nontrivial invariants exist sufficient to integrate the equation. For example, it was demonstrated by Lie [6] that the nonlinear Klein-Gordon equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = f(u), \quad u : \mathbf{R}^2 \longrightarrow \mathbf{R},$$

is Darboux integrable if and only if $f(u) = \alpha e^{\beta u}, \alpha, \beta \in \mathbf{R}$. A proof of Lie's result can be found in [2].

§3 **Lie algebra structure for Monge characteristics.** From the brief description of Darboux's method given in the last section it is not difficult to see that when $m > 1$, the process of completing the integration once invariants have been found will in general be very difficult. In this case and as a first step in dealing with this problem one may make use of a fundamental discovery of E. Vessiot [4] which shows that there is an intimate relationship between the Monge characteristics of scalar hyperbolic equations ($m = 1$) and finite Lie algebras of vector fields. In [7], Vessiot's results were extended to the case $m = 2$. Specifically, we studied equations (2.1) in the case $m = 2$ where each system of characteristics on $J^2(\mathbb{R}^2, \mathbb{R}^2)$ have two two nontrivial invariants. Thus we studied equations which satisfy the condition of Darboux integrability as given in section 2. In this paper we build on the results of [7]. Let $E(2;3,3)$ denote the subset of equations (2.1) with $m = 2$ and with 2 nontrivial invariants for each of ${}_1\tilde{\Omega}_{\mathcal{E}^2_2}$ and ${}_2\tilde{\Omega}_{\mathcal{E}^2_2}$. Note that the elements of $\tilde{\Omega}_{\mathcal{E}^2_2}$ and ${}_i\tilde{\Omega}_{\mathcal{E}^2_2}, i = 1,2$, are 6-generated and 3-generated distributions respectively on $\mathcal{E}^2_2 \subseteq J^2(\mathbb{R}^2, \mathbb{R}^2)$.

Consider the structure of the Vessiot distribution ${}_1\tilde{\Omega}_{\mathcal{E}^2_2} \oplus_2 \tilde{\Omega}_{\mathcal{E}^2_2}$. Calculation shows that

$$\left.\begin{array}{l} [\mathcal{D}_1, \mathcal{D}_3] \equiv \dfrac{\partial}{\partial z^1_1}, [\mathcal{D}_1, \mathcal{D}_5] \equiv \dfrac{\partial}{\partial z^2_1}, \\ [\mathcal{D}_2, \mathcal{D}_4] \equiv \dfrac{\partial}{\partial z^1_2}, [\mathcal{D}_2, \mathcal{D}_6] \equiv \dfrac{\partial}{\partial z^2_2}, \\ \text{All other Lie brackets in } {}_1\tilde{\Omega}_{\mathcal{E}^2_2} \oplus {}_2\tilde{\Omega}_{\mathcal{E}^2_2} \equiv 0 \end{array}\right\} \bmod \tilde{\Omega}_{\mathcal{E}^2_2}.$$

We also note the conditions

$$[\mathcal{D}_5, [\mathcal{D}_1, \mathcal{D}_3]] \equiv 0, [\mathcal{D}_3, [\mathcal{D}_1, \mathcal{D}_3]] \equiv 0, [\mathcal{D}_6, [\mathcal{D}_2, \mathcal{D}_4]] \equiv 0, [\mathcal{D}_4, [\mathcal{D}_2, \mathcal{D}_6]] \equiv 0 \bmod \tilde{\Omega}_{\mathcal{E}^2_2}.$$

We are led naturally to study the class of C^∞ distributions of vector fields Π on $\mathbb{R}^{12}$ of the form

$$\Pi \ni \pi = \pi_1 \oplus \pi_2 : \{y_1, y_3, y_5\} \oplus \{y_2, y_4, y_6\}$$

satisfying the following properties

$$(1)\quad \left.\begin{array}{ll} [y_1, y_3] \equiv Z_3, & [y_1, y_5] \equiv Z_5 \\ [y_2, y_4] \equiv Z_4, & [y_2, y_6] \equiv Z_6 \\ \multicolumn{2}{l}{\text{All other Lie brackets in } \pi \equiv 0} \end{array}\right\} \bmod \pi$$

$$(2)\quad \left.\begin{array}{l} [y_5, [y_1, y_3]] \equiv 0, [y_3, [y_1, y_3]] \equiv 0 \\ [y_6, [y_2, y_4]] \equiv 0, [y_4, [y_2, y_6]] \equiv 0 \end{array}\right\} \bmod \pi$$

and finally, because we wish ultimately to study the class of p.d.e. $E(2;3,3)$

(3) π_2 has exactly 3 independent invariants, ξ, u_1 and u_2,
π_1 has exactly 3 independent invariants, η, v_1, and v_2,
while π has no invariants .

We note that $\{\pi, Z_3, ..Z_6\}$ is a 10-dimensional C^∞ distribution.

Let ξ, u_1, u_2 be as in (3) and choose any other 9 nine coordinates for $\mathbb{R}^{12}, x_1, .., x_9$. Suppose y_1, y_3 and y_5 have local expression of the form

$$\left.\begin{aligned} y_1 &= \omega_0^1 \frac{\partial}{\partial \xi} + \omega_1^1 \frac{\partial}{\partial u_1} + \omega_2^1 \frac{\partial}{\partial u_2} + \xi_1^\alpha \frac{\partial}{\partial x_\alpha} \\ y_3 &= \omega_0^3 \frac{\partial}{\partial \xi} + \omega_1^3 \frac{\partial}{\partial u_1} + \omega_2^3 \frac{\partial}{\partial u_2} + \xi_3^\alpha \frac{\partial}{\partial x_\alpha} \\ y_5 &= \omega_0^5 \frac{\partial}{\partial \xi} + \omega_1^5 \frac{\partial}{\partial u_1} + \omega_2^5 \frac{\partial}{\partial u_2} + \xi_5^\alpha \frac{\partial}{\partial x_\alpha} \end{aligned}\right\}$$

where $\omega_j^i, \xi_k^\alpha \in C^\infty(\mathbb{R}^{12})$. There are three possibilities for the matrix

$$A = (\omega_j^i).$$

Case I : Rank A = 1. The canonical form for y_1, y_2, y_3 is

$$y_1 = \frac{\partial}{\partial \xi} + \omega_1^1 \frac{\partial}{\partial u_1} + \omega_2^1 \frac{\partial}{\partial u_2} + \xi_1^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_3 = \xi_3^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_5 = \xi_5^\alpha \frac{\partial}{\partial x_\alpha}$$

Case II : Rank A = 2. The canonical form is

$$y_1 = \frac{\partial}{\partial \xi} + \omega_2^1 \frac{\partial}{\partial u_2} + \xi_1^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_3 = \frac{\partial}{\partial u_1} + \omega_2^3 \frac{\partial}{\partial u_2} + \xi_3^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_5 = \xi_5^\alpha \frac{\partial}{\partial x_\alpha}$$

Case III : Rank A = 3. The canonical form is

$$y_1 = \frac{\partial}{\partial u} + \xi_1^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_3 = \frac{\partial}{\partial u_1} + \xi_3^\alpha \frac{\partial}{\partial x_\alpha}, \quad y_5 = \frac{\partial}{\partial u_2} + \xi_5^\alpha \frac{\partial}{\partial x_\alpha}$$

We easily rule out case I since because of $[\pi_2, \pi_1] \equiv 0$ we have

$$y_{2j}\omega_1^1 = 0, \ \ y_{2j}\omega_2^1 = 0, \ \ 1 \le j \le 3,$$

whence ω_1^1 and ω_2^1 are invariants of π_2. It follows by property (3) that there exist $\phi_1, \phi_2 \in C^\infty(\mathbb{R}^3)$ such that

$$\omega_1^1 = \phi_1(\xi, u_1, u_2), \ \ \omega_2^1 = \phi_2(\xi, u_1 u_2).$$

This in turn shows that any nonconstant solution of

$$\frac{\partial f}{\partial \xi} + \phi_1 \frac{\partial f}{\partial u_1} + \phi_2 \frac{\partial f}{\partial u_2} = 0$$

will be an invariant of π which contradicts properly (3).

By a similar argument one shows that in case II, there exists functions $\psi_1, \psi_3 \in C^\infty(\mathbb{R}^3)$ such that

$$\omega_2^1 = \psi_1(\xi, u_1, u_2), \;\; \omega_2^3 = \psi_3(\xi, u_1, u_2).$$

It then follows that the canonical form of case II is unadmissible whenever there exist nonconstant solutions of

$$\lambda_1 f = \frac{\partial f}{\partial \xi} + \psi_1 \frac{\partial f}{\partial u_2} = 0, \;\; \lambda_2 f = \frac{\partial f}{\partial u_1} + \psi_3 \frac{\partial f}{\partial u_2} = 0.$$

Now since $\{\lambda_1, \lambda_2\}$ is 2-generated on $\mathbb{R}^3$, case II is inadmissible whenever $\{\lambda_1, \lambda_2\}$ is completely integrable. We now restrict our attention to that subset of the class Π of distributions where each of π_1 and π_2 have the canonical form of case III. Denote this subclass of Π by Π_1 and we have the following structure theorem for Π_1 which is proved in [7].

THEOREM 3.1. *Let $\pi \in \Pi_1$ then there exist coordinates on $\mathbb{R}^{12}$ such that $\pi = \pi_1 \oplus \pi_2$ is expressed locally as*

$$\pi_1 : \left\{ \mathcal{L} = \frac{\partial}{\partial \xi} + \sum_{\alpha=1}^{6} (\phi_\alpha^1(\xi, u_1) + \phi_\alpha^2(\xi, u_2)) L_\alpha, \;\; \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right\}$$

$$\pi_2 : \left\{ \mathcal{R} = \frac{\partial}{\partial \eta} + \sum_{\beta=1}^{6} (\psi_\beta^2(\eta, v_1) + \psi_\beta^2(\eta, v_2)) R_\beta, \;\; \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}$$

where π_1 and π_2 are each defined on $U \times W$ and $V \times W$ respectively and where U and V are manifolds with local coordinates (ξ, u_1, u_2) and (η, v_1, v_2) respectively. W is the manifold of remaining variables on $\mathbb{R}^{12}$ and $\phi_\alpha^i, \psi_\beta^j, \in C^\infty(\mathbb{R}^2)$, $1 \le i, j \le 2$, $1 \le \alpha, \beta \le 6$. Finally $\mathbf{L} = \{L_\alpha\}_{\alpha=1}^6, \mathbf{R} = \{R_\alpha\}_{\alpha=1}^6$ are simply transitive Lie algebras defined on W satisfying $[\mathbf{L}, \mathbf{R}] = 0$.

Now theorem 3.1 asserts the existence of an underlying Lie algebra structure for the class of distributions Π_1. Later we will state a theorem which will enable us to choose the functions in the distributions of theorem 3.1 in order that they correspond to the Monge characteristics of some partial differential equation $e \in E(2;3,3)$ (definition 3.2 and theorem 3.4). This then establishes a Lie algebra structure for a subset (at least) of the Monge characteristics for $E(2;3,3)$ once invariants have been chosen as new variables. Now if indeed a $\pi \in \Pi_1$ is associated (see definition 3.2) with a p.d.e. $e \in E(2;3,3)$ then (η, v_1, v_2) (say) are the functionally independent invariants of ${}_1\tilde{\Omega}_{\mathcal{E}_2^2}$ (say). Hence ${}_1\tilde{\Omega}_{\mathcal{E}_2^2}$ is a vector field system tangent to the submanifolds of $\mathcal{E}_2^2$ defined by $\eta =$ constant, $v_1 =$ constant, $v_2 =$ constant. More precisely, if we let $\mathfrak{F}_i$ denote the foliation of $\mathcal{E}_2^2$ induced by the invariants of ${}_i\tilde{\Omega}_{\mathcal{E}_2^2}$ then each leaf of $\mathfrak{F}_1 \simeq U \times W \simeq_{\text{locally}} \mathbb{R}^9$. Similarly each leaf of $\mathfrak{F}_2 \simeq V \times W \simeq_{\text{locally}} \mathbb{R}^9$.

Our aim now is to show how theorem 3.1 and its consequence can be used to study the Darboux integrability of the class $E(2;3,3)$. Ultimately we wish to show how to obtain local explicit general solutions for the class $E(2;3,3)$. We denote the class of distributions Π_1 which are the Monge characteristics for some p.d.e. $e \in E(2;3,3)$ by $\sum(2;3,3)$ and each of the subdistributions π_1 and π_2 by σ_1 and σ_2 respectively. Now our approach to the integration problem may be summarized by the following commuting diagram

$$\begin{array}{ccc} E(2;3,3) \ni e & \longrightarrow & (\text{explicit local solution of } e) \\ \downarrow & & \uparrow \\ \sum(2;3,3) \ni \sigma_G & \longrightarrow & (\text{local integral submanifold for } \sigma_G) \end{array}$$

The idea is that one *begins* with a Lie group G. As we will show, this determines a distribution $\sigma_G \in \sum(2;3,3)$ which is the Vessiot distribution of some $e \in E(2;3,3)$. We will show that the integral submanifolds of σ_G may be explicitly constructed and that these give the general solution of the equation e. Of course from the point of view of applied mathematics the question of interest is the converse : given $e \in E(2;3,3)$ find the corresponding $\sigma_G \in \sum(2;3,3)$ for some G. At present however it is not clear how this problem may be solved.

DEFINITION 3.2. A distribution of vector fields Λ on $\mathbb{R}^n$ is said to be *associated* with a p.d.e. (2.1) if and only if there exists a local diffeomorphism $\mathcal{P} : \mathbb{R}^n \longrightarrow J^2(\mathbb{R}^2, \mathbb{R}^m)$ such that

$$\mathcal{P}_*(\Lambda) = \tilde{\Omega}_{\mathcal{E}^2_m}$$

where $\mathcal{E}^2_m \subseteq J^2(\mathbb{R}^2, \mathbb{R}^m)$ is the submanifold defining the p.d.e.

DEFINITION 3.3. The *characteristic system* of a distribution of vector fields Λ denoted $char\Lambda$, is the set of all $\lambda \in \Lambda$ such that $[\lambda, \Lambda] \subseteq \Lambda$.

We now state and prove a theorem which gives sufficient conditions for a distribution of the form given in theorem 3.1 to be associated with an equation $e \in E(2;3,3)$.

THEOREM 3.4. *Let $\sigma \in \sum(2;3,3)$ be a distribution of the form given by theorem 3.1. That is*

$$\sigma : \{\mathcal{L}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\} \oplus \{\mathcal{R}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\}$$

Suppose the subdistribution

$$\Gamma : \left\{[\frac{\partial}{\partial u_j}, \mathcal{L}], [\frac{\partial}{\partial v_j}, \mathcal{R}], \frac{\partial}{\partial u_j}, \frac{\partial}{\partial v_j}\right\}_{j=1}^{2} \subseteq \sigma^{(1)}$$

has four functionally independent invariants x_1, x_2, z^1, z^2 *such that* $\mathcal{L}x_2 = \mathcal{R}x_1 = 0$ *and* $(\mathcal{L}x_1)(\mathcal{R}x_2) \neq 0$. *Define the functions*

$$(*) \qquad \begin{cases} z_1^A = \frac{\mathcal{L}z^A}{\mathcal{L}x_1}, \ z_2^A = \frac{\mathcal{R}z^A}{\mathcal{R}x_2}, \ z_{11}^A = \frac{\mathcal{L}z_1^A}{\mathcal{L}x_1} \\ \\ z_{22}^A = \frac{\mathcal{R}z_2^A}{\mathcal{R}x_2}, \ z_{12}^A = \frac{\mathcal{R}z_1^A}{\mathcal{R}x_2} = \frac{\mathcal{L}z_2^A}{\mathcal{L}x_1}, \ A = 1, 2. \end{cases}$$

Then whenever the functions $(x_1, x_2, z^A, z_1^A, z_{11}^A, z_{22}^A)$ *are functionally independent they define a map*

$$\mathcal{P} : U \times V \times W \longrightarrow J^2(\mathbb{R}^2, \mathbb{R}^2)$$

such that

$$(3.1) \qquad \mathcal{P}_* \sigma = \tilde{\Omega}_{\mathcal{E}_2^2}$$

where $\mathcal{E}_2^2 \subset J^2(\mathbb{R}^2, \mathbb{R}^2)$ *is defined by the functions* z_{12}^A.

Theorem 3.4 enables us to explicitly construct the equation or family of equations associated with a given $\sigma \in \sum(2; 3, 3)$. Now since the integration problem for the p.d.e. is equivalent to constructing the integral submanifolds of the Vessiot distribution $\tilde{\Omega}_{\mathcal{E}_2^2}$, the explicit knowledge of $\mathcal{P}$ means we can transfer the integration problem to σ which as we will show is easier.

Proof of theorem 3.4. The proof proceeds by attempting to solve the partial differential equations defining the equivalence (3.1). The existence of the map $\mathcal{P}$ implies the existence of a $g \in GL(6, \mathbb{R})$ satisfying

$$(3.2) \qquad \mathcal{P}_* \begin{pmatrix} \mathcal{L} \\ \mathcal{R} \\ \partial/\partial u_1 \\ \partial/\partial u_2 \\ \partial/\partial v_1 \\ \partial/\partial v_2 \end{pmatrix} = g \begin{pmatrix} D_1^{(2)} \\ D_2^{(2)} \\ \partial/\partial z_{11}^1 \\ \partial/\partial z_{11}^2 \\ \partial/\partial z_{22}^1 \\ \partial/\partial z_{22}^2 \end{pmatrix}.$$

where $D_1^{(2)}, D_2^{(2)}$ are the total differential operators of definition 2.1.

We have

$$(3.3) \qquad \begin{aligned} \mathcal{P}_*(Q_\alpha) = (Q_\alpha x_1)\frac{\partial}{\partial x_1} + (Q_\alpha x_2)\frac{\partial}{\partial x_2} + (Q_\alpha z^A)\frac{\partial}{\partial z^A} + (Q_\alpha z_1^A)\frac{\partial}{\partial z_1^A} \\ + (Q_\alpha z_2^A)\frac{\partial}{\partial z_2^A} + (Q_\alpha z_{11}^A)\frac{\partial}{\partial z_{11}^A} + (Q_\alpha z_{22}^A)\frac{\partial}{\partial z_{22}^A} \end{aligned}$$

where $\{Q_\alpha\}_{\alpha=1}^6$ is the set of vector fields $\{\mathcal{L}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \mathcal{R}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\}$.

Let

$$g = (\omega_{\alpha\beta})_{1\leq\alpha,\beta\leq 6} : U \times V \times W \longrightarrow GL(6,\mathbb{R}). \tag{3.4}$$

Then from (3.2), (3.3) and (3.4) we have the equations defining the equivalence as follows

$$\mathcal{L}x_1 = \omega_{11},\ \mathcal{L}x_2 = \omega_{12},\ \mathcal{L}z^A = \omega_{11}z_1^A + \omega_{12}z_2^1,\ \mathcal{L}z_1^A = \omega_{11}z_{11}^A + \omega_{12}f^A,$$

$$\mathcal{L}z_2^A = \omega_{11}f^A + \omega_{12}z_{22}^A,\ \mathcal{L}z_{11}^1 = \omega_{13},\ \mathcal{L}z_{11}^2 = \omega_{14},\ \mathcal{L}z_{22}^1 = \omega_{15},\ \mathcal{L}z_{22}^2 = \omega_{16},$$

$$\mathcal{R}x_1 = \omega_{21},\ \mathcal{R}x_2 = \omega_{22},\ \mathcal{R}z^A = \omega_{21}z_1^A + \omega_{22}z_2^A,\ \mathcal{R}z_1^A = \omega_{21}z_{11}^A + \omega_{22}f^A,$$

$$\mathcal{R}z_2^A = \omega_{21}f^A + \omega_{22}z_{22}^A,\ \mathcal{R}z_{11}^1 = \omega_{23},\ \mathcal{R}z_{11}^2 = \omega_{24},\ \mathcal{R}z_{22}^1 = \omega_{25},\ \mathcal{R}z_{22}^2 = \omega_{26};$$

$$\frac{\partial}{\partial u_A}x_1 = \omega_{2+A,1},\ \frac{\partial}{\partial u_A}x_2 = \omega_{2+A,2},\ \frac{\partial}{\partial u_A}z^B = \omega_{2+A,1}z_1^B + \omega_{2+A,2}z_2^B,$$

$$\frac{\partial}{\partial u_A}z_1^B = \omega_{2+A,1}z_{11}^B + \omega_{2+A,2}f^B,\ \frac{\partial}{\partial u_A}z_2^B = \omega_{2+A,1}f^B + \omega_{2+A,2}z_{22}^B \tag{3.5}$$

$$\frac{\partial}{\partial u_A}z_{11}^1 = \omega_{2+A,3},\ \frac{\partial}{\partial u_A}z_{11}^2 = \omega_{2+A,4},\ \frac{\partial}{\partial u_A}z_{22}^1 = \omega_{2+A,5},\ \frac{\partial}{\partial u_A}z_{22}^2 = \omega_{2+A,6};$$

$$\frac{\partial}{\partial v_A}x_1 = \omega_{4+A,1},\ \frac{\partial}{\partial v_A}x_2 = \omega_{4+A,2},\ \frac{\partial}{\partial v_A}z^B = \omega_{4+A,1}z_1^B + \omega_{4+A,2}z_2^B,$$

$$\frac{\partial}{\partial v_A}z_1^B = \omega_{4+A,1}z_{11}^B + \omega_{4+A,2}f^B,\ \frac{\partial}{\partial v_A}z_2^B = \omega_{4+A,1}f^B + \omega_{4+A,2}z_{22}^B,$$

$$\frac{\partial}{\partial v_A}z_{11}^1 = \omega_{4+A,3},\ \frac{\partial}{\partial v_A}z_{11}^2 = \omega_{4+A,4},\ \frac{\partial}{\partial v_A}z_{22}^1 = \omega_{4+A,5},\ \frac{\partial}{\partial v_A}z_{22}^2 = \omega_{4+A,6}$$

We now notice that considerable simplification is gained if we seek diffeomorphisms $\mathcal{P}$ such that g lies in the subgroup defined by matrices $g \in GL(6,\mathbb{R})$ of the form

$$g = \begin{pmatrix} \omega_{11} & 0 & \omega_{13} & \omega_{14} & \cdots & \omega_{16} \\ 0 & \omega_{22} & \omega_{23} & \omega_{24} & \cdots & \omega_{26} \\ 0 & 0 & \omega_{33} & \omega_{34} & \cdots & \omega_{36} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & \omega_{63} & \omega_{64} & \cdots & \omega_{66} \end{pmatrix}.$$

In this case equations (3.5) became

(3.6)$_1$

$$\mathcal{L}x_1 = \omega_{11},\ \mathcal{L}x_2 = 0,\ \mathcal{L}z^A = \omega_{11}z_{11}^A,\ \mathcal{L}z_2^A = \omega_{11}f^A,\ \mathcal{L}z_{11}^1 = \omega_{13},\ \mathcal{L}z_{11}^2 = \omega_{14},$$
$$\mathcal{L}z_{22}^1 = \omega_{15},\ \mathcal{L}z_{22}^2 = \omega_{16};$$

(3.6)$_2$

$$\mathcal{R}x_1 = 0,\ \mathcal{R}x_2 = \omega_{22},\ \mathcal{R}z^A = \omega_{22}z_2^A,\ \mathcal{R}z_1^A = f^A,\ \mathcal{R}z_2^A = \omega_{22}z_{22}^A,$$
$$\mathcal{R}z_{11}^1 = \omega_{23},\ \mathcal{R}z_{11}^2 = \omega_{23},\ \mathcal{R}z_{11}^2 = \omega_{24},\ \mathcal{R}z_{22}^1 = \omega_{25},\ \mathcal{R}z_{22}^2 = \omega_{26};$$

(3.6)$_3$

$$\frac{\partial}{\partial u_A}x_1 = \frac{\partial}{\partial v_A}x_1 = \frac{\partial}{\partial u_A}x_2 = \frac{\partial}{\partial v_A}x_2 = \frac{\partial}{\partial u_A}z^B = \frac{\partial}{\partial v_A}z^B = \frac{\partial}{\partial u_A}z_1^B$$
$$= \frac{\partial}{\partial v_A}z_1^B = \frac{\partial}{\partial u_A}z_2^B = \frac{\partial}{\partial v_A}z_2^B = 0;$$

(3.6)$_4$

$$\frac{\partial}{\partial u_A}z_{11}^1 = \omega_{2+A,3},\ ,\ \frac{\partial}{\partial u_A}z_{11}^2 = \omega_{2+A,4},\ \frac{\partial}{\partial u_A}z_{22}^1 = \omega_{2+A,5},\ \frac{\partial}{\partial u_A}z_{22}^2 = \omega_{2+A,6}$$
$$\frac{\partial}{\partial v_A}z_{11}^1 = \omega_{4+A,3},\ \frac{\partial}{\partial v_A}z_{11}^2 = \omega_{4+A,4},\ \frac{\partial}{\partial v_A}z_{22}^1 = \omega_{4+A,5},\ \frac{\partial}{\partial v_A}z_{22}^2 = \omega_{4+A,6}$$

The system (3.6) consists of 54 equations for 28 unknowns so it is overdetermined and we proceed to find all integrability conditions. From the first equation of (3.6)$_1$ and that of (3.6)$_3$ we have

$$\frac{\partial \omega_{11}}{\partial u_1} = \frac{\partial}{\partial u_1}\mathcal{L}x_1 - \mathcal{L}\frac{\partial x_1}{\partial u_1} = \left[\frac{\partial}{\partial u_1}, \mathcal{L}\right] x_1.$$

In the same way we find

$$\left.\begin{aligned} &\left[\frac{\partial}{\partial u_2}, \mathcal{L}\right] x_1 = \frac{\partial \omega_{11}}{\partial u_2},\ \left[\frac{\partial}{\partial v_A}, \mathcal{L}\right] x_1 = \frac{\partial \omega_{11}}{\partial v_A}, \\ &\left[\frac{\partial}{\partial u_A}, \mathcal{R}\right] x_1 = \left[\frac{\partial}{\partial v_A}, \mathcal{R}\right] x_1 = 0, \\ &\left[\frac{\partial}{\partial u_A}, \mathcal{L}\right] x_2 = \left[\frac{\partial}{\partial v_A}, \mathcal{L}\right] x_2 = 0, \\ &\left[\frac{\partial}{\partial u_A}, \mathcal{R}\right] x_2 = \frac{\partial \omega_{22}}{\partial u_A},\ \left[\frac{\partial}{\partial v_A}, \mathcal{R}\right] x_2 = \frac{\partial \omega_{22}}{\partial v_A}. \end{aligned}\right\} \tag{3.7}$$

Also, from (3.6)$_3$, we find

$$\frac{\partial}{\partial u_1}\left(\mathcal{L}z^A\right) - \mathcal{L}\left(\frac{\partial z^A}{\partial u_1}\right) = \left[\frac{\partial}{\partial u_1}, \mathcal{L}\right] z^A = \frac{\partial \omega_{11}}{\partial u_1}z_1^A + \omega_{11}\frac{\partial z_1^A}{\partial u_1}.$$

Hence by assumption on $\mathcal{P}, \omega_{11} \neq 0$ and we have

$$0 = \frac{\partial z_1^A}{\partial u_1} = \frac{1}{\omega_{11}} \left\{ \left[\frac{\partial}{\partial u_1}, \mathcal{L} \right] z^A - \frac{\partial \omega_{11}}{\partial u_1} z_1^A \right\}. \tag{3.8}$$

Now (3.8) is one of the integrability conditions for set of equations (3.6). In the same way we construct the remaining independent integrability conditions. These are

$$\begin{aligned}
\frac{\partial z_1^A}{\partial u_2} &= \frac{1}{\omega_{11}} \left\{ \left[\frac{\partial}{\partial u_2}, \mathcal{L} \right] z^A - \frac{\partial \omega_{11}}{\partial u_2} z_1^A \right\} = 0 \\
\frac{\partial z_2^A}{\partial u_1} &= \frac{1}{\omega_{22}} \left\{ \left[\frac{\partial}{\partial u_1}, \mathcal{R} \right] z^A - \frac{\partial \omega_{22}}{\partial u_1} z_2^A \right\} = 0 \\
\frac{\partial z_2^A}{\partial u_2} &= \frac{1}{\omega_{22}} \left\{ \left[\frac{\partial}{\partial u_2}, \mathcal{R} \right] z^A - \frac{\partial \omega_{22}}{\partial u_2} z_2^A \right\} = 0 \\
\frac{\partial z_1^A}{\partial v_1} &= \frac{1}{\omega_{11}} \left\{ \left[\frac{\partial}{\partial v_1}, \mathcal{L} \right] z^A - \frac{\partial \omega_{11}}{\partial v_1} z_1^A \right\} = 0 \\
\frac{\partial z_1^A}{\partial v_2} &= \frac{1}{\omega_{11}} \left\{ \left[\frac{\partial}{\partial v_2}, \mathcal{L} \right] z^A - \frac{\partial \omega_{11}}{\partial v_2} z_1^A \right\} = 0 \\
\frac{\partial z_2^A}{\partial v_1} &= \frac{1}{\omega_{22}} \left\{ \left[\frac{\partial}{\partial v_1}, \mathcal{R} \right] z^A - \frac{\partial \omega_{22}}{\partial v_1} z_2^A \right\} = 0 \\
\frac{\partial z_2^A}{\partial v_2} &= \frac{1}{\omega_{22}} \left\{ \left[\frac{\partial}{\partial v_2}, \mathcal{R} \right] z^A - \frac{\partial \omega_{22}}{\partial v_2} z_2^A \right\} = 0
\end{aligned} \tag{3.9}$$

Thus the middle two equations of (3.7), equation (3.8) and equations(3.9) is the complete set of integrability conditions for equations (3.6). Taking into account these equations, it is not difficult to see that if x_1, x_2, z^1 and z^2 are chosen to be any functionally independent invariants of the subdistribution

$$\Gamma : \left\{ \left[\frac{\partial}{\partial u_A}, \mathcal{L} \right], \ \left[\frac{\partial}{\partial v_A}, \mathcal{R} \right], \ \frac{\partial}{\partial u_A}, \ \frac{\partial}{\partial v_A} \right\}$$

of $\sigma^{(1)}$ such that $\mathcal{L}x_2 = \mathcal{R}x_1 = 0$ and $(\mathcal{L}x_1)(\mathcal{R}x_2) \neq 0$ then all integrability conditions will be satisfied. In this case from equations $(3.6)_1$ we obtain the equations $(*)$ in the statement of the theorem.

Finally, we need to check that if new coordinates are chosen in this way, the condition

$$\frac{\mathcal{R}z_1^A}{\mathcal{R}x_2} = \frac{\mathcal{L}z_2^A}{\mathcal{L}x_1} \tag{3.10}$$

is satisfied. From equations $(*)$, theorem 3.4, we have

$$\mathcal{R}z_1^A = \mathcal{R}\left(\frac{\mathcal{L}z^A}{\mathcal{L}x_1}\right) = \frac{(\mathcal{L}x_1)(\mathcal{R}\mathcal{L}z^A) - (\mathcal{L}z^A)(\mathcal{R}\mathcal{L}x_1)}{(\mathcal{L}x_2)^2},$$

and

$$\mathcal{L}z_2^A = \mathcal{L}\left(\frac{\mathcal{R}z^A}{\mathcal{R}x_2}\right) = \frac{(\mathcal{R}x_2)(\mathcal{L}\mathcal{R}z^A) - (\mathcal{R}z^A)(\mathcal{L}\mathcal{R}x_2)}{(\mathcal{R}x_2)^2}.$$

Making use of the fact that $[\mathcal{L}, \mathcal{R}] = 0$ and $\mathcal{R}x_1 = \mathcal{L}x_2 = 0$ gives

$$\mathcal{L}z_2^A = \frac{\mathcal{L}\mathcal{R}z^A}{\mathcal{R}x_2}, \ \ \mathcal{R}z_1^A = \frac{\mathcal{R}\mathcal{L}z^A}{\mathcal{L}x_1}.$$

Making use of $[\mathcal{L}, \mathcal{R}] = 0$ once again proves (3.10) □

Theorem 3.4 provides a simple algorithm by which a diffeomorphism may be explicitly constructed so that given $\sigma \in \sum(2;3,3)$ the associated equation $e \in E(2;3,3)$ may be found. Of course we have not proved that the subdistribution $\Gamma \subset \sigma^{(1)}$ always has the required invariants. Theorem 3.4 amounts to sufficient conditions for equivalence with the important feature of being *constructive*. We will show in §5 by direct computation that a diffeomorphism can in fact very easily be found in this way.

§4 **Integral submanifolds of Lie-Vessiot distributions.** Theorems 3.1 and 3.4 solve in part the classification problem for equations $E(2;3,3)$ in the sense each equation $e \in E(2;3,3)$ is in correspondence with a pair of six-dimensional Lie algebras of vector fields on $\mathbb{R}^6$. The classification of six-dimensional Lie groups will now give the class of p.d.e in $E(2;3,3)$ associated with distributions $\sum(2;3,3)$, however we will not here consider this classification problem. Instead, in this section, we give results which will enable us to integrate the distributions of the form expressed in theorem 3.1 which we will call *Lie-Vessiot distributions*. Then, because of theorem 3.4, this result will enable us to solve the associated partial differential equations.

We begin with a theorem due to Vessiot.

THEOREM 4.1. *(Vessiot, [8]) Let G be an r-parameter Lie group with composition function*

$$c : G \times G \longrightarrow G.$$

Let $\mathbf{L} = \{L_\alpha\}_{\alpha=1}$ and $\mathbf{R} = \{R_\alpha\}_{\alpha=1}^r$ be bases for the left and right -invariant vector fields on G. Then the vector field system

$$V : \left\{ \frac{\partial}{\partial z_+} + \sum_{\alpha=1}^{r} \phi_\alpha(z_+) L_\alpha, \frac{\partial}{\partial z_-} + \sum_{\alpha=1}^{4} \psi_\alpha(z_-) R_\alpha \right\}$$

on $\mathbb{R}^2 \times G$ has integral submanifold

$$g : (z_+, z_-) \longrightarrow \mathbf{R}^2 \times G$$

defined by

$$g(z_+, z_-) = (z_+, z_-, c_1(\mathbf{a}(z_+), \mathbf{b}(z_-)), \ldots, c_r(\mathbf{a}(z_+), \mathbf{b}(z_-)),$$

where the curves $\mathbf{a}, \mathbf{b} : \mathbb{R} \longrightarrow G$ *in* G *satisfy the ordinary differential equations*

$$\begin{aligned} \frac{da_\alpha}{dz_+} &= \sum_{\beta=1}^{r} \phi_\beta(z_+)\lambda_{\beta\alpha}(\mathbf{a}), \\ \frac{db_\alpha}{dz_-} &= \sum_{\beta=1}^{r} \psi_\beta(z_-)\rho_{\beta\alpha}(\mathbf{b}), \end{aligned} \qquad 1 \le \alpha \le 6$$

and where the functions $\lambda_{\beta\alpha}$ *and* $\rho_{\beta\alpha} : G \longrightarrow \mathbb{R}$ *are defined by*

$$L_\beta = \lambda_{\beta\alpha}(\mathbf{a})\frac{\partial}{\partial a_\beta}, \quad R_\beta = \rho_{\beta\alpha}(\mathbf{b})\frac{\partial}{\partial b_\beta},$$

the left and right invariant vector fields on G.

Proof. V is generated on $\mathbb{R}^2 \times G$ by

$$\left\{\frac{\partial}{\partial z_+} + \phi_\alpha(z_+)\lambda_{\alpha\beta}(\mathbf{a})\frac{\partial}{\partial a_\beta}, \ \frac{\partial}{\partial z_-} + \psi_\alpha(z_-)\rho_{\alpha\beta}(\mathbf{a})\frac{\partial}{\partial a_\beta}\right\}$$

The annihilator of V is V^* generated by

$$\{da_\beta - \phi_\alpha\lambda_{\alpha\beta}dz_+ - \psi_\alpha\rho_{\alpha\beta}dz_-\}.$$

Suppose $g : (z_+, z_-) \longmapsto (z_+, z_-, a_1(z_+, z_-), \dots, a_r(z_+, z_-))$ is the integral submanifold. Then from the condition $g^*V^* \equiv 0$ on $\mathbb{R}^2$, the a_β must satisfy

$$\frac{\partial a_\beta}{\partial z_+} - \phi_\alpha\lambda_{\alpha\beta} = 0, \tag{4.1a}$$

$$\frac{\partial a_\beta}{\partial z_-} - \phi_\alpha\rho_{\alpha\beta} = 0. \tag{4.1b}$$

Now let

$$a_\beta(z_+, z_-) = c_\beta(p_1(z_+), \dots p_r(z_+); q_1(z_-), \dots, q_r(z_-)) \tag{4.2}$$

be a trial solution for (4.1). The set of functions (4.2) may be thought of as defining a pair of Lie transformation groups

$$\begin{aligned} c_L &: G \times \mathcal{C}_q \longrightarrow \mathcal{C}_q \\ c_R &: \mathcal{C}_p \times G \longrightarrow \mathcal{C}_p \end{aligned}$$

where $\mathcal{C}_p : p_1(z_+), \dots, p_r(z_+)$ and $\mathcal{C}_q : q_1(z_-), \dots q_r(z_-)$ are curves in G. That is, in c_R, as z_- varies elements of the Lie group G act on $\mathcal{C}_p$ by right-translation and in c_L elements of the group act on $\mathcal{C}_q$ by left-translation as z_+ varies. Hence substitution of (4.2) into (4.1a) is equivalent (since there are no z_- differentiations) to applying an element of the transformation group c_R to (4.1a). But the Lie algebra of the transformation group c_R is a curve in the right invariant vector fields $\mathbf{R}$ on G; and since

$$\left[\mathbf{R}, \frac{\partial}{\partial z_+} + \phi_\alpha L_\alpha\right] = 0$$

(4.1a) is invariant under the action of c_R. It follows that substitution of (4.2) into (4.1a) has the effect of replacing each $a_\beta(z_+, z_-)$ in (4.1a) by $p_\beta(z_+)$. A similar argument involving transformation group c_L shows that in (4.1b) each $a_\beta(z_+, z_-)$ is replaced by $q_\beta(z_-)$ and the theorem is proved. □

Example. Let $G = (\mathbb{R}^2, c)$ where

$$c : G \times G \longrightarrow G$$

is defined by

$$c : (\mathbf{a}, \mathbf{b}) \longrightarrow (a_1 b_1, a_2 b_1 + b_2).$$

Define left & right G-actions on $\mathbb{R}^2$:

$$c_L : G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad c_R : \mathbb{R}^2 \times G \longrightarrow \mathbb{R}^2$$

by

$$c_L(\mathbf{a}, \mathbf{x}) = c(\mathbf{a}, \mathbf{x}) = (a_1 x_1, a_2 x_1 + x_2),$$

and

$$c_R(\mathbf{x}, \mathbf{b}) = c(\mathbf{x}, \mathbf{b}) = (b_1 x_1, b_1 x_2 + b_2).$$

Calculating the Lie algebras of c_L and c_R provides isomorphic copies in $\mathbb{R}^2$ of left and right invariant vector fields on G as follows

$$\mathbf{L} : L_1 = x_1 \frac{\partial}{\partial x_1}, \quad L_2 = x_1 \frac{\partial}{\partial x_2},$$
$$\mathbf{R} : R_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad R_2 = \frac{\partial}{\partial x_2}.$$

The vector field system to be integrated is

$$V : \left\{ \frac{\partial}{\partial z_+} + \phi_1(z_+) L_1 + \phi_2(z_+) L_2, \ \frac{\partial}{\partial z_-} + \psi_1(z_-) R_1 + \psi_2(z_-) R_2 \right\}$$

defined on $\mathbb{R}^2 \times \mathbb{R}^2$. The integral submanifold for V is a map

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2$$

defined by

$$g : (z_+, z_-) \longrightarrow (z_+, z_-, a_1(z_+)b_1(z_-),\ \ a_2(z_+)b_1(z_-) + b_2(z_-)),$$

where $a_\alpha(z_+)$ and $b_\beta(z_-)$ satisfy the o.d.e's

$$\begin{aligned} \frac{da_1}{dz_+} - \phi_1 a_1 = 0, \ \frac{da_2}{dz_+} - \phi_2 a_1 = 0, \\ \frac{db_1}{dz_-} - \psi_1 b_1 = 0, \ \frac{db_2}{dz_-} - \psi_1 b_2 - \psi_2 = 0. \end{aligned}$$

We can now give the main theorem on the integration of Lie-Vessiot distributions

THEOREM 4.2. *Let G be a six-parameter Lie group with composition function*

$$c : G \times G \longrightarrow G.$$

Let

$$L_\alpha = \lambda_{\alpha\beta}(w_1, \dots, w_6)\frac{\partial}{\partial w_\beta}, \ \ R_\alpha = \rho_{\alpha\beta}(w, \dots, w_6)\frac{\partial}{\partial w_\beta},$$

$1 \leq \alpha, \beta \leq 6$, be isomorphic copies in $\mathbb{R}^6$ of the left and right-invariant vector fields respectively on G (assumed to be simply transitive). Suppose $\sigma \in \sum(2;3,3)$ is associated with some $e \in E(2;3,3)$ and that σ involves the Lie algebras $\{L_\alpha\}_{\alpha=1}^6$ and $\{R_\alpha\}_{\alpha=1}^6$ as in theorem 3.1. Then, the integral submanifold for σ is a map

$$\mathfrak{I} : \mathbb{R}^2 \longrightarrow U \times V \times W$$

defined by

$$\begin{aligned} \mathfrak{I} : (\xi, \eta) &\longmapsto (\xi, \eta, f_1(\xi), f_2(\xi), g_1(\eta), g_2(\eta), w_1(\xi, \eta), \dots, w_6(\xi, \eta)) \\ &= (\xi, \eta, u_1, u_2, v_1, v_2, w_1, w_2, \dots w_6), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} &f_i, g_i : \mathbb{R} \longrightarrow \mathbb{R} \text{ are arbitrary } C^\infty \text{ functions and} \\ &w_\alpha(\xi, \eta) = c_\alpha(a_1(\xi), a_2(\xi), \dots a_6(\xi); b_1(\eta), b_2(\eta), \dots, b_6(\eta)). \end{aligned}$$

Finally, the $a_\alpha(\xi)$ and $b_\alpha(\eta)$ satisfy the ordinary differential equations

$$\left.\begin{aligned} \frac{da_\alpha}{d\xi} &= \phi_\beta(\xi, f_1(\xi), f_2(\xi))\lambda_{\beta\alpha}(a), \\ \frac{db_\alpha}{d\eta} &= \psi_\beta(\eta, g_1, (\eta), g_2(\eta))\rho_{\beta\alpha}(b). \end{aligned}\right\} \tag{4.4}$$

Proof. The distribution to be integrated is locally generated on $U \times V \times W$ by

$$\sigma : \left\{ \frac{\partial}{\partial \xi} + \phi_\alpha(\xi, u_1, u_2) L_\alpha, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \eta} + \psi_\alpha(\eta, v_1, v_2) R_\alpha, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}$$
$$= \left\{ \mathcal{L}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial u_2}, \ \mathcal{R}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}$$

which is 6-generated on a 12-dimensional manifold. Now by Vessiot's theory on the integration of vector field systems [9], in order to find integral submanifolds for σ we must look for *nonsingular involutions* of maximum order (see also [7], [8]). We have the following structure for σ:

$$\left.\begin{array}{c} \left[\mathcal{L}, \dfrac{\partial}{\partial u_i}\right], \ \left[\mathcal{R}, \dfrac{\partial}{\partial v_i}\right] \not\equiv 0, \\ \text{all other Lie brackets } \equiv 0. \end{array}\right\} \mod \sigma \tag{4.5}$$

This structure is imposed on σ by the structure conditions on $\sum(2;3,3)$. The nonsingular involutions of order 2 are given by subdistributions $\{S_1, S_2\} \subset \sigma$
where

$$S_1 = k_0 \mathcal{L} + k_1 \frac{\partial}{\partial u_1} + k_2 \frac{\partial}{\partial u_2} + m_0 \mathcal{R} + m_1 \frac{\partial}{\partial v_1} + m_2 \frac{\partial}{\partial v_2}$$
$$S_2 = \alpha_0 \mathcal{L} + \alpha_1 \frac{\partial}{\partial u_1} + \alpha_2 \frac{\partial}{\partial u_2} + \beta_0 \mathcal{R} + \beta_1 \frac{\partial}{\partial v_1} + \beta_2 \frac{\partial}{\partial v_2}$$

and

$$k_0, \ldots, \beta_2 \in C^\infty(U \times V \times W)$$

which satisfy

$$[S_1, S_2] \equiv 0 \mod \sigma. \tag{4.6}$$

Condition (4.6) becomes, taking into account (4.5)

$$\begin{aligned} &(k_0\alpha_1 - \alpha_0 k_1) \left[\mathcal{L}, \frac{\partial}{\partial u_1}\right] + (k_0 \alpha_2 - k_2 \alpha_0) \left[\mathcal{L}, \frac{\partial}{\partial u_2}\right] \\ &\quad + (m_0 \beta_1 - m_1 \beta_0) \left[\mathcal{R}, \frac{\partial}{\partial v_1}\right] + (m_0 \beta_2 - m_2 \beta_0) \left[\mathcal{R}, \frac{\partial}{\partial v_2}\right] = 0. \end{aligned} \tag{4.7}$$

Again, by the structure conditions on $\sum(2;3,3)$ all vector fields $\left[\mathcal{L}, \frac{\partial}{\partial u_i}\right]$, $\left[\mathcal{R}, \frac{\partial}{\partial v_i}\right]$, $i = 1, 2$, must be linearly independent and hence (4.7) becomes

$$0 = k_0\alpha_1 - \alpha_0 k_1 = k_0 \alpha_2 - k_2 \alpha_0 = m_0 \beta_1 - m_1 \beta_0 = m_0 \beta_2 - m_2 \beta_0. \tag{4.8}$$

We now assume that S_1 (say) is fixed and solve the $\alpha_0, \ldots, \beta_2$ in terms of $k_0, \ldots, m_2$ to ensure that condition (4.6) holds. This leads to the linear algebraic system

$$
\text{(4.9)} \qquad \begin{bmatrix} -k_1 & k_0 & 0 & 0 & 0 & 0 \\ -k_2 & 0 & k_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m_1 & m_0 & 0 \\ 0 & 0 & 0 & -m_2 & 0 & m_0 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = 0.
$$

Now in the nonsingular case, the functions $k_0, k_1, \ldots, m_2$ are assumed to be in general position and hence the first Vessiot index is $q_1 = 4$. There will therefore be exactly two independent solutions S_2 for each S_1 (one of them has to be S_1 itself). Furthermore, there can be no higher order involutions for this would mean that the second Vessiot index q_2 would have to satisfy $q_2 > 4$ and $q_2 + 2 < 6$ by [9, p356; 7, pp 113-116]. Solving (4.9) we find that

$$
S_2 = \frac{\alpha_0}{k_0}\left(k_0\mathcal{L} + k_1\frac{\partial}{\partial u_1} + k_2\frac{\partial}{\partial u_2}\right) + \frac{\beta_0}{m_0}\left(m_0\mathfrak{R} + m_1\frac{\partial}{\partial v_1} + m_2\frac{\partial}{\partial v_2}\right)
$$

and $k_0, m_0 \neq 0$ while $q_1 = 4$. Hence S_2 is any vector field in the linear span (over $C^\infty(U \times V \times W)$) of

$$
\left\{k_0\mathcal{L} + k_1\frac{\partial}{\partial u_1} + k_2\frac{\partial}{\partial u_2}, m_0\mathfrak{R} + m_1\frac{\partial}{\partial v_1} + m_2\frac{\partial}{\partial v_2}\right\}.
$$

Hence the *involution of order 2*, $\{S_1, S_2\}$ can be taken in resolved form

$$
\Delta_2 : \left\{\mathcal{L} + \omega_1^1\frac{\partial}{\partial u_1} + \omega_2^1\frac{\partial}{\partial u_2},\ \mathfrak{R} + \omega_1^2\frac{\partial}{\partial v_1} + \omega_2^2\frac{\partial}{\partial v_2}\right\}.
$$

By Vessiot's existence theorem [9], it follows that there are choices of the ω_j^i in Δ_2 so that it be Frobenius-integrable. It follows that the most general integral submanifolds for σ are 2-dimensional. We therefore seek a map

$$
\mathfrak{I} : \mathbb{R}^2 \longrightarrow U \times V \times W
$$

such that

$$
\mathfrak{I}^*\Delta_2^* \equiv 0 \quad \text{on} \quad \mathbb{R}^2
$$

where Δ_2^* is the anihilator of Δ_2. By inspection of Δ_2^*, we take ξ and η to be independent coordinates on $\mathbb{R}^2$ and the map $\mathcal{J}$ has the form

$$\begin{aligned}\mathcal{J} : (\xi, \eta) \longmapsto & (\xi, \eta, f_1(\xi, \eta), f_2(\xi, \eta), g_1(\xi, \eta) \\ & g_2(\xi, \eta), H_1(\xi, \eta), \ldots, H_6(\xi, \eta)), \\ & = (\xi, \eta, u_1, u_2, v_1, v_2, w_1, \ldots, w_6)\end{aligned}$$

$f_i, g_i, H_\alpha \in C^\infty(\mathbb{R}^2)$. It remains to determine the precise form of these functions. Now Δ_2^* is generated by

$$\begin{aligned}&\theta_1 = du_1 - \omega_1^1 d\xi, \;\; \theta_2 = du_2 - \omega_2^1 d\eta, \\ &\pi_1 = dv_1 - \omega_1^2 d\xi, \;\; \pi_2 = dv_2 - \omega_2^2 d\eta, \\ &\sigma_\beta = dw_\beta - \phi_\alpha(\xi, u_1, u_2)\lambda_{\alpha\beta}(w) d\xi - \psi_\alpha(\eta, v_1, v_2)\rho_{\alpha\beta}(w) d\eta.\end{aligned}$$

sectioning $\theta_1, \ldots, \pi_2$ by $\mathcal{J}$ gives

$$\frac{\partial f_1}{\partial \eta} = \frac{\partial f_2}{\partial \eta} = \frac{\partial g_1}{\partial \xi} = \frac{\partial g_2}{\partial \xi} = 0. \tag{4.10}$$

Hence given Δ_2, we can section $\theta_1, \ldots, \pi_2$ by choosing the ω_j^1 to be functions of ξ only and ω_j^2 functions of η only and such that f_i and g_i satisfy

$$\frac{df_i}{d\xi} = \omega_i^1(\xi), \quad \frac{dg_i}{d\eta} = \omega_i^2(\eta).$$

We next proceed to section the σ_β. Now the distribution of vector fields dual to the σ_β is generated by

$$V : \left\{ \frac{\partial}{\partial \xi} + \phi_\alpha(\xi, f_1(\xi), f_2(\xi)) L_\alpha, \frac{\partial}{\partial \eta} + \psi_\alpha(\eta, g_1(\eta), g_2(\eta)) R_\alpha \right\}$$

where we have made use of the result of sectioning $\theta_1, \ldots, \pi_2$. Notice that V has the precise form of distributions studied by Vessiot (theorem 4.1). An application of theorem 4.1 proves theorem 4.2. □

With theorem 4.2 we have completed the promised algorithm for solving equations in the class $E(2; 3, 3)$ which are associated with Lie-Vessiot distributions (see diagram, page 9). As mentioned earlier, we have as yet no results which begin with a given $e \in E(2; 3, 3)$ and which then associates a $\sigma_G \in \sum(2; 3, 3)$ for some G. We hope however that the forward problem will lend some insight into the inverse problem. Before going on to apply these results to an example, we point out that the construction described in this paper splits the integration problem for equations $E(2; 3, 3)$ from that of integrating a 6-generated distribution on $\mathcal{E}_2^2 \simeq \mathbb{R}^{12}$, to a 3-generated distribution on $\mathbb{R}^9$. This is because if the Monge characteristics are tangent to lower dimensional submanifolds in $\mathcal{E}_2^2$ (that is ${}_1\tilde{\Omega},{}_2\tilde{\Omega}$ possess non-trivial invariants) then (theorem 3.1) there exist coordinates in which the Vessiot distribution has the local representation

$$\sigma = \sigma_1 \oplus \sigma_2 : \left\{\mathcal{L}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right\} \oplus \left\{\mathcal{R}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\right\}$$

where σ_1 and σ_2 are each defined on 9-dimensional manifolds with an intersection consisting of a 6-dimensional manifold.

Now theorem 4.2 shows that each of σ_1 and σ_2 have genus 1 and that the integral submanifolds for σ may be constructed explicitly once those of σ_1 and σ_2 are known. Specifically, the group composition function $c : G \times G \longrightarrow G$ determines a *nonlinear map* for the integral submanifolds of σ_1 and σ_2. Finally, only one of σ_1 or σ_2 need be integrated since the Lie algebras $\mathbf{L}$ and $\mathbf{R}$ are isomorphic and the isomorphism is usually known explicitly. We now briefly discuss the above remarks in terms of the maps $\mathcal{P}$ and $\mathcal{J}$ found earlier. We have

$$0 \equiv \mathcal{J}^*(\sigma_G^\perp) \equiv \mathcal{J}^*\mathcal{P}^*(\Omega_{\mathcal{E}_2^2}^\perp).$$

Hence

$$\mathcal{P}_0\mathcal{J} : \mathbf{R}^2 \longrightarrow \mathcal{E}_2^2$$

is the integral submanifold for the Vessiot distribution $\tilde{\Omega}_{\mathcal{E}_2^2}$ associated with σ_G. Thus $\mathcal{P}_0\mathcal{J}$ is the 2-jet extension of the solution of the partial differential equation associated with $\tilde{\Omega}_{\mathcal{E}_2^2}$. Now its easy to see from theorem 4.2 that the integral submanifold of σ_1 is a map

$$\mathcal{J}_1 : (\xi) \longrightarrow U \times W$$

of the form

$$\mathcal{J}_1 : \xi \longrightarrow (\xi, f_1(\xi), f_2(\xi), a_1(\xi), \ldots a_6(\xi))$$

and that of σ_2 is a map

$$\mathcal{J}_2 : (\eta) \longrightarrow V \times W$$

of the form

$$\mathcal{J}_2 : \eta \longrightarrow (\eta, g_1(\eta), g_2(\eta), b_1(\eta), \ldots, b_6(\eta))$$

where $a_\alpha(\xi)$ and $b_\alpha(\eta)$ satisfy the equations (4.4). Hence assuming we have integrated σ_1 and σ_2 we identify the map $\mathcal{J}$ with $\mathcal{J}_1 \otimes_G \mathcal{J}_2 : (\xi, \eta) \longrightarrow (\xi, \eta, f_i(\xi), g_i(\eta), c(\mathbf{a}, \mathbf{b}))$ where c is the group composition function. It is in this sense that we say that the group G associated with σ_G acts as a nonlinear map for integral submanifolds of the Monge characteristics.

Now in the scalar case, Vessiot has shown that σ_1 and σ_2 are each locally diffeomorphic to the third order contact structure $\Omega^3(\mathbf{R}, \mathbf{R})^\perp$. In that case one integrates σ_1 and σ_2 by explicitly constructing the diffeomorphisms $\psi^1 : U \times W \longrightarrow J^3(\mathbb{R},\mathbb{R})$ and $\psi^2 : V \times W \longrightarrow J^3(\mathbb{R},\mathbb{R})$ such that

$$\psi^1_* \sigma^2_1 = \Omega^3(\mathbb{R},\mathbb{R})^{\perp},$$
$$\psi^2_* \sigma_2 = \Omega^3(\mathbb{R},\mathbb{R})^{\perp}.$$

Then if $f, g \in C^\infty(\mathbb{R})$ we obtain integral submanifolds for σ_1 and σ_2 as follows

$$\begin{aligned}(\psi^1)^{-1}{}_o j^3 f : \mathbb{R} &\longrightarrow U \times W \\ : t &\longrightarrow (\xi(j^3 f), u_1(j^3 f), u_2(j^3 f), a_1(j^3 f), a_6(j^3 f)) \\ &\text{for } \sigma_1\end{aligned}$$

and

$$\begin{aligned}(\psi^2)^{-1}{}_o j^3 g : \mathbb{R} &\longrightarrow V \times W \\ : s &\longrightarrow (\eta(j^3 g), v_1(j^3 g), v_2(j^3 g), b_1(j^3 g), b_6(j^3 g)) \\ &\text{for } \sigma_2\end{aligned}$$

The integral submanifold for σ is then given by

$$\mathcal{I} : (t,s) \longmapsto (\xi, \eta, u_1, u_2, v_1, v_2, c(\mathbf{a}(j^3 f), \mathbf{b}(j^3 g))$$

where $c : G \times G \longrightarrow G$ is the group composition function. The solution of the partial differential equation is then given as before. A natural question is to extend this result to the vector case. It turns out however in the vector case that σ_1 and σ_2 area not diffeomorphic to contact structure in general as we will show (Proposition 5.1).

§5 A system of semilinear hyperbolic partial differential equations associated with the Lie algebra $so(3,1;\mathbb{R})$. By way of illustrating the results of the previous sections of this paper we give in this section an example of a Darboux integrable partial differential equation associated with the Lie algebra $so(3,1;\mathbb{R})$. This example will also serve to demonstrate that in contrast to the case $m = 1$, the Monge characteristics σ_1 and σ_2 are not diffeomorphic to contact structures in general.

Consider a local projective action on $\mathbb{C}$ of the form

$$z \longmapsto \frac{a_1 z + a_2}{a_3 z + 1}, \quad \begin{bmatrix} a_1 & a_2 \\ a_3 & 1 \end{bmatrix} \in GL(2;\mathbb{C}).$$

The composition function for the Lie group is

$$c : G \times G \longrightarrow G : (a,b) \longmapsto \left(\frac{a_1 b_1 + a_2 b_3}{a_3 b_2 + 1}, \frac{a_1 b_2 + a_2}{a_3 b_2 + 1}, \frac{a_3 b_1 + b_3}{a_3 b_2 + 1} \right), \tag{5.1}$$

The Lie algebra $\mathfrak{g}$ of G has structure

$$\mathfrak{g} : [K_1, K_2] = -K_2, [K_1, K_3] = K_3, \ [K_3, K_3] = -2K_1,$$

and $\mathfrak{g} \simeq sl(2; \mathbb{C})$. We now investigate the real six-parameter Lie group $G^{\mathbf{R}}$ with composition function (5.1). The Lie algebras associated with left and right translations by $G^{\mathbf{R}}$ are obtained by direct computation:

$$\mathbf{L} : \begin{cases} L_1 = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} + w_2 \frac{\partial}{\partial w_4} + w_5 \frac{\partial}{\partial w_5}, \\ L_2 = w_3 \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} + w_6 \frac{\partial}{\partial w_4}, \\ L_3 = (w_4 w_5 - w_1 w_2) \frac{\partial}{\partial w_1} + (w_5^2 - w_2^2) \frac{\partial}{\partial w_2} + (w_1 - w_2 w_3 + w_5 w_6) \frac{\partial}{\partial w_3}, \\ \quad - (w_2 w_4 + w_1 w_5) \frac{\partial}{\partial w_4} - 2 w_2 w_5 \frac{\partial}{\partial w_5} + (w_4 - w_3 w_5 - w_2 w_6) \frac{\partial}{\partial w_6}, \\ L_4 = -w_4 \frac{\partial}{\partial w_1} - w_5 \frac{\partial}{\partial w_2} + w_1 \frac{\partial}{\partial w_4} + w_2 \frac{\partial}{\partial w_5}, \\ L_5 = -w_6 \frac{\partial}{\partial w_1} + w_3 \frac{\partial}{\partial w_4} + \frac{\partial}{\partial w_5}, \\ L_6 = (w_1 w_5 + w_2 w_4) \frac{\partial}{\partial w_1} + 2 w_2 w_5 \frac{\partial}{\partial w_2} - (w_4 - w_2 w_6 - w_3 w_5) \frac{\partial}{\partial w_3}, \\ \quad + (w_4 w_5 - w_1 w_2) \frac{\partial}{\partial w_4} + (w_5^2 - w_2^2) \frac{\partial}{\partial w_5} + (w_1 - w_2 w_3 + w_5 w_6) \frac{\partial}{\partial w_6}. \end{cases}$$

The Lie algebra $\mathbf{R}$ associated with right translations by $G^{\mathbf{R}}$ is obtained from $\mathbf{L}$ via the permutation

$$P : w_1 \to w_1, \ w_2 \to w_3, \ w_3 \to w_2, \ w_4 \to w_4, \ w_5 \to w_6, \ w_6 \to w_5.$$

That is, $P_*(L_\alpha) = R_\alpha$. $\mathbf{L}$ and $\mathbf{R}$ are simply transitive Lie algebras isomorphic to $so(3,1;\mathbb{R})$, the Lie algebra of the homogeneous Lorentz group. Our interest now is in obtaining a system of equations $e \in E(2;3,3)$ associated with the Lie algebras $\mathbf{L}$ and $\mathbf{R}$. There are in fact a number of systems which may be so constructed but we will not here deal with this classification problem. Instead, we make the ansatz $\sigma = \sigma_1 \oplus \sigma_2 \in \sum(2;3,3)$:

$$\sigma_1 : \left\{ \frac{\partial}{\partial \xi} + L_3 + L_4 + u_1 L_1 + u_2 L_2, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2} \right\},$$
$$\sigma_2 : \left\{ \frac{\partial}{\partial \eta} + R_2 + R_4 + v_1 R_1 + v_2 R_3, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}.$$

It is desirable to check some necessary conditions if $\sigma_1 \oplus \sigma_2$ is associated with a p.d.e. $e \in E(2;3,3)$.

DEFINITION 5.1. Given a vector field system V on $\mathbf{R}^n$, the i^{th} *derived system* $V^{(i)}$ of V is defined inductively by

$$V^{(i+1)} = [V^{(i)}, V^{(i)}] \oplus V^{(i)}, V^{(o)} = V.$$

It is known [Hermann, 10] that dim char $V^{(i)}$ are numerical invariants for the equivalence of two vector field systems with respect the pseudogroup of diffeomorphisms of $\mathbf{R}^n$. If $\tilde{\Omega}_{\mathcal{E}^2_2}$ is the Vessiot distribution of some equation (2.1) with $m = 2$, then dim char $\tilde{\Omega}_{\mathcal{E}^2_2} = 0$, dim char $\tilde{\Omega}^{(1)}_{\mathcal{E}^2_2} = 4$, dim char $\tilde{\Omega}^{(j)}_{\mathcal{E}^2_2} = 12, j \geq 2$. For the vector field system σ associated with $so(3,1;\mathbf{R})$ we have

$$\sigma^{(1)} : \left\{ \frac{\partial}{\partial \xi} + L_3 + L_4, L_1, L_2, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \eta} + R_2 + R_4, R_1, R_3, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}$$

$$\sigma^{(2)} : \left\{ \frac{\partial}{\partial \eta} + L_3 + L_4, L_1, L_2, L_3, -2L_1 + L_5, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \eta} + R_2 + R_4, R_1, R_2, R_3, \right.$$
$$\left. -2R_1 + R_6, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}$$

and we note that char $\sigma = \{0\}$, char $\sigma^{(1)} = \left\{ \frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i} \right\}^2_{i=1}$, char $\sigma^{(2)} = \Gamma(T(U \times V \times W))$ and hence the necessary conditions are satisfied.

We wish now to show how the partial differential equation associated with σ may be found. Making use of Theorem 3.4, we need to construct the invariants of the subdistribution $\Gamma \subseteq \sigma^{(1)}$ generated by

$$\Gamma : \left\{ L_1, L_2, R_1, R_3, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\}.$$

To do this it is convenient in this case to construct the completely integrable codistribution dual to Γ which we denote by Γ^*. Since Γ is an 8-generated completely integrable distribution of vector fields on $U \times V \times W (\simeq_{\text{loc}} \mathbf{R}^{12}), \Gamma^*$ is 4-generated

$$\Gamma^* : \begin{cases} \omega_+ = d\xi, \omega_- = d\eta, \\ \omega_1 = dw_1 - w_3 dw_2 - w_2 dw_3 - \left(\frac{w_1 - w_2 w_3}{w_5}\right) dw_5 - \left(\frac{w_1 - w_2 w_3}{w_6}\right) dw_6, \\ \omega_2 = dw_4 - w_6 dw_2 - w_5 dw_3 - \left(\frac{w_4 - w_2 w_6}{w_5}\right) dw_5 - \left(\frac{w_4 - w_3 w_5}{w_6}\right) dw_6. \end{cases}$$

An easy calculation shows that

$$\omega_1 = w_5 w_6 d\left(\frac{w_1 - w_2 w_3}{w_5 w_6}\right), \omega_2 = w_5 w_6 d\left(\frac{w_4 - w_2 w_6 - w_3 w_5}{w_5 w_6}\right)$$

and hence the four functionally independent invariants of Γ are

$$x_1 = \xi, \;\; x_2 = \eta, \;\; z^1 = \frac{w_1 - w_2 w_3}{w_5 w_6}, \;\; z^2 = \frac{w_4 - w_2 w_6 - w_3 w_5}{w_5 w_6}.$$

If we now make use of the equations $(*)$ in the statement of theorem 3.4, we obtain the local diffeomorphism $\mathcal{P}$ such that

$$\mathcal{P}_* \sigma = \tilde{\Omega}_{\mathcal{E}^2_2}$$

and the associated equation $e \in E(2;3,3)$ is explicitly known. We will not attempt to write these equations down as they are extremely complicated. We wish only to show that the construction can in principle be carried out. We will end by giving a result concerning the canonical form of the σ_1 and σ_2 associated with $so(3,1;\mathbb{R})$. Let e_L be the hyperbolic partial differential equation with Monge characteristics σ_1 and σ_2 associated with $so(3,1;\mathbb{R})$.

PROPOSITION 5.1. *The Monge characteristics σ_1 and σ_2 associated with the Darboux integrable equation $e_L \in E(2;3,3)$ are not diffeomorphic to contact structures.*

Proof. The k^{th} order contact structure $\Omega^k(\mathbb{R},\mathbb{R}^q)^\perp$ on $J^k(\mathbb{R},\mathbb{R}^q)$ is locally generated by

$$\left\{\frac{\partial}{\partial x} + z_{11}\frac{\partial}{\partial z_{10}} + z_{12}\frac{\partial}{\partial z_{11}} + \cdots + z_{1k}\frac{\partial}{\partial z_{1,k-1}} + z_{21}\frac{\partial}{\partial z_{20}} + \cdots + z_{2k}\frac{\partial}{\partial z_{2,k-1}} + \cdots \right.$$
$$\left. + z_{q1}\frac{\partial}{\partial z_{q0}} + \cdots + z_{qk}\frac{\partial}{\partial z_{q,k-1}}, \frac{\partial}{\partial z_{1k}}, \frac{\partial}{\partial z_{2k}}, \ldots, \frac{\partial}{\partial z_{qk}}\right\}.$$

Calculation gives the following structure properties for σ_1(Similarly for σ_2):

1. $\dim \sigma_1 = 3, \dim \sigma_1^{(1)} = 5, \dim \sigma_1^{(2)} = 7, \dim \sigma_1^{(3)} = 9$
2. $\dim\ \mathrm{char}\ \sigma_1 = 0, \dim\ \mathrm{char}\ \sigma_1^{(1)} = 2, \dim\ \mathrm{char}\ \sigma_1^{(2)} = 3, \dim\ \mathrm{char}\ \sigma_1^{(3)} = 9.$

Now from property (1), if σ_1 were diffeomorphic to a contact structure, the only candidate would be $\Omega^3(\mathbb{R},\mathbb{R}^2)^\perp$. However, $\dim\ \mathrm{char}\ (\Omega^3(\mathbb{R},\mathbb{R}^2)^\perp)^{(2)} = 4 \neq \dim\ \mathrm{char}\ \sigma_1^{(2)} = 3$ by property (2). □

Proposition 5.1 leaves open the interesting question of the appropriate normal forms for the Monge characteristics in the vector case.

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