ASYMPTOTICS FOR A PARABOLIC DOUBLE OBSTACLE PROBLEM

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Abstract. We consider a parabolic double obstacle problem which is a version of the Allen–Cahn equation $u_t = \Delta u - \varepsilon^{-2}\psi'(u)$ in $\Omega \times (0,\infty)$, where Ω is a bounded domain, ε is a small constant, and ψ is a double well potential; here we take ψ such that $\psi(u) = (1-u^2)$ when $|u| \leq 1$ and $\psi(u) = \infty$ when $|u| > 1$. We study the asymptotic behavior, as $\varepsilon \to 0$, of the solution of the double obstacle problem. Under some natural restrictions on the initial data, we show that after a short time (of order $\varepsilon^2|\ln\varepsilon|$), the solution takes value 1 in a region Ω_t^+ and value -1 in Ω_t^-, where the region $\Omega \setminus (\Omega_t^+ \cup \Omega_t^-)$ is a thin strip and is contained in a $O(\varepsilon|\ln\varepsilon|)$ neighborhood of a hypersurface Γ_t which moves with normal velocity equal to its mean curvature. We also study the asymptotic behavior, as $t \to \infty$, of the solution in the one–dimensional case. In particular, we prove that the ω–limit set consists of a singleton.

1 Introduction.

The Allen–Cahn equation

$$u_t = 2\alpha k \Delta u - \alpha\psi'(u) \qquad \text{in } \mathcal{R}^3 \times (0,\infty) \tag{1.1}$$

was introduced by Allen–Cahn [1979] in order to describe the motion of an 'anti-phase' boundary separating two phases of a polycrystalline material. Each phase is characterized by the value taken by the order parameter u at the minima of the double equal well potential $\psi : \mathcal{R} \to \mathcal{R}$. With the scaling

$$\alpha = \frac{1}{\varepsilon^2}, \qquad k = \frac{\varepsilon^2}{2}, \qquad \varepsilon \to 0, \tag{1.2}$$

it was proposed that the interface evolves according to the mean curvature flow

$$V = -`K^m \tag{1.3}$$

where V is the velocity of the interface and K^m is the sum of its principle curvatures. Formal asympototics showing this were carried out by Rubinstein–Sternberg–Keller [1989]. Rigorous results have been obtained by Fife–Hsiao [1988], Carr–Pego

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[1989, 1990], Fusco–Hale [1989], and Fusco [1990] in the one dimensional case and by Bronsard–Kohn [1991], DeMottoni–Schatzman [1989], Chen [1990], and Evans–Soner–Souganides [1990] in the higher dimensional case.

The purpose of this paper is to study the version of (1.1) obtained by replacing the smooth double well potential ψ by

$$\psi(u) = \begin{cases} \frac{1}{2}(1-u^2) & |u| \leq 1, \\ +\infty & |u| > 1. \end{cases} \tag{1.4}$$

With the scaling (1.2) it follows that we study the parabolic double obstacle problem: find $u(x,t)$, $(x,t) \in \Omega \times (0,T)$, such that for a.e. (x,t),

$$\begin{cases} (u_t - \Delta u - \frac{1}{\varepsilon^2}u)\,\mathrm{Sign}\, u \leq 0, \\ |u| \leq 1, \\ (u_t - \Delta u - \frac{1}{\varepsilon^2}u)(|u|-1) = 0. \end{cases} \tag{1.5}$$

In particular we impose the boundary condition

$$\frac{\partial u}{\partial n} = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \tag{1.6}$$

where Ω is a bounded domain in $\mathcal{R}^N$ with a sufficient smooth boundary $\partial\Omega$. (Either Ω is convex and $\partial\Omega$ is Lipschitz or $\partial\Omega$ is smooth.)

We note that (1.5) and (1.6) is a gradient flow for the energy functional

$$\mathrm{E}(v) := \int_\Omega \left\{ |\nabla v|^2 + \frac{1}{\varepsilon^2}(1-v^2) \right\} dx, \tag{1.7}$$

$$v \in \mathcal{K} := \left\{ \eta \in H^1(\Omega) \,:\, |\eta(x)| \leq 1 \text{ a.e. } x \in \Omega \right\}. \tag{1.8}$$

The existence theory for (1.5) and (1.6) together with asymptotic long time behavior and a comparison lemma is given in §2. Using the method of Chen [1990], the connection with smooth mean curvature flow is rigorously established in §3. Finally in §4 detailed and precise results concerning the asymptotic long time behavior in one space dimension are proved.

For other connections between double obstacle problems and curvature dependent phase boundary motion we refer to Blowey and Elliott [1991a,b,c].

2 The Double Obstacle Problem.

In this section we briefly expose the existence and regularity theory for the parabolic variational inequality version of the Allen–Cahn equation.

2.1. EXISTENCE.

Let $\mathcal{K}$ be as in (1.8) and

$$\mathcal{G} := \Big\{\eta \in L^\infty(\Omega) \, : \, |\eta(x)| \le 1 \text{ a.e. } x \in \Omega\Big\}.$$

Given $g \in \mathcal{G}$ and $T > 0$ we say that $u \in C[0,T;\, L^2(\Omega)]$ is a solution to the double obstacle problem provided

$$\left\{\begin{array}{l} \quad u \in L^2(0,T;\, H^1(\Omega)), \qquad u_t \in L^2(0,T;\, (H^1(\Omega))'), \\ \quad u(\cdot,t) \in \mathcal{K} \text{ a.e. } t \in (0,T), \qquad u(\cdot,0) = g(\cdot), \\ \displaystyle\int_0^T \Big\{\langle u_t, \eta - u\rangle + (\nabla u, \nabla(\eta - u)) - \frac{1}{\varepsilon^2}(u, \eta - u)\Big\}\, dt \ge 0 \\ \qquad\qquad\qquad\qquad \forall \eta \in L^2(0,T;\, H^1(\Omega)), \quad \eta(t) \in \mathcal{K} \quad \forall t. \end{array}\right. \tag{2.1}$$

We have the following theorem concerning existence and uniqueness.

THEOREM 2.1. *(1) For each $g \in \mathcal{G}$ there exists a unique solution to (2.1) and the mapping $g \to u(t)$ is continuous in $L^2(\Omega)$ for each $t > 0$.*

(2) For each $T > 0$ and $\tau > 0$, the following regularity holds:

$$\begin{array}{l} \sqrt{t}u_t \in L^2(0,T;\, L^2(\Omega)), \\ u \in C[\tau,T;\, H^1(\Omega)] \cap L^2(\tau,T;\, H^2(\Omega)) \cap C^{1+\alpha,(1+\alpha)/2}(\Omega \times (\tau,T)). \end{array}$$

(3) There exist $\rho\,(= \rho(\varepsilon))$ and $t^*\,(= t^*(\varepsilon))$ such that for each $g \in \mathcal{G}$

$$\begin{array}{rl} \|u(t)\|_{H^1(\Omega)} \le \rho & \qquad \forall\, t \ge t^*, \\ \|u_t\|_{L^2(\Omega)} + \|u(t)\|_{H^2(\Omega)} \le \rho & \qquad \forall\, t \ge t^*. \end{array}$$

(4) For any $0 < \tau < t$,

$$E(u(t)) + 2\int_\tau^t \|u_t(s)\|\, ds = E(u(\tau)).$$

(5) u solves the parabolic problem

$$\left\{\begin{array}{ll} u_t - \Delta u = \beta \in \frac{1}{\varepsilon^2}\beta(u) & \quad \textit{in } \ \Omega \times (0,\infty), \\ \partial u/\partial n = 0 & \quad \textit{on } \ \partial\Omega \times (0,\infty), \\ u(x,0) = g(x) & \quad \textit{on } \ \Omega \times \{0\}, \end{array}\right. \tag{2.2}$$

where $\beta(u)$ is a maximal graph given by

$$\beta(u) = \left\{\begin{array}{ll} \infty & \quad \textit{if } \ u < -1, \\ [-1,0] & \quad \textit{if } \ u = -1, \\ u & \quad \textit{if } \ u \in (-1,1), \\ [0,1] & \quad \textit{if } \ u = 1, \\ -\infty & \quad \textit{if } \ u > 1. \end{array}\right. \tag{2.3}$$

Proof. Existence and regularity are proven using the standard regularization technique for parabolic variational inequalities. For each $\delta > 0$, let $\beta^\delta \in C^\infty(\mathcal{R})$ satisfy

$$\begin{cases} \beta^\delta(r) = r & \forall\, |r| \le 1-\delta, \\ \beta^\delta(\pm 1) = 0, & \\ (\beta^\delta(r))' \le 1 & \forall\, |r| \le 1. \end{cases} \tag{2.4}$$

It follows that

$$\begin{aligned} &r\beta^\delta(r) \le r^2 \qquad \forall\, |r| \le 1, \\ &\lim_{\delta\to 0} \beta^\delta(r) = r \qquad \forall\, |r| < 1. \end{aligned} \tag{2.5}$$

For convenience we define

$$B_\delta(u) = \int_0^u \beta^\delta(r)\, dr$$

so that

$$0 \le B_\delta(u) \le u^2/2 \qquad \forall\, |u| \le 1. \tag{2.6}$$

Standard theory for quasi-linear parabolic equations yields the existence of a unique solution to

$$\begin{cases} {u^\delta}_t - \Delta u^\delta = \frac{1}{\varepsilon^2}\beta^\delta(u^\delta), & x \in \Omega, t > 0, \\ \partial u^\delta/\partial n = 0, & x \in \Omega, t > 0 \\ u^\delta(x,0) = g(x), & x \in \Omega \end{cases} \tag{2.7}$$

which satisfies, by maximum principle because of (2.4b) and (2.5),

$$|u^\delta(x,t)| \le 1 \qquad \forall\, (x,t) \in \Omega \times (0,\infty). \tag{2.8}$$

Multiplying (2.7a) by u^δ and integrating yield

$$\frac{1}{2}\frac{d}{dt}\|u^\delta\|^2 + \|\nabla u^\delta\|^2 = \frac{1}{\varepsilon^2}\langle \beta^\delta(u^\delta), u^\delta\rangle \le \frac{|\Omega|}{\varepsilon^2}$$

so that

$$\int_0^t \|\nabla u^\delta(s)\|^2\, ds \le |\Omega|\Big(\frac{1}{2} + \frac{t}{\varepsilon^2}\Big). \tag{2.9}$$

Multiplying (2.7a) by ${u^\delta}_t$ and integrating yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^\delta\|^2 + \|{u^\delta}_t\|^2 = \frac{1}{\varepsilon^2}\frac{d}{dt}\int_\Omega B_\delta(u^\delta)\, dx$$

so that for each $t > \tau > 0$,

$$2\int_\tau^t \|{u^\delta}_t(s)\|^2\, ds + \mathrm{E}^\delta(u^\delta(t)) = \mathrm{E}(u^\delta(\tau)) \tag{2.10}$$

where

$$\mathrm{E}^\delta(\eta) = \int_\Omega \left\{ |\nabla\eta|^2 + \frac{1}{\varepsilon^2} - \frac{2}{\varepsilon^2} B_\delta(\eta) \right\} dx. \tag{2.11}$$

It follows from (2.10), (2.6) that

$$2\int_\tau^t \|u^\delta{}_t(s)\|^2\, ds + \|\nabla u^\delta(t)\|^2 \le \|\nabla u^\delta(\tau)\|^2 + \frac{|\Omega|}{\varepsilon^2}. \tag{2.12}$$

Similarly multiplying (2.7a) by $tu^\delta{}_t$ and integrating yield

$$\begin{aligned}
\int_0^t s\|u^\delta{}_t(s)\|^2\, ds + \frac{1}{2}t\|\nabla u^\delta(t)\|^2 &= \frac{1}{2}\int_0^t \|\nabla u^\delta(s)\|^2\, ds \\
&\quad + \frac{1}{\varepsilon^2} t \int_\Omega B_\delta(u^\delta(t)) - \frac{1}{\varepsilon^2}\int_0^t\int_\Omega B_\delta(u^\delta) \\
&\le \frac{|\Omega|}{2}\left(\frac{1}{2} + \frac{t}{\varepsilon^2}\right) + \frac{|\Omega| t}{2\varepsilon^2}.
\end{aligned}$$

by (2.6) and (2.9). It follows that

$$\|\nabla u^\delta(t)\|^2 \le |\Omega|\left(\frac{1}{2t} + \frac{2}{\varepsilon^2}\right).$$

Therefore there exists t^* such that for $t > t^*/2$,

$$\|u^\delta(t)\|_{H^1(\Omega)} \le \frac{3|\Omega|}{\varepsilon^2}. \tag{2.13}$$

Differentiating (2.7a) with respect to t, multiplying by $(t - t^*/2)u^\delta{}_t$, and using (2.4c) yield

$$\frac{d}{dt}\Big((t - t^*/2)\|u^\delta{}_t\|^2\Big) \le \left(1 + \frac{2(t - t^*/2)}{\varepsilon^2}\right)\|u^\delta{}_t\|^2 \qquad \forall\, t > t^*/2,$$

and integrating in time and using (2.12), (2.13) we obtain for $t > t^*/2$,

$$(t - t^*/2)\|u^\delta{}_t(t)\|^2 \le \left(1 + \frac{2(t - t^*/2)}{\varepsilon^2}\right)\frac{1}{2}\left(\frac{3|\Omega|}{\varepsilon^2} + \frac{|\Omega|}{\varepsilon^2}\right).$$

Since (2.8) holds we have from (2.7a), (2.5) that

$$\|u^\delta(t)\|_{H^2(\Omega)} \le C(\|u^\delta{}_t\| + 1/\varepsilon^2).$$

For different initial data g_1 and g_2 it is an easy calculation to show

$$\|u^\delta{}_1(t) - u^\delta{}_2(t)\| \le e^{c_\varepsilon t}\|g_1 - g_2\| \tag{2.14}$$

for the two solutions $\{u^\delta{}_i\}_{i=1}^2$ and C_ε a positive constant independent of δ and $\{g_i\}$.

The statement of the theorem is a standard consequence of a compactness argument with

$$u(t) = \lim_{\delta \to 0} u^{\delta}(t),$$
$$\beta(u) \ni \beta = \mathrm{w}^{*}\text{—}\lim_{\delta \to 0} \beta_{\delta}(u^{\delta}).$$

2.2. ASYMPTOTIC BEHAVIOR AS $t \to \infty$.

Let $\{\mathcal{S}(t)\}_{t \geq 0}$ be the family of solution operators defined by $\mathcal{S}(t)g := u(t)$ where $u(\cdot)$ solves (2.1) with g as initial data. It follows from Theorem 2.1 that $\{\mathcal{S}(t)\}_{t\geq 0}$ forms a continuous semigroup on $\mathcal{G} \subset L^2(\Omega)$. We denote by $\mathcal{E}$ the set of equilibria defined by

$$\mathcal{E} \equiv \Big\{ v \in \mathcal{K} \,|\, \varepsilon^2(\nabla v, \nabla \eta - \nabla v) \geq (v, \eta - v), \;\; \forall \eta \in \mathcal{K} \Big\}. \tag{2.15}$$

For each $g \in \mathcal{G}$ we define the ω–limit set $\omega(g)$ by

$$\omega(g) = \Big\{ v \in \mathcal{G} \,|\, \exists \{t_n\} \to \infty \;\; \text{such that} \;\; \|\mathcal{S}(t_n)g - v\| \to 0 \;\; \text{as} \;\; n \to \infty \Big\}. \tag{2.16}$$

THEOREM 2.2. *The semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ possesses a global attractor $\mathcal{A}$ which is bounded in $\mathcal{K}$. Furthermore for any $g \in \mathcal{G}$ it holds that $\omega(g) \subset \mathcal{E}$ and there exists a constant e such that $e = \mathcal{E}(v)$, $\forall\, v \in \omega(g)$.*

Proof. We follow the exposition given in Temam (1988). It follows from Theorem 2.1 that $\{\mathcal{S}(t)\}_{t \geq t^*}$ is uniformly compact and that there exists a ρ such that $\mathcal{B} = \{\eta \in \mathcal{K} \,|\, \|\eta\|_{H^1(\Omega)} \leq \rho\}$ is an absorbing set. Hence the existence of a global attractor is an immediate consequence of Temam [1988; Theorem 1.1, p23].

Since $\mathrm{E}(\cdot) \geq 0$, it follows from the results (2), (3), and (4) of Theorem 2.1 that $\mathrm{E}(\mathcal{S}(t)g)$ is a bounded decreasing continuous function on (t^*, ∞) and so has a unique limit $e = \lim_{t\to\infty} \mathrm{E}(\mathcal{S}(t)g)$. Since $\cup_{t \geq t^*} \mathcal{S}(t)g$ is relatively compact, there exists a cluster point $v\,(\in \mathcal{K}) = \lim_{k\to\infty} \mathcal{S}(t_k)g \in \omega(g)$ and because $\mathrm{E}(\cdot)$ is continuous on $H^1(\Omega)$ and $\mathcal{S}(t)g \in C[\tau, T;\, H^1(\Omega)]$ for all $T > 0$, it follows that $\mathrm{E}(v) = \lim_{k\to\infty} \mathrm{E}(\mathcal{S}(t_k)g) = e$. Hence, $\mathrm{E}(v) = e$ for all $v \in \omega(g)$ and since $\omega(g)$ is invariant, $\mathcal{S}(t)v \in \omega(g)$ for all $t \geq 0$. Therefore $\mathrm{E}(\mathcal{S}(t)v) = e = \mathrm{E}(v)$ for all $t > 0$ and so result (4) implies that $\frac{d}{dt}\mathcal{S}(t)v = 0$ or $v \in \mathcal{E}$. □

2.3. COMPARISON LEMMA.

LEMMA 2.3. *Let $w \in L^2(0,T;\, H^1(\Omega))$ and $w_t \in L^2(0,T;\, (H^1(\Omega))')$ satisfy*

$$\begin{aligned} w(x,t) &\leq 1, \qquad (x,t) \in \Omega \times (0,T), \\ w(x,0) &\leq g(x), \qquad x \in \Omega \end{aligned}$$

and

$$\begin{aligned} \int_0^T \Big\{ \langle w_t, \eta \rangle \;\; &+ (\nabla w, \nabla \eta) - \tfrac{1}{\varepsilon^2}(f, \eta) \Big\} \leq 0 \\ &\forall \eta \in L^2(0,T;\, H^1(\Omega)), \;\; \eta \geq 0 \;\; \text{a.e.} \;\; \Omega \times (0,T) \end{aligned} \tag{2.17}$$

where $f(w-u)_+ \leq w(w-u)_+$ and u solves (2.1) with initial data $g \in \mathcal{G}$. Then

$$w \leq u \qquad \text{a.e. } \Omega \times (0,T).$$

Proof. Taking $\eta = (w-u)_+$ in (2.17) and $\eta = u + (w-u)_+$ in (2.1c) gives a pair of inequalities which when added yield

$$\int_0^T \left\{ \frac{1}{2}\frac{d}{dt}\|(w-u)_+\|^2 + \|\nabla(w-u)_+\|^2 \right\} dt \leq \int_0^T \frac{1}{\varepsilon^2}\|(w-u)_+\|^2 dt.$$

The assertion follows from Gronwall's inequality. □

In the sequel, we shall also use the following comparison lemma.

LEMMA 2.4. Let u be the solution of (2.1) and Ω' be an open subset of Ω with Lipschitz boundary. Assume that a function $v \in W^{2,1}_\infty(\Omega' \times (\tau, T))$ satisfies

$$\begin{cases} v_t - \Delta v \leq \frac{1}{\varepsilon^2}\beta(v) & \text{in } \Omega' \times (\tau, T), \\ v(x,t) \leq 1 & \text{on } \overline{\Omega'} \times [\tau, T], \\ v(x,t) \leq u(x,t) & \text{on } \partial\Omega' \times [\tau, t] \cup \Omega' \times \{0\}. \end{cases}$$

Then

$$v(x,t) \leq u(x,t) \qquad \forall (x,t) \in \Omega' \times (\tau, t).$$

Here $f \leq \beta(v)$ means that $f \leq v$ if $v \in (-1,1)$, $f \leq 1$ if $v = 1$ and $f \leq 0$ if $v = -1$.

Since u solves the parabolic problem (2.2), the proof follows from standard routine techniques, namely, subtracting the parabolic equations from each other, multiplying the resulting equation by $(v-u)_+$, integrating and using Gronwall's inequality.

The comparison lemmas for supersolutions can be stated in a corresponding way and are omitted.

3 Asymptotic behavior of $\varepsilon \to 0$.

In this section we consider the asymptotic behavior of the solution, denoted by u^ε, to the double obstacle problem (2.1). It is convenient to begin by describing the evolution of an interface according to its mean curvature.

3.1. MOTION BY MEAN CURVATURE.

The family $\{\Gamma(t)\}_{t\in[0,T]}$ of sufficiently smooth hypersurfaces in $\mathcal{R}^N$ is said to form a motion by mean curvature flow starting from Γ_0 if the following holds:

(i) $\Gamma(t)$ is the boundary of an open set $A(t) \in \mathcal{R}^N$ for $t \in [0,T]$;

(ii) $\Gamma(0) = \Gamma_0$;

(iii) The speed V of the hypersurface $\Gamma(t)$ in the outward normal direction to $A(t)$ is $-K^m$, the sum of the principle curvatures (mean curvature) of $\Gamma(t)$ where K^m is positive if $A(t)$ is convex.

It is convenient to describe such a flow by introducing the signed distance function $d(x,t) : \mathcal{R}^N \times [0,T] \to \mathcal{R}^1$ from x to $\Gamma(t)$ so that $d(x,t)$ is negative in $A(t)$ and positive in $\mathcal{R}^N \setminus \overline{A(T)}$. It follows that

$$\begin{aligned} \Gamma(t) &= \{x \in \mathcal{R}^N \mid d(x,t) = 0\}, & |\nabla d(x,t)| &= 1 \text{ a.e. } x \in \mathcal{R}^N, \\ d_t(x,t) &= -V(x,t), & \Delta d(x,t) &= K^m(x,t) \quad \forall x \in \Gamma(t). \end{aligned}$$

The following facts are known.

(1) If $\Gamma_0 \in C^{m+\alpha}$ $(m \geq 2)$ is the boundary of a bounded open set then there exists a unique solution of the motion by mean curvature flow starting from Γ_0 in a time interval $(0,T]$ for some $T > 0$. Furthermore the signed distance function $d \in C^{m+\alpha,\,(m+\alpha)/2}$ in a neighborhood of $\cup_{t\in[0,T]}(\Gamma(t) \times \{t\})$.

(2) If $N = 2$ or Γ_0 is convex then the solution can be extended up to $T = T^*$ where T^* is the time at which $\Gamma(\cdot)$ shrinks to a single point.

For more details, we refer to Hamilton [1982], Gage–Hamilton [1986], Grayson [1987], Huisken [1984], Chen–Giga–Goto [1991], and Evans–Spruck [1991].

3.2. Approximation of mean curvature flow by the double obstacle problem.

Let $\{\Gamma(t)\}_{t\in[0,T]}$ be a mean curvature flow such that for each $t \in [0,T]$, $\Gamma(t)$ lies in the interior of a bounded open set Ω and $\operatorname{dist}(\Gamma(t), \partial\Omega) \geq \delta > 0$ for some fixed δ. We shall assume that the associated signed distance function d satisfies

$$\sup_{t\in[0,T]} \sup_{|d|\leq c_0} |\nabla(d_t - \Delta d)| \leq D_0. \tag{3.1}$$

THEOREM 3.1. *Let $\gamma(\varepsilon)$ satisfy*

$$\lim_{\varepsilon\to 0} \gamma(\varepsilon) = 0$$

and assume that there exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{\pi}{2}\varepsilon \leq \gamma(\varepsilon) \quad \text{and} \quad \frac{\pi}{2}\varepsilon + 2\gamma(\varepsilon)e^{2D_0T} \leq c_0. \tag{3.2}$$

Let $u^\varepsilon(x,t)$ be the unique solution of (2.1) with initial data $g^\varepsilon \in \mathcal{G} \cap C(\overline{\Omega})$ satisfying

$$\{x : d(x,0) > \gamma(\varepsilon)\} \subset \{x : g^\varepsilon(x) = 1\}, \tag{3.3}$$

$$\{x : d(x,0) < -\gamma(\varepsilon)\} \subset \{x : g^\varepsilon(x) = -1\}. \tag{3.4}$$

Then for all $t \in [0,T]$, it holds that

$$\{x : d(x,t) \geq \gamma(\varepsilon)(1 + 2e^{2D_0t})\} \subset \{x : u^\varepsilon(x,t) = 1\}, \tag{3.5}$$

$$\{x : d(x,t) \leq -\gamma(\varepsilon)(1 + 2e^{2D_0t})\} \subset \{x : u^\varepsilon(x,t) = -1\}. \tag{3.6}$$

Proof. The fact that $\{\Gamma(t)\}$ is a mean curvature flow implies

$$d_t - \Delta d = 0 \quad \text{on} \quad \Gamma(t),\ t \in (0,T]$$

and the assumption (3.1) yields

$$|d_t - \Delta d| \leq D_0 |d|$$

whenever $|d| \leq c_0$.

Set

$$z^\varepsilon(x,t) := d(x,t) - 2\gamma(\varepsilon) e^{2D_0 t}$$

and

$$\Omega^\varepsilon(t) := \{x \in \Omega \,:\, |z^\varepsilon(x,t)| < \varepsilon\pi/2\}.$$

It follows from (3.2) that

$$|d(x,t)| \leq c_0 \qquad \forall\, x \in \Omega^\varepsilon(t),\ t \in (0,T].$$

Hence on $\Omega^\varepsilon(t)$,

$$\begin{aligned} z^\varepsilon_t - \Delta z^\varepsilon &= d_t - \Delta d - 4D_0\gamma(\varepsilon)e^{2D_0 t} \\ &\leq D_0(|d| - 4\gamma(\varepsilon)e^{2D_0 t}) \\ &\leq D_0(\varepsilon\pi/2 - 2\gamma(\varepsilon)) \leq 0 \end{aligned}$$

by (3.2).

We now introduce $U \in C^{1,1}(\mathcal{R})$ given by

$$U(s) = \begin{cases} 1 & s \geq \pi/2, \\ \sin s & s \in (-\pi/2, \pi/2), \\ -1 & s \leq -\pi/2. \end{cases}$$

Set

$$v^\varepsilon(x,t) := U(z^\varepsilon(x,t)/\varepsilon).$$

Observe that $v^\varepsilon(x,0) > -1$ if and only if $d(x,0) - 2\gamma(\varepsilon) > -\varepsilon\pi/2$. Thus $v^\varepsilon(x,0) > -1$ implies that $d(x,0) > \gamma(\varepsilon)$ and so by (3.3) $g^\varepsilon(x) = 1$. Hence it holds that

$$-1 \leq v^\varepsilon(x,0) \leq g^\varepsilon(x) = u^\varepsilon(x,0).$$

Let

$$f(x,t) = \begin{cases} v^\varepsilon(x,t) & v^\varepsilon(x,t) > -1, \\ 0 & v^\varepsilon(x,t) = -1. \end{cases} \tag{3.7}$$

For any $\varphi \in C^1(\Omega \times (0,T))$ and $\varphi \geq 0$ we have

$$
\begin{aligned}
&\int_0^T \int_\Omega \left\{ v_t^\varepsilon \varphi + \nabla v^\varepsilon \nabla \varphi - \frac{1}{\varepsilon^2} f \varphi \right\} dx dt \\
&= \int_0^T \int_{\Omega^\varepsilon(t)} \left\{ v_t^\varepsilon \varphi + \nabla v^\varepsilon \nabla \varphi - \frac{1}{\varepsilon^2} f \varphi \right\} dx dt - \iint_{\{v^\varepsilon=-1\}\cup\{v^\varepsilon=1\}} \frac{1}{\varepsilon^2} f \varphi \, dx dt \\
&\leq \int_0^T \int_{\Omega^\varepsilon(t)} \left\{ v_t^\varepsilon - \Delta v^\varepsilon - \frac{1}{\varepsilon^2} v^\varepsilon \right\} \varphi \, dx dt,
\end{aligned}
$$

where we have used the fact that on $\partial\Omega^\varepsilon(t)$, $|z^\varepsilon| = \varepsilon\pi/2$ and $\nabla v^\varepsilon = \frac{1}{\varepsilon}\nabla z^\varepsilon U'(z^\varepsilon/\varepsilon) = 0$.

Calculation yields

$$
v_t^\varepsilon - \Delta v^\varepsilon - \frac{1}{\varepsilon^2} v^\varepsilon = U' \frac{z_t^\varepsilon}{\varepsilon} - U' \frac{\Delta z^\varepsilon}{\varepsilon} - U'' \frac{|\nabla z^\varepsilon|^2}{\varepsilon^2} - \frac{1}{\varepsilon^2} U
$$

and since on $\Omega^\varepsilon(t)$, $U'' = -U$, $U' > 0$ and $|\nabla z^\varepsilon| = 1$, we find

$$
v_t^\varepsilon - \Delta v^\varepsilon - \frac{1}{\varepsilon^2} v^\varepsilon = \frac{1}{\varepsilon} U'(z_t^\varepsilon - \Delta z^\varepsilon) \leq 0 \quad \forall x \in \Omega^\varepsilon(t).
$$

Therefore

$$
\int_0^T \int_\Omega \left\{ v_t^\varepsilon \varphi + \nabla v^\varepsilon \nabla \varphi - \frac{1}{\varepsilon^2} f \varphi \right\} dx dt \leq 0.
$$

By (3.7), $f(v^\varepsilon - u^\varepsilon)_+ \leq v^\varepsilon (v^\varepsilon - u^\varepsilon)_+$. It then follows by the comparison Lemma 2.3 that

$$
u^\varepsilon(x,t) \geq v^\varepsilon(x,t) \qquad \forall (x,t) \in \Omega \times [0,T].
$$

Since $d(x,t) \geq \gamma(\varepsilon) + 2\gamma(\varepsilon) e^{2D_0 t}$ implies that $z^\varepsilon(x,t) \geq \varepsilon\pi/2$ which again implies that $v^\varepsilon(x,t) = 1$, it follows that (3.5) holds. The proof of (3.6) is identical. □

REMARK 3.2. It follows from Theorem 3.1 that the set

$$
\Gamma^\varepsilon(t) = \{x \in \Omega : u^\varepsilon(x,t) = 0\}
$$

satisfies the estimate

$$
\text{dist}\,(\Gamma(t), \Gamma^\varepsilon(t)) \leq c\gamma(\varepsilon) \qquad \forall t \in [0,T]
$$

for some c independent of ε and t. It follows that choosing

$$
\gamma(\varepsilon) \leq \frac{\pi}{2}\varepsilon + c\varepsilon
$$

the estimate

$$
\text{dist}\,(\Gamma(t), \Gamma^\varepsilon(t)) \leq \bar{c}\varepsilon
$$

holds.

3.3. Generation and propagation of an interface.

In this subsection, we shall show that, for more general initial data than considered in §3.2, after a short time an interface will be formed near the set $\Gamma_0 = \{x \in \Omega : g(x) = 0\}$ and will propagates according to motion by mean curvature.

Lemma 3.3. *Let $g \in \mathcal{G} \cap W^{1,\infty}(\Omega)$. Then there exist positive constants $\{c_1, \varepsilon_1, \tau_1\}$ such that if $\varepsilon \in (0, \varepsilon_1)$ then*

$$u^\varepsilon(x,t) \geq 1 - \varepsilon \qquad \forall x \in \Omega_\varepsilon^+,\ t \in [\tau_1 \varepsilon^2 |\ln \varepsilon|, (\tau_1 + 1)\varepsilon^2 |\ln \varepsilon|], \tag{3.8}$$

$$u^\varepsilon(x,t) \leq -1 + \varepsilon \qquad \forall x \in \Omega_\varepsilon^-,\ t \in [\tau_1 \varepsilon^2 |\ln \varepsilon|, (\tau_1 + 1)\varepsilon^2 |\ln \varepsilon|] \tag{3.9}$$

where

$$\begin{aligned} \Omega_\varepsilon^+ &:= \{x \in \Omega : g(x) > c_1 \varepsilon |\ln \varepsilon|\}, \\ \Omega_\varepsilon^- &:= \{x \in \Omega : g(x) < -c_1 \varepsilon |\ln \varepsilon|\}. \end{aligned}$$

Proof. Let $\phi \in C^0(\mathcal{R})$ satisfy

$$\begin{gathered} \phi(0) = \phi(1) = \phi(-2) = 0, \quad \phi'(0) > 0,\ \phi'(1) < 0,\ \phi'(-2) < 0, \\ \phi(s) \leq s\ \forall s \in [-1,1]; \quad \phi(s) > 0\ \forall s \in (0,1); \quad \phi(s) < 0\ \forall s \in (-2,0). \end{gathered}$$

Set $v^\varepsilon(x,t)$ to be the unique solution of

$$\begin{array}{rll} v_t^\varepsilon - \Delta v^\varepsilon & = \frac{1}{\varepsilon^2}\phi(v^\varepsilon), & \text{in } \Omega \times (0,T), \\ v^\varepsilon(x,0) & = g^\varepsilon(x) & \text{on } \Omega, \\ \partial v^\varepsilon / \partial n & = 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

where $g^\varepsilon \in \mathcal{G}$ and

$$g^\varepsilon \begin{cases} = g(x) & \text{if } g(x) \leq 1/2, \\ = 3/4 & \text{if } g(x) \geq 7/8, \\ \leq g(x) & \text{if } g(x) \in (1/2,\ 7/8). \end{cases}$$

It follows that, by Theorems 1 and 2 of Chen [1991], there exist positive constants $\{c_1, \varepsilon_1, \tau_1\}$ such that if $\varepsilon \in (0, \varepsilon_1)$ then

$$v^\varepsilon(x,t) \geq 1 - \varepsilon,\ \ \forall t \in (\tau_1 \varepsilon^2 |\ln \varepsilon|, (\tau_1 + 1)\varepsilon^2 |\ln \varepsilon|),\ x \in \{y \in \Omega : g^\varepsilon \geq c_1 \varepsilon |\ln \varepsilon|\} = \Omega_\varepsilon^+.$$

Furthermore, since $|v^\varepsilon| \leq 1$ and $\phi(v^\varepsilon) \leq \beta(v^\varepsilon)$, it follows by the comparison lemma that

$$u^\varepsilon(x,t) \geq v^\varepsilon(x,t) \qquad \forall (x,t) \in \Omega \times [0,T].$$

Thus (3.8) is proved and a similar argument proves (3.9). □

Lemma 3.4. *For any $c > 0$ let*

$$Q_{c\varepsilon} := \{(x,t) : |x| \leq c\varepsilon,\ 0 < t < c^2 \varepsilon^2\},$$

$$\partial_T Q_{c\varepsilon} = \{(x,t) : |x| = c\varepsilon,\ 0 \le t \le c^2\varepsilon^2\} \cup \{(x,0) : |x| < c\varepsilon\}.$$

Then there exists a positive constant c such that if for some $x_0 \in \Omega$ and $\delta > 0$, the conditions $(x_0,\delta) + Q_{c\varepsilon} \in \Omega \times (0,T)$ and

$$u^\varepsilon(x + x_0, t + \delta) \ge 1/2 \qquad \forall (x,t) \in \partial_T Q_{c\varepsilon}$$

are satisfied, then

$$u^\varepsilon(x_0, \delta + c^2\varepsilon^2) = 1.$$

Proof. Let $\xi(\cdot) \in C^\infty(\mathcal{R})$ satisfy

$$\xi(s) = 1/2 \quad \forall s \le 0; \qquad \xi(s) = 1 \quad \forall s \ge 1;$$
$$0 \le \xi'(s) \le 2, \quad |\xi''(s)| \le 2 \quad \forall s \in \mathcal{R}^1.$$

Define

$$v^\varepsilon(x,t) = \xi\Big(\frac{t}{c^2\varepsilon^2}\Big)\xi\Big(y(x)\Big)$$

where

$$y(x) = (\sqrt{2}c\varepsilon - \sqrt{|x|^2 + c^2\varepsilon^2})/(c\varepsilon(\sqrt{2}-1)).$$

We wish to compare $v^\varepsilon(\cdot,\cdot)$ with $u^\varepsilon(x_0 + \cdot, \delta + \cdot)$ in $Q_{c\varepsilon}$. Clearly,
(i) $1/4 \le v^\varepsilon(x,t) \le 1 \qquad \forall (x,t) \in \overline{Q_{c\varepsilon}}$;
(ii) $v^\varepsilon(x,t) \le 1/2 \le u^\varepsilon(x_0 + x, \delta + t) \qquad \forall (x,t) \in \partial_T Q_{c\varepsilon}$;
(iii) $v^\varepsilon(0, c^2\varepsilon^2) = 1$.
Computation yields

$$\begin{aligned} v_t^\varepsilon - \Delta v^\varepsilon - \frac{1}{\varepsilon^2}v^\varepsilon &= \frac{1}{c^2\varepsilon^2}\xi'\Big(\frac{t}{c^2\varepsilon^2}\Big)\xi(y) + \frac{Nc^2\varepsilon^2 + (N-1)|x|^2}{(\sqrt{2}-1)c\varepsilon(|x|^2 + c^2\varepsilon^2)^{3/2}}\xi\Big(\frac{t}{c^2\varepsilon^2}\Big)\xi'(y) \\ &\quad - \frac{|x|^2}{(\sqrt{2}-1)^2c^2\varepsilon^2(|x|^2 + c^2\varepsilon^2)}\xi\Big(\frac{t}{c^2\varepsilon^2}\Big)\xi''(y) - \frac{1}{\varepsilon^2}v^\varepsilon \\ &\le \frac{1}{c^2\varepsilon^2} + \frac{N}{(\sqrt{2}-1)c^2\varepsilon^2} + \frac{1}{(\sqrt{2}-1)^2c^2\varepsilon^2} - \frac{1}{4\varepsilon^2} \le 0 \end{aligned}$$

for all $(x,t) \in Q_{c\varepsilon}$ if c is sufficiently large. It follows by the comparison principle that

$$v^\varepsilon(x,t) \le u^\varepsilon(x_0 + x, \delta + t) \qquad \forall (x,t) \in Q_{c\varepsilon}$$

and in particular,

$$u^\varepsilon(x_0, \delta + c^2\varepsilon^2) \ge v^\varepsilon(0, c^2 v^\varepsilon) = 1.$$

The assertion of the lemma thus follows. $\square$

First applying Lemma 3.3 and then applying Lemma 3.4 with $\delta = \tau_1\varepsilon^2|\ln\varepsilon|$ yield

$$u^\varepsilon(x, \tau_1\varepsilon^2|\ln\varepsilon| + c^2\varepsilon^2) = 1 \qquad \forall x \in \{x \in \Omega_\varepsilon^+ : \operatorname{dist}(x, \partial\Omega_\varepsilon^+) \ge c\varepsilon\},$$

$$u^\varepsilon(x, \tau_1\varepsilon^2|\ln\varepsilon| + c^2\varepsilon^2) = -1 \qquad \forall\, x \in \{x \in \Omega_\varepsilon^- \,:\, \mathrm{dist}\,(x, \partial\Omega_\varepsilon^-) \geq c\varepsilon\}.$$

Finally applying Theorem 2.1 with $\gamma(\varepsilon) = C\varepsilon|\ln\varepsilon|$, we conclude the following.

THEOREM 3.5. *Assume that $g \in \mathcal{G} \cap W^{1,\infty}(\Omega)$ and*

$$|\nabla g| > c \qquad \forall\, x \in \{y \in \Omega \,:\, |g(y)| \leq c\} \tag{3.10}$$

for some $c > 0$. Let $\Gamma_0 = \{y \in \Omega : g(y) = 0\}$ and assume that $\Gamma_0 \subset \Omega$ and $\Gamma_0 \in C^{3+\alpha}$. Then there exists positive constants C and ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ and $t \in (C\varepsilon^2|\ln\varepsilon|, T]$,

$$u^\varepsilon(x,t) = 1 \;\; \text{if} \;\; d(x,t) > C\varepsilon|\ln\varepsilon|,$$

$$u^\varepsilon(x,t) = -1 \;\; \text{if} \;\; d(x,t) < -C\varepsilon|\ln\varepsilon|$$

where $d(x,t)$ and T are as in §3.1.

REMARK 3.6. In Theorem 3.5, condition (3.10) is only used to ensure that $\Omega \setminus (\Omega_\varepsilon^+ \cup \Omega_\varepsilon^-)$ is contained in a $O(\varepsilon|\ln\varepsilon|)$ neighborhood of Γ_0, whereas the regularity assumption on Γ_0 is to ensure the existence of a smooth solution to the motion by mean curvature equation.

4 Asymptotic behavior as $t \to \infty$ in one space dimension.

4.1. STATEMENT OF MAIN RESULT.

In this section we shall restrict our attention to one space dimension and set $\Omega = (0,1)$. We shall prove the following theorem.

THEOREM 4.1. *Let $g \in \mathcal{G}$ and let $u_g(t) = \mathcal{S}(t)g$ be the unique solution of the double obstacle problem (2.1). Then there exists $f \in C^{1,1}[0,1] \in \mathcal{E}$ such that*

$$\lim_{t\to\infty} u_g(x,t) = f(x) \qquad \forall\, x \in [0,1].$$

We have already established in §2 that $\omega(g) \in \mathcal{E}$, so in order to prove the theorem we need only show that $\omega(g)$ consists of a single element. It is convenient to characterize $\mathcal{E}$ as follows (c.f. Blowey–Elliott [1991a]).

Any $f \in \mathcal{E}$ satisfies: $f \in C^{1,1}[0,1]$ and

$$\begin{cases} \varepsilon^2 f_{xx} + f &= 0 \qquad \text{for} \;\; x \in \Omega_I := \{x \in (0,1) : |f(x)| < 1\}. \\ f'(x) &= 0 \qquad \text{for} \;\; x \in \partial\Omega_T \cup \partial\Omega. \end{cases} \tag{4.1}$$

Let K be the largest integer strictly less than $1/(\varepsilon\pi)$. It is obvious that if $K+1 = 1/\varepsilon\pi$, then

$$f(x) = a\cos x/\varepsilon \qquad \forall\, a \in [-1,1]$$

is a solution to (4.1) and furthermore for any $\varepsilon > 0$,

$$f(x) \equiv 0, \quad f(x) \equiv -1, \quad f(x) \equiv 1$$

are equilibria. It is convenient to set

$$\mathcal{E}_0 := \{f : f(x) \equiv 1 \ \text{ or } \ f(x) \equiv -1\}$$

and

$$\mathcal{E}^c = \begin{cases} \{f : f(x) \equiv 0\} & \text{if } 1 < (K+1)\varepsilon\pi, \\ \{f : f(x) = a \cos x/\varepsilon, \, |a| \le 1\} & \text{if } 1 = (K+1)\varepsilon\pi. \end{cases}$$

Any other solution f of (4.1) has at least one layer connecting $f = 1$ to $f = -1$ in which $|f(x)| < 1$. By the definition of K, there can be at most K layers in a solution other than $f(x) = \cos x/\varepsilon \in \mathcal{E}^c$ when $1 = (K+1)\varepsilon\pi$. Therefore for each $k \in \{1, 2, \dots, K\}$, we let $\mathcal{E}_k$ be the set of k-layered solutions characterized by the $k+1$ parameters $\{a; s_1, \dots, s_k\}$ in the following way. $f \equiv f[a; s_1, \dots, s_k] \in \mathcal{E}_k$ if and only if

$$\begin{cases} |a| = 1, \quad 0 \le s_1, \quad s_k \le 1 - \varepsilon\pi, \quad s_j \le s_{j+1} - \varepsilon\pi \ \forall j = 1, \cdots, k-1; \\ A_1^k = a, \quad A_{j+1}^k = -A_j^k, \quad j = 1, 2, \dots, k; \\ f(x) = A_j^k \cos(x - s_j^k)/\varepsilon \quad \text{if} \quad s_j^k \le x \le s_j^k + \varepsilon\pi; \\ f(x) = a \ \text{ if } \ 0 \le x \le s_1, \qquad f(x) = A_{k+1} \ \text{ if } \ s_k + \varepsilon\pi \le x \le 1; \\ f(x) = A_{j+1} \ \text{ if } \ s_j + \varepsilon\pi \le x \le s_{j+1}, \quad j = 1, 2, \cdots, k-1. \end{cases} \tag{4.2}$$

This leads to the following conclusion.

LEMMA 4.2.

$$\mathcal{E} = \cup_{k=0}^{K} \mathcal{E}_k \cup \mathcal{E}^c.$$

4.2. STABILITY OF THE EQUILIBRIA.

In this subsection we study the stability of the equilibria in $\mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_K$ and $\mathcal{E}^c$ respectively. A direct calculation yields the following lemma.

LEMMA 4.3. *(i)* $\inf_{\eta \in H^1(\Omega) \cap \mathcal{G}} E(\eta) = 0$;
(ii) $E(f) = \frac{k\pi}{\varepsilon} =: e_k \quad \forall f \in \mathcal{E}_k, \ 0 \le k \le K$;
(iii) $E(f) = \frac{1}{\varepsilon^2} =: e^c \quad \forall f \in \mathcal{E}^c$;
(iv) $0 = e_0 < e_1 < \cdots < e_K < e^c = 1/\varepsilon^2$.
(v) The sets $\{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_K, \mathcal{E}^c\}$ *are isolated in* $L^\infty(0,1)$ *in the sense that there exists* $\rho > 0$ *such that*

$$dist\,(\mathcal{E}_i, \mathcal{E}_j) > \rho, \quad dist\,(\mathcal{E}_j, \mathcal{E}^c) > \rho, \qquad \forall i, j, i \ne j.$$

(vi) The set $\mathcal{E}^c$ is connected and for each $k = 1, \cdots, K$, $\mathcal{E}_k$ consists of two isolated connected subsets

$$\{f[1; s_1, \cdots, s_k] : 0 \le s_1, \quad s_k \le 1 - \varepsilon\pi, \quad s_j \le s_{j+1} - \varepsilon\pi \ \forall j = 1, \cdots, k-1\}$$

and

$$\{f[-1; s_1, \cdots, s_k] : 0 \le s_1, \quad s_k \le 1 - \varepsilon\pi, \quad s_j \le s_{j+1} - \varepsilon\pi \ \forall j = 1, \cdots, k-1\}.$$

(vii) The k-layered equilibria have the energy π/ε associated with each layer (interface) connecting $+1$ to -1; there is no energy associated with the set $\{x : |f(x)| = 1\}$.

Since from §2 the Lyapunov functional $\mathrm{E}(\cdot)$ is constant on $\omega(g)$, the following holds:

COROLLARY 4.4. *$\omega(g)$ is contained in one and only one of $\{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_K, \mathcal{E}^c\}$.*

DEFINITION 4.5. (1) $f \in \mathcal{E}$ is called *asymptotically stable* if there exists $\delta > 0$ such that

$$\lim_{t\to\infty} u_g(\cdot, t) = f, \quad \forall g \in B_\delta(f) := \{h \in \mathcal{G} : \|f - h\|_{L^\infty} \le \delta\}.$$

(2) $f \in \mathcal{E}$ is called *stable* if for every $\delta > 0$ there exists an $\eta > 0$ such that

$$\overline{\lim_{t\to\infty}} \max_{x\in[0,1]} |u_g(x,t) - f(x)| \le \delta \quad \forall g \in B_\eta(f).$$

(3) If $f \in \mathcal{E}$ is not stable then it is said to be *unstable*.

(4) An unstable $f \in \mathcal{E}$ is called *M–unstable* if there exists a sequence $g_n \to f$ in $L^\infty(0,1)$ such that

$$\lim_{n\to\infty} \overline{\lim_{t\to\infty}} \|u_{g_n}(\cdot, t) - f\|_{L^\infty} > 0$$

and a sequence $f_n \to f$ in $L^\infty(0,1)$ such that

$$\lim_{n\to\infty} \overline{\lim_{t\to\infty}} \|u_{f_n}(\cdot, t) - f(\cdot)\|_{L^\infty} = 0.$$

LEMMA 4.6. *If $f \in \mathcal{E}_0$ then f is asymptotically stable.*

Proof. For every $a \in (0,1) \cup (-1,0)$, set

$$w(a,t) = \begin{cases} a e^{t/\varepsilon^2} & \text{if } 0 \le t \le \varepsilon^2 \ln 1/a, \\ \operatorname{sign} a & \text{if } t > \varepsilon^2 \ln 1/a. \end{cases}$$

It follows that $w(a,t) = \mathcal{S}(t)a$ and we can prove, by the comparison Lemma 2.2, that

$$w(\inf_{x\in[0,1]} g(x), t) \le u_g(x,t) \le w(\sup_{x\in[0,1]} g(x), t)$$

and therefore

$$\text{if } g > 0 \text{ then } \lim_{t\to\infty} u_g(\cdot,t) = 1,$$

$$\text{if } g < 0 \text{ then } \lim_{t\to\infty} u_g(\cdot,t) = -1,$$

from which, the lemma follows. □

Remark 4.7. This Lemma is true for the multi-dimensional problem. The proof is identical.

LEMMA 4.8. *Each element of $\mathcal{E}^c$ is M–unstable.*

Proof. Set

$$K' = \begin{cases} K+1 & \text{if } \varepsilon\pi(K+1) = 1, \\ K & \text{if } \varepsilon\pi(K+1) > 1, \end{cases}$$

$$U = \{\eta : \eta(x) = \sum_{j=0}^{K'} a_j \cos(j\pi x)\},$$

and

$$S = \{\eta : \eta(x) = \sum_{j=K'+1}^{\infty} a_j \cos(j\pi x)\}.$$

Let $f \in \mathcal{E}^c$ and

$$g = f + h \in \mathcal{G}$$

where $h\,(\not\equiv 0) \in U$. A direct calculation yields

$$\mathrm{E}(g) = \frac{1}{\varepsilon^2} - \frac{a_0^2}{\varepsilon^2} + \sum_{j=1}^{K'} \frac{a_j^2}{2}\Big(j^2\pi^2 - \frac{1}{\varepsilon^2}\Big) < \frac{1}{\varepsilon^2} = \mathrm{E}(f).$$

(If $K'\varepsilon\pi = 1$ and $f = \cos x/\varepsilon$, then take $a_{K'} < 0$ and $a_0 \neq 0$ so that $f + h \in \mathcal{G}$.)

It follows that

$$\lim_{t\to\infty} \mathrm{E}(u_g(\cdot,t)) \le \mathrm{E}(g) < \mathrm{E}(f).$$

Therefore U is an unstable manifold.

On the other hand, let $h \in S$ be chosen so that

$$g = f + h \in \mathcal{G} \quad \text{and} \quad |g(x)| < 1 \text{ a.e. }.$$

It follows that

$$\widetilde{u_g} := f(x) + \sum_{j=K'+1}^{\infty} a_j e^{(1/\varepsilon^2 - j^2\pi^2)t} \cos(j\pi x)$$

solves

$$\begin{aligned} &u_t - u_{xx} = u/\varepsilon^2, \qquad x \in (0,1), t > 0; \\ &u(x,0) = g(x), \quad x \in (0,1); \qquad u_x(0,t) = u_x(1,t) = 0, \quad t > 0. \end{aligned}$$

Since $1/\varepsilon^2 - j^2\pi^2 \le \pi^2((K'+1)^2 - j^2) < 0$, it follows that

$$\lim_{t\to\infty} \widetilde{u_g}(\cdot,t) = f.$$

Suppose that $f \neq \pm\cos x/\varepsilon$ then there exists $\delta > 0$ such that $\|f\|_\infty < 1-\delta$. Choose $h \in S$ such that

$$\sum_{j=K'+1}^{\infty} |a_j| \le \delta/2.$$

It follows that

$$\|\widetilde{u_g}\|_\infty < 1 - \delta/2 \qquad \forall t > 0$$

which implies that $\widetilde{u_g}(\cdot,t) = u_g(\cdot,t)$ and the assertion of the lemma follows.

When $f = \pm\cos x/\varepsilon$, the sequence $\{\pm(1-1/n)\cos x/\varepsilon\}_{n=1}^{\infty}$ is a sequence of equilibria which converges to f as $n \to \infty$. This completes the proof of the lemma. □

In order to discuss the stability of $\{\mathcal{E}_k\}_{k=1}^{K}$, it is convenient to introduce

$$\mathcal{E}_k^s = \Big\{f[a;s_1,\ldots,s_k] \in \mathcal{E}_k : 0 < s_1,\ s_k < 1-\varepsilon\pi,\ s_j < s_{j+1} - \varepsilon\pi\ \forall j = 1,\cdots,k-1\Big\},$$

$$\mathcal{E}_k^u = \Big\{f[a;s_1,\ldots,s_k] \in \mathcal{E}_k : s_1 = 0 \text{ or } \ s_k = 1-\varepsilon\pi \ \text{ or } \ s_{j+1}^k = s_j^k + \varepsilon\pi \ \text{ for some } j\Big\},$$

so that

$$\mathcal{E}_k = \mathcal{E}_k^s \cup \mathcal{E}_k^u, \qquad \mathcal{E}_k^s \cap \mathcal{E}_k^u = \emptyset.$$

LEMMA 4.9. *For each $k \in \{1,\ldots,K\}$, every element of $\mathcal{E}_k^u$ is M–unstable.*

Proof. We first consider the case $k = 2$. Without loss of generality let $f = f[1;s,s+\varepsilon\pi] \in \mathcal{E}_2^u$. Then $f_n := f[1;s,s+\varepsilon\pi+1/n]$ (or $f_n := f[1;s-1/n,s+\varepsilon\pi]$ when $s = 1-2\varepsilon\pi$) is a sequence of equilibria which converges to f as $n \to \infty$. It remains to show that f is unstable.

Let $g \not\equiv f$ satisfy

$$f \le g \le 1.$$

By comparison,

$$u_g(\cdot,t) \ge f(\cdot) \qquad \forall t \ge 0.$$

so that

$$u_g(x,t) = 1 \quad \text{when} \quad |x-(s+\varepsilon\pi)| \ge \varepsilon\pi.$$

Let $\alpha \in (\pi/2,\pi)$ and set

$$h(x) := \begin{cases} 0 & x-(x+\varepsilon\pi) \le -\alpha\varepsilon, \\ \sin\Big[\pi(x-(s+\varepsilon\pi)+\alpha\varepsilon)/(2\alpha\varepsilon)\Big] & |x-(s+\varepsilon\pi)| < \alpha\varepsilon, \\ 0 & x-(s+\varepsilon\pi) \ge \alpha\varepsilon. \end{cases}$$

Clearly,

$$\int_s^{s+2\varepsilon\pi} \left\{ f'h' - \frac{1}{\varepsilon^2} fh \right\} dx = - \int_s^{s+2\varepsilon\pi} \left(f'' + \frac{1}{\varepsilon^2} f \right) h \, dx = 0.$$

Take $\delta \in (0,1)$ small so that

$$g = f + \delta h \in \mathcal{G}.$$

Direct calculation yields

$$\begin{aligned}
\mathrm{E}(g) &= \mathrm{E}(f) + 2\delta \int_s^{s+2\varepsilon\pi} \left\{ f'h' - \frac{1}{\varepsilon^2} fh \right\} dx + \delta^2 \int_s^{s+2\varepsilon\pi} \left\{ h'^2 - \frac{1}{\varepsilon^2} h^2 \right\} dx \\
&= \mathrm{E}(f) + \delta^2 \int_{(s+\varepsilon\pi)-\varepsilon\alpha}^{(s+\varepsilon\pi)+\varepsilon\alpha} h \Big(-h'' - \frac{1}{\varepsilon^2} h \Big) \, dx \\
&= \mathrm{E}(f) + \delta^2 \int_{(s+\varepsilon\pi)-\varepsilon\alpha}^{(s+\varepsilon\pi)+\varepsilon\alpha} \frac{h^2}{\varepsilon^2} \Big(\frac{\pi^2}{4\alpha^2} - 1 \Big) \, dx < \mathrm{E}(f).
\end{aligned}$$

Therefore if $\tilde{f} \in \omega(g)$, then $\mathrm{E}(\tilde{f}) < \mathrm{E}(f)$. This establishes the instability of f. The proof for $k = 1$ and $k > 2$ is identical. □

LEMMA 4.10. *For each $k \in \{1, \ldots, K\}$, the set $\mathcal{E}_k^s$ is stable .*

Proof. Let $f \in \mathcal{E}_k$ and $g \in B_\eta(f)$. The Sobolev inequality yields

$$\|f - u_g(\cdot, t)\|_\infty \le \sqrt{2} \|f - u_g\|^{1/2} \|f - u_g\|_{H^1}^{1/2} \le \sqrt{2\eta\rho e^{C_\varepsilon t/2}} \qquad \forall t \ge t^* \tag{4.3}$$

where in the second inequality we have used (2.14) and Theorem 2.1(3).

For simplicity, we shall assume that $k = 2$. Since for $f \in \mathcal{E}_k^s$, the layers are isolated and lie in the interior of $(0,1)$, the argument we use for $k = 2$ can be readily extended to the general case. So without loss of generality, we assume that $f = f[-1; s_1, s_2] \in \mathcal{E}_2^s$ where $0 < s_1 < s_2 - \varepsilon\pi < 1 - 2\varepsilon\pi$.

For any μ satisfying $0 < \mu < \frac{1}{5} \min\{\varepsilon\pi, s_1, s_2 - s_1 - \varepsilon\pi, 1 - s_2 - \varepsilon\pi\}$, let $w^\mu(y,t)$ be the solution of

$$\begin{cases} w_t^\mu - w_{yy}^\mu = 1/(2\varepsilon^2), & (y,t) \in (-\mu, \mu) \times (0, \infty), \\ w^\mu(-\mu, t) = w^\mu(\mu, t) = 0, & t \in (0, \infty), \\ w^\mu(y, 0) = 0, & y \in (-\mu, \mu). \end{cases}$$

One can easily verify that

$$w^\mu(0,t) > w^\mu(y,\tau) \qquad \forall\, 0 \le \tau < t, \ y \in (-\mu, \mu).$$

Set $M(\mu) = w^\mu(0,1)$. Taking smaller μ if necessary, we can assume, without loss of generality, that $0 < M(\mu) \le 1/2$. For every $x_0 \in (s_1 + \varepsilon\pi + \mu, s_2 - \mu)$, consider the function W^μ defined by

$$W^\mu(x,t) = 1 - M(\mu) + w^\mu(x - x_0, t).$$

Clearly,
$$1/2 \leq W^\mu < 1 \qquad \forall t \in [0,1),\ x \in I := (x_0-\mu, x_0+\mu).$$
It follows that
$$W_t^\mu - W_{xx}^\mu - \frac{1}{\varepsilon^2}W^\mu \leq w_t^\mu - w_{yy}^\mu - \frac{1}{2\varepsilon^2} = 0, \qquad (x,t) \in I \times (0,1).$$
Taking η small such that $\sqrt{2\eta\rho}e^{C_\varepsilon(t^*+1)/2} \leq M(\mu)$, one obtains that for all $(x,t) \in \partial I \times (0,1) \cup I \times \{0\}$,
$$W^\mu(x,t) = 1 - M(\mu) \leq 1 - \eta\rho e^{C_\varepsilon(t^*+t)} \leq u_g(x,t)$$
by (4.3). The comparison Lemma then implies that
$$W^\mu(x,t) \leq u_g(x, t^*+t) \qquad \forall (x,t) \in I \times [0,1].$$
Consequently,
$$u_g(x_0, t^*+1) = 1 \qquad \forall x_0 \in (s_1 + \varepsilon\pi + \mu, s_2 - \mu).$$

Define
$$\eta(\mu) = \frac{1}{2\rho e^{C_\varepsilon(t^*+1)}} \min\left\{M^2(\mu),\ 4\sin^4\frac{\mu}{2\varepsilon}\right\}.$$
We claim that if $\eta \in (0, \eta(\mu))$ then the inequality
$$u_g(x, t^*+1) \geq f[-1; s_1+2\mu, s_2-2\mu](x) \qquad \forall x \in (0,1) \tag{4.4}$$
holds. In fact, since (4.4) holds in the interval $I_1 := [0, s_1+2\mu] \cup [s_2 - 2\mu + \varepsilon\pi, 1]$ (where $f[-1; s_1+2\mu, s_2-2\mu] = -1$) and in the interval $I_2 := [s_1+\varepsilon\pi+\mu, s_2-\mu]$ (where $u_g(x,t^*+1) = 1$), we need only consider the case when $x \in [0,1] \setminus (I_1 \cup I_2) = (s_1+2\mu, s_1+\varepsilon\pi+\mu) \cup (s_2-\mu, s_2+\varepsilon\pi-2\mu) \equiv I_3 \cup I_4$. When $x \in I_3$, we can use the definition of f in (4.2) to calculate
$$\begin{aligned}
&f[-1; s_1, s_2](x) - f[-1; s_1+2\mu, s_2-2\mu](x) \\
&\geq \min\Big\{ \min_{y\in[s_1+2\mu, s_1+\varepsilon\pi]} \Big(-\cos\frac{y-s_1}{\varepsilon} + \cos\frac{y-s_1-2\mu}{\varepsilon}\Big), \\
&\qquad\qquad \min_{y\in[s_1+\varepsilon\pi, s_1+\varepsilon\pi+\mu]} \Big(1 + \cos\frac{y-s_1-2\mu}{\varepsilon}\Big)\Big\} \\
&= 2\sin^2\tfrac{\mu}{2\varepsilon}.
\end{aligned}$$

It follows that
$$\begin{aligned}
&u_g(x, t^*+1) - f[-1; s_1+2\mu, s_2-2\mu](x) \\
&\geq f[-1; s_1, s_2](x) - f[-1; s_1+2\mu, s_2-2\mu](x) - \|u_g(\cdot, t^*+1) - f[-1; s_1, s_2]\|_{L^\infty} \\
&\geq 2\sin^2\tfrac{\mu}{2\varepsilon} - \sqrt{2\eta\rho}e^{C_\varepsilon(t^*+1)/2} \geq 0
\end{aligned}$$

by (4.3) the the definition of $\eta(\mu)$. The case when $x \in I_4$ can be treated similarly. This establishes (4.4).

It follows from (4.4) that, for all $(x,t) \in (0,1) \times [t^* + 1, \infty)$,

$$u_g(\cdot,t) \geq \mathcal{S}(t - t^* - 1) f[-1; s_1 + 2\mu, s_2 - 2\mu] = f[-1; s_2 + 2\mu, s_2 - 2\mu].$$

Similarly, we can show that

$$u_g(x,t) \leq f[-1; s_1 - 2\mu, s_2 + 2\mu](x) \qquad \forall (x,t) \in (0,1) \times [t^* + 1, \infty)$$

provided that $\eta \in (0, \eta(\mu))$. Therefore,

$$\overline{\lim_{t\to\infty}} \left\| u_g - f \right\|_\infty \leq \left\| f[-1; s_1 - 2\mu, s_2 + 2\mu] - f[-1; s_1 + 2\mu, s_2 - 2\mu] \right\|_\infty.$$

Clearly, as $\eta \to 0$, we can let $\mu \to 0$ so that the right hand side goes to zero. Hence,

$$\lim_{\eta\to 0} \overline{\lim_{t\to\infty}} \sup_{g \in B_\eta(f)} \|u_g(\cdot,t) - f\|_\infty = 0.$$

The assertion of the lemma follows. □

Remark 4.11. From (4.3), we see that the L^∞ stability is equivalent to L^2 stability.

4.3. Completion of the proof of Theorem 4.1.

It is sufficient to show that ω consists of a singleton. Suppose that f_1 and f_2 belong to $\omega(g)$. By Corollary 4.4, both f_1 and f_2 belong to the same set being one of $\{\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_K, \mathcal{E}^c\}$.

In the case $f_1 \in \left(\cup_{i=1}^K \mathcal{E}_k^s \right) \cup \mathcal{E}_0$, it follows, by lemma 4.1 and 4.10, that $f_1 = f_2$.

Now we consider the case $f_1 \in \mathcal{E}_k^u$, $k \in \{1, \cdots, K\}$. Write $f_1 = f[a; s_1, \cdots, s_k]$ and $f_2 = f[\bar{a}; \bar{s}_1, \cdots, \bar{s}_k]$. We want to show that $a = \bar{a}$, $s_1 = \bar{s}_1, \cdots$, and $s_k = \bar{s}_k$.

Since $\omega(g)$ is connected, f_1 and f_2 belong to the same connected component of $\mathcal{E}_k$, so that $a = \bar{a}$. Without loss of generality we assume that $a = \bar{a} = -1$.

We proceed to show that $s_1 = \bar{s}_1$. Assume that this is not true, say, $\bar{s}_1 > s_1 \geq 0$. Set $\mu = (\bar{s}_1 - s_1)/5$. We can use the same argument as in the proof of Lemma 4.10 to deduce that

$$u_g(\cdot,t) \leq f[-1; \bar{s}_1 - 2\mu] \qquad \forall t > t_n + t^* + 1$$

where t_n is the time such that $\|u_g(\cdot,t_n) - f_2\|_\infty \leq \eta(\mu)$. It follows that

$$f \leq f[-1, \bar{s}_1 - 2\mu] \qquad \forall f \in \omega(g),$$

contradicting to the fact that $f_1 \in \omega(g)$. Therefore $s_1 = \bar{s}_1$. Successively repeating the same argument for $i = 2, \cdots, k$, it follows that $s_i = \bar{s}_i$ for all $j = 1, \cdots, k$, and therefore $f_1 = f_2$.

It remains to consider the case $f_1, f_2 \in \mathcal{E}^c$. We need only to consider the case when $1 = (K+1)\varepsilon\pi$ since otherwise $\mathcal{E}^c$ consists of a single element $f \equiv 0$. The following lemma completes the proof of the theorem.

LEMMA 4.12. *Suppose that $1/\varepsilon\pi$ is an integer and $\omega(g) \in \mathcal{E}^c$. Then there exists a constant $a \in [-1,1]$ such that*

$$\lim_{t\to\infty} u_g(x,t) = a\cos\frac{x}{\varepsilon} \qquad x \in [-1,1].$$

Proof. Since $\omega(g)$ is connected, we need only consider the case when there exists $a \in (-1,1)$ such that $a\cos x/\varepsilon \in \omega(g)$ because otherwise $\omega(g) = \{\cos x/\varepsilon\}$ or $\omega(g) = \{-\cos x/\varepsilon\}$ and therefore the lemma holds automatically.

Noting that $u_g(\cdot,t)$ is in $H^2((0,1))$, we can write u_g as the uniformly convergent Fourier series

$$u_g(x,t) = \frac{1}{2}a_0(t) + \sum_{k=1}^{\infty} a_k(t)\cos(k\pi x)$$

where

$$a_k(t) = 2\int_0^1 u_g(x,t)\cos(k\pi x)\,dx, \qquad k = 0,1,2,\cdots,$$

and, by using Theorem 1 (5),

$$\begin{aligned}
\frac{d}{dt}a_k(t) &= 2\int_0^1 \frac{\partial}{\partial t}u_g(x,t)\cos(k\pi x)\,dx \\
&= 2\int_0^1 \left(u_{xx} + \frac{1}{\varepsilon^2}u\right)\cos(k\pi x)\,dx + \frac{1}{\varepsilon^2}\int_0^1 (\beta(u)-u)\cos(k\pi x)\,dx \\
&= \left(\frac{1}{\varepsilon^2} - k^2\pi^2\right)a_k(t) + \frac{1}{\varepsilon^2}e_k(t) \qquad (4.5)
\end{aligned}$$

where

$$e_k(t) = \int_0^1 (\beta(u)-u)\cos(k\pi x)\,dx.$$

The function e_k satisfies

$$|e_k(t)| \le m(t) := \sup_{\tau\ge t}\ \text{measure}\ \{x : |u_g(x,\tau)| = 1\}.$$

We claim that

$$\lim_{t\to\infty} m(t) = 0. \qquad (4.6)$$

In fact, otherwise measure $\{x : |u_g(x,\tau_n)| = 1\} > \delta$ for some positive δ and for a sequence τ_n converging to ∞. It then follows that there exists an $f \in \omega(g)$ such that $|f(\alpha)| = 1$ for some $\alpha \notin \{0, \varepsilon\pi, 2\varepsilon\pi, \cdots, K\varepsilon\pi, 1\}$. Hence $f \notin \mathcal{E}^c$, which is a contradiction. Therefore (4.6) must hold.

Solving the ordinary differential equation (4.5) yields

$$a_k(t) = a_k(t_0)e^{(1/\varepsilon^2 - k^2\pi^2)(t-t_0)} + \frac{1}{\varepsilon^2}\int_{t_0}^{t} e^{(1/\varepsilon^2 - k^2\pi^2)(t-\tau)} e_k(\tau)\, d\tau \qquad \forall t \geq t_0 > 0.$$

It follows that for $k \geq K+2$,

$$\begin{aligned} |a_k(t)| &\leq a_k(t_0)e^{(1/\varepsilon^2 - k^2\pi^2)(t-t_0)} + \frac{1}{(k\varepsilon\pi)^2 - 1} m(t_0) \\ &\leq 2e^{\pi^2[(K+1)^2 - k^2](t-t_0)} + \frac{m(t_0)}{k^2[(K+1)^{-2} - k^{-2}]} \\ &\leq C\Big(e^{-kt_0} + m(t_0)/k^2\Big) \qquad \forall t \geq 2t_0, \end{aligned} \tag{4.7}$$

where in the second inequality we have used the fact that $|a_k| \leq 2$ which follows from the definition of a_k.

Direct calculation yields

$$\mathrm{E}(u_g(\cdot,t)) = \frac{1}{\varepsilon^2} - \frac{a_0^2(t)}{4\varepsilon^2} + \sum_{k=1}^{K} \frac{a_k^2(t)}{2}\Big(k^2\pi^2 - \frac{1}{\varepsilon^2}\Big) + \sum_{k=K+2}^{\infty} \frac{a_k^2(t)}{2}\Big(k^2\pi^2 - \frac{1}{\varepsilon^2}\Big)$$

Since $\mathrm{E}(u_g(\cdot,t)) \searrow e^c = 1/\varepsilon^2$ and

$$0 \leq \sum_{k=K+2}^{\infty} a_k^2(2t)\Big(k^2\pi^2 - \frac{1}{\varepsilon^2}\Big) \leq C\Big[\frac{e^{-t}}{1-e^{-t}} + m(t)\Big] \to 0 \text{ as } t \to \infty,$$

it follows that

$$\lim_{t\to\infty}\Big[-\frac{1}{4\varepsilon^2}a_0^2(t) + \sum_{k=1}^{K}\frac{a_k^2(t)}{2}\Big(k^2\pi^2 - \frac{1}{\varepsilon^2}\Big)\Big] = 0;$$

i.e.,

$$\lim_{t\to\infty} a_k(t) = 0 \qquad \forall\, k = 0, 1, \cdots, K. \tag{4.8}$$

Write u_g as

$$u_g = u^+ + u^0 + u^-$$

where

$$\begin{aligned} u^+(x,t) &= \frac{a_0(t)}{2} + \sum_{k=1}^{K} a_k(t)\cos(k\pi x), \\ u^0(x,t) &= a_{K+1}(t)\cos((K+1)\pi x) = a_{K+1}(t)\cos\frac{x}{\varepsilon}, \\ u^-(x,t) &= \sum_{k=K+2}^{\infty} a_k(t)\cos(k\pi x). \end{aligned}$$

It follows from (4.8) that

$$\lim_{t\to\infty} \|u^+(\cdot,t)\|_\infty = 0 \tag{4.9}$$

and from (4.7) that

$$\|u^-(\cdot,2t)\|_\infty \le \sum_{k=K+2}^{\infty} |a_k(2t)| \le C[\frac{e^{-t}}{1-e^{-t}} + m(t)] \to 0 \qquad \text{as} \quad t\to\infty. \qquad (4.10)$$

Let N be large enough such that

$$\|u_g(x,t_N) - a\cdot\cos(x/\varepsilon)\|_\infty \le \frac{1}{8}(1-|a|),$$
$$\|u^\pm(\cdot,t)\|_\infty \le \frac{1}{8}(1-|a|) \qquad \forall\, t\ge t_N.$$

It follows that $|u_g(x,t_N)| \le 1-\frac{7}{8}(1-|a|)$. Let d be the largest constant such that

$$\|u_g(\cdot,t)\|_\infty < 1-\frac{1}{8}(1-|a|) \qquad \forall\, t\in[t_N,t_N+d).$$

Clearly, $d>0$ and $e_k(t)=0$ for all $t\in[t_N,t_N+d)$. It follows that $a_{K+1}(t)=a_{K+1}(t_N)$ for all $t\in[t_N,t_N+d]$. Therefore, for all $t\in[t_N,t_N+d]$,

$$\begin{aligned} u_g(x,t) &= u^+(x,t)+a_{K+1}(t_N)\cos(x/\varepsilon)+u^-(x,t). \\ &= u_g(x,t_N)+(u^+(x,t_N+d)-u^+(x,t_N))+(u^-(x,t_N+d)-u^-(x,t_N)). \end{aligned}$$

This implies that $d=\infty$ since otherwise

$$\begin{aligned} 1-\tfrac{1}{8}(1-|a|) &= \|u_g(\cdot,t_N+d)\|_\infty \\ &\le \|u(\cdot,t_N)\|_\infty + \|u^+(\cdot,t_N)\|_\infty + \|u^+(\cdot,t_N+d)\|_\infty \\ &\qquad + \|u^-(\cdot,t_N)\|_\infty + \|u^-(\cdot,t_N+d)\|_\infty \\ &\le 1-\tfrac{7}{8}(1-|a|)+4\cdot\tfrac{1}{8}(1-|a|) = 1-\tfrac{3}{7}(1-|a|), \end{aligned}$$

a contradiction. Therefore,

$$u_g(x,t) = u^+(x,t)+a_{K+1}(t_N)\cos\frac{x}{\varepsilon}+u^-(x,t) \qquad \forall\, t\in(t_N,\infty).$$

Consequently, by (4.9) and (4.10),

$$\lim_{t\to\infty} u_g(x,t) = a_{K+1}(t_N)\cos\frac{x}{\varepsilon} \qquad \forall\, x\in[0,1].$$

The assertion of the lemma follows. This also completes the proof of Theorem 4.1. □

Remark 4.12. The fact that $\|u_g\|<1$ in $[t_N,\infty)$ implies $a_k(t)=a_k(t_N)e^{1/\varepsilon^2-k^2\pi^2}$ for all $t\ge t_N$, so that $a_k(t)=0$ for all $k=0,1,\cdots,K$ and $t\ge t_N$; that is, $u^+\equiv 0$ for $t\ge t_N$.

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