CONTINUATION TO GRADIENT FLOWS

By

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IMA Preprint Series # 666

July 1990

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Abstract. We show that any isolated invariant set S in a Riemannian manifold can be continued to an isolated invariant set in a gradient flow, and hence to one in a Morse-Smale gradient flow. For the Morse-Smale case, a result of Floer allows one to compute the homology index of S by counting connecting orbits between the critical points. For an attractor-repeller pair in S, the same counting procedure allows one to define a map from the homology index of the repeller to the homology index of the attractor, and this map can also be used to compute the homology index of S.

1. Introduction. In [**1**], Conley introduced the idea of continuation of an isolated invariant set. Roughly, if one has a parameterized family of flows, and N is an isolating neighborhood for each flow in the parameter interval, then the sets isolated by N along the interval are said to be related by continuation. A fundamental result of [**1**] is that sets related by continuation have the same Conley index. This continuation property is especially useful for computations: one can continue a complicated invariant set to a simpler one and perform the computations in the simpler setting.

The first part of this paper is devoted to showing that any isolated invariant set on a manifold can be continued to an isolated invariant set in a gradient flow. In the nondegenerate (i.e. Morse-Smale) case, such flows are particularly simple: all bounded orbits are either critical points or orbits connecting two critical points. The Morse-Smale case can always be acheived by an arbitrarily small perturbation (i.e. continuation). In this case, one can compute the $\mathbf{Z}_2$ homology of the index by counting the number of connecting orbits between points of adjacent indices (see [**2**]).

If we have an attractor-repeller pair in the invariant set, then we can continue the attractor and repeller to Morse-Smale gradients. The continuation, plus counting the number of connecting orbits, gives a degree -1 map from the homology index of the original repeller to the homology index of the original attractor. We will show that this map allows us to form a chain complex whose homology is isomorphic to the homology index of the original invariant set. Thus the continuation gives us a new approach to the flow defined boundary map and two set connection matrix for the Conley index theory.

The rest of this section contains background definitions and results from the theory of the Conley index. In the second section we prove that any isolated invariant set can be continued to a gradient. The third section contains background material on attractor-repeller pairs, Morse decompositions, and index filtrations, and in the fourth section we use continuation to define the boundary map for attractor-repeller pairs.

Let ϕ_t be a smooth flow on a finite dimensional manifold M. Everything in this section is based on the ideas of Conley as described in [**1**]. Other references are given where appropriate. For $N \subset M$, we let $I(N)$ be the maximal invariant set contained in N, i.e. $I(N) = \{x \in N \mid \phi_t x \in N \text{ for all } t \in \mathbf{R}\}$. If $I(N)$ is strictly interior to N, then N is called an isolating neighborhood, and S is called an isolated invariant set if there is a compact neighborhood N of S such that $S = I(N)$. In this situation, Robbin and Salamon constructed a Lyaponov function.

Research supported in part by the National Science Foundation

THEOREM 1.1 [6]. *Let N be an isolating neighborhood of S. Then there is a neighborhood U of N and a smooth function $f : U \to \mathbf{R}$ satisfying*

(1) *$f(x) = 0$ for all $x \in S$.*
(2) *$\frac{d}{dt}\big|_{t=0} f(\phi_t x) < 0$ for all $x \in U$.*

A compact pair of spaces (N_1, N_0) is called an index pair for S if

(1) $N_1 \setminus N_0$ is a neighborhood of S and $cl(N_1 \setminus N_0)$ is an isolating neighborhood for S.
(2) N_0 is positively invariant relative to N_1, i.e. if $x \in N_0$ and $\phi_{[0,T]}x \subset N_1$, then $\phi_{[0,T]}x \subset N_0$.
(3) If $x \in N_1$ and $\phi_{[0,\infty)}x \not\subset N_1$, then there is a T with $\phi_{[0,T]}x \subset N_1$ and $\phi_T x \in N_0$.

Property 3 says that everything in N_1 which leaves in forward time leaves via N_0. Index pairs exist in arbitrarily small isolating neighborhoods of S. Robbin and Salamon used the Lyaponov function of Theorem 1.1 to construct index pairs. If (N_1, N_0) is an index pair for S, we consider the quotient N_1/N_0 as a pointed space with the equivalence class of N_0 as the distinguished point. The following result is fundamental.

THEOREM 1.2. *If (N_1, N_0) and (N_1', N_0') are two index pairs for S, then there is a flow defined homotopy equivalence between the pointed spaces N_1/N_0 and N_1'/N_0'.*

A nice proof of this fact is in the paper by Salamon [**7**]. We let $h(S)$ denote the homotopy type of N_1/N_0 for any index pair, and call $h(S)$ the Conley index of S. The homology index $CH(S)$ is the Čech homology of any quotient of index pairs, with the identifications arising from the homotopy equivalences of Theorem 1.2.

EXAMPLE 1.3. If $\{p\}$ is a hyperbolic critical point, then $\{p\}$ is an isolated invariant set with $h(\{p\})$ a pointed k-sphere where k is the number of eigenvalues of the linerization at p with positive real part.

A parameterized family of of flows on M is a collection of flows $\{\, \phi_t^\lambda \mid \lambda \in I \,\}$ indexed by $I = [0,1]$ such that $\Phi_t(x, \lambda) = (\phi_t^\lambda x, \lambda)$ is a smooth flow on $M \times I$. We say S^0, an invariant set for ϕ_t^0, and S^1, an invariant set for ϕ_t^1, are related by continuation if there is an isolated invariant set $\Sigma \subset M \times I$ for Φ_t such that $S^0 = \Sigma \cap \{(x,0)\}$ and $S^1 = \Sigma \cap \{(x,1)\}$. The following facts will be useful.

PROPOSITION 1.4. *If N is an isolating neighborhood for ϕ_t^λ for all $\lambda \in I$, let $\Sigma = I(N \times I)$ in Φ_t. Then Σ defines a continuation from $\Sigma \cap \{(x,0)\}$ to $\Sigma \cap \{(x,1)\}$.*

PROPOSITION 1.5. *For any parameterized family, if N is an isolating neighborhood for ϕ_t^0, then N is an isolating neighborhood for ϕ_t^λ for λ small enough.*

The reason continuations are interesting is the following theorem.

THEOREM 1.6. *If S^0 and S^1 are related by continuation, then $h(S^0) = h(S^1)$.*

Proposition 1.5 says that if we have an isolated invariant set S and make a small pertubation of the flow, then the new flow will have an isolated invariant set S' near S, and theorem 1.6 says $h(S) = h(S')$.

2. Continuation to a gradient.

THEOREM 2.1. *Let X be a smooth vector field on a Riemannian manifold M, and let S be an isolated invariant set in ϕ_t, the flow generated by X, with isolating neighborhood N. Then S can be continued to an isolated invariant set in a gradient flow. Moreover, this can be done without changing X on $M \setminus N$.*

PROOF: Let $\langle \cdot, \cdot \rangle_x$ denote the Riemannian metric in T_xM, and let f be the Lyaponov function from Theorem 1.1 defined on an open neighborhood U of N. $\nabla f(x) \neq 0$ if $x \notin S$ and for $x \in U \setminus S$, $\langle \nabla f(x), X(x) \rangle_x < 0$ since

$$0 > \left. \frac{d}{dt} \right|_{t=0} f(\phi_t x) = \langle \nabla f(\phi_0 x), X(\phi_0 x) \rangle_{\phi_0 x} = \langle \nabla f(x), X(x) \rangle_x .$$

Let N' be a compact neighborhood of S such that $cl(N') \subset int(N)$ and choose a smooth function $\rho : M \to [0,1]$ such that

$$\rho(x) = \begin{cases} 1, & \text{for } x \in N' \\ 0, & \text{for } x \in M \setminus N. \end{cases}$$

Consider the one parameter family of vector fields defined by

$$X_\lambda(x) = \rho(x) \left[(1-\lambda) X(x) - \lambda \nabla f(x) \right] + (1 - \rho(x)) X(x)$$

which generates a parameterized family of flows ϕ_t^λ. Note that

$$\begin{aligned} X_0 &= X, \\ X_1 &= -\rho \nabla f + (1-\rho) X, \\ X_1 &= -\nabla f \quad \text{on } N', \\ X_\lambda &= X \quad \text{on } M \setminus N. \end{aligned}$$

For any $\lambda \in [0,1]$, f is still decreasing along orbits in $U \setminus S$ since

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} f(\phi_t^\lambda x) &= \langle \nabla f(x), \rho(x) \left[(1-\lambda) X(x) - \lambda \nabla f(x) \right] + (1-\rho(x)) X(x) \rangle_x \\ &= (1-\lambda) \rho(x) \langle \nabla f(x), X(x) \rangle_x - \lambda \rho(x) \langle \nabla f(x), \nabla f(x) \rangle_x + \\ &\qquad + (1-\rho(x)) \langle \nabla f(x), X(x) \rangle_x \\ &< 0. \end{aligned}$$

It is easy to see that N and N' are isolating neighborhoods for each λ, and $I(N) = I(N') = S$ for each ϕ_t^λ. If $x \in N \setminus S$ and $f(x) \leq 0$, follow the orbit forward under ϕ_t^λ. Since f is strictly decreasing along the orbit and all critical points of f are contained in the set $\{f = 0\}$, the orbit must leave N in forward time. If $f(x) \geq 0$, then follow the backward orbit, and it must leave N. Of course, the same applies for $x \in N' \setminus S$. Proposition 1.4 implies that S in the flow ϕ_t^0 is related by continuation to the set $I(N)$ in the gradient flow ϕ_t^1. ∎

If we let $I^\lambda(N)$ denote the maximal invariant set in N for the flow ϕ_t^λ, the above argument shows that $I^1(N) \subset I^0(N)$.

By making an arbitrarily small pertubation of f on N', we can obtain a new function with the property that the resulting gradient flow is Morse-Smale. This fact, combined with Proposition 1.5 yields the following corollary.

COROLLARY 2.2. *Let X be a smooth vector field on a Riemannian manifold M, and let S be an isolated invariant set in ϕ_t, the flow generated by X, with isolating neighborhood N. Then S can be continued to an isolated invariant set in a Morse-Smale gradient flow without changing the vector field on $M \setminus N$.*

EXAMPLE 2.3. Consider $\mathbf{R}^2$ with the usual metric, and consider the vector field in polar coordinates

$$\dot{r} = r(r-1)$$
$$\dot{\theta} = 1.$$

Let $N = \{ (r,\theta) \mid 1/2 \leq r \leq 3/2 \}$ isolate the periodic orbit $\{r = 1\}$. A function with the properties of Theorem 1.1 is, e.g. $f(r,\theta) = -(r-1)^2$. The gradient field will have the form

$$\dot{r} = 2(r-1)$$
$$\dot{\theta} = 0.$$

A small perturbation to hyperbolic critical points and transverse intersections would be

$$\dot{r} = 2(r-1)$$
$$\dot{\theta} = \epsilon \sin\theta.$$

3. Attractor-repeller pairs and Morse decompositions. We first review some facts from Conley index theory (again, the main reference is [**1**], and see [**7**]). Let S be an isolated invariant set. $A \subset S$ is called an *attractor* in S if there is an S-neighborhood U of S such that $\omega(U) = A$. Similarly, a repeller is a set which is the ω^*- (i.e. α-) limit set of an S neighborhood of itself. If A is an attractor in S, let $A^* = \{ x \in S \mid \omega(x) \cap A = \emptyset \}$. Then A^* is a repeller in S called the repeller dual to A. It follows that $S = A \cup A^* \cup C(A^*, A)$ where $C(A^*, A) = \{ x \in S \mid \omega(x) \subset A, \omega^*(x) \subset A^* \}$ is the set of connecting orbits from A^* to A. A *Morse decomposition* of S is a finite collection $\{ M_p \mid p = 0, \ldots, n \}$ of disjoint, nonempty compact invariant subest of S such that for each $x \in S \setminus \cup_{i=0}^{n} M_i$, there is an $i < j$ such that $x \in C(M_j, M_i)$. Given a Morse decomposition, for $i \leq j$ let $M_{ij} = \cup_{k=i}^{j} M_i \cup (\cup_{k<l} C(M_l, M_k))$. Then M_{ij} is an isolated invariant set with $M_{ii} = M_i$. It is possible to construct a filtration $N_{-1} \subset N_0 \subset \cdots \subset N_n$ of a neighborhood of S such that (N_j, N_{i-1}) is an index pair for M_{ij}.

We now want to make a construction which was formalized by Floer in the gradient case [**2**], but is implicit in Morse theory. Assume that there is a Morse decomposition of S in an n-manifold which consists of hyperbolic critical points, and that stable and unstable manifolds intersect transversally. For critical points x of index k and y of index $k-1$, the set of connecting orbits is finite. Let $n(x,y)$ be the number of orbits (mod 2). Form a graded $\mathbf{Z}_2$ vector space C,

$$C_k = \bigoplus_{h(x)=\Sigma_k} \mathbf{Z}_2\langle x \rangle$$

generated by the critical points and graded by their indices. Define a map $\partial_k : C_k \to C_{k-1}$ via

$$\partial_k(x) = \sum_{h(y)=\Sigma^{k-1}} n(x,y)\langle y\rangle$$

THEOREM 3.1. *$\partial_{k-1} \circ \partial_k = 0$, and $H(C,\partial) = \ker \partial_k / im\, \partial_{k+1} \cong CH(S)$.*

PROOF: Let $M_j = \{\text{critical points } x \mid h(x) = \Sigma_j\}$. Transversality implies that $\{M_j \mid j = 0,\dots,n\}$ is a Morse decomposition of S. Choose an index filtration $\{N_{-1},\dots,N_n\}$ for this Morse decomposition. A theorem of McCord [**5**] shows that ∂_k agrees with the boundary map of the triple

$$H_*(N_k, N_{k-1}; \mathbf{Z}_2) \xrightarrow{\partial'} H_*(N_{k-1}, N_{k-2}; \mathbf{Z}_2),$$

and standard arguments show that the result holds for ∂'. ∎

For a discussion of the history of Theorem 3.4, as well as the details of the proof, see the paper by Salamon [**8**].

EXAMPLE 3.2 In the situation of example 2.3, there is a point of index 1 and one of index 2. $\partial = 0$, so $CH_k(S) \cong \mathbf{Z}_2$ for $k = 1,2$, and is 0 otherwise.

4. The boundary map for attractor-repeller pairs. Suppose (A, A^*) is an attractor-repeller pair in $S \subset M$. We define a map

$$\Delta : CH(A^*) \to CH(A).$$

First choose an isolating neighborhood N of S, and disjoint isolating neighborhoods N_A, $N_{A^*} \subset N$ of A and A^*. Continue A and A^* to isolated invariant sets A_1 and A_1^* in Morse-Smale gradient flows. By making a small perturbation if necessary, we may assume that the connections from points in A_1^* to points in A_1 are transverse, and the hyperbolic critical points in A_1 and A_1^* form a Morse decomposition of $S_1 = I(N)$ in the altered flow. There is a map ∂ as in §3. We can think of ∂ as a matrix, and in this case, $\partial = \partial_S$ has the following form:

$$\partial_S = \begin{bmatrix} \partial_A & \partial_{(A,A^*)} \\ 0 & \partial_{A^*} \end{bmatrix}$$

where ∂_A represents maps between points in A_1, ∂_{A^*} represents maps between points in A_1^*, and $\partial_{(A,A^*)}$ represents maps from points in A_1^* to points in A_1. $\partial_{(A,A^*)}$, from points in A_1 to points in A_1^*, is zero since (A, A^*) is an attractor- repeller pair in S. $\partial_S^2 = 0$ implies

$$\partial_A\partial_{(A,A^*)} + \partial_{(A,A^*)}\partial_{A^*} = 0.$$

Let $C(S)$ denote the $\mathbf{Z}_2$ vector space generated by the collection of rest points on which ∂ acts, $C(A)$ denote the vector space generated by points in A_1 and $C(A^*)$ denote the space generated by the points in A_1^*. There are obvious projections $\pi_A : C(S) \to C(A)$ and $\pi_{A^*} : C(S) \to C(A^*)$. By Theorem 3.1 and Theorem 1.6, there are isomorphisms

$$\Psi_A : \ker \partial_A / \text{im}\, \partial_A \to CH(A)$$
$$\Psi_{A^*} : \ker \partial_{A^*} / \text{im}\, \partial_{A^*} \to CH(A^*)$$
$$\Psi_S : \ker \partial_S / \text{im}\, \partial_S \to CH(S).$$

There are obvious projections

$$p_A : \ker \partial_A \to \ker \partial_A / \operatorname{im} \partial_A,$$

etc., and choose maps

$$\mu_A : \ker \partial_A / \operatorname{im} \partial_A \to \ker \partial_A$$

such that $p_A \mu_A = \mathrm{id}$, and similarly for A^* and S. Note that for any $x \in \ker \partial_A$, $(\mu_A p_A x) - x \in \operatorname{im} \partial_A$. Now define

$$\Delta(A, A^*)x = \Psi_A p_A \partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1} x.$$

This depends on the continuations, of course, but not on μ_{A^*} as one can easily verify. We now make a chain complex

$$C\Delta = \left(\begin{bmatrix} CH(A) \\ CH(A^*) \end{bmatrix}, \begin{bmatrix} 0 & \Delta(A, A^*) \\ 0 & 0 \end{bmatrix} \right).$$

THEOREM 4.1. $H(C\Delta) \cong CH(S)$.

PROOF: By theorem 3.1, it suffices to show that $H(C\Delta) \cong H(C(S), \partial_S)$. Note that $\partial_S(A^*) \subset \pi_{A^*}(\ker \partial_A)$ since if $\beta = \partial_{A^*} b$, then $(\partial_{(A,A^*)} b, \beta) \in \ker \partial_S$. (The matrices act on column vectors, but we write them as rows in the text to save space). Define a map $\alpha' : \pi_{A^*}(\ker \partial_S) \to \pi_A(C(S))$ as follows. Let $\beta_1, \ldots, \beta_l$ be a basis for $\ker \partial_{A^*}$, and extend this to a basis $\beta_1, \ldots, \beta_l, \ldots, \beta_m$ of $\pi_{A^*}(\ker \partial_S)$. We first define α' on $\ker \partial_{A^*}$. For $i = 1, \ldots, l$, choose a b_i such that $\partial_{A^*} b_i = \beta_i$. Define $\alpha'(\beta_i) = \partial_{(A,A^*)} b_i$. For $i = l+1, \ldots, m$, let $\alpha'(\beta_i)$ be any a such that $\partial_A a + \partial_{(A,A^*)} \beta_i = 0$. Extend α' to $\pi_{A^*}(\ker \partial_S)$ by linearity. Note that $\partial_A \alpha'(\beta) + \partial_{(A,A^*)} \beta = 0$ for any $\beta \in \pi_{A^*}(\ker \partial_S)$. Also, for any $(\alpha, \beta) \in \ker \partial_S$, $\alpha - \alpha'(\beta) \in \ker \partial_A$.

Now define a map

$$\begin{gathered} \hat{\Phi} : \ker \partial_S \to \ker \Delta \\ (\alpha, \beta) \mapsto (\Psi_A p_A(\alpha - \alpha'(\beta)), \Psi_{A^*} p_{A^*} \beta). \end{gathered}$$

$\Psi_{A^*} p_{A^*} \beta \in \ker \Delta(A, A^*)$ since

$$\begin{aligned} \Psi_{A^*} p_{A^*} \beta &= \Psi_A p_A \partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1} \Psi_{A^*} p_{A^*} \beta \\ &= \Psi_A p_A \partial_{(A,A^*)} (\beta + \partial_{A^*} a) \quad \text{for some } a \in C(A) \\ &= \Psi_A p_A (-\partial_A \alpha - \partial_A \partial_{(A,A^*)} a) \\ &= 0. \end{aligned}$$

$\hat{\Phi}$ is onto: suppose $(x, y) \in \ker \Delta$. Then x is arbitrary, and $\Delta(A, A^*)y = 0$, so $\partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1} y \in \operatorname{im} \partial_A$. It follows that $\mu_{A^*} \Psi_{A^*}^{-1} y \in \pi_{A^*}(\ker \partial_S)$, so $\hat{\Phi}(\mu_A \Psi_A^{-1} x + \alpha'(\mu_{A^*} \Psi_{A^*}^{-1} y), \mu_{A^*} \Psi_{A^*}^{-1} y) = (x, y)$.

Define Φ to be the composition

$$\ker \partial_S \xrightarrow{\hat{\Phi}} \ker \Delta \xrightarrow{proj} H(C\Delta).$$

Φ is onto since $\hat{\Phi}$ and the projection are. We will have $H(C\Delta) \cong H(C(S), \partial_S)$ if we can show $\ker \Phi \cong \operatorname{im} \partial_S$. Suppose $(\alpha, \beta) \in \ker \Phi$. Then $(\Psi_A p_A(\alpha - \alpha'(\beta)), \Psi_{A^*} p_{A^*} \beta) \in \operatorname{im} \Delta$. Thus $\Psi_{A^*} p_{A^*} \beta = 0$, so $\beta = \partial_{A^*} b$, and we can choose b such that $\alpha'(\beta) = \partial_{(A,A^*)} b$. Also,

$$\begin{aligned} \Psi_A p_A(\alpha - \alpha'(\beta)) &= \Delta(A^*, A) z \\ &= \Psi_A p_A \partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1} z \end{aligned}$$

so $\alpha - \alpha'(\beta) = \partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1} z + \partial_A a$ for some a. It follows that $\partial_S(a, \mu_{A^*} \Psi_{A^*}^{-1} z + b) = (\alpha, \beta)$, so $(\alpha, \beta) \in \operatorname{im} \partial_S$.

Suppose $(\alpha, \beta) \in \operatorname{im} \partial_S$. Then $\alpha = \partial_A x + \partial_{(A,A^*)} y$ and $\beta = \partial_{A^*} y$ for some $x \in C(A)$ and $y \in C(A^*)$. Let b be as in the definition of α', i.e. $\beta = \partial_{A^*} b$ and $\alpha'(\beta) = \partial_{(A,A^*)} b$. Note that $p_{A^*} \beta = 0$, and $y - b \in \ker \partial_{A^*}$. We have

$$\begin{aligned} \Delta(0, \Psi_{A^*} p_{A^*}(y - b)) &= (\Psi_A p_A \partial_{(A,A^*)} \mu_{A^*} \Psi_{A^*}^{-1}(y - b), 0) \\ &= (\Psi_A p_A \partial_{(A,A^*)}(y - b + \partial_{A^*} c, 0) \quad \text{for some } c \\ &= (\Psi_A p_A(\alpha - \partial_A x - \alpha'(\beta), 0) \\ &= (\Psi_A p_A(\alpha - \alpha'(\beta)), \Psi_{A^*} p_{A^*} \beta). \end{aligned}$$

Thus $\hat{\Phi}(\alpha, \beta) \in \operatorname{im}\Delta$, so $(\alpha, \beta) \in \ker \Phi$, and we have $H(C\Delta) \cong H(C, \partial) \cong CH(S)$. ∎

The proof that $H(C\Delta) \cong H(C, \partial)$ actually works for any field.

This is a different proof of the existence of a connection matrix for an attractor-repeller pair, and it uses only the basic Conley index theroy from [**1**]. Fransosa ([4]) has shown by other methods that in an arbitrary Morse decomposition, there are boundary maps like ∂' in the proof of Theorem 3.1 between adjacent sets, and one can define maps between nonadjacent sets to obtain a connection matrix Δ which satisfies the conclusion of theorem 4.1. It is not known whether the procedure used here can be extended to arbitrary Morse decompositions.

EXAMPLE 4.2 Suppose A and A^* are hyperbolic periodic orbits, for each orbit the Poincaré map preserves orientation on the unstable eigenspace (i.e the space spanned by the eigenvectors whose corresponding eigenvalues have modulus > 1), A has k such eigenvalues and A^* has $k+1$ such eigenvalues. Let N_A and N_{A^*} be tubular neighborhoods of A and A^*, respectively. $W^s(A)$ denotes the stable manifold of A, and $W^u(A^*)$ denotes the unstable manifold of A^*. $W^s(A) \cap N_A$ is the cartesian product of an $(n-k-1)$-ball and a circle, and its boundary $B^+ = S^{n-k-2} \times S^1$. Similarly, the boundary of $W^u(A^*) \cap N_{A^*}$, $B^- = S^k \times S^1$. We can continue A to points r of index k and s of index $k+1$, which are connected by two orbits as in Example 2.3. Similarly, we can continue A^* to two points, p of index $k+1$ and q of index $k+2$, which are connected by two orbits. Moreover, we can arrange the continuation so that $W^s(A) = W^s(r) \cup W^s(s)$, $W^s(A^*) = W^s(p) \cup W^s(q)$,

$W^s(s) \cap B^+ = \{(x,\theta) \in S^{n-k-2} \times S^1 \mid \theta = 0\}$, and $W^u(p) \cap B^- = \{(x,\theta) \in S^k \times S^1 \mid \theta = 0\}$ (see [**3**] for details). The possible nonzero maps in ∂ are $\partial(s,q)$ and $\partial(r,p)$. $W^u(A^*) \cap B^+$ is closed, embedded 1-manifold, so it must be a collection of circles. If $W^u(A^*)$ and $W^s(s)$ intersect transversally in B^+, then the number of intersections gives us the entry $\partial(s,q)$. If the intersection is not transverse, any perturbation to a transverse intersection will give us the same number of intersections (mod 2), (namely, the intersection number of the 1-manifold $W^u(A^*) \cap B^+$ and the $n-k-1$-manifold $W^s(s)$ in B^+). Similarly, the intersection number of $W^u(p) \cap B^-$ and $W^s(A) \cap B^-$ gives us the entry $\partial(r,p)$.

$CH(A)$ is nontrivial in dimensions k and $k+1$. $CH(A^*)$ is nontrivial in dimensions $k+1$ and $k+2$. Since the maps ∂_A and ∂_{A^*} are both zero, it is easy to see that $\Delta_{k+2} : CH_{k+2}(A^*) \to CH_{k+1}(A) = \partial(s,q)$ and $\Delta_{k+1} : CH_{k+1}(A^*) \to CH_k(A) = \partial(r,p)$.

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Keywords. continuation, Conley index, gradient flow
1980 *Mathematics subject classifications*: primary 58F12; secondary 58F09, 34C40

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