IDEMPOTENCE FOR SIGN PATTERN MATRICES

By

Carolyn Eschenbach

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CAROLYN ESCHENBACH*

Abstract. We first characterize the n-by-n irreducible sign pattern matrices A that are sign idempotent, that is, $A = A^2$. To identify the n-by-n reducible sign idempotent patterns, we develop the upper diagonal completion process to find the sign pattern of each off-diagonal block in an upper block triangular sign pattern matrix A, so that $A = A^2$. Next we analyze the completion process qualitatively, and then discuss a graph theoretic interpretation of it. We then formulate the first characterization of sign idempotent patterns in terms of the upper diagonal completion process. Finally we establish a graph theoretic characterization of sign idempotent patterns.

0. Introduction. Qualitative matrix analysis involves the study of properties that are either required or allowed based just upon knowledge of the signs of the entries of a matrix. A matrix whose entries consist of the symbols +, −, and 0 is called a *sign pattern matrix*. For each n-by-n sign pattern matrix A, there is a natural class of real matrices whose entries have the signs indicated by A. If $A = (a_{ij})$ is an n-by-n sign pattern matrix, then the *sign pattern class of* A is defined by

$$Q(A) = \{B = (b_{ij}) \text{ real} \mid \operatorname{sgn} b_{ij} = a_{ij} \text{ for all } i \text{ and } j \text{ in } \{1, 2, \ldots, n\}\}.$$

We shall be interested in the cycles and chains in a sign pattern matrix, since every real matrix associated with it has the same qualitative cycle and chain structure. If $A = (a_{ij})$ is an n-by-n sign pattern matrix, then a product of the form $\lambda = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k j}$, in which the index set $\{i_1, i_2, \ldots, i_k, j\}$ consists of distinct indices is called a *chain of length* k *from* i_1 *to* j. If the index $j = i_1$, then γ is called a *simple cycle of length* k. A cycle or a chain is said to be *negative* (respectively, *positive*) if it contains an *odd* (respectively, *even*) number of negative entries and no entries equal to zero.

Recall that a real n-by-n matrix B is said to be *idempotent* if $B = B^2$. Analogously a square sign pattern matrix A is said to be *sign idempotent* if $B^2 \varepsilon Q(A)$ whenever $B \varepsilon Q(A)$; henceforth we write $A = A^2$. One important reason for studying sign idempotence, is that powers of sign idempotent matrices preserve not only the sign pattern, but also the cycle structure of the matrix. In qualitative matrix analysis, the answers to many require and allow questions depend entirely upon the cycle structure of the matrix (see [EJ1], [EJ2]).

If A and C are n-by-n sign pattern matrices, then $A + C$ exists, that is, $A + C$ is qualitatively defined if $a_{ij} c_{ij} \neq -$ for all i and j in $\{1, 2, \ldots, n\}$. The product AC exists if no two terms in the sum

$$\sum_{k=1}^{n} a_{ik} c_{kj} \text{ are oppositely signed, for all } i \text{ and } j \text{ in} \{1, 2, \ldots, n\}.$$

*Department of Mathematics and Computer Science, Georgia State University, Atlanta, GA 30303.

In order to simplify our notation, in the remainder of this paper, we let $P = (p_{ij})$ be the product matrix A^2. If P is defined for an n-by-n sign pattern matrix $A = (a_{ij})$, then

$$p_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}.$$

It is worth mentioning that if A is a sign idempotent matrix, then its sign pattern class $Q(A)$ forms a semigroup under matrix multiplication.

Our objective is to characterize n-by-n sign patterns that are sign idempotent. The entrywise nonzero qualitative matrices (sign patterns consisting of $+$'s and $-$'s only) for which $A = A^2$ are classified in theorem 4.1 of [MQ]. It is shown that if A is an entrywise nonzero sign pattern matrix, then $A = A^2$ if and only if A is cyclically positive, that is, all simple cycles in A are positive. It is also shown that if $a_{ij} = 0$, for some indices i and j in $\{1, 2, \ldots, n\}$, then $A = A^2$ only if A is partly *decomposable* (see theorem 4.3 in [MQ]), that is, there are n-by-n permutation matrices Q_2 and Q_1 such that

$$Q_1 A Q_2^T = \begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix},$$

(A_1 and A_2 square). In the special case when $Q_2 = Q_1^T$, then A is said to be *reducible*. (If no such permutation matrices exist, then A is said to be *indecomposable*.) We generalize these results in this paper by classifying all sign patterns that are sign idempotent.

By a *partial block sign pattern matrix* we mean a rectangular array whose entries are sign pattern matrices, the entries in certain locations of which are particular, known sign patterns. We refer to the known entries as "specified blocks" and the remaining entries as unspecified blocks. By a *completion* of a partial block matrix $A = (A_{ij})$, we mean a conventional sign pattern matrix $\widehat{A} = (\widehat{A}_{ij})$, where each $\widehat{A}_{ij}$ is a sign pattern matrix of the same dimension as A_{ij}, and if A_{ij} is specified, then $\widehat{A}_{ij} = A_{ij}$. A qualitative matrix completion problem, then, is to find a completion of a partial sign pattern matrix in a certain class or with a particular property. Our interest here is to find completions of certain partial block sign pattern matrices that are sign idempotent.

We first need some graph theoretic notions to describe our results and proofs. The *directed graph* (digraph) $D(A)$ of an n-by-n sign pattern matrix A is the directed graph on n vertices $1, 2, \ldots, n$, such that there is a directed edge in $D(A)$ from i to j, denoted by (i, j), if and only if $a_{ij} \neq 0$. The set of all vertices is called the *vertex set* V, and the set of all edges is called the *edge set* E.

1. Irreducible Sign Idempotent Patterns. To simplify notation, in the remainder of this paper we let the index set $\{1, 2, \ldots, n\}$ be represented by N. We let the class of sign idempotent matrices be denoted by SI. The first two lemmas are clear, and we state them without proof.

1.1 LEMMA. *The class SI is closed under the following operations:*

(i) *signature similarity;*

(ii) *permutation similarity; and*

(iii) *transposition.*

Part (ii) of lemma 1.1 allows us to assume that a matrix is in Frobenius normal form when investigating sign idempotent matrices.

1.2 LEMMA. *If $A = (a_{ij})$ is an n-by-n ($n \geq 2$) sign idempotent matrix and if $a_{ij} = 0$, for some i and j in N, then each product $a_{ik}a_{kj} = 0$, for all k in N.*

An n-by-n ($n \geq 2$) matrix $A = (a_{ij})$ is said to be *transitive* if $a_{ik} \neq 0$ and $a_{kj} \neq 0$ imply that $a_{ij} \neq 0$, for any i, j and k in N. An immediate consequence of Lemma 1.2 is that a sign idempotent matrix is transitive.

1.3 LEMMA. *If A is an n-by-n ($n \geq 2$) irreducible sign idempotent matrix, then A is entrywise nonzero.*

Proof. Suppose $A = (a_{ij})$ is an n-by-n irreducible sign idempotent matrix. For any indices i and j in N, the irreducibility of A implies that there is a path form i to j, say, $a_{ik_1}a_{k_1k_2}\ldots a_{k_mj} \neq 0$, where each k_h, $h = 1, 2, \ldots, m$, is in N. By repeatedly using the transitive property of A, it follows that $a_{ij} \neq 0$. Since i and j are arbitrary indices in N, we conclude A is entrywise nonzero. □

From lemma 1.3, we know that an irreducible n-by-n ($n \geq 2$) sign idempotent matrix is entrywise nonzero; and from theorem 4.1 in [MQ], we know that an entrywise nonzero sign pattern matrix is sign idempotent if and only if it is cyclically positive. Theorem 3.4 in [FP] proves that a $(1, -1)$ real matrix that is cyclically positive is diagonally similar to the all 1's matrix. Consequently an irreducible sign idempotent may be taken to be an entrywise positive sign pattern matrix. We state this formally as follows:

1.4 THEOREM. *If A is an n-by-n ($n \geq 2$) irreducible sign pattern matrix, then A is sign idempotent if and only if it is entrywise positive (up to equivalences, as stated in lemma 1.1).*

2. Reducible Sign Idempotent Matrices and the Upper Diagonal Completion Process. If A is a reducible sign idempotent matrix in Frobenius normal form, then it is clear that each irreducible diagonal block of A is sign idempotent. In the remainder of this paper, we use the results of section 1, and assume that each irreducible diagonal block A_{ii} of A is entrywise positive. Further in terms of block multiplication, $P_{ij} = A_{ii}A_{ij} + A_{i,i+1}A_{i+1,j} + \cdots + A_{ij}A_{jj} = A_{ij}$. The key question here is: What are the possible sign patterns of the off-diagonal blocks A_{ij}, so that the above equation is satisfied? The remainder of this paper is dedicated to answering this question, and to this end, we state our first lemma.

2.1 LEMMA. *Suppose A is an n-by-n reducible, sign idempotent matrix. If A_{ii} and A_{jj} are entrywise positive n_i-by-n_i, respectively, n_j-by-n_j matrices, and if the n_i-by-n_j matrix A_{ij} contains a zero entry, then A_{ij} is a 0-block.*

Proof. Assume A_{ij} contains a zero entry, say, $(A_{ij})_{kr} = 0$, for some k in N_i and some r in N_j. Then $(P_{ij})_{kr} = 0$, and it follows that $(A_{ii})_{ks}(A_{ij})_{sr} = 0$, for all s in N_i. However $(A_{ii})_{ks} = +$, for all s in N_i, implies that $(A_{ij})_{sr} = 0$, for all s in N_i. Consequently the s^{th} column of A_{ij} is an entrywise 0-column. Similarly $(A_{ij})_{km}(A_{jj})_{mr} = 0$, for all m in N_j. Since $(A_{jj})_{mr} = +$, for all m in N_j, we conclude that $(A_{ij})_{km} = 0$, for all m in N_j. Thus the k^{th} row of A_{ij} is entrywise zero. Since $(A_{ij})_{kr}$ is an arbitrary 0-entry in the r^{th} column of A_{ij}, we conclude that every row of A_{ij} is a 0-row, that is, A_{ij} is a 0-block. □

2.2 LEMMA. *If A_{ii} and A_{jj} are n_i-by-n_i, respectively, n_j-by-n_j entrywise positive matrices, and A_{ij} is an n_i-by-n_j entrywise nonzero matrix, then*

(i) *$A_{ii}A_{ij}$ is defined if and only if each column of A_{ij} contains only $+$'s or only $-$'s; and*

(ii) *$A_{ij}A_{jj}$ is defined if and only if each row of A_{ij} contains only $+$'s or only $-$'s.*

Proof. For simplicity, let $A_{ii} = H = (h_{ij})$ and $A_{ij} = B = (b_{ij})$. Then $(b_{1j}b_{2j}\dots b_{n_ij})^T$ is the j^{th} column of B, for all j in N_j, and $(h_{11}h_{12}\dots h_{1n_i})$ is the first row of H. Consequently

$$(HB)_{1j} = \sum_{k=1}^{n_i} h_{1k}b_{kj}$$

is defined if and only if $b_{kj} = h_{1k}$, or $b_{kj} = -h_{1k}$ for all k in N_i. Thus $(\operatorname{col} j)_B = \pm(\text{row } 1)_H^T$. We omit the proof of (ii), since the argument used is similar to the one used to prove (i). □

Combining lemmas 2.1 and 2.2 we obtain

2.3 LEMMA. *If A is an n-by-n reducible sign pattern matrix such that A_{ii} and A_{jj} are entrywise positive diagonal blocks, then A is sign idempotent only if the sign pattern of A_{ij} is obtained as follows:*

(i) *A_{ij} contains only $+$'s or only $-$'s; or*

(ii) *$A_{ij} = 0$.*

2.4 LEMMA. *Suppose A is an n-by-n reducible sign pattern matrix, where A_{ii} is an n_i-by-n_i entrywise positive matrix, and $A_{jj} = (0)$. If A_{ij} contains a 0-entry, then A_{ij} is an entrywise 0-column matrix.*

Proof. To simplify notation, let $A_{ii} = H = (h_{ij})$ and $A_{ij} = B = (b_{ij})$. Assume $b_{rj} = 0$, for some r in N_j. By sign idempotence, we know that the product entry $(HB)_{rj} = 0$, and

$$\sum_{k=1}^{n_i} h_{rk}b_{kj} = (HB)_{rj} = 0.$$

However $h_{rk} \neq 0$ implies that $b_{kj} = 0$, for all k in N_i. □

Since a matrix is sign idempotent only if each off-diagonal block P_{ij} is defined, we obtain the following:

2.5 LEMMA. *Suppose A is an n-by-n reducible sign pattern matrix. If A_{ii} is entrywise positive and $A_{jj} = (0)$, then A is sign idempotent only if $A_{ij} = 0$ or A_{ij} contains only +'s or only −'s.*

We should remark that if $A_{ii} = (0)$ and A_{jj} is an n_j-by-n_j entrywise positive matrix, then we could state results for A_{ij} analogous to those given in lemmas 2.4 and 2.5. In the remainder of this paper, we refer to the latter results as 2.4 (ii) and 2.5 (ii).

Suppose $A = (a_{ij})$ is an n-by-n reducible, sign idempotent matrix in Frobenius normal form containing m diagonal blocks. Further assume A contains t consecutive 1-by-1 0-diagonal blocks, where $A_{ii}(1 \leq i < m)$ is the first of the t consecutive 0-diagonal blocks. It is not difficult to show that the t adjacent 1-by-1 0-diagonal blocks are the diagonal entries of a t-by-t 0-diagonal block in A. We relabel and denote this t-by-t 0-block by A_{ii}, and the diagonal 0-entries in A_{ii} by $A_{i_1i_1}, A_{i_2i_2}, \ldots, A_{i_ti_t}$. Repeat this labeling procedure, if necessary, until all consecutive 1-by-1 0-diagonal blocks in A have been relabeled as described above. Call this the *modified Frobenius normal form of A*. We note that the diagonal blocks of a matrix in modified Frobenius normal form are entrywise positive matrices or entrywise 0 matrices. Since a reducible sign pattern matrix A in Frobenius normal form can be relabeled as described above, we may assume, without loss of generality, that A is in modified Frobenius normal form.

If A is an m-by-m block reducible matrix, then the off-diagonal blocks $A_{i(i+k)}$ lie on the k^{th} superdiagonal, for all $k = 1, 2, \ldots, (m-1)$. Due to the triangular structure of A, each $P_{i(i+k)}$ in the product matrix $P = A^2$ is independent of all terms above the k^{th} superdiagonal. This independence allows us to complete the sign pattern of A so that $A = A^2$ as described in the following:

2.6. The Upper Diagonal Completion Process. Suppose $A = (A_{ij})$ is an m-by-m reducible, partial block sign pattern matrix in modified Frobenius normal form. Determine the sign pattern of each off-diagonal block as follows:

(i) Start with the 1^{st} superdiagonal. Determine the sign patterns of each off-diagonal block $A_{i,i+1}$ using lemma 2.3 if A_{ii} and $A_{i+1,i+1}$ are entrywise positive (up to equivalences); lemma 2.5 for each diagonal block of $A_{i+1,i+1}$ if A_{ii} is equivalent to an entrywise positive matrix and $A_{i+1,i+1}$ is a 0-block; or lemma 2.5 (ii) for each diagonal block of A_{ii} if A_{ii} is a 0-block and $A_{i+1,i+1}$ is equivalent to an entrywise positive matrix. Move up to the next diagonal (if there is one).

(ii) For each unspecified entry $A_{i,i+k}$ on the k^{th} superdiagonal, $k = 2, 3, \ldots, (m-1)$, let $A_{i,i+k} = A_{i,i+1}A_{i+1,i+k}$. When all blocks are specified on this diagonal, move up to the next diagonal, if there is one, increase k by 1, for all $k = 2, 3, \ldots, (m-2)$, and repeat (ii).

2.7 Example. Let

$$A = \left[\begin{array}{ccc|cc|cc|cccc|ccc}
+ & + & + & - & + & - & - & - & + & + & - & - & - & - \\
+ & + & + & - & + & - & - & - & + & + & - & - & - & - \\
+ & + & + & - & + & - & - & - & + & + & - & - & - & - \\
\hline
 & & & 0 & 0 & + & + & + & - & - & + & + & + & + \\
 & & & 0 & 0 & - & - & - & + & + & - & - & - & - \\
\cline{4-14}
 & & & & & + & + & + & - & - & + & + & + & + \\
 & & & & & + & + & + & - & - & + & + & + & + \\
\cline{6-14}
 & & & & & & & 0 & 0 & 0 & 0 & + & + & + \\
 & & & & & & & 0 & 0 & 0 & 0 & - & - & - \\
 & & 0 & & & & & 0 & 0 & 0 & 0 & - & - & - \\
 & & & & & & & 0 & 0 & 0 & 0 & + & + & + \\
\cline{8-14}
 & & & & & & & & & & & + & + & + \\
 & & & & & & & & & & & + & + & + \\
 & & & & & & & & & & & + & + & +
\end{array}\right]$$

The sign pattern of each off-diagonal block in A was determined using the nonzero options in the upper diagonal completion process. □

2.8 THEOREM. *Suppose A is a reducible sign pattern matrix A, in modified Frobenius normal form, each of whose nonzero diagonal blocks is entrywise positive. Then A is sign idempotent only if each off-diagonal block A_{ij} is obtained using 2.6.*

Proof. The result follows from lemmas 2.3, 2.5 and 2.5 (ii). □

To establish that completing each off-diagonal block of a reducible sign pattern matrix A by the upper diagonal completion process is sufficient for sign idempotence, we use a graph theoretic approach in the next section.

3. A Graph Theoretic Interpretation of Reducible Sign Idempotent Matrices. If A is a reducible sign idempotent matrix in modified Frobenius normal form, and if A_{ii} and A_{jj} are entrywise positive, then according to the upper diagonal completion process, A_{ij} is a signed matrix or a 0-block. If A_{ii} is a 0-block, each column of A_{ij} is signed; and if A_{jj} is a 0-block, each row of A_{ij} is signed. We show that we can perform a sequence of signature similarities on A, where a qualitative signature matrix is a diagonal sign pattern matrix with an entrywise nonzero diagonal, so that A is signature similar to a matrix, say, SAS, each of whose off-diagonal blocks is a positively signed matrix, or a 0-block.

3.1 LEMMA. *Suppose A is a reducible sign pattern matrix in modified Frobenius normal form containing m diagonal blocks. If each nonzero diagonal block is entrywise positive, and each off-diagonal block is determined using 2.6, then A is signature similar to an m-by-m upper block triangular matrix, each of whose block entries is a positively signed matrix, or a 0-block.*

Proof. Let Q be the qualitative identity matrix, that is, the signature matrix with a positive diagonal. Let S_1 be the block signature matrix defined as follows:

$$S_{11} = Q_{n_1} \text{ (here } Q \text{ is } n_1\text{-by-}n_1) \text{ and },$$
$$S_{jj} = \begin{cases} Q_{n_1} & \text{if } A_{1j} \text{ is 0 or positively signed} \\ -Q_{n_1} & \text{if } A_{1j} \text{ is negatively signed} \end{cases},$$

for all $j = 2, 3, \ldots, m$. Then A is signature similar to the matrix $S_1 A S_1$, where $(S_1 A S_1)_{ij} = S_{11} A_{1j} S_{jj}$

$$\begin{aligned} &= Q_{n_1} A_{1j} Q_{n_1} \quad = A_{1j}, \text{ if } A_{1j} \text{ is positively signed or 0} \\ &= Q_{n_1} A_{1j}(-Q_{n_1}) = -A_{1j}, \text{ if } A_{1j} \text{ is negatively signed.} \end{aligned}$$

Clearly $S_1 A S_1$ is a block upper triangular matrix whose first row consists of positively signed matrices and/or 0-blocks. Suppose S_i is the block signature matrix defined by

$$S_{11} = Q_{n_1}; S_{22} = Q_{n_2}; \ldots, S_{ii} = Q_{n_i}; \text{ and}$$
$$S_{jj} = \begin{cases} Q_{n_i} & \text{if } [(S_{i-1} \ldots S_2 S_1) A (S_1 S_2 \ldots S_{i-1})]_{ij} \text{ is positively signed or 0} \\ -Q_{n_i} & \text{if } [(S_{i-1} \ldots S_2 S_1) A (S_1 S_2 \ldots S_{i-1})]_{ij} \text{ is negatively signed .} \end{cases}.$$

for all $j = i+1, i+2, \ldots, m$. Assume the first k rows of the signature similarity $(S_k \ldots S_2 S_1) A (S_1 S_2 \ldots S_k)$ consist of positively signed matrices and/or 0-blocks, for all $k \leq i$. Define the block signature matrix S_{i+1} by $S_{11} = Q_n$; $S_{22} = Q_{n_1}; \ldots, S_{(i+1)(i+1)} = Q_{n_{(i+1)}}$; and

$$S_{jj} = \begin{cases} Q_{n_j} & \text{if } [(S_i \ldots S_2 S_1) A (S_1 S_2 \ldots S_i)]_{ij} \text{ is positively signed or 0} \\ -Q_{n_j} & \text{if } [(S_i \ldots S_2 S_1) A (S_1 S_2 \ldots S_i)]_{ij} \text{ is negatively signed} \end{cases},$$

for all $j = i+2,\ i+3, \ldots, m$. To simplify notation, let $P = (S_i \ldots S_2 S_1)A(S_1 S_2 \ldots S_i)$. Suppose $(S_{i+1}PS_{i+1})_{hk}$ is an arbitrary entry in the first i rows of $S_{i+1}PS_{i+1}(h \leq i)$. Then $(S_{i+1}PS_{i+1})_{hk} = S_{hh}P_{hk}S_{kk}$. By hypothesis, P_{hk} is positively signed or an n_h-by-n_k 0-block. Consequently $(S_{i+1}PS_{i+1})_{hk} = Q_{n_h}P_{hk}Q_{n_k} = P_{hk}$ is positively signed or a 0-block. Thus the first i rows of $(S_{i+1}PS_{i+1})$ consist of positively signed matrices or 0-blocks. Now let $(S_{i+1}PS_{i+1})_{i+1,k}$ be an arbitrary entry in the $(i+1)^{\text{st}}$ row of $S_{i+1}PS_{i+1}$. If $P_{i+1,k}$ is positively signed or a 0-block, $(S_{i+1}PS_{i+1})_{i+1,k} = Q_{n_k}P_{i+1,k}Q_{n_k} = P_{i+1,k}$, which is clearly positively signed or a 0-block. If $P_{i+1,k}$ is negatively signed, then $(S_{i+1}PS_{i+1})_{i+1,k} = Q_{n_{(i+1)}}P_{(i+1)k}(-Q_{n_k}) = -P_{i+1,k}$, which is positively signed. It follows that the first $(i+1)$ rows of $S_{i+1}PS_{i+1} = (S_{i+1} \ldots S_2 S_1)A(S_1 S_2 \ldots S_{i+1})$ consist of positively signed matrices or 0-blocks. By induction, we conclude that A is signature similar to the entrywise nonnegative matrix $(S_m \ldots S_2 S_1)A(S_1 S_2 \ldots S_m)$. □

We are now prepared to interpret the upper diagonal completion process graph theoretically. Let A be a reducible sign pattern matrix having m diagonal blocks. If each block entry in A is a positively signed matrix or a 0-block, then we form the m-by-m *reduced matrix* $R = (r_{ij})$ of A as follows:

$$r_{ij} = \begin{cases} + & \text{if } A_{ij} \text{ is positively signed} \\ 0 & \text{if } A_{ij} \text{ is a 0-block} \end{cases} .$$

for all i and j in m. The directed graph of the reduced matrix R is called the *reduced directed graph of A, denoted by $RD(A)$*. In somewhat different terms, the reduced graph of a nonnegative matrix is defined in [S]. We say $RD(A)$ is *transitively closed* if for any (i,j) and (j,k) in the edge set E, the edge (i,k) is in E (see Example 8.7 in [S]).

3.2 LEMMA. *Suppose A is a reducible sign pattern matrix in modified Frobenius normal form, each of whose nonzero diagonal blocks is entrywise positive. If each off-diagonal block is obtained using the upper diagonal completion process, then the $RD(A)$ is transitively closed.*

Proof. Assume A is a sign pattern matrix that satisfies the conditions stated in the lemma. For contradiction, suppose that the $RD(A)$ is not transitively closed. Then there is a k such that $i+1 \leq k \leq j-1$, where the edges (i,k), (k,j) (i,j) satisfy one of the two cases below.

Case (i). Suppose the edges (i,k) and (k,j) are in E, and (i,j) is not in E. Then $A_{ij} = 0$, and $A_{ik}A_{kj}$ is positively signed. By 2.6 (ii) we have the following:

$$\begin{aligned} A_{ij} &= A_{i,i+1}A_{i+1,j} \\ &= A_{i,i+1}A_{i+1,i+2}A_{i+2,j} = A_{i,i+2}A_{i+2,j} \\ &= A_{i,i+1}A_{i+1,i+2}A_{i+2,i+3}A_{i+3,j} \\ &\ \vdots \\ &= A_{ik}A_{kj} = \cdots = A_{i,j-1}A_{j-1,j}. \end{aligned}$$

However $A_{ij} = A_{ik}A_{kj}$ implies that $A_{ij} \neq 0$, and, consequently, $(i,j)\varepsilon E$, which contradicts the assumption that $(i,j) \not\varepsilon E$.

Case (ii). Suppose that $(i,j)\varepsilon E$ and either $(i,k)\varepsilon E$ or $(k,j)\varepsilon E$, but not both. Assume (i,j) and (i,k) are in E, but $(k,j) \not\varepsilon E$. Then A_{ij} and A_{ik} are positively signed, and $A_{kj} = 0$. From case (i), we have $A_{ij} = A_{ik}A_{kj} \neq 0$, for all $k = i+1, \ldots, j-1$. Thus $A_{ik} \neq 0$ and $A_{kj} \neq 0$, and there are edges (i,k) and (k,j) in E, which contradicts the assumption that $(k,j) \not\varepsilon E$. A similar argument holds if (i,j) and (k,j) are in E, but $(i,k) \not\varepsilon E$. Consequently we conclude that the $RD(A)$ is transitively closed. □

3.3 LEMMA. *Suppose A is an m-by-m reducible sign pattern matrix in modified Frobenius normal form, each nonzero diagonal block is entrywise positive, and the sign pattern of each off-diagonal block is determined using the upper diagonal completion process. Then A is sign idempotent.*

Proof. We use the results of lemma 3.2 to assume, without loss of generality, that the $RD(A)$ is transitively closed. First assume $A_{ij} = 0$ for any i and j in M; and for contradiction, suppose there is a k such that $i+1 \leq k \leq j-1$, and $A_{ik}A_{kj} \neq 0$. Then (i,k) and (k,j) are in the edge set E of the directed graph $RD(A)$. Since $RD(A)$ is transitively closed, $(i,j)\varepsilon E$, which implies that $A_{ij} \neq 0$. However this contradicts the assumption that $A_{ij} = 0$. Thus

$$P_{ij} = A_{ii}A_{ij} + 0 + \cdots + 0 + A_{ij}A_{jj} = 0 + 0 + \cdots + 0 = 0 = A_{ij}.$$

Now suppose that $A_{ij} \neq 0$.

Case (i). Assume A_{ii} and A_{jj} are entrywise positive. Then $A_{ii}A_{ij}$ and $A_{ij}A_{jj}$ are positively signed; and for each k such that $i+1 \leq k \leq j-1$, $A_{ik}A_{kj} = A_{ij}$ follows by the same argument as given in the proof of lemma 3.2, case (i), and we conclude that $P_{ij} = A_{ij}$.

Case (ii). Assume A_{ii} is entrywise positive and A_{jj} is a 0-block. Then $A_{ii}A_{ij}$ is positively signed; and, as in case (i), $A_{ik}A_{kj} = A_{ij}$ for all k such that $i+1 \leq k \leq j-1$, implies that $P_{ij} = A_{ij}$.

Case (iii). Assume A_{ii} is a 0-block, and A_{jj} is entrywise positive. Then reversing the roles of A_{ii} and A_{jj} in case (ii) implies that $P_{ij} = A_{ij}$.

Case (iv). Assume A_{ii} and A_{ij} are 0-blocks. By step (ii) of the completion process, we know $A_{ij} = A_{i,i+1}A_{i+1,j} \neq 0$. Further by the argument given in case (i) of lemma 3.3, $A_{ik}A_{kj} = A_{ij}$ for all k such that $i+1 \leq k \leq j-1$, and we conclude that $P_{ij} = A_{ij}$. Cases (i)–(iv) imply that $P_{ij} = A_{ij}$, for any indices i and j in M, and it follows that A is sign idempotent. □

At the end of Section 2, in theorem 2.8 we proved that it is necessary to determine the sign pattern of each off-diagonal block A_{ij} of a reducible sign idempotent matrix using the upper diagonal completion process. Lemma 3.2 implies that the completion process is sufficient for sign idempotence. Consequently we have proved the following:

3.4 THEOREM. *A reducible sign pattern matrix A, in modified Frobenius normal form, each of whose nonzero diagonal blocks is entrywise positive, (up to equivalence, as in 1.1) is sign idempotent if and only if the sign pattern of each off-diagonal block is obtained using the upper diagonal completion process.*

3.5 Example. The matrix in example 2.7 is sign idempotent since the sign pattern of each off-diagonal block was determined using the completion process. □

A second characterization of sign idempotence in graph theoretic terms is given below.

3.6 THEOREM. *A reducible sign pattern matrix A, in modified Frobenius normal form, each of whose nonzero diagonal blocks is entrywise positive, (up to equivalence) is sign idempotent if and only the $RD(A)$ is transitively closed.*

Proof. Necessity of the condition stated in Theorem 3.5 follows from theorem 3.4 and lemma 3.2. Conversely assume the $RD(A)$ is transitively closed. Then sufficiency follows from the argument given in the proof of lemma 3.3. □

4. Remarks. Suppose P is a property a real matrix may or may not have. A sign pattern matrix A is said to *require* P if *every* real matrix in $Q(A)$ has property P, or to *allow* P if *some* real matrix in $Q(A)$ has property P [J], [EJ1] or [EJ2]. Let A be a given sign pattern matrix, and let P be the property "B in $Q(A)$ implies B^2 is in $Q(A)$." Then sign idempotent patterns require P, that is, every matrix in the sign pattern class of A has property P. However A could be sign idempotent, and not allow idempotence. For example, if

$$A = \begin{bmatrix} + & - \\ 0 & + \end{bmatrix},$$

then A is sign idempotent. However simple algebraic calculations show that no real matrix in the sign pattern class of A is idempotent. Identifying the sign idempotent sign patterns and arbitrary sign patterns that allow idempotence are provocative open questions for future research.

It is known from theorem 5.1 and Corollary 5.3 of [ES1] (see also [ES2]), in somewhat different terms, that an irreducible cyclically nonnegative matrix is signature-similar to an entrywise nonnegative matrix. Since most interesting properties are preserved under signature similarity, the class of cyclically nonnegative matrices is a natural generalization of the class of entrywise nonnegative matrices. From sections 1 and 2, we know that the class SI is a subclass of the class of cyclically nonnegative sign patterns. One interesting question for future study is to investigate this class and determine if it contains any subclasses of particular importance. Another open question is to extend the notion of sign idempotence, and characterize the sign patterns for which k is the smallest positive integer such that $A = A^{k+1}$.

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REFERENCES

[BW] Beineke, Lowell W. and Robin J. Wilson, *Selected Topics in Graph Theory*, Academic Press, 1978.

[EJ1] Eschenbach, Carolyn A. and Charles R. Johnson, *A Combinatorial converse to Perron-Frobenius Theorem*, Linear Algebra and Its Applications, 136 (1990), pp. 173–180.

[EJ2] Eschenbach, Carolyn A. and Charles R. Johnson, *Sign Patterns that Require Real, Nonreal or Pure Imaginary Eigenvalues*, Linear and Multilinear Algebra, vol. 29 (1991), pp. 299–311.

[ES1] Engel, G.M. and H. Schneider, *Cyclic and diagonal Products on a Matrix*, Linear Algebra and its Applications, 7 (1973), pp. 301–335.

[ES2] Engel, G.M. and H. Schneider, *Diagonal Similarity and Equivalence on Matrices over Groups with 0*, Czechoslovak Math. Journal, 25 (100) (1975), pp. 389–403.

[FP] Fiedler, Miroslav and Vlastimil Ptak, *Cycle Products and an Inequality for Determinants*, Czechoslovak Math. Journal, 19 (94) (1969), pp. 428–450.

[J] Johnson, C.R., *Combinatorial Matrix Analysis: An Overview*, Linear Algebra and Its Applications, 107 (1988), pp. 3–15.

[MQ] Maybee, John S. and J. Quirk, *Qualitative Problems in Matrix Theory*, SIAM Review, 11 (1969), pp. 30–51.

[S] Schneider, Hans, *The Influence of the Marked Reduced Graph of a Nonnegative Matrix on the Jordan Form and on Related Properties: A Survey*, Linear Algebra and Its Applications, 84 (1986), pp. 161–189.

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