No-Scale Supergravity, Inflation, and Dark Matter

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Finally, I would like to finish with a quote:

“All that is gold does not glitter,
Not all those who wander are lost”

- J. R. R. Tolkien, ‘The Lord of the Rings’
Dedication

To my wife, Marija, my parents, Ernestas and Jolanta, my brother, Nerijus, and my grandparents, Eugenija and Albinas. I also want to dedicate this thesis to those who did not believe in me—I proved you wrong.
Abstract

After reviewing the motivations for cosmological inflation formulated in the formalism of supersymmetry, we argue that the appropriate framework is that of no-scale supergravity. We first discuss how to realize the most viable model of cosmic inflation, which is known as the Starobinsky model of inflation, in the context of no-scale supergravity. The Starobinsky model is based on $R + R^2$ gravity and predicts the scalar tilt value of $n_s \sim 0.965$ and the tensor-to-scalar ratio $r \sim 0.0035$ for 55 e-folds of inflation, as favored by Planck, WMAP, and BICEP/Keck data on the cosmic microwave background (CMB). We then show that several different no-scale supergravity models yield the Starobinsky model of inflation and that all these models are equivalent to each other and arise from the underlying symmetry properties of supergravity. We discuss generalized no-scale models, Minkowski and de Sitter solutions, and supersymmetry breaking. We then show how to construct unified no-scale models of inflation, that unify modulus fixing, supersymmetry breaking, and a small cosmological constant. We develop phenomenological and cosmological aspects of these no-scale attractor models that underpin their physical applications and discuss cosmological constraints from entropy considerations. Finally, we use the recent BICEP/Keck data on the CMB to constrain the attractor models of inflation as formulated in no-scale supergravity.
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Chapter 1

Introduction

1.1 Motivation

The Standard Model of particle physics is arguably one of the greatest theories describing elementary particles and their interactions. However, this theory has its shortcomings because it only explains three of the four fundamental forces and does not describe the gravitational force. Additionally, it does not explain why the expansion of the universe is accelerating or the existence of dark matter. One of the most attractive extensions to the Standard Model is known as supersymmetry. According to the Haag–Lopuszanski–Sohnius theorem [1], the theory of supersymmetry is the only viable way to consistently combine spacetime with internal symmetries. Crucially, supersymmetry resolves the well-known hierarchy problems within the Standard Model [2, 3, 4] by ensuring that quadratic divergences cancel out for all orders in perturbation theory. Importantly, the cancellation of quadratic divergences implies that the electroweak scale is stable against the radiative corrections [5].

The first realistic supersymmetric extension of the Standard Model was proposed by Pierre Fayet in 1977 [6, 7]. This model is known as the Minimal Supersymmetric Standard Model (MSSM) and it introduces the minimum number of new particles in the system that are consistent phenomenologically [8]. Supersymmetry combines bosons with fermions into pairs and supersymmetric transformations turn bosonic states into fermionic states, and vice versa. Therefore, every Standard Model particle has a supersymmetric particle which is known as a superpartner that is yet to be discovered.
In addition to providing the solution to the hierarchy problem, the Minimal Supersymmetric Standard Model has other important theoretical motivations. In the MSSM, the gauge coupling constants associated with the Standard Model $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge group symmetry unify at high energies, typically around $10^{16}$ GeV. Therefore, the three fundamental forces merge into a single grand unified force at high energies [9, 10, 11]. Another important feature of MSSM is that the lightest supersymmetric particle (LSP) [12, 13] is also a weakly interacting massive particle (WIMP) [14, 15], which is stable and is a viable dark matter candidate [16]. Lastly, the experimentally determined Higgs mass of 125 GeV [17, 18, 19] is within the favored range of masses in MSSM [20, 21, 22].

Cosmologists were puzzled for many years about why the universe appears to be so uniform and nearly flat on a large scale. The theory that can answer these questions is known as cosmic inflation [23], which states that in the very early universe, a small region of space has undergone a very short but dramatic expansion. Moreover, inflation also explains the suppressing by large factors of the densities of unobserved massive relics that arise in Grand Unified Theories (GUTs) [24, 25]. The most important experimental evidence supporting cosmic inflation comes from the quantum fluctuations in the cosmic microwave background (CMB) radiation, and these experimental results can be used to test different models of inflation. The observed CMB spectrum can be explained by the so-called slow-roll approximation [26, 27]. It is assumed that the scalar field driving inflation, known as the inflaton, rolls slowly down its effective potential while driving the exponential expansion of the universe. The quantum fluctuations generated by the scalar inflaton field should be predominantly Gaussian and the gravitational background would also be subject to small tensorial quantum fluctuations. In particular, the most recent Planck measurements of the CMB data predict that the CMB spectrum is almost scale-invariant with $0.958 < n_s < 0.975$ at the 95% C.L., and set stringent upper limits on the tensor-to-scalar-ratio, with $r < 0.06$, in agreement with the slow-roll inflationary paradigm [28].

It was argued in [29] that cosmic inflation cries out for supersymmetry. Importantly, in a supersymmetric theory, a gauge singlet field provides a natural candidate that can be associated with the scalar inflaton field. Supersymmetry is also a perfect framework to resolve other problems related to the Big Bang cosmology, such as the baryon
asymmetry problem; the observed matter asymmetry can be explained by the supersymmetric electroweak baryogenesis [30, 31, 32, 33], Affleck-Dine mechanism [34, 35], or supersymmetric leptogenesis [36, 37, 38].

Nevertheless, supersymmetric theories do not incorporate probably the most important force that binds our universe together—the force of gravity. One of the most successful attempts to unify all four fundamental forces of nature is known as supergravity [39, 40, 41, 42, 43, 44], which combines the principles of supersymmetry with the classical description of gravity, characterized by Einstein’s theory of general relativity. Therefore, supergravity is the appropriate theoretical framework for any cosmological scenario involving supersymmetry.

The theories of supergravity are the gauge theories of local supersymmetry. The simplest model of minimal supergravity ($N = 1$ supergravity) [45] with additional matter multiplets suffers from divergences at one-loop level [46, 47]. However, some theories of supergravity emerge as the low energy limits of string theories, which are the leading candidates for the theory of everything (TOE). Supersymmetry requires that the spin-2 graviton field has a spin-3/2 superpartner known as the gravitino to ensure that supergravity is a consistent field theory containing a spin-3/2 particle [48, 49, 50]. In a generic supergravity theory, the number of gravitino fields in the theory is equal to the number of supersymmetries. Additionally, a complete model of supergravity must include the spontaneous breaking of local supersymmetry to ensure the mass splitting between the particles and their superpartners. Another important point is that the scalar sector of supergravity theories usually has many flat directions which are necessary for successful inflation.

Generic models of supergravity also contain many weakly-coupled fields known as the moduli, which are typically associated with weak scale masses. When one of the scalar fields obtains a large vacuum expectation value (VEV) during inflation it decays very late and generally produces enormous amounts of entropy, washing away any baryon asymmetry. Their decays into supersymmetric particles may also lead to an excessive dark matter abundance in the form of the lightest supersymmetric particle (LSP). This is known as the Polonyi problem, or more generally the moduli problem [51, 52, 53, 54, 55]. The gravitino potentially poses similar cosmological problems for models of supergravity [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67]. If the post-inflationary
gravitino production is large and the gravitino is the LSP, the relic gravitino abundance could exceed the permitted density of cold dark matter, which depends on its mass. On the other hand, if the gravitino is not the LSP, the fact that the gravitino couplings to other particles are Planck-suppressed implies that its lifetime for decays into other particles may be quite long: \( \tau \sim M_P^2 / m_{3/2}^3 \), where \( M_P = 1/\sqrt{8\pi G} \simeq 2.435 \times 10^{18} \text{GeV} \) is the reduced Planck mass and \( m_{3/2} \) is the mass of the gravitino. In this case, the gravitino mass and abundance are constrained by the experimental limits arising from big bang nucleosynthesis (BBN) [63, 68, 69, 70].

Therefore, there is a significant theoretical hurdle to overcome to construct viable models of supergravity [71, 72]. Generic models of supergravity are problematic, and more realistic models that contain matter fields lead to effective potentials that are not flat and unsuitable for slow-roll inflation. They also typically have anti-de Sitter (AdS) vacua with field energies \( -O(m_{3/2}^2 M_P^2) \) which does not agree with the observations. Moreover, to understand the nature of gravity in the subatomic realm, the classical description of gravity is not sufficient, and we need to construct a quantum theory of gravity. It is widely believed that string theory is the most promising candidate for a theory of quantum gravity. If string theory provides a theoretical framework that encompasses all four fundamental forces, then we must answer the following question: what is the connection between string theory and supergravity? Therefore, another desideratum for any complete particle physics model, and in particular any model of inflation, is that it is compatible with ultraviolet completion in some string theory models. So what are the supergravity models that do not have AdS vacua and naturally emerge as the low energy effective field theory of superstring theory?

The supergravity models that can easily avoid these problems are known as no-scale models of supergravity [43, 73, 74, 75, 76]. They are named so because their effective potentials contain flat directions not associated with any specific dynamical scale at the tree level. These models emerge from string theory models as their effective low-energy theories [77, 78, 79]. Therefore, no-scale supergravity is an appropriate framework to study the cosmology of the very early universe.

In this thesis, we focus on the phenomenological and cosmological aspects of no-scale supergravity models. Typically, no-scale models are based on an \( SU(N,1)/SU(N) \times U(1) \) non-compact Kähler geometry. It is not difficult to break local supersymmetry
in no-scale supergravity models. After the supersymmetry is broken, the no-scale type Kähler potential leads to a scalar effective potential with a semi-positive definite minimum. This potential has flat directions which are suitable for inflation. The effective potential of the hidden sector scalars is also flat, which implies that their vacuum expectation values cannot be determined at the tree level. Therefore, for these models the Planck scale $M_P$ is the only input scale. In no-scale models of supergravity, the soft supersymmetry parameters are all determined by the gravitino mass (or supersymmetry breaking scale), which is a common feature of many models of supergravity [80].

Furthermore, no-scale models of inflation suffer from the following two problems: the unstable directions in the field space that are remedied by introducing the higher-order corrections in the Kähler potential [81, 82], and the inflaton stability at the tree level, which leads to highly Planck-suppressed inflaton decay rate and a low reheating temperature that could lead to cosmological problems [83]. In this work, we address both problems.

The measurements of the cosmic microwave background (CMB) favor models of inflation with a small tensor-to-scalar ratio $r$, as predicted by the Starobinsky $R + R^2$ model [84]. It has been shown previously that various models based on no-scale supergravity with different forms of superpotential make predictions similar to those of the Starobinsky model [85]. Therefore, we first present a unified and general treatment of Starobinsky avatars of no-scale supergravity and demonstrate equivalences between different models.

Next, we show the uniqueness of superpotentials leading to Minkowski vacua ($V = 0$) of single-field no-scale supergravity models, and the construction of de Sitter/anti-de Sitter (dS/AdS) solutions using pairs of these single-field Minkowski superpotentials. We then expand the construction to two- and multi-field no-scale supergravity models, providing also a geometrical interpretation.

We then expand these Starobinsky-like models and present how to construct minimal no-scale models that unify modulus fixing, Starobinsky-like inflation, an adjustable scale for supersymmetry breaking, and the possibility of a small cosmological constant, a.k.a. dark energy. We extend these constructions to inflationary models based on generalized no-scale structures. In particular, we consider alternative values of the curvature parameter, $\alpha < 1$, as may occur if not all the complex Kähler moduli contribute
to driving inflation, as well as $\alpha > 1$, as may occur if complex structure moduli also contribute to driving inflation. In all cases, we combine these Starobinsky-like models of inflation with supersymmetry breaking and a present-day cosmological constant, allowing for additional contributions to the vacuum energy from stages of gauge symmetry breaking. We develop phenomenological and cosmological aspects of these no-scale attractor models that underpin their physical applications. We also discuss cosmological constraints from entropy considerations and the density of dark matter on the mechanism for stabilizing the modulus field via higher-order terms in the no-scale Kähler potential. Finally, we use the recent BICEP/Keck data [86] on the cosmic microwave background, in combination with previous WMAP and Planck data [28], and impose the constraints on the tilt in the scalar perturbation spectrum, $n_s$, as well as the tensor-to-scalar ratio, $r$, for the attractor models of inflation. These constrain the number of $e$-folds of inflation, $N_*$, the magnitude of the inflaton coupling to matter, $y$, and the reheating temperature, $T_{\text{RH}}$, which we evaluate in attractor models of inflation as formulated in no-scale supergravity.

1.2 Structure of this thesis

This thesis contains a total of 10 chapters. In Chapter 1, we discuss the current problems of the Standard Model of particle physics and explain briefly the motivation for supersymmetric extensions. In particular, we discuss the importance of unifying the supersymmetric theories with general relativity into the theories of supergravity. We argue that supergravity is the appropriate framework to study the early universe cosmology and cosmic inflation and discuss briefly the no-scale models of supergravity.

In Chapter 2, we provide a short introduction to cosmology and inflation. We first recall the key concepts of the Big Bang cosmology and discuss the importance of beyond the Standard Model physics (BSM) that is necessary to explain the current astrophysical observations. We discuss the dynamics of the expanding universe that are described by Friedmann equations and then review the most important aspects of cosmic inflation and reheating of the universe. Finally, we discuss the cosmic microwave background (CMB) observables that can be associated with the inflationary model predictions.
Chapter 3 discusses the motivation for supersymmetry and supergravity, and summarizes the key concepts of the Minimal Supersymmetric Standard Model (MSSM). We then show how to combine supersymmetry and general relativity into theories of supergravity, and briefly discuss some important problems that arise in theories of supergravity. We then show in Chapter 4 how to construct models of supergravity. In particular, we focus on no-scale models of supergravity and demonstrate how to construct plateau-like inflationary models within this framework. We pay particular attention to the predictions for inflationary observables and focus on models that mimic those of the Starobinsky $R + R^2$ model.

In Chapter 5, we discuss in detail the natural relation between the no-scale supergravity and Starobinsky-like models of inflation. To specify, we show how to construct Starobinsky-like inflationary models with different forms of superpotential. Importantly, we use the underlying $SU(2,1)/SU(2) \times U(1)$ coset symmetry to demonstrate the equivalences between different models. Then in Chapter 6, we show how to construct Minkowski vacua solutions in single- and multi-field no-scale supergravity models. We combine unique Minkowski vacua solutions and show how to construct successfully the de Sitter/anti-de Sitter (dS/AdS) vacua solutions, and then also demonstrate that all these solutions have a simple geometric interpretation. We then briefly consider the construction of the inflationary models within this framework.

We combine the key ideas that were discussed before and in Chapter 7, we show how to successfully build no-scale models of inflation that unify the modulus fixing (field stabilization), supersymmetry breaking, and a small and positive cosmological constant. We call such models unified no-scale attractors. We show that such models can be constrained by the CMB measurements. Chapter 8 is devoted to the study of the phenomenological and cosmological aspects of unified no-scale attractors. We pay particular attention to the reheating following inflaton decay and the production of gravitinos and supersymmetric dark matter. We then show in Chapter 9 how such unified no-scale attractor models can be constrained using the recent release of the BICEP/Keck data. Finally, the conclusions of this thesis are presented in Chapter 10.
The results presented in this thesis are based on the following publications:


Chapter 2

Cosmology and Inflation

This chapter contains a brief introduction to the Big Bang and inflationary cosmology. We discuss the shortcoming of the standard Big Bang model and then show how the theory of cosmic inflation solves them. We then show how to relate the slow-roll models of inflation to the density perturbations in the cosmic microwave background (CMB). Finally, we show how the entropy generation during reheating epoch affects the number of $e$-folds of inflation.

2.1 The Modern Picture of Cosmology

To understand the existence of the observable universe and answer such compelling questions as “How did we get here?” or “How did matter form?”, one first needs to create a testable model of the universe that agrees with the experiment. Observational evidence shows that our visible universe is homogeneous and isotropic when averaged over large scales. While the universe is uniform on large scales, there are many different structures, such as galaxies and stars, and inhomogeneities on small scales. Another essential feature of the standard picture of cosmology is that the universe is expanding. This result was discovered by Edwin Hubble in 1929 [87] when he observed that distant galaxies were receding (redshifting) from us.

To quantify the effect of expansion, it is convenient to introduce the scale factor $a(t)$, whose value today is equal to unity by convention: $a(t_0) \equiv 1$. The comoving distance between two points remains constant as the universe expands, whereas the
physical distance increases with time and is proportional to the scale factor \( a(t) \). The isotropic and homogeneous universe can be characterized by the maximally-symmetric Friedmann-Robertson-Walker (FRW) metric

\[
ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - k r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

where \((t, r, \theta, \phi)\) are the comoving coordinates, and \(k\) is the spatial curvature that can be positive (+1), negative (−1), or zero (0). The time coordinate in the FRW metric is chosen so that the observer remains at rest in the comoving frame, i.e., \((r, \theta, \phi) = \text{const.}\)

To describe the change in the scale factor and the expansion of the universe, it is useful to introduce the Hubble parameter

\[
H \equiv \frac{\dot{a}}{a},
\]

where an overdot indicates a derivative with respect to time.

Because the physical coordinates depend on the scale factor, \(x_{\text{phys}} = a(t)x\), the physical velocity can be expressed as

\[
v_{\text{phys}} = \frac{dx_{\text{phys}}}{dt} = a(t) \frac{dx}{dt} + \frac{da}{dt} x \equiv v_{\text{pec}} + H x_{\text{phys}},
\]

where \(v_{\text{pec}} \equiv a(t)\dot{x}\) is the peculiar velocity. This expression is known as the Hubble–Lemaître law. The Hubble parameter is determined by measuring the slope of the best fit line in the Hubble diagram illustrated in Fig. 2.1.

The Hubble parameter is typically parameterized in the following way

\[
H_0 \equiv 100 \, h \, \text{km s}^{-1} \, \text{Mpc}^{-1} = \frac{h}{0.98 \times 10^{10} \, \text{years}},
\]

where 1 Mpc = 3.085 × 10^{19} km, and most recent Planck measurements of the cosmic microwave background (CMB) radiation yield \(h \simeq 0.674\) [28].

Now that we introduced the concepts of the expanding universe and scale factor, we can discuss the physics of the early universe. The leading cosmological theory that explains how the universe began is called the Big Bang theory. Around 13.8 billion years ago, the universe was concentrated in an extremely high density and temperature Big
Figure 2.1: A Hubble diagram (velocity vs distance) from the Hubble Space Telescope (HST) Key project for 23 galaxies with Cepheid distances [88]. In this plot, the galaxy velocities have been corrected using the analysis presented in [89]. The Cepheid distances include the metallicity effects on period-luminosity relations. A formal fit predicts a slope of $H_0 = 75 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

**Bang singularity.** Consequently, the universe began expanding rapidly from a singularity. During the initial period of expansion ($t < 10^{-43}$ s), known as the Planck epoch, the four fundamental forces of nature were unified as one. Eventually, as the temperature of the universe cooled, the gravitational force separated from the other three, and the universe entered the grand unification epoch at around $t \simeq 10^{-43}$ s. The grand unification epoch was succeeded by a period of cosmic inflation at $t \simeq 10^{-37}$ s when a phase transition occurred, and the universe began expanding exponentially. The inflationary paradigm resolves many problems associated with the Big Bang model, and is discussed extensively in Section 2.3. At the end of cosmic inflation, the scalar field particle that drives inflation, known as the inflaton, begins oscillating about its potential minimum.
and ultimately decays into elementary particles and reheat the universe.

As the temperature of the universe decreased to $T \sim 1$ MeV, the electrons, positrons, and photons were in thermal equilibrium, and the baryons were non-relativistic. As the universe cooled below the typical nuclear binding energies, the neutrons and protons combined in nuclear fusion reactions to form deuterium and other elements. At roughly 3 minutes after the Big Bang, the light elements, such as hydrogen, helium, and lithium, were created in a process known as Big Bang Nucleosynthesis (BBN). The theory of Big Bang Nucleosynthesis predicts that the visible matter in the universe should consist of approximately 25% helium, about 75% hydrogen, 0.01% deuterium and helium-3, and some trace amounts of other elements. The primordial abundances predicted by BBN as a function of the baryon-to-photon ratio $\eta$ are shown in Fig. 2.2.

However, the standard model of Big Bang nucleosynthesis constrains the baryon density of the universe to $\sim 5\%$ of the total critical density of the universe, which means that dark matter and dark energy constitute the remaining 95% of the total energy content. Since the Standard Model of particle physics does not have a viable dark matter candidate and cannot explain dark energy, we are motivated to study theories beyond the Standard Model (BSM), such as theories of supersymmetry and supergravity, and we discuss them in detail in Chapter 3.

Another important phenomenon in Big Bang cosmology is the cosmic microwave background (CMB) radiation. At relatively high temperatures above 1 eV, the photons are coupled to the thermal plasma through electron-photon interactions (Compton scattering). When the temperature of the primordial plasma drops to order of $T_{\text{dec}} \sim 1$ eV, the interaction rate becomes smaller than the expansion rate, $\Gamma_\gamma(T_{\text{dec}}) \lesssim H(T_{\text{dec}})$, and the photons cease interacting with the thermal bath. The photon decoupling occurs at time $t \sim 380000$ years after the Big Bang, which corresponds to the redshift $z_{\text{dec}} \sim 1100$, and the photons travel freely through space.

The cosmic microwave background spectra can be used to analyze the time of last scattering (the period of photon decoupling). The CMB radiation was first discovered by Penzias and Wilson in 1965 by a ground-based telescope [92], which provided even stronger evidence in favor of the Big Bang model. Most importantly, the cosmic microwave background spectrum is consistent with the theoretical predictions that it should have a blackbody spectrum in all directions, with spectral radiance peaking in
Figure 2.2: The primordial abundances of $^4\text{He}$, $^3\text{He}$, and $^7\text{Li}$ as a function of baryon-to-photon ratio $\eta$ as predicted by Big Bang Nucleosynthesis. The bands corresponds to the 95% CL range [90]. The CMB predictions at 95% CL are shown by narrow vertical bands and the wider bands show the BBN $\text{D} + ^4\text{He}$ concordance range at 95% CL. From Particle Data Group 2020 update [91].
the microwave range and a temperature of about $T \simeq 2.7255\text{K}$ \cite{28}.

From the CMB power spectrum analysis, we learned that the very early universe was extremely uniform. However, in 1992 NASA satellite mission COBE measured the CMB temperature fluctuations in the sky and discovered the limit $\delta T / T < 10^{-4}$. These temperature anisotropies are measured as a two-dimensional field in the sky, which is then expanded in terms of spherical harmonics (instead of the usual Fourier analysis). In this case, the power spectrum is a function of spherical multipole moments, $\ell$. The temperature power spectrum from \textit{Planck} 2018 data is shown in Fig. 2.3.

![Planck 2018 temperature power spectrum](image)

**Figure 2.3:** \textit{Planck} 2018 temperature power spectrum. Top panel: CMB anisotropies measured by the Planck satellite as a function of multipole moments $\ell$ (red points). The light blue line shows the best-fit base-$\Lambda$CDM theoretical power spectrum to the TT,TE,EE+lowE+lensing likelihoods. The error bars indicate the $\pm 1\sigma$ diagonal uncertainties. The variance of the temperature fluctuations $D_{\ell}^{TT}$ is given by $D_{\ell}^{TT} \equiv \ell(\ell + 1)C_{\ell}^{TT}/(2\pi)$, where $C_{\ell}^{TT}$ is the multipole associated with the two-point temperature anisotropy correlation function. Bottom panel: residuals with respect to this model. From Planck Collaboration \cite{28}.
Now we can summarize the standard model of cosmology: our universe is spatially flat (Euclidean) that is expanding, whose constituents today are dominated by non-baryonic cold dark matter (CDM) and a cosmological constant $\Lambda$ (dark energy). The primordial adiabatic fluctuations are generated in the early universe by cosmic inflation and are Gaussian-distributed. Importantly, they are responsible for the structure formation in the early universe. This model is known as the $\Lambda$CDM model of cosmology.

Next, we briefly discuss the key ingredients of the $\Lambda$CDM model. The existence of cold dark matter has been known for more than eight decades, and dark matter hypothesis is supported by observational evidence [93, 94, 95]. However, despite many years of intensive searches, the particle nature of dark matter remains a mystery. Another important component of $\Lambda$CDM is the dark energy or the cosmological constant. From the measurements of the type Ia distant supernovae [96, 97], we know that the expansion of the universe is accelerating, and it implies that the present-day energy density is dominated by dark energy, $\Lambda$. The concept of cosmological constant was first introduced by Einstein, who introduced this term in the field equations to make sure that the universe remained static but later abandoned this idea. Current measurements reveal that 68% of our universe’s energy density is dominated by dark energy by while the remaining 32% is dominated by a combination of cold dark matter and baryonic matter [28]. Finally, the initial primordial seeds of structure formation are believed to have been generated by cosmic inflation. It also solves the horizon and flatness problems of the standard Big Bang model, and we discuss it in detail in Section 2.3. The $\Lambda$CDM model is specified by six parameters, given in Table 2.1. See [98, 99] for a more detailed treatment of the early universe and cosmology.

### 2.2 Dynamics of the Universe

In the previous section, we discussed that the kinematics of the universe is characterized by a Friedmann-Robertson-Walker metric (2.1). The expanding universe dynamics are governed by the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$  \hspace{1cm} (2.5)
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Planck 2018 Data (68% CL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baryon Density</td>
<td>$\Omega_b h^2$</td>
</tr>
<tr>
<td>Cold Dark Matter Density</td>
<td>$\Omega_c h^2$</td>
</tr>
<tr>
<td>Sound Horizon at Recombination</td>
<td>$100\theta_{MC}$</td>
</tr>
<tr>
<td>Reionization Optical Depth</td>
<td>$\tau$</td>
</tr>
<tr>
<td>Amplitude of Curvature Perturbations</td>
<td>$\ln(10^{10}A_S)$</td>
</tr>
<tr>
<td>Scalar Spectral Index</td>
<td>$n_s$</td>
</tr>
</tbody>
</table>

Table 2.1: 68% confidence limits for the ΛCDM model from Planck cosmic microwave background power spectra, in combination with lensing reconstruction and external data (BAO+JLA+$H_0$). The cold dark matter and baryon densities are multiplied by the reduced Hubble constant squared $h^2$, which is defined in Eq. (2.4). The sound horizon at recombination $\theta_{MC}$ (the distance that the sound could travel from the initial time $t = 0$ to the time at recombination) is used in the CosmoMC analysis [100, 101]. The reionization optical depth $\tau$ is a unitless parameter that measures the line-of-sight free-electron opacity to CMB radiation. The amplitude of curvature perturbations is defined in Eq. (2.86) and scalar spectral index is defined in Eq. (2.87), and they are measured at the pivot scale $k_s = 0.05\text{Mpc}^{-1}$.

where $G_{\mu\nu}$ is the Einstein tensor, $\Lambda$ is the cosmological constant, $g_{\mu\nu}$ is the metric tensor, $G$ is the gravitational constant, and $T_{\mu\nu}$ is the stress-energy tensor that incorporates all the fluids present, such as matter or radiation. This expression relates the Einstein tensor, which measures the spacetime curvature of the Friedmann-Robertson-Walker universe, to the stress-energy tensor, which measures the energy density and pressure of the universe. The Einstein tensor $G_{\mu\nu}$ can be expressed as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

(2.6)

where $R_{\mu\nu}$ is the Ricci tensor and $R$ is the Ricci scalar. We find that the non-zero components of the Ricci tensor for FRW space are

$$R_{00} = -3 \frac{\ddot{a}}{a},$$

(2.7)

$$R_{ij} = - \left[ \frac{\ddot{a}}{a} + 2 \frac{\dot{a}^2}{a^2} + 2 \frac{k}{a^2} \right] g_{ij},$$

(2.8)
and the Ricci scalar is given by

\[ R = -6 \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right]. \tag{2.9} \]

Next, we treat the homogeneous and isotropic universe as a perfect fluid, so that the stress-energy tensor becomes

\[ T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}, \tag{2.10} \]

where \( p \) is the pressure, \( \rho \) is the energy density of the fluid, and \( u_{\mu} = (1, 0, 0, 0) \) is the four-velocity relative to the observer, which leads to the energy component

\[ T_{00} = \rho + p, \tag{2.11} \]

and the pressure component

\[ T_{ij} = -p g_{ij}. \tag{2.12} \]

Using these expressions, we find that the 0 – 0 energy component of the Einstein equation (2.5) leads to the so-called first Friedmann equation

\[ H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{\rho}{3M_P^2} - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{2.13} \]

and the \( i – i \) component gives

\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\frac{p}{M_P^2} + \Lambda. \tag{2.14} \]

Here the quantity \( M_P \equiv 1/\sqrt{8\pi G} \) is known as the reduced Planck mass. The difference between Eqs. (2.14) and (2.13) leads to the second Friedmann equation

\[ \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{1}{6M_P^2}(\rho + 3p), \tag{2.15} \]

which is also known as the Friedmann acceleration equation.

From the conservation of the stress-energy tensor (2.10), we recover the 1st law of
thermodynamics
\[ d(\rho a^3) = -pd(a^3), \] (2.16)

which further implies the following energy-momentum conservation law
\[ \dot{\rho} + 3H(\rho + p) = 0. \] (2.17)

To simplify these equations, we introduce the equation of state parameter of a perfect fluid
\[ w \equiv \frac{p}{\rho}, \] (2.18)

and the energy-momentum conservation law becomes\(^1\)
\[ \dot{\rho} + 3H(1 + w)\rho = 0, \] (2.19)

and the energy density becomes proportional to \( \rho \propto a^{-3(1+w)} \). Below we show some important relations for the perfect fluids of interest, that are derived from Eqs. (2.13) and (2.19):

**Radiation**: \( w = \frac{1}{3}, \quad \rho \propto a^{-4}, \quad a \propto t^{1/2}, \quad H = \frac{1}{2t}, \) (2.20)

**Matter**: \( w = 0, \quad \rho \propto a^{-3}, \quad a \propto t^{2/3}, \quad H = \frac{2}{3t}, \) (2.21)

**Dark Energy**: \( w = -1, \quad \rho \propto \text{const.}, \quad a \propto e^{\sqrt{\Lambda}/3t}, \quad H = \frac{\sqrt{\Lambda}}{3}, \) (2.22)

where we set \( k = 0 \) at early times, and when the universe is dominated by radiation or matter, the cosmological constant contribution \( \Lambda \) can be neglected.

We introduce the *critical density today* \([91]\)
\[ \rho_{\text{cr,0}} = 3H_0^2M_P^2 = 1.88 \times 10^{-29} h^2 \text{ g cm}^{-3} \]
\[ = 1.05 \times 10^{-5} h^2 \text{ GeV cm}^{-3} \]
\[ = 2.78 \times 10^{11} h^2 M_{\odot} \text{ Mpc}^{-3}. \] (2.23)\(^2\)

\(^1\)We note that here we assume that the equation of state parameter does not depend on time.

\(^2\)We note that here we assume that the equation of state parameter does not depend on time.
The critical density can be used to define the dimensionless density parameters

\[ \Omega_{a,0} \equiv \frac{\rho_{a,0}}{\rho_{cr,0}}, \quad a = r, m, k, \Lambda, \quad (2.26) \]

where the subscript 0 indicates that the values are evaluated today at time \( t_0 \). The curvature density parameter can be defined as

\[ \Omega_k \equiv \frac{k}{H^2 a^2} = \Omega - 1, \quad (2.27) \]

where \( \Omega \equiv \Omega_r + \Omega_m + \Omega_\Lambda \). Because the denominator \( H^2 a^2 > 0 \), the sign of the curvature \( k \) can be connected to the dimensionless density parameter:

- **Closed Universe**: \( k = +1 \iff \Omega > 1 \),
- **Flat Universe**: \( k = 0 \iff \Omega = 0 \),
- **Open Universe**: \( k = -1 \iff \Omega < 1 \). \quad (2.28)

We can rewrite the Friedmann equation (2.13) in terms of dimensionless density parameters

\[ H^2(a) = H_0^2 \left[ \Omega_{r,0} \left( \frac{a_0}{a} \right)^4 + \Omega_{m,0} \left( \frac{a_0}{a} \right)^3 + \Omega_{k,0} \left( \frac{a_0}{a} \right)^2 + \Omega_{\Lambda,0} \right], \quad (2.29) \]

or if we set the present-day scale factor \( a_0 \equiv 1 \), we obtain

\[ \frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda. \quad (2.30) \]

Astrophysical observations show that the current dimensionless density parameters are [28]

\[ \Omega_{m,0} = 0.32, \quad \Omega_{r,0} = 9.2 \times 10^{-5}, \quad |\Omega_{k,0}| = 0.001, \quad \Omega_{\Lambda,0} = 0.68. \quad (2.31) \]

Since the curvature density parameter is small, we assume that \( \Omega_k \simeq 0 \), and we treat our universe as flat and set \( k = 0 \) in our calculations. The evolution of the energy densities as functions of scale factor is shown in Fig. 2.4.
2.3 Cosmic Inflation

The standard cosmological model of Big Bang successfully explains many observed properties of the universe. However, the Big Bang theory needs very specific initial conditions and faces multiple problems. They are listed below:

1. The Horizon Problem. The standard Big Bang cosmology predicts that the very early universe contained many causally disconnected regions. But from observations we know that the universe is statistically homogeneous and isotropic on large scales—disjoint patches have almost identical densities and temperatures with \( T = 2.7255 \pm 0.0006 \text{K} \) [102, 103]. It is also consistent with the notion of cosmological principle since the forces should act uniformly throughout the universe and should not contain any irregularities [104]. When the early universe was dominated by matter and radiation, two separated regions could not have been
in thermal equilibrium unless they moved away from each other faster than the speed of light, which is inconsistent with the principles of relativity.

2. The Flatness Problem. The initial conditions of the universe appear to be extremely fine-tuned and even a small deviation from such values would have critical effects on the present-day universe. Importantly, to agree with the observations the energy density of the universe is supposed to be a very specific and fine-tuned value. This problem can be understood better from Eqs. (2.13) and (2.23), which lead to following expression

\[ \frac{\rho_{cr}}{3M_P^2} = \frac{\rho}{3M_P^2} - \frac{k}{a^2}, \]  

(2.32)

that can be rearranged into

\[ (\Omega^{-1} - 1) \rho a^2 = -3kM_P^2. \]  

(2.33)

From this equation we see that because the right-hand side is always equal to a constant, the left-hand side should also remain constant during the evolution of the universe. When the universe is dominated by matter or radiation, the energy density \( \rho \) decreases faster than \( a^2 \), and the parameter \( \rho a^2 \) decreases. It is estimated that since the beginning of the Big Bang, the parameter \( \rho a^2 \) has decreased by a factor of \( \sim 10^{60} \), which would require a fine tuning of the critical density parameter \( \Omega \) by 60 orders of magnitude.

3. The Magnetic Monopole Problem. At the very early universe, right after the Planck epoch, the gravitational force separated from other fundamental forces, and the remaining three forces—the electromagnetic force, the strong nuclear force, and the weak nuclear force—were unified as one. This epoch is known as the grand unified epoch and it is described by the Grand Unified Theory. However, when the temperature of the universe fell, the strong nuclear force separated from other forces and broke the full gauge symmetry down to \( SU(3) \times SU(2) \times U(1) \) symmetry, which would lead to the production of magnetic monopoles [24, 25]. These stable magnetic monopoles should exist in the present-day universe and they would dominate the primary constituent of the energy density of the universe.
Therefore, a viable mechanism explaining why we do not observe these exotic relics is needed.

4. **Primordial Fluctuations.** Although the universe is uniform on very large scales, there is plenty of structure in the universe on the small scale. Cosmologists believe that some density variations were created in the very early universe by some mechanism and the initial density inhomogeneities grew via the Jeans gravitational instability into large structures that we observe today [105].

The most successful theory that solves these problems associated with the Big Bang model is known as *cosmic inflation* [23, 84].\(^2\) According to the inflationary paradigm, the very early universe undergoes a (near-)exponential expansion of space. During the inflationary epoch, the scale factor grows exponentially, \(a(t) \propto e^{Ht}\), and \(\ddot{a} > 0\). Such accelerated expansion flattens the Friedmann-Robertson-Walker metric (for \(k = +1\) or \(-1\)) and solves the flatness problem. We rewrite equation (2.27) in the following form

\[
\Omega - 1 = \frac{k}{a^2}. \tag{2.34}
\]

The observational constraints today require \(|\Omega_0 - 1| < 0.005\) [28], which means that this value was significantly smaller in the early universe. During an extended accelerated expansion of the universe, \(\dot{a} > 0\) and \(\ddot{a} > 0\), which would drive the value of \(\Omega\) close to unity and ensure that \(|\Omega - 1|\) was sufficiently small in the past.

Cosmic inflation also solves the horizon problem. Before the inflationary epoch, the universe was in causally connected near equilibrium region of size \(< H^{-1}\). When the universe has undergone inflation and its size increased dramatically over a very short period of time, the regions at large distances remained smooth. It also explains why the CMB is so uniform. Moreover, a phase of accelerated exponential expansion also substantially dilutes the abundance of relics that were produced before inflation and solves the monopole problem. The same dilution also affects any matter and radiation content that was created before inflation. Finally, another important consequence of inflation is that it naturally provides the primordial fluctuations, which act as the seeds needed for the creation of structure in the universe (see the discussion below). The initial conditions of inflation and their origin is discussed in [106, 108, 109, 110, 111, 112, 113].

\(^2\)For a detailed review on inflation, see [106, 107]
Next, we turn to study the physics of inflation. The early models of inflation, introduced by Alan Guth in 1981 [23], included tunneling out of a false vacuum state, but this model was plagued by the so-called “graceful exit problem”, which means that inflation would never end and the universe would not thermalize and reheat. The solution to this problem was proposed by introducing the slow-roll inflation [26, 27], and hereafter we focus on such models.

As mentioned earlier, during inflation, the conditions $\dot{a} > 0$ and $\ddot{a} > 0$ must be satisfied, and equation (2.19) implies the condition $w < -1/3$. The simplest way to satisfy these conditions is to introduce a scalar field $\phi(t)$ that drives inflation, which is known as the inflaton. In here we focus on single-field models of inflation (for multi-field models, see [114, 115, 116, 117, 118, 119, 120, 121]). Assuming the FRW background, the inflaton action is given by

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right],$$

(2.35)

where $g = \text{det} \{g_{\mu\nu}\}$ is the determinant of the metric tensor, $\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is the kinetic term, and $V(\phi)$ is the scalar potential of the inflaton. Varying this action, we find the Klein-Gordon equation of motion

$$\ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \nabla^2 \phi + V'(\phi) = 0.$$

(2.36)

We introduce the expression for the perturbed inflaton field

$$\phi(x, t) = \phi_0(t) + \delta \phi(x, t),$$

(2.37)

where $\phi_0(t)$ is the infinite wavelength 'classical' field, that corresponds to the inflaton field expectation value on the initial isotropic and homogeneous state and $\delta \phi(x, t)$ corresponds to the quantum fluctuations around the classical field $\phi_0$ [122]. At the beginning of inflation, the scale factor grows exponentially and the gradient term becomes suppressed by a factor of $e^{-2Ht}$, and for slow-roll models of inflation it can be safely neglected.\(^3\) Ignoring the quantum fluctuations, and approximating $\phi(x, t) \simeq \phi_0(t) \equiv \phi(t)$,

\(^3\)We note that the gradient term is important when one considers the non-perturbative particle production.
we can rewrite the Klein-Gordon equation of motion (2.36) for the homogeneous inflaton field as
\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0, \tag{2.38} \]
and the scalar field evolution is primarily driven by the gradient term \( V'(\phi) = dV/d\phi \), subject to the damping term \( 3H \dot{\phi} \). For a homogeneous background, from the stress-energy tensor (2.10) we find
\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \tag{2.39} \]
\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi), \tag{2.40} \]
and the equation of state parameter (2.18) becomes
\[ w = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \tag{2.41} \]

We introduce the Hubble slow-roll parameter
\[ \epsilon_H \equiv -\frac{\dot{H}}{H^2} = 1 - \frac{a(t) \ddot{a}(t)}{a(t)^2}, \tag{2.42} \]
and since during exponential expansion \( \ddot{a} > 0 \) and hence \( \dot{H} > -H^2 \), it implies that during inflation \( \epsilon_H < 1 \). Using the Friedmann equation \( 3H^2 M_P^2 = \rho \) with Eq. (2.38), we find that the condition for inflation becomes
\[ \epsilon_H = \frac{3 \dot{\phi}^2}{2V + \dot{\phi}^2} = \frac{3}{2} (1 + w) < 1, \tag{2.43} \]
which implies that inflation requires \( \dot{\phi}^2 < V \) or \( w < -1/3 \).

Next, we discuss briefly the slow-roll approximation. It is commonly assumed that during the slow-roll inflation, the inflaton acceleration term \( \ddot{\phi} \) in the Klein-Gordon equation (2.38) can be neglected. This means that the Hubble damping term is approximately balanced by the potential gradient term
\[ 3H \dot{\phi} \simeq -V'(\phi), \tag{2.44} \]
and the Friedmann equation becomes

\[ H^2 \simeq \frac{V(\phi)}{3M_P^2}, \]

which shows that the Hubble parameter is dominated by the potential energy of the inflaton.

It is important to know when the slow-roll condition is violated. Since slow-roll inflation implies \( \ddot{a} \gg a \) and hence \( \epsilon_H = -\dot{H}/H^2 \ll 1 \), we can introduce the first slow-roll parameter \[ \epsilon_V \equiv \frac{M_P^2}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \ll 1. \] (2.46)

Another important condition that must be satisfied during the slow-roll is \( |\dot{\phi}| \ll |3H\dot{\phi}|, |V'(\phi)| \), which implies that the ratio \( -\dot{\phi}/(H\dot{\phi}) \ll 1 \), and the second slow-roll parameter is given by

\[ \eta_V \equiv \frac{M_P^2}{2} \left( \frac{V''(\phi)}{V(\phi)} \right) \ll 1, \]

and the slow-roll conditions \( \epsilon_V, |\eta_V| \ll 1 \) require that the inflationary potential is flat.

We also note that up to the first-order in the slow-roll approximation, the parameters \( \epsilon_H \simeq \epsilon_V \) coincide.

In the slow-roll regime one can readily determine the Hubble parameter as a function of the scalar field value. In particular, we introduce the quantity called the number of e-folds that measures the total logarithmic expansion of the universe as the inflaton field \( \phi(t) \) rolls

\[ N_\ast \equiv \ln \left( \frac{a_{end}}{a_\ast} \right) = \int_{t_\ast}^{t_{end}} \dot{H}dt \simeq -\frac{1}{M_P^2} \int_{\phi_\ast}^{\phi_{end}} \frac{V(\phi)}{V'(\phi)} d\phi = -\int_{\phi_\ast}^{\phi_{end}} \frac{1}{\sqrt{2\epsilon_V} M_P} d\phi, \]

(2.48)

where the integral limits are defined between the end of inflation and the pivot scale of \( k_\ast = 0.05 \text{ Mpc}^{-1} \) used in the Planck analysis, assuming that the condition \( \epsilon_V < 1 \) is satisfied within those boundaries. Typically, the number of e-folds needed to solve the horizon, flatness, and other problems associated with the Big Bang cosmological model is 50 – 60 e-folds, and we discuss it more extensively in Section 2.5.
Since $\epsilon_V$ and $\eta_V$ are approximately constant during slow-roll inflation, we have

$$N_* \simeq \frac{1}{\sqrt{2\epsilon_V} M_P} \frac{\Delta \phi}{M_P}.$$ \hspace{1cm} (2.49)

This approximation implies that in order to get 50 – 60 e-folds, either the slow-roll parameter has to be very small, $\epsilon_V < 0.01$, or the change in the scalar inflaton field value is large with respect to the Planck scale, $\Delta \phi > M_P$.

### 2.4 Reheating

For most conventional models of inflation, the inflaton field exits the plateau, rolls down to a minimum and starts oscillating, ultimately decaying into some elementary particles and reheating our universe [125, 126, 58]. As mentioned in the previous section, the end of inflation occurs when the kinetic energy of the inflaton starts dominating over the potential energy, which corresponds to the condition $\epsilon_H < 1$. Here we focus on simple quadratic chaotic inflation models [127]. These models resolve the problems associated with the old and new inflation scenarios. Typically, they assume that before the Planck time $t \sim t_p \sim 1/M_P$, the inflaton field $\phi$ domains existed with a chaotic initial distribution in the very early universe. Furthermore, this assumption implies that at Planck time, there exist homogeneous and isotropic domains of space with the field values $\phi \gtrsim M_P$, and that the majority of the physical volume of the universe is created in a state where $V(\phi) \sim M_P$, $\phi \gg M_P$, and $\dot{\phi} = 0$ [128, 129]. These are the initial conditions that we use when discussing cosmic inflation.

We first assume a general potential minimum of the form\(^4\)

$$V(\phi) \simeq \frac{1}{2} m^2 \phi^2,$$ \hspace{1cm} (2.50)

where $m$ is the effective inflaton mass and $\phi \lesssim M_P$. The equation of motion (2.38) becomes

$$\ddot{\phi} + 3H \dot{\phi} = -m^2 \phi,$$ \hspace{1cm} (2.51)

\(^4\)For a detailed analysis of coherent inflaton oscillations about a scalar potential minimum of the form $V \sim \phi^k$, where $k = 2, 4, \ldots$, see [130].
and if we solve it for $\phi$, we find

$$\phi(t) \propto \frac{\sin(mt)}{mt}, \quad H \simeq \frac{2}{3t},$$

(2.52)

where the frequency of the oscillations is $\omega = m$, and the inflaton field behaves as pressureless matter, with $\phi \sim t^{-1} \sim a^{-3/2}$. Therefore, at the end of inflation the universe enters a matter-dominated phase dominated by coherent inflaton oscillations.

Since we know that our universe is not empty, the inflaton should couple to the Standard Model particles. During the coherent oscillations, the inflaton decays (defragmentations) and its energy is transferred to the elementary particles. Assuming that the inflaton decay is slow, we can introduce the *Friedmann-Boltzmann* set of equations

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi \rho_\phi,$$

(2.53)

$$\dot{\rho}_R + 4H\rho_R = \Gamma_\phi \rho_\phi,$$

(2.54)

$$\rho_\phi + \rho_R = 3H^2 M_P^2,$$

(2.55)

where $\Gamma_\phi$ is the decay/scattering rate of the inflaton. The energy density of the produced relativistic products as a function of temperature are given by

$$\rho_R(T) = \frac{\pi^2}{30} g_*(T) T^4,$$

(2.56)

where the *effective number of degrees of freedom* is defined as

$$g_*(T) = \sum_{i=b} g_i + \frac{7}{8} \sum_{i=f} g_i,$$

(2.57)

assuming that the relativistic products are not in thermal equilibrium with the photons, $m_{b,f} \ll T$. The relative factor of 7/8 arises from the difference between the *Bose-Einstein* and *Fermi-Dirac statistics*. For the temperatures that are larger than $T \gtrsim 100$ GeV, all Standard Model particles are relativistic, and $g_* = 106.75$. When the relativistic species are no longer in thermal equilibrium with the photons, $g_*(T)$ can be expressed as

$$g_*(T) = \sum_{i=b} g_i \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{i=f} g_i \left( \frac{T_i}{T} \right)^4.$$  

(2.58)
The entropy density of the relativistic products is given by

\[ s_R(T) = \frac{2\pi^2}{45} g_{*S}(T)T^3, \]  

(2.59)

where

\[ g_{*S}(T) = \sum_{i=b} g_i \left( \frac{T_i}{T} \right)^3 + \frac{7}{8} \sum_{i=f} g_i \left( \frac{T_i}{T} \right)^3, \]  

(2.60)

is the effective number of degrees of freedom in entropy. We note that for most of the thermal history of the universe, \( g_{*S}(T) = g_*(T) \), and it only becomes different below the temperatures of \( T \lesssim 1 \text{ MeV} \). In Fig. (2.5) we show the evolution of \( g_*(T) \) and \( g_{*S}(T) \) as a function of temperature for the Standard Model particle content.

We define reheating at time \( t_{\text{reh}} \) when the energy density of the oscillating inflaton becomes equal to that of the produced radiation

\[ \rho_\phi(t_{\text{reh}}) = \rho_R(t_{\text{reh}}), \quad w = \frac{1}{6}, \]  

(2.61)
and from Eq. (2.59), the reheating temperature is given by
\[ T_{\text{reh}} = \left( \frac{30 \rho_R(t_{\text{reh}})}{\pi^2 g_*(t_{\text{reh}})} \right)^{1/4}, \] (2.62)
and the upper limit on the reheating temperature after inflation is given by [132]
\[ T_{\text{reh}} \simeq 0.2 \left( \frac{100}{g_*} \right)^{1/4} \sqrt{\Gamma_\phi M_P}. \] (2.63)

Here we only discuss the perturbative inflaton decay. However, the inflaton can also decay non-perturbatively in the process known as parametric resonance and the produced particles are created far from thermal equilibrium [133, 134, 135, 136].

As mentioned earlier, the inflaton condensate loses its energy while decaying perturbatively to elementary particles. We introduce the following Lagrangian
\[ L_{\text{int}} \supset -y \phi \bar{f} f, \] (2.64)
where \( y \) is a Yukawa-like coupling and \( f \) corresponds to the Standard Model fermions. When the inflaton mass is sufficiently heavy and \( m \gg m_f \), the inflaton decay rate becomes [125]
\[ \Gamma_\phi = \frac{y^2}{8\pi} m. \] (2.65)

Afterward, the decay products thermalize and interact with each other, producing other Standard Model particles. Importantly, the particles must thermalize before the standard Hot Big Bang era begins. For more detailed discussions on the inflaton decays and reheating, see [137, 138, 139, 140, 141, 142].

2.5 Evolution of Perturbations created by Inflation

Our next aim is to calculate the number of \( e \)-folds \( N_* \) before the end of inflation in a more realistic and complete cosmological scenario that includes a reheating phase. During the inflationary epoch, the scale factor increases exponentially according to \( a \sim e^{Ht} \). The Hubble radius \( 1/H \) sets the distance that the light can travel in Hubble time \( t \sim 1/H \). Therefore, the size of the universe increases by a factor of \( a \sim e \) during one Hubble
time, and the regions that are larger than $1/H$ become causally disconnected. Since the physical distance is proportional to the scale factor $x_{\text{phys}} \propto a \sim e^{Ht}$, a comoving distance $x = x_{\text{phys}}/a$ is no longer in causal contact when the comoving Hubble radius becomes $H^{-1} = (aH)^{-1}$, where the comoving scale is given by

$$k = aH.$$  \hspace{1cm} (2.66)

Importantly, during inflation the comoving Hubble radius $H^{-1}$ shrinks, and it implies that the comoving scales (modes) $k^{-1}$ exit the horizon during inflation and re-enter at late times during the Hot Big Bang phase. The evolution of the Fourier modes $k$ is shown in Fig. 2.6.

Next, we follow the analysis presented in [143]. We can express the number of

![Figure 2.6: The characteristic evolution of the comoving scales $k$ throughout the cosmological history. During inflation, the comoving Hubble radius or the horizon $H^{-1}$ shrinks, and later grows after the end of inflation. It implies that some comoving scales $k$ exit the horizon during inflation and later re-enter the horizon during the Hot Big Bang stage. Adapted from [131].](image-url)
e-folds (2.48) after the comoving scale $k_*$ re-enters the horizon

$$e^{N_*(k)} = \frac{a_{\text{end}}}{a_*}, \quad (2.67)$$

where $a_{\text{end}}$ is the scale factor at the end of inflation and $a_*$ is the scale factor at the horizon crossing. Assuming instantaneous transitions for the end of reheating epoch and the matter-radiation equality, which we label with a subscript ‘eq’, we can write

$$\frac{k_*}{a_0H_0} = \frac{a_*H_*}{a_0H_0} = e^{-N_*(k)} \frac{a_{\text{end}}}{a_{\text{reh}}} \frac{a_{\text{reh}}}{a_{\text{eq}}} \frac{H_\ast}{H_{\text{eq}}} \frac{a_{\text{eq}}}{a_0H_0}, \quad (2.68)$$

where we used $k_* = a_*H_*$. We introduce the related factors [110]

$$\frac{a_{\text{eq}}H_{\text{eq}}}{a_0H_0} = 219 \Omega_0 h. \quad (2.69)$$

and

$$H_{\text{eq}} = 5.25 \times 10^6 h^3 \Omega_0^2 H_0. \quad (2.70)$$

Eq. (2.67) can be expressed as

$$\frac{k_*}{a_0H_0} = e^{-N_*(k)} \frac{a_{\text{end}}}{a_{\text{reh}}} \left( \frac{\rho_{\text{end}}}{\rho_{\text{reh}}} \right)^{1/4} \frac{H_*}{\rho_{\text{reh}}^{1/4}} \left( \frac{\rho_{\text{eq}}^{1/4}}{a_{\text{reh}} a_0H_0} \right), \quad (2.71)$$

where $\rho_{\text{end}}$ and $\rho_{\text{reh}}$ are the energy densities at the end of inflation and reheating, respectively, and if we solve this expression for the number of e-folds, we find

$$N_* = -\ln \left( \frac{k_*}{a_0H_0} \right) + \frac{1}{3} \ln \frac{\rho_{\text{reh}}}{\rho_{\text{end}}} + \frac{1}{4} \ln \frac{\rho_{\text{eq}}^{1/4}}{\rho_{\text{reh}}^{1/4}} + \ln \sqrt{\frac{V_*}{3M_*^2 H_{\text{eq}}}} - 1 \ln 219 \Omega_0 h^2, \quad (2.72)$$

or equivalently

$$N_* = -\ln \left( \frac{k_*}{a_0H_0} \right) + \ln \left( \frac{H_*}{\rho_{\text{end}}^{1/4}} \right) + \ln \left( \frac{a_{\text{end}} \rho_{\text{end}}^{1/4}}{a_{\text{reh}} \rho_{\text{reh}}^{1/4}} \right) + \ln \left( \frac{\rho_{\text{reh}}^{1/4} a_{\text{reh}}}{a_0H_0} \right), \quad (2.73)$$
where we used $H_*^2 = V_*/3M_P^2$. The second term can be rewritten as

$$\ln \left( \frac{H_*}{\sqrt[4]{\rho_{\text{end}}}} \right) \simeq -\ln \sqrt{3} + \frac{1}{4} \ln \left( \frac{V_*^2}{\rho_{\text{end}}M_P^4} \right), \quad (2.74)$$

and the third term can be expressed as [144]

$$\ln \left( \frac{a_{\text{end}}^{1/4}}{a_{\text{reh}}^{1/4}} \right) \equiv \ln R_{\text{rad}}, \quad (2.75)$$

where we defined a new parameter $R_{\text{rad}}$.

The quantity $R_{\text{rad}}$ has a simple physical interpretation. If we recast the Friedmann equation (2.19) as a function of $e$-folds

$$\frac{d\rho(N)}{dN} = -3(1 + w), \quad (2.76)$$

then its solution is given by

$$\rho_{\text{reh}} = \rho_{\text{end}} \exp \left( -3 \int_{N_{\text{end}}}^{N_{\text{reh}}} (1 + w_{\text{reh}}(n)) \, dn \right). \quad (2.77)$$

If we combine this solution with (2.75), we find

$$\ln R_{\text{rad}} = \frac{1 - 3w_{\text{int}}}{12(1 + w_{\text{int}})} \ln \left( \frac{\rho_{\text{reh}}}{\rho_{\text{end}}} \right), \quad (2.78)$$

where the quantity

$$w_{\text{int}} \equiv \frac{1}{N_{\text{rad}} - N_{\text{end}}} \int_{N_{\text{end}}}^{N_{\text{reh}}} w(n) \, dn \quad (2.79)$$

is the equation of state parameter averaged over the number of $e$-folds during reheating. Therefore, the parameter $R_{\text{reh}}$ only depends on the cosmological history during reheating, and it measures when the ‘true’ reheating has happened and the equation of state of the effective fluid is $w \simeq 1/3$, i.e., the universe is fully dominated by radiation.

Assuming that the entropy between the end of reheating and today is conserved,
which implies that $sa^3 = \text{const.}$, we can express the fourth term in Eq. (2.73) as

$$\ln \left( \frac{\rho_{\text{reh}} a_{\text{reh}}}{a_0 H_0} \right) = \ln \left[ \frac{1}{\sqrt{3}} \left( \frac{\pi^2}{30} \right)^{1/4} \left( \frac{43}{11} \right)^{1/3} \frac{T_0}{H_0} \right],$$

where we used

$$\rho_{\text{reh}} = \frac{\pi^2}{30} g_{\text{reh}} T_{\text{reh}}^4, \quad s_{\text{reh}} = \frac{2\pi^2}{45} g_{\text{reh}} T_{\text{reh}}^3,$$

and

$$s_0 = \frac{2\pi^2}{45} \left( \frac{43}{11} \right) T_0^3,$$

$$N_* = \ln \left[ \frac{1}{\sqrt{3}} \left( \frac{\pi^2}{30} \right)^{1/4} \left( \frac{43}{11} \right)^{1/3} \frac{T_0}{H_0} \right] - \ln \left( \frac{k_*}{a_0 H_0} \right)$$

$$+ \frac{1}{4} \ln \left( \frac{V_*^2}{M_p^4 \rho_{\text{end}}} \right) + \frac{1 - 3w_{\text{int}}}{12 (1 + w_{\text{int}})} \ln \left( \frac{\rho_{\text{rad}}}{\rho_{\text{end}}} \right) - \frac{1}{12} \ln g_{\text{reh}},$$

of if we use the present Hubble parameter and photon temperature $H_0 = 67.36 \text{ km s}^{-1} \text{ Mpc}^{-1}$ [28] and $T_0 = 2.7255 \text{ K}$ [103], we have

$$N_* = 66.89 - \ln \left( \frac{k_*}{a_0 H_0} \right) + \frac{1}{4} \ln \left( \frac{V_*^2}{M_p^4 \rho_{\text{end}}} \right) + \frac{1 - 3w_{\text{int}}}{12 (1 + w_{\text{int}})} \ln \left( \frac{\rho_{\text{rad}}}{\rho_{\text{end}}} \right) - \frac{1}{12} \ln g_{\text{reh}}.$$

The number of $e$-folds can be determined very precisely for a given inflationary model and a value of the pivot scale $k_*$. Typically, the entropy generation during the reheating epoch leads to a well-known range

$$50 < N_* < 60,$$

and in Chapter 9, we analyze the reheating mechanism for different models of inflation and justify this range.

### 2.6 Inflationary Observables

Quantum fluctuations during inflation generate scalar density and tensor metric perturbations that leave imprints on the CMB and large-scale structure [145, 146, 147,
The overall scale of the potential is related to the amplitude of the power spectrum of scalar perturbations in the CMB, $A_s$, [28] and other CMB observables include the tilt in the spectrum of scalar perturbations, $n_s$, and the tensor-to-scalar ratio, $r$. It is convenient to characterize these in terms of the slow-roll parameters $\epsilon_V$ and $\eta_V$, given by Eqs. (2.46) and (2.47). The tilt in the spectrum of scalar perturbations, $n_s$, and the tensor-to-scalar ratio, $r$ are the principal CMB observables [28, 86], and can be expressed as follows in terms of the slow-roll parameters at the pivot scale $k_s = 0.05 \text{Mpc}^{-1}$:

\begin{align}
\text{Amplitude of scalar perturbations } A_s : A_s &= \frac{V_s}{24\pi^2 \epsilon_s M_P^4} \simeq 2.1 \times 10^{-9}, \quad (2.86) \\
\text{Scalar spectral tilt } n_s : n_s &\simeq 1 - 6\epsilon_s + 2\eta_s \\
&= 0.965 \pm 0.004 \, (68\% \, \text{CL}), \quad (2.87) \\
\text{Tensor-to-scalar ratio } r : r &\simeq 16\epsilon_s < 0.061 \, (95\% \, \text{CL}). \quad (2.88)
\end{align}

In addition to the above expressions, we note that the number of $e$-folds of inflation between the initial and final inflaton field values is given by Eq. (2.48). The region of the $(n_s, r)$ plane allowed by the Planck data is shown in Fig. 2.7, and it displays that the plateau type models of inflation are strongly favored whereas the simple monomial inflationary models disfavored or completely excluded.
Figure 2.7: Marginalized joint 68% and 95% CL regions for the CMB observables $n_s$ and $r$ at $k = 0.002\,\text{Mpc}^{-1}$ from Planck alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68% and 95% CL regions assume $dn_s/d\ln k = 0$. From Planck Collaboration [152].
Chapter 3

Supersymmetry and Supergravity

In this chapter, we discuss the motivation for supersymmetry and provide a summary of supersymmetry and supergravity. We first argue that supersymmetry is an unavoidable extension of the Standard Model and is arguably one of the leading candidates for physics beyond the Standard Model. We then introduce the key concepts of supersymmetry, including the Minimal Supersymmetric Standard Model (MSSM), and combine supersymmetry and general relativity into a supergravity theory. Finally, we discuss the aspects of supersymmetry breaking and review the problems in supergravity.

3.1 Why do we need supersymmetry?

The Standard Model of particle physics is a remarkably successful theory that has passed every experimental test with flying colors. With the discovery of the Higgs boson in 2012 by the ATLAS and CMS collaborations at the LHC [17, 18, 19], the Standard Model of particle physics was completed. However, we know that the Standard Model is not the ultimate theory and has its shortcomings. It only incorporates three of the four known fundamental forces, the electromagnetic, weak nuclear, and strong nuclear forces, and it does not include the existence of the gravitational force. Additionally, it does not explain why the expansion of the universe is accelerating or the origin of dark matter. Finally, the Standard Model cannot explain why there is more matter than antimatter. Therefore, it is clear that the Standard Model is an incomplete theory, which cannot explain all the observed features of the universe, including the features
of the ΛCDM cosmological model, and we must consider theories that lie beyond the Standard Model of particle physics.

To understand one of the main problems with the Standard Model, we first introduce the Higgs potential \( V_H(\phi) \), which characterizes the Higgs mechanism of spontaneous symmetry breaking

\[
V_H(\phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2.
\]  

(3.1)

Here \( \phi \) corresponds to the complex scalar Higgs field, \( \mu^2 < 0 \) is the Higgs mass parameter squared that is negative, and the positive term \( \lambda \) characterizes the quartic self-interaction of the Higgs field. From the Standard Model predictions, the vacuum expectation value (VEV) for \( \phi \) at the minimum of the potential cannot vanish. We find that the Higgs field obtains the following VEV value

\[
\langle \phi \rangle = \frac{\sqrt{-\mu^2}}{2 \lambda} \equiv \frac{v}{\sqrt{2}},
\]

(3.2)

and from the experimental measurements, it is determined that \( \langle \phi \rangle \simeq 174 \text{ GeV} \). Assuming that the Standard Model is an effective field theory, this measurement implies that \( \lambda \simeq 0.13 \) and \( \mu^2 = -88.4 \text{ GeV} \) [91].

Next, from the Standard Model, we know that the electroweak processes are typically characterized by the electroweak scale \( v \simeq 246 \text{ GeV} \). However, from Einstein’s principles of general relativity, it is believed that the quantum gravitational effects become important at the Planck scale or slightly below it, \( M_P \simeq 1/\sqrt{8 \pi G} \simeq 2.4 \times 10^{18} \text{ GeV} \). The fact that the ratio \( M_P/v \simeq \mathcal{O}(10^{16}) \) is so enormous leads to the infamous “hierarchy problem” [2, 3, 4], and it serves as one of the main motivations for studying physics beyond the Standard Model.

Furthermore, it is well known that in general, any field theories with fundamental scalar fields are extremely sensitive to the effective cutoff of the high energy scale \( \Lambda_{UV} \) [3]. Therefore, for the Higgs mass squared in the Standard Model, this results in quadratically divergent contributions that would lead to a sick theory since it receives significant quantum corrections from every particle that couples to it, either directly or indirectly.

We consider one-loop correction arising from the scalar field \( S \), shown by Feynman
diagram 3.1a, which is given by

$$\delta m^2_{H,S} = \lambda_S \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_S} = \frac{\lambda_S}{16\pi^2} \left( \Lambda^2_{UV} - 2m_S^2 \ln \left( \frac{\Lambda_{UV}}{m_S} \right) + \mathcal{O}\left( \frac{m_S^2}{\Lambda_{UV}} \right) \right), \quad (3.3)$$

where for the Higgs loop, $\lambda_S \sim m_H^2/v^2$. Similarly, the one-loop correction to the Higgs mass due to Dirac fermion $F$ is illustrated by Feynman diagram 3.1b, and can be expressed as

$$\delta m^2_{H,F} = i\lambda^2_F \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \frac{i}{(k - m_F)} \frac{i}{(k - m_F)} \right], \quad (3.4)$$

$$= \frac{\lambda^2_F}{4\pi^2} \left( -\Lambda^2_{UV} + 6m_F^2 \ln \left( \frac{\Lambda_{UV}}{m_F} \right) - 2m_F^2 + \mathcal{O}\left( \frac{m_F^2}{\Lambda_{UV}} \right) \right), \quad (3.5)$$

where the largest contribution comes from the heaviest top quark, with $\lambda_F^2 \sim m_t^2/v^2$. If we combine the dominant scalar and fermion one-loop corrections, given by Eqs. (3.3) and (3.4), respectively, we find

$$\delta m^2_{H,S} + \delta m^2_{H,F} \simeq \frac{3\Lambda^2_{UV}}{8\pi^2 v^2} \left( m_H^2 - 4m_F^2 - \mathcal{O}\left( \ln \left( \frac{\Lambda_{UV}}{\mu} \right) \right) \right). \quad (3.6)$$

It was shown in [153] that it is also important to include the Higgs mass one-loop corrections arising from the vector bosons, specifically, the corrections due to $Z$ and $W$ gauge bosons. If we also assume that the UV cutoff is equal to the Planck scale,
Λ_{UV} \sim M_P$, we obtain the following Higgs mass one-loop correction \[154\]

$$\delta m_H^2 \simeq \frac{3M_P^2}{8\pi^2 v^2} \left( m_H^2 - 4m_t^2 + m_Z^2 + 2m_W^2 \right),$$

(3.7)

which leads to disastrously large one-loop mass corrections. The Veltman condition is satisfied when $4m_t^2 = m_H^2 + 2m_W^2 + m_Z^2$, and it was argued in \[155\] that this condition might be satisfied when the energies are close to the Planck scale $M_P$. However, when evaluated at the electroweak scale, this quantum correction is more than 30 orders of magnitude larger than the Higgs mass squared. Since all the Standard Model particles obtain their mass from the VEV $\langle \phi \rangle$, the entire particle spectrum is sensitive to the UV cutoff $\Lambda_{UV}$.

One possibility to fix this problem is to choose a lower $\Lambda_{UV}$ scale which is not necessarily associated with the Planck scale. But this assumption would still imply that there is some new physics at the scale $\Lambda_{UV}$. Furthermore, one would also need to include any virtual effects associated with the BSM heavy particles, and choosing a different $\Lambda_{UV}$ scale does not solve the hierarchy problem because the biggest issue is not just the quadratic divergences, but instead the quadratic sensitivity to high-scale masses. A more thorough discussion of the hierarchy problem is reviewed in \[156\].

### 3.2 Theory of Supersymmetry

In order to remedy the hierarchy problem and deal with the enormously large one-loop corrections, we must introduce a systematic way to cancel out the dangerous contributions to $\delta m_H^2$. An important clue can be noticed from Eq. (3.7), which suggests that there must be some fundamental symmetry relating bosons to fermions so that the loop corrections are either negligible or zero. If each of the Standard Model quarks and leptons has two complex scalars with $\lambda_S = |\lambda_F|^2$, then the quadratic divergences shown by Figures (3.1a) and (3.1b) will cancel \[157, 158, 159, 160, 161, 162\]. Therefore, it is necessary to assume that there exists a symmetry that relates fermions and bosons, known as supersymmetry.

To understand the key principles of supersymmetry, we first discuss the well-known Coleman-Mandula no-go theorem \[163\]. The theorem concerns the possible symmetries
in a relativistic theory of interactions, and it states that the only possible Lie group symmetries must be a direct product of the Poincaré group and an internal symmetry group. It was later found by Wess and Zumino [164, 165] that the Coleman-Mandula theorem could be avoided if the extended class of symmetries, called supersymmetries, were considered. Importantly, because the generators are of fermionic nature, which are characterized by the anticommutation relations, it provides a possible way around the Coleman-Mandula no-go theorem. Additional restrictions on supersymmetries were later discussed by Haag, Lopuszanski, and Sohnius [1].

We introduce a fermionic generator $Q$, which acts on bosonic and fermionic states in the following way:

$$Q |B \rangle = |F \rangle, \quad Q |F \rangle = |B \rangle,$$

where $|B \rangle$ is the Boson eigenstate and $|F \rangle$ is the fermion eigenstate. Because the spinors are complex objects, the hermitian conjugate of $Q$ also generates symmetry. Therefore, $Q$ and $Q^\dagger$ are fermionic operators, that carry a spin angular momentum of $1/2$, and imply that supersymmetry is a spacetime symmetry. They obey the following commutation and anticommutation relations [156]

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_\alpha^{\mu} P_{\mu}, \quad (3.9)$$
$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \quad (3.10)$$
$$[Q_\alpha, P^\mu] = [\bar{Q}_{\dot{\alpha}}, P^\mu] = 0, \quad (3.11)$$
$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^\beta Q_\beta, \quad (3.12)$$
$$[\bar{Q}_{\dot{\alpha}}, M^{\mu\nu}] = (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}_{\dot{\beta}} \bar{Q}_{\dot{\beta}}, \quad (3.13)$$

where $P^\mu$ and $M^{\mu\nu}$ are the Poincaré generators, $\bar{Q}_{\dot{\alpha}} = (Q_\alpha)^\dagger$ are the spinor generators, and

$$\sigma^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \quad (3.14)$$
$$\bar{\sigma}^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu). \quad (3.15)$$

The commutation and anticommutation relations (3.9) - (3.13) characterize the super-Poincaré algebra [166]. In general, the spinor generators can be written as $Q_\alpha^M$, $\bar{Q}_{\dot{\alpha}}^M$, where $M = 1, \ldots, \mathcal{N}$, and here we only consider the minimal $\mathcal{N} = 1$ supersymmetry.
with four supercharges.\(^1\) The single-particle states of a supersymmetric theory fall into *irreducible representations* of the *superalgebra*, known as *supermultiplets*. Each supermultiplet contains both boson and fermion states, and they can be referred to as *superpartners* of each other.

Next, we illustrate how to obtain an important result, which shows that in any supermultiplet, the number of fermions \(n_F\) is equal to the number of bosons \(n_B\). We consider the fermion number operator \((-1)^F\), which acts on bosonic and fermionic eigenstates in the following way

\[
(-1)^F |B\rangle = |B\rangle, \quad (-1)^F |F\rangle = -|F\rangle,
\]

and we show that the fermion number operator anticommutes with the spinor generator \(Q_\alpha\):

\[
Q_\alpha |F\rangle = -Q_\alpha (-1)^F |F\rangle = (-1)^F Q_\alpha |F\rangle = (-1)^F |B\rangle = |B\rangle,
\]

and it leads to

\[
\{(-1)^F, Q_\alpha\} = 0.
\]

Next, we consider the following trace identity

\[
\text{Tr}\left\{-(-1)^F \{Q_\alpha, \bar{Q}_\beta\}\right\} = \text{Tr}\left\{-(-1)^F Q_\alpha \bar{Q}_\beta + (-1)^F \bar{Q}_\beta Q_\alpha\right\} = 0,
\]

where we have used the anticommutation relation (3.18). If we use the relation (3.9), we find

\[
\text{Tr}\left\{-(-1)^F \{Q_\alpha, \bar{Q}_\beta\}\right\} = \text{Tr}\left\{-(-1)^F 2(\sigma^\mu)_{\alpha\beta} P_\mu\right\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu \text{Tr}\left\{(-1)^F\right\}.
\]

\(^1\)The extended supersymmetries and gauge theories with \(\mathcal{N} > 1\) are discussed in [167, 168, 169].
If we assume that the momentum eigenvalues $p_\mu \neq 0$, it leads to $\text{Tr}\{(−1)^F\} = 0$, and

$$\text{Tr}\{(−1)^F\} = \sum_{\text{bosons}} \langle B| (−1)^F |B\rangle + \sum_{\text{fermions}} \langle F| (−1)^F |F\rangle$$

$$= \sum_{\text{bosons}} \langle B|B\rangle - \sum_{\text{fermions}} \langle F|F\rangle = 0,$$  \hspace{0.5cm} (3.21)

and we finally derive the result which shows that the supermultiplets contain equal number of fermionic and bosonic degrees of freedom

$$n_B = n_F,$$  \hspace{0.5cm} (3.22)

which holds for a given $p^\mu \neq 0$ in each supermultiplet.

### 3.3 The Minimal Supersymmetric Standard Model

After introducing the key principles of the super-Poincaré algebra, we can apply them to extend the Standard Model. The simplest possibility to satisfy the constraint (3.22) is to consider a supermultiplet with a single Weyl spinor, that has two degrees of freedom $n_F = 2$, and combine it with two real scalar degrees of freedom, each with $n_B = 1$. We call this supermultiplet combination of a single Weyl spinor with a complex scalar field a chiral (or matter) supermultiplet.

Another possible combination that satisfies the condition (3.22) contains a massless spin-1 vector boson, which has two degrees of freedom $n_B = 2$, with a massless spin-1/2 Weyl spinor, which has two helicity states with $n_F = 2$. We call such a combination of spin-1 gauge bosons and spin-1/2 Weyl spinors, called gauginos, a gauge supermultiplet. Other combinations, that satisfy the expression (3.22) are possible but we do not discuss them here. Therefore, in a minimal supersymmetric extension of the Standard Model [6, 7, 170], all the Standard Model particles are in either a chiral or gauge supermultiplet, which must also contain a superpartner whose spin is different by 1/2.

In Section 3.4, we combine gravity with supersymmetric theories. In such a case, the spin-2 graviton, which has 2 helicity states with $n_B = 2$, has a spin-3/2 superpartner called the gravitino. If the supersymmetry is unbroken, the gravitino is massless and it has two helicity states with $n_F = 2$. 
It is also important to discuss the properties of the Higgs boson in supersymmetric theories. The Higgs scalar boson must reside in a chiral supermultiplet, with its supersymmetric partner called higgsino, which has spin-1/2. Furthermore, it turns out that just a single Higgs doublet is not sufficient and we must introduce a second Higgs doublet. If there were only one Higgs chiral supermultiplet, the electroweak gauge symmetry would have a gauge anomaly, and it would be inconsistent with the principles of quantum field theory. Another way to understand it is that there would be no cancellation of the chiral anomalies due to higgsino, and we could not correctly break the $SU(2) \times U(1)$ symmetry, which is a necessary ingredient of the Standard Model. Therefore, a single complex Higgs doublet is insufficient because it can only provide four degrees of freedom. A more detailed discussion about the supersymmetric Higgs sector can be found in [156].

Now we can discuss how can the simplest supersymmetric theory be realized in nature. We have discussed earlier in this section that it is possible to classify all Standard Model particles and their superpartners in supermultiplets. We summarize the particle content of the Minimal Supersymmetric Standard Model (MSSM) in Tables 3.1 and 3.2, classified in terms of their transformation properties under the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, which combines the degrees of freedom associated with the quarks ($u_L, d_L$) and the leptons ($\nu, e_L$) into $SU(2)_L$ doublets.

The matter fields are characterized by chiral superfields, defined in terms of left-handed Weyl spinors, and the conjugates of right-handed leptons and quarks (together with their superpartners) are shown in Table 3.1.

It is clear that the vector bosons of the Standard Model must reside in gauge supermultiplets, together with their supersymmetric partners known as gauginos. After the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$ is broken, the $W^0, B^0$ gauge states mix to give mass eigenstates $Z_0$ and $\gamma$. The corresponding gaugino mixtures of $\tilde{W}^0$ and $\tilde{B}^0$ are called zino ($\tilde{Z}^0$) and photino ($\tilde{\gamma}$), and in case of unbroken supersymmetry, these mixtures would correspond to mass eigenstates $m_Z$ and 0. Gauge supermultiplets of MSSM are summarized in Table 3.2.

We note that the Higgs chiral multiplet $H_d$, with the degrees of freedom $H^0_d, H^-_d, \tilde{H}^0_d, \tilde{H}^-_d$, contains the same Standard Model quantum numbers as the left-handed leptons and sleptons $L_i$, such as ($\tilde{\nu}, \tilde{e}_L, \nu, e_L$). However, one cannot take a neutrino and a Higgs
names spin 0 spin 1/2 $SU(3)_C, SU(2)_L, U(1)_Y$

<table>
<thead>
<tr>
<th>Names</th>
<th>spin 0</th>
<th>spin 1/2</th>
<th>$SU(3)_C, SU(2)_L, U(1)_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>squarks, quarks ($\times 3$ families)</td>
<td>$Q$</td>
<td>$(\bar{u}_L \ d_L)$</td>
<td>$(u_L \ d_L)$</td>
</tr>
<tr>
<td></td>
<td>$\bar{u}$</td>
<td>$\bar{u}_R^+ \ u_R^+$</td>
<td>$\bar{u}_R^+ \ u_R^+$</td>
</tr>
<tr>
<td></td>
<td>$d$</td>
<td>$\bar{d}_R^+ \ d_R^+$</td>
<td>$\bar{d}_R^+ \ d_R^+$</td>
</tr>
<tr>
<td>sleptons, leptons ($\times 3$ families)</td>
<td>$L$</td>
<td>$(\bar{\nu} \ \bar{e}_L)$</td>
<td>$(\nu \ e_L)$</td>
</tr>
<tr>
<td></td>
<td>$e$</td>
<td>$\bar{e}_R^{\ast} \ e_R^+$</td>
<td>$\bar{e}_R^{\ast} \ e_R^+$</td>
</tr>
<tr>
<td>Higgs, higgsinos</td>
<td>$H_u$</td>
<td>$(H_u^+ \ H_u^0)$</td>
<td>$(H_u^+ \ H_u^0)$</td>
</tr>
<tr>
<td></td>
<td>$H_d$</td>
<td>$(H_d^0 \ H_d^-)$</td>
<td>$(\bar{H}_d^0 \ H_d^-)$</td>
</tr>
</tbody>
</table>

Table 3.1: Chiral supermultiplets in the Minimal Supersymmetric Standard Model.

<table>
<thead>
<tr>
<th>Names</th>
<th>spin 1/2</th>
<th>spin 1</th>
<th>$SU(3)_C, SU(2)_L, U(1)_Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>gluino, gluon</td>
<td>$\tilde{g}$</td>
<td>$g$</td>
<td>$(8, 1, 0)$</td>
</tr>
<tr>
<td>winos, W bosons</td>
<td>$\tilde{W}^\pm \ \tilde{W}^0$</td>
<td>$W^\pm \ W^0$</td>
<td>$(1, 3, 0)$</td>
</tr>
<tr>
<td>bino, B boson</td>
<td>$\tilde{B}^0$</td>
<td>$B^0$</td>
<td>$(1, 1, 0)$</td>
</tr>
</tbody>
</table>

Table 3.2: Gauge supermultiplets in the Minimal Supersymmetric Standard Model.

Scalar to be superpartners of each other, because that would imply that the sneutrino is the same particle as the Higgs boson, and it leads to many phenomenological issues. Thus, all supersymmetric particles of the Standard Model are necessarily new particles.

Now that we introduced the MSSM supermultiplets, the additional superparticles guarantee the one-loop Higgs mass correction

$$\delta m_H^2 \simeq \frac{M_P^2}{8\pi^2} \left( \lambda_S - |\lambda_F|^2 \right)$$

(3.23)

cancels, as long as $\lambda_S = |\lambda_F|^2$.

However, if supersymmetry was not broken, then the supersymmetric particles would have identical masses as their Standard Model counterparts, and the supersymmetric particles would have been detected a long time ago. This serves as an important clue as to why the supersymmetry must be broken. Importantly, if we want to make sure that the radiative corrections are still canceled, equation (3.23) must always still hold.
when the supersymmetry is broken, with \( \lambda_S = |\lambda_F|^2 \), and we examine it in detail in Section 3.5.

### 3.3.1 Global Supersymmetry

Now we can discuss the construction of supersymmetric Lagrangians. We do not use the superfield formalism here, which is explored in [171, 172]. We first show the general Lagrangians that are applied to the special case of MSSM and consider the simplest supersymmetric theory in four dimensions, which consists of a single left-handed two-component Weyl fermion, \( \psi \), with its complex scalar field superpartner, \( \phi \). We introduce the following Lagrangian for free chiral supermultiplet, which consists of only kinetic energy terms and is known as the non-interacting Wess-Zumino model [164]

\[
\mathcal{L}_{\text{free}} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}},
\]

where

\[
\mathcal{L}_{\text{scalar}} = -\partial^\mu \phi^\dagger \partial_\mu \phi, \quad \mathcal{L}_{\text{fermion}} = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi. \tag{3.25}
\]

Since a supersymmetric transformation should convert the scalar boson field \( \phi \) to a quantity that is proportional to a fermion, \( \psi_\alpha \), it implies that the scalar field transforms according to

\[
\delta \phi = \epsilon \psi, \quad \delta \phi^\dagger = \epsilon^\dagger \psi^\dagger, \tag{3.26}
\]

where \( \epsilon^\alpha \) is an infinitesimal two-component Weyl fermion parameter. When discussing supersymmetry and MSSM, we only consider global supersymmetry transformations, where \( \epsilon^\alpha \) is constant and satisfies condition \( \partial_\mu \epsilon^\alpha = 0 \). Because \( \psi \) has dimensions of [mass]\(^{3/2} \) and the dimensions of \( \phi \) are [mass], for four-dimensional theories this implies that \( \epsilon \) must have the dimension of [mass]\(^{-1/2} \). Thus, the variation of Eq. (3.25) can be expressed as

\[
\delta \mathcal{L}_{\text{scalar}} = -\epsilon \partial^\mu \psi \partial_\mu \phi^\dagger - \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi. \tag{3.27}
\]

We want this contribution to be canceled by \( \delta \mathcal{L}_{\text{fermion}} \). Comparing the expression (3.27) with \( \mathcal{L}_{\text{ferm}} \), we see that \( \delta \psi \) should be linear in \( \epsilon^\dagger \) and proportional to \( \phi \), and should
contain a single spacetime derivative. Therefore, we consider

\[ \delta \psi_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi, \quad \delta \psi_\dagger_\dot{\alpha} = i(\epsilon \sigma^\mu)_\alpha \partial_\mu \phi^*, \] (3.28)

which leads to

\[ \delta \mathcal{L}_{\text{fermion}} = -\epsilon \sigma^\mu \bar{\sigma}^\nu \partial_\nu \psi \partial_\mu \phi^* + \bar{\psi} \dagger \bar{\sigma}^\nu \sigma^\mu \epsilon \dagger \partial_\mu \partial_\rho \phi. \] (3.29)

Combining Eqs. (3.27) and (3.29), we find that the variation of the action vanishes

\[ \delta S = \int d^4x \ (\delta \mathcal{L}_{\text{scalar}} + \delta \mathcal{L}_{\text{fermion}}) = 0. \] (3.30)

However, if we would carry out the algebra we would find that constraint \( \delta \mathcal{L}_{\text{scalar}} + \delta \mathcal{L}_{\text{fermion}} = 0 \) is only satisfied on-shell, which would lead to problems in a quantum mechanical framework. To fix this problem, we introduce a new complex auxiliary field \( F \), which has dimensions of \([\text{mass}]^2\) and allows for symmetry algebra to close off-shell

\[ \mathcal{L}_{\text{auxiliary}} = F^* F. \] (3.31)

We note that the auxiliary field does not contain a kinetic term, and the above equation implies the equations of motion \( F = F^* = 0 \). It is important to ensure that \( F \) transforms into equations of motions associated with \( \psi \)

\[ \delta F = -i \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi, \quad \delta F^* = i \partial_\mu \psi \dagger \bar{\sigma}^\mu \epsilon, \] (3.32)

and the auxiliary part becomes

\[ \delta \mathcal{L}_{\text{auxiliary}} = -i \epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi F^* + i \partial_\mu \bar{\psi} \dagger \bar{\sigma}^\mu \epsilon F, \] (3.33)

which vanishes on-shell but not for off-shell configurations. Now the full transformation laws for \( \psi \) and \( \psi^\dagger \) are given by

\[ \delta \psi_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi + \epsilon_\alpha F, \quad \delta \psi_\dagger_\dot{\alpha} = i(\epsilon \sigma^\mu)_\dot{\alpha} \partial_\mu \phi^* + \epsilon^\dagger_\dot{\alpha} F^*, \] (3.34)

and this additional contribution to \( \delta \mathcal{L}_{\text{fermion}} \) cancels with \( \delta \mathcal{L}_{\text{auxiliary}} \). The full modified Lagrangian \( \mathcal{L} = \mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{fermion}} + \mathcal{L}_{\text{auxiliary}} \) is invariant under supersymmetry
transformations.

Next, we want to consider a supersymmetry Lagrangian in a more realistic physical scenario such as MSSM with interactions of chiral supermultiplets, including both gauge and non-gauge interactions. We start with the Lagrangian that characterizes the free chiral supermultiplets

$$
\mathcal{L}_{\text{free}} = -\partial^\mu \phi^* \partial_\mu \phi_i + i\psi_\dagger \bar{\sigma}^\mu \partial_\mu \psi_i + F^{*i} F_i.
$$

(3.35)

It remains invariant under the supersymmetry transformations

$$
\delta \phi_i = \epsilon \psi_i, \quad \delta \phi^* = \epsilon^{\dagger} \psi_i^\dagger, \\
\delta (\psi_i)_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \phi_i + \epsilon_\alpha F_i, \quad \delta (\psi^\dagger_i)_\dot{\alpha} = i(\epsilon \sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^{*i} + \epsilon_{\dot{\alpha}} F^{*i}, \\
\delta F_i = -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi_i, \quad \delta F^{*i} = i\partial_\mu \psi^{\dagger i} \bar{\sigma}^\mu \epsilon.
$$

(3.36) – (3.38)

We introduce the following interaction Lagrangian

$$
\mathcal{L}_{\text{int}} = \left( -\frac{1}{2} W^{ij} \psi_i \psi_j + W^{*i} F_i \right) + \text{h.c.},
$$

(3.39)

where $W$ is a holomorphic object called the superpotential. In general, it can be expressed as

$$
W = \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k,
$$

(3.40)

where $M^{ij}$ is a symmetric mass matrix for the fermion fields and $y^{ijk}$ is a totally symmetric Yukawa-like coupling.

Thus the most general form of non-gauge interactions for chiral supermultiplets can be described by a single holomorphic function known as the superpotential $W$. Now we can eliminate the auxiliary fields $F_i$ and $F^{*i}$ using their equations of motion. The terms in the Lagrangian that contain the auxiliary fields are given by $F_i F^{*i} + W^i F_i + W^* F^{*i}$, and they lead to the following equations of motion

$$
F_i = -W_i^*, \quad F^{*i} = -W^i.
$$

(3.41)
The full Lagrangian density becomes

\[
\mathcal{L}_{\text{chiral}} = -\partial^\mu \phi^s i \partial_{\mu} \phi^i + i \psi^i \tilde{\sigma}^\mu \partial_{\mu} \psi^i - \frac{1}{2} \left( W^{ij} \psi_i \psi_j + W^*_{ij} \psi^i \psi^j \right) - W^i W_i^*,
\]

(3.42)

where the non-propagating auxiliary fields \( F_i \) and \( F^*_i \) were eliminated. From the Lagrangian, we find that the effective scalar potential can be written in terms of the superpotential

\[
V(\phi, \phi^*) = W^k W_k^* = F^*k F_k = M_{ik} M_{kj} \phi^*_i \phi_j + \frac{1}{2} M^*_{im} y^{jkni} \phi_i \phi^*_j \phi^*_k + \frac{1}{4} y_{ijklm} \phi^*_i \phi_j \phi^*_k \phi^*_l.
\]

(3.43)

We note that this effective scalar potential is always positive (bounded from below).

Our next step is to build the supersymmetric Lagrangian for a gauge supermultiplet. A gauge supermultiplet contains a massless gauge boson field \( A^a_\mu \) and a two-component Weyl fermion gaugino \( \lambda^a \). The vector supermultiplet gauge transformations are given by

\[
A^a_\mu \rightarrow A^a_\mu + \partial_\mu \Lambda^a + g f_{abc} A^b_\mu \Lambda^c,
\]

(3.44)

\[
\lambda^a \rightarrow \lambda^a + g f_{abc} \lambda^b \Lambda^c,
\]

(3.45)

where \( \Lambda^a \) is a gauge parameter, \( g \) is the gauge coupling, and \( f_{abc} \) are the gauge group structure constants.

Following the same discussion as for the chiral supermultiplets, the supersymmetry required for on-shell degrees of freedom amounts for two bosonic helicity states for \( A^a_\mu \) and two fermionic helicity states for \( \lambda^a \). However, off-shell \( \lambda^a \) has four real fermionic degrees of freedom, whereas \( A^a_\mu \) only has three bosonic degrees of freedom. Therefore, we introduce one real bosonic auxiliary field \( D^a \) to ensure that supersymmetry is consistent off-shell. Just as the chiral auxiliary field \( F_i \), the gauge auxiliary field has no kinetic term and has the dimensions of \([\text{mass}]^2\).

The gauge supermultiplet Lagrangian is given by

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} + i \tilde{\lambda}^a \tilde{\sigma}^\mu \nabla_\mu \lambda^a + \frac{1}{2} D^a D^a,
\]

(3.46)
where $F^a_{\mu\nu}$ is the Yang-Mills field strength given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu, \quad (3.47)$$

and the gaugino covariant derivative is

$$\nabla_\mu \lambda^a = \partial_\mu \lambda^a + gf^{abc}A^b_\mu \lambda^c. \quad (3.48)$$

For the full supersymmetric gauge and chiral interactions, we find the supersymmetry transformations are given by [156]

$$\delta A^a_\mu = -\frac{1}{\sqrt{2}} \left( \epsilon^\dagger \bar{\sigma}_\mu \lambda^a + \lambda^a \bar{\sigma}_\mu \epsilon \right), \quad (3.49)$$

$$\delta \lambda^a_\alpha = \frac{i}{2\sqrt{2}} (\sigma^\mu \bar{\sigma}^\nu)_{\alpha} F^a_{\mu\nu} + \frac{1}{\sqrt{2}} \epsilon_\alpha D^a, \quad (3.50)$$

$$\delta D^a = \frac{i}{\sqrt{2}} \left( -\epsilon^\dagger \bar{\sigma}_\mu \nabla_\mu \lambda^a + \nabla_\mu \lambda^a \bar{\sigma}_\mu \epsilon \right), \quad (3.51)$$

$$\delta \phi_i = \epsilon \psi_i, \quad (3.52)$$

$$\delta \psi^i_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \nabla_\mu \phi_i + \epsilon_\alpha F^i, \quad (3.53)$$

$$\delta F^i = -i\epsilon^\dagger \bar{\sigma}^\mu \nabla_\mu \psi_i + \sqrt{2}g(T^a \phi)_i \epsilon^\dagger \lambda^a. \quad (3.54)$$

The full renormalizable supersymmetry Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{gauge}} - \sqrt{2}g(\phi^* T^a \psi)\lambda^a - \sqrt{2}g\lambda^a (\psi^\dagger T^a \phi) + g(\phi^* T^a \phi)D^a, \quad (3.55)$$

where $\mathcal{L}_{\text{chiral}}$ is given by Eq. (3.42) (with ordinary derivatives replaced by covariant derivatives) and $\mathcal{L}_{\text{gauge}}$ is given by Eq. (3.46). If we combine the last term with the $D^a D^a / 2$ term in $\mathcal{L}_{\text{gauge}}$, then the equation of motion for $D^a$ is given by

$$D^a = -g(\phi^* T^a \phi). \quad (3.56)$$

If we combine all the key components of the chiral and gauge supersymmetry Lagrangian, we obtain the following complete scalar potential

$$V(\phi, \phi^*) = F^a F^i + \frac{1}{2} \sum_a D^a D^a = W_i^2 W^i + \frac{1}{2} \sum_a g_a^2 (\phi^* T^a \phi)^2, \quad (3.57)$$
where we refer to the first term as the $F$-term and the second term as the $D$-term. For most cases that we will be studying, we only consider the $F$-term contributions, and we set the $D$-term contributions to zero.

To introduce the supersymmetric couplings between the chiral superfields, we consider the following MSSM superpotential form

$$W_{\text{MSSM}} = \bar{u} y_u Q H_u - \bar{d} y_d Q H_d - \bar{e} y_e L H_d + \mu H_u H_d,$$

(3.58)

where for simplicity we suppress the family and gauge indices. The chiral supermultiplets $\bar{u}, \bar{d}, \bar{e}, Q, L, H_u, H_d$ appearing in the MSSM superpotential are included in Table 3.1. The Yukawa matrices $y_u, y_d, y_e$ are the $3 \times 3$ matrices in family space. The $\mu$-term in the superpotential (3.58) is responsible for the masses of Higgs bosons, and it must be of order the electroweak scale $v$. The appearance of the $\mu$-term and its smallness relative to the Planck scale $M_P$ leads to the puzzle, which is called the $\mu$-problem. Possible solutions to the $\mu$-problem include the Kim-Nilles mechanism [173] and the Giudice-Masiero mechanism [174]. From the superpotential (3.58), one can notice that the second Higgs doublet is necessary to give masses to the remaining quarks and leptons because the superpotential must be holomorphic in the scalar fields treated as complex variables.

It is important to note that the dimensionful coupling in the supersymmetric part of MSSM Lagrangian depends on $\mu$, and leads to the following masses

$$-L_{\text{higgsino mass}} = \mu \left( \bar{H}_u^+ H_d^- - \bar{H}_u^0 H_d^0 \right) + \text{h.c.},$$

(3.59)

and the Higgs mass squared terms are

$$-L_{\text{supersymmetric Higgs mass}} = |\mu|^2 \left( |H_u^0|^2 + |H_u^+|^2 + |H_d^0|^2 + |H_d^-|^2 \right).$$

(3.60)

Because the above equation is always positive, the electroweak symmetry breaking requires some additional terms in the Lagrangian to ensure that the supersymmetry-breaking squared-mass soft term is negative, and we discuss in the next section.

Next, we explore the baryon number $B$ and the lepton number $L$ conservation in the MSSM. The $B-$ and $L-$ violating processes have not been observed experimentally,
and it is important to ensure that these processes are not allowed in supersymmetry. In
the Standard Model there are no renormalizable Lagrangian terms that could violate
$B$ or $L$. However, we know that they are violated by non-perturbative electroweak
effects [175]. Thus, in the MSSM we introduce a new type of symmetry that ensures
that there are no $B$ and $L$ violating terms in the renormalizable superpotential, called
$R$-parity [176] (or matter parity [160, 177]).

$R$-parity is characterized by the following expression

$$P_R = (-1)^{3(B-L)+2s}, \quad (3.61)$$

where $s$ is the spin of the particle, and it can be argued that $R$-parity must be con-
served [156]. The $R$-parity assignment for the Standard Model particles is even $R$-parity
($P_R = +1$), whereas all the supersymmetric particles, or sparticles, have odd $R$-parity
($P_R = -1$). The $R$-parity conservation implies that:

1. There exists a lightest supersymmetric particle (LSP) with $P_R = -1$ which is
   completely stable and serves as a viable dark matter candidate [12, 13];

2. Each sparticle except for the LSP must decay into a state with an odd number of
   LSPs in the final state;

3. In collider experiments, sparticles are produced in pairs. Importantly, the MSSM
   must respect the $R$-parity conservation.

### 3.4 Supergravity

#### 3.4.1 Local Supersymmetry

Our next step is to combine the principles of supersymmetry with general relativity. This
can be accomplished by promoting supersymmetry, which is a theory of global symmetry,
to a theory of local symmetry. A supersymmetric theory with a local symmetry is known
as supergravity. In this case, the parameter $\epsilon(x)$ is now a function of spacetime, and we
modify the commutation relation (3.9) in the following way

$$[\epsilon_1(x)Q, \bar{Q}\bar{\epsilon}_2(x)] = 2\epsilon_1(x)\sigma^\mu\bar{\epsilon}_2(x)P_\mu. \quad (3.62)$$
Note the difference in the right-hand side when compared to the global symmetry case, and in the local symmetry case, it characterizes an infinitesimal element of the group of local diffeomorphisms on spacetime.

Here we recall some important aspects of supergravity presented in [44, 41, 178, 179]. A supersymmetric theory with local symmetry requires that the space-time is treated as a dynamic object. It can be shown that the full theory that is invariant under local supersymmetry transformations would lead to Einstein’s theory of gravity. Since we are considering it in a supersymmetric context, a spin-2 graviton must have a supersymmetric spin-3/2 particle, known as gravitino. However, we know that if the supersymmetry is unbroken, the spin-2 graviton and spin-3/2 gravitino have degenerate mass.

To simplify our discussion, we limit our framework to the simplest $N = 1$ four-dimensional theory of supergravity. Therefore, our main goal is to construct the following Lagrangian

$$L_{\text{supergravity}} = L_{\text{EH}} + L_{3/2},$$

where the graviton part is described by the Einstein-Hilbert Lagrangian

$$L_{\text{EH}} = -\frac{M_P^2}{2} \Box R,$$

and $L_{3/2}$ is the Rarita-Schwinger Lagrangian that characterizes the spin-3/2 particles, and we discuss it below.

Next, we introduce the vielbein formalism which is needed to discuss the spinor transformations in curved backgrounds [180]. One can consider a system of locally inertial coordinates, and translate it back to the original non-inertial coordinate frame. If we denote a coordinate frame that is inertial at point $x_0$ by $y(x_0; x)$, we can define the so-called vielbein as

$$e'_\mu(x) = \left. \frac{\partial y^a(x_0; x)}{\partial x^\mu} \right|_{x=x_0}.$$ 

For general coordinates transformations $x \to x'$, the vielbein transforms as a covariant vector

$$e'_{\mu} (x') = \frac{\partial x'^\nu}{\partial x^\mu} e^a_\nu (x).$$
and the Lorentz transformation is given by

\[ e'_\mu(x) = e'_\mu(x) \Lambda^b_b. \tag{3.67} \]

We can express the spacetime metric in terms of the vielbein and the Minkowski metric \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \) as

\[ g_{\mu \nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}. \tag{3.68} \]

We introduce the full locally supersymmetric pure supergravity action \([181, 45, 44]\)

\[ S = -\frac{M_P^2}{2} \int d^4x \, e \, R - \frac{1}{2} \int d^4x \, \epsilon^{\mu \nu \rho \sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu D_\rho \psi_\sigma. \tag{3.69} \]

The first term in the pure supergravity action (3.69) is the Einstein-Hilbert action and the second term is the Rarita-Schwinger action that characterizes the spin-3/2 particle. Here \( e \equiv |\det e^a_\mu| \), where \( e^a_\mu \) is the vielbein, \( \epsilon^{\mu \nu \rho \sigma} \) is the Levi-Civita symbol, and the covariant derivative is defined as

\[ D_\mu \equiv \partial_\mu + \frac{1}{2} \omega^{mn}_\mu \sigma_{mn}, \tag{3.70} \]

where \( \omega^{mn}_\mu \) is the spin connection

\[ \omega^{mn}_\mu = \frac{1}{2} e^{vm}(\partial_\mu e^m_n - \partial_n e^m_\mu) - \frac{1}{2} e^{vn}(\partial_\mu e^m_n - \partial_n e^m_\mu) - \frac{1}{2} e^{pm} e^{\sigma n}(\partial_\rho e_{\sigma c} \partial_\sigma e_{pc} \epsilon^c_\mu), \tag{3.71} \]

and

\[ \sigma_{mn} \equiv \frac{1}{4} [\gamma_m, \gamma_n]. \tag{3.72} \]

This action is locally symmetric and contains the following transformations

\[ \delta e^m_\mu = \frac{1}{2M_P} e^m \gamma^\mu \psi_\mu, \tag{3.73} \]

\[ \delta \psi_\mu = M_P \left( \partial_\mu + \frac{1}{2} \omega^{mn}_\mu \sigma_{mn} \right) \epsilon, \tag{3.74} \]

\[ \delta \omega^{mn}_\mu = 0. \tag{3.75} \]
However, we need to consider more realistic theories of supergravity that include the coupling of matter and gauge fields to supergravity, and we discuss it below.

### 3.4.2 General Theories of Supergravity

Next, we introduce the essential features of general supergravity models with matter fields.\(^2\) The geometry of the scalar field space is described by the Kähler potential \(K(\Phi_i, \bar{\Phi}_j)\) \([182, 183]\), where the fields \(\Phi_i\) are complex scalar fields and the fields \(\bar{\Phi}_j\) are the complex conjugate fields. A general action is given by

\[
S = \int d^4x \sqrt{-g} \left( L_{\text{kin}} - V(\Phi_i, \bar{\Phi}_j) \right).
\]

(3.76)

In theories of supergravity, the kinetic terms of the Lagrangian are characterized by the following expression

\[
L_{\text{kin}} = K^{ij} \partial_\mu \Phi_i \partial^\mu \bar{\Phi}_j,
\]

(3.77)

where \(K^{ij} = \partial^2 K / \partial \Phi_i \partial \bar{\Phi}_j\) is known as the Kähler metric. Therefore, we can rewrite the action (3.76) as:

\[
S = \int d^4x \sqrt{-g} \left( K^{ij} \partial_\mu \Phi_i \partial^\mu \bar{\Phi}_j - V(\Phi_i, \bar{\Phi}_j) \right).
\]

(3.78)

We note that this general supergravity action is invariant under Kähler transformations

\[
K(\Phi_i, \bar{\Phi}_j) \rightarrow K(\Phi_i, \bar{\Phi}_j) + f(\Phi_i) + \bar{f}(\bar{\Phi}_j),
\]

(3.79)

\[
W(\Phi_i) \rightarrow e^{-f(\Phi_i)} W(\Phi_i).
\]

(3.80)

To introduce the gauge field interactions into our theory of supergravity, we need to introduce the superpotential \(W\), which is a holomorphic function of \(\Phi_i\). We define the Kähler function as:

\[
G \equiv K + \ln W + \ln \bar{W},
\]

(3.81)

where in this expression we have set \(M_P = 1\). Using \(G\), we can construct its derivatives with respect to the scalar fields and complex conjugate fields: \(G^i = \partial G / \partial \Phi_i\); \(G^\bar{i} = \partial G / \partial \bar{\Phi}_{\bar{i}}\).
\[ \partial G / \partial \bar{\Phi}_i; \text{ and } G^{i\bar{j}} = \partial^2 G / \partial \mu_i \partial^\mu \bar{\Phi}_j. \]

It is important to note that the Kähler metric \( G^{i\bar{j}} = K^{i\bar{j}} \) does not depend on the superpotential \( W \).

To derive the effective scalar potential in supergravity, we introduce the bosonic action of the \( \mathcal{N} = 1 \) supergravity Lagrangian \([39, 40, 42, 41]\)

\[
e^{-1} \mathcal{L}_B = -e^G \left( G^{i\bar{j}} G^{i\bar{j}} - 3 \right) - \frac{1}{2} g^2 \text{Re} f_{ab}^{-1} \left( G^{i\bar{j}} T^{\alpha\beta}_i \phi_j \right) \left( G^{k\bar{l}} T^{\mu\nu}_k \phi_l \right) - \frac{1}{4} \text{Re} f_{ab} F_{\mu\nu}^a F^{b\mu\nu} + i \frac{1}{4} \text{Im} f_{ab} F_{\mu\nu}^a \tilde{F}^{b\mu\nu} + G^{i\bar{j}} D_i \phi_j - \frac{1}{2} R, \tag{3.82}
\]

where \( K_{i\bar{j}} \) is the inverse Kähler metric, and the remaining parts of the full \( \mathcal{N} = 1 \) supergravity Lagrangian, including the fermionic Lagrangian, are given in Appendix B.

In supergravity, the \( F \)-term contribution to the effective scalar potential is given by

\[
V_F = e^G \left[ G^{i\bar{j}} G^{i\bar{j}} - 3 \right], \tag{3.83}
\]

or in another form

\[
V_F = K^{i\bar{j}} F_i F_j - 3 e^K |W|^2, \tag{3.84}
\]

where

\[
F_i = -e^{G/2} G^{i\bar{j}} G^{\bar{j}i} = -e^K K_{i\bar{j}} (\bar{W}^{\bar{j}} + \bar{W} K^{\bar{j}}). \tag{3.85}
\]

If we include the \( D \)-term contributions from gauge interactions, we find that the effective scalar potential can be expressed as

\[
V_D = \frac{1}{2} \text{Re} [f_{ab} \tilde{D}^a \tilde{D}^b], \tag{3.86}
\]

where \( f_{ab} \) is the gauge kinetic function that is dimensionless and for non-abelian fields, \( f_{ab} \propto \delta_{ab} \), \( \tilde{D}^a = f_{ab}^{-1} \tilde{D}^b \), and

\[
\tilde{D}^a \equiv -G^i (T^a)_i \phi_j = -\tilde{\phi}_j (T^a)_i \beta_i = -K^i (T^a)_i \phi_j = -\tilde{\phi}_j (T^a)_i K^i, \tag{3.87}
\]

are real parameters of supersymmetry breaking. Therefore, the full effective scalar potential is given by

\[
V = V_F + V_D. \tag{3.88}
\]

For most of our calculations, we use the following \( \mathcal{N} = 1 \) supergravity effective scalar
potential form

\[ V = e^K \left[ (K_{i\bar{j}}D^iW\bar{D}^j\bar{W} - 3|W|^2) + \frac{g^2}{2} (\text{Re} f^{-1}_{ab}) K^i (T^a)^i_{\bar{j}} \phi_j K^l (T^b)^l_{\bar{k}} \phi_k \right], \quad (3.89) \]

where the Kähler derivatives are defined as

\[ D^i W = K^i W + W^i. \quad (3.90) \]

### 3.5 Supersymmetry Breaking

In both realistic supersymmetry and supergravity theories, one must incorporate supersymmetry breaking which ensures that particles and their corresponding sparticles do not have degenerate masses. From our experience with the quantum field theory, we expect that supersymmetry breaking is spontaneous. Furthermore, the Lagrangian density must be invariant under supersymmetry, but its vacuum state is not. This would ensure that supersymmetry is hidden from low-energy collider experiments.

#### 3.5.1 Supersymmetry Breaking with Global Symmetry

To maintain naturally the hierarchy between the electroweak and some high energy mass scale \( \Lambda_{\text{UV}} \), one needs to introduce soft supersymmetry-breaking terms (positive mass dimension). It means that dimensionless supersymmetry-breaking couplings should not appear in the Lagrangian density. Therefore, we introduce soft supersymmetry breaking, which is described by the following Lagrangian

\[ \mathcal{L} = \mathcal{L}_{\text{SUSY}} + \mathcal{L}_{\text{soft}}, \quad (3.91) \]

where \( \mathcal{L}_{\text{SUSY}} \) contains all the Yukawa and gauge interactions and preserves supersymmetry invariance, and \( \mathcal{L}_{\text{soft}} \) violates supersymmetry and contains only terms with positive mass dimension. The Lagrangian density form (3.91) leads to Higgs radiative corrections, given by \( \delta m^2_H \propto m^2_{\text{soft}} \), which vanishes in the limit \( m_{\text{soft}} \to 0 \), and quadratically divergent terms always maintain the cancellation. Most importantly, \( m_{\text{soft}} \) also characterizes the mass splitting between the Standard Model particles and their superpartners.

We introduce the general form of the soft supersymmetry-breaking terms in the
Lagrangian

\[ \mathcal{L}_{\text{soft}} = -\left( \frac{1}{2} M^a \lambda^a + \frac{1}{6} a^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + t^i \phi_i \right) + \text{h.c.} - (m^2)^i_j \phi^j \phi_i, \tag{3.92} \]

where \( M^a \) are the gaugino masses, \( (m^2)^j_i \) and \( b^{ij} \) are the scalar squared-mass terms, \( a^{ijk} \) and \( c^{jk} \) are the trilinear couplings, and \( t^i \) are the 'tadpole' couplings. We note that in MSSM the tadpole couplings do not exist since that would require \( \phi_i \) to be a gauge singlet. It has been shown by [184] that the Lagrangian density (3.92) is free of quadratic divergences related to the scalar masses.

We specify the soft supersymmetry breaking terms for the MSSM. The gaugino masses are given by

\[ \mathcal{L}_{\text{soft, gaugino}}^{\text{MSSM}} = -\frac{1}{2} \left( M_3 \bar{g} g + M_2 \bar{W} W + M_1 \bar{B} B + \text{h.c.} \right), \tag{3.93} \]

where \( M_3, M_2, \) and \( M_1 \) are the gluino, wino, and bino masses, respectively. The Lagrangian density for squared mass terms is \( \delta \mathcal{L} = -\delta m^2 |\phi|^2 + (B_0 W_2(\phi) + \text{h.c.}) \), where the bilinear couplings that arise from the quadratic contribution and couple to the quadratic part of the superpotential \( W_2(\phi) \) are called the \( B\)-terms. In this case, the Lagrangian density is

\[ \mathcal{L}_{\text{soft, quadratic}}^{\text{MSSM}} = -\bar{Q}^{\dagger} m_Q^2 Q - \bar{L}^{\dagger} m_L^2 L - \bar{\tilde{u}} m_u^2 \tilde{u} - \bar{d} m_d^2 \tilde{d} - \bar{e} m_e^2 \tilde{e} \]

\[ -m_{H_u}^2 H_u^* H_u - m_{H_d}^2 H_d^* H_d - (B_0 \mu H_u H_d + \text{h.c.}). \tag{3.94} \]

Here \( m_Q^2, m_L^2, m_U^2, m_D^2, m_E^2 \) are the \( 3 \times 3 \) Hermitian matrices in family space. Finally, we introduce the \( A\)-terms that are the proportional to the cubic part of the superpotential \( W_3(\phi) \), \( \delta \mathcal{L} = A_0 (W_3(\phi) + \text{h.c.}) \), with the full Lagrangian density

\[ \mathcal{L}_{\text{soft, cubic}}^{\text{MSSM}} = -\left( \bar{u} A_u \bar{Q} H_u - \bar{\tilde{d}} A_d \bar{\tilde{Q}} H_d - \bar{\tilde{e}} A_e \bar{\tilde{L}} H_d + \text{h.c.} \right). \tag{3.95} \]

Here, \( A_u, A_d, A_e \) are the Hermitian \( 3 \times 3 \) matrices in family space of dimension [mass].
We summarize the results below:

\[ M_1, M_2, M_3, A_u, A_d, A_e \sim m_{\text{soft}}, \]
\[ m_Q^2, m_L^2, m_{\tilde{u}}^2, m_{\tilde{d}}^2, m_{\tilde{e}}^2, m_{H_u}^2, m_{H_d}^2, B_0 \sim m_{\text{soft}}^2. \]  

(3.96)

Here we do not discuss the subtleties of the renormalization group equations (RGEs) and unification of gauge couplings in the MSSM, and they can be found in [156, 8, 185, 9, 10, 11, 186].

Next, we explore the aspects of global supersymmetry breaking. As we discussed before, in MSSM the supersymmetry is broken explicitly. One can readily show that the unbroken global supersymmetry immediately leads to a Minkowski vacuum

\[ H = P^0 = \frac{1}{4} \left( Q_1 Q_1^\dagger + Q_2 Q_2^\dagger + Q_1^\dagger Q_2^\dagger + Q_2^\dagger Q_1^\dagger \right), \]  

(3.97)

and for unbroken supersymmetry we find \( H \langle 0 \rangle = E_{\text{vac}} = 0 \). However, if the supersymmetry is broken, we use the commutation relation (3.9) and obtain

\[ E_{\text{vac}} = \langle 0 | P^0 | 0 \rangle = \frac{1}{4} \sum_{\alpha} \langle 0 | \{ Q_\alpha, \bar{Q}_\alpha \} | 0 \rangle = \frac{1}{4} \sum_{\alpha} \langle 0 | Q_\alpha \bar{Q}_\alpha + \bar{Q}_\alpha Q_\alpha | 0 \rangle \geq 0, \]  

(3.98)

which shows that the vacuum state energy is always positive. This condition implies that vanishing vacuum energy leads to unbroken global supersymmetry. Most importantly, we show in the next subsection that this condition is not necessary for local supersymmetry breaking.

The spontaneous symmetry breaking of global symmetry implies the existence of a massless Nambu-Goldstone boson, together with its supersymmetric massless Weyl fermion superpartner, known as goldstino. Here we do not discuss all the details related to global supersymmetry breaking and we only focus on \( F \)-term supersymmetry breaking. For a more detailed discussion on the Fayet-Iliopoulos (\( D \)-term) supersymmetry breaking, see [187, 188, 156].

Models, where the supersymmetry breaking occurs due to a non-vanishing \( F \)-term VEV, are known as O’Raifeartaigh (\( F \)-term) supersymmetry breaking models [189]. Typically, in such models, the \( D \)-term is set to zero, and the effective scalar potential \( V = \sum_i |F_i|^2 \) will have a positive minimum with broken supersymmetry.
The simplest example includes the following superpotential form

$$W = -k\Phi_1 + m\Phi_2\Phi_3 + \frac{y}{2}\Phi_1\Phi_3^2.$$  \hspace{1cm} (3.99)

We note that this superpotential has a linear term which is necessary to ensure the \textit{F}-term breaking at tree-level. We choose \(k, m\), and \(y\) to be real and positive. Combining the superpotential (3.99) with Eq. (3.57), we find

$$V_{\text{tree-level}} = |F_1|^2 + |F_2|^2 + |F_3|^2,$$

$$F_1 = k - \frac{y}{2}\phi_3^2, \quad F_2 = -m\phi_3^*, \quad F_3 = -m\phi_2^* - y\phi_1^*\phi_3^*.$$  \hspace{1cm} (3.100)

Because one cannot have \(F_1 = F_2 = 0\) simultaneously, it implies that supersymmetry is broken and \(V_{\text{tree-level}} > 0\). If \(m^2 > yk\), then the absolute minimum of the scalar effective potential is at \(\phi_2 = \phi_3 = 0\) and \(\phi_1\) is undetermined, and we refer to it as a \textit{flat direction} in the scalar potential. These flat directions can be removed by introducing one-loop quantum corrections [190].

### 3.5.2 Supersymmetry Breaking with Local Symmetry

Next, we discuss the supersymmetry breaking in theories of supergravity. From the effective scalar potential (3.84) it can be seen that we no longer need to satisfy the inequality \(E_{\text{vac}} > 0\) to break local supersymmetry, which was a requirement for global supersymmetry breaking, and now the ground state could be negative.

From Noether’s theorem, we find the following expression for the conserved supercurrent

$$J^\mu_{\alpha} = (\sigma^\nu\bar{\sigma}^\mu\psi_i)_\alpha \nabla_\nu\bar{\phi}^i + i \left(\sigma^\mu\psi^i\right)_\alpha \bar{W}^i - \frac{1}{2\sqrt{2}} \left(\sigma^\nu\bar{\sigma}^\rho\sigma^\mu\lambda^a\right)_\alpha F^a_{\nu\rho} + \frac{i}{\sqrt{2}} g_4\bar{\phi}T^a\phi \left(\sigma^\mu\lambda^a\right)_\alpha,$$  \hspace{1cm} (3.101)

and if we assume that the only non-vanishing contribution arises from an \(F\)-term, \langle \mathcal{F} \rangle, with goldstino superpartner \(\tilde{G}\), the supercurrent implies

$$0 = \partial_\mu J^\mu_{\alpha} = -i\langle \mathcal{F} \rangle (\sigma^\mu\partial_\mu\tilde{G}^\dagger)_\alpha + \partial_\mu J^\alpha + \cdots,$$  \hspace{1cm} (3.102)

where \(J^\mu_{\alpha}\) is the supercurrent of other supermultiplets, and the ellipses are the remaining
goldstino supermultiplet contributions to $\partial_{\mu}J_{\alpha}^{\mu}$. This expression implies the following goldstino equation of motion

$$\mathcal{L}_{\text{goldstino}} = i\bar{G}^i \sigma^\mu \partial_{\mu} G_i - \frac{1}{\langle F \rangle} (\bar{G} \partial_{\mu} j^\mu + \text{h.c.}), \quad (3.103)$$

which characterizes the goldstino interactions with fermion-boson pairs.

For supergravity models, when supersymmetry is unbroken, the graviton and gravitino are both massless. However, once the spontaneous supersymmetry breaking occurs, the gravitino ‘eats’ a goldstino, which corresponds to its longitudinal modes with the helicity $\pm 1/2$. This is known as the super-Higgs mechanism. Therefore, gravitino acquires a mass and now has four helicity states (two helicity states arise from the “would-be goldstino”). The gravitino mass is denoted $m_{3/2}$ and for $F$-term breaking models it is proportional to [191, 40]

$$m_{3/2} \sim \langle F \rangle. \quad (3.104)$$

Another way to understand the local symmetry breaking is to assume that fields that acquire a VEV are gauge singlets. It implies that local supersymmetry is broken if

$$- \sqrt{2} e^{G/2} G_{ij} G^j \epsilon(x) \neq 0, \quad (3.105)$$

which then leads to a more simple condition

$$\langle G^i \rangle \neq 0. \quad (3.106)$$

Therefore, the supergravity formulation allows obtaining a Minkowski vacuum with broken local supersymmetry. To summarize, when we have broken global supersymmetry, we acquire a massless goldstino, while for broken local supersymmetry, the super-Higgs mechanism ensures that the massless goldstino mode becomes the longitudinal component of massive gravitino.

We define Goldstino as

$$\eta = G^i \chi_i, \quad (3.107)$$

and the gravitino and the chiral fermion masses and kinetic terms can be expressed
\[ e^{-1} \mathcal{L}_F^C = -\frac{e^{-1}}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma + \frac{i}{2} e^{G/2} \bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu + \frac{i}{\sqrt{2}} e^{G/2} \bar{\psi}_\mu \gamma^\mu \eta \\
+ \frac{1}{3} \left( \frac{i}{2} \bar{\eta}^\mu \partial_\mu \eta - e^{G/2} \bar{\eta} \eta \right) + \frac{i}{2} \left( G^{ij} - \frac{1}{3} G^i G^j \right) \bar{\chi}_i \gamma^\mu \partial_\mu \chi_j \]
\[ + \frac{1}{2} e^{G/2} \left( -G^{ij} - \frac{1}{3} G^i G^j + G^{ijk} G_{ik} G^j \right) \bar{\chi}_i \chi_j + \cdots. \tag{3.108} \]

From this expression, it can be readily shown that for a Minkowski vacuum \( V = 0 \) and \( V^i = 0 \), there are no contributions to the Goldstino mass from the subtracted fermion mass matrix. If we use the field redefinition

\[ \psi'_\mu = \psi_\mu - \frac{i}{3\sqrt{2}} \gamma_\mu \eta - \frac{\sqrt{2}}{3} e^{-G/2} \partial_\mu \eta, \tag{3.109} \]

where \( \psi'_\mu \) is now associated with the massive spin-3/2 field, the Lagrangian density (3.108) becomes

\[ e^{-1} \mathcal{L}_F^C = -\frac{e^{-1}}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}'_\mu \gamma_5 \gamma_\nu \partial_\rho \psi'_\sigma + \frac{i}{2} e^{G/2} \bar{\psi}'_\mu \sigma^{\mu\nu} \psi'_\nu + \frac{i}{\sqrt{2}} e^{G/2} \bar{\psi}'_\mu \gamma^\mu \eta \\
+ \frac{1}{3} \left( \frac{i}{2} \bar{\eta}'^\mu \partial_\mu \eta - e^{G/2} \bar{\eta}' \eta \right) + \frac{i}{2} \left( G^{ij} - \frac{1}{3} G^i G^j \right) \bar{\chi}_i \gamma^\mu \partial_\mu \chi_j \]
\[ + \frac{1}{2} e^{G/2} \left( -G^{ij} - \frac{1}{3} G^i G^j + G^{ijk} G_{ik} G^j \right) \bar{\chi}_i \chi_j + \cdots. \tag{3.110} \]

Therefore, we find that the mass of the gravitino is given by

\[ m_{3/2} = \langle e^{G/2} \rangle. \tag{3.111} \]

We apply the general supergravity formalism introduced in this section to our unified theories of no-scale supergravity, and we discuss it further in Chapter 4.

### 3.6 Problems in Supergravity

Now that we introduced the key concepts of supersymmetry and supergravity, we are ready to study their role in cosmology. Most importantly, the supergravity framework allows us to discuss the inflationary epoch in a supersymmetric context. Typically, the inflationary scale is of order \( \sim \mathcal{O}(M_P^{-5}) \), and quantum gravity effects could potentially
be significant. Therefore, the inflationary models in the supergravity framework must have a gauged supersymmetry.

Since any generic supersymmetric theory has a plethora of flat directions, after supersymmetry breaking they are lifted and acquire a slope. Importantly, the fields that have flat directions could potentially be a good inflaton field candidate but this typically leads to a well-known problem in models of supergravity, known as the $\eta$ problem, and we discuss it below.

### 3.6.1 The $\eta$ problem

The $\eta$ problem arises in any kind of general theory of supergravity. Since the effective scalar potential contains a factor $e^K$, the squared scalar field masses are $\propto H^2 \propto V$ [71, 72]. Typically, a generic supergravity inflationary model has problems associated with the inflaton mass that is too large.

As we discussed in Chapter 2, the most recent Planck measurements of the CMB require the inflationary potential to be flat. The simplest way to explain the $\eta$ problem is with an example. Let’s consider the following Kähler potential form

$$K = \phi \bar{\phi} + \cdots,$$  

(3.112)

and the effective scalar potential (3.83) becomes

$$V = e^{\vert \phi \vert^2 \times}$$

$$\{ [W_\phi + W(\bar{\phi} + \cdots)] (1 + \cdots) [\bar{W}_\phi + \bar{W}(\phi + \cdots)] - 3\vert W \vert^2 \}$$  

(3.113)

Therefore, when $\phi = 0$, the exponential term in the scalar potential contributes to the effective squared mass of all scalar fields in the theory. It implies that

$$\frac{V''}{V} = 1 + \cdots,$$  

(3.114)

but the slow-roll inflation requires that $\vert V''/V \vert \ll 1$. Thus, a viable model of inflation must ensure that there is a cancellation between the exponential term and the terms inside the curly brackets.
It can also be applied to any generic theory of supergravity with an arbitrary Kähler potential. The effective scalar potential (3.83) implies that the slow-roll parameters (2.46) and (2.47) are given by

$$\epsilon_V = \frac{1}{2} (K_\Phi + \cdots)^2, \quad \eta_V = K_{\Phi\bar{\Phi}} + \cdots,$$

(3.115)

where here the subscript shows the differentiation with respect to the inflaton field, $\phi$. Therefore, it is clear from the above equations that models based on minimal supergravity lead to $\eta$ problem.

One of the viable solutions to the $\eta$ problem is to consider the Kähler potential that contain a shift symmetry, for example $K \propto (\Phi - \bar{\Phi})^2$. In such a case, the exponential term $e^K$ no longer depends on the linear combination $\Phi + \bar{\Phi}$, which is associated with the inflaton field [193], and the dangerous contribution is avoided. It is also possible to avoid the $\eta$ problem by considering the non-minimal Kähler potentials [72].

3.6.2 Gravitino Problem

During reheating, gravitinos and light weakly-interacting fields (moduli) are typically produced via the scattering and decay of particles in the thermal bath [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67]. Since these fields are quite light, they usually decay very late and produce too much entropy, which leads to various cosmological problems. One of the most important consequences is that they can begin dominating the total energy density of the universe and dilute the existing baryon asymmetry. Another important outcome is that if they decay too late, they lead to the overproduction of the lightest supersymmetric particle (LSP) [194, 195].

Another way to understand why the gravitino poses important cosmological problems in supergravity models can be related to the mass of the gravitino. If the gravitino is the lightest supersymmetric particle (LSP), then the relic gravitino abundance could exceed the allowed cold dark matter density, depending on its mass. On the other hand, if the gravitino is not the LSP, the fact that the gravitino couplings to other particles are Planck-suppressed implies that its lifetime for decays into other particles may be quite long: $\tau \sim M_P^2/m_{3/2}^3$. In this case, the mass and abundance of the gravitino are constrained by experimental limits on late-decaying particles, particularly the ones arising
from the Big Bang nucleosynthesis (BBN) [63, 68, 69, 196, 197, 198, 199].

Another problem associated with the moduli fields arises because when the supersymmetry is broken, the vacuum expectation value of the moduli field shifts from zero, and the moduli field subsequently starts oscillating about a new non-zero minimum. The energy density of the oscillating moduli behaves like matter and the decay of this oscillating field leads to a significant release of entropy

\[ \frac{s_f}{s_i} \sim \frac{M_P}{m_{3/2}}, \]  

(3.116)

where \( s_i, f \) are the entropy densities before and after the moduli decay. Usually, this additional contribution to the total entropy density washes out the baryon asymmetry.

We include some examples and discuss the gravitino and Polonyi problem in detail in the next chapter, and in Chapter 8.
Chapter 4

Building Models of No-Scale Supergravity

In this chapter, we discuss how to construct the models of supergravity, that contain a plateau-like inflationary model and avoid successfully the $\eta$ and gravitino problems. Afterward, we argue that the appropriate framework is that of no-scale supergravity since it avoids the aforementioned cosmological problems. We then show how to construct within this framework inflationary models whose predictions for the tilt in the spectrum of scalar perturbations, $n_s$, and the ratio, $r$, of tensor and scalar perturbations coincide with those of the $R + R^2$ model of inflation first proposed by Alexei Starobinsky back in 1980 [84].

4.1 Early Models of Supergravity

As we discussed in Chapter 2, most early formulations of inflationary models were based on one or multiple effectively elementary scalar fields, though a proposal by Starobinsky was based on an extension of the Einstein-Hilbert action - which is linear in the Ricci tensor $R$ - to include an additional $R^2$ term. However, it was subsequently noticed that the $R + R^2$ model is equivalent to the Einstein-Hilbert action complemented by a scalar field. This connection can be shown using the conformal transformations, although it is important to note a very specific effective potential form is needed [200, 201]. Typically, in scalar field models, one finds that the change in the effective scalar inflaton field is
\( \sim \mathcal{O}(M_P) \). It implies that some assumptions related to gravity are necessary for their formulation.

In Chapter 2, we also discussed the phenomenological interpretation of models of inflation that became clear with the realization that the scalar inflaton field would be subject to quantum fluctuations. At the time, observational upper limits on possible perturbations in the cosmological microwave background (CMB) already imposed strong constraints on quantum fluctuations, which implied that the inflaton mass must be significantly below the Planck scale.

This important finding posed a cosmological hierarchy problem: what physical mechanism could explain that the inflaton mass is \( \ll M_P \), which resembles the electroweak hierarchy problem. It was later suggested that this inflationary problem requires stabilization by supersymmetry [202, 29, 203, 204].

Many inflationary models share similar generic features arising from quantum fluctuations. The perturbations in the scalar inflaton field are expected to be predominantly Gaussian in all models whose inflaton field rolls slowly down its effective potential, their spectrum would not in general be scale-invariant but exhibit a small tilt, \( n_s \neq 1 \), and the gravitational background would also be subject to small tensorial fluctuations, with a ratio \( r \) relative to the larger scalar perturbations.

Since supergravity is a natural extension of supersymmetry that incorporates gravity, it is the appropriate framework to study inflation and cosmology in a supersymmetric context. Therefore, it is important to understand the recipe that can be used to write down supersymmetric models of inflation that are compatible with the CMB data. Most importantly, how can this be achieved within the supergravity framework?

However, building such models could be challenging from a theoretical perspective. Typically, generic supergravity models with matter fields do not contain a plateau-like effective scalar potential that would satisfy the slow-roll conditions, and usually have an anti-de Sitter (AdS) vacua with field energies \( -\mathcal{O}(m_{3/2}^2 M_P^2) \), where \( m_{3/2} \) is the gravitino mass. Another issue is that any inflationary model should be compatible with ultraviolet completion in some string models. How can we construct such models of supergravity that avoid all of these problems?

One of the most successful supergravity models that overcome these issues are known as *no-scale supergravity models* [43, 73, 74, 74, 75, 76]. They are named so because their
effective potentials feature directions with no specific dynamical scale selected at the classical tree-level, which have been derived from string models as their effective low-energy theories [77, 78]. In this chapter, we review the key aspects related to the construction of no-scale models of supergravity and the discussion presented here is primarily based on [205].

4.2 Minimal Supergravity

Here we extend our discussion presented in Chapter 3. We recall the bosonic Lagrangian of a generic $\mathcal{N} = 1$ theory of supergravity up to second order in the derivatives of the physical scalar fields

$$ e^{-1} \mathcal{L}_B \supset -\frac{1}{2} R + G^{ij} \partial_\mu \phi_i \partial^\mu \phi_j - V - \frac{1}{4} \text{Re} f_{ab} F^a_{\mu\nu} F^{b\mu\nu} - \frac{1}{4} \text{Im} f_{ab} F^a_{\mu\nu} \tilde{F}^{b\mu\nu}, \quad (4.1) $$

where the first term is the usual Einstein-Hilbert term, $V$ is the effective scalar potential given by (3.83), and $f_{ab}$ is the gauge kinetic function, which is in general a function of the chiral fields. The full bosonic action is given by Eq. (3.82). We note that we do not include the $D$-term contribution for gauge non-singlet chiral fields.

**Minimal supergravity (mSUGRA)** is characterized by a Kähler potential of the form

$$ K = |\phi_i|^2, \quad (4.2) $$

in which case the effective potential (3.83) can be written in the form

$$ V(\phi_i, \bar{\phi}_j) = e^{\phi_i} \left[ |W_i + \bar{\phi}_j W|^2 - 3|W|^2 \right], \quad (4.3) $$

where $W_i \equiv \partial W / \partial \phi^i$.

From Eq. (4.3), we can see that, unlike the effective scalar potential in global supersymmetry, where $V = |W_i|^2$, the minimal supergravity potential is not positive semi-definitive. Indeed, the negative term $\propto |W|^2$ in (4.3) generates in general AdS vacua with depth $-O(m_{3/2}^2 M_p^2)$, inducing a cosmological instability. More generally, Eq. (4.3) does not have flat directions in field space except under special conditions such as those we discuss below. This is an issue for constructing models of inflation because,
as we discuss in more detail below, a period of inflation that is long enough to solve the horizon and flatness problems should satisfy slow-roll conditions that require the scalar potential to have a flat direction. As we discussed before, the minimal supergravity also leads to the $\eta$ problem.

Restricting our attention to theories of inflation in the context of supersymmetry, for the reasons discussed earlier we focus on supergravity models. These include some Planck-scale effects which may be important for inflation and, as already discussed, make possible the breaking of supersymmetry while (almost) canceling the cosmological constant.

We begin with the simplest such model, which is based on a single chiral superfield, $\phi$, the inflaton [58], in minimal supergravity with $K = \phi \bar{\phi}$. One can consider a general polynomial form for the superpotential [206], the simplest being [207]

$$W = M^2 (1 - \phi)^2, \tag{4.4}$$

which leads to

$$V = M^4 e^{4|\phi|^2} \left[ 1 + |\phi|^2 - (\phi^2 + \bar{\phi}^2) - 2|\phi|^2(\phi + \bar{\phi}) + 5|\phi|^4 + |\phi|^6 \right]. \tag{4.5}$$

This can be expanded about the origin in the real direction $\phi = \bar{\phi}$ to give

$$V \simeq M^4 \left( 1 - 4\phi^3 + \frac{13}{2} \phi^4 + \ldots \right), \tag{4.6}$$

which is shown in Fig. 4.1. This is an example of new inflation [26, 27] driven by the cubic term, and the mass scale $M \sim 10^{-4}$ is determined by the normalization of the CMB fluctuation spectrum [203]. We note that, although the theory defined by (4.4) is constructed to avoid the $\eta$ problem, a generic inflationary model is in general plagued by the problem of large masses. This simple model is an example of “accidental” inflation [208], as the ratio of the constant and linear terms in (4.4) must be equal to 1 to very high accuracy to avoid the $\eta$ problem.

It is possible to construct many more examples of inflationary models by adding an
auxiliary chiral field, \( S \) \([193, 209, 210, 211]\). Consider, for example

\[
K = S \bar{S} - \frac{1}{2} (\phi - \bar{\phi})^2, \tag{4.7}
\]

which can still be viewed as minimal because \( K^{ij} = \delta^{ij} \). Then, for the simple choice

\[
W = S f(\phi), \tag{4.8}
\]

one finds

\[
V = |f(\phi)|^2, \tag{4.9}
\]

for \( S = \text{Im} \phi = 0 \), and one can easily generate any scalar potential that is a perfect square.

### 4.3 No-Scale Supergravity

Next, we discuss no-scale supergravity models, of which the simplest is the single-field example with \([74]\)

\[
K = -3 \ln(T + \bar{T}), \tag{4.10}
\]
where $T$ may be identified with the volume modulus in a string compactification. This leads to the Einstein-Hilbert Lagrangian for gravity accompanied by the following kinetic term for the modulus field, derived from Eq. (4.1)

$$L_{\text{kin}} = \frac{3}{(T + \bar{T})^2} \partial^\mu T \partial_\mu \bar{T} = \frac{1}{12} (\partial_\mu K)^2 + \frac{3}{4} e^{2K/3} |\partial_\mu (T - \bar{T})|^2, \quad (4.11)$$

where we note that, up to a factor of $\sqrt{6}$, $K$ has a canonical kinetic term. In the absence of a superpotential for the modulus, the effective scalar potential vanishes

$$V = 0, \quad (4.12)$$

which satisfies trivially the flatness condition, in particular the absence of negative-energy AdS solutions.

In the minimal no-scale model (4.2), the single complex field $T$ parametrizes a non-compact SU(1,1)/U(1) coset space. It can be generalized by including matter fields $\phi^i$ that parametrize, together with $T$, an SU(N,1)/SU(N) × U(1) coset space, defined by the Kähler potential [75] \(^1\)

$$K = -3 \ln (T + \bar{T} - |\phi_i|^2/3). \quad (4.13)$$

In this case, we find the following scalar-field Lagrangian

$$L = \frac{1}{12} (\partial_\mu K)^2 + e^{K/3} |\partial_\mu \phi^i|^2 + \frac{3}{4} e^{2K/3} |\partial_\mu (T - \bar{T})| + \frac{1}{3} (\bar{\phi}_i \partial_\mu \phi^i - \phi^i \partial_\mu \bar{\phi}_i)^2 - V, \quad (4.14)$$

where the effective scalar potential can be written as

$$V = e^{\frac{2}{3}K} \hat{V} = \frac{\hat{V}}{((T + \bar{T}) - \frac{1}{3} |\phi^i|^2)^2}, \quad (4.15)$$

\(^1\)There are other generalizations based on other non-compact coset spaces, which also appear in some string models.
with

\[ \hat{V} \equiv |W_i|^2 + \frac{1}{3}(T + \bar{T})|W_T|^2 + \frac{1}{3}\left(W_T(\bar{\phi}_iW^i - 3) + \text{h.c.}\right). \tag{4.16} \]

We see that when \( W_T = 0 \), the potential takes a form related to that in global supersymmetry, though with a proportionality factor of \( e^{2K/3} \), as seen in Eq. (4.15), where \( K \) is the canonically-redefined modulus. Hence large mass terms are not generated [212], and the \( \eta \) problem is avoided [213].

### 4.4 Supersymmetry Breaking

Next, we extend our discussion on supersymmetry breaking in models of supergravity. The simplest example of a model that breaks supersymmetry and allows \( V = 0 \) is the Polonyi model [214]. The model is based on adding a single chiral superfield that breaks supersymmetry spontaneously through the super-Higgs mechanism [40, 215], which has two physical scalar fields whose fermionic partners are eaten by the gravitino. In the simplest version of the model, the superpotential is separable in the Polonyi field, \( z \), and the matter fields, \( \phi^i \),

\[ W(z, \phi^i) = f(z) + g(\phi^i), \tag{4.17} \]

with the particular choice

\[ f(z) = \mu(z + \zeta), \tag{4.18} \]

where \( \zeta \) is a constant. Ignoring for the moment the matter fields, the potential for \( z \) is

\[ V(z, \bar{z}) = e^{z\bar{z}}\mu^2 \left[1 + \bar{z}(z + \zeta)|^2 - 3(z + \zeta)^2\right], \tag{4.19} \]

where we have assumed a minimal supergravity framework, i.e., \( K = z\bar{z} \). Minimizing the potential and requiring that \( V(\langle z \rangle) = 0 \) (which needs fine-tuning), we find that the VEV of the Polonyi field is \( \langle z \rangle = \sqrt{3} - 1 \) and \( \zeta = 2 - \sqrt{3} \). The Polonyi potential is shown in Fig. 4.2.

In this example, the masses of the two real scalars, denoted by \( A \) and \( B \), are

\[ m_A^2 = 2\sqrt{3}m_{3/2}^2, \quad m_B^2 = 2(2 - \sqrt{3})m_{3/2}^2, \tag{4.20} \]
where the gravitino mass

\[ m_{3/2} = e^{(G)/2} = e^{2-\sqrt{3}} \mu. \]  

These satisfy the mass relation \( m_A^2 + m_B^2 = 4m_{3/2}^2 \), which is a consequence of the supertrace formula in supergravity [40].

Including now the matter fields, one can calculate their soft supersymmetry-breaking mass terms [216] by evaluating the potential at \( \langle z \rangle \) and dropping terms in the potential that are of dimension higher than four, as these would be suppressed by the Planck mass. The scalar potential then becomes

\[ V = m_{3/2}e^{(2-\sqrt{3})} \left( \phi^i \frac{\partial g}{\partial \phi^i} - \sqrt{3}g + \text{h.c.} \right) + m_{A/2}^2 \phi^i \bar{\phi}_i. \]  

Rescaling the superpotential by a factor \( e^{\sqrt{3}-2} \), and noting that \( \sum \phi \partial g/\partial \phi = 3g \) for trilinear terms and \( \sum \phi \partial g/\partial \phi = 2g \) for bilinear terms, we can read off the soft masses

\[ m_0 = m_{3/2}, \quad B_0 = (2 - \sqrt{3})m_{3/2}, \quad A_0 = (3 - \sqrt{3})m_{3/2}, \]  

where \( m_0 \) is a universal soft scalar mass, \( A_0 \) is universal soft trilinear term, and \( B_0 \) is a universal soft bilinear term.
This simple paradigm for supersymmetry breaking has important consequences for the minimal supersymmetric extension of the Standard Model (MSSM). The soft masses in Eq. (4.23) represent universal boundary conditions for all scalar masses, $A$-terms, and $B$-terms. In the constrained MSSM (CMSSM) [217, 218, 219, 220, 221, 222], all scalars are assumed to be universal at some high energy scale often taken to be the GUT scale, $A$-terms are left free but universal, and the $B$-term (there is only one in the MSSM) is obtained from the minimization of the Higgs potential when $\tan \beta$ (the ratio of the two Higgs VEVs) is taken as a free parameter.

A supersymmetry-breaking gaugino mass requires a non-trivial gauge kinetic function for a canonically-normalized gauge field,

\[
m_{1/2} = \left| \frac{1}{2} e^{G/2} \frac{f_i}{\text{Re} f} (G^{-1})^j G^j \right|,
\]

(4.24)

where we have assumed a universal gauge kinetic function, $f_{\alpha \beta} = f \delta_{\alpha \beta}$.

In addition to the gaugino mass, $m_{1/2}$, $m_0$, $A_0$, and $\tan \beta$ make up the four continuous free input parameters of the CMSSM. The boundary conditions in Eq. (4.23) are more restrictive, as $m_0$, $A_0$, and $B_0$ are all determined by the gravitino mass. Indeed, the relation $B_0 = A_0 - m_0$, is a common feature of many models based on supergravity [80, 223].

However, this simple paradigm for supersymmetry breaking could still lead to potential problems. In particular, the potential shown in Fig. 4.2 has a serious cosmological problem [51, 224, 54, 52, 53] of excess entropy production. Since we expect the parameter $\mu$ to be of order the weak scale whereas the VEV of $z$ is of order the Planck scale, the potential is very flat. This means that if $z$ is displaced from its minimum after inflation (and we would expect an $\mathcal{O}(M_P)$ displacement), the subsequent evolution of $z$ would lead to huge entropy production. The problem appears when the $z$ field begins oscillating about its minimum, which occurs when the Hubble parameter drops to $H \sim m_z \sim \mu$, where $m_z$ corresponds to $m_A$ or $m_B$ in Eq. (4.20). At this time, the universe becomes matter-dominated by Polonyi oscillations until they decay when $H \sim \Gamma_z \sim m_z^3/M_P^2$. This leads to late reheating and an entropy increase by a factor $M_P/\mu \sim 10^{16}$. Furthermore, the late decay almost inevitably leads to an overproduction of cold dark matter in the form of the lightest supersymmetric particle (LSP) [194, 195].
We note, however, that the Polonyi problem can be alleviated by a mechanism of strong stabilization [55], and we discuss it below.

It is very easy to break supersymmetry in models of no-scale supergravity. Even in the minimal SU(1,1)/U(1) case, simply taking a constant superpotential, \( W = \mu \), leads to a non-zero gravitino mass:

\[
m_{3/2} = \frac{\mu}{(T + \bar{T})^{3/2}},
\]

whereas the scalar potential vanishes (as in Eq. (4.12)). Hence the magnitude of the gravitino mass is undetermined so long as the modulus \( T \) remains unfixed. On the other hand, in this case there is no supersymmetry breaking in the matter sector:

\[
m_0 = 0, \quad B_0 = 0, \quad A_0 = 0.
\]

We discuss later other mechanisms for breaking supersymmetry in the matter sectors of no-scale models, and how the Polonyi and gravitino problems may be avoided.

### 4.5 Inflationary Tests for Models of Supergravity

We can now test the simple models of inflation discussed in the previous section. For the potential determined by Eq. (4.4), the amplitude of density fluctuations (2.86) implies that \( M \sim 10^{-4} M_P \), as expected. In this case, the slow-roll parameter \( \epsilon \approx 3.6 \times 10^{-10} \) is very small, yielding a value of the tensor-to-scalar ratio that is allowed, but unobservably small. However, the scalar tilt in this model is \( n_s \approx 1 + 2\eta \approx 0.928 \), which is strongly excluded by the Planck data [28].

In contrast to the model discussed above, one of the first models of inflation, namely the Starobinsky model [84], yields a value of \( n_s \) that is in excellent agreement with observation, and a value of \( r \) that is testable in the next generation of CMB experiments. As originally written, the model was based on an \( R + R^2 \) theory of gravity. However, a suitable conformal transformation [200] brings the theory into the form of a theory

\footnote{We note that simply adding a constant superpotential in minimal supergravity does not break supersymmetry. Rather, minimization of the potential in this case leads to an AdS vacuum that preserves supersymmetry.}
with an Einstein-Hilbert action for gravity, and a canonical scalar field with a scalar potential of the form

\[ V = 3M^2 e^{-\frac{\sqrt{2} \phi}{3}} \sinh^2 \left( \frac{\phi}{\sqrt{6}} \right) = \frac{3}{4} M^2 \left(1 - e^{-\frac{\sqrt{2} \phi}{3}}\right)^2, \]  

(4.27)

as depicted in Fig. 4.3.

\[ \frac{V}{M^4} \]

\[ \begin{align*}
V/\text{M}^4 & \quad 1.0 \quad 0.8 \quad 0.6 \quad 0.4 \quad 0.2 \\
\phi & \quad 0 \quad 5 \quad 10
\end{align*} \]

Figure 4.3: The effective scalar potential in the Starobinsky model of inflation [84].

This model was the first to predict a slightly red scalar perturbation spectrum \((n_s < 1)\) [145, 146]. It is easy to determine in analytic form the slow-roll parameters for this potential [85]:

\[ A_s = \frac{3M^2}{8\pi^2} \sinh^4 \left( \frac{\phi}{\sqrt{6}} \right), \]  

(4.28)

\[ \epsilon = \frac{1}{3} \cosh^2 \left( \frac{\phi}{\sqrt{6}} \right) e^{-\frac{\sqrt{2} \phi}{3}}, \]  

(4.29)

\[ \eta = \frac{1}{3} \cosh^2 \left( \frac{\phi}{\sqrt{6}} \right) \left(2e^{-\frac{\sqrt{2} \phi}{3}} - 1\right). \]  

(4.30)

For \(N_\ast = 55\), we find \(M = 1.25 \times 10^{-5} M_P\), \(n_s = 0.965\), and \(r = 0.0035\). The line between the two green dots in Fig. 4.4 corresponds to the predictions of the Starobinsky model for \(N_\ast = 50\) to 60. We make the connection between this model and no-scale
Figure 4.4: Plot of the CMB observables $n_s$ and $r$. The red shadings correspond to the 68% and 95% confidence level regions from Planck data combined with BICEP2/Keck results [28]. The pairs of dots are the predictions of the $\alpha$-Starobinsky potential (7.60), discussed later, for $N_* = 50$ (left) and 60 (right). The upper (lower) pair of yellow (green) dots are the predictions when $\alpha = 100$ ($\alpha = 1$, corresponding to the Starobinsky model (4.27)), while the lower end of the swath represents the cosmological observables in the limit $\alpha \to 0$. The 68% upper bound $r_{0.002} \lesssim 0.041$, indicated by the blue star, is attained for $\alpha \sim 27$ when $n_s \sim 0.967$, for a nominal choice of $N_* \simeq 55$.

supergravity in the next section.

4.6 A No-Scale Starobinsky model

As already mentioned, the Starobinsky inflationary model based on the potential in Eq. (4.27) had its origins in higher-derivative gravity. However, it can be seen to arise rather simply and directly within the no-scale supergravity framework [85].

Our starting point is the no-scale supergravity scalar potential given in Eqs. (4.15) and (4.16). We see immediately that it is not possible to construct a Starobinsky-like model using only a single field, $T$. In that case, the scale invariance of the Starobinsky
potential at large field values would require a constant potential at large $T$, which is possible only if the superpotential scales as $W \sim T^{3/2}$ [225]. However, in that case, the leading term in $\dot{V}$ would be negative, $\dot{V} \sim -\frac{3}{2}T^2$. Therefore a minimal model requires two fields, which we take as $T$ and the inflaton, $\phi$.

For now, we assume that the modulus is fixed by some unspecified mechanism with $\langle T \rangle = 1/2$ for illustration (the value of $\langle T \rangle$ is unimportant and its stabilization is discussed below). Further, we postulate the following Wess-Zumino form for the superpotential [85]

$$f(\phi) = \frac{M}{2} \phi^2 - \frac{\lambda}{3} \phi^3,$$

which is only a function of $\phi$. In this case, $W_T = 0$, and we see from Eq. (4.16) that $\dot{V} = |W_\phi|^2$. The resulting potential is

$$V(\phi) = M^2 \frac{\phi^2 |1 - \lambda \phi/M|^2}{(1 - |\phi|^2/3)^2}.$$

We can rewrite the kinetic terms in Eq. (4.14) as

$$\mathcal{L}_{KE} = (\partial_\mu \bar{\phi}, \partial_\mu T) \left( \frac{3}{(T + T - |\phi|^2/3)^2} \right) \begin{pmatrix} \frac{T+T}{3} & -\phi/3 \\ -\phi/3 & 1 \end{pmatrix} \begin{pmatrix} \partial^\mu \phi \\ \partial^\mu T \end{pmatrix},$$

indicating that neither $T$ nor $\phi$ is normalized canonically. When $T$ is fixed, we can define the canonically-normalized field $\chi$:

$$\chi \equiv \sqrt{3} \tanh^{-1} \left( \frac{\phi}{\sqrt{3}} \right).$$

Decomposing $\chi$ into its real and imaginary parts: $\chi = (x + iy)/\sqrt{2}$, we find that the potential is minimized for $y = 0$, and that the potential in the real direction takes the same form as the Starobinsky potential in Eq. (4.27) when $\lambda = M/\sqrt{3}$.

To get a feel for how “accidental” this result is, the potential is plotted for several values of $\lambda/M \simeq 1/\sqrt{3}$ in Fig. 4.5. Requiring $N_* = 50$ to 60 $e$-folds specifies the value of the field $x$ at the beginning of inflation. For example, the nominal choice $N_* = 55$ corresponds to $x = 5.35$ and, as one can see in Fig. 4.5, inflation is still possible for
\[ \sqrt{3}\lambda/M \text{ slightly less than 1. However, deviations from 1 by more than a few parts in } 10^{-4} \text{ would not provide a suitable inflationary potential, as seen in Fig. 4.6.} \]

Figure 4.5: Starobinsky-like inflationary potential in the no-scale supergravity model with superpotential (4.31) for choices of \( \lambda \sim M/\sqrt{3} \), as indicated.

Fig. 4.6 displays the predictions for \((n_s, r)\) in this model for five choices of the coupling \( \lambda \) that yield \( n_s \in [0.93, 1.00] \) and \( N_* \in [50, 60] \). The last 50 to 60 e-folds of inflation arise as \( x \) rolls to zero from \( \sim 5.1-5.8 \), with the exact value depending on \( \lambda \) and \( N_* \). We see again that the values of \( \lambda \) are constrained to be close to the critical value \( M/\sqrt{3} \), for which we find extremely good agreement with the Planck determination of \( n_s \). The values of \( r \) are well below the current experimental limit [86], varying over the range 0.0012 – 0.0084, in the models displayed, with \( r \simeq 0.003 \) for \( \lambda = M/\sqrt{3} \).

In the later sections, we discuss more formal aspects of no-scale supergravity and some important aspects of the phenomenology of no-scale models of inflation. However, we comment first how modifying the no-scale Kähler potential can affect the observables, causing deviations from the Starobinsky predictions [225].

The Starobinsky potential can be expressed in the simple form

\[
V = A \left(1 - e^{-Bx}\right)^2, \tag{4.35}
\]
Figure 4.6: Predictions for the tilt $n_s$ in the spectrum of scalar perturbations and for the tensor-to-scalar ratio $r$, compared with the 68 and 95% C.L. regions from Planck data combined with BICEP2/Keck results [28]. In the main panel the lines are labelled by the values of $\sqrt{3}\lambda/M$ (in Planck units) assumed in each case. In the inset, the same cases are shown on a log scale to display more clearly the values of $r$.

with $B = \sqrt{2/3}$, and we note that the inflationary predictions are derived in the large-field regime where the leading non-constant term is $\propto e^{-Bx}$. In Ref. [225], the phenomenological generalizations of Eq. (4.35) were considered. We can expand it as

$$V = A (1 - \delta e^{-Bx} + O(e^{-2Bx})) ,$$

(4.36)

with $\delta$ and $B$ treated as free parameters that may deviate from the Starobinsky values.
\( \delta = 2 \) and \( B = \sqrt{2/3} \). At leading order when \( e^{-Bx} \) is small, one finds

\[
\begin{align*}
n_s &= 1 - 2B^2 \delta e^{-Bx} + \cdots , \\
r &= 8B^2 \delta^2 e^{-2Bx} + \cdots , \\
N_s &= \frac{1}{B^2 \delta} e^{+Bx} + \cdots ,
\end{align*}
\]

and we have

\[
\begin{align*}
n_s &= 1 - 2 \frac{N_s}{\delta} + \cdots , \\
r &= \frac{8}{B^2 N_s^2} + \cdots .
\end{align*}
\]

Requiring \( N_s = 50 \) to 60 yields the characteristic predictions \( n_s = 0.960 \) to 0.967, independent of \( B \), and the Starobinsky choice \( B = \sqrt{2/3} \) yields \( r = 12/N_s^2 = 0.0048 \) to 0.0033. Since the experimental upper limit is \( r < 0.06 \) \[28\], it is clear that substantial deviations from the Starobinsky value of \( B \) are currently allowed. We note finally that the predictions (4.38) are independent of \( \delta \).

Different values of \( B \) may be obtained by considering generalized \( \text{Kähler} \) potentials \[225\] that include an inflationary sector:

\[
K \ni K_{\text{inf}} = -3 \alpha \ln(T + \bar{T}) ,
\]

where \( \alpha \neq 1 \) in general. In such a case

\[
B = \sqrt{\frac{2}{3\alpha}} ,
\]

leading to the prediction

\[
r \simeq \frac{12}{N_s^2 \alpha} .
\]

For example, a simple string compactification contains 3 complex moduli \( T^i \) whose product is the volume modulus \( T \) considered above. Inflation might be driven by just one or a pair of the \( T^i \), in which case \( \alpha = 1/3 \) or \( 2/3 \). In these examples \( r \) is decreased below the Starobinsky prediction, rendering its experimental detection more challenging. However, it is interesting that, within the class of string-inspired no-scale models discussed here, measuring \( r \) might cast light on the dynamics of string compactification \[225\].

One may also consider models with larger values of \( \alpha \), as illustrated in Fig. 2.7, where
we see that values of $\alpha \lesssim 27$ are allowed by the current upper limit on $r$. However, we note that, for large values of $\alpha \gg 1$, sub-asymptotic corrections to the leading-order predictions (4.37)-(4.38) become important, causing the curvature in the predicted bands in Fig. 2.7.

We discuss such models with $\alpha \neq 1$ in more detail in subsequent sections.
Chapter 5

Classification of Starobinsky-like Inflationary Models

In the previous chapter, we showed how to construct different supergravity models. In particular, we focused on no-scale supergravity models. We also discussed that the measurements of the cosmic microwave background (CMB) [28] continue to be consistent with the inflationary paradigm. We argued that one simple model is consistent with all the experimental measurements, namely the original Starobinsky $R + R^2$ model [84].

In view of the virtues of low-energy supersymmetry [44, 226], such as making the hierarchies between the electroweak, inflationary and gravitational scales less unnatural [227], providing a plausible candidate for cold dark matter [12, 13], facilitating the unification of the strong, weak and electromagnetic interactions [228, 9], and predicting successfully the mass of the Higgs boson [20, 21, 22] and that its couplings should resemble those in the Standard Model [229], we think that an inflationary model should be (approximately) supersymmetric [29]. In the context of cosmology, one must combine supersymmetry with gravity, so the appropriate framework is supergravity [44]. Moreover, in order to avoid unacceptable anti-de Sitter minima in the effective potential, we are led to consider no-scale supergravity [43, 73], as appears generically in the effective low-energy limit of string compactifications [77]. The question then arises whether one can find in the no-scale framework models that yield Starobinsky-like predictions for the inflationary observables.
As we discussed in the previous chapter, a positive answer to this question was found in [85], where it was shown that the simplest Wess-Zumino form of superpotential, \( W = \mu \phi^2 + \lambda \phi^3 \), would yield Starobinsky-like inflation for suitable values of the superpotential parameters \( \mu \) and \( \lambda \). An alternative realization of the \( R + R^2 \) theory in the framework of no-scale supergravity, with a very different form of superpotential, had been given previously in [230], though without discussing the connection to models of inflation. Following [85], a number of other Starobinsky-like inflationary models were obtained as avatars of no-scale supergravity in [225]. However, the existence of an underlying connection between all these no-scale realizations of Starobinsky-like inflation remained an open question.

5.1 No-Scale Supergravity Models

Here we address this question systematically using the underlying symmetries of no-scale supergravity. The original minimal no-scale supergravity model contained a single chiral field \( T \), with dynamics described by the Kähler potential (4.10). The field \( T \) parameterizes a non-compact \( SU(1,1)/U(1) \) coset manifold [43, 73, 74], and \( \bar{T} \) is its complex conjugate. As was shown in [77], this type of Kähler structure emerges naturally in simple compactifications of string theory, where \( T \) can be identified as a modulus field. Generalizations of the minimal no-scale model (4.10) can be constructed with higher-dimensional non-compact coset manifolds such as \( SU(N,1)/SU(N) \times U(1) \) [75].\(^1\) As was discussed in [225], it is not possible to construct a Starobinsky-like model using the minimal no-scale Kähler potential (4.10)\(^2\) and the models constructed in [230, 225, 85, 82, 231] were based on a non-compact \( SU(2,1)/SU(2) \times U(1) \) coset manifold.

This theory may be parameterized in terms of two chiral fields either of two equivalent forms:

\[
K = -3 \ln \left( T + \bar{T} - \frac{\vert \phi \vert^2}{3} \right) \quad \text{or} \quad K = -3 \ln \left( 1 - \frac{\vert y_1 \vert^2}{3} - \frac{\vert y_2 \vert^2}{3} \right). \quad (5.1)
\]

One can transform between these two equivalent forms using the underlying non-compact

\(^1\)There are also generalizations of (4.10) in which the supergravity fields parameterize a direct product of such non-compact coset manifolds, but we do not discuss them here.

\(^2\)However, it is still possible to construct viable inflationary models with a de Sitter plateau in the minimal no-scale supergravity, and such models were considered in [231, 232, 233].
SU(2,1)/SU(2) × U(1) symmetry. When one does so, the superpotential is in general modified. In this chapter, we study how SU(2,1)/SU(2) × U(1) transformations can be used to exhibit equivalences between Starobinsky-like inflationary models with superpotentials that appear a priori to be distinct.

We first recall some important features of supergravity models, which we interpret as low-energy effective theories at energies significantly smaller than the Planck scale. The geometric properties of the scalar field space are characterized by the Kähler potential \( K(\Phi_i, \bar{\Phi}_j) \), where the fields \( \Phi_i \) are complex scalar fields and the fields \( \bar{\Phi}_j \) are their Hermitian conjugates. The kinetic terms of the Lagrangian are given by

\[
L_{\text{kin}} = K^{i\bar{j}} \partial_{\mu} \Phi_i \partial^\mu \bar{\Phi}_j ,
\]

where \( K^{i\bar{j}} \equiv \partial^2 K / \partial \Phi_i \partial^{\mu} \bar{\Phi}_j \) is the Kähler metric. To define the supergravity dynamics of the complex scalar fields \( \Phi_i \), one introduces the superpotential \( W \), which is a holomorphic function of the \( \Phi_i \). The Kähler function is then defined as \( G \equiv K + \ln W + \ln \bar{W} \), and the corresponding supergravity action is given by Eq. (3.78), where the effective scalar potential is (3.83), and \( K_{i\bar{j}}^{ij} \) is the inverse Kähler metric.

The next step is to apply the general formalism within a specific supergravity model. Here we consider no-scale supergravity, which was first described in [43, 73] in its minimal version. We consider here the following generalized Kähler potential for \( N \) complex scalar fields [75] that parametrize a non-compact SU(N,1)/SU(N) × U(1) coset manifold:

\[
K = -3 \ln \left( T + \bar{T} - |\phi_i|^2 \right) ,
\]

where \( T \) is the complex scalar field which can be associated with the volume modulus in a compactified string model, and \( i = 1, 2, \ldots, N-1 \). The minimal model with no chiral matter fields has \( N = 1 \), in which case the Kähler potential (5.3) can be written in terms of a single volume modulus field \( T \), as in (4.10), which parametrizes the non-compact SU(1,1)/U(1) coset manifold. Using expressions (5.2) and (3.83), the kinetic terms of the Lagrangian together with the effective scalar potential are given by:

\[
L_{\text{kin}} = \frac{3}{(T + \bar{T})^2} \partial_\mu T \partial^\mu T ,
\]
and
\[ V = \frac{\dot{V}}{(T + \overline{T})^2}, \quad \text{where} \quad \dot{V} = \frac{1}{3}(T + \overline{T})|W_T|^2 - (W\overline{W}_T + \overline{W}W_T). \quad (5.5) \]

To understand how the identical inflationary potentials might be recovered starting from distinct superpotentials \( W(T) \), it is useful to recall the general SU(1,1) isometric transformations for the volume modulus \( T \): \([74, 225]\)

\[ T \rightarrow \frac{\alpha T + i\beta}{i\gamma T + \delta}, \quad \text{where} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R} \quad \text{and} \quad \alpha \delta + \beta \gamma = 1. \quad (5.6) \]

Not all the general isometric transformations respect the invariance of the Kähler potential. Indeed, only imaginary translations \( T \rightarrow T + i\beta \) together with inversions \( T \rightarrow \frac{\beta}{\gamma}T \) leave it invariant up to a Kähler transformation. For example, in the case of an inversion, we have
\[ K \rightarrow K + f(T) + \overline{f}(\overline{T}) \] and
\[ W \rightarrow e^{-f} W \text{ with } f = -\ln \sqrt{\gamma/\beta T}. \]

However, the superpotential does not remain invariant under these transformation laws and, therefore, the effective scalar potential also acquires a new form. Nevertheless, the complex scalar fields should be redefined to obtain canonically-normalized kinetic terms, and for \( \gamma/\beta = 4 \), we recover the initial form of the effective scalar potential in terms of canonical fields.

The symmetry properties of the non-compact SU(N,1)/SU(N)×U(1) coset space can be understood better by adopting a more symmetric representation \([75]\), redefining the corresponding complex scalar fields in the following way:

\[ T = k \left( \frac{1 - \frac{y_N}{\sqrt{3}}}{1 + \frac{y_N}{\sqrt{3}}} \right), \quad \text{and} \quad \phi^i = \sqrt{k} \left( \frac{y_i}{1 + \frac{y_N}{\sqrt{3}}} \right), \quad \text{with} \quad i = 1, \ldots, N - 1, \quad (5.7) \]

where \( k \) is an arbitrary constant. The field redefinition (5.7) also transforms the effective superpotential into the following form:

\[ W(\Phi^i) \rightarrow \tilde{W}(y_i) = \left[ \frac{1}{\sqrt{k}} \left( 1 + \frac{y_N}{\sqrt{3}} \right) \right]^3 W(y_i), \quad \text{with} \quad i = 1, \ldots, N. \quad (5.8) \]

Using these symmetric field redefinitions we obtain the following expression for the
The symmetric form of the Kähler potential (5.9) informs us that the U(1) phase transformation $y^{i} \to e^{i\theta}y^{i}$ is trivial and can be discarded. Therefore, it is sufficient to consider the transformation laws of the non-compact SU(N,1)/SU(N) coset manifold. With this simplification, we can define an $N \times N$ complex matrix $U$ that parameterizes the SU(N,1)/SU(N) coset space. The complex matrix $U$ must satisfy the following conditions:

$$U^\dagger gU = g, \quad \text{and} \quad U^\dagger U = I, \quad (5.10)$$

where the diagonal matrix $g$ of the SU(N,1) group is given by:

$$g = diag(1, \ldots, 1, -1), \quad (5.11)$$

and $I$ is the $N \times N$ identity matrix. Because the complex matrix $U$ has $\det(U) = 1$, it can be shown that $U \in SL(N, \mathbb{C})$, and can be related to the projective special linear group $PSL(N, \mathbb{C})$ using the following relation:

$$PSL(N, \mathbb{C}) = \frac{SL(N, \mathbb{C})}{Z(SL(N, \mathbb{C}))}, \quad (5.12)$$

where $Z(SL(N, \mathbb{C}))$ is the center of the special linear group $SL(N, \mathbb{C})$. Hence the general invariance laws of the Kähler potential are described by projective linear transformations that are elements of the projective linear group $PSL(N, \mathbb{C})$, and the projective linear
transformations are given by:

\[ \left[ \phi_1, \ldots, \phi_N, 1 \right] \rightarrow \left[ \frac{(U\Phi)_1}{(U\Phi)_{N+1}}, \ldots, \frac{(U\Phi)_N}{(U\Phi)_{N+1}}, 1 \right], \quad \text{with} \quad \Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \\ 1 \end{pmatrix}, \quad (5.13) \]

where the \((U\Phi)_i\) are the respective row vectors.

Returning to the SU(1, 1)/U(1) coset space and redefining the complex scalar field \(T\) in terms of variable \(y_1\) from the equation (5.7), the Kähler potential (4.10) now acquires the symmetric form

\[ K = -3 \ln \left( 1 - \frac{|y_1|^2}{3} \right). \quad (5.14) \]

To show that the result (5.13) is equivalent to the isometric transformations for the volume modulus \(T\) (5.6), we first find the complex matrix \(U\) that parametrizes the SU(1, 1) group. The complex 2 × 2 matrix \(U\) can be expressed as

\[ U = \begin{pmatrix} a & \lambda \\ \lambda^* & a \end{pmatrix}, \quad \text{where} \quad a \in \mathbb{R}_{>0}, \quad \lambda \in \mathbb{C}, \quad a^2 - |\lambda|^2 = 1. \quad (5.15) \]

We find from equation (5.13) that the complex field \(y_1\) follows the transformation law

\[ \frac{y_1}{\sqrt{3}} \rightarrow \frac{ay_1/\sqrt{3} + \lambda}{\lambda^*y_1/\sqrt{3} + a} \quad (5.16) \]

and now, if we plug equation (5.16) back into the symmetric Kähler potential (5.14), we see that it remains invariant. Finally, if we use the field redefinitions (5.7) and transform the field \(y_1\) back to the volume modulus field \(T\) using \(y_1/\sqrt{3} \rightarrow (1 - 2T)/(1 + 2T)\), we recover successfully the isometric transformations (5.6) with \(\alpha = 2a - 2Re\lambda, \beta = -Im\lambda, \gamma = 4Im\lambda, \) and \(\delta = 2a + 2Re\lambda.\)

In the single-field theory based on SU(1,1)/U(1), de Sitter solutions are possible for
$W = T^3 - 1$ [74, 232, 234, 235], corresponding to $W = (\sqrt{3}y^3 + 7y^2 + 9\sqrt{3}y + 7)/8$ in the symmetric basis. However, this would not lead to a Starobinsky-like inflationary potential [225]. Nevertheless, the mathematical framework introduced in this Section can be applied to non-minimal no-scale supergravity models, and in next section we explore how the Starobinsky model of inflation arises in SU(2,1)/SU(2) × U(1) supergravity models. We also discuss de Sitter/anti-de Sitter solution in great detail in the next chapter.

### 5.2 SU(2,1)/SU(2) × U(1) No-Scale Supergravity

We consider now the simplest non-minimal no-scale supergravity model, based on the non-compact SU(2,1)/SU(2) × U(1) coset. In this case the Kähler potential may be written in the following form:

$$K = -3 \ln \left( T + \bar{T} - \left| \phi \right|^2 \right),$$  \hspace{1cm} (5.17)

where now the complex scalar fields $(T, \phi)$ parametrize the non-compact SU(2,1)/SU(2) × U(1) coset manifold. Using the Kähler potential (5.17) in conjunction with expression (5.2), we find that the kinetic terms of the Lagrangian are given by:

$$L_{kin} = \left( \partial_\mu \bar{\phi}, \partial_\mu T \right) \left( \frac{3}{(T + \bar{T} - |\phi|^2/3)^2} \right) \left( \begin{array}{cc} T + \bar{T} & -\phi \\ \bar{T} & 1 \end{array} \right) \left( \begin{array}{c} \partial_\mu \bar{\phi} \\ \partial_\mu T \end{array} \right).$$  \hspace{1cm} (5.18)

The effective potential (3.83), which in this case is expressed in terms of a general superpotential $W(T, \phi)$, becomes

$$V = \frac{\hat{V}}{(T + \bar{T} - \frac{\phi \bar{\phi}}{3})^2},$$  \hspace{1cm} (5.19)

with

$$\hat{V} \equiv \left| \frac{\partial W}{\partial \phi} \right|^2 + \frac{1}{3} (T + \bar{T}) |W_T|^2 + \frac{1}{3} (W_T (W_{\phi} \bar{\phi} - 3 \bar{W}) + h.c.) ,$$  \hspace{1cm} (5.20)

where $W_T = \partial W/\partial T$ and $W_{\phi} = \partial W/\partial \phi$. 
Because the general SU(2,1)/SU(2)×U(1) no-scale models are parametrized by two complex scalar fields (T, ϕ), or by four real scalar fields, in order to recover the Starobinsky inflationary potential it will be necessary to fix one of the complex scalar fields and hence break the coset space symmetry. Thus, our first step toward deriving the expressions for general superpotentials \( W(T, \phi) \) capable of yielding a Starobinsky inflationary potential is to find all possible canonically-normalized scalar field expressions. For convenience, we expand fully the kinetic terms \( (5.18) \) and fix either field \( T \) or \( \phi \), in which case the kinetic cross-terms can be discarded, leaving us with

\[
L_{\text{kin}} = \frac{3}{(T + \bar{T} - \frac{\phi \bar{\phi}}{3})^2} \partial_{\mu}T \partial^{\mu}\bar{T} + \frac{(T + \bar{T})}{(T + \bar{T} - \frac{\phi \bar{\phi}}{3})^2} \partial_{\mu}\phi \partial^{\mu}\bar{\phi}.
\]  

\( (5.21) \)

Let us first follow the same treatment as in [85] and assume that the \( T \) field is fixed, with a vacuum expectation value of \( \langle \text{Re} T \rangle = \frac{k}{2} \) and \( \langle \text{Im} T \rangle = 0 \), in which case the kinetic terms \( (5.21) \) become

\[
L_{\text{kin}} = \frac{k}{(k - \frac{\phi \bar{\phi}}{3})^2} \partial_{\mu}\phi \partial^{\mu}\bar{\phi}.
\]  

\( (5.22) \)

To understand better how to normalize canonically the field \( \phi \), we assume that the imaginary part of the matter field \( \phi \) is fixed to \( \langle \text{Im} \phi \rangle = 0 \) by the dynamics of the potential. The details of this assumption will be discussed later.

Then we find from equation \( (5.22) \) the following redefinition for a canonically-normalized field

\[
\phi = \pm \sqrt{3k} \tanh \left( \frac{x}{\sqrt{\phi}} \right),
\]  

\( (5.23) \)

where \( x \) is a real scalar field. When considering the general expression for the superpotential \( W(T, \phi) \) that is to yield a Starobinsky inflationary potential, it is important to respect the canonical formulation of the model by using the redefinition \( (5.23) \) of the matter field \( \phi \).

Alternatively, instead of fixing the field \( T \), one can perform the analogous procedure of setting the vacuum expectation value of the matter field to \( \langle \phi \rangle = 0 \). In this case, the
kinetic terms of the Lagrangian (5.21) are expressed as
\[ \mathcal{L}_{\text{kin}} = \frac{3}{(T + \bar{T})^2} \partial_\mu T \partial^\mu \bar{T}. \] (5.24)

Similarly, we assume a vacuum expectation value for \( T \) with \( \langle \text{Im } T \rangle = 0 \), so that the volume modulus \( T \) is now a real field and the redefined canonically-normalized field can be expressed as
\[ T = \frac{k}{2} e^{\pm \sqrt{3} \rho}, \] (5.25)

where the field \( \rho \) is now real and the coefficient in front of (5.24) is chosen to be compatible with the symmetric field redefinitions (5.7). Hence we can see that, by fixing one of the complex scalar fields (\( T, \phi \)) and then performing a canonical field redefinition (5.22) or (5.24), the SU(2,1)/SU(2) × U(1) symmetry can be broken into one of four different branches according to the breaking diagram shown in Fig. 5.1.

5.2.1 The SU(2,1)/SU(2) × U(1) Coset Space in the Symmetric Basis

When seeking models leading to the Starobinsky inflationary potential, one may consider any of the four possible branches shown in Fig. 5.1. However, it is possible to perform a SU(2,1)/SU(2) × U(1) projective linear transformation on a general superpotential with fully dynamical fields to connect the different superpotential branches, as we now show following the description introduced previously. To show this, we adopt the symmetric approach and switch from the basis \( (T, \phi) \) to the basis \( (y_1, y_2) \) using the general field redefinitions (5.7), the following relations
\[ y_1 = \sqrt{k} \left( \frac{2\phi}{k + 2T} \right), \quad y_2 = \sqrt{3} \left( \frac{k - 2T}{k + 2T} \right), \] (5.26)

and the inverse relations
\[ T = \frac{k}{2} \left( \frac{1 - y_2/\sqrt{3}}{1 + y_2/\sqrt{3}} \right), \quad \phi = \sqrt{k} \left( \frac{y_1}{1 + y_2/\sqrt{3}} \right). \] (5.27)
Figure 5.1: This diagram illustrates how, starting from a general two-field superpotential \( W(T, \phi) \), it is possible to break the \( SU(2, 1)/SU(2) \times U(1) \) symmetry by fixing one of the complex scalar fields and then casting the dynamical field in canonical form by a suitable field redefinition, yielding four distinct branches of models with Starobinsky-like effective scalar potentials.

In the \((y_1, y_2)\) basis the Kähler potential (5.3) takes the symmetric form

\[
K = -3 \ln \left( 1 - \frac{|y_1|^2 + |y_2|^2}{3} \right). \tag{5.28}
\]
The expressions for the canonically-normalized field redefinitions (5.23) and (5.25) in the \((y_1, y_2)\) symmetric basis are the following in the different branches

**Branch I:**
\[
\phi = +\sqrt{3}k \tanh \left( \frac{x}{\sqrt{6}} \right); \quad \langle T \rangle = \frac{k}{2} \rightarrow \left( y_1 = +\sqrt{3} \tanh \left( \frac{x}{\sqrt{6}} \right); \quad \langle y_2 \rangle = 0 \right),
\]
(5.29)

**Branch II:**
\[
\phi = -\sqrt{3}k \tanh \left( \frac{x}{\sqrt{6}} \right); \quad \langle T \rangle = \frac{k}{2} \rightarrow \left( y_1 = -\sqrt{3} \tanh \left( \frac{x}{\sqrt{6}} \right); \quad \langle y_2 \rangle = 0 \right),
\]
(5.30)

**Branch III:**
\[
\langle \phi \rangle = 0; \quad T = \frac{k}{2} e^{\pm \sqrt{3} \rho} \rightarrow \left( y_1 = 0; \quad y_2 = -\sqrt{3} \tanh \left( \frac{\rho}{\sqrt{6}} \right) \right),
\]
(5.31)

**Branch IV:**
\[
\langle \phi \rangle = 0; \quad T = \frac{k}{2} e^{-\sqrt{3} \rho} \rightarrow \left( y_1 = 0; \quad y_2 = +\sqrt{3} \tanh \left( \frac{\rho}{\sqrt{6}} \right) \right).
\]
(5.32)

After recovering the canonically-normalized kinetic terms in the \((y_1, y_2)\) symmetric basis using equations (5.29)-(5.32), we consider general superpotential expressions \(W(y_1, y_2)\).

The SU(2,1)/SU(2) × U(1) coset space can be parameterized with the following complex matrix \(U\):
\[
U = \begin{pmatrix}
\alpha & \beta & 0 \\
-\beta^* & \alpha^* & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{where} \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{5.33}
\]

Using \(U\) (5.33) together with equation (5.13), we obtain the following transformation laws for the fields \(y_1\) and \(y_2\)
\[
y_1 \rightarrow \alpha y_1 + \beta y_2, \quad y_2 \rightarrow -\beta^* y_1 + \alpha^* y_2. \tag{5.34}
\]

When we apply the transformation laws (5.34), the Kähler potential (5.28) remains invariant while the general superpotential \(W(y_1, y_2)\) transforms non-trivially. It is usually more convenient to work in the symmetric \((y_1, y_2)\) basis when considering the general superpotential at the starting point of the analysis. The essence of our approach is to start with a general superpotential \(W(y_1, y_2)\), corresponding to one of the four different branches in Fig. 5.1, and then apply the transformation laws (5.34) to obtain
the corresponding superpotential in a different branch. It is important to note that all
the transformations must be performed for dynamical fields, following which three of
the four real fields are fixed while the remaining field is put into the canonical form
according to the branch rules. The transformation relations between different branch
superpotentials in the \((y_1, y_2)\) symmetric basis are illustrated in Fig 5.2.

Figure 5.2: This diagram shows the transformation laws between the superpotentials in
different branches, together with the field fixings and canonical field redefinitions that
yield the same Starobinsky-like effective scalar potential.

If one specific superpotential form is known, by performing the indicated field trans-
formations one can find the corresponding superpotential in a different branch. Thus,
for SU(2,1)/SU(2)×U(1) no-scale supergravity models there are four different forms of
superpotential \(W(y_1, y_2)\) that, by following the corresponding branch parametrization
rules (5.29)-(5.32), yield identical effective scalar potentials. In the next section we find all four different branch superpotential expressions that reduce to the Starobinsky inflationary potential in one of the two real fields. The same procedure could also be followed to recover a different effective scalar potential, and analogous transformation rules for general SU(N,1)/SU(2)×U(1) no-scale supergravity models could also be derived.

5.3 Starobinsky Superpotentials: General Classification

We consider now the most general superpotential that allows us to recover the Starobinsky inflationary potential in SU(2,1)/SU(2)×U(1) no-scale supergravity models. As was mentioned in the previous section, by applying the transformation laws (5.29)-(5.32), which are also depicted in Fig 5.2, one can transform between the different superpotential branches.

For a general superpotential \( W(y_1, y_2) \), we have the following expression for the effective potential:

\[
V = \frac{\hat{V}}{\left(1 - \frac{|y_1|^2 + |y_2|^2}{3}\right)^2},
\]

(5.35)

where

\[
\hat{V} = |W_1|^2 + |W_2|^2 - \frac{1}{3} |3W - W_1 y_1 - W_2 y_2|^2,
\]

(5.36)

with \( W_1 = \partial W/\partial y_1 \) and \( W_2 = \partial W/\partial y_2 \).

- In order to find the general expressions for all four different superpotential branches that will allow us to recover the Starobinsky inflationary potential with canonically-normalized kinetic terms, we start with a general superpotential expression for Branch I, which can be expressed as:

**Branch I:**

\[
W(y_1, y_2) = ay_1 + by_1^2 + cy_1^3 + dy_2 + ey_2 y_1 + fy_2^2 y_1^2 + g(y_1, y_2),
\]

(5.37)

where the additional term \( g(y_1, y_2) \) obeys the following conditions: \( g(y_1, 0) = 0 \), \( \partial g/\partial y_1 = 0 \), and \( \partial g/\partial y_2 = 0 \). Terms containing factors \( y_2^n \), with \( n > 1 \) may also appear in \( g \) but, since we will require \( \langle y_2 \rangle = 0 \), these terms do not contribute to \( V \). We have not
included a constant term in (5.37) (or in the general form for $W$ for the other branches) to avoid supersymmetry breaking of order the inflationary scale.

• If we perform the transformation (5.34), with $\alpha = -1$ and $\beta = 0$, we obtain

\[ \text{Branch II: } W(y_1, y_2) = -ay_1 + by_1^2 - cy_1^3 - dy_2 + ey_2y_1 - fy_2y_1^2 + g(y_1, y_2). \]  

(5.38)

• If instead we apply the transformation with $\alpha = 0$ and $\beta = -1$ to the general expression for Branch I (5.37) we obtain

\[ \text{Branch III: } W(y_1, y_2) = -ay_2 + by_2^2 - cy_2^3 + dy_1 - cy_1y_2 + fy_1y_2^2 + h(y_1, y_2), \]  

(5.39)

where, as before, the additional term $h(y_1, y_2)$ must now satisfy the following conditions: $h(0, y_2) = 0$, $\partial h / \partial y_1(0, y_2) = 0$ and $\partial h / \partial y_2(0, y_2) = 0$, when $\langle y_1 \rangle = 0$.

• Finally, applying either the same transformation to Branch II or applying the previous transformation with $\alpha = -1$ and $\beta = 0$ to Branch III, we obtain

\[ \text{Branch IV: } W(y_1, y_2) = ay_2 + by_2^2 + cy_2^3 - dy_1 - cy_1y_2 - fy_1y_2^2 + h(y_1, y_2). \]  

(5.40)

Using the form of Branch I superpotential (5.37), we can derive $\hat{V}$ from (5.36) and match that with a known solution from [225] in which

\[ \hat{V} = |y_1|^2 |1 - y_1 / \sqrt{3}|^2, \]  

(5.41)

corresponding to the Wess-Zumino model found in [85]. Matching the coefficients leads to the following four sets of solutions:

\[
\begin{aligned}
& a = 0, \quad e = \pm \sqrt{1 - 4b^2}, \quad f = \pm \frac{\sqrt{1 - 4b^2} - 2b^2}{\sqrt{3}}, \\
& c = \pm \frac{b(\sqrt{1 - 4b^2} - 2b^2)}{3\sqrt{3}}, \quad d = 0, \quad e = \pm \sqrt{1 - 4b^2}, \quad f = \pm \frac{\sqrt{1 - 4b^2} - 2b^2}{\sqrt{3}},
\end{aligned}
\]  

(5.42)
where all the coefficients are expressed in terms of an arbitrary free parameter \( b \). There are in addition the two solutions:

\[
\begin{align*}
    b &= -\frac{\sqrt{3}a}{2a^2 + 3}, \\
    c &= \frac{16a^6 + 72a^4 + 108a^2 + 27}{36a (2a^2 + 3)^2}, \\
    d &= \pm ia, \\
    e &= \pm 2i\frac{\sqrt{3}a}{2a^2 + 3}, \\
    f &= \pm i\frac{4a^2 (2a^2 + 3)^2 + 27}{12a (2a^2 + 3)^2},
\end{align*}
\]

where now the coefficients are expressed in terms of an arbitrary free parameter \( a \).

Eqs. (5.42) and (5.43) encompass all the Branch I solutions that yield the Starobinsky inflationary potential with canonically-normalized kinetic terms.

For illustration, Figs. 5.3 and 5.4 show the couplings necessary for Starobinsky solutions as function of a single free coupling. In the two panels of Fig. 5.3, the free coupling is \( b \), corresponding to the solutions in Eq. (5.42). We show only two of the solutions, as the remaining involve only a change in sign for each coupling. In Fig. 5.4, the free coupling is \( a \), corresponding to Eq. (5.43).

![Figure 5.3: The superpotential couplings, \( c, e, \) and \( f \) as functions of \( b \) that correspond to the Starobinsky solutions in (5.42).](image)

One can then use the SU(2,1) field transformations given in Eq. (5.34) to find the corresponding Starobinsky-like model in the other branches, as illustrated in Fig. 5.2. For example, if we transform to Branch II, we can reapply the process using now (5.38) in (5.36) and match to

\[
\hat{V} = |y_1|^2 [1 + y_1 / \sqrt{3}]^2.
\]

Solving for the superpotential coefficients \( a, b, c, \cdots \), leads to exactly the same sets of
Figure 5.4: The superpotential couplings, $b, c, d, e, \text{ and } f$ as functions of $a$ that correspond to the Starobinsky solutions in (5.43).

solutions as given in Eqs. (5.42) and (5.43). Similarly using the general superpotentials for Branches III and IV, matching to $\hat{V}$ (with $y_1 \rightarrow y_2$), we again would obtain the solutions in (5.42) and (5.43).

Thus, Eqs. (5.37)-(5.40), together with (5.42) and (5.43) provide all of the solutions that yield the Starobinsky inflationary potential with canonically-normalized kinetic terms. The solutions (5.42) and (5.43) provide all solutions within a branch, and the SU(2,1) transformations allow us to rotate between branches with the same form of solutions.

When considering the inflationary dynamics, we have assumed that three out of four complex scalar fields have been stabilized. To achieve this result, we can include a sufficient stabilization mechanism, which was described in [225, 81], and tackle the stability problem by adding a quartic stabilization term in the Kähler potential that will leave the effective scalar potential unaffected. For Branch I and II solutions, we assumed that $\langle y_2 \rangle = 0$, therefore, to stabilize the inflaton field $y_1$, we could add a quartic stabilization term to the Kähler potential

$$K = -3 \ln \left( 1 - \frac{|y_1|^2}{3} - \frac{|y_2|^2}{3} + \frac{|y_2|^4}{\Lambda^2} \right),$$ (5.45)

where we assume that $\Lambda \lesssim M_p$. However, the quartic stabilization term might not always stabilize the potential in both real and imaginary directions of $y_2$. To tackle this problem, we can either include an additional quartic stabilization term in the Kähler potential or use the additional term $g(y_1, y_2)$ in the general Branch I (5.37) and Branch
II (5.38) expressions, which does not affect the inflationary potential when \( \langle y_2 \rangle = 0 \). Analogously, for the general Branch III and IV expressions, we assumed that \( \langle y_1 \rangle = 0 \), and the stabilization can be achieved by introducing a quartic stabilization term in the Kähler potential for the field \( y_1 \).

Further, it can be readily shown that after fixing the volume modulus field \( \langle T \rangle = k/2 \) for Branch I or Branch II scalar potentials, the mass of the imaginary component \( m_{Im,\phi}^2 \) will be positive and independent of the free arbitrary parameter \( b \) for the solutions (5.42), while for the complex set of solutions (5.43), \( m_{Im,\phi}^2 \geq 0 \) will be obtained after setting the arbitrary free parameter \( a \) to be purely imaginary, where the value of |\( a \)| will determine the curvature in the imaginary direction. Therefore, after fixing the field \( T \), the imaginary part of the field \( \phi \) will be fixed by the dynamics of the potential to \( \langle Im \phi \rangle = 0 \) and this was shown concretely for the Wess-Zumino model in [85]. Similarly, for Branch III and IV effective potentials, we fix the matter field \( \langle \phi \rangle = 0 \). In an identical manner, after fixing the field \( \phi \), the imaginary part of the field \( T \) will be fixed dynamically to \( \langle Im T \rangle = 0 \) and the imaginary mass \( m_{Im,T}^2 \) will be positive for any free arbitrary parameter \( b \) for the four solutions (5.42), and any purely imaginary parameter \( a \) for the two solutions (5.43).

One can now observe that the four different branch solutions (5.37)-(5.40) exhibit similar characteristics. The crucial difference is that for Branch I and II solutions the inflaton is identified with a matter field while for Branch III and IV solutions it is identified with a modulus fields. Therefore, discrete SU(2,1)/SU(2) coset transformations are a powerful tool that can be employed to change the field that will be identified as inflaton, and this will have important consequences on how the inflationary sector couples to matter. We do not investigate such possibilities here, and the phenomenological aspects of our models are discussed in the subsequent chapters.

We show next that our general expressions include the SU(2,1)/SU(2) \( \times U(1) \) no-scale inflationary models considered previously in the literature: solutions known previously are related through rotations within a branch and/or SU(2,1) transformations. This classification also allows us to find new, compact forms of superpotential that also yield the Starobinsky potential.
5.4 Specific Examples of Starobinsky-Like Models

We now show how some specific examples of models yielding a Starobinsky-like effective potential fit within this general classification.

5.4.1 Wess-Zumino Superpotential

It was shown previously [85] that it is possible to obtain the Starobinsky inflationary potential for a matter field $\phi$ using a simple Wess-Zumino superpotential containing only a quadratic and cubic coupling, that can be written as

$$ W = M \left[ \frac{\sqrt{k} \phi^2}{2} - \frac{\phi^3}{3\sqrt{3}} \right], \quad (5.46) $$

where we include a constant $\sqrt{k}$, so that our Wess-Zumino superpotential expression is compatible with the transformation laws (5.26) and (5.27). If we switch to the $(y_1, y_2)$ symmetric basis we obtain the following expression [225]:

$$ W = M \left[ \frac{y_1^2}{2} - \frac{y_1^3}{3\sqrt{3}} + \frac{y_1^2 y_2}{2\sqrt{3}} \right]. \quad (5.47) $$

To recover the Starobinsky inflationary potential, we assume that $y_2$ is fixed so that $\langle y_2 \rangle = 0$, while $W, W_1$ and $W_2$ are all non-zero. If one then uses (5.35) and (5.36), the effective potential becomes:

$$ V = \frac{M^2 |y_1|^2 \left| 1 - \frac{y_1}{\sqrt{3}} \right|^2}{\left( 1 - \frac{|y_1|^2}{3} \right)^2} = \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{2/3}x} \right)^2. \quad (5.48) $$

With the Branch I canonical field redefinition for the symmetric field $y_1$ (5.29), we obtain the Starobinsky inflationary potential. If we compare the Wess-Zumino superpotential in the symmetric basis (5.47) to the general Branch I superpotential expression (5.37), we find the following values of the arbitrary coefficients:

$$ a, d, e = 0; \quad b = \frac{1}{2}; \quad c = -\frac{1}{3\sqrt{3}}; \quad f = \frac{1}{2\sqrt{3}}, \quad (5.49) $$
which satisfies the general set of coefficient conditions (5.42).

5.4.2 Cecotti Superpotential

The Wess-Zumino model with general coefficients given by Eq. (5.49) is just one particular solution in the general class given by (5.42). We may consider instead the solution with

\[ a, b, c, d = 0; \quad e = -1; \quad f = \frac{1}{\sqrt{3}}. \]  

(5.50)

We may now perform the \( SU(2, 1) \) transformation with \( \alpha = 0 \) and \( \beta = -1 \) or \( y_1 \to -y_2 \) and \( y_2 \to y_1 \) from Branch I to Branch III. Then, from Eq. (5.39), we have

\[ W = M \left[ y_1 y_2 + \frac{y_1 y_2^2}{\sqrt{3}} \right], \]  

(5.51)

which when transformed to the \((T, \phi)\) basis gives

\[ W = \sqrt{3} M \phi \left( T - \frac{k}{2} \right), \]  

(5.52)

which is precisely the Cecotti \([230]\) form for the superpotential giving rise to the \( R + R^2 \) theory and Starobinsky inflation when \( \langle y_1 \rangle = \langle \phi \rangle = 0 \). Indeed, evaluating the scalar potential from (5.51), we obtain again the Starobinsky form

\[ V = \frac{M^2 |y_2|^2 \left( 1 + \frac{|y_2|}{\sqrt{3}} \right)^2}{\left( 1 - \frac{|y_2|^2}{3} \right)^2} = \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{2/3} \rho} \right)^2, \]  

(5.53)

using the Branch III field redefinition in (5.31).

5.4.3 Related Superpotentials

As one can imagine, through a combination of \( SU(2, 1) \) transformations and choice of solutions from (5.42) or (5.43), several other models can be generated. For example, the transformation with \( \alpha = -1 \) and \( \beta = 0 \) or \( y_1 \to -y_1 \) and \( y_2 \to -y_2 \) takes us from Branch I to Branch II or from Branch III to Branch IV. If we now transform the Wess-Zumino
superpotential (5.47) to Branch II in this way, we obtain

\[ W = M \left[ \frac{y_2^2}{2} + \frac{y_1^3}{3\sqrt{3}} - \frac{y_1 y_2}{2\sqrt{3}} \right]. \]  

(5.54)

Transforming to the \((T, \phi)\) basis, we obtain:

\[ W = M \left[ \frac{T \phi^2}{\sqrt{k}} + \frac{\phi^3}{3\sqrt{3}} \right], \]  

(5.55)

and the Starobinsky potential is found when \(T\) is fixed to \(\langle T \rangle = k/2\). The Branch III version of the Wess-Zumino model is given by

\[ W = M \left[ \frac{y_2^2}{2} + \frac{y_2^3}{3\sqrt{3}} + \frac{y_1 y_2^2}{2\sqrt{3}} \right], \]  

(5.56)

which, when transformed to the \((T, \phi)\) basis, becomes:

\[ W = M \left[ \frac{1}{16k^{3/2}} (k - 2T)^2 \left( 2T + 2\sqrt{3}\sqrt{k}\phi + 5k \right) \right], \]  

(5.57)

as originally found in [225]. In this case, we must fix \(\langle \phi \rangle = 0\) to obtain the Starobinsky potential. There is a Branch IV analogue also given in [225] as the reversed Wess-Zumino solution with superpotential given by

\[ W = M \left[ \frac{y_2^2}{2} - \frac{y_2^3}{3\sqrt{3}} - \frac{y_1 y_2^2}{2\sqrt{3}} \right]. \]  

(5.58)

Transforming it to the \((T, \phi)\) basis, we find

\[ W = M \left[ \frac{1}{16k^{3/2}} (k - 2T)^2 \left( 10T - 2\sqrt{3}\sqrt{k}\phi + 5k \right) \right]. \]  

(5.59)

Thus, we have identified four different forms of Wess-Zumino superpotential that, by following the corresponding branch field parametrization rules, yield the Starobinsky inflationary potential. All four of these stem from the same solution of (5.42) with \(b = 1/2\). It is also clear from this specific example that some superpotential branches correspond to simpler expressions. Thus, for a general analysis, it is more convenient to choose the superpotential branch that has the simplest superpotential expression in
the \((T, \phi)\) basis.

Similarly, if we take the Cecotti form (5.51) and make the same sets of transformations, we obtain solutions in the other branches. Using \(\alpha = -1\) and \(\beta = 0\) or \(y_1 \rightarrow -y_1\) and \(y_2 \rightarrow -y_2\) we obtain the Branch IV solution

\[
W = M \left[ y_1y_2 - \frac{y_1y_2^2}{\sqrt{3}} \right], \quad (5.60)
\]

which becomes [225]

\[
W = \sqrt{3}MT\phi \left( 1 - \frac{2T}{k} \right) \quad (5.61)
\]

in the \((T, \phi)\) basis. On the other hand, if we start with the Cecotti superpotential (5.51) and apply the transformations \(\alpha = 0\), \(\beta = -1\), or \(y_1 \rightarrow -y_2\) and \(y_2 \rightarrow y_1\), we obtain the following Branch I superpotential in the symmetric \((y_1, y_2)\) basis:

\[
W = M \left[ -y_1y_2 - \frac{y_1^2y_2}{\sqrt{3}} \right], \quad (5.62)
\]

and in the \((T, \phi)\) basis we have [225]

\[
W = -\frac{1}{4}M\phi \left( 1 - \frac{2T}{k} \right) \left( 2\sqrt{3}T + \sqrt{3}k + 2\sqrt{k}\phi \right). \quad (5.63)
\]

Finally, we give the remaining Cecotti superpotential form, which belongs to Branch II

\[
W = M \left[ -y_1y_2 + \frac{y_1^2y_2}{\sqrt{3}} \right]. \quad (5.64)
\]

If we consider the same superpotential in the \((T, \phi)\) frame, we find

\[
W = -\frac{1}{4}M\phi \left( 1 - \frac{2T}{k} \right) \left( 2\sqrt{3}T + \sqrt{3}k - 2\sqrt{k}\phi \right). \quad (5.65)
\]

Therefore, we once again managed to recover four different superpotential forms that with corresponding field fixing and canonical field redefinitions all yield the Starobinsky inflationary potential.

We stress that the models highlighted above are relatively simple models based on solutions of Eq. (5.42) with the specific values \(b = 0\) or \(1/2\). There are of course
a continuous family of solutions with \( b \in [-1/2, 1/2] \). Furthermore, the 4 branches we have focused on are obtained from only a discrete subset of the possible SU(2,1) transformations (5.34), where we have chosen \( \alpha = 0, -1 \) and/or \( \beta = 0, -1 \). Any pair of values of \( \alpha \) and \( \beta \) with \( |\alpha|^2 + |\beta|^2 = 1 \) will yield additional solutions based on Eq. (5.42), so long as the appropriate combination of \( y_1 \) and \( y_2 \) (\( T \) and \( \phi \)) is held fixed.

### 5.4.4 Complex Superpotential

There remains one additional class of solutions of the general Starobinsky superpotential expressions associated with Eq. (5.43). In this case, we can build a superpotential, where all six coefficients are non-zero and some of the arbitrary coefficients will be complex. For example, if we choose the free parameter to be \( a = \sqrt{3}i \), then the remaining coefficients become: \( b = -i \); \( c = \frac{i}{2\sqrt{3}} \); \( d = \frac{\sqrt{3}}{2} \); \( e = -2 \); \( f = \frac{\sqrt{3}}{2} \). With this coefficient choice the general Branch I superpotential (5.37) acquires the following form

\[
W = M \left[ \frac{\sqrt{3}iy_1}{2} - iy_2^2 + \frac{iy_1^3}{2\sqrt{3}} + \frac{\sqrt{3}y_2}{2} - 2y_1y_2 + \frac{\sqrt{3}y_2y_1^2}{2} \right]. \tag{5.66}
\]

When transforming from the symmetric basis \((y_1, y_2)\) to the \((T, \phi)\) basis, for convenience we will set the coefficient \( k = 1 \) for the transformation laws (5.26), and obtain

\[
W = \frac{1}{48} \left( -18T(4T^2 + 2T - 1) - 12((6 + 4i)T + (-3 + 2i))\phi^2 \right.
\]
\[
+24\sqrt{3}T((4 + i)T + i)\phi + 8i\sqrt{3}\phi^3 - (24 - 6i)\sqrt{3}\phi + 9 \right). \tag{5.67}
\]

Although this Branch I superpotential in the \((T, \phi)\) (5.67) is somewhat complicated, we can find a simpler form by transforming the Branch I complex superpotential (5.66) to a Branch III superpotential. By using the transformation laws \( y_1 \to -y_2 \) and \( y_2 \to y_1 \), one obtains

\[
W = M \left[ -\frac{\sqrt{3}iy_2}{2} - iy_1^2 - \frac{iy_2^3}{2\sqrt{3}} + \frac{\sqrt{3}y_1}{2} + 2y_1y_2 + \frac{\sqrt{3}y_1y_2^2}{2} \right], \tag{5.68}
\]
and then transforming it to the \((T, \phi)\) basis:

\[
W = \sqrt{3}M \left[ \frac{\sqrt{3}i}{4} (2T - 1) - \phi(T - 1) \right].
\] (5.69)

Hence, by performing the field transformation, we managed to obtain a compact superpotential form (5.69), which would be a more convenient form if it were chosen as the starting point of the analysis.

5.5 Concluding Remarks

In this chapter, we have developed a general classification of models formulated in the framework of \(SU(2,1)/SU(2) \times U(1)\) no-scale supergravity that have a Starobinsky-like effective scalar potential. We have exhibited four different branches of such models, characterized by different choices of field expectation values and canonical field redefinitions, as illustrated in Fig. 5.1. These branches are obtained from discrete \(SU(2,1)/SU(2)\) transformations where either \(\phi\) or \(T\) are held fixed. The branches are in fact related by the continuous set of transformations which require a linear combination of \(\phi\) and \(T\) to be fixed in order to obtain a Starobinsky scalar potential. Within each branch, there are six classes of Starobinsky-like models, as shown in Eqns. (5.42) and (5.43). The solutions in the different branches are related via redefinitions of the \(SU(2,1)/SU(2) \times U(1)\) coset fields, as illustrated in Fig. 5.2. We have also shown how Starobinsky-like models known previously [230, 225, 85] are embedded within this general framework, and given some examples of additional Starobinsky-like models.

Our classification serves as a demonstration that Starobinsky-like inflation is a relatively generic feature of no-scale supergravity models, unlike simple polynomial models of inflationary potentials, for example. This may be encouraging for string theorists, in view of the facts that CMB data favour Starobinsky-like models and that no-scale models emerge as generic low-energy effective field theories derived from string theory. Until now, the derivation from string theory of a specific superpotential yielding a Starobinsky-like inflationary model has proved elusive. However, the results of this chapter may help by exhibiting the general form of such superpotentials, thereby extending the target to be aimed at, which is considerably larger than the specific examples
known previously [230, 225, 85].

The analysis presented in this chapter may serve as a useful framework for the analysis of present and future CMB data. Any deviations from the specific parameter relations in (5.42) and (5.43) would yield potentially observable deviations from the predictions of the Starobinsky model of inflation. Although the present CMB data are completely consistent with Starobinsky-like inflation, one should be on the lookout for any possible deviations from this paradigm. If observed, they might help identify the context in which Starobinsky-like inflation should be embedded. In addition, we want to emphasize that discrete $SU(2,1)/SU(2)$ coset transformations are by no means limited to just the Starobinsky-like inflationary models and the same transformation laws can be successfully applied to any arbitrary models based on a non-compact $SU(2,1)/SU(2)$ coset manifold. Our analysis could serve as a useful guide in this respect. We look forward to the next generation of CMB data following those from the Planck satellite [28].

In closing, we recall that there are two observables in slow-roll inflation, the scalar tilt, including $n_s$ as well as the scalar-to-tensor ratio $r$. Predictions for these quantities are, in general, sensitive to the number of $e$-folds of inflation, which depends in turn on the rate of inflaton decay [137]. One might expect that this would be different if the inflaton is identified with a modulus field or a matter field (or some combination of the two). Future measurements of $n_s$, in particular, could help break the observational degeneracy between different Starobinsky-like models within our general classification.
Chapter 6

Minkowski and de Sitter Vacua in No-Scale Models

We inhabit a universe with small but non-vanishing vacuum energy that is increasingly well described by a de Sitter geometry that is almost Minkowski at sub-cosmological scales [91]. Moreover, we argued that the early universe underwent a period of near-exponential inflationary expansion, that might correspond to a near-de Sitter (dS) geometry. These observations motivate the construction of models that accommodate dS and Minkowski spaces, and may be used to explore transitions between them.

We expect that physics below the Planck scale is approximately supersymmetric [44, 226], in which case the appropriate theoretical framework for studying such cosmological issues is supersymmetry [29], more specifically $\mathcal{N} = 1$ supergravity in order to accommodate chiral matter fields and general relativity. Generic supergravity models are well known to possess anti-de Sitter (AdS) vacua and have effective potentials that are far from flat, the $\eta$ problem [71, 72]. However, no-scale supergravity avoids these problems, and can accommodate flat potentials that may have vanishing energy density, corresponding to Minkowski vacua, or have constant positive energy densities, corresponding to dS vacua [74, 235].

Another reason for favoring no-scale supergravity is that it emerges as the natural framework for the low-energy effective field theory derived from strings [77]. This was
first shown in the context of a simplified model of compactification with a single volume modulus, but this first example has been extended to multifield models, including compactifications with three complex Kähler moduli and a complex coupling modulus, as well as some number of complex structure moduli [236].

Several issues then arise within this broader theoretical context. How unique are no-scale supergravity models with Minkowski or de Sitter solutions? What are the relationships between them? Can they be given simple geometrical interpretations? How may constructions with a single complex modulus field be generalized to two- or multifield supergravity models? Can the de Sitter models be used to construct inflationary models predicting perturbations that are consistent with observations, e.g., resembling the successful [28, 86] predictions of the Starobinsky model [84] as in [85]? How may the universe evolve from a (near-)de Sitter inflationary state towards the (near-)Minkowski contemporary epoch with its (small) cosmological constant, a.k.a. dark energy?

Aspects of these questions have been discussed previously. In [235], the dS vacua were constructed in two- and multifield models as could occur in string compactifications, discussed the conditions for their stability, and gave examples with only integer powers of the chiral fields in the superpotential. In the previous chapter, we presented a general discussion of two-field no-scale supergravity models of inflation yielding predictions similar to those of the Starobinsky model, using the non-compact $SU(2,1)/SU(2) \times U(1)$ symmetry to catalogue them in six equivalence classes. In [237], we constructed within this framework a specific minimal $SU(2,1)/SU(2) \times U(1)$ no-scale model that incorporates Starobinsky-like inflation, supersymmetry breaking and dark energy. This construction was generalized in [238] to inflationary models based on generalized no-scale structures with different values of the Kähler curvature $R$, as may occur if different numbers of complex moduli contribute to driving inflation, and we discuss it in subsequent chapters.

In this chapter, we discuss the uniqueness of superpotentials leading to Minkowski, dS and AdS vacua of single-field no-scale supergravity models, and how pairs of Minkowski superpotentials can be used to construct dS/AdS solutions. Expanding on previous work which showed how this construction may be extended to two- and multifield no-scale
supergravity models, we show how matter fields can be incorporated in a multifield construction of Minkowski, dS and AdS vacua. We also provide a geometrical visualization of the construction. We also mention how Starobinsky-like inflationary models can be constructed in this framework, and comment on the inclusion of additional twisted or untwisted moduli fields.

6.1 Vacua Solutions with Moduli Fields

We first recall some general properties of no-scale supergravity models, which emerge naturally from generic string compactifications in the low-energy effective limit [77]. The simplest $\mathcal{N} = 1$ no-scale supergravity models are characterized by the Kähler potential (4.10) where field $T$ is a complex chiral field that can be identified as the volume modulus field, and $\bar{T}$ is its conjugate field. The minimal no-scale Kähler potential (4.10) describes a non-compact $SU(1,1)/U(1)$ coset manifold and its higher-dimensional generalizations [75] will be considered in the following sections. Furthermore, the Kähler curvature of a general Kähler manifold is given by the expression $R_{i\bar{j}} \equiv \partial_i \partial_{\bar{j}} \ln K_{i\bar{j}}$, and the scalar curvature obeys the relation

$$R \equiv \frac{R_{i\bar{j}}}{K_{i\bar{j}}},$$

(6.1)

where $K_{i\bar{j}}$ is the inverse Kähler metric. If we consider the maximally-symmetric $SU(1,1)/U(1)$ Kähler manifold (4.10), the Kähler curvature reduces to the familiar result $R = \frac{2}{3}$. The Kähler potential (4.10) can be modified by introducing a curvature parameter $\alpha$

$$K = -3 \alpha \ln(T + \bar{T}),$$

(6.2)

which also parametrizes a non-compact $SU(1,1)/U(1)$ coset manifold, but with a positive constant curvature $R = \frac{2}{3\alpha}$ if we assume that $\alpha > 0$. This unique structure was first discussed in [74], and similar models were studied in [232, 239, 240], where they were termed $\alpha$-attractors.
6.1.1 Review of Earlier Work

As was shown in [74, 234, 232], one can consider combining cubic and constant superpotential terms to acquire a de Sitter vacuum solution. Choosing the following superpotential form:

\[ W = 1 - T^3, \]  
(6.3)

together with the Kähler potential (4.10), and imposing the condition \( T = \overline{T} \), the effective scalar potential (3.83) yields a de Sitter vacuum solution \( V = \frac{3}{2} \). However, the superpotential (6.3) leads to an unstable vacuum solution, since the mass-squared of the imaginary component of the scalar field is negative: \( m_{mT}^2 = -2 \). As we discuss in more detail below, the problem of instabilities can be addressed by adding a quartic term to the Kähler potential [74, 225, 238].

A detailed analysis of the general de Sitter vacua constructions for multi-moduli models was conducted in [235] and for convenience, we recall some of the key results. The Minkowski vacua solutions for a single complex chiral field \( T \) were found by considering the Kähler potential (6.2) with a monomial superpotential of the following form

\[ W = \lambda \cdot T^{n \pm}, \]  
(6.4)

where \( n \pm \) are two possible solutions given by

\[ n \pm = \frac{3}{2} \left( \alpha \pm \sqrt{\alpha} \right). \]  
(6.5)

Along the real \( T \) direction, \( V = 0 \). The scalar mass-squared in the imaginary direction is:

\[ m_{mT}^2 = 2^{2-3\alpha} \cdot \lambda^2 \cdot \frac{(\alpha - 1)}{\alpha} \cdot T^{\pm 3\sqrt{\alpha}}, \]  
(6.6)

where the choice \( T^{\pm 3\sqrt{\alpha}} \) corresponds to the two possible solutions \( n \pm \) (6.5). As can be seen from (6.6), to obtain a stable Minkowski vacuum solution, the stability condition \( \alpha \geq 1 \) has to be satisfied. For cases when \( 0 < \alpha < 1 \), quartic stabilization terms in the imaginary direction must be introduced in the Kähler potential (6.2).

As was shown in [74, 235, 238], de Sitter vacua solutions can be obtained from the
Kähler potential (6.2) by choosing a superpotential of the form:

\[ W = \lambda_1 T^{n_-} - \lambda_2 T^{n_+}, \]  

(6.7)

where \( n_\pm \) is given by (6.5). In this case, along the real \( T \) direction the effective scalar potential (3.83) becomes:

\[ V = 3 \cdot 2^{2-3\alpha} \cdot \lambda_1 \lambda_2. \]  

(6.8)

One of the most fascinating features of the de Sitter vacua construction (6.7) is that it is obtained by combining two distinct Minkowski vacua solutions (6.4). In the next sections, we show that there is a deeper connection between dS/AdS and Minkowski vacua solutions, and that this relation is not accidental.

Superpotential classes yielding constant scalar potentials were first considered in [74], namely:

1) \[ W = \lambda \text{ with } \alpha = 1, \]

(6.9)

2) \[ W = \lambda T^{3\alpha/2}, \]

(6.10)

3) \[ W = \lambda T^{3\alpha/2}(T^{3\sqrt{\alpha}/2} - T^{-3\sqrt{\alpha}/2}). \]

(6.11)

Comparing solution 1) to (6.7), we see that it can be recovered by setting \( \alpha = 1, \lambda_1 = \lambda \) and \( \lambda_2 = 0 \). Because \( \lambda_2 \) is chosen to be zero, we find a Minkowski vacuum: \( V = 0 \). Solution 3) is identical to (6.7) with \( \lambda_1 = \lambda_2 = -\lambda \). Solution 2) can also be obtained from (6.7) with the aid of a Kähler transformation:

\[ K \rightarrow K + f(T) + \bar{f}(\bar{T}) \]  

(6.12)

and

\[ W(T) \rightarrow \bar{W}(T) = e^{-f(T)}W(T), \]  

(6.13)

with

\[ f(T) = \ln \left( \frac{1 + T^{3\sqrt{\alpha}}}{2T^{3\sqrt{\alpha}/2}} \right), \]  

(6.14)

and applying the transformation laws (6.12), (6.13) with (6.14) and \( \lambda_1 = -\lambda_2 = \lambda/2 \), we recover solution 2) which is in fact an AdS vacuum solution \( V = -(3/2^{3\alpha}) \lambda^2 \), which
is always negative.

While the scalar potential is flat in the real direction, the scalar mass-squared of the imaginary component is given by:

\[ m_{ImT}^2 = \frac{2^{2-3\alpha} \left[ \lambda_1^2 (\alpha - 1) T^{-3\sqrt{\alpha}} - 2\lambda_1 \lambda_2 (\alpha + 1) + \lambda_2^2 (\alpha - 1) T^{3\sqrt{\alpha}} \right]}{\alpha}. \]  

(6.15)

For \( \alpha > 0 \), in the absence of stabilization terms, there are always some field values for which the instability in the imaginary direction persists. The problem of instability can be remedied by modifying the Kähler potential (6.2) and introducing quartic stabilization terms in the imaginary direction [74, 225, 238]

\[ K = -3\alpha \ln (T + T + \beta (T - T)^4), \]  

(6.16)

with \( \beta > 0 \). The newly-introduced quartic stabilization term does not alter the potential in the real direction, while it stabilizes the mass of the imaginary component (6.15) so that:

\[ m_{ImT}^2 = \frac{2^{2-3\alpha}}{\alpha} \left[ \lambda_1^2 (\alpha - 1 + 96\beta T^3) T^{-3\sqrt{\alpha}} - 2(\alpha + 1 - 96\beta T^3) \lambda_1 \lambda_2 \right. \]
\[ \left. + \lambda_2^2 (\alpha - 1 + 96\beta T^3) T^{3\sqrt{\alpha}} \right]. \]  

(6.17)

### 6.1.2 Uniqueness of Vacua Solutions

By solving an inhomogeneous differential equation, we now show that the monomial Minkowski superpotential solutions (6.4) are the only possible unique solutions that yield \( V = 0 \), while the combination of two distinct Minkowski solutions (6.7) yield dS/AdS vacuum solutions.

We consider a general superpotential expression \( W(T) \), which is a function of volume modulus \( T \) only, and solve the general homogeneous differential equation, which is equivalent to finding Minkowski vacuum solutions. As before, we assume that the VEV of the imaginary component \( \langle ImT \rangle = 0 \), so that \( T = \bar{T} \) and \( W(T) = \overline{W(T)} \). Using the Kähler potential (6.2) and the effective scalar potential (3.83), we find

\[ V = (2T)^{-3\alpha} \cdot \left[ \frac{(3\alpha W - 2TW')^2}{3\alpha} - 3W^2 \right], \]  

(6.18)
where $W \equiv W(T)$ and $W' \equiv \frac{dW(T)}{dT}$. In order to find Minkowski vacuum solutions, we set Eq. (6.18) to zero:

$$\frac{(3\alpha W - 2TW')^2}{3\alpha} - 3W^2 = 0. \quad (6.19)$$

Solving the homogeneous differential equation (6.19), we obtain two distinct Minkowski solutions

$$W = \lambda_i \cdot T^{\frac{3}{2}(\alpha \mp \sqrt{\alpha})}, \quad (6.20)$$

where $\lambda_i$ is an arbitrary constant. To find the dS/AdS vacuum solutions, we set the differential equation (6.18) equal to a constant and solve the following inhomogeneous equation

$$\frac{(3\alpha W - 2TW')^2}{3\alpha} - 3W^2 = \Lambda \cdot (2T)^{3\alpha}, \quad (6.21)$$

where $\Lambda$ is an arbitrary constant. We look for a particular superpotential solution to the inhomogeneous equation (6.21) of the following form

$$W = \lambda_1 \cdot T^{\frac{3}{2}(\alpha \mp \sqrt{\alpha})} - \lambda_2 \cdot T^m. \quad (6.22)$$

Inserting the expressions (6.22) into (6.21), we find that $m = n_{\mp} = \frac{3}{2} (\alpha \mp \sqrt{\alpha})$ is a particular solution of the inhomogeneous differential equation and the general solution has the following form

$$W = \lambda_1 \cdot T^{n_{\mp}} - \lambda_2 \cdot T^n, \quad (6.23)$$

where we have defined the constant $\Lambda = 3 \cdot 2^{2-3\alpha} \cdot \lambda_1 \lambda_2$. Thus, we have constructed the unique combination of two Minkowski solutions that yields dS/AdS solutions.\(^1\)

6.1.3 Generalized Solutions and Vacuum Stability

Before concluding this section, we introduce a formalism with which the construction of Minkowski-dS-AdS solutions can be generalized and applied to more complicated

\(\text{\(^1\)We note that these solutions correspond to flat directions in the real field direction. It is possible and relatively straightforward to construct minima with non-zero vacuum energy. In particular, it is well known that supergravity models with unbroken supersymmetry generally lead to AdS vacua.} \)
Kähler manifolds. Let us write

\[ K = -3 \alpha \ln(\mathcal{V}), \quad (6.24) \]

where \( \mathcal{V} \) is the argument inside the logarithm. For the simplest minimal no-scale \( SU(1,1) \) supergravity case with a single volume modulus field \( T \), we have \( \mathcal{V} \equiv T + \overline{T} \). As before, and in all the cases that we consider, we assume that the VEV of the imaginary part of the complex field is fixed to zero: \( \langle \text{Im} T \rangle = 0 \), which can always be achieved by introducing quartic stabilization terms in Eq. (6.24).

For the single field case, the effective scalar potential (3.83) becomes

\[ V = \frac{\dot{V}}{3 \alpha}, \text{ with } \dot{V} = \frac{|\mathcal{V} \cdot \overline{W}_T - 3 \alpha W|^2}{3 \alpha} - 3|W|^2. \quad (6.25) \]

In the real direction, where \( T = \overline{T} \), we define

\[ \mathcal{V} \rightarrow \xi, \quad (6.26) \]

so that the argument inside the logarithm becomes \( \xi = 2T \).

From our previous discussion, we already know which superpotential forms reduce to Minkowski solutions. We introduce the following notation, which will be used for all our Kähler coset manifolds:\(^2\)

\[ W_M \equiv \lambda \cdot \xi^{n_{\pm}}, \text{ with } V = 0, \quad (6.27) \]

where, as usual, the two possible choices \( n_{\pm} \) are given by Eq. (6.5). Note that, for this construction to work, we must impose the constraint \( \xi > 0 \) and the positive curvature condition \( \alpha > 0 \), which are necessary features of the no-scale structure.\(^3\)

With this redefinition, the scalar mass-squared in the imaginary field direction given in Eq. (6.6) becomes

\[ m_{\text{Im} T}^2 = \frac{4 \lambda^2 (\alpha - 1) \xi_{\pm}^3 \sqrt{\alpha}}{\alpha}, \quad (6.28) \]

\(^2\)Note that we are using a trick in our definition of the superpotential. Strictly speaking, \( \xi \) is defined by the argument of the log in \( K \) when all fields are taken as real. However, in the superpotential we are assuming that \( \xi \) is a function of (complex) superfields and ignore the restriction to real fields.

\(^3\)Note also that the definition of \( \lambda \) here differs from that in Eq. (6.4) by a constant factor of \( 2^{n_{\pm}} \).
where the sign depends on the choice of the Minkowski vacuum solution in (6.27). We later show that the same Minkowski mass expression (6.28) holds for any Kähler potential form, and hence that the solution is stable when $\alpha \geq 1$. When $0 < \alpha < 1$, Minkowski vacuum solutions become unstable and we must introduce the quartic stabilization terms in the imaginary direction.

Similarly, de Sitter/anti-de Sitter vacua solutions are constructed by combining two distinct Minkowski solutions (6.27),

$$W_{dS/AdS} = \lambda_1 \cdot \xi^{n-} - \lambda_2 \cdot \xi^{n+},$$  \hspace{1cm} (6.29)

and we call such constructions Minkowski pairs. The dS/AdS vacuum solution (6.29) yields an effective scalar potential (3.83):

$$V = 12 \lambda_1 \lambda_2,$$  \hspace{1cm} (6.30)

which allows three different types of vacua:

- de Sitter vacuum solutions when $\lambda_1$ and $\lambda_2$ are $\neq 0$ and have the same sign.
- anti-de Sitter vacuum solutions when $\lambda_1$ and $\lambda_2$ are $\neq 0$ and have opposite signs.
- Minkowski vacuum solutions when either $\lambda_1$ or $\lambda_2$ is set to zero.

The generalization of the scalar mass in the imaginary direction $m_{ImT}^2$ in equation (6.15) is given by

$$m_{ImT}^2 = \frac{4 \left[ \lambda_1^2 (\alpha - 1) \xi^{-3\sqrt{\alpha}} - 2(\alpha + 1)\lambda_1 \lambda_2 + \lambda_2^2 (\alpha - 1) \xi^{3\sqrt{\alpha}} \right]}{\alpha},$$  \hspace{1cm} (6.31)

which should always be positive, $m_{ImT}^2 \geq 0$ for stability in the imaginary field direction. Recalling that dS vacuum solutions are acquired when $\lambda_1$ and $\lambda_2$ have the same sign, we introduce the ratio coefficient $\gamma = \lambda_1/\lambda_2$, which must always be positive.

To visualize this condition, we plot in Fig. 6.1 the $(\alpha, \gamma)$ plane with $T$ on the vertical axis, and the size of $\log \left( m_{ImT}^2 / 4 \lambda_2^2 \right)$ indicated by color coding. The boundary of the colored region corresponds to the critical value of $m_{ImT}^2 / 4 \lambda_2^2 = 0$, and it indicates when $m_{ImT}^2$ becomes unstable. Interestingly, the same general expression (6.31) holds
also for more complicated forms of $\xi$. It is important to note that Fig. 6.1 shows two colored regions which are separated by a gap, and indicates that the dS vacuum becomes unstable in the imaginary direction for certain values of $T$ and $\alpha$.

Figure 6.1: Illustration of the value of the expression (6.31) as a function of $(\alpha, \gamma, T)$, as shown by the color coding for $\log\left(m_{\text{Im} T}^2/4\lambda_2^2\right)$ on the right-hand side.

To understand the occurrence of the dS vacuum instability, we consider two specific cases with different values of $\alpha$, where for illustrative purposes we choose $\lambda_1 = \lambda_2 = 1$, and we use the field parametrization $T = (x + iy)/\sqrt{2}$. The effective scalar potential is plotted in the left panel of Fig. 6.2 for $\alpha = 1$, which is characteristic of solutions with $\alpha \leq 1$. We see that dS vacuum solutions are always unstable in the imaginary field direction, so these solutions must be stabilized. In the right panel of Fig. 6.2, we show the scalar potential with $\alpha = 3$, which is characteristic of solutions with $\alpha > 1$. Here,
Figure 6.2: The effective scalar potential $V(x, y)$ without quartic stabilization terms in the imaginary direction ($\beta = 0$), for the cases $\alpha = 1$ (left panel) and $\alpha = 3$ (right panel).

we see that vacuum solutions might fall into an AdS vacuum, which corresponds to the gap region shown in Fig. 6.1. In both cases, the potential is completely flat along the line $y = 0$ corresponding to the dS solution up to the point where $x = 0$ (the potential is not defined at $x \leq 0$).

To address the stability issue, we consider the modified K"ahler potential (6.16), where if we compare it to the general K"ahler potential (6.24), we see that in the real direction the argument inside the logarithm remains unchanged, with $\xi = 2T$.

The generalization of the mass squared in Eq. (6.31) is:

$$m^2_{ImT} = \frac{4}{\alpha} \left[ \lambda_1^2 (\alpha - 1 + 12 \beta \cdot \xi^3) \xi^{-3\sqrt{\alpha}} - 2 (\alpha + 1 - 12 \beta \cdot \xi^3) \lambda_1 \lambda_2 \right. \left. + \lambda_2^2 (\alpha - 1 + 12 \beta \cdot \xi^3) \xi^{3\sqrt{\alpha}} \right], \quad (6.32)$$

where it can readily be seen from the numerator of (6.32) that, by choosing a value of $\beta$ that is large enough, we can always make the imaginary field direction stable.\(^4\) We plot in Fig. 6.3 the unstable cases considered previously with $\alpha = 1$ and $\alpha = 3$, which have been each stabilized with the choice $\beta = 2$. Once again, the potential along $y = 0$ is flat.

\(^4\)A similar expression when $\gamma = 1$ can be found in [235].
Figure 6.3: The effective scalar potential $V(x, y)$ for the two values $\alpha = 1$ (left panel) and $\alpha = 3$ (right panel), now stabilized by quartic terms in the imaginary direction with $\beta = 2$.

6.2 Multi-Moduli Models

6.2.1 Minkowski Vacuum for Two Moduli

Our next step is to extend this formulation to the two- and multi-moduli cases. As before, we first construct the general Minkowski vacuum solutions and then use Minkowski superpotential pairs to obtain dS/AdS solutions. We begin by considering the following two-field Kähler potential

$$K = -3\alpha_1 \ln(V_1) - 3\alpha_2 \ln(V_2).$$

(6.33)

For now, we consider $V_1 = T_1 + \tilde{T}_1$ and $V_2 = T_2 + \tilde{T}_2$. Along the real directions, $T_1 = \tilde{T}_1$ and $T_2 = \tilde{T}_2$, we adopt the following notation

$$V_1 \rightarrow \xi_1, \quad V_2 \rightarrow \xi_2,$$

(6.34)

and we choose the following ansatz that yields Minkowski vacuum solutions:

$$W_M = \lambda \cdot \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2}.$$  

(6.35)
Inserting the superpotential (6.35) into the expression (3.83) for the effective scalar potential, we obtain

\[ V = \lambda^2 \cdot \xi_1^{2n_1-3\alpha_1} \cdot \xi_2^{2n_2-3\alpha_2} \cdot \left( \frac{(2n_1-3\alpha_1)^2}{3\alpha_1} + \frac{(2n_2-3\alpha_2)^2}{3\alpha_2} - 3 \right). \] (6.36)

To recover Minkowski vacua, we set \( V = 0 \), which holds when the following expression is satisfied [235]:

\[ \frac{(2n_1-3\alpha_1)^2}{3\alpha_1} + \frac{(2n_2-3\alpha_2)^2}{3\alpha_2} = 3. \] (6.37)

For ease of illustration, we introduce the following parametrization:

\[ r_1 \equiv \frac{2n_1-3\alpha_1}{3\sqrt{\alpha_1}}, \quad r_2 \equiv \frac{2n_2-3\alpha_2}{3\sqrt{\alpha_2}}, \] (6.38)

in terms of which the general expression (6.37) becomes:

\[ r_1^2 + r_2^2 = 1. \] (6.39)

Solving the constraint (6.37) for \( n_1 \) and \( n_2 \), we find:

\[ n_1 = \frac{3}{2} \left( \alpha_1 \pm \sqrt{1 - \frac{(2n_2-3\alpha_2)^2}{9\alpha_2} \cdot \sqrt{\alpha_1}} \right), \] (6.40)

and

\[ n_2 = \frac{3}{2} \left( \alpha_2 \pm \sqrt{1 - \frac{(2n_1-3\alpha_1)^2}{9\alpha_1} \cdot \sqrt{\alpha_2}} \right), \] (6.41)

which can be parametrized using (6.38):

\[ n_1 = \frac{3}{2} (\alpha_1 + r_1 \sqrt{\alpha_1}) \quad \text{and} \quad n_2 = \frac{3}{2} (\alpha_2 + r_2 \sqrt{\alpha_2}), \] (6.42)

where the values of \( r_1 \) and \( r_2 \) are constrained by expression (6.39), and must satisfy the condition \( r_i \in \{-1, 1\} \). It can already be seen from these equations that the circular parametrization (6.38) simplifies our expressions significantly, and it will be useful in
establishing a geometric connection. We must also satisfy the following inequalities:

\[ \alpha_i > 0, \quad \text{with } i = 1, 2. \]  

(6.43)

We see from (6.42) that we can consider a total of four different sign combinations that yield \( V = 0 \). The corresponding expressions for the imaginary masses-squared are given by:

\[ m_{Im}^2 T_i = 4 \lambda^2 \cdot \xi_1^{3r_1 \sqrt{\alpha_1}} \cdot \xi_2^{3r_2 \sqrt{\alpha_2}} (\alpha_i - r_i^2), \quad \text{with } i = 1, 2, \]  

(6.44)

where stability in the imaginary direction is obtained when the condition \( \alpha_i - r_i^2 \geq 0 \) is satisfied. If we this combine this inequality with the constraint (6.39), we obtain another stability condition in terms of the curvature parameters:

\[ \alpha_1 + \alpha_2 \geq 1. \]  

(6.45)

6.2.2 Minkowski Pair Formulation for Two Moduli

Applying the same approach that we used for the case of a single modulus, we now show how to construct Minkowski pairs for the two-field case and recover dS/AdS vacuum solutions with \( V = 12 \lambda_1 \lambda_2 \) (as in (6.30)) along the direction where all fields are real. The general dS/AdS vacuum solutions for the two-field case are given by

\[ W_{dS/AdS} = \lambda_1 \cdot \xi_1^{n_1} \cdot \xi_2^{n_2} - \lambda_2 \cdot \xi_1^{\bar{n}_1} \cdot \xi_2^{\bar{n}_2}, \]  

(6.46)

where we define

\[ \bar{n}_1 \equiv \frac{3}{2} (\alpha_1 + \bar{r}_1 \sqrt{\alpha_1}) \quad \text{and} \quad \bar{n}_2 \equiv \frac{3}{2} (\alpha_2 + \bar{r}_2 \sqrt{\alpha_2}), \]  

(6.47)

with the expressions for \( n_{1,2} \) being given by Eq. (6.42) and \( \bar{r}_i = -r_i \). We note that the powers (6.47) describe the antipode of a point lying on the surface of a circle described by the coordinates \((r_1, r_2)\), and we discuss the geometric interpretation of our models in the next section.

The scalar masses recovered from the dS/AdS superpotential (6.46) have complicated expressions that we do not list here. However, we note that we can always modify
the initial Kähler potential (6.33) by including higher-order corrections in the imaginary direction:

\[ K = -3 \alpha_1 \ln \left( T_1 + \overline{T}_1 + \beta_1 \left( T_1 - \overline{T}_1 \right)^4 \right) - 3 \alpha_2 \ln \left( T_2 + \overline{T}_2 + \beta_2 \left( T_2 - \overline{T}_2 \right)^4 \right) , \tag{6.48} \]

where these quartic terms easily remedy the stability problems [235]. If we compare it to the general two-field Kähler potential in Eq. (6.33), along the real directions, \( T_1 = \overline{T}_1 \) and \( T_2 = \overline{T}_2 \), we recover \( \xi_1 = 2T_1 \) and \( \xi_2 = 2T_2 \). In the next section, we extend this formulation to the \( N \)-field case.

6.2.3 Minkowski Pair Formulation for Multiple Moduli

We now show how to generalize our formulation and construct successfully the Minkowski pair superpotential for cases with \( N > 2 \) moduli. We first introduce the following Kähler potential

\[ K = -3 \sum_{i=1}^{N} \alpha_i \ln (\mathcal{V}_i) , \tag{6.49} \]

where \( \mathcal{V}_i = T_i + \overline{T}_i \). Next, we impose the condition that all our fields are real, therefore \( T_i = \overline{T}_i \), which leads to

\[ \mathcal{V}_i \rightarrow \xi_i , \quad \text{for } i = 1, 2, ..., N . \tag{6.50} \]

Minkowski vacuum solutions are obtained with the choice

\[ W_M = \lambda \cdot \prod_{i=1}^{N} \xi_i^{n_i} \tag{6.51} \]

in the general \( N \)-field case. Inserting the superpotential (6.51) into Eq. (3.83), we find

\[ V = \lambda^2 \cdot \prod_{i=1}^{N} \xi_i^{2n_i - 3\alpha_i} \cdot \left( \sum_{i=1}^{N} \frac{(2n_i - 3\alpha_i)^2}{3\alpha_i} - 3 \right) , \tag{6.52} \]
and it can be seen from Eq. (6.52) that in order to obtain Minkowski vacuum solutions, \( V = 0 \), we must satisfy the constraint

\[
\sum_{i=1}^{N} \frac{(2n_i - 3\alpha_i)^2}{3\alpha_i} = 3. \tag{6.53}
\]

Once again, we introduce the following parametrization

\[
r_i \equiv \frac{2n_i - 3\alpha_i}{3\sqrt{\alpha_i}}, \quad \text{for } i = 1, 2, \ldots, N, \tag{6.54}
\]

and combining the equations (6.53) and (6.54) we obtain

\[
\sum_{i=1}^{N} r_i^2 = 1. \tag{6.55}
\]

Therefore, Eq. (6.55) parametrizes the \( N \)-field Minkowski solutions as lying on the surface of an \((N-1)\)-sphere.

Solving Eq. (6.54) for \( n_i \), we obtain

\[
n_i = \frac{3}{2} \left( \alpha_i + r_i \sqrt{\alpha_i} \right), \quad \text{for } i = 1, 2, \ldots, N, \tag{6.56}
\]

where \( r_i \in \{-1, 1\} \) and \( \alpha_i > 0 \). For the \( N \)-moduli case, we obtain the following expression for the scalar masses-squared in the imaginary directions

\[
m_{m_{T_i}}^2 = \frac{4\lambda^2 (\alpha_i - r_i^2) \prod_{i=1}^{N} \xi_i^{-3r_i \sqrt{\alpha_i}}}{\alpha_i}, \quad \text{with } i = 1, 2, \ldots, N. \tag{6.57}
\]

To obtain a stable solution in the imaginary direction, we must satisfy the condition \( \alpha_i - r_i^2 \geq 0 \). If we use the constraint of the \((N-1)\)-sphere (6.55), we obtain the following stability condition

\[
\sum_{i=1}^{N} \alpha_i \geq 1. \tag{6.58}
\]
Following the procedure described previously, we combine a pair of Minkowski solutions (6.51) and introduce the following dS/AdS superpotential

\[ W_{dS/AdS} = \lambda_1 \cdot \prod_{i=1}^{N} \xi_i^{n_i} - \lambda_2 \cdot \prod_{i=1}^{N} \xi_i^{h_i}, \]  

(6.59)

where \( \bar{n}_i = \frac{3}{2} (\alpha_i + \bar{r}_i \sqrt{\alpha_i}) \), with \( \bar{r}_i = -r_i \). This superpotential form also yields the familiar dS/AdS vacuum result \( V = 12 \lambda_1 \lambda_2 \).

It proves difficult to perform a detailed stability analysis for \( N \)-moduli models, because this would involve finding the eigenvalues of an \( N \times N \) matrix. Nevertheless, one can always introduce higher-order corrections in the Kähler potential (6.49):

\[ K = -3 \sum_{i=1}^{N} \alpha_i \ln \left( T_i + T_i + \beta_i (T_i - T_i)^4 \right), \]  

(6.60)

where the quartic terms stabilize the imaginary directions [235]. If we compare the multi-moduli Kähler potential (6.49) with (6.60), we see that along the real directions, \( T_i = T_i \), and we recover \( \xi_i = 2T_i \).

6.2.4 Geometric Interpretation

We now discuss the geometric interpretation of this Minkowski pair formulation. From equations (6.53)-(6.55), it is clear that our parametrization describes Minkowski superpotential solutions (6.51) that lie on the surface of an \( (N-1) \)-sphere that is embedded in Euclidean \( N \)-space. We first return to the two-moduli case, in which the Eq. (6.55) reduces to (6.39), and all Minkowski solutions lie on a circle embedded in 2-dimensional space. We define the radius vector of points on a circle \( r \) by

\[ r = (r_1, r_2), \quad \text{with} \quad r_1^2 + r_2^2 = 1. \]  

(6.61)

As expected, equation (6.61) includes 4 possible sign combinations corresponding to different quadrants of a circle. To construct successfully a Minkowski pair superpotential that yields a dS/AdS vacuum solution, we must combine any chosen point on the circle
with its antipodal point, given by the vector

$$\bar{r} = -r = -(r_1, r_2) .$$  \hspace{1cm} (6.62)

In this way, we can construct an infinite number of distinct Minkowski superpotential pairs by considering different point/antipode combinations lying on the surface of a circle. The Minkowski pair construction on a circle is illustrated in Fig. 6.4. For any value of $\alpha > 0$, Eq. (6.46) will yield a dS or AdS solution so long as $n_i = \frac{3}{2} (\alpha_i + r_i \sqrt{\alpha_i})$ and $\bar{n}_i = \frac{3}{2} (\alpha_i + \bar{r}_i \sqrt{\alpha_i})$.

We can readily generalize this framework to the $N$-moduli case, in which we define the radius vector $r$ to lie on the surface of an $(N-1)$-sphere, and it is expressed as:

$$r = (r_1, r_2, ..., r_N), \quad \text{with} \quad \sum_{i=1}^{N} r_i^2 = 1 ,$$  \hspace{1cm} (6.63)

while the antipodal vector $\bar{r}$ is given by

$$\bar{r} = -(r_1, r_2, ..., r_N) .$$  \hspace{1cm} (6.64)

As an illustration, we consider the three-field case: $N = 3$. In this case, the Minkowski solutions lie anywhere on the surface of the unit sphere. Then dS and AdS solutions can be obtained from any point on the sphere, by combining it with this antipodal point with $r_i \rightarrow -r_i$. In Fig. 6.5 we show an example where four different Minkowski vacuum solutions are combined into 2 distinct Minkowski pairs lying on the surface of a sphere.

We have seen how all Minkowski pair solutions lie on the surface of an $(N-1)$-sphere of unit radius, and recall the general expressions for the corresponding powers, $n_i$ and $\bar{n}_i$, of $\xi$ given earlier: We show in Fig. 6.6 Minkowski pair solutions for these powers as functions of $|r_i|$ and $\alpha_i$. The lower yellow sheet illustrates the possible choices for $n_i$, while the upper blue sheet illustrates the possible choices for $\bar{n}_i$. If we are only concerned with Minkowski solutions, we can freely choose any point lying on either the upper or lower sheet, which leads to $V = 0$. In order to construct successfully a Minkowski pair, we need to combine our chosen point with the corresponding point on the opposite sheet, which will yield the dS/AdS solution $V = 12 \lambda_1 \lambda_2$. 

Figure 6.4: Depiction of Minkowski pairs on a circle. The circle is split into four quadrants and two distinct Minkowski pairs are shown lying in different quadrants. The red dots show a Minkowski pair solution $r = (\sqrt{3}/2, 1/2)$ and $\bar{r} = (-\sqrt{3}/2, -1/2)$, which lies in the first and third quadrants of the circle, while the blue dots show a Minkowski pair solution $r = (-1/2, \sqrt{3}/2)$ and $\bar{r} = (1/2, -\sqrt{3}/2)$, which lies in the second and fourth quadrants of the circle.

Having established successfully a geometric connection between unique vacuum solutions, in the remaining sections we show that identical patterns emerge for Kähler
Figure 6.5: Illustration of Minkowski pairs on the surface of a sphere. The sphere is split into eight octants and two distinct Minkowski pairs lying in different octants are shown. The red dots represent a Minkowski pair solution $r = (1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ and $\bar{r} = (-1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, which lies in the fourth and sixth octants of the sphere, while the blue dots represent a Minkowski pair solution $r = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $\bar{r} = (-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$, which lies in the first and seventh octants of the sphere.

potential forms with untwisted and twisted matter fields.
Figure 6.6: Illustration of Minkowski pair formulation on the $n_i$ (yellow) and $\bar{n}_i$ (blue) sheets. The Minkowski pairs are depicted by red dots and their coordinates are given by $(1, \frac{1}{2}, \frac{3}{4})$ with $(1, \frac{1}{2}, \frac{9}{4})$ and $(2, \frac{3}{4}, 3 - \frac{9}{8}\sqrt{2})$ with $(2, \frac{3}{4}, 3 + \frac{9}{8}\sqrt{2})$.

6.3 Minkowski Pairs with Matter Fields

6.3.1 The Untwisted Case

In this section, we extend our formulation to no-scale models with untwisted matter fields. We begin by considering the following Kähler potential, which parametrizes a non-compact $\frac{SU(2,1)}{SU(2) \times U(1)}$ coset space:

$$K = -3\alpha \ln \left( T + \bar{T} - \frac{\phi \bar{\phi}}{3} \right),$$  \hspace{1cm} (6.65)
where $\alpha$ is a curvature parameter, $T$ can be interpreted as a volume modulus, and $\phi$ is a matter field. Moreover, we impose the conditions $T = \bar{T}$ and $\phi = \bar{\phi}$ by fixing the VEVs of the imaginary components of the fields to zero, along the lines discussed above. Clearly, Eq. (6.65) can be written in the form of Eq. (6.24) with $V$ set equal to the argument of the log in (6.65), $V = T + \bar{T} - \frac{\phi \bar{\phi}}{3}$.

Once again, when we restrict to real fields, and in this case set $T = \bar{T}$ and $\phi = \bar{\phi}$ we obtain

$$V \rightarrow \xi, \quad \text{with} \quad \xi = 2T - \frac{\phi^2}{3}.$$ \hfill (6.66)

We then consider the following form of superpotential:

$$W_M = \lambda \cdot \xi^n,$$ \hfill (6.67)

which leads to the following effective scalar potential:

$$V = \lambda^2 \xi^{2n-3\alpha} \cdot \left( \frac{(2n - 3\alpha)^2}{3\alpha} - 3 \right).$$ \hfill (6.68)

To obtain a Minkowski vacuum: $V = 0$, we solve the constraint for $n$, and recover the familiar result given in Eq. (6.5) for the case with a single modulus. Using the superpotential in Eq. (6.67), we find the following scalar masses-squared for the imaginary components of the fields $T$ and $\phi$:

$$m^2_{ImT} = \frac{4\lambda^2(\alpha - 1)\xi^{\pm3\sqrt{\alpha}}}{\alpha}, \quad m^2_{Im\phi} = \frac{4\lambda^2(\sqrt{\alpha} \pm 1)\xi^{\pm3\sqrt{\alpha}}}{\sqrt{\alpha}},$$ \hfill (6.69)

whereas, as anticipated, the masses-squared of the real components are $m^2_{ReT} = 0$ and $m^2_{Re\phi} = 0$. It can be seen from Eqs. (6.69) that stability in the imaginary directions for both fields requires that the inequalities $\alpha \geq 1$.

To construct the SU(2,1) Minkowski pair formulation, we follow the previous discussion and use the same superpotential as in Eq. (6.29)

$$W_{dS/AdS} = \lambda_1 \cdot \xi^{n-} - \lambda_2 \cdot \xi^{n+}.$$ \hfill (6.70)

Doing so, we recover dS/AdS vacuum solutions given by Eq. (6.30). In this case, the
masses-squared for the imaginary field components are given by

\begin{equation}
    m^2_{T_{Im}} = \frac{4}{\alpha} \left( \lambda_1^2 (\alpha - 1) \xi^{-3} \sqrt{\alpha} - 2(\alpha + 1)\lambda_1 \lambda_2 + \lambda_2^2 (\alpha - 1) \xi^3 \sqrt{\alpha} \right)
\end{equation}

and

\begin{equation}
    m^2_{T_{Im,\phi}} = \frac{4}{\sqrt{\alpha}} \left( \frac{\lambda_1^2 (\sqrt{\alpha} - 1) \xi^{-3} \sqrt{\alpha} + 4\lambda_1 \lambda_2 + \lambda_2^2 (\sqrt{\alpha} + 1) \xi^3 \sqrt{\alpha}}{\sqrt{\alpha}} \right).
\end{equation}

We do not discuss here the stabilization of these components, but we can always include quartic stabilization terms in the Kähler potential (6.65), as discussed previously.

Having established the principles in the case of the $SU(2,1)$ Kähler potential with an untwisted matter field $\phi$, we can generalize our formulation to no-scale models that parametrize a non-compact $SU(N,1)$ coset manifold. Following the same recipe considered in previous sections, we start with the Kähler potential (6.24), and we define the argument inside the logarithm as

\begin{equation}
    V \equiv T + \bar{T} - \sum_{j=1}^{N-1} \frac{|\phi_j|^2}{3}.
\end{equation}

Furthermore, we fix the VEVs of the imaginary fields to zero, so that $T = \bar{T}$ and $\phi_j = \bar{\phi}_j$. Using the same notation

\begin{equation}
    V \rightarrow \xi, \text{ when } T = \bar{T} \text{ and } \phi_j = \bar{\phi}_j,
\end{equation}

the argument inside the logarithm in the Kähler potential becomes

\begin{equation}
    \xi = 2T - \sum_{j=1}^{N-1} \frac{|\phi_j|^2}{3}.
\end{equation}

With this definition of $\xi$, Minkowski vacuum solutions are found for the same choice of superpotential given in Eq. (6.67). The masses-squared of the imaginary components, with $m^2_{T_{Im}}$ and $m^2_{T_{Im,\phi}}$ are given by (6.69).

At this point, it should not be surprising that by combining two distinct Minkowski solutions we can form a Minkowski superpotential pair given by Eq. (6.70). This dS/AdS superpotential yields identical scalar masses-squared for the imaginary components,
with \( m^2_{\text{Im} T} \) given by (6.71) and \( m^2_{\text{Im} \phi_j} \) given by (6.72).

Finally, we can also extend our formulation to more complicated Kähler potentials that take the form

\[
K = -3 \prod_{i=1}^{M} \alpha_i \ln(\mathcal{V}_i), \quad \text{with } \mathcal{V}_i = T_i + \overline{T}_i - \sum_{j=1}^{N-1} \frac{|\phi_{ij}|^2}{3}.
\]

(6.76)

We again assume that \( T_i = \overline{T}_i \) and \( \phi_{ij} = \overline{\phi}_{ij} \), which leads to

\[
\mathcal{V}_i \rightarrow \xi_i, \quad \text{when } T_i = \overline{T}_i \text{ and } \phi_{ij} = \overline{\phi}_{ij}.
\]

(6.77)

Thus, we obtain the following Minkowski pair superpotential

\[
W_{dS/AdS} = \lambda_1 \cdot \prod_{i=1}^{M} \xi_i^n - \lambda_2 \cdot \prod_{i=1}^{M} \overline{\xi_i}^n, \quad \text{with } V = 12 \lambda_1 \lambda_2,
\]

(6.78)

which coincides with the multi-moduli case considered previously.

### 6.3.2 The Twisted Case

An analogous Minkowski pair formulation can also be considered in the case of twisted matter fields. We consider the corresponding Kähler potential

\[
K = -3 \alpha \ln (T + \overline{T}) + \varphi \overline{\varphi},
\]

(6.79)

where we introduce the notation \( \varphi \) for twisted matter fields. To this end, we first find a relatively simple superpotential form that yields Minkowski solutions, and consider the following Ansatz:

\[
W_M = \lambda \cdot (2T)^n \cdot e^{-\varphi^2/2}.
\]

(6.80)

Combining it with the effective scalar potential in Eq. (3.83), and setting \( T = \overline{T} \) and \( \varphi = \overline{\varphi} \) by fixing the VEVs of the imaginary components of the fields to zero, we obtain

\[
V = \lambda^2 \cdot (2T)^{2n-3\alpha} \cdot \left( \frac{(2n-3\alpha)^2}{3\alpha} - 3 \right).
\]

(6.81)
From the form of the scalar potential, we see that it does not depend on \( Re \phi \). To obtain a Minkowski vacuum solution, we find the same solutions found in Eq. (6.5) for \( n \). This yields the following scalar masses-squared for the imaginary components:

\[
m_{Im T}^2 = \frac{4\lambda^2(\alpha - 1)}{\alpha} \cdot (2T)^{\pm 3\sqrt{\alpha}} \tag{6.82}
\]

and

\[
m_{Im \varphi}^2 = 4 \lambda^2 (2T)^{\pm 3\sqrt{\alpha}}. \tag{6.83}
\]

We can see from Eqs. (6.82) and (6.83) that \( Im \varphi \) is always stable, and that \( Im T \) is stable when \( \alpha \geq 1 \).

Similarly, we also consider the following Ansatz:

\[
W_M = \lambda \cdot (2T)^n \cdot e^{+\varphi^2/2}. \tag{6.84}
\]

If we combine this with Eq. (3.83), and set \( T = T \) and \( \varphi = -\bar{\varphi} \), we obtain the same effective scalar potential (6.81) with solutions for \( n \) given by Eq. (6.5). In this case, the scalar potential does not depend on \( Im \varphi \), and the scalar masses-squared are given by Eqs. (6.82) and (6.83).\(^5\)

Therefore, there are two ways to construct Minkowski vacuum solutions with twisted matter fields that do not depend on either the real or imaginary components of \( \varphi \).

Next, we construct the dS/AdS superpotential by combining two distinct Minkowski solutions

\[
W_{dS/AdS} = (\lambda_1 \cdot (2T)^{n_-} - \lambda_2 \cdot (2T)^{n_+}) \cdot e^{-\varphi^2/2}, \tag{6.85}
\]

where we choose a Minkowski pair construction which does not depend on \( Re \varphi \), and, if we assume that \( T = T \) and \( \varphi = \bar{\varphi} \), the effective scalar potential (3.83) is given by Eq. (6.30) once again. In the case of the superpotential (6.85), the scalar masses-squared of the imaginary field components are:

\[
m_{Im T}^2 = \frac{4 \left( \lambda_1^2 (\alpha - 1)(2T)^{-3\sqrt{\alpha}} - 2(\alpha + 1)\lambda_1 \lambda_2 + \lambda_2^2 (\alpha - 1)(2T)^{3\sqrt{\alpha}} \right)}{\alpha} \tag{6.86}
\]

\(^5\)It is important to note that in this case the effective scalar potential has curvature in the real direction and the scalar mass-squared expression (6.83) becomes \( m_{Re \varphi}^2 \).
and
\[ m^2_{Im\,\varphi} = 4 \left( \lambda_1^2 (2T)^{-3\sqrt{\alpha}} + 4\lambda_1 \lambda_2 + \lambda_2^2 (2T)^{3\sqrt{\alpha}} \right). \] (6.87)

It is important to note that for de Sitter solutions, while \( m^2_{Im\,\varphi} \) is always positive, \( m^2_{Im\,T} \) is not and may require quartic stabilization terms in the imaginary direction for the field \( T \).

Analogously, one can also consider the following dS/AdS superpotential form
\[ W_{dS/AdS} = \lambda_1 \cdot (2T)^n - \lambda_2 \cdot (2T)^{n+} \cdot e^{+\varphi^2/2}, \] (6.88)

where, after setting \( T = \bar{T} \) and \( \phi = -\bar{\phi} \), we obtain the dS/AdS scalar potential \( V = 12\lambda_1 \lambda_2 \), with the scalar masses-squared given by (6.86) and (6.87).

This analysis with a single twisted matter field can be generalized to include multiple fields. We consider the following Kähler potential form
\[ K = -3\alpha \ln(V) + \sum_{j=1}^{N} |\varphi_j^2|. \] (6.89)

In this case, all of the previous results hold after the simple substitution of \( \varphi^2 \rightarrow \sum \varphi_j^2 \).

Another possible generalization is to consider Kähler potentials of the form \( K = \sum_i K_i + \sum_j |\varphi_j|^2 \), where each \( K_i \) is of no-scale type
\[ K = -3 \sum_{i=1}^{M} \alpha_i \ln(V_i) + \sum_{j=1}^{N} |\varphi_j|^2, \quad \text{with} \ V_i = T_i + \bar{T}_i. \] (6.90)

As before, we assume that:
\[ V_i \rightarrow \xi_i, \quad \text{when} \ T_i = \bar{T}_i. \] (6.91)

In this case, with a superpotential of the form
\[ W_M = \lambda \cdot \prod_{i=1}^{M} \xi_i^{n_i} \cdot \exp \left( -\frac{1}{2} \sum_{j=1}^{N} \omega_j \varphi_j^2 \right), \] (6.92)

where \( \omega_j \) can take a value of either +1 or -1, we get a Minkowski solution \( V = 0 \) after
setting $T_i = \bar{T}_i$ and $\varphi_j = \omega_j \bar{\varphi}_j$. Similarly, we can obtain dS/AdS solutions $V = 12 \lambda_1 \lambda_2$ along the direction $T_i = \bar{T}_i$, $\varphi_j = \omega_j \bar{\varphi}_j$ from the superpotential

$$W_{dS/AdS} = \left( \lambda_1 \cdot \prod_{i=1}^{M} \xi_{i}^{n_i} - \lambda_2 \cdot \prod_{i=1}^{M} \bar{\xi}_{i}^{n_i} \right) \cdot \exp \left( -\frac{1}{2} \sum_{j=1}^{N} \omega_j \varphi_j^2 \right). \quad (6.93)$$

### 6.3.3 The Combined Case

Finally, we note that one can consider more complicated cases combining twisted and untwisted matter fields by following the principles discussed earlier in this section. The only difference is that one needs to modify the Kähler potential in Eq. (6.90) and introduce untwisted matter fields $\phi_{ik}$:

$$K = -3 \sum_{i=1}^{M} \alpha_i \ln(V_i) + \sum_{j=1}^{N} |\varphi_j|^2, \quad \text{with} \quad V_i = T_i + \bar{T}_i + \sum_{k=1}^{P-1} |\phi_{ik}|^2, \quad (6.94)$$

If we assume that all our fields are fixed to be real, this leads to

$$V_i \rightarrow \xi_i, \text{ when } T_i = \bar{T}_i, \phi_{ik} = \bar{\phi}_{ik}, \text{ and } \varphi_j = \bar{\varphi}_j, \quad (6.95)$$

and for this case Minkowski solutions are given by superpotential (6.92) and dS/AdS solutions are given by (6.93).

### 6.4 Applications to Inflationary Models

#### 6.4.1 Inflation with an Untwisted Matter Field

We now indicate briefly how to construct inflationary models in this framework [237, 238]. For simplicity, we use a non-compact $SU(2) \times U(1)$ Kähler potential (6.65), and we associate the matter field $\phi$ with the inflaton. If we set $\alpha = 1$, the Kähler potential (6.65) becomes

$$K = -3 \ln \left( T + \bar{T} - \frac{\phi \bar{\phi}}{3} \right). \quad (6.96)$$
Next, we introduce a unified superpotential that combines the Minkowski pair superpotential $W_{\text{dS}}$ with an inflationary superpotential $W_I = f(\phi)$:

$$W = W_I + W_{\text{dS}} = Mf(\phi) + \lambda_1 - \lambda_2 \left(2T - \frac{\phi^2}{3}\right)^3 . \quad (6.97)$$

We also require that supersymmetry is broken at the minimum through the Minkowski pair superpotential $W_{\text{dS}}$ instead of the inflationary superpotential $W_I$. Therefore, we impose the conditions that $f(0) = f'(0) = 0$. Again, we assume that $T = \bar{T}$ and $\phi = \bar{\phi}$, and the superpotential (6.97) then yields the following effective scalar potential

$$V = 12\lambda_1 \lambda_2 + 12\lambda_2 Mf(\phi) + \frac{f'(\phi)^2}{\left(2T - \frac{\phi^2}{3}\right)^2}, \quad (6.98)$$

where we can safely neglect the mixing terms between $\lambda_2$ and $M$, leading to the approximation

$$V \simeq 12\lambda_1 \lambda_2 + \frac{M^2 f'(\phi)^2}{\left(2T - \frac{\phi^2}{3}\right)^2} . \quad (6.99)$$

Supersymmetry is broken by an $F$-term, which is given by

$$\sum_{i=1}^2 |F_i|^2 = F_T^2 = (\lambda_1 + \lambda_2)^2 , \text{ with } m_{3/2} = \lambda_1 - \lambda_2 . \quad (6.100)$$

### 6.4.2 Inflation with a Twisted Matter Field

Following the same approach, we now show how to construct viable inflationary models with a twisted inflaton field. We use a non-compact $\frac{SU(1,1)}{U(1)} \times U(1)$ Kähler potential form (6.79), and we associate the matter field $\varphi$ with the inflaton. We set $\alpha = 1$, and Eq. (6.79) reduces to

$$K = -3\ln (T + \bar{T}) + \varphi \bar{\varphi} . \quad (6.101)$$
Next, we introduce the following unified superpotential form\(^6\)

\[
W = W_I + W_{dS} = \left( M f(\varphi) + \lambda_1 - \lambda_2 (2T)^3 \right) \cdot e^{-\varphi^2/2},
\]

(6.102)

where the inflationary superpotential is given by \(W_I = M f(\varphi) \cdot e^{-\varphi^2/2}\). We again require supersymmetry to be broken through the Minkowski pair superpotential \(W_{dS}\), and we impose the conditions that at the minimum we must have \(f(0) = f'(0) = 0\). The superpotential form (6.102) leads to the following effective scalar potential

\[
V = 12\lambda_1\lambda_2 + 12\lambda_2 M f(\varphi) + \frac{M^2 f'(\varphi)^2}{8T^3}.
\]

(6.103)

If we neglect the mixing terms between \(\lambda_2\) and \(M\), and fix \(\langle T \rangle = \frac{1}{2}\), we can approximate

\[
V \simeq 12\lambda_1\lambda_2 + M^2 f'(\varphi)^2,
\]

(6.104)

and supersymmetry breaking is characterized by the same expression given in Eq. (6.100). In order to construct a Starobinsky-like inflationary potential that is a function of the field \(\varphi\), we use the following canonical field redefinition

\[
\varphi = \frac{x + iy}{\sqrt{2}}.
\]

(6.105)

and we assume that \(\varphi = \overline{\varphi} = \frac{x}{\sqrt{2}}\). We then introduce the following inflationary superpotential form

\[
W_I = \frac{3}{4} M \left( \frac{2\varphi}{\sqrt{3}} + e^{-\frac{2\varphi}{\sqrt{3}}} - 1 \right) e^{-\varphi^2/2},
\]

(6.106)

and assume that \(\varphi = \overline{\varphi} = \frac{x}{\sqrt{2}}\) and \(T = \overline{T} = 1/2\), which yields the Starobinsky inflationary potential with a positive cosmological constant at the minimum

\[
V = 12\lambda_1\lambda_2 + 3\lambda_2 M \left( \sqrt{6} x + 3e^{-\sqrt{\frac{2}{3}} x} - 3 \right) + \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{\frac{2}{3}} x} \right)^2,
\]

(6.107)

\(^6\)Similarly, we can consider a unified superpotential form with \(W_{dS}\) given by (6.88). In this case, inflation will be driven by \(Im \varphi\).
or if we neglect the mixing terms between $\lambda_2$ and $M$, we obtain

$$V \simeq 12 \lambda_1 \lambda_2 + \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{\frac{2}{3} x}} \right)^2.$$  \hspace{1cm} (6.108)

### 6.5 Concluding Remarks

In this chapter, we have exhibited the unique choice of superpotential leading to a Minkowski vacuum in a single-field no-scale supergravity model, and also shown how to construct dS/AdS solutions using pairs of these single-field Minkowski superpotentials. We have then extended these constructions to two- and multifield no-scale supergravity models, providing also a geometrical interpretation of the dS/AdS solutions in terms of combinations of superpotentials that are functions of fields at antipodal points on hyperspheres. As we have also shown, these constructions can be extended to scenarios with additional twisted or untwisted fields, and we have also discussed how Starobinsky-like inflationary models can be constructed in this framework.

The models described in this chapter provide a general framework that is suitable for constructing unified supergravity cosmological models that include a primordial near-dS inflationary epoch that is consistent with CMB measurements, the transition to a low-energy effective theory incorporating soft supersymmetry breaking at some scale below that of inflation, and a small present-day cosmological constant (dark energy). As such, this framework is suitable for constructing complete models of cosmology and particle physics below the Planck scale.

For the future, two general classes of issues stand out. One is the construction of specific models for sub-Planckian physics, which should address the incorporation of Standard Model (and possibly other) matter and Higgs degrees of freedom. Should these be described by twisted or untwisted fields, and how are they coupled to the inflaton? Specific answers to some of these issues have been proposed in [241, 242].

Another set of issues concerns the interface with string theory. For example, although no-scale supergravity theories arise generically in the low-energy limits of string compactifications, many different non-compact coset manifolds may be realized. Which of these is to be preferred? Another set of questions concerns the specific forms of superpotential that are needed to obtain a Minkowski or dS vacuum. In this chapter, we
have constructed them from a bottom-up approach, and demonstrated their uniqueness. How could one hope to obtain them in a top-down approach, starting from a specific string model?

This question is particularly acute in the case of dS vacuum solutions, since swampland conjectures [243] suggest that string theory may not possess such vacua. At the time of writing controversy still swirls about these conjectures, and in this chapter, we have taken the pragmatic approach of exploring what such solutions would look like. As such, our solutions may suggest avenues to explore in searching for them, or at least the obstacles to be overcome.
Chapter 7

Unified No-Scale Attractors

Physics contains many hierarchies of mass scales, starting from the Planck scale $M_P \sim 10^{18}$ GeV at which the effects of quantum gravity must become important, through the energy scale of cosmological inflation, which is $\lesssim 10^{13}$ GeV, through the electroweak scale $\sim 100$ GeV down to the energy scale of dark energy, a.k.a. the cosmological constant, which is $\sim 10^{-3}$ eV. What are the origins of these hierarchies, and how can they be stabilized in a natural way despite the depredations of quantum corrections? Diverse origins have been proposed, and in this chapter we focus on the question of how they can be accommodated within a simple framework that incorporates a mechanism for stabilizing hierarchies of mass scales.

That framework is provided by supersymmetry, which could stabilize the electroweak hierarchy if the supersymmetry-breaking scale is $\lesssim 1$ TeV, and could also stabilize the parameters of an inflationary scalar potential at some scale $\ll M_P$ [29]. On the other hand, simple supersymmetry is insufficient by itself to render natural the small magnitude of the cosmological constant, and it would need to be supplemented by dynamical mechanisms to generate the hierarchies of mass scales. In the context of cosmology, supersymmetry must be combined with general relativity within some form of supergravity theory [44].

No-scale supergravity has been shown to yield Starobinsky-like models of inflation [84], under suitable conditions on the theoretical parameters [85], and we have recently characterized general conditions under which de Sitter (dS) vacua can be accommodated within no-scale supergravity [235]. Upgrading such models to something
resembling the Standard Model (SM) in a more realistic way requires a deeper discussion on how matter fields should be incorporated, that should also include a mechanism for supersymmetry breaking. Previously, this has often been done by invoking some variant of the Polonyi model in which supersymmetry breaking originates dynamically within a hidden sector [214, 216, 55, 244, 245].

In this chapter, we build upon [235, 246], generalizing the characterization of de Sitter (dS) no-scale supergravity models with SU(2,1)/SU(2) × U(1) Kähler manifolds. Without extending the field content, we introduce a mechanism that allows for a transition between two dS vacua, one that can accommodate Starobinsky-like inflation and one with an amount of vacuum energy that could be very small, like the present cosmological constant (dark energy), without invoking any external mechanism such as uplifting by fibres [247]. As we show, this class of models also allows for supersymmetry breaking with a magnitude suitable for stabilizing the electroweak hierarchy, without invoking any hidden sector à la Polonyi. Additionally, a mechanism proposed previously [81, 225] can be used to fix the compactification modulus of the SU(2,1)/SU(2) × U(1) model.

7.1 De Sitter Vacua and No-Scale Attractors

The successes of inflation motivate attempts to relate it to the Standard Model (SM) of laboratory particle physics on the one hand and, on the other hand, to a candidate quantum theory of everything including gravity, such as string theory. The characteristic energy scale of inflation is presumably intermediate between those of the SM and quantum gravity, and models of inflation may provide a welcome bridge between them.

In this chapter, we discuss no-scale supergravity models based on a Kähler potential that could be written in the form [74, 75]

\[ K = -3 \alpha \ln \left( T + \bar{T} - \frac{\phi^2}{3} \right), \]

parametrizing a \( \frac{SU(2,1)}{SU(2) \times U(1)} \) coset Kähler manifold with the choice \( \alpha = 1 \), corresponding to an Einstein space with curvature \( R = 2/\alpha = 2 \). In this case, the prefactor 3 guarantees that in the absence of a superpotential the effective potential vanishes. Hence it
is *a fortiori* independent of the field $T$, which may correspond to a generic compactification volume modulus, and also $\phi$, which represents a generic chiral matter field. The no-scale model (7.1) with $\alpha = 1$ was the starting-point for our recent classification of Starobinsky-like models of inflation [246] and their extension to include supersymmetry breaking and dark energy [237]. The possibility of constructing inflationary models with different values of $\alpha$ was first discussed in [225], where it was noted that for $\alpha \lesssim \mathcal{O}(1)$ such models predict

$$
\begin{align*}
    n_s &\simeq 1 - \frac{2}{N_*} , \quad r \simeq \frac{12\alpha}{N_*^2} ,
\end{align*}
$$

(7.2)

where $N_*$ is the number of $e$-folds of inflation. Similar models were later discussed in more detail in [232, 239, 248, 249], where they were termed $\alpha$-attractors. Here we give a general treatment of the construction of no-scale Starobinsky-like inflationary models with $\alpha \neq 1$.

There are two directions in which the no-scale framework can be generalized: one is to consider Kähler manifolds that are direct products of irreducible components, i.e., their Kähler potentials take the form $K = \sum_n K_n$ where each $K_n$ is of no-scale type, and the other is to consider no-scale coset manifolds parametrized by multiple chiral fields, e.g., of the forms $\frac{SU(N,1)}{SU(N) \times U(1)}$: $N > 2$. In the former case, each of the $K_n$ is of logarithmic form, but they may have different prefactors. For example, in the case of multiple $\frac{SU(1,1)}{U(1)}$ cosets one may postulate the Kähler potential form (6.49), where the quantities $\alpha_i$ are positive, in general. Alternatively, the single-coset Kähler potential would be generalized to multiple chiral fields (6.76), and these two options may be combined.

In Witten’s original model for string compactification based on the dimensional reduction of 10-dimensional supergravity [77], there was a single compactification volume modulus $T$ with a Kähler potential of the form (7.1) with $\alpha = 1$. However, in a more general class of string compactifications one expects three complex Kähler moduli $T_n : n = 1, 2, 3$ with a combined Kähler potential of the form (6.49) with $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. This opens up two additional options for inflation, using either a single complex modulus with $\alpha = \frac{1}{3}$, or a pair linked together as an ‘area’ modulus in a single logarithmic Kähler potential with an effective prefactor $\alpha = \frac{2}{3}$. These were the possibilities mentioned in [225].
However, there are other chiral fields that one expects to appear in the effective supergravity theory derived from general string theory compactifications. For example, one generally encounters a complex dilaton/axion field $S$ and some number of complex structure moduli $U^a$. In general, the dynamical framework for each of these is again a logarithmic non-compact coset Kähler potential, and a popular example is the $STU$ model

$$K = - \sum_{n=1}^{3} \ln (T^n + \bar{T}_n) - \ln (S + \bar{S}) - \sum_{a=1}^{3} \ln (U^a + \bar{U}_a),$$

(7.3)
in which case the values of $3\alpha_i$ in (6.49) can take integer values $\leq 7$, and can arise if inflation is driven by some linked combination of the $S, T^n$ and $U^a$ fields, as discussed in [236].

In Chapter 5, we presented a general classification of Starobinsky-like inflationary avatars of $\text{SU}(2) \times \text{U}(1)$ no-scale supergravity in the case $\alpha = 1$, discussing explicit forms of superpotentials and relations between them. In this chapter, we extend these constructions to models with $\alpha \neq 1$, and also consider the generalization to models based on $\text{SU}(N) \times \text{U}(1)$ no-scale supergravity.

To understand better the geometric properties of this framework, we recall that the Kähler curvature for an $\text{SU}(N) \times \text{U}(1)$ Kähler manifold can be calculated from the following expression [75]

$$R^j_i = (\log \det G^b_a)^j_i,$$

(7.4)

and the scalar curvature is then\(^1\)

$$R = (G^{-1})^i_j R^j_i.$$  

(7.5)

For a Kähler potential of the form in Eq. (6.76), we find

$$R = \frac{N(N + 1)}{3\alpha},$$

(7.6)

which reduces to the familiar result $R = 2/3$ in minimal $\text{SU}(1) \times \text{U}(1)$ no-scale supergravity when $N = 1$ and $\alpha = 1$. Note that, because $R$ depends on the total number of chiral

\(^1\)For $R$ we use the convention that $R > 0$ for a hyperbolic manifold and $R < 0$ for a spherical manifold as in [74].
fields in the theory, we cannot determine the Kähler curvature $R$ from the cosmological observables $n_s$ and $r$ alone, though one can in general write

$$R \simeq \frac{N(N+1)(1-n_s)^2}{r},$$  \hspace{1cm} (7.7)

for $\alpha \ll O(1)$.

It is well-known that the no-scale formalism we have been discussing is suitable for producing inflationary models of the Starobinsky type [85, 225, 246]. Indeed, for $\alpha = 1$, there are many known examples of superpotentials that lead to a Starobinsky potential for inflation. The Wess-Zumino model [85], in which the inflaton is associated with a matter field, and the Cecotti model [230], in which the inflaton is associated with $T$, are the two cases most often considered. It has also been shown [246] that these models can be related through the underlying $\frac{SU(N,1)}{SU(N) \times U(1)}$ symmetry of the non-compact no-scale coset space, and we show later how models with a matter inflaton can be extended to arbitrary values of the curvature parameter, $\alpha$.

The original Starobinsky model is characterized by the following action

$$S = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left( R + \mu \frac{R^2}{M^2} \right),$$  \hspace{1cm} (7.8)

where $\mu = \frac{1}{6}$ and $M \ll M_P$. One then makes the following Weyl transformation

$$\tilde{g}_{\mu\nu} = e^{2\Omega} g_{\mu\nu} = \left( 1 + \frac{\phi}{3M^2} \right) g_{\mu\nu},$$  \hspace{1cm} (7.9)

and uses the field redefinition $\phi' = \sqrt{\frac{3}{2}} \ln \left( 1 + \frac{\phi}{3M^2} \right)$. The action (7.8) then becomes

$$S = \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left( \tilde{R} - \partial_\mu \phi' \partial^\mu \phi' - \frac{3}{2} M^2 \left( 1 - e^{-\sqrt{\frac{3}{2}} \phi'} \right)^2 \right),$$  \hspace{1cm} (7.10)

where $\phi'$ is a canonical field. From Eq. (7.10), we can see that the Starobinsky inflationary potential will be given by

$$V = \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{\frac{3}{2}} \phi'} \right)^2.$$  \hspace{1cm} (7.11)
It is interesting to note the correspondence between the $R^2$ (de Sitter) and $R + R^2$ (Starobinsky) theories of gravity and no-scale supergravity [234]. The supergravity Lagrangian written in terms of an Einstein-Hilbert action requires a conformal transformation, which can be expressed in terms of the Kähler potential $2\Omega = -K/3$ [39]. The details of the correspondence then lie in the choice of the superpotential. The pure $R^2$ case requires the choice of $W$ given in (6.3), and the superpotential for the $R + R^2$ models is discussed in the next section.

We note here that the Starobinsky potential can be generalized by changing the Kähler curvature to $\alpha \neq 1$. In this case, the corresponding conformal transformation is related to the Kähler potential by $2\Omega = -K/3\alpha$ and the $\alpha$-Starobinsky scalar potential becomes (again with a suitable choice of superpotential)

$$V = \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{\frac{2}{3\alpha}} \phi'} \right)^2. \quad (7.12)$$

The cosmological observables for $\alpha$-Starobinsky potential (7.12) are given by (7.2). Our goal in the following is to unify no-scale models that incorporate the $\alpha$-Starobinsky inflationary model (or a general no-scale attractor inflationary potential) at a scale $\mathcal{O}(10^{13})$ GeV, with an adjustable scale for supersymmetry breaking and a cosmological constant $\mathcal{O}(10^{-120})$. To achieve such unification, we must stabilize strongly the volume modulus $T$, an aspect that is deferred to the next section.

For illustration, we plot $\alpha$-Starobinsky potential forms with different values of $\alpha$ in Fig. 7.1. We can see from the Figure that increasing the value of the curvature parameter $\alpha$ stretches the Starobinsky potential horizontally, reducing the flatness of the plateau at any fixed value of $\phi'$.

We show in Fig. 4.4 the predicted cosmological observables for these $\alpha$-Starobinsky potentials in the $(n_s, r)$ plane, together with the results of the Planck collaboration combined with other CMB data, indicated by blue shadings corresponding to the 68% and 95% confidence level regions [28].\(^2\) As the curvature parameter $\alpha$ increases, the value of the scalar tilt $n_s$ changes only slightly and stays within the range $\sim 0.96 - 0.97$.

\(^2\)Note that the $\alpha$-Starobinsky predictions include corrections beyond the small-$\alpha$ values shown in (7.2), since they were calculated by numerical integration of the equation of motion for the inflaton field. Similar predictions were shown in Fig. 1 of [239], but corrections of higher order in $\alpha$ are absent from the corresponding $\alpha$-attractor predictions in Fig. 1 of [236] and Fig. 8 of [28].
Figure 7.1: The $\alpha$-Starobinsky potentials for different values of the curvature parameter $\alpha$. The blue line corresponds to the original Starobinsky inflationary potential with $\alpha = 1$, the yellow line corresponds to $\alpha = 5$, the green line corresponds to $\alpha = 10$, and the orange line corresponds to $\alpha = 30$.

while the tensor-to-scalar ratio $r$ increases with the value of $\alpha$. The CMB data set a 68% upper bound on the tensor-to-scalar ratio $r \sim 0.055$, which is attained for $\alpha \sim 51$ when $n_s \sim 0.967$ for a nominal choice of $N_* \simeq 55$, as indicated by the blue star. The green dots and line at small $r$ show the prediction of the original Starobinsky model, corresponding to the case $\alpha = 1$. It is apparent that future measurements of $r$ will be able to constrain $\alpha$ more significantly, and that more precise measurements of $n_s$ could in principle constrain $n_s$, thereby $N_*$ and hence the post-inflationary history of the universe, which is sensitive to the decay of the inflaton into low-mass particles [137].


7.2 Unified No-Scale Models

7.2.1 Supersymmetry Breaking and Inflationary Dynamics

We now show how one can combine the dark energy sector $W_{dS}$ with various inflationary models to obtain a unified no-scale supergravity model. Because we want to consider Minkowski pair models that break supersymmetry, we impose the condition that $W_I$ does not break the supersymmetry at the minimum. For simplicity, we consider models based on a non-compact $\frac{SU(2,1)}{SU(2) \times U(1)}$ coset space with curvature parameter set to $\alpha = 1$, i.e., the Kähler potential given in Eq. (7.1). In order to incorporate supersymmetry breaking via the Minkowski pair superpotential $W_{dS}$, we need to consider specific inflationary superpotential forms that do not lead to unphysical supersymmetry-preserving AdS vacua states. It can readily be shown that the Minkowski pair construction combined with an arbitrary superpotential that is a function of the volume modulus $T$ only, $W_I = f(T)$, can give dS vacuum states with broken supersymmetry. However, we do not consider such models here because, as discussed in [225], superpotentials of this form cannot lead to Starobinsky-like inflationary potentials with unbroken supersymmetry at the minimum.

It was shown in [85] that the Starobinsky model of inflation can be obtained from the Wess-Zumino form of superpotential, which is a function of the matter-like field $\phi$ only. Thus, we are led to consider the following inflationary superpotential form:

$$W_I = f(\phi),$$

where we assume that the minimum is located at $\langle \phi \rangle = 0$. The superpotential (7.13) yields the following compact form of scalar potential:

$$V = \frac{f'(\phi)^2}{\left(2T - \frac{\phi^2}{F}\right)^2}. \quad (7.14)$$

and requiring vanishing vacuum energy at the minimum imposes the condition $f'(0) = 0$. If we also require the minimum to be supersymmetric, we have $D_TW_I = \partial_TW_I + K_TW_I \simeq -3f(\phi) = 0$, which implies that $f(0) = 0$.\(^3\)

\(^3\)We note that $D\phi W = 0$ is automatically satisfied if $\phi = 0$ and $f'(0) = 0$ at the minimum.
We can now combine the inflaton superpotential (7.13) with the Minkowski pair superpotential $W_{dS}$, that was discussed in the previous chapter, obtaining

$$W = W_I + W_{dS} = f(\phi) + \lambda_1 - \lambda_2 \left(2T - \frac{\phi^2}{3}\right)^3,$$

(7.15)

which yields the following effective scalar potential

$$V = 12\lambda_1\lambda_2 + 12\lambda_2 f(\phi) + \frac{f'(\phi)^2}{2 \left(2T - \frac{\phi^2}{3}\right)^2},$$

(7.16)

Taking derivatives with respect to the fields $T$ and $\phi$, we obtain

$$V_T = -\frac{4f'(\phi)^2}{\left(2T - \frac{\phi^2}{3}\right)^3},$$

(7.17)

and

$$V_\phi = 6f'(\phi) \left(3 \left(2T - \frac{\phi^2}{3}\right) f''(\phi) + 2\phi f'(\phi)\right) \left(\frac{9 \left(2T - \frac{\phi^2}{3}\right)^3}{2 \left(2T - \frac{\phi^2}{3}\right)^2} + 2\lambda_2\right).$$

(7.18)

We can see from these expressions that, as long as the condition $f'(0) = 0$ is satisfied, the position of the minimum does not shift when the dark energy superpotential $W_{dS}$ is introduced. In addition, at the minimum the effective scalar potential (7.16) reduces to the dS vacua solutions $V = 12\lambda_1\lambda_2$. While the $F$-term for $\phi$ remains zero (when $\phi = 0$ and $f'(0) = 0$ at the minimum), the $F$-term for $T$ is proportional to $D_T W = -3(f(\phi) + \lambda_1 + \lambda_2) = -3(\lambda_1 + \lambda_2)$, indicating that supersymmetry is broken. Thus, we have shown that supersymmetry breaking with a positive cosmological constant can be achieved if one considers an inflationary superpotential of the form $W_I = f(\phi)$.

However, the dark energy sector cannot be combined with a general superpotential of the form $W_I = F(T, \phi)$. To illustrate that, we consider the following superpotential form

$$W_I = F(T, \phi) = f(T) \cdot \phi.$$  

(7.19)
We obtain from (7.19) the following scalar potential

\[ V = \frac{2}{3}T \phi^2 f'(T)^2 - \frac{4}{3} \frac{\phi^2 f(T) f'(T) + f(T)^2}{(2T - \frac{\phi^2}{3})^2}, \]  

(7.20)

whose derivative with respect to \( \phi \) is given by

\[ V_\phi = \frac{4\phi \left( -6f(T) \left( 2T + \frac{\phi^2}{3} \right) f'(T) + 3T \left( 2T + \frac{\phi^2}{3} \right) f'(T)^2 + 3f(T)^2 \right) }{9 \left( 2T - \frac{\phi^2}{3} \right)^3}. \]  

(7.21)

We see from this that the minimum must be located at \( \langle \phi \rangle = 0 \), and we assume that the minimum is also at \( \langle T \rangle = 1/2 \). Using this condition, we obtain the following expression for the derivative with respect to \( T \) at the minimum:

\[ V_T = 2f \left( 1/2 \right) \left[ f' \left( 1/2 \right) - 2f \left( 1/2 \right) \right], \]  

(7.22)

which shows that at the minimum we must also satisfy the condition \( f(1/2) = 0 \), which also guarantees \( V = 0 \) at the minimum.

Next, we combine the inflationary superpotential (7.19) with the dark energy sector

\[ W = W_I + W_{dS} = f(T) \cdot \phi + \lambda_1 - \lambda_2 \left( 2T - \frac{\phi^2}{3} \right)^3, \]  

(7.23)

which yields the following effective scalar potential

\[ V = \left( \frac{1}{2T - \frac{\phi^2}{3}} \right)^2 \left[ \frac{2}{3} T \phi^2 f'(T)^2 + 12 \lambda_2 \left( 2T - \frac{\phi^2}{3} \right)^2 \left( \phi f(T) + \lambda_1 \right) 
\]  

\[ + f(T)^2 - 2\phi f'(T) \left( \frac{1}{3} \left( 2\phi f(T) + 3\lambda_1 \right) + \lambda_2 \left( 2T - \frac{\phi^2}{3} \right)^3 \right) \right]. \]  

(7.24)

The minimum of this more complicated scalar potential is shifted to a new position. To find this shift, we consider small perturbations around the initial minimum, given by

\[ f^{(n)} \left( 1/2 + \delta t \right) \simeq f^{(n)} \left( 1/2 \right) + f^{(n+1)} \left( 1/2 \right) \cdot \delta t, \]  

with \( n = 0, 1, 2 \).  

(7.25)
and

\[ \langle \phi \rangle + \delta \phi = \delta \phi. \]  

(7.26)

Using these perturbations in the scalar potential (7.24), we find:

\[ \delta \phi \simeq \frac{3(\lambda_1 + \lambda_2)}{f'(1/2)} \delta T \simeq -\frac{3(\lambda_1 + \lambda_2)^2}{f'(1/2)^2}, \]

(7.27)

which yields the following effective scalar potential at the minimum:

\[ V \simeq -3(\lambda_1 - \lambda_2)^2, \]

(7.28)

We see from (7.28) that, when \( \lambda_1 \neq \lambda_2 \), the minimum always shifts to a supersymmetry-preserving AdS vacuum, at least for small \( \lambda_{1,2} \).

The Cecotti form of the superpotential \( W_I = \sqrt{3}M\phi(T - 1/2) \) [230] falls into the category of superpotentials that is not suitable for combination with our Minkowski pair formulation of broken supersymmetric dS solutions. We argued previously in Chapter 5 for an equivalence among the many avatars of superpotentials yielding Starobinsky inflation based on an the underlying \( SU(2,1) \) invariance. However, when \( W_{dS} \) is added to the theory, this invariance is broken and the avatars are no longer equivalent. Therefore, in the remainder of the chapter, we combine the dark energy superpotential \( W_{dS} \) with inflationary superpotentials that are functions of matter fields only. More general forms for \( W(T, \phi) \) may be possible, but we have not explored the general conditions on \( W \) and its derivatives for models other than the three sets of models that have attracted most interest in this context.

We note that the inflationary superpotential \( W_I \) contains a single parameter, denoted by \( M \), which is not of order unity. In the Wess-Zumino model discussed in more detail below, \( M \) corresponds to the inflaton mass and its magnitude is set by the normalization of the scalar density perturbations [28], so that \( M \simeq 1.2 \times 10^{-5}M_P \simeq 3 \times 10^{13} \text{ GeV} \). In contrast, the constants in \( W_{dS} \) must be significantly smaller. Supersymmetry breaking is characterized by an \( F \)-term of the form in Eq. (8.43) and is given by

\[ \sum_{i=1}^{2} |F_i|^2 = F_T^2 = \frac{(\lambda_1 + \lambda_2)^2}{\alpha}, \]

(7.29)
where the supersymmetry breaking is generated through an $F$-term for $T$. For $\alpha = 1$, $F_T = (\lambda_1 + \lambda_2)$, and the gravitino mass is given by

$$m_{3/2} = e^{G/2} = e^{K/2} W = (\lambda_1 - \lambda_2)$$

at the minimum, and is independent of $\alpha$. Thus we expect the difference of the two parameters to be of order $10^{-16}$ in Planck units. Hence the terms in potential (7.16) that are coupled to $\lambda_{1,2}$ can be safely neglected during inflation, and do not affect the slow-roll dynamics.

We recall that the vacuum energy density at the minimum is given by $\Lambda = 12\lambda_1\lambda_2$. Thus, in the absence of any phase transitions, we must require $12\lambda_1\lambda_2 \sim \mathcal{O}(10^{-120})$, which is possible if one of the two constants is hierarchically much smaller than the other, e.g., $\lambda_1 \sim 10^{-15}$ and $\lambda_2 \sim 10^{-105}$. However, we know that the vacuum energy today (i.e., the cosmological constant) is a sum of contributions that have changed during phase transitions throughout the history of the universe. For example, the electroweak transition would make a contribution $\sim -\mathcal{O}(10^{-60})$, which could be cancelled to a sufficiently small value if $\lambda_1 \sim 10^{-15}$ and $\lambda_2 \sim 10^{-45}$. However, a grand unified (GUT) transition would plausibly make a contribution $\sim -\mathcal{O}(10^{-30})$, corresponding to vacuum energy of $\sim 10^{11}$ GeV as is typical in flipped SU(5) models where the vacuum energy is related to $m_{3/2}^2 M_{GUT}^2$ (see, e.g., [250]) which could be cancelled by $\lambda_1 \sim \lambda_2 \sim 10^{-15}$ to provide a suitable cosmological constant today. Without loss of generality, we can define

$$\lambda_1 = \tilde{\lambda}_1 M^3 \quad \lambda_2 = \tilde{\lambda}_2 M^3,$$

and we return below to the possibility of a cancellation when we discuss the Wess-Zumino model in more detail.

### 7.2.2 Multi-field No-Scale Attractors with Supersymmetry Breaking

It is relatively straightforward to generalize the previous results to multi-field no-scale attractors. We consider the Kähler potential

$$K = -3\alpha_1 \ln(V_1) - 3 \sum_{n=2}^{P+1} \alpha_n \ln(V_n),$$

(7.32)
where
\[ V_1 = T + \bar{T} - \sum_{i=1}^{N-1} \frac{|\phi_i|^2}{3}, \quad V_n = T_n + \bar{T}_n : n > 1, \]  
(7.33)

which parametrizes a non-compact \( \frac{SU(N,1)}{SU(N)\times U(1)} \times \left[ \frac{SU(1,1)}{U(1)} \right]^P \) coset space. We now study its Minkowski vacuum solutions. Just as was done before, we restrict our attention to the real directions of the chiral fields, so that
\[ V_i \rightarrow \xi_i, \quad \text{when} \quad T_n = \bar{T}_n, \quad \phi_i = \bar{\phi}_i. \]  
(7.34)

As previously, this requirement can always be achieved dynamically by introducing higher-order stabilization terms, as we discuss in more detail when we consider multi-field inflationary models in the next section.

As shown in the previous chapter, to recover successfully Minkowski vacuum solutions for the \( \frac{SU(N,1)}{SU(N)\times U(1)} \times \left[ \frac{SU(1,1)}{U(1)} \right]^P \) coset space, we must consider the following superpotential form
\[ W_M = \lambda \cdot \prod_{n=1}^{P+1} \xi_n^{\frac{3}{2}(\alpha_n - r_n\sqrt{\alpha_n})}, \quad \text{with} \quad \sum_{n=1}^{P+1} r_n^2 = 1, \]  
(7.35)

where, as in Eq. (6.4), we include only one of the two solutions, \( \lambda \) refers to either \( \lambda_1 \) or \( \lambda_2 \), and \( \alpha_n > 0 \). Clearly we can interpret the Minkowski superpotential (7.35) as being specified on the surface of a \( P \)-sphere with a radius 1.

In order to obtain de Sitter vacua solutions for this \( \frac{SU(N,1)}{SU(N)\times U(1)} \times \left[ \frac{SU(1,1)}{U(1)} \right]^P \) coset space, which may be interpreted as solutions with dark energy, we combine two antipodal points lying on the \( P \)-sphere
\[ W_{dS} = \lambda_1 \cdot \prod_{n=1}^{P+1} \xi_n^{\frac{3}{2}(\alpha_n - r_n\sqrt{\alpha_n})} - \lambda_2 \cdot \prod_{n=1}^{P+1} \xi_n^{\frac{3}{2}(\alpha_n - \bar{r}_n\sqrt{\alpha_n})}, \]  
(7.36)

where \( r = (r_1, r_2, ..., r_{P+1}) \), and the antipodal vector is given by \( \bar{r} = -(r_1, r_2, ..., r_{P+1}) = -r \). For convenience, we adopt a notation in which the antipodal vector is written as \( \bar{r} = -r \), so that all our expressions can be expressed in terms of the radial components \( r_n \).
Next, we combine the Kähler potential (7.32) with the dS vacuum solutions (7.36). The matter fields \( \phi_i \), which will be associated with inflaton fields, are described by the no-scale Kähler potential

\[
K_1 = -3\alpha_1 \ln(V_1),
\]

where the function \( V_1 \) is given by (7.33). The multi-field no-scale attractors will be characterized by a curvature parameter \( \alpha_1 \).

As discussed in the previous section, we assume that the superpotential associated with inflation is a function of matter fields only. Thus, the unified superpotential can be expressed as

\[
W = W_I + W_{dS} = f(\phi) + \lambda_1 \cdot \prod_{n=1}^{P+1} \xi_n^2(\alpha_n - r_n \sqrt{\alpha_n}) - \lambda_2 \cdot \prod_{n=1}^{P+1} \xi_n^2(\alpha_n + r_n \sqrt{\alpha_n}),
\]  

(7.37)

where \( W_I = f(\phi) \), and \( \phi \equiv \{ \phi_1, \phi_2, ..., \phi_{N-1} \} \). Exactly as before, we require that at the minimum \( W_I = f(0) = 0 \) and \( f'(0) = 0 \). We assume that the chiral fields at the minimum obtain vacuum expectation values \( \langle T^n \rangle = \frac{1}{2} \) and \( \langle \phi^i \rangle = 0 \), and in this case the gravitino mass becomes \( m_{3/2} = \lambda_1 - \lambda_2 \), also as before, while the \( F \)-term giving rise to supersymmetry breaking (8.43) is given by

\[
\sum_{i=1}^{P+1} |F_i|^2 = \sum_{i=1}^{P+1} \frac{r_i^2}{\alpha_i} (\lambda_1 + \lambda_2)^2.
\]  

(7.38)

Note that we have only included the sum over moduli in (7.38). It is relatively easy to see that the \( F \)-terms associated with matter fields (including the inflaton) are all zero. Next, we use the Kähler potential expression (7.32) with the superpotential form (7.37), and obtain the following effective scalar potential

\[
V = 12\lambda_1 \lambda_2 + 3f(\phi)^2 \left( \sum_{n=1}^{P+1} \alpha_n - 1 \right) \prod_{n=1}^{P+1} \xi_n^{-3\alpha_n} + \prod_{n=1}^{P+1} \xi_n^{-3\alpha_n} \cdot \prod_{n=1}^{P+1} \xi_n^{-\frac{3}{2}(\alpha_n - r_n \sqrt{\alpha_n})} - \lambda_1 \left( 1 + \sum_{n=1}^{P+1} r_n \sqrt{\alpha_n} \right) \cdot \prod_{n=1}^{P+1} \xi_n^{-\frac{3}{2}(\alpha_n + r_n \sqrt{\alpha_n})}
\]

\[
-6f(\phi) \cdot \left[ \lambda_1 \left( 1 + \sum_{n=1}^{P+1} r_n \sqrt{\alpha_n} \right) \cdot \prod_{n=1}^{P+1} \xi_n^{-\frac{3}{2}(\alpha_n - r_n \sqrt{\alpha_n})} \right] - \lambda_2 \left( 1 - \sum_{n=1}^{P+1} r_n \sqrt{\alpha_n} \right) \cdot \prod_{n=1}^{P+1} \xi_n^{-\frac{3}{2}(\alpha_n + r_n \sqrt{\alpha_n})},
\]  

(7.39)

where \( \partial_i f(\phi) \equiv \frac{\partial f(\phi)}{\partial \phi_i} \). We can identify four distinct contributions to \( V \). The first term
is once again the vacuum energy density after inflation. The second term is proportional
to $f^2$ and is potentially dangerous, as it could seriously impact the inflaton potential.
The third term is the generalization of the inflaton potential, and the final term is
related to the supersymmetry-breaking terms. These terms can be neglected during
inflation, as $\lambda_i \ll M$.

In order to safeguard Starobinsky-like inflation, we must ensure the absence of the
second term. There are two ways to achieve this. The first is rather obvious, namely
we could require

$$\sum_{n=1}^{P+1} \alpha_n = 1. \quad (7.40)$$

However, there is a more elegant (and general) solution to this problem, which at the
same time simplifies the scalar potential. The key is to couple our inflationary potential
to the Minkowski vacuum solution, given by (7.35). Thus, we consider the following
superpotential form

$$W = W_{dS} + W_M \cdot W_I, \quad (7.41)$$

which amounts to adding $f(\phi)$ to either $\lambda_1$ or $\lambda_2$ in Eq. (7.35). Thus, we can couple
the inflationary superpotential in two different ways: either

$$W = (f(\phi) + \lambda_1) \cdot \prod_{n=1}^{P+1} \xi_n^{\alpha_n - r_n \sqrt{\alpha_n}} - \lambda_2 \cdot \prod_{n=1}^{P+1} \xi_n^{\alpha_n + r_n \sqrt{\alpha_n}}, \quad (7.42)$$

which yields the following scalar potential:

$$V = 12\lambda_1\lambda_2 + 12\lambda_2 f(\phi) + \frac{\prod_{n=1}^{P+1} \xi_n^{-3\alpha_n}}{\alpha_1} \cdot \xi_1 \cdot \sum_{i=1}^{N-1} \partial_i f(\phi)^2, \quad (7.43)$$

or as a second possibility

$$W = \lambda_1 \cdot \prod_{n=1}^{P+1} \xi_n^{\alpha_n - r_n \sqrt{\alpha_n}} - (f(\phi) + \lambda_2) \cdot \prod_{n=1}^{P+1} \xi_n^{\alpha_n + r_n \sqrt{\alpha_n}}, \quad (7.44)$$
which gives the following scalar potential

\[
V = 12\lambda_1\lambda_2 + 12\lambda_1 f(\phi) + \frac{\prod_{n=1}^{P+1} \xi_{-3\alpha_n}}{\alpha_1} \cdot \sum_{i=1}^{N-1} \partial_i f(\phi)^2.
\]  

(7.45)

We note that the scalar potentials (7.43) and (7.45) have relatively simple forms. Once again, the first term in each equation is the vacuum energy density after inflation, and the second term, while proportional to \(f(\phi)\), is rendered harmless as it is proportional to one of the two small coefficients \(\lambda_i\). The third term in each case leads to \(\alpha\)-Starobinsky inflation. We need only impose the conditions that \(f(\phi)\) and \(f'(\phi)\) are zero at the minimum.

As mentioned previously, to stabilize the moduli fields \(T_n\) dynamically at their vacuum expectation values to \(\langle T_n \rangle = \frac{1}{2}\), we introduce quartic stabilization terms [81, 225] in the Kähler potential. Thus, the complete Kähler potential based on non-compact \(\frac{SU(N,1)}{SU(N) \times U(1)} \times \frac{SU'(1,1)}{U(1)}\) coset space takes the form

\[
K = -3\alpha_1 \ln \left[ T + T^\dagger + \beta^R \left( T + T^\dagger - 1 \right)^4 + \beta^I \left( T - T^\dagger \right)^4 - \sum_{i=1}^{N-1} \frac{\left| \phi_i \right|^2}{3} \right] \\
-3 \sum_{n=2}^{P+1} \alpha_n \ln \left[ T^n + T_n^\dagger + \beta_n^R \left( T^n + T_n^\dagger - 1 \right)^4 + \beta_n^I \left( T^n - T_n^\dagger \right)^4 \right].
\]  

(7.46)

Inflation is described by the generalization of \(-3\alpha_1 \ln(\mathcal{V}_1)\), where we assume that inflation is driven by the matter fields \(\phi^i\). The term proportional to \(\beta^R\) fixes \(\langle T \rangle = \frac{1}{2}\) as needed to generate the Starobinsky potential. Since the potential is actually flat along the real \(T^n\) directions, we use the terms proportional to the \(\beta_n^R\) to fix the remaining real parts of the moduli to the same value.\(^4\) Hence, during inflation, we obtain \(\xi_n = 1\) for \(n \geq 2\), and the scalar potential forms (7.43) and (7.45) can be approximated by

\[
V \simeq \frac{\sum_{i=1}^{N-1} \partial_i f(\phi)^2}{\alpha_1 \xi_1^{3\alpha_1-1}},
\]  

(7.47)

\(^4\)The actual fixed values are unimportant, and can be fixed to a set of constants \(c_n\).
which, after fixing the volume modulus to $\langle T \rangle = \frac{1}{2}$, becomes

$$V \simeq \frac{\sum_{i=1}^{N-1} \partial_i f(\phi)^2}{\alpha_1 \left( 1 - \sum_{i=1}^{N-1} \frac{\phi_i^2}{3} \right)^{3/2}}. \quad (7.48)$$

The next step is to obtain the kinetic terms for our unified multi-field no-scale attractor models. After setting the moduli to their vacuum values, the Kähler potential (7.32) yields

$$L_{\text{kin}} = \frac{\alpha_1}{\left( 1 - \sum_{i=1}^{N-1} \frac{\phi_i^2}{3} \right)^{3/2}} \left[ \sum_{i=1}^{N+1} \left( 1 - \sum_{j=1, j \neq i}^{N-1} \frac{\phi_j^2}{3} \right) \cdot (\partial_\mu \phi_i)^2 + \frac{2}{3} \sum_{i,j=1; i \neq j}^{N-1} (\phi_i \partial_\mu \phi_i) \cdot (\phi_j \partial_\mu \phi_j) \right]. \quad (7.49)$$

Although the kinetic terms of the Lagrangian (7.49) may appear complicated, the Lagrangian is still highly symmetric.

## 7.3 No-Scale $\alpha$-Starobinsky Models with Supersymmetry Breaking

In this section, we examine in more detail no-scale $\alpha$-Starobinsky models characterized by a non-compact $\text{SU}(2,1)/\text{SU}(2) \times \text{U}(1)$ Kähler potential (7.1). As discussed before, we assume that the no-scale Kähler potential is modified to include a quartic stabilization term which fixes the VEV of the volume modulus $\langle T \rangle = \frac{1}{2}$. During inflation, the imaginary part of $\phi$ picks up a mass, so we can take $\langle \text{Im} \phi \rangle = 0$, and we associate the real part of the field $\phi$ with the inflaton. We make the following field redefinition to obtain a canonically-normalized field

$$\phi = \sqrt{3} \tanh \left( \frac{x}{\sqrt{6\alpha}} \right). \quad (7.50)$$

It was shown in [85] for $\alpha = 1$ that Starobinsky inflation can be derived from an $\text{SU}(2,1)/\text{SU}(2) \times \text{U}(1)$ Kähler potential with a Wess-Zumino superpotential

$$W_I = M \left( \frac{\phi^2}{2} - \frac{\phi^3}{3\sqrt{3}} \right), \quad (7.51)$$
which leads to the effective scalar potential given in Eq. (7.11) with the replacement $\phi' \rightarrow x$.

The Wess-Zumino superpotential (7.51) can be combined with the supersymmetry breaking and dark energy sector $W_{dS}$ using Eq. (7.31)

$$W = W_I + W_{dS} = M \left( \frac{\phi^2}{2} - \frac{\phi^3}{3\sqrt{3}} \right) + \lambda_1 M^3 - \tilde{\lambda}_2 M^3 \left( 2T - \frac{\phi^2}{3} \right)^3, \text{ for } \alpha = 1. \quad (7.52)$$

The unified Wess-Zumino model (7.52) with the fields fixed at $\langle T \rangle = \frac{1}{2}$ and $\langle \text{Im } \phi \rangle = 0$ then yields the following scalar potential

$$V = 12\tilde{\lambda}_1 \tilde{\lambda}_2 M^6 + 12\tilde{\lambda}_2 M^4 \left( \frac{\phi^2}{2} - \frac{\phi^3}{3\sqrt{3}} \right) + 3M^2 \left( \frac{\phi}{\sqrt{3} + \phi} \right)^2, \quad (7.53)$$

which, after canonical field redefinition (7.50), becomes:

$$V = 12\tilde{\lambda}_1 \tilde{\lambda}_2 M^6 + 6\tilde{\lambda}_2 M^4 \tanh^2 \left( \frac{x}{\sqrt{6}} \right) \left( 3 - 2 \tanh \left( \frac{x}{\sqrt{6}} \right) \right) + \frac{3}{4} M^2 \left( 1 - e^{-\sqrt{\frac{2}{3}x}} \right)^2. \quad (7.54)$$

The first term in (7.54) corresponds to the cosmological constant, $\Lambda = 12\tilde{\lambda}_1 \tilde{\lambda}_2 M^6$. As mentioned earlier, we expect that the vacuum energy density is modified by (negative) contributions from phases transitions occurring after inflation. For example, for $\tilde{\lambda}_{1,2} \sim \mathcal{O}(1)$, we would require a contribution of order $M^6 \sim 10^{-30}$ to cancel the term in (7.54) to eventually yield a cosmological constant of order $10^{-120}$ today. Interestingly, the GUT phase transition in a flipped SU(5) × U(1) model occurs after inflation [250] and contributes $\Delta V \sim -M_{\text{susy}}^2 M_{\text{GUT}}^2 \sim -(\tilde{\lambda}_1 - \tilde{\lambda}_2)^2 M^6 M_{\text{GUT}}^2$ and would indicate that perhaps $\tilde{\lambda}_1 / \tilde{\lambda}_2 \sim (M_{\text{GUT}} / M_P)^2$ or equivalently for $\tilde{\lambda}_2 / \tilde{\lambda}_1$.

The second term in (7.54) corresponds to a perturbation of the inflaton potential and has a negligible effect on the inflationary dynamics, because it is scaled by $M^4$ relative to the inflationary potential (the third term in (7.54)) which scales as $M^2$. Therefore, at large $x$ $\Delta V$ adds a relatively small amount $6\tilde{\lambda}_2 M^4$ to the Starobinsky plateau value of $(3/4)M^2$.

Our next goal is to extend this formalism to general cases with $\alpha \neq 1$, and construct generalized $\alpha$-Starobinsky inflationary models consistent with supersymmetry breaking and a positive cosmological constant. We follow the treatment presented in the previous
section and, for simplicity, consider only the cases where the inflationary superpotential \( W_I \) is coupled to a Minkowski vacuum solution associated with the lower power \( \xi^n \).

Our goal therefore, is to determine the superpotential which generates a Starobinsky potential for any value of \( \alpha \). To this end, we parametrize the superpotential with a function \( f(\phi) \) as

\[
W_I = \sqrt{\alpha} f(\phi) \cdot \left( 2T - \frac{\phi^2}{3} \right)^{\frac{3}{2} (\alpha - \sqrt{\alpha})}.
\]

(7.55)

In the real direction (\( \phi = \bar{\phi} \) and \( T = \bar{T} \)), this reduces to the relatively simple form

\[
V = \left( 2T - \frac{\phi^2}{3} \right)^{1 - 3\sqrt{\alpha}} \cdot f'(\phi)^2,
\]

(7.56)

where \( f'(\phi) = df/d\phi \). Setting the potential (7.56) to the Starobinsky potential, e.g., as in the third term of Eq. (7.53), we can determine \( f(\phi) \) from

\[
f'(\phi) = \frac{\sqrt{3}M \phi}{(\phi + \sqrt{3})} \left( 1 - \frac{\phi^2}{3} \right)^{(1 - 3\sqrt{\alpha})/2},
\]

(7.57)

where we have set \( \langle T \rangle = 1/2 \) to find \( f(\phi) \). The solution to this 1st order equation has the form of a hypergeometric function

\[
f(\phi) = M \left[ \frac{3 - 3^{-m} (3 - \phi^2)^{m+1}}{2 (m+1)} - \frac{\phi^2 F_1 \left( \frac{3}{2}, -m; \frac{5}{2}; \frac{\phi^2}{3} \right)}{3\sqrt{3}} \right],
\]

(7.58)

where \( m = \frac{3}{2} (\sqrt{\alpha} - 1) \). Remarkably, when this expression for \( f(\phi) \) is used in (7.55), we obtain the following scalar potential

\[
V = \frac{3M^2 \phi^2}{(\phi + \sqrt{3})^2},
\]

(7.59)

which, in terms of canonically-normalized fields (7.50), yields the \( \alpha \)-Starobinsky model of inflation

\[
V = \frac{3}{4}M^2 \left( 1 - e^{-\sqrt{\frac{2}{m^2}}} \right)^2.
\]

(7.60)

It is important to note that the hypergeometric superpotential (7.58) is a function of
a matter field $\phi$ only, therefore it can be successfully combined with the dark energy sector $W_{\text{dS}}$.

Despite its rather cumbersome form, the expression in (7.58) simplifies dramatically for certain values of $\alpha$. For example, for $\alpha = 1$, the superpotential is simply our original Wess-Zumino superpotential given in Eq. (7.51) as, in this case, $m = 0$ and $2F_1 \left( \frac{3}{2}, 0; \frac{5}{2}; \frac{\phi^2}{3} \right) = 1$. A relatively simple form for $f(\phi)$ also arises for $\alpha = 25/9$

$$f = M \left( \frac{\phi^5}{15\sqrt{3}} - \frac{\phi^4}{12} - \frac{\phi^3}{3\sqrt{3}} + \frac{\phi^2}{2} \right),$$

Other polynomial forms arise when $\alpha = 49/9$ and 9. However, for any $\alpha$, the scalar potential always reduces to the $\alpha$-Starobinsky potential (7.60). We note that the full superpotential, $W_I$ is a polynomial whenever $9\alpha$ is an odd perfect square other than 1.

Because the potential (7.60) depends on $\alpha$, the evolution of inflaton field and resulting slow-roll parameters differ when $\alpha$ is varied. We show in Fig. 7.2 the slow-roll evolution of the field $x$ for different values of $\alpha$. Once again, we assume that all models have strongly stabilized moduli, so that we can safely treat them as single-field models of inflation. We consider the following $\alpha$-Starobinsky cases with four different values of $\alpha$ that give $N_*= 55$:

- $\alpha = 1$, $x(0) = 5.347$, $r = 0.0035$, $n_s = 0.965$.
- $\alpha = 5$, $x(0) = 8.003$, $r = 0.0138$, $n_s = 0.966$.
- $\alpha = 10$, $x(0) = 9.181$, $r = 0.0230$, $n_s = 0.967$.
- $\alpha = 30$, $x(0) = 12.354$, $r = 0.0430$, $n_s = 0.967$.

As could be expected from the form of the scalar potential, 55 $e$-folds of inflation can be obtained using increasing initial field values $x(0)$ as $\alpha$ is increased. While $n_s$ varies little as $\alpha$ increases, the tensor-to-scalar ratio, $r$, increases from its nominal Starobinsky inflation value of $r = 0.0035$ to $r = 0.0430$ when $\alpha = 30$. This effect is clearly seen in Fig. 4.4.
7.4 Concluding Remarks

Intriguingly, observations of the CMB are highly consistent with the original Starobinsky model of inflation, whereas many other models proposed subsequently have fallen by the wayside. The range of $n_s$ preferred by the data is highly consistent with the Starobinsky prediction $n_s = 1 - 2/N_*$, where $N_* \sim 50$ to 60 is the number of $e$-folds of inflation, and the current upper limit on $r \sim 0.06$ is also consistent with the Starobinsky prediction $\sim 0.003$, albeit with considerable leeway. One of our reasons for being intrigued by the Starobinsky model is that its predictions are shared by simple models based on no-scale supergravity (7.1), as was first discussed in [85] for the case $\alpha = 1$. As we have emphasized here, some form of no-scale supergravity emerges naturally as the effective low-energy theory derived in compactified string models, thus offering a specific bridge between cosmological observables and string theory.

As was first emphasized in [225], one may consider generalizations of the original
model (7.1) with $\alpha \neq 1$, depending on the (combination of) compactification modulus field(s) providing the inflaton, yielding the predictions (7.2) for $n_s$ and $r$. Similar models were discussed from a more general point of view in [239, 232, 236], where they were dubbed $\alpha$-attractors. The predictions of such models are compatible with the CMB data for a large range of possible values of $\alpha$, as seen in Fig. 4.4. In this connection, we are encouraged by the recent approval of the LiteBIRD space mission, which is projected to be able to measure $r$ to an accuracy of $\pm 0.001$, sufficient to measure $\alpha$ with interesting precision, and thereby provide an entrée into the phenomenology of string compactification.

In this chapter we developed a framework for this prospective phenomenology that extends beyond the scope of Kähler manifolds with the $\text{SU}(2,1)\text{SU}(2)\times\text{U}(1)$ coset structure discussed in the previous chapters. Here we have extended these earlier constructions in two main directions: to inflationary models based on generalized no-scale structures with $\text{SU}(N,1)\text{SU}(N)\times\text{U}(1)$ coset structures and to models with different values of $\alpha$ and hence $r$, as may occur if the inflaton corresponds to only a subset of the complex Kähler moduli, or if complex structure moduli also help drive inflation. As in the previous case, key building blocks in these generalizations are played by minimal superpotentials that yield Minkowski vacua and can, in pairs, yield either de Sitter or anti-de Sitter vacuum states.

In the next chapter, we present more detailed phenomenological investigations of such generalized $\alpha$-no-scale models, with a view to kickstarting the exploration of cosmological string phenomenology that will be opened up by LiteBIRD and other CMB experiments. We have emphasized the important role that could be played by measurements of $r$ in constraining the geometry of the underlying no-scale Kähler manifold but, before closing, we stress also the importance of measurements of the scalar tilt, $n_s$. As seen in (7.1), this is sensitive to the number of e-folds of inflation, $N_*$, and hence to the post-inflationary history of the universe. In particular, it is sensitive to the amount of post-inflationary reheating, and hence to the coupling of the inflaton to lighter degrees of freedom. Thus, it complements the measurement of $r$ by being sensitive to the superpotential of the inflationary model. Combining the constraints on $r$ and $n_s$ could provide unique insights into the underlying string compactification.
Chapter 8

Phenomenology and Cosmology of Unified No-Scale Attractors

In Chapter 5, we showed that there are many inflationary avatars within the no-scale framework, including the no-scale attractor models. We extended these models in Chapters 6 and 7, and combined supersymmetry breaking and dark energy with a Starobinsky-like model of inflation.

This latter point is crucial, as constructing models with acceptable phenomenology, cosmology and supersymmetry breaking has been notoriously difficult, particularly when combined with models of inflation that are consistent with the CMB data. Although first steps were made in [237], many more detailed issues remain to be studied. The main goal of this chapter is to develop further the phenomenology and cosmology of no-scale attractor models, bridging the gap between string inspiration and a viable scenario incorporating dark matter and the SM.

In particular, we show how to construct various successful no-scale attractor models of inflation, characterize different possibilities for supersymmetry breaking, and discuss cosmology following inflation and the constraints imposed by entropy considerations and the dark matter density on mechanisms for field stabilization via higher-order terms in the Kähler potential.

We distinguish two types of no-scale attractor models: in the first type the inflaton

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is identified with a volume modulus field, denoted by $T$, and in the second type the inflaton is identified with a matter field, denoted by $\phi$. In $T$-type models supersymmetry is broken in a hidden Polonyi sector [214], whereas in $\phi$-type models supersymmetry is broken by the $T$ field, in which case there is no need for an additional sector to break supersymmetry. Supersymmetry breaking in the $T$-type models was studied in [244, 55], and their crucial feature is that they favor boundary conditions with universal soft scalar masses, as in minimal supergravity (mSUGRA) models. The $\phi$-type models were discussed in [244, 237, 238], and they open up various less constrained phenomenological possibilities, including sources for non-universal scalar masses, as we discuss in this chapter. In principle, these different boundary conditions for supersymmetry breaking have distinctive phenomenological and cosmological features that may be used to distinguish between models. We show that in $\phi$-type models it is possible to relate the scale of supersymmetry breaking to the inflationary scale without fine-tuning.

We focus primarily on $\alpha$-Starobinsky models of inflation based on a scalar potential given by Eq. (7.12), where $\phi'$ is a canonically-normalized inflaton field and $M$ is the inflaton mass. In such an $\alpha$-Starobinsky model, the cosmological observables $(n_s, r)$ (where $n_s$ is the tilt in the scalar perturbation spectrum and $r$ is the tensor-to-scalar ratio) depend on the Kähler curvature parameter $\alpha$. A first discussion of such models was presented in [225], and in the previous chapter we showed that for $\alpha \lesssim \mathcal{O}(1)$, $\alpha$-Starobinsky models of inflation predict (7.2), where $N_*$ is the number of $e$-folds of inflation. Eliminating $N_*$ from (7.2), the curvature parameter can be expressed as follows in terms of the cosmological observables

$$\alpha \simeq \frac{r}{3(1-n_s)^2}. \quad (8.1)$$

for $\alpha \lesssim \mathcal{O}(1)$.

It is possible to find analytic formulae for $n_s$ and $r$ that hold without any restriction on $\alpha$, but they involve special functions and are given in Appendix C. We show in Fig. 8.1 curves of $n_s$ and $r$ as functions of $\alpha$ for $N_* = 50, 55,$ and $60$. The current observation range\(^1\) $n_s \in [0.961, 0.969]$ does not constrain $\alpha$ significantly. However, the current

\(^1\)We are applying Planck results [28] based on the combination of TT,TE,EE+lowE+lensing data for $n_s$ and the combination of BICEP2, Keck Array, and Planck data [86] for $r$.\]
Planck upper limit $r < 0.06$ imposes the upper limit $\alpha \lesssim 46$ (88) for $N_* = 50$ (60). Future CMB observations [251] should be able to probe the tensor-to-scalar ratio down to an upper limit $r \lesssim 0.001$, which is sufficient to determine accurately the curvature parameter $\alpha$, with a lower limit $\alpha \gtrsim 0.3$ when $N_* \simeq 60$, and underpin cosmological string phenomenology. In the following sections we discuss the distinctive phenomenological features and cosmological aspects of both twisted and untwisted models.

Figure 8.1: Plots of $n_s$ (left panel) and $r$ (right panel) as functions of $\alpha$ for the representative values $N_* = 50, 55, 60$.

### 8.1 $T$-type No-Scale Attractor Scenarios

#### 8.1.1 General Framework

We begin with general no-scale attractors based on the non-minimal $SU(2,1)/SU(2) \times U(1)$ Kähler potential (6.33), where the volume modulus $T$ drives inflation. We consider the following general form of inflationary superpotential [248]

\[
W_I = \sqrt{\alpha} M \phi f(T) \left(2T \right)^{3\alpha -1},
\]

where $f(T)$ is an arbitrary function of the volume modulus $T$ only, and $M$ is the inflaton mass scale. For $\alpha = 1$, this reduces to the supergravity version of the $R + R^2$ model discussed in [225, 248, 230, 82]. If we combine the superpotential (8.2) with the Kähler potential (6.33), the effective scalar potential (3.83) in the real $T$ direction becomes

\[
V = M^2 f(T)^2,
\]
where we have assumed that the vacuum expectation value (VEV) of the matter-like field $\phi$ is fixed to $\langle \phi \rangle = 0$, which can achieved by introducing higher-order stabilization terms in the Kähler potential (6.33), as we discuss below.

It was discussed in Chapter 7 that one can obtain Starobinsky-like models of inflation from such a superpotential (8.2) when the volume modulus $T$ is associated with the inflaton. The $\alpha$-Starobinsky model can be obtained by considering the function:

$$f(T) = \frac{\sqrt{3}}{2T} \left( T - \frac{1}{2} \right),$$

(8.4)

in which case the effective potential (8.3) becomes

$$V = \frac{3M^2}{4T^2} \left( T - \frac{1}{2} \right)^2.$$

(8.5)

Defining a canonically-normalized field $\rho \equiv \sqrt{3\alpha \ln 2T}$, the scalar potential (8.5) can be rewritten in the $\alpha$-Starobinsky inflationary form given in Eq. (7.12) with $x = \rho$ driving inflation.

More generally, this framework can be applied to any form of effective scalar potential that vanishes when the volume modulus $T$ obtains a vacuum expectation value, as long as $f(\langle T \rangle) = 0$.

### 8.1.2 Supersymmetry Breaking with a Polonyi Field

In this section, we discuss possible patterns of supersymmetry breaking in $T$-type models, which we accomplish by introducing a Polonyi field $Z$ [214] with a non-vanishing $F$-term.\footnote{The representative example above has $\langle T \rangle = 1/2$, but this choice is arbitrary and models with other values of $\langle T \rangle$ yield similar results.}

We can consider the Kähler potential (6.33) with either an untwisted and strongly-stabilized Polonyi field [244, 252, 253, 55, 254, 255], given by

$$K = -3\alpha \ln \left( T + \bar{T} - \frac{|\phi|^2}{3} + \frac{|\phi|^4}{\Lambda_\phi^2} - \frac{|Z|^2}{3} + \frac{|Z|^4}{\Lambda_Z^2} \right),$$

(8.6)
or a twisted and strongly-stabilized Polonyi field,

\[ K = -3\alpha \ln \left( T + \frac{1}{3} \left( \frac{\phi^2}{3} + \frac{|\phi|^4}{\Lambda_{\phi}^2} \right) + |Z|^2 - \frac{|Z|^4}{\Lambda_Z^2} \right), \quad (8.7) \]

where we have also introduced a quartic stabilization term for the matter-like field \( \phi \), which fixes dynamically its VEV to \( \langle \phi \rangle = 0 \) during inflation [225, 81]. We can consider a general form of the function \( f(T) \) (8.2) with \( f(\langle T \rangle) = f(1/2) = 0 \), which we express as \( f(T) = c(T) (T - \frac{1}{2}) \), where \( c(\langle T \rangle) \equiv c \) and \( f'(\langle T \rangle) = c \). For example, the superpotential (8.4) with \( \langle T \rangle = 1/2 \) gives \( c = \sqrt{3} \).

Next, we introduce the following Polonyi superpotential [214]

\[ W_{P} = \mu (Z + b), \quad (8.8) \]

which is responsible for supersymmetry breaking, and \( b \) is a constant. In the absence of strong stabilization, minimization of the Polonyi potential at zero vacuum energy leads to the solution that \( \langle z \rangle = \sqrt{3} - 1 \) and \( b = 2 - \sqrt{3} \). In models with a strongly-stabilized Polonyi field, the minimum of the potential with zero vacuum energy is near the origin and \( \langle z \rangle \propto \Lambda_Z^2 \) with \( b = 1/\sqrt{3} \) (for \( \alpha = 1 \)) [35, 55]. If we consider the combined superpotential \( W_I + W_P \), where \( W_I \) is given by (8.2), the minimum of the effective scalar potential shifts [244, 252, 253] and we find new VEVs for our fields. The shifted VEVs for Kähler potentials (8.6) and (8.7) are given by

\[
\text{Untwisted Case :} \\
\langle T \rangle \simeq \frac{1}{2} + \left( \frac{2\alpha - 1}{\alpha c^2} \right) \Delta^2, \\
\langle \phi \rangle \simeq \frac{\sqrt{3}}{c} \Delta, \\
\langle Z \rangle \simeq \frac{\sqrt{\alpha}}{6\sqrt{3}} \Lambda_Z^2, \\
b \simeq \frac{1}{\sqrt{3\alpha}} - \left( \frac{1 + 3\alpha(\alpha - 1)}{2\sqrt{3\alpha^3/2c^2}} \right) \Delta^2,
\]

\[
\text{Twisted Case :} \\
\langle T \rangle \simeq \frac{1}{2} + \frac{2\alpha}{c^2} \Delta^2, \\
\langle \phi \rangle \simeq \frac{\sqrt{3\alpha}}{c} \Delta, \\
\langle Z \rangle \simeq \frac{1}{2\sqrt{3}} \Lambda_Z^2, \\
b \simeq \frac{1}{\sqrt{3}} - \frac{\sqrt{3\alpha^2}}{2c^2} \Delta^2,
\]

where we define \( \Delta \equiv \mu/M \) and assume that \( \Delta, \Lambda_Z \ll 1 \). It is important to note that the VEVs of the shifted fields and the induced soft parameters depend on the
curvature parameter $\alpha$ in the Kähler potentials (8.6) and (8.7). If we consider the original model [230] with the choice $\alpha = 1$, we recover the results in [244, 252, 253].

As mentioned at the beginning of this section, supersymmetry is broken through a non-vanishing $F$-term for the Polonyi field $Z$, which is given by

Untwisted Case : $\sum_{i=1}^{3} |F_i|^2 = |e^{G/2} (K^{-1})_i^{j} G_j|^2 \simeq |F_Z|^2 \simeq \frac{\mu^2}{\alpha} \simeq 3m_{3/2}^2$, \quad \text{(8.10)}$

Twisted Case : $\sum_{i=1}^{3} |F_i|^2 = |e^{G/2} (K^{-1})_i^{j} G_j|^2 \simeq |F_Z|^2 \simeq \mu^2 \simeq 3m_{3/2}^2$. \quad \text{(8.11)}$

where the gravitino mass $m_{3/2}$ is given simply by

Untwisted Case : $\quad m_{3/2} \simeq \frac{\mu}{\sqrt{3\alpha}}$, \quad \text{(8.12)}

Twisted Case : $\quad m_{3/2} \simeq \frac{\mu}{\sqrt{3}}$. \quad \text{(8.13)}$

Further, we introduce a canonical parameterization of the complex Polonyi field $Z$

Untwisted Case : $\quad Z = \sqrt{3} \tanh\left( \frac{z}{\sqrt{6\alpha}} \right)$, \quad \text{(8.14)}

Twisted Case : $\quad Z = \frac{z}{\sqrt{2}}$, \quad \text{(8.15)}$

and we assume that the imaginary component of the complex field $Z$ vanishes, which is achieved dynamically with the help of a stabilization term parameterized by $\Lambda_Z$. The mass of the canonically-normalized Polonyi field $z$ is then given by

Untwisted Case : $\quad m_z^2 \simeq \frac{36m_{3/2}^2}{\Lambda_Z^2}$, \quad \text{(8.16)}$

Twisted Case : $\quad m_z^2 \simeq \frac{12m_{3/2}^2}{\Lambda_Z^2}$, \quad \text{(8.17)}$

which is heavier than the gravitino mass $m_{3/2}$ in both cases when $\Lambda_Z \lesssim \mathcal{O}(1)$. This mass hierarchy between $z$ and the gravitino is instrumental in alleviating [55] the so-called cosmological moduli problem [51, 224]. Using the field VEVs (8.9), we can express the
Goldstino field as

\[
\text{Untwisted Case : } \quad \eta = \sum_{i=1}^{3} G^i \chi_i \simeq \sqrt{3} \alpha \chi_z, \quad (8.18)
\]

\[
\text{Twisted Case : } \quad \eta = \sum_{i=1}^{3} G^i \chi_i \simeq \sqrt{3} \chi_z, \quad (8.19)
\]

where we see that the Goldstino is the fermionic partner of supersymmetry-breaking Polonyi field \( Z \), as expected.

### 8.1.3 Incorporation of Matter Particles

We are now in a position to extend the model to include a general superpotential form that incorporates matter-like fields \( X_i \) such as appear in the SM

\[
W = W_I(T, \phi) + W_P(Z) + W_2(X_i) + W_3(X_i), \quad (8.20)
\]

where \( W_I \) is our inflationary superpotential and we have introduced general bilinear and trilinear couplings \( W_2,3 \). The kinetic terms for the matter fields may originate as untwisted or twisted fields. In the case of untwisted matter fields, their contributions to the Kähler potential lies inside the logarithmic term in either Eqs. (8.6) or (8.7)

\[
\text{Untwisted Matter Fields : } \quad K \supset -\frac{|X_i|^2}{3}. \quad (8.21)
\]

For twisted matter fields, the contribution to \( K \)

\[
\text{Twisted Matter Fields : } \quad K \supset |X_i|^2, \quad (8.22)
\]

sits outside the logarithmic terms in Eqs. (8.6) and (8.7).

Having introduced matter fields (twisted and/or untwisted) and supersymmetry breaking via a Polonyi sector, we are now in a position to calculate the soft supersymmetry breaking terms for each of the four possible cases. In each case, the soft
supersymmetry-breaking terms in the Lagrangian are written as

\[ L \supset -m_0^2 |X_i|^2 - B_0 W_2 - A_0 W_3. \]  

(8.23)

For an untwisted Polonyi field, characterized by the Kähler potential (8.6), we find the following expressions for the induced soft terms

\begin{align*}
\text{Untwisted Matter Fields :} & \quad \text{Twisted Matter Fields :} \\
m_0^2 &= (\alpha - 1) m_{3/2}^2, & m_0^2 &= m_{3/2}^2, \\
B_0 &= -m_{3/2}, & B_0 &= -m_{3/2}, \\
A_0 &= 0, & A_0 &= 0.
\end{align*}

(8.24)

As one can see, the only dependence in the soft supersymmetry breaking terms on the curvature parameter \( \alpha \) appears in the soft scalar masses for untwisted matter fields. When \( \alpha = 1 \), we have vanishing input scalar masses, which must then be generated by RGE evolution (typically above the GUT scale [256]). When \( \alpha = 2 \), we obtain \( m_0 = m_{3/2} \), \( B_0 = -m_{3/2} \), and \( A_0 = 0 \), which is the pattern of soft terms when matter fields are twisted as well. In this case, we recover minimal supergravity (mSUGRA) [216, 80, 223] boundary conditions, given by \( A_0 = B_0 + m_0 \), with \( A_0 = 0 \) as in models of pure gravity mediation (PGM) [257, 258].

In the untwisted case, imposing \( \alpha \geq 1 \) would avoid tachyonic soft supersymmetry-breaking scalar masses and the associated issue of vacuum stability. However, while this is condition is sufficient, it is not necessary [259, 260]. It is possible that soft supersymmetry-breaking scalar masses are negative at the input universality scale but no physical tachyonic scalars are found when the soft supersymmetry-breaking parameters are run down to the weak scale. In fact, in studies of supersymmetric models with non-universal Higgs masses [261, 262], it was found that frequentist fits including many phenomenological and cosmological observables were best fit with \( m_0^2 < 0 \) [263]. These models are however, potentially problematic due to the presence of charge- and/or colour-breaking minima [264, 265]. However, if the electroweak vacuum is long-lived, the relevance of other vacua becomes a cosmological question related to our position in field space after inflation. For a discussion of cosmological issues associated with such tachyonic soft supersymmetry-breaking mass parameters, see [266].
For the case with a twisted Polonyi field, characterized by the Kähler potential (8.7), we find

Untwisted Matter Fields:

\[ m_0^2 = \alpha m_3^{3/2}, \]
\[ B_0 = -m_3^{3/2}, \]
\[ A_0 = 0, \]

Twisted Matter Fields:

\[ m_0^2 = m_3^{3/2}, \]
\[ B_0 = -m_3^{3/2}, \]
\[ A_0 = 0. \]  

(8.25)

The soft terms for twisted matter fields are unchanged from Eq. (8.24) and, once again, the only dependence on \( \alpha \) appears for untwisted matter fields though, in this case, because \( m_0 = \sqrt{\alpha} m_3^{3/2} \) there is no restriction on \( \alpha \) other than its positivity.

Note that we have not included here any modular weights in either the kinetic terms for twisted fields, or superpotential terms. These will be included in the next section for \( \phi \)-type attractor models of inflation. As explained in [244], the soft terms induced in \( T \)-type models are independent of all of the modular weights, which is not be the case for the \( \phi \)-type models, as we discuss in the next section.\(^4\)

These key results are summarized in Table 8.1 below, and we briefly discuss cosmological aspects of such \( T \)-type models in the sections below. They were covered in detail in [55].

### 8.2 \( \phi \)-type No-Scale Attractor Scenarios

#### 8.2.1 General Framework

In this section, we discuss models of inflation where a matter-like field \( \phi \) is interpreted as the inflaton. As discussed in [225, 232, 267], inflationary models based on the minimal single field no-scale Kähler potential (6.2) entail uplifting a Minkowski vacuum via supersymmetry breaking, which leads to an extremely heavy gravitino. An alternative way to construct viable inflationary models is to consider higher-dimensional non-compact coset manifolds [75, 268]. There is a long history of constructing inflationary models this way [233]. However, in many of the early models, the predictions of cosmological observables fall outside the range now determined by CMB observations [28]. In [85], it

\(^4\)However, the shifted minimum in Eq. (8.9) does depend on possible weights for the Polonyi field \( Z \), as discussed in Appendix D.
was shown that a simple Wess-Zumino superpotential can produce Starobinsky-like inflation, which leads to a spectral tilt, \( n_s \), in good agreement with CMB measurements, and a tensor-to-scalar ratio, \( r \), within reach of future experiments. The connection between Starobinsky inflation, \( R + R^2 \) gravity, and no-scale supergravity was further developed in [234].

Here, we consider two possible non-minimal no-scale models, in which we introduce a single additional matter-like field, to be interpreted as the inflaton field. The inflaton may be included as an untwisted matter-like field, which parameterizes together with \( T \) a non-compact \( SU(2,1)/SU(2) \times U(1) \) coset space [85, 225].

Untwisted Inflaton Field: \[ K = -3 \alpha \ln \left( T + \bar{T} + \frac{(T + \bar{T} - 1)^4}{\Lambda_T^2} + \frac{d(T - \bar{T})^4}{\Lambda_T^2} - \frac{|\phi|^2}{3} \right), \]
or as a twisted matter-like field, which parameterizes together with $T$ an $SU(1,1) \times U(1)$ space [269, 270, 121, 271]

Twisted Inflaton Field:

$$K = -3 \alpha \ln \left( T + \bar{T} + \frac{(T + \bar{T} - 1)^4}{\Lambda_T^2} + \frac{d (T - \bar{T})^4}{\Lambda_T^2} \right) + |\phi|^2.$$  

(8.27)

In both cases, we include in the Kähler potential quartic stabilization terms for the volume modulus $T$, with $\Lambda_T < 1$. These stabilize the volume modulus $T$ in both the real and imaginary directions, and ensure that the VEV of the volume modulus is fixed dynamically to $\langle T \rangle = \frac{1}{2}$.

A superpotential that is a function only of a matter-like field can be used to break supersymmetry and introduce a massive gravitino without invoking a hidden Polonyi sector [244]. In such unified no-scale attractor models [237, 238], the volume modulus $T$ plays the role of a Polonyi-like field that breaks supersymmetry, and only two complex fields are necessary for $\phi$-type models.\(^5\) Furthermore, in addition to inflation and supersymmetry breaking, these models can account for a small residual (though fine-tuned) cosmological constant. The superpotential for such models can be written as [237]

$$W = W_I + W_{dS},$$  

(8.28)

where $W_I$ characterizes inflation and $W_{dS}$ is responsible for supersymmetry breaking and a (small) positive cosmological constant that appears at the end of inflation [75, 235]. The forms of the superpotentials $W_I$ and $W_{dS}$ depend whether the inflaton is twisted or untwisted and therefore combined with the Kähler potential in either Eq. (8.26) or (8.27), respectively [271]

Untwisted Inflaton Field:

$$W_I = \sqrt{\alpha} M f(\phi) \cdot \left( 2T - \frac{\phi^2}{3} \right)^{n_-},$$  

(8.29)

$$W_{dS} = \lambda_1 M^3 \cdot \left( 2T - \frac{\phi^2}{3} \right)^{n_-} - \lambda_2 M^3 \cdot \left( 2T - \frac{\phi^2}{3} \right)^{n_+},$$  

(8.30)

\(^5\)In [245], a term linear in $\phi$ is included, which plays the role of the Polonyi field, and Starobinsky-like inflation is possible so long as the gravitino mass $m_{3/2} \lesssim 1 \text{ PeV}$. The soft supersymmetry breaking parameters for this model were derived in [272].
and

Twisted Inflaton Field : \( W_I = M f(\phi) \cdot (2T)^{n_-} \cdot e^{-\frac{\phi^2}{2}}, \) \hspace{1cm} (8.31)

\( W_{dS} = \left[ \lambda_1 M^3 \cdot (2T)^{n_-} - \lambda_2 M^3 \cdot (2T)^{n_+} \right] \cdot e^{-\frac{\phi^2}{2}}, \) \hspace{1cm} (8.32)

where \( n_{\pm} = \frac{3}{2} (\alpha \pm \sqrt{\alpha}), \ M \simeq 1.2 \times 10^{-5} M_P \simeq 3 \times 10^{13} \text{ GeV} \) is the inflaton mass for Starobinsky-like inflation, and one of the couplings \( \lambda_i \) must be tuned to obtain a small vacuum density, whereas the other may be of order 1.

After the volume modulus \( T \) is stabilized by the quartic terms in the Kähler potential forms (8.26) and (8.27), with a VEV \( \langle T \rangle = \frac{1}{2} \), the inflaton field \( \phi \) is stabilized in the imaginary direction throughout inflation in both cases, and we have \( \phi = \bar{\phi} \). Note that, despite the presence of supersymmetry breaking and a non-zero final vacuum energy density, Starobinsky-inflation is reproduced for an appropriate choice of \( f(\phi) \).

If we combine the Kähler potential (8.26) with the superpotentials (8.29) and (8.30), the effective scalar potential (3.83) becomes

Untwisted Inflaton Field : \( V \simeq \Lambda + M^2 \left( 1 - \frac{\phi^2}{3} \right)^{1-3\sqrt{\alpha}} \ f'(\phi)^2, \) \hspace{1cm} (8.33)

and similarly, if we combine the Kähler potential (8.27) with the superpotentials (8.31) and (8.32), equation (3.83) gives

Twisted Inflaton Field : \( V \simeq \Lambda + M^2 \ f'(\varphi)^2, \) \hspace{1cm} (8.34)

where

\( \Lambda = 12 \lambda_1 \lambda_2 M^6. \) \hspace{1cm} (8.35)

In both cases, the cosmological constant \( \Lambda \) depends on two constants, \( \lambda_{1,2} \). It should be noted that we neglected the contribution of a term which is proportional to \( \lambda_2 M^4 f \). This term vanishes at the minimum and, because \( \lambda_2 M^4 \ll M^2 \), it does not affect the inflationary dynamics [238].

Being proportional to \( M^6 \), the cosmological constant is of order \( 10^{-30} \lambda_2 \) in Planck units, when we assume \( \lambda_1 \sim \mathcal{O}(1) \). We expect that the final vacuum energy density is modified by (negative) contributions from phase transitions occurring after inflation.
For $\lambda_2 \sim \mathcal{O}(1)$, we would require a contribution of order $M^6 \sim 10^{-30}$ to cancel the term in (8.35) to eventually yield a cosmological constant of order $10^{-120}$ today. For example, the GUT phase transition in a flipped SU(5) $\times$ U(1) model occurs after inflation [242, 250, 273, 274] and contributes $\Delta V \sim -M^2_{\text{phys}}M^2_{\text{GUT}} \sim -(\lambda_1 - \lambda_2)^2 M^6 M^2_{\text{GUT}}$, indicating that perhaps $\lambda_2 \sim (M_{\text{GUT}}/M_P)^2$ for $\lambda_1 \sim 1$.

In the untwisted case, the $\alpha$-Starobinsky inflationary potential

$$V \simeq \Lambda + \frac{3}{4} M^2 \left(1 - e^{-\sqrt{\frac{2}{3}} x}\right)^2,$$  

(8.36)

can be obtained from Eq. (8.33) with the choice of $f(\phi)$ which satisfies [238]

$$f'(\phi) = \sqrt{3} \phi \left(1 - \frac{\phi^2}{3}\right)^{(3\sqrt{\alpha}-1)/2},$$  

(8.37)

and a field redefinition

$$\phi = \sqrt{3} \tanh \left(\frac{x}{\sqrt{6}\alpha}\right).$$  

(8.38)

The superpotential function $f(\phi)$ derived from Eq. (8.37) with boundary condition $f(0) = 0$ is in general a hypergeometric function, which assumes a polynomial form whenever $9\alpha$ is an odd perfect square other than 1. For example, when $\alpha = 1$, $f(\phi)$ is of the Wess-Zumino form [85]

$$f(\phi) = \left(\frac{\phi^2}{2} - \frac{\phi^3}{3\sqrt{3}}\right).$$  

(8.39)

In the twisted case, the $\alpha$-Starobinsky inflationary potential (8.36) can be obtained from Eq. (8.34) with the choice of $f(\varphi)$ which satisfies

$$f'(\varphi) = \frac{\sqrt{3}}{2} \left(1 - e^{-\frac{2\varphi}{\sqrt{\alpha}}}\right),$$  

(8.40)

and a field redefinition

$$\varphi = x/\sqrt{2}.$$  

(8.41)
In this case, there is a relatively simple form for \( f(\varphi) \) for all \( \alpha \)

\[
f(\varphi) = \frac{3\sqrt{\alpha}}{4} \left( \frac{2\varphi}{\sqrt{3}\alpha} + e^{-\frac{2\varphi}{\sqrt{3}\alpha}} - 1 \right).
\] (8.42)

### 8.2.2 Supersymmetry Breaking

The unified no-scale attractor models with an untwisted or a twisted inflaton field both yield a de Sitter vacuum at the minimum, and the two formulations can be considered as equivalent for cosmological purposes. At the end of inflation, supersymmetry is broken through an \( F \)-term for \( T \), which is given by [237, 238]

\[
\sum_{i=1}^{2} |F_i|^2 = F_T^2 \simeq \frac{(\lambda_1 + \lambda_2)^2}{\alpha} M^6,
\] (8.43)

and the gravitino mass is given simply by

\[
m_{3/2} = e^{G/2} = e^{K/2} W = (\lambda_1 - \lambda_2) \frac{M^3}{M_P^2},
\] (8.44)

which is independent of the curvature parameter \( \alpha \). Thus, in our framework the \( F \)-term is a function of the curvature \( \alpha \) whereas the gravitino mass \( m_{3/2} \) is not. Moreover, as we have mentioned before, our framework incorporates supersymmetry breaking without introducing an additional Polonyi sector [214] or external uplifting by fibres [247].

In order to obtain a gravitino mass \( m_{3/2} \simeq \mathcal{O}(1) \text{ TeV} \), we choose \( \lambda_2 \ll \lambda_1 \). Then we can write

\[
m_{3/2} = (\lambda_1 - \lambda_2) \frac{M^3}{M_P^2} \simeq \lambda_1 \frac{M^3}{M_P^2},
\] (8.45)

and we can re-express the \( F \)-term for \( T \) (8.43) as \( F_T \simeq \frac{m_{3/2}}{\sqrt{\alpha}} \). We note that, by scaling \( W_{dS} \) with \( M^3 \), we obtain a TeV mass scale for supersymmetry breaking without fine-tuning, and relate the supersymmetry-breaking scale to the inflation scale \( M \) (see also [252, 253]).

Using (8.44) and (8.45), we find that the squared masses of the real and imaginary
components of the volume modulus $T$ are given by

\[ m^2_{\text{Re}T} \simeq \frac{48 m^2_{3/2}}{\alpha \Lambda^2}, \quad m^2_{\text{Im}T} \simeq \frac{48d m^2_{3/2}}{\alpha \Lambda^2}, \quad (8.46) \]

which depend on the supersymmetry-breaking parameter $\lambda_1$, the stabilization constant $\Lambda_T$ and the curvature parameter $\alpha$. For $\Lambda_T \ll 1$, we have a built-in hierarchy between the modulus and the gravitino mass scales. It is important to note that, in the absence of supersymmetry breaking, $m_{3/2} = 0$, and both $T$ components remain massless.

Finally, the squared mass of the inflaton is given by

\[ m^2_\phi \simeq M^2 f''(0)^2, \quad (8.47) \]

where we may assume that $f''(0) \sim O(1)$, in which case the inflaton mass is $m_\phi = M \simeq O(10^{-5})$.

### 8.2.3 Incorporating Matter Particles

In order to incorporate Standard Model-like particles in $\phi$-type models of unified no-scale attractors, we illustrate different possible superpotential structures that couple the hidden and visible matter sectors.

We consider the following general superpotential form

\[ W = W_I + W_{dS} + W_{SM}, \quad (8.48) \]

where in the case of an untwisted inflaton, the superpotentials $W_I$ and $W_{dS}$ are given by (8.29) and (8.30) and for a twisted inflaton field $W_I$ and $W_{dS}$ are given by (8.31) and (8.32), and $W_{SM}$ describes the Standard Model-like interactions, given by

\[ W_{SM} = \sqrt{\alpha} \left[ \left( T + \frac{1}{2} \right)^\beta W_2(X_i) + \left( T + \frac{1}{2} \right)^\gamma W_3(X_i) \right] \cdot Y^n, \quad (8.49) \]

where we have introduced bilinear and trilinear couplings $W_{2,3}$ with non-zero modular
weights $\beta$ and $\gamma$, and

\[
\text{Untwisted Inflaton Field : } Y = 2T - \frac{\phi^2}{3}, \quad (8.50)
\]
\[
\text{Twisted Inflaton Field : } Y = 2T. \quad (8.51)
\]

It should also be noted that we couple the Standard Model-like sector to $(2T - \phi^2/3)^{n_-}$ for the case with an untwisted inflaton field and $(2T)^{n_+}$ for the case with a twisted inflaton field, where $n_- = \frac{3}{2} (\alpha - \sqrt{\alpha})$. One may also consider couplings to $Y^{n_+}$, where $n_+ = \frac{3}{2} (\alpha + \sqrt{\alpha})$, and find similar results.

As in the previous section, matter fields may appear either as untwisted in the Kähler potential as in Eq. (8.21) or as twisted fields in the Kähler potential

\[
K \supset |X_i|^2 (T + \overline{T})^{-n_i}, \quad (8.52)
\]

which sits outside the logarithmic term and where we have included a kinetic modular weight, $n_i$. For $T$-type inflation, soft mass terms do not depend on modular weights and so these were neglected in writing the superpotential in Eq. (8.20). On the other hand, in $\phi$-type models the modular weights do enter into the soft supersymmetry breaking terms, and the weights $\beta$ and $\gamma$ are included in Eq. (8.49) separately for bilinear and trilinear couplings. We obtain the following induced soft terms for untwisted matter fields (8.21) and twisted matter fields (8.22)

\[
\text{Untwisted Matter Fields : } \begin{align*}
    m_0^2 &= (\alpha - 1) m_3^{2/2}, \\
    B_0 &= (2\sqrt{\alpha} - 2 - \beta) m_3^{2/2}, \\
    A_0 &= (3\sqrt{\alpha} - 3 - \gamma) m_3^{2/2},
\end{align*}
\]

\[
\text{Twisted Matter Fields : } \begin{align*}
    m_0^2 &= \left(\frac{\alpha - n_i}{\alpha}\right) m_3^{2/2}, \\
    B_0 &= (2\sqrt{\alpha} - 2n_i - \beta) m_3^{2/2}, \\
    A_0 &= (3\sqrt{\alpha} - 3n_i - \gamma) m_3^{2/2}.
\end{align*}
\]

(8.53)

For $\alpha = 1$, these results reduce to those found in [244].

The induced soft terms (8.53) allow us to consider various phenomenological scenarios. Let us first consider $\alpha = 1$. For untwisted matter fields, we obtain $m_0 = 0$, $B_0 = -\beta m_3/2$, and $A_0 = -\gamma m_3/2$. If we set $\beta = \gamma = 0$, we recover standard no-scale soft terms with $A_0 = B_0 = m_0 = 0$ [43, 73, 74]. For twisted matter fields with $n_i = 0$
or universal, one finds non-zero universal soft mass terms as in the constrained minimal
supersymmetric extension of the Standard Model (CMSSM) [275, 217, 276, 277], and if
\( \beta = \gamma \) one obtains soft terms of the mSUGRA type with \( A_0 = B_0 + m_0 \), all proportional
to the gravitino mass. If \( \beta \) and \( \gamma \) vanish, we have \( A_0 = 3m_0 \) and \( B_0 = 2m_0 \). For
\( \beta = \gamma = 3 \), we obtain PGM [257, 258, 278, 279] soft terms, given by \( m_0 = m_{3/2} \), \( A_0 = 0 \)
and \( B_0 = -m_{3/2} \). Finally, we note that scalar mass universality is lost if the modular
weights \( n_i \) are not universal.

As in the case of \( T \)-type models, for untwisted matter fields with \( \alpha < 1 \) or for twisted
matter fields with \( \alpha < n_i \), there is the possibility that \( m_0^2 < 0 \). However, as discussed
above, such models are not necessarily excluded by cosmological considerations. For
\( \alpha > 1 \), one finds non-zero scalar masses even in the untwisted case. Finally, we point
out that these results do not depend whether the inflaton is twisted or not. Our key
results for \( \phi \)-type unified no-scale models of inflation are summarized in Table 8.2.

<table>
<thead>
<tr>
<th>Twisted/Untwisted Inflaton Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )-term</td>
</tr>
<tr>
<td>( m_{\Re T}^2 )</td>
</tr>
<tr>
<td>( m_{\Im T}^2 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Matter Fields</th>
<th>Untwisted</th>
<th>Twisted</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_0^2 )</td>
<td>((\alpha - 1)m_{3/2})</td>
<td>(\frac{(\alpha - n_i)}{\alpha}m_{3/2})</td>
</tr>
<tr>
<td>( B_0 )</td>
<td>((2\sqrt{\alpha} - 2 - \beta)m_{3/2})</td>
<td>((2\sqrt{\alpha} - 2n_i - \beta)m_{3/2})</td>
</tr>
<tr>
<td>( A_0 )</td>
<td>((3\sqrt{\alpha} - 3 - \beta)m_{3/2})</td>
<td>((3\sqrt{\alpha} - 3n_i - \gamma)m_{3/2})</td>
</tr>
</tbody>
</table>

Table 8.2: Model parameters and soft supersymmetry-breaking quantities in \( \phi \)-type
unified no-scale attractor scenarios with either untwisted or twisted matter fields. The
quantities \( \beta, \gamma \) and \( n_i \) are modular weights introduced in (8.49) and (8.52), respectively.
8.3 Cosmological Scenarios, Entropy and Dark Matter Production

Supersymmetry breaking has often been a source of cosmological angst. The so-called Polonyi problem, or more generally the moduli problem, arises when scalars with weak scale masses but with Planck scale vacuum expectation values are displaced from their minima after inflation [51, 224, 54, 52]. Their evolution and late decay generally produce enormous amounts of entropy, washing away any baryon asymmetry. Their decays into supersymmetric particles may also lead to an excessive dark matter abundance [194, 195] in the form of the lightest supersymmetric particle (LSP), if R-parity is preserved. In this section, we discuss these cosmological issues in $T$- and $\phi$-type no-scale attractor models.

8.3.1 Post-inflationary Dynamics in $T$-Type Models

As we have seen, in $T$-type models inflation is driven by a volume modulus $T$ whose dynamics is characterized by a function $f(T)$, and supersymmetry is broken through a Polonyi field $Z$. When the inflaton rolls down to a Minkowski minimum, given by the left side of (8.9) for the case with untwisted Polonyi field, and by the right side of (8.9) for the case with a twisted Polonyi field, the fields (both the inflaton and $Z$) begin to oscillate and their subsequent decay begins the process of reheating of the universe. For example, during inflation, a twisted Polonyi field $Z$ is displaced to a minimum determined by Hubble induced mass corrections $\sim H^2 Z \bar{Z}$, which is much smaller than the VEV of $\langle Z \rangle$ at the true minimum, i.e., $\langle Z \rangle_{\text{Inf}} \ll \langle Z \rangle$ [280, 281, 55]. When the Hubble parameter becomes smaller than the Polonyi mass, $m_z$, more precisely when $H \lesssim \frac{2}{3} m_z$, the inflationary minimum of $Z$ starts moving adiabatically toward the true minimum. As a result, the initial amplitude of the field is very small and roughly proportional to

$$\langle z \rangle_{\text{Max}} \sim \Lambda_Z^2.$$  \hfill (8.54)

The decay of the inflaton is model-dependent, but the main decay channel of the Polonyi field $Z$ is into a pair of gravitinos, potentially exacerbating the gravitino problem [83, 282]. The strongest limit on $\Lambda_Z$ comes from the decay of $Z$ to gravitinos and their
subsequent decay to the LSP, $\chi$. In [55], it was found that $\Lambda \lesssim 3 \times 10^{-4}$ for $m_\chi = 100$ GeV, and $m_{3/2} = 10^{-15} M_P$, and scales as $(m_{3/2}/m_\chi)^{1/5}$. A more complete and detailed treatment of cosmological consequences of $T$-type models is presented in [55]. The results derived there apply to the $T$-type models discussed here, and we do not discuss them further in this chapter.

8.3.2 Post-inflationary Dynamics in $\phi$-Type Models

As we have also seen, in $\phi$-type models inflation is driven by a matter-like field $\phi$, and inflationary dynamics is characterized by a function $f(\phi)$ [237]. In this case, supersymmetry is broken by the volume modulus $T$, which acts like a Polonyi field. The inflaton field $\phi$ exits the high-lying de Sitter plateau and rolls down toward a low-lying global de Sitter minimum, characterized by $\Lambda = 12 \lambda_1 \lambda_2 M^6$. When the inflaton reaches this de Sitter minimum, located at $\langle T \rangle = \frac{1}{2}$ and $\langle \phi \rangle = 0$, it starts oscillating about the minimum, with an initial maximum amplitude given by

$$\langle T \rangle_{\text{Max}} \simeq \frac{\Lambda T}{4\sqrt{3}}, \quad \langle \phi \rangle_{\text{Max}} \simeq O(10^{-1}). \quad (8.55)$$

Note the magnitude of the initial amplitude of $T$ oscillations is a key difference between these models and the $T$-type models. In the latter, the maximum amplitude for strongly-stabilized Polonyi oscillations, given in Eq. (8.54) is $\propto \Lambda_Z^2$. In this case, the initial amplitude is significantly larger, and we expect stronger constraints on $\Lambda_T$. The main decay channel of the volume modulus $T$ is into a pair of gravitinos.

We consider now various post-inflationary scenarios in $\phi$-type models, and calculate the corresponding upper limits on the modulus stabilization parameter $\Lambda_T$. At the end of the inflationary epoch, when the inflaton rolls down toward a global minimum and starts oscillating about it, and the universe enters a period of matter-dominated expansion. As the Hubble damping parameter decreases, the volume modulus $T$ starts oscillating about its minimum. These oscillations may occur before or after reheating of the universe, and we treat these cases separately below.

When the inflationary period is over (i.e., the near-exponential expansion ends), the energy density of the universe is dominated by the oscillations of the inflaton about its minimum. We define the start of inflaton oscillations as the moment when the Hubble
parameter is of order the inflaton mass, or more precisely when \( H = \frac{2}{3}m_\eta \), where \( \eta \) (\( \phi \) in the untwisted case or \( \varphi \) in the twisted case) is the inflaton. If we define \( R_\eta \) as the scale factor when inflaton oscillations begin, we can write the energy density and Hubble parameter as

\[
\begin{align*}
\rho_\eta &\simeq \frac{4}{3}m_\eta^2 M_P^2 \left( \frac{R_\eta}{R} \right)^3, \\
H &\simeq \frac{2}{3}m_\eta \left( \frac{R_\eta}{R} \right)^{3/2}.
\end{align*}
\]

(8.56) \hspace{1cm}  (8.57)

During this matter-dominated epoch, the Hubble expansion rate keeps decreasing until the field \( T \) begins oscillating about the minimum. These oscillations start when the Hubble parameter approaches the mass of \( T \) or when \( H \simeq \frac{2}{3}m_T \), where \( m_T \) is the canonically-normalized mass of the volume modulus. This value of the Hubble parameter can be reached before or after the inflaton \( \eta \) decays, i.e., before or after reheating. When the field \( T \) begins oscillating about the supersymmetry-breaking minimum, the energy density of oscillations is given by

\[
\rho_T \simeq \frac{1}{2}m_T^2 M_P^2 \langle T \rangle^2_{\text{Max}} \left( \frac{R_T}{R} \right)^3,
\]

where \( \langle T \rangle_{\text{Max}} \) is the maximum amplitude of the oscillations of the volume modulus and \( R_T \) is the cosmological scale factor at the onset of \( T \) oscillations, which may begin before or after inflaton decays.

We distinguish the following scenarios for the possible evolution of the universe.

**Scenario I: Oscillations of the volume modulus \( T \) begin before inflaton decay**

We assume that the inflaton couples only through gravitational-strength interactions, in which case the inflaton decay rate can be estimated as

\[
\Gamma_\eta = d_\eta^2 \frac{m_\eta^3}{M_P^2},
\]

(8.59)

where \( d_\eta \) is a gravitational-strength coupling. \(^6\) The inflaton decays when the Hubble parameter decreases to the critical value \( \Gamma_\eta = \frac{3}{4}H \) where, using the expressions (8.57)

\(^6\)If there are non-gravitational couplings leading to inflaton decay, we can write \( d_\eta = d_\eta M_P / m_\eta \), where \( d_\eta \) is the non-gravitational coupling.
and (8.59), we obtain the following scale factor ratio

$$\frac{R_{d\eta}}{R_\eta} = \left( \frac{M_P}{d_\eta m_\eta} \right)^{4/3},$$

(8.60)

where $R_{d\eta}$ is the cosmological scale factor at the time of inflaton decay.

Using equations (8.46, 8.55, 8.58), we can express the energy density stored in $T$ oscillations (8.58) as

$$\rho_T \simeq \frac{1}{2} \alpha \frac{m_{3/2}^2 M_P^3}{R_T} \left( \frac{R_T}{R} \right)^3.$$

(8.61)

To find the scale factor at the beginning of $T$ oscillations, we use (8.46, 8.57) and $H \simeq \frac{2}{3} m_T$ to estimate\(^7\)

$$R_T \simeq \alpha^{1/3} \left( \frac{\Lambda_T}{4 \sqrt{3} m_{3/2}} \right)^{2/3} R_\eta.$$

(8.62)

We seek the limits on $\Lambda_T$ that ensure that entropy production from $T$ decay does not cause excessive dilution of the primordial baryon-to-entropy ratio, $n_B/s \simeq 8.7 \times 10^{-11}$.

In this scenario, we are assuming that oscillations of the volume modulus $T$ begin before inflaton decay, i.e., $R_T < R_{d\eta}$. Using the scale factor ratios (8.62) and (8.60), we find

$$\frac{R_T}{R_{d\eta}} = \alpha^{1/3} \left( \frac{d_\eta^2 \Lambda_T M_P}{4 \sqrt{3} m_{3/2}} \right)^{2/3} \left( \frac{m_\eta}{M_P} \right)^2 < 1,$$

(8.63)

which leads to the following condition for Scenario I

$$\Lambda_T \lesssim \alpha^{-1/2} \frac{4 \sqrt{3} m_{3/2} M_P^2}{d_\eta^2 m_\eta^3}.$$  

(8.64)

Therefore, if we choose $m_\eta \sim 10^{-5} M_P$, $m_{3/2} \sim 10^{-15} M_P \sim \mathcal{O}(1) \text{TeV}$, and $\alpha = 1$, we find $T$ oscillations begin before inflaton decay when $d_\eta^2 \Lambda_T \lesssim \sqrt{48}$, a condition which is almost always satisfied. We must now distinguish between the possibilities that $T$ decays before the inflaton (I a), after the inflaton but before $T$ oscillations dominate the energy density (I b), and when they do dominate the energy density (I c).

\(^7\)It should be noted that $T$ oscillations might also begin after the inflaton decays, and we discuss this possibility later.
We assume instantaneous inflaton decay and thermalization of inflaton decay products. When the inflaton decays the universe becomes radiation-dominated, and the energy density and Hubble parameter have the following expressions

\[ \rho_r = \frac{4}{3} d_\eta^{-4/3} m_\eta^{2/3} M_P^{10/3} \left( \frac{R_\eta}{R} \right)^4, \]  

(8.65)

\[ H = \frac{2}{3} d_\eta^{-2/3} m_\eta^{1/3} M_P^{2/3} \left( \frac{R_\eta}{R} \right)^2, \]  

(8.66)

and the reheating temperature is given by

\[ T_{RH} = d_\eta \left( \frac{40}{\pi^2 g_\eta} \right)^{1/4} \frac{m_\eta^{3/2}}{M_P^{1/2}}, \]  

(8.67)

where \( g_\eta = g(T_{RH}) \) is the effective number of degrees of freedom at reheating, for which we take the MSSM value \( g(T_{RH}) = 915/4 \). We note that, in this model, the reheating temperature is model-dependent and depends on how the inflaton is coupled to the Standard Model, and hence also its decay rate defined in Eq. (8.59). The preferred values of \( N_* \) and \( n_s \) also depend on the decay rate, though very weakly as discussed in detail in [137].

To calculate the decay rate of the volume modulus to gravitinos, we use the following fermion-scalar interaction supergravity Lagrangian (see e.g. [44]), where we work in the unitary gauge:

\[ L_{F,\text{int}} \supset \frac{i}{2} \bar{\chi}_{iL} \Phi_j \chi^k \left( -G^i_{kj} + \frac{1}{2} G^i_k G^j \right) + \frac{1}{2} e^{G/2} \left( -G^{ij} - G^i G^j \right) + G^{ij}_k (G^{-1})^k_l G^l \bar{\chi}_{iL} \chi_{jR} + \text{h.c.} \]  

(8.68)

In this case, the modulino \( \chi_T \) becomes the longitudinal component of the gravitino, and the dominant two body decay \( T \to \psi_3/2 \bar{\psi}_3/2 \) coupling can be obtained from the following interaction term in the Lagrangian

\[ L_{F,2} \supset \frac{1}{2} m_{3/2} G_T^T (G^{-1})^T_T G^T \psi_{3/2} \bar{\psi}_{3/2} = 36 \alpha \frac{m_{3/2}}{M_P \Lambda_T^2} T \bar{\psi}_{3/2} \psi_{3/2}. \]  

(8.69)
The decay rate obtained from this interaction term is given by

$$\Gamma_{T}^{(\text{Total})} \simeq \Gamma(T \to \psi_{3/2}\bar{\psi}_{3/2}) = \alpha^{3/2} \frac{648\sqrt{3}m_{3/2}^{3}}{\pi M_{P}^{2}\Lambda_{T}^{5}}. \quad (8.70)$$

We now discuss in turn the specific cases I a), b), c) mentioned above.

**I a): The volume modulus $T$ decays before the inflaton**

In this scenario, the decay of the field $T$ is characterized by $\Gamma_{T} = \frac{3}{2}H$, and the Hubble parameter for the matter-dominated universe is given by (8.57)

$$\frac{R_{dT}}{R_{\eta}} = \frac{1}{108\alpha} \left(\frac{\pi m_{\eta}\Lambda_{T}^{5}M_{P}^{2}}{m_{3/2}^{3}}\right)^{2/3}, \quad (8.71)$$

where $R_{dT}$ is the cosmological scale factor when the volume modulus $T$ decays. To obtain the case when the volume modulus $T$ decays before $\eta$, we must impose $R_{dT} < R_{d\eta}$, and if we use the scale factor ratio (8.60) with (8.71), we find the following upper bound for $\Lambda_{T}$

$$\Lambda_{T} \lesssim 3.2 \alpha^{3/10} \left(\frac{m_{3/2}^{3}}{d_{\eta}^{2}m_{\eta}^{3}}\right)^{1/5}. \quad (8.72)$$

The representative values $m_{\eta} \sim 10^{-5}M_{P}$, $m_{3/2} \sim 10^{-15}M_{P}$, $d_{\eta} \sim 1$, and $\alpha = 1$ yield $\Lambda_{T} < 3.2 \times 10^{-6}$. In this case the entropy released after $T$ decay is negligible.

**I b): The volume modulus $T$ decays after the inflaton, but never dominates**

If the decay of $T$ occurs after inflaton reheating, the value of the scale factor at $T$ decay is computed using the radiation-dominated form of $H$ given in Eq. (8.66). Then, using $\Gamma_{T} = 2H_{r}$, we find following scale factor ratio

$$\frac{R_{dT}}{R_{\eta}} = \alpha^{-3/4} \left(\frac{\pi}{486\sqrt{3}}\right)^{1/2} \frac{m_{\eta}^{1/6}\Lambda_{T}^{5/2}M_{P}^{A/3}}{d_{\eta}^{1/3}m_{3/2}^{3/2}}. \quad (8.73)$$

In order to realize this scenario, we must impose $R_{dT} > R_{d\eta}$, which holds when Eq. (8.72) is violated. However in this case, we are also assuming that $T$ never comes to dominate the energy density, i.e., $\rho_{T}(R_{dT}) < \rho_{r}(R_{dT})$. From the expressions above, we have

$$\frac{\rho_{T}}{\rho_{r}} = \alpha^{-3/4} \frac{d_{\eta}}{128} \left(\frac{\pi}{486\sqrt{3}}\right)^{1/2} \left(\frac{m_{\eta}\Lambda_{T}^{3/2}}{m_{3/2}^{3/2}}\right)^{3/2} < 1 \quad (8.74)$$
which gives an upper bound on $\Lambda_T$

$$\Lambda_T < 5.5\alpha^{1/6} d_\eta^{-2/9} \left( \frac{m_{3/2}}{m_\eta} \right)^{1/3} ,$$

so that for $m_\eta \sim 10^{-5} M_P$, $m_{3/2} \sim 10^{-15} M_P$, $d_\eta \sim 1$, and $\alpha = 1$, $\Lambda_T < 2.5 \times 10^{-3}$.

**I c): The volume modulus $T$ decays after the inflaton, and dominates at decay**

For $\Lambda_T$ larger than the upper limit in Eq. (8.75), oscillations of the volume modulus $T$ will dominate the energy density, i.e., $\rho_T > \rho_r$ before decay. In this case, the Hubble parameter is given by

$$H_T = \frac{1}{\sqrt{6\alpha}} m_{3/2} \left( \frac{R_T}{R} \right)^{3/2} ,$$

as the universe becomes matter-dominated again.

The volume modulus $T$ decays when $H_T \simeq \frac{2}{3} \Gamma_T$, and using the expressions (8.76) and (8.70), we find the following scale factor ratio

$$\frac{R_{dT}}{R_T} = \alpha^{-4/3} \left( \frac{\pi}{1296 \sqrt{2}} \frac{\Lambda_T^5 M_P^2}{m_{3/2}^2} \right)^{2/3} .$$

Because $T$ dominates when it decays, its decay products increase the entropy. The entropy densities in radiation and $T$ are given by

$$s_r = \frac{4}{3} \left( \frac{g_\eta \pi^2}{30} \right)^{1/4} \rho_r^{3/4} , \quad s_T = \frac{4}{3} \left( \frac{g_T \pi^2}{30} \right)^{1/4} \rho_T^{3/4} ,$$

yielding the following entropy ratio:

$$\frac{s_T}{s_r} = \alpha^{-3/4} \left( \frac{g_T}{g_\eta} \right)^{1/4} \left( \frac{d_\eta \sqrt{2} \Lambda_T^{9/2}}{2304 \times 3^{1/4} \sqrt{2}} \right) \left( \frac{m_\eta}{m_{3/2}} \right)^{3/2} .$$

To avoid a Polonyi-like problem, we must limit the amount of entropy production $s_T/s_r = \Delta_s$. For a given value of $\Delta_s$, we can derive an upper limit on $\Lambda_T$ (assuming $g_\eta = g_T$)

$$\Lambda_T \lesssim 2\alpha^{1/6} \sqrt{3} \left( \frac{256}{\pi} \right)^{1/9} d_\eta^{-2/9} \Delta_s^{2/9} \left( \frac{m_{3/2}}{m_\eta} \right)^{1/3} .$$

If we consider the representative choices $m_\eta \sim 10^{-5} M_P$, $m_{3/2} \sim 10^{-15} M_P$, $d_\eta \sim O(1)$,
and $\alpha = 1$, we find the upper bound $\Lambda_T \lesssim 0.003\Delta_s^{2/9} \sim 0.007$ for $\Delta_s \leq 100$, thereby mitigating the entropy production problem.

II): Oscillations of the volume modulus $T$ begin after inflaton decay

We can also consider the case when the damped oscillations of $T$ occur only after the inflaton $\eta$ decays. In this case, we use the Hubble parameter for a radiation-dominated universe (8.66) together with the expression $H = \frac{2}{3} m_T$ to obtain

$$R_T/R_\eta \simeq d_\eta^{-1/3} \left(\frac{\alpha}{48}\right)^{1/4} m_\eta^{1/6} m_P^{1/3} \Lambda_T^{1/2} m_3^{1/2}. \quad (8.81)$$

Using Eq. (8.60) to obtain $R_T/R_{d\eta}$, we can confirm that we are in scenario II when $R_T > R_{d\eta}$, which occurs when the inequality in Eq. (8.64) is violated. We can now distinguish two cases depending on whether $T$ dominates at the time of its decay, or not.

II a): The volume modulus $T$ decays before the universe becomes dominated by $T$ oscillations

This case is similar to scenario I b). The ratio $R_{dT}/R_\eta$ is given by Eq. (8.73), but the ratio of energies is computed using (8.81) as opposed to (8.62), giving

$$\frac{\rho_T}{\rho_r} = \frac{1}{\alpha \left(\frac{\sqrt{\pi}}{576}\right)^{1/8}} \frac{m_P^4 \Lambda_T^{4/3}}{m_3^{4/3}} < 1, \quad (8.82)$$

where the inequality should be satisfied to avoid $T$ domination, and give the limit

$$\Lambda_T < 2\alpha^{1/4} \left(\frac{7776}{\pi}\right)^{1/8} \left(\frac{m_3^{2/3}}{m_P}\right)^{1/4}. \quad (8.83)$$

Note that the limit is independent of $m_\eta$ in this case, in contrast to the limit in Eq. (8.75). The range for $\Lambda_T$ must therefore satisfy the inequality in Eq. (8.83) and violate that in Eq. (8.64). This is possible if

$$m_3^{2/3} < \alpha \left(\frac{3}{8\pi}\right)^{1/6} d_\eta^{8/3} m_\eta^4 m_P^2, \quad (8.84)$$

and corresponds to very small gravitino masses.

II b): The volume modulus $T$ decays after the universe becomes dominated by $T$
oscillations

This case is similar to I c). The ratio of the entropies is now

$$\frac{s_T}{s_r} = \frac{1}{\alpha} \left( \frac{g_T}{g_\eta} \right)^{1/4} \left( \frac{\sqrt{\pi}}{1152 \sqrt{2}} \right) \left( \frac{\Lambda_{T}^4 M_P}{m_{3/2}} \right).$$  \hspace{1cm} (8.85)

The limit on $\Lambda_T$ to avoid excessive entropy production is

$$\Lambda_T < 2\alpha^{1/4} \sqrt{3} \left( \frac{128}{\pi} \right)^{1/8} \Delta_s^{1/4} \left( \frac{m_{3/2}}{M_P} \right)^{1/4}$$  \hspace{1cm} (8.86)

where we are again allowing for entropy increase by a maximum factor of $\Delta_s$. This case is realized when the inequality in (8.86) is satisfied, but that in (8.64) is not. This can occur when

$$m_{3/2} < \alpha^{1/6} \left( \frac{1}{2\pi} \right)^{1/6} d_\eta^{8/3} \frac{m_\eta^4}{M_P^2} \Delta_s^{1/3}. \hspace{1cm} (8.87)$$

### 8.3.3 Limits on $\Lambda_T$ from Dark Matter Production

In this section we consider the limits on $\Lambda_T$ that avoid the overproduction of the lightest supersymmetric particle (LSP) and related cosmological problems.

We consider first inflaton decay to gravitinos. We calculate the inflaton to two gravitino decay rate using the Lagrangian (8.68), obtaining the decay rate $\Gamma(\phi \to \psi_{3/2} \psi_{3/2}) \sim \langle G_\phi \rangle$. As previously, we assume that $\lambda_2 \ll \lambda_1$. For $\phi$-type models, at the minimum when $\langle T \rangle = \frac{1}{2}$ and $\langle \phi \rangle = 0$, we have $\langle G_\phi \rangle = 0$. Similarly, for the case with a twisted inflaton field, we have $\langle G_\phi \rangle = 0$. Therefore, the decays of the inflaton to a pair of gravitinos are negligible, unless there is an additional coupling of $\phi$ to $T$ [244].

If the $T$ field decays into a gravitino pair, we have

$$n_{3/2} = 2n_T, \hspace{1cm} (8.88)$$

where $n_{3/2}$ and $n_T$ are the number densities of the produced gravitinos and of the decaying $T$ field respectively. Using the following approximations

$$n_\chi \simeq n_{3/2}, \quad s_0 \simeq 7n_\gamma, \hspace{1cm} (8.89)$$
where \( n_\chi \) is the cold dark matter number density, and \( s_0 \) is the total entropy density today, we obtain the following estimate of the cold dark matter abundance (\( \rho_c \) is the critical density)\)

\[
\Omega_\chi \simeq \frac{7m_\chi n_{3/2}^\gamma}{s_0 \rho_c} \simeq 2.75 \times 10^{10} h^{-2} \left( \frac{m_\chi}{100 \text{ GeV}} \right) \left( \frac{n_{3/2}}{s_0} \right) .
\]

(8.90)

Planck 2018 data impose the constraint [28]\)

\[
\Omega_\chi h^2 \lesssim 0.12 ,
\]

(8.91)

which leads to the following upper limit on the gravitino-to-entropy ratio:

\[
\frac{n_{3/2}}{s_0} \lesssim 4.4 \times 10^{-12} \left( \frac{100 \text{ GeV}}{m_\chi} \right) ,
\]

(8.92)

which is different for scenarios I and II. We first consider gravitinos produced by \( T \) decay and subsequently consider the thermal production of gravitinos which is controlled by the reheat temperature and the value of \( d_\eta \).

**Scenario I:** The number density \( n_T = \rho_T / m_T \) depends whether \( T \) decays before the inflaton (I a), or after the inflaton but does not dominate the energy density (I b), or dominates the energy density (I c). For (I a), we find:

(I a) : \( n_T = \alpha^{7/2} \frac{2187\sqrt{3}}{2\pi^2} m_{3/2}^{3/2} \frac{1}{M_P^2 \Lambda_T^9} \) when \( R_{dT} < R_{d\eta} \).

(8.93)

whereas for (I b) and (I c), we obtain

(I b) : \( n_T = \alpha^{11/4} \frac{729 \times 3^{3/4}}{32\sqrt{2}\pi^{3/2}} d_\eta^2 \frac{m_{3/2}^{7/2}}{M_P^2 \Lambda_T^{9/2}} \) when \( R_{dT} > R_{d\eta} \), does not dominate,

(8.94)

and

(I c) : \( n_T = \alpha^{7/2} \frac{1}{\sqrt{48}} \left( \frac{1296}{\pi} \right)^2 \frac{m_{3/2}^{5/2}}{M_P^2 \Lambda_T^{9/2}} \) when \( R_{dT} > R_{d\eta} \), dominates.

(8.95)
We obtain the same gravitino-to-entropy ratio in all three cases, which is given by

\[
\frac{n_{3/2}}{s_0} \simeq \sqrt{\alpha} \frac{1}{512} \left( \frac{45}{2\pi^2} \right)^{1/4} d_\eta g_\eta^{-1/4} \frac{m_{3/2}^3 / \Lambda_T^3}{m_{3/2} M_P^{1/2}}.
\]  

(8.96)

The relevant dark matter yield is given by \(n_\chi / s \simeq n_{3/2} / s\). In this case, the density parameter \(\Omega_\chi h^2 \simeq m_\chi n_\chi / \rho_c\) can be expressed as

\[
\Omega_\chi h^2 \simeq 6.6 \times 10^7 \sqrt{\alpha} d_\eta g_\eta^{-1/4} \frac{m_{3/2}^3 / \Lambda_T^3}{m_{3/2} M_P^{1/2}} \left( \frac{m_\chi}{100 \text{ GeV}} \right),
\]  

(8.97)

where we assume that the LSP mass is not much smaller than that of the gravitino. If we use the nominal value \(\Omega_\chi h^2 \simeq 0.12\) with (8.97), we find

\[
\Lambda_T \lesssim 1.2 \times 10^{-3} \alpha^{-1/6} d_\eta^{-1/3} g_\eta^{1/12} \frac{m_{3/2} M_P^{1/2}}{m_{3/2} M_P^{1/2}} \left( \frac{m_\chi}{100 \text{ GeV}} \right)^{-1/3}.
\]  

(8.98)

This is the upper bound on \(\Lambda_T\) that is imposed by consistency with the current dark matter density given by the most recent Planck data [28], making the plausible assumption that the entropy released by the decay of the gravitino is negligible.

We also consider the thermal production of gravitinos. Gravitinos are produced during reheating, and the abundance of gravitinos scales with the reheating temperature \(T_{RH}\). The gravitino-to-entropy ratio arising from thermal production can be related to the inflaton decay rate, and hence the inflaton decay coupling \(d_\eta\), and is given by [67, 68, 283, 284, 285, 69, 138]

\[
\frac{n_{3/2}}{s_0} \simeq 2.6 \times 10^{-4} d_\eta \left( 1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2} \right) \left( \frac{m_\eta}{M_P} \right)^{3/2},
\]  

(8.99)

where contributions to the production of transverse modes (1) and longitudinal models (.56 \(m_{1/2}^2 / m_{3/2}^2\)) are included. If we use Eq. (8.99) in the limit (8.92), we obtain an upper limit on the coupling \(d_\eta\)

\[
d_\eta < 1.7 \times 10^{-8} \left( \frac{M_P}{m_\eta} \right)^{3/2} \left( \frac{100 \text{ GeV}}{m_\chi} \right) \left( 1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2} \right)^{-1}.
\]  

(8.100)
If \( m_\eta = 3 \times 10^{13} \text{ GeV} \) and assuming conservatively the lower limit \( m_\chi \gtrsim 100 \text{ GeV} \) and that \( m_{1/2} \ll m_{3/2} \), we obtain
\[
d_\eta \lesssim 0.4. \tag{8.101}
\]
This limit must be respected independently of any assumptions on the stabilization of the volume modulus. Note that \( d_\eta < 0.4 \) corresponds to a limit \( y < 2.4 \times 10^{-5} \) where \( y \) is a conventionally defined inflaton coupling, \( y = \sqrt{8\pi d_\eta m_\eta/M_P} \), in agreement with past results [286, 241, 242].

Considering the ratio of the yield produced by \( T \) decays (8.96) to the thermal yield (8.99), we find
\[
\frac{\langle n_{3/2}/s_0 \rangle_{\text{Decay}}}{\langle n_{3/2}/s_0 \rangle_{\text{Thermal}}} \simeq 2.4 \alpha^{1/2} \left(1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2}\right)^{-1} \frac{M_P \Lambda_T^3}{m_{3/2}}, \tag{8.102}
\]
and thermal production is subdominant when
\[
\Lambda_T \gtrsim 0.75 \alpha^{-1/6} \left(1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2}\right)^{1/3} \left(\frac{m_{3/2}}{M_P}\right)^{1/3}. \tag{8.103}
\]

**Scenario II:** We find the following number density \( n_T = \rho_T/m_T \) for case (II a):

\[
(\text{II a}) : \quad n_T \simeq \alpha^{5/2} \frac{2187}{16 \sqrt{2\pi^3/2}} \frac{m_{3/2}^4}{M_P \Lambda_T^5} \quad \text{when} \quad R_{dT} > R_{d\eta}, \quad \text{does not dominate.} \tag{8.104}
\]

For case (II b), we have the same result as in Eq. (8.95) for \( n_T \), but we find the gravitino-to-entropy ratio to be same for cases II (a, b):
\[
\frac{n_{3/2}}{s_0} \simeq \alpha^{1/4} \frac{135}{256} \left(\frac{135}{2\pi^2}\right)^{1/4} g_\eta^{-1/4} \frac{M_P^{1/2} \Lambda_T^{5/2}}{m_{3/2}^{1/2}}. \tag{8.105}
\]

In this case, we find the following density parameter
\[
\Omega_\chi h^2 \simeq 1.7 \times 10^8 \alpha^{1/4} g_\eta^{-1/4} \frac{M_P^{1/2} \Lambda_T^{5/2}}{m_{3/2}^{1/2}} \left(\frac{m_\chi}{100 \text{ GeV}}\right). \tag{8.106}
\]
We next compare this production of gravitinos in moduli decays with thermal production, using the following expression for the gravitino-to-entropy ratio from thermal production

\[
\left( \frac{n_{3/2}/s_0}{(n_{3/2}/s_0)_{\text{Thermal}}} \right)^{\text{Decay}} \simeq 6.2 \alpha^{1/4} d^{-1} \left( 1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2} \right)^{-1} \frac{M_P^2 \Lambda_T^{5/2}}{m_{1/2}^{1/2} m_{3/2}^{3/2}},
\]

and thermal production is subdominant when

\[
\Lambda_T \gtrsim 0.5 \alpha^{-1/10} d^{-2/5} \left( 1 + 0.56 \frac{m_{1/2}^2}{m_{3/2}^2} \right)^{2/5} \frac{M_{1/2}^1 m_{3/2}^{3/5}}{M_P^{4/5}}.
\]

Finally, we note that there is a lower limit on \( \Lambda_T \) coming from the postulated form of the stabilization terms in Eqs. (8.26) and (8.27). Since the stabilization terms in the Kähler potential should be treated as an effective interaction by integrating out fields with masses, \( \Lambda_T M_P \), we should require \( \Lambda_T M_P > \sqrt{F_T} \) [252, 253, 254], and using (8.43) we find the limit

\[
\Lambda_T > \alpha^{-1/4} \left( \frac{m_{3/2}}{M_P} \right)^{1/2},
\]

that is imposed by the effective interaction assumption.

We display in Fig. 8.2 compilations of the constraints on \( \Lambda_T \) as functions of the gravitino mass for three different values of the coupling \( d_\eta \): 0.4 (which is the largest value allowed by our analysis - see Eq. (8.101)), \( 10^{-3} \) and \( 10^{-5} \), illustrating their impacts on the various scenarios discussed above. For all our plots we set the inflaton mass equal to its value in the Starobinsky model, \( m_\eta = 3 \times 10^{13} \) GeV, and the curvature parameter \( \alpha = 1 \). The following are the interpretations of the lines and shadings in the various panels. The red lines mark the upper limit on \( \Lambda_T \) that is imposed by the avoidance of entropy overproduction, assuming \( \Delta_s \leq 100 \). Above the region labelled Scenario II b) in the top panel, this line is given by Eq. (8.86) and scales as \( \Delta_s^{1/4} \), and above the regions labelled Scenario I c) the red line is determined from Eq. (8.80) and scales as \( \Delta_s^{2/9} \). The dashed green line in the top panel corresponds to the condition that \( T \) oscillations
begin before inflaton decay - see Eq. (8.64) - and separates Scenarios I) and II). Scenario II) is visible only in the upper panel, for large $d_\eta$, and is realized in the region shaded green between the green dotted line and the entropy overproduction line. The part of the solid purple line crossing the green region corresponds to Eq. (8.83) and marks the boundaries between Scenarios II a) and II b) (below and above, respectively). The part of the purple line that crosses the yellow region corresponds to Eq. (8.75) and marks the boundaries between Scenarios I c) and I b) (above and below, respectively). The largest value of $m_{3/2}$ in region II a) is given by Eq. (8.84) and that in region II b) is given by Eq. (8.87). Variants of Scenario I) are realized in the regions shaded yellow and blue. The dashed blue line corresponds to Eq. (8.72) and marks the boundary between Scenarios I b) (above) and I a) (below): the latter region is shaded blue. The solid grey line represents the effective interaction condition (8.110), below which our parametrization of the dynamics responsible for $T$ stabilization is invalid.

The strongest upper limits on $\Lambda_T$ come from the production of dark matter. The thermal production of gravitinos leads to a dark matter abundance which is independent of $\Lambda_T$ (and $m_{3/2}$ when $m_{1/2} \ll m_{3/2}$ - see Eqs. (8.100)), and depends only on the coupling $d_\eta$. For $d_\eta \simeq 0.4$, thermal production contributes $\Omega_\chi h^2 = 0.12$ everywhere in the plane. The solid black lines in the three panels show the constraint (8.98) on the contribution to $\Omega_\chi h^2 = 0.12$ from $T$ decay, which decreases at lower $\Lambda_T$. At lower values of $d_\eta$, as in the middle and bottom panels of Fig. 8.2, the thermal contribution always gives a dark matter density below the Planck limit. We show as solid orange lines in these panels, the values of $m_{3/2}, \Lambda_T$ for which the thermal and non-thermal contributions are equal. This line is independent of $d_\eta$ and given by Eq. (8.102). Below the orange line, the thermal contribution dominates the final dark matter abundance. For $d_\eta = 0.4$, as in the upper panel, the orange and black lines coincide. Note that we have fixed $m_\chi = 100$ GeV everywhere in this figure and the relic density scales as $m_\chi$, so that the limit on $\Lambda_T$ scales as $m_\chi^{-1/3}$. Recall also that we have fixed $\alpha = 1$, though the limit on $\Lambda_T$ scales weakly as $\alpha^{-1/6}$.

We see that there are allowed regions in Fig. 8.2 below the dark matter density constraint and above the effective interaction limit. These include regions realized in Scenarios I a) and I b), but not Scenarios I c) and II). If $m_{3/2} \gtrsim 1$ TeV, the only Scenario that can be realized is I a), in which the volume modulus $T$ decays before the inflaton.
Figure 8.2: Plots of the constraints on the modulus stabilization parameter, $\Lambda_T$, as functions of the gravitino mass, $m_{3/2}$, for models with $m_\eta = 3 \times 10^{13}$ GeV, $\alpha = 1$ and $d_\eta = 0.4$ (top), $10^{-3}$ (middle) and $10^{-5}$ (bottom). The regions shaded green, yellow and blue correspond to Scenarios II a, b), I b, c), and I a), respectively, whereas the grey regions are excluded by the effective interaction condition (8.110). Regions between this and the dark matter density constraint (solid black line) are allowed by all the constraints.
However, these restrictions on the possible scenarios might not apply if the dark matter density constraint was weakened, e.g., if the assumption of R-parity conservation was relaxed.

8.4 Concluding Remarks

In this chapter, we have presented some important phenomenological and cosmological aspects of the no-scale attractor models of inflation that we have introduced previously. These models are based on no-scale supergravity, and include mechanisms for modulus fixing, inflation, supersymmetry breaking and dark energy. As we have discussed, there are models in which inflation is driven by either a modulus field (T-type), in which supersymmetry is broken by a Polonyi field, or a matter field (ϕ-type) with supersymmetry broken by the modulus field. We have derived the possible patterns of soft supersymmetry breaking in these different types of models, which depend on the chosen Kähler geometries for the matter and inflaton fields, i.e., the parameter α and whether the matter and inflaton are twisted or untwisted. The results are tabulated in Tables 8.1 and 8.2 for the T- and ϕ-type models, respectively. The patterns of soft supersymmetry breaking found in our analysis include those postulated in the CMSSM, mSUGRA, minimal no-scale supergravity and pure gravity mediation models. Within the framework of no-scale attractor models, phenomenological analyses of the pattern of soft supersymmetry breaking could help to pin down the model type and its Kähler geometry. We find that there is a direct relation between the scale of supersymmetry breaking and the inflaton mass in ϕ-type models.

We have also discussed cosmological constraints on the models from entropy considerations, the density of dark matter, and field stabilization. These constraints restrict the possible ranges of the quartic parameters in the Kähler potential that are used to stabilize the Polonyi field in T-type models and the modulus field in the ϕ-type models. We focus on the ϕ-type models, in particular, with the results shown in Fig. 8.2. As we see there, the avoidance of entropy and particularly dark matter overproduction require the corresponding stabilization parameter $\Lambda_T$ to be a few orders of magnitude below the Planck scale. We see in Fig. 8.2 that there are allowed regions for some of the cosmological scenarios discussed in the text, which could be expanded for small LSP
masses or if R-parity is broken.

The key Kähler geometry parameter for no-scale attractor models of inflation is the parameter $\alpha$ appearing in Eq. (6.2), which may be related to the form of string compactification. This parameter can be determined by measurements of the CMB observables $r$ and $n_s$, as seen in Eq. (8.1). It is intriguing that this same parameter enters the values of the soft supersymmetry-breaking parameters in Tables 8.1 and 8.2, offering the possibility of correlating directly collider and CMB measurements. This is a concrete example how no-scale attractor models could, in the future, serve as bridges between early-universe cosmology, collider physics and string theory.
Chapter 9

BICEP/Keck Constraints on Attractor Models of Inflation

In this chapter, we discuss how to constrain the attractor models of inflation, that we discussed in the previous chapters, using the CMB constraints.

Successive releases of data on perturbations in the cosmic microwave background (CMB) [28] have provided increasingly strong upper limits on the tensor-to-scalar ratio, $r$, and hence sharpened focus on models of inflation that favour small values of $r$, such as the original Starobinsky model [84] that predicts $r \sim 0.004$ for 55 e-folds. The recent release of the BICEP/Keck [86] data has followed this trend, imposing the bound $r_{0.05} < 0.036$ at the 95% C.L. where the subscript denotes the pivot scale in Mpc$^{-1}$. Moreover, the combination of WMAP, Planck and BICEP/Keck data constrains the scalar tilt to the limited range $0.958 < n_s < 0.975$ at the 95% C.L. for $r = 0.004$. A further analysis by [287] used BB autocorrelation data from [288] and allowed a free reionization optical depth, and obtained a lower limit on the scalar-to-tensor ratio to $r_{0.05} < 0.032$, with a slightly relaxed range on the spectral tilt $0.956 < n_s < 0.974$ at the 95% C.L. for $r = 0.004$.

The Starobinsky model is not alone in accommodating the upper limit on $r$. For example, Higgs inflation predicts a similar value of $r$ [289], and similar potentials appear naturally in the context of supergravity, including no-scale supergravity [43, 76]. In particular, the simplest no-scale supergravity models characterized by a Kähler potential
of the form $K = -3 \ln(T + \bar{T} - |\phi|^2/3)$, where $T$ and $\phi$ are complex scalar fields, predict a Starobinsky-like value of $r$ \cite{85}, but the no-scale supergravity framework can also accommodate other possibilities \cite{205}.

For example, generalizing $-3 \to -3\alpha$ as the coefficient of the logarithm modifies the prediction for $r$ by a factor $\alpha$, as was first pointed out in \cite{225} and subsequently in \cite{239}. Such a modification of the simplest no-scale model is a natural possibility in compactified string models, where $T$ may be interpreted as the volume modulus \cite{77}, which is a product of three independent compactification moduli $T_i : i = 1, 2, 3$. Models in which inflation is driven by one (two) of these moduli correspond to $\alpha = 1/3$ ($2/3$) \cite{225}. Larger values of $\alpha$ are also possible, since string compactifications also have complex structure moduli that can contribute to the inflationary dynamics \cite{290}.

A common feature of these no-scale supergravity models is a quadratic singularity in the kinetic term for the inflaton. This feature leads generically to an effective potential for the canonically normalized inflaton field with a plateau that leads to a quasi-de Sitter inflationary epoch similar to that in Starobinsky inflation. This property was abstracted from the no-scale models in \cite{290}, where they were baptized “attractor” models. Two specific types of attractor potential can be distinguished \cite{225, 239, 291}: \cite{1}

\begin{align*}
V &= \frac{3}{4} \lambda M_P^4 \left(1 - e^{-\sqrt{5\alpha} \frac{\varphi}{M_P}}\right)^2, \quad (\alpha\text{-Starobinsky}) \\
V &= \frac{3}{4} \lambda M_P^4 \operatorname{tanh}^2 \left(\frac{\varphi}{\sqrt{6\alpha} M_P}\right), \quad (T\text{-Model})
\end{align*}

(9.1) (9.2)

where $\varphi$ is the canonically normalized inflaton field and $\lambda$ is the potential scale determined from the CMB normalization and the inflaton field value at horizon crossing.\footnote{The normalization of the potentials is chosen so that the inflaton normalization scale coincides in both cases, and is given by Eq. (9.7). This choice does not affect the CMB observables $n_s$ and $r$.} For the attractor models discussed here, increasing the value of $\alpha$ reduces the flatness of the plateau at the inflaton field value at the horizon crossing of the CMB scale, $\varphi_*$, which affects the cosmological observables $n_s$ and $r$. It was argued in \cite{225, 239, 291, 205, 238} that broad classes of attractor models lead to identical predictions of $n_s$ and $r$ in the limit of a large number of $e$-folds, $N_*$.\footnote{We note that the potentials (9.1) and (9.2) are identical at zeroth and first order in $e^{-\sqrt{5\alpha} \frac{\varphi}{M_P}}$, but...} In the context of supergravity, the parameter $\alpha$...
determines the curvature of the internal Kähler manifold: $R = 2/\alpha$.

In this chapter, we explore the impact of the latest BICEP/Keck/WMAP/Planck constraints in the $(n_s, r)$ plane on the $\alpha$-Starobinsky and T-model inflationary attractors (see also [293]) from both [86] and [287]. From the analysis in [86], we find that the region of CMB parameters favoured at the 68% C.L. by the combination of CMB data favours $N_{0.05} \gtrsim 50.9\, (52.6)$ in the $\alpha$-Starobinsky (T-models), corresponding to an inflaton decay coupling $y \gtrsim 1.7 \times 10^{-6}\, (1.7 \times 10^{-4})$ for $\alpha = 1$, with an order of magnitude sensitivity to $\alpha \in (0.1, 5)$. In contrast, the analysis in [287] yields substantially weaker bounds, $N_{0.05} \gtrsim 47.9\, (49.4)$ in the $\alpha$-Starobinsky (T-models), corresponding to an inflaton decay coupling $y \gtrsim 1.9 \times 10^{-10}\, (1.2 \times 10^{-8})$ for $\alpha = 1$. Additionally, supergravity models must avoid overproducing gravitinos and supersymmetric dark matter [55, 294]. We find that based on [86], $\alpha$-Starobinsky models that respect these constraints fall inside the region favoured by the CMB data at the 68% C.L. only for $\alpha \in (0.67, 12)$, and that T-models fall inside this region only for $\alpha \in (1.3, 5.1)$. At the 95% C.L. these range are $(0, 26)$ and $(0, 11)$, respectively. Based on [287], the 68% C.L. range is $(0.4, 12)$ and $(0.5, 7)$ for the $\alpha$-Starobinsky and T-models, respectively, and the 95% C.L. ranges are $(0, 24)$ and $(0, 12)$.

9.1 Inflationary Dynamics

The dynamics of the inflaton is characterized by the action (2.35) where the effective scalar potential is given by Eq. (9.1) or (9.2). We use for our analysis the conventional slow-roll parameters, which are given in single-field inflationary models by (2.46, 2.47). In the slow-roll approximation, the number of $e$-folds can be computed using (2.48). The end of inflation occurs when $\ddot{\alpha} = 0$, i.e., $\varphi_{\text{end}}^2 = V(\varphi_{\text{end}})$.

\begin{itemize}
  \item In general, the Kähler curvature $R$ depends on the total number, $n$, of chiral fields describing the theory [238, 76], $R = n(n + 1)/3\alpha$, and this result holds for two chiral fields, which is the minimal number needed to construct a plateau-like potential in no-scale supergravity [225].
  \item The corresponding 95% limits are $N \gtrsim 45.9\, (47.5)$ and $y \gtrsim 3.8 \times 10^{-13}\, (3.6 \times 10^{-11})$, respectively.
  \item In this case, the corresponding 95% limits are $N \gtrsim 42.9\, (44.6)$ and $y \gtrsim 2.8 \times 10^{-17}\, (4.0 \times 10^{-15})$, respectively.
  \item Here the lower bound $\alpha > 0$ arises because $\alpha = 0$ leads to a completely flat potential that is not suitable for inflation.
\end{itemize}
The principal CMB observables, namely, the scalar tilt, \( n_s \), the tensor-to-scalar ratio, \( r \), and the amplitude of the curvature power spectrum, \( A_S \), can be expressed in terms of the slow-roll parameters (2.88). In the large \( N_\ast \) limit, the inflationary attractor potentials (9.1) and (9.2) predict (7.2), where the approximation holds for \( \alpha \lesssim O(1) \) in \( \alpha \)-Starobinsky models, and the full analytical expression can be found in [294].

Using expression (2.48), we can calculate the approximate value of the inflaton field at the horizon exit scale \( k_\ast \) [137] when \( \alpha = 1 \),

\[
\frac{\varphi_\ast}{M_P} \simeq \sqrt{\frac{3}{2}} \left[ 1 + \frac{3}{4N_\ast - 3} \right] \times \ln \left( \frac{4N_\ast}{3} + e^{\sqrt{\frac{2}{3}} \frac{\varphi_\ast}{M_P}} - \sqrt{\frac{2}{3}} \frac{\varphi_\ast}{M_P} \right),
\]

\( \quad \text{(\( \alpha \)-Starobinsky)} \quad (9.3) \)

\[
\frac{\varphi_\ast}{M_P} \simeq \sqrt{\frac{3}{2}} \cosh^{-1} \left( \frac{4N_\ast}{3} + \cosh \left( \sqrt{\frac{2}{3}} \frac{\varphi_\ast}{M_P} \right) \right),
\]

\( \quad \text{(T-Model)} \quad (9.4) \)

with

\[
\frac{\varphi_\ast}{M_P} \simeq \sqrt{\frac{3}{2}} \ln \left[ \frac{2}{11} \left( 4 + 3\sqrt{3} \right) \right],
\]

\( \quad \text{(\( \alpha \)-Starobinsky)} \quad (9.5) \)

\[
\frac{\varphi_\ast}{M_P} \simeq \sqrt{\frac{3}{2}} \ln \left[ \frac{1}{11} \left( 14 + 5\sqrt{3} \right) \right], \quad \text{(T-Model)} \quad (9.6)
\]

where \( \varphi_\ast \) was calculated using the expression \( \epsilon = (1 + \sqrt{1 - \eta/2})^2 \), and the full analytical approximations for \( \varphi_\ast \) and \( \varphi_\ast \) can be found in Appendix E, where they are given by Eqs. (E.2)-(E.5). Combining the expressions above with expression (2.86) for the curvature power spectrum, we find that the inflaton normalization scale is proportional to \( \lambda \), which is in turn proportional to \( \alpha \) and given by

\[
\lambda \simeq \frac{24\alpha \pi^2 A_{S\ast}}{N_{S\ast}^2}.
\]

(9.7)
We now calculate the number of \( e \)-folds, \( N_* \), assuming that there is no additional entropy injection between the end of reheating and when the horizon scale \( k_* \) reenters the horizon [144, 143], and the expression for \( N_* \) is given by Eq. (2.83), and the present Hubble parameter and photon temperature are given by \( H_0 = 67.36 \text{ km s}^{-1} \text{ Mpc}^{-1} \) [28] and \( T_0 = 2.7255 \text{ K} \) [103]. Here, \( \rho_{\text{end}} \) and \( \rho_{\text{rad}} \) are the energy density at the end of inflation and at the beginning of the radiation domination era when \( w = p/\rho = 1/3 \), respectively, \( a_0 = 1 \) is the present day scale factor, \( g_{\text{reh}} = 915/4 \) is the effective number of relativistic degrees of freedom in the minimal supersymmetric standard model (MSSM) at the time of reheating, and the equation of state parameter averaged over the \( e \)-folds during reheating is given by Eq. (2.79).

Using the numerical values given above with the Planck pivot scale \( k_* = 0.05 \text{ Mpc}^{-1} \), we find the following value for the sum of the first two lines in (2.79): \( N_* \approx 61.04 + \cdots \). Mechanisms for producing a baryon asymmetry (such as leptogenesis) are simplified when \( T_{\text{reh}} \gtrsim \text{the electroweak scale} \). Accordingly, we also display results for a reheating temperature \( T_{\text{reh}} = T_{\text{EW}} \approx 100 \text{ GeV} \), whilst acknowledging that lower reheating temperatures are possible. For \( T_{\text{reh}} = T_{\text{EW}} \) we take the Standard Model value for \( g_{\text{reh}} = 427/4 \), and find \( N_{\text{EW}} = 61.10 + \cdots \). The minimum reheating temperature that is compatible with Big Bang Nucleosynthesis (BBN) is \( T_{\text{reh}} \gtrsim \mathcal{O}(1) \text{ MeV} \). Using \( T_{\text{BBN}} = 2 \text{ MeV} \) in our numerical analysis, corresponding to \( g_{\text{reh}} = 10.75 \), the sum of the first two lines of (2.79) takes the following numerical value: \( N_{\text{BBN}} \approx 61.29 + \cdots \).

To calculate the values of \( N_* \), \( N_{\text{EW}} \) and \( N_{\text{BBN}} \) numerically, we use the following equations that govern the cosmic background dynamics:

\[
\dot{\rho}_\varphi + 3H \rho_\varphi = -\Gamma_\varphi \rho_\varphi, \tag{9.8}
\]

\[
\dot{\rho}_r + 4H \rho_r = \Gamma_\varphi \rho_\varphi, \tag{9.9}
\]

\[
\rho_\varphi + \rho_r = 3M_P^2 H^2, \tag{9.10}
\]

\[
dt (N w_{\text{int}}) = H w, \tag{9.11}
\]

\[\text{We note that when we calculate the tensor-to-scalar ratio } r_{0.002} \text{ numerically, we evaluate } N_* \text{ at the pivot scale } k_* = 0.002 \text{ Mpc}^{-1}.\]
where $\rho_\phi$ and $\rho_r$ are the energy densities of the inflaton and produced radiation, respectively, and $\Gamma_\phi$ is the inflaton decay rate given by

$$\Gamma_\phi = \frac{y^2}{8\pi} m_\phi,$$

(9.12)

where $y$ is a Yukawa-like coupling, and we find the following masses in the inflationary attractor potentials (9.1) and (9.2):

$$m_\phi = \sqrt{\frac{\lambda}{\alpha}} M_P ,$$

($\alpha$-Starobinsky) (9.13)

$$m_\phi = \frac{1}{2} \sqrt{\frac{\lambda}{\alpha}} M_P .$$

(T-Model) (9.14)

### 9.2 Reheating

The reheating process occurs after the end of inflation in a matter-dominated background. As the inflaton starts to decay, the dilute plasma reaches a maximum temperature, $T_{\text{max}}$ [295, 138], and subsequently starts falling as $T \propto a^{-3/8}$. The reheating temperature is defined through [140, 296]

$$\frac{\pi^2 g_{\text{reh}} T_{\text{reh}}^4}{30} = \frac{12}{25} (\Gamma_\phi M_P)^2 ,$$

(9.15)

when the energy density of the inflaton is equal to the energy density of radiation, corresponding to

$$T_{\text{reh}} \simeq 1.9 \times 10^{15} \text{ GeV} \cdot y \cdot g_{\text{reh}}^{-1/4} \left( \frac{m_\phi}{3 \times 10^{13} \text{ GeV}} \right)^{1/2} .$$

(9.16)

In order to evaluate the constraint on $T_{\text{reh}}$ from overproduction of supersymmetric dark matter in scenarios where the gravitino is lighter than $T_{\text{reh}}$, we use the expression [138, 297]$^9$

$$Y_{3/2}(T) = 0.00336 \left( 1 + 0.51 \frac{m_{1/2}^2}{m_{3/2}^2} \right) \left( \frac{\Gamma_\phi}{M_P} \right)^{1/2} ,$$

(9.17)

$^9$We use here an analytical approximation since there is only a 0.03 % difference between the analytical and fully numerical calculation.
where \( Y_{3/2} \equiv n_{3/2}/n_{\text{rad}} \) is the gravitino yield, \( n_{\text{rad}} = \zeta(3)T^3/\pi^2 \), \( m_{3/2} \) the gravitino mass, and \( m_{1/2} \) the gluino mass \([60, 65, 67]\). Disregarding the term \( m_{1/2}^2/m_{3/2}^2 \) in (9.17) and using the observed dark matter density today, \( \Omega_{\text{CDM}} h^2 \approx 0.12 \), we find the following upper limit on the Yukawa-like inflaton coupling, assuming that the gravitino decays after the lightest supersymmetric particle (LSP) decouples,

\[
|y| < 9.2 \times 10^{-8} \sqrt{\frac{M_P}{m_\varphi}} \left( \frac{100 \text{ GeV}}{m_{\text{LSP}}} \right),
\]

where \( m_{\text{LSP}} \) is the mass of the LSP and the inflaton masses for the different inflationary attractor potentials are given by Eqs. (9.13) and (9.14).\(^{10}\) We note that, since \( m_\varphi \propto 1/\sqrt{\alpha} \), \( |y| \propto \alpha^{1/4}. \)

In high-scale supersymmetry models in which the gravitino mass may be significantly larger than the electroweak scale and the other supersymmetric particles are heavier than the inflaton, the gravitino, which is now the LSP, is pair-produced via its longitudinal components \([299]\). In such a scenario, we find \([139]\]

\[
\Omega_{3/2} h^2 \approx 0.12 \left( \frac{|y|}{3.0 \times 10^{-7}} \right)^{19/5} \left( \frac{m_\varphi}{3 \times 10^{13} \text{ GeV}} \right)^{67/10} \times \left( \frac{0.1 \text{ EeV}}{m_{3/2}} \right)^3 \left( \frac{0.030}{\alpha_3} \right)^{16/5},
\]

where \( m_{3/2} \) is the gravitino mass and \( \alpha_3 \) is the strong coupling. Using the observed dark matter abundance today to constrain \( \Omega_{3/2} h^2 \), we find that avoiding overproduction of dark matter imposes the following bound:

\[
|y| < 6.6 \times 10^{-16} \left( \frac{M_P}{m_\varphi} \right)^{67/38} \left( \frac{m_{3/2}}{0.1 \text{ EeV}} \right)^{15/19}.
\]

We note that in a non-supersymmetric theory there would, in general, be a lower limit on \( y \) due to the fact that it generates radiative corrections \( \propto y^4 \) in the effective inflaton potential \([300]\). However, this is not the case in supersymmetric models such as those discussed above, where these radiative corrections cancel down to the level of the

\(^{10}\)If the gravitino is the LSP, the second term in the brackets in (9.17) must be taken into account, and the constraint on \( y \) depends on the ratio \( m_{1/2}/m_{3/2} \).

\(^{11}\)For another recent analysis of gravitino constraints in light of the BICEP/Keck results, see \([298]\).
relatively small supersymmetry-breaking effects [29].

9.3 Results

We solve the cosmic background equations (9.8)-(9.11) numerically to determine the number of e-folds \( N_\star, N_{\text{EW}}, \) and \( N_{\text{BBN}} \). In the \( \alpha = 1 \) case, the procedure of calculating the analytical approximations for \( N_\star \) is given in Appendix E (see Eqs. (E.11) and (E.12)). The full numerical computation of the CMB observables is discussed in Appendix F.

Figure 9.1 summarizes our numerical results based on the analysis of [86]: those for \( \alpha \)-Starobinsky models are shown in the upper pair of panels and those for T-models in the lower pair. For each of the two models, we derive limits on \( N_\star \) from the requirements that \( T_{\text{reh}} > 2 \) MeV (100 GeV) and the supersymmetric relic density when \( m_{\text{LSP}} = 100 \) GeV. The former gives a lower limit to \( N_\star \), while the latter gives an upper limit. We also derive the corresponding limits on \( y \). These are compared to the 68\% and 95\% C.L. limits on \( N \) and \( y \) from the BICEP/Keck constraints on \( n_s \). For \( \alpha = 1 \), we find the following limits:

\( \alpha \)-Starobinsky:

\[
41.8(45.6) < N_\star < 51.8,
1.7 \times 10^{-18}(1.6 \times 10^{-13}) < |y| < 2.6 \times 10^{-5},
N_{68\%} = 50.9, \quad N_{95\%} = 45.9,
T_{\text{reh,68\%}} = 8.7 \times 10^8 \text{ GeV}, \quad T_{\text{reh,95\%}} = 2.4 \times 10^2 \text{ GeV},
y_{68\%} = 1.7 \times 10^{-6}, \quad y_{95\%} = 3.8 \times 10^{-13},
\]

(9.21)

T-Model:

\[
42.0(45.8) < N_\star < 52.1,
2.3 \times 10^{-18}(2.2 \times 10^{-13}) < |y| < 3.6 \times 10^{-5},
N_{68\%} = 52.6, \quad N_{95\%} = 47.5,
T_{\text{reh,68\%}} = 5.9 \times 10^9 \text{ GeV}, \quad T_{\text{reh,95\%}} = 1.4 \times 10^4 \text{ GeV},
y_{68\%} = 1.7 \times 10^{-4}, \quad y_{95\%} = 3.6 \times 10^{-11}.
\]

(9.22)
We note that the first two lines do not depend on the BICEP/Keck constraints, since these limits are derived from the conditions $T_{\text{reh}} > 2$ MeV (100 GeV) (smaller limit) and $m_{\text{LSP}} = 100$ GeV (larger limit). The dark (light) blue regions in the left panels are the 68 (95) % C.L. regions of the $(n_s, r_{0.002})$ planes favoured by a global analysis of the CMB and BAO data.

We also show in the left panels of Fig. 9.1 dotted contours corresponding to 60 and 50 e-folds, solid lines corresponding to the maximum number of e-folds consistent with $y \leq 1$, and the minimum number of e-folds consistent with $T_{\text{reh}} > T_{\text{BBN}}$ and $T_{\text{EW}}$, as well as the dark matter density constraints for a LSP mass of 100 GeV. The corresponding limit for a gravitino mass of $10^8$ GeV in the high-scale supersymmetry case would lie roughly midway between the $m_{\text{LSP}} = 100$ GeV and $N_s = 50$ lines. For the $\alpha$-Starobinsky (T-models) we shade in red (orange) the preferred region respecting the constraint $T_{\text{reh}} > T_{\text{EW}}$ and the relic density constraint with $m_{\text{LSP}} = 100$ GeV. In the upper left panel we also show lines corresponding to $\alpha = 1$ and 12, the latter being the largest value allowed at the 68% C.L. for $m_{\text{LSP}} = 100$ GeV, and $\alpha = 26$, the largest value allowed at the 95% C.L. for $m_{\text{LSP}} = 100$ GeV. We see in the lower left panel that values of $\alpha \in (1.3, 5.1)$ are consistent with the data at the 68% C.L. if $m_{\text{LSP}} = 100$ GeV, and values of $\alpha \leq 11$ are allowed at the 95% C.L.

The right panels of Fig. 9.1 show the $(y(T_{\text{reh}}), N_{0.05}(n_s))$ planes for the $\alpha$-Starobinsky models and T-models. The left-most vertical lines (red) correspond to the minimum values of $y$ allowed by BBN, the middle vertical lines (grey) correspond to $T_{\text{reh}} = T_{\text{EW}}$, and the right-most vertical lines (purple) correspond to the maximum values allowed for $m_{\text{LSP}} = 100$ GeV. We assume $\alpha = 1$ when plotting the parameters and constraints. The constraints would each move to the right (towards larger values of $y$ and $T_{\text{reh}}$) with decreasing values of $\alpha$, though their dependences are weak. The diagonal lines are the predictions of the $\alpha$-Starobinsky and T-models for $\alpha = 0.1$ (dashed lines), 1 (solid lines) and 5 (dotted lines). Finally, we show as horizontal lines the lower limits on $r_{0.05}$ at the 68 and 95% C.L. We see that the 68% lower limit of $N_{0.05}$ requires $y > 1.7 \times 10^{-6}$ in the $\alpha$-Starobinsky model and $y > 1.7 \times 10^{-4}$ for the T-Starobinsky model, both for $\alpha = 1$. This implies a lower limit to the reheating temperature of $8.7 \times 10^8$ GeV and $5.9 \times 10^{10}$ GeV for the $\alpha$-Starobinsky models and T-models, respectively. This limit is relaxed at the 95% C.L., where the lower limit on the reheating temperature drops to
Figure 9.1: Illustrations of the impacts of the BICEP/Keck and other constraints on α-Starobinsky models (upper panels) and T-models (lower panels) based on the analysis of [86]. The left panels compare the observational 68% and 95% C.L. constraints in the \((n_s, r)\) plane (using pivot scales 0.002 for \(r\) and 0.05 for \(n_s\)) with the model predictions for different numbers of e-folds \(N_{50,60}\), showing also the predictions for an inflaton coupling \(y = 1\), the constraints from \(T_{\text{reh}} \geq T_{\text{BBN}}\) and \(T_{\text{EW}}\), and the constraints if the LSP mass is 100 GeV. The right panels display \((y, N)\) planes (using the pivot scale 0.05), showing the relations between \(y\) and \(T_{\text{reh}}\) and between \(N\) and \(n_s\), and the values \(\alpha = 0.1, 1, 5\) (dashed, solid and dotted black lines). We also include lower limits on \(y\) from BBN (red line), \(T_{\text{reh}} = T_{\text{EW}}\) (grey line), and gravitino production (purple line) for \(\alpha = 1\), which increase for smaller \(\alpha\), and 68% and 95% C.L. lower limits on \(N_{0.05}\) from BICEP/Keck and other data (blue lines).

\(2.4 \times 10^2\) GeV in the α-Starobinsky models and \(1.4 \times 10^4\) GeV for the T-models.

We assumed in the above analysis that generation of a factor \(\Delta\) of entropy subsequent
to inflaton decay could be neglected. However, this may not be the case, e.g., in models with additional phase transitions at temperatures between $T_{\text{reh}}$ and $T_{\text{EW}}$, such as those based on flipped SU(5) GUTs [250]. In this case there would be a modification to the calculation of $N_*$ in Eq. (2.79) in the form of an extra term $-\frac{1}{3} \ln \Delta$ in the right-hand side. This would in turn modify the left panels of Fig. 9.1, e.g., the $T_{\text{BBN}}$ and $T_{\text{EW}}$ constraints would move to lower $n_s$, as would the $y = 1$ line, whereas the $N_{50}$ and $N_{60}$ lines would be unchanged, as would the LSP density constraint. As entropy generation would allow a higher initial gravitino abundance, and thus a higher reheating temperature, the contribution to $N_*$ from reheating is exactly compensated by the contribution from $\Delta$. In addition, the lines of fixed $\alpha$ are unchanged. The net result would be to expand the favoured regions of the $(n_s, r_{0.002})$ planes towards lower values of $n_s$, while keeping the same overlaps with the regions of the planes favoured by the BICEP/Keck and other constraints at the 68% C.L. However, this would require higher reheating temperatures.

Fig. 9.2 shows analogous results based on the analysis in [287]. Since this work provides limits on $r$ using $0.05 \text{ Mpc}^{-1}$ for the pivot scale, we have recalculated the theory curves accordingly, although the difference is quite small. What is more striking is the difference in the 68% and 95% lower limits to $n_s$. These are shifted slightly to smaller values and, as one can see in Fig. 9.2, a large portion of the red-shaded region (between $T_{\text{EW}}$ and the 100 GeV relic density limit) now overlaps the 68% C.L. observational region (dark blue). In the right panels, we see that the weaker lower limits on $n_s$ reduce the lower limits on $N_{0.05}$ and hence allow a smaller inflaton coupling to matter and a lower reheat temperature. However, the allowed ranges for $\alpha$ are only slightly modified: (0.4, 12) and (0, 24) for the $\alpha$-Starobinsky model at 68% and 95% C.L., respectively, and (0.5, 7) and (0, 12) for the T-model.
The modified limits analogous to Eqs. (9.21) and (9.22) for $\alpha = 1$ are

$\alpha$-Starobinsky :

\begin{align*}
N_{68\%} &= 47.9, \quad N_{95\%} = 42.9, \\
T_{\text{reh}, 68\%} &= 9.8 \times 10^4 \text{ GeV}, \quad T_{\text{reh}, 95\%} = 0.031 \text{ GeV}, \\
y_{68\%} &= 1.9 \times 10^{-10}, \quad y_{95\%} = 2.8 \times 10^{-17},
\end{align*}

(9.23)

T-Model :

\begin{align*}
N_{68\%} &= 49.4, \quad N_{95\%} = 44.6, \\
T_{\text{reh}, 68\%} &= 4.4 \times 10^6 \text{ GeV}, \quad T_{\text{reh}, 95\%} = 2.0 \text{ GeV}, \\
y_{68\%} &= 1.2 \times 10^{-8}, \quad y_{95\%} = 4.0 \times 10^{-15}.
\end{align*}

(9.24)

The limits on $N_*$ and $y$ from limits to $T_{\text{reh}}$ and the relic density are unaffected by the choice of data analysis and are not repeated.

### 9.4 Concluding Remarks

As can be seen from the left panels of Figs. 9.1 and 9.2, the primary driver of the upper limits on $\alpha$ is the new upper limit on $r$, whereas the constraint on $n_s$ is the primary driver of the lower limit on the number of $e$-folds. In both the $\alpha$-Starobinsky and T-models, there is also an upper limit on the number of $e$-folds due to requiring the inflaton decay coupling $y \lesssim \mathcal{O}(1)$, namely, $N_* \lesssim 56$ as seen in the right panels of the figures, which restricts $n_s$ to the left halves of the preferred ovals in the left panels of Figs. 9.1 and 9.2.

In both cases, couplings near or at this upper limit lead to observables closest to the central value of the confidence contours. This indicates that the updated constraints in $n_s$ favour scenarios for which radiation domination is almost immediately reached after the end of inflation. We note that such a thermal history is always realized regardless of the inflaton-Standard Model couplings if the inflationary potential is quartic near its minimum, as is the case of Higgs inflation [289], WIMPflation [301] or T-models of the form $V \sim \tanh^4(\varphi/\sqrt{6}\alpha M_P)$ [130, 140]. For quartic minima, $N_* \simeq 56$, independent of the reheating temperature.
Figure 9.2: Illustrations of the impacts of the BICEP/Keck and other constraints on \( \alpha \)-Starobinsky models (upper panels) and T-models (lower panels) based on the analysis of [287]. The left panels compare the observational 68% and 95% C.L. constraints in the \((n_s, r)\) plane (using pivot scales 0.05 for both \(r\) and \(n_s\)) with the model predictions for different numbers of e-folds \(N = 50, 60, \ldots\), showing also the predictions for an inflaton coupling \(y = 1\), the constraints from \(T_{\text{reh}} \geq T_{\text{BBN}}\) and \(T_{\text{EW}}\), and the constraints if the LSP mass is 100 GeV. The right panels display \((y, N)\) planes (using the pivot scale 0.05), showing the relations between \(y\) and \(T_{\text{reh}}\) and between \(N\) and \(n_s\), and the values \(\alpha = 0.1, 1, 5\) (dashed, solid and dotted black lines). We also include lower limits on \(y\) from BBN (red line), \(T_{\text{reh}} = T_{\text{EW}}\) (grey line), and gravitino production (purple line) for \(\alpha = 1\), which increase for smaller \(\alpha\), and 68% and 95% C.L. lower limits on \(N_{0.05}\) from BICEP/Keck and other data (blue lines).

The values of the effective Yukawa coupling \(y\) disfavoured by electroweak scale gravitino overproduction, shown in purple in Figs. 9.1 and 9.2, correspond coincidentally to
the domain of non-perturbative particle production (preheating). Indeed, for \( y \gtrsim 10^{-5} \), efficient parametric resonance will be present during the early stages of reheating, for either fermionic or bosonic inflaton decay products \([302, 303, 274, 304, 142]\). However, this effect is not necessarily reflected in the CMB observables. In the case of fermionic preheating, the expansion history during reheating (and hence \( w_{\text{int}} \) and \( \rho_{\text{rad}} \)) is not affected unless \( y \sim O(1) \). The resulting Pauli suppression of particle production simply reduces the energy density of radiation relative to the value predicted by (9.9) for a time much shorter than the duration of reheating \([142]\). Hence our results for \( N^* \) shown in the left panels of Fig. 9.1 would be mostly unchanged in this fermionic case. In the case of bosonic preheating, the efficiency of non-perturbative particle production depends on the resonance band structure of the coupling. If the backreaction regime is reached, transient radiation-dominated stages can occur during reheating, modifying \( w_{\text{int}} \) and hence our predictions \([305, 142]\). However, we do not delve here into this model-dependent issue. Finally, for attractors with quadratic minima, the self-interaction of the inflaton does not disrupt the matter-like oscillation of the inflaton condensate during reheating \([306]\).

Turning to the future, we note that the experiments CMB-S4 \([307]\) and LiteBIRD \([308]\) will target primarily the search for B-modes in the CMB and will impose strong constraints on \( r \), with the potential to reduce substantially the uncertainty in \( r \), by a factor \( O(2) \). Such a measurement will reduce the uncertainty in \( \alpha \) to a similar value, constraining significantly string models of inflation. Unfortunately, the ability of these experiments to constrain \( n_s \) is limited. However, this is an important objective for the future, as \( n_s \) is related directly to the magnitude of the coupling between the inflaton and matter, whose understanding will be key for connecting the theory of inflation to laboratory physics.
Chapter 10

Conclusions and Discussion

In this thesis, we have argued that no-scale models of supergravity are an attractive framework that can be used to construct and study various models of inflation and early universe cosmology. Importantly, there are many motivations for supersymmetry and supergravity, including the possible solution to the hierarchy problem, unification of the gauge couplings at high energy, the stability of the Higgs vacuum, and providing viable dark matter candidates. Therefore, we have argued that cosmic inflation must involve Planck-scale physics, and that inflation must be studied in a supersymmetric context, in particular, inflationary models should be constructed from supergravity. Generic models of supergravity typically run into various problems, such as the $\eta$ problem and gravitino/moduli problem. However, we showed in this thesis that no-scale supergravity models easily avoid these particular problems.

Since we argued that supergravity is needed to construct realistic models of inflation, we first reviewed the construction of inflationary models that were based on no-scale supergravity. However, supergravity, even no-scale supergravity, is not sufficient as a framework to specify the inflationary model. As is usually the case, theory must rely on experiment to make progress. Therefore, we focused on model predictions for inflationary observables, the scalar tilt $n_s$ and the tensor-to-scalar ratio $r$, and considered different models of inflation.

It is a remarkable feature of no-scale supergravity that the Starobinsky model of inflation, which is based on $R + R^2$ gravity and was one of the first models of inflation,
can easily be constructed in no-scale supergravity framework as is in excellent agreement with the most recent Planck measurements of the tensor-to-scalar ratio $r \simeq 0.0035$ and the tilt of the scalar perturbations $n_s = 0.965$. In this thesis, we constructed different Starobinsky-like models of inflation and studied their cosmological observables, supersymmetry-breaking aspects, and various cosmological and phenomenological implications.

It was shown in [85] that one can easily construct the Starobinsky inflationary potential starting with a very simple Wess-Zumino superpotential form. It was quickly realized that this superpotential form is not unique, and other superpotential forms could also yield the Starobinsky potential [225]. We discussed in Chapter 5 that all these models are related by the underlying $SU(2,1)/SU(2) \times U(1)$ coset symmetry. Therefore, the underlying $SU(2,1)/SU(2) \times U(1)$ symmetry can be used to construct various simple superpotential forms that would yield the Starobinsky potential. Most importantly, this formalism can be extended to various $SU(N,1)/SU(N) \times U(1)$ field coset spaces or more complicated coset structures.

We studied in Chapter 6 the question of generating Minkowski and de Sitter vacuum solutions in no-scale supergravity. We showed that de Sitter solutions can be constructed by combining two special cases of Minkowski superpotential forms, and generalized these solutions to various no-scale supergravity structures, including the cases with $N > 1$ moduli. Such $N$-field Minkowski and de Sitter vacuum solutions can be parametrized by coordinates compactified on the surface of an $(N-1)$-sphere.

We then combined the de Sitter vacuum solutions with a Starobinsky-like inflationary potential, and incorporated an adjustable scale for supersymmetry breaking and the possibility of a small cosmological constant (dark energy). This simple no-scale supergravity model is suitable for sub-Planckian physics capable of including modulus fixing, Starobinsky-like inflation at a scale of $\mathcal{O}(10^{13})$ GeV, supersymmetry breaking at a TeV scale, and a small positive cosmological constant (dark energy density). This model can be understood as an effective field theory model that should ultimately be derived from a suitable variant of string theory.

In a more realistic model that includes the relevant dynamics, the renormalization group evolution of the supersymmetry breaking terms would be able to drive the effective Higgs mass-squared negative, triggering electroweak symmetry breaking and the
corresponding change in the vacuum energy density. Thus, one can choose the free parameters so that the dark energy density obtains the observed value of $O(10^{-120})$ GeV$^4$ after this contribution is included. Therefore, the electroweak scale could be generated dynamically in this framework.

We have extended these general formulations in Chapter 7 and generalized our results to more complicated structures with an arbitrary curvature parameter $\alpha$. Typically, such Starobinsky-like models of inflation lead to the tensor-to-scalar ratio $r = 3(1 - n_s)^2$, where $n_s$ is the tilt of the scalar perturbations. In particular, we have shown how this scenario can be unified with modulus fixing, supersymmetry breaking, and dark energy. We considered alternative values of the curvature parameter, $\alpha < 1$, as may occur if not all the complex Kähler moduli contribute to driving inflation, as well as $\alpha > 1$, as may occur if complex structure moduli also contribute to driving inflation. In all cases, we combine these $\alpha$-Starobinsky models with supersymmetry breaking and a present-day cosmological constant, allowing for additional contributions to the vacuum energy from stages of gauge symmetry breaking.

We also studied in detail some simple unified constructions and develop phenomenological and cosmological aspects of these no-scale attractor models that underpin their physical applications in Chapter 8. Typically, in these models, the inflaton mass, $M$, determines the supersymmetry breaking scale, with $m_{3/2} \sim M^3/M_P^2$. We consider two different types of models: when inflation is driven by a modulus field (T-type), with supersymmetry broken by a Polonyi field, or a matter field ($\phi$-type), with supersymmetry broken by the modulus field. We derived the possible patterns of soft supersymmetry-breaking terms, which depend in T-type models whether the Polonyi and/or matter field are twisted (outside the logarithm of the Kähler potential) or untwisted (inside the logarithm of the Kähler potential), and in $\phi$-type models on whether the inflaton and/or other matter fields are twisted or not. In $\phi$-type models, we were able to directly relate the scale of supersymmetry breaking to the inflaton mass. We also discussed cosmological constraints from entropy considerations and the density of dark matter on the mechanism for stabilizing the modulus field via higher-order terms in the no-scale Kähler potential.

Finally, in Chapter 9, we studied the impact of the recent BICEP/Keck data on the cosmic microwave background on the attractor models of inflation. We discussed
the $\alpha$-Starobinsky models and T-models, and both of them can be easily constructed in no-scale supergravity framework. Importantly, the BICEP/Keck data on the cosmic microwave background, in combination with previous WMAP and Planck data, impose strong new constraints on the tilt in the scalar perturbation spectrum, $n_s$, as well as the tensor-to-scalar ratio, $r$. These constrain the number of $e$-folds of inflation, $N_*$, the magnitude of the inflaton coupling to matter, $y$, and the reheating temperature, $T_{\text{reh}}$, which we evaluate in attractor models of inflation as formulated in no-scale supergravity.

We found that the 68% C.L. region of $(n_s, r)$ favours large values of $N_*$, $y$, and $T_{\text{reh}}$ that are constrained by the production of gravitinos and supersymmetric dark matter.

Most importantly, the future experiments CMB-S4 [307] and LiteBIRD [308] should be able to measure $r$ to an accuracy of $\pm 0.001$. Such future observations will be sufficient to measure the value of $\alpha$ in attractor models of inflation, and such a measurement will constrain the magnitude of the couplings between the matter sector and the inflationary sector, whose understanding is crucial to establish the connection between the low-energy laboratory physics scale and high-energy inflationary scale.
References


Appendix A

Definitions and Conventions

In this appendix, we summarize the notation and conventions that were used in this thesis. In this work we use natural units, where

$$\hbar = c = k_B = 1.$$ \hfill (A.1)

We also use the reduced Planck units, which are given by

$$M_P = \frac{1}{\sqrt{8\pi G}} \simeq 2.435 \times 10^{18} \text{GeV},$$ \hfill (A.2)

where $G = 6.67430 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$ is the gravitational constant, and in some instances we set it to unity.

For the Minkowski metric, we primarily use the “positive energy” convention that is most commonly used in particle physics and quantum field theory. The Minkowski metric is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$ \hfill (A.3)

where the Greek indices range from 0 to 3.
A.1 Four-vectors

We denote the space-time coordinates by the contravariant four-vector $x^\mu$

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (t, \mathbf{x}) \, ,$$  \hfill (A.4)

and the corresponding covariant four-vector is given by

$$x_\mu = \eta_{\mu\nu}x^\nu = (t, -x, -y, -z) = (t, -\mathbf{x}) \, .$$  \hfill (A.5)

The four gradients are defined as

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right) = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \, ,$$  \hfill (A.6)

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right) \, .$$  \hfill (A.7)

The scalar product of two four-vectors is defined as

$$A \cdot B = \eta_{\mu\nu}A^\mu B^\nu = A_\mu B^\mu = A_0B_0 - \mathbf{A} \cdot \mathbf{B} \, .$$  \hfill (A.8)

A.2 Spinors and Dirac Algebra

The Dirac matrices must satisfy the following anticommutation relation that is necessary to satisfy the Clifford algebra:

$$\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbf{1}_4 \, .$$  \hfill (A.9)

In the standard Dirac-Pauli representation, the $\gamma$-matrices are give by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \, .$$  \hfill (A.10)
Note that $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. The standard representation of Pauli matrices $\sigma_i$ is given by

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (A.11)

The Pauli matrices satisfy the constraint

\[
\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k,
\] (A.12)

where $\epsilon_{ijk}$ is a totally antisymmetric Levi-Civita symbol, with $\epsilon_{123} = +1$. The Hermitian conjugates of the $\gamma$-matrices are given by

\[
\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu, \quad \gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i.
\] (A.13)

Other important matrices and their properties include:

\[
\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \quad \{\gamma_5, \gamma_\mu\} = 0,
\] (A.14)

and

\[
\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] = \frac{i}{2} (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).
\] (A.15)

Since the four-dimensional spinors correspond to an irreducible representation of the Lorentz group, they can be split into left-handed and right-handed spinors. The left-handed $(0, \frac{1}{2})$ spinor is defined by $\psi_\alpha$, and the right-handed $(\frac{1}{2}, 0)$ is defined by $\bar{\psi}_\dot{\alpha}$. By convention, the left-handed Weyl spinors do not carry daggers and right-handed Weyl spinors do carry daggers.

The two representations can be related by Hermitian conjugation

\[
\bar{\psi}_\dot{\alpha}^\dagger \equiv (\psi_\alpha)^\dagger \equiv (\bar{\psi}_\dot{\alpha})^\dagger,
\] (A.16)

and vice versa

\[
(\psi_\alpha^\dagger)^\dagger = \psi_\alpha.
\] (A.17)

Here the spinor indices range from $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$. 


The spinor indices are raised and lowered using the antisymmetric symbol
\[ \epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \quad \epsilon_{11} = \epsilon_{22} = \epsilon^{11} = \epsilon^{22} = 0, \] (A.18)
according to
\[ \xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta, \quad \xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta, \quad \chi_\dot{\alpha} = \epsilon_{\dot{\alpha}\dot{\beta}} \chi^{\dot{\beta}}, \quad \chi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \chi_{\dot{\beta}}. \] (A.19)

We use the convention that suppresses the repeated indices:
\[ \xi \chi \equiv \xi_\alpha \chi^\alpha = \xi^\alpha \epsilon_{\alpha\beta} \chi_\beta = -\chi_\beta \epsilon_{\beta\alpha} \xi^\alpha = \xi^\beta \epsilon_{\beta\alpha} \xi_\alpha = \xi^\beta \xi_\beta \equiv \chi \xi. \] (A.20)

In this representation, a four-component Dirac fermion \( \Psi(x) \) contains two spinor fields with two components:
\[ \Psi(x) = \begin{pmatrix} \xi_\alpha(x) \\ \chi^{\dot{\alpha}}(x) \end{pmatrix}. \] (A.21)

The Hermitian conjugate of a four-component Dirac spinor is given by
\[ \bar{\Psi} = \Psi^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_\alpha \\ \xi^{\dot{\alpha}} \end{pmatrix}. \] (A.22)

To project the left- and right-handed states, it is convenient to introduce the following projection operators
\[ P_L = (1 - \gamma_5)/2, \quad P_R = (1 + \gamma_5)/2, \] (A.23)
and if we use them on a Dirac spinor \( \Psi \):
\[ P_L \Psi(x) = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}, \quad P_R \Psi(x) = \begin{pmatrix} 0 \\ \chi^{\dot{\alpha}} \end{pmatrix}. \] (A.24)

Here we do not include all the important identities that arise in the Weyl spinor representation, and a complete treatment and discussion of the two-component Weyl fermion notation can be found in [156].
Appendix B

$\mathcal{N} = 1$ Supergravity

In this appendix, we introduce the full $\mathcal{N} = 1$ supergravity Lagrangian. To simplify the discussion, we split the supergravity Lagrangian into purely bosonic Lagrangian $\mathcal{L}_B$ and purely fermionic Lagrangian $\mathcal{L}_F$. We follow the treatment presented in [192, 41, 44].

The covariant derivatives $D_\mu$ are defined as

$$D_\mu \equiv \partial_\mu + \frac{1}{2} \omega^{mn}_\mu \sigma_{mn},$$

where the spin connection $\omega^{mn}_\mu$ is given by Eq. (3.71) and $\sigma_{mn}$ is given by Eq. (3.72). The left- and right-handed components of the Majorana spinors can be expressed as

$$\chi_i L = P_L \chi_i, \quad \bar{\chi}_i L = \chi_i P_R,$$
$$\chi_i R = P_R \chi_i, \quad \bar{\chi}_i R = \chi_i P_L.$$

The Kähler function is given by $G = K + \ln |W|^2$, and $K^{ij} = \partial^2 K / \partial \phi_i \partial \phi_j$ is the Kähler metric and $K_{ij}$ is the inverse Kähler metric. The dual of the field strength is given by

$$\tilde{F}_a^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{a \rho \sigma}.$$

We start with the bosonic Lagrangian, which can be expressed as

$$\mathcal{L}_B = \mathcal{L}_B^C + \mathcal{L}_B^F,$$
where the part that does not depend on the gauge kinetic function is given by

\[ e^{-1} \mathcal{L}_B^C = -\frac{R}{2} + G^{ij} \partial_\mu \phi_i \partial^\mu \phi_j - e^G \left[ G^i K_{ij} \bar{G}^j - 3 \right], \tag{B.5} \]

and the gauge part that depends on the gauge kinetic function is

\[ e^{-1} \mathcal{L}_B^G = -\frac{1}{4} (\text{Re} \ f_{ab}) F_{a\mu\nu} F_{b}^{\mu\nu} + \frac{i}{4} (\text{Im} \ f_{ab}) \tilde{F}_{a\mu\nu} \tilde{F}_{b}^{\mu\nu} - \frac{g^2}{2} (\text{Re} \ f_{ab}^{-1}) G^i (T_a^j)_{l} \phi_j \phi_l (T_b^k)_{l} \phi_l. \tag{B.6} \]

Next, we introduce the kinetic terms of the fermion Lagrangian

\[ \mathcal{L}_{F,\text{kin}} = \mathcal{L}_{F,\text{kin}}^C + \mathcal{L}_{F,\text{kin}}^G, \tag{B.7} \]

where

\[ e^{-1} \mathcal{L}_{F,\text{kin}}^C = -\frac{e^{-1}}{2} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\nu \tilde{D}_\rho \psi_\sigma - \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \left( G^\sigma \phi_i - G^i \phi_i \right) + \left[ \frac{i}{2} G^{ij} \bar{\chi}_{iR} \gamma^\mu D_\mu \chi_{jR} + \frac{i}{2} \bar{\chi}_{iR} \tilde{D}_\phi \chi_{jR} \left( -G^{ijk} + \frac{1}{2} G^{ik} G^{jl} \right) \right. \]

\[ + \left. \frac{1}{\sqrt{2}} G^{ij} \bar{\psi}_{iR} \tilde{D} \phi_j \bar{\lambda} \chi_{jL} + \text{h.c.} \right], \tag{B.8} \]

and

\[ e^{-1} \mathcal{L}_{F,\text{kin}}^G = \frac{1}{2} \text{Re} \ f_{ab} \left( \frac{i}{2} \bar{\lambda}_a \tilde{D}_\lambda \bar{\lambda}_b - \frac{1}{4} \bar{\lambda}_a \gamma^\mu \sigma^{\mu\nu} \bar{\psi}_\mu \tilde{F}_{b\nu\rho} - \frac{i}{2} \bar{G}^i \tilde{D}_\mu \bar{\lambda}_a R \bar{\lambda}_b \right) + \frac{1}{8} \text{Im} \ f_{ab} e^{-1} D_\mu \left( e \bar{\lambda}_a \gamma_5 \gamma_\lambda \lambda_b \right) - \frac{i}{4\sqrt{2}} (f_{ab})^i \bar{\chi}_{iL} \sigma^{\mu\nu} F_{a\mu\nu} \lambda_b R + \text{h.c.} \tag{B.9} \]

The fermionic interaction Lagrangian is given by

\[ \mathcal{L}_{F,\text{int}} = \mathcal{L}_{F,\text{int}}^C + \mathcal{L}_{F,\text{int}}^G. \tag{B.10} \]
where

\[
e^{−1}L_{F,\text{int}}^{C} = \frac{i}{2} e^{G/2} \bar{\psi}_{\mu L} \sigma^{\mu \nu} \psi_{\nu R} + \frac{1}{2} e^{G/2} \left( -G^{ij} - G^{i} G^{j} + G^{ij} G_{kl} G^{kl} \right) \bar{\chi}_{i R} \chi_{j L} \\
+ \frac{i}{\sqrt{2}} e^{G/2} G^{i} \bar{\psi}_{\mu L} \gamma^{\mu} \chi_{j L} + \frac{i}{16} G^{ij} \bar{\chi}_{i L} \gamma_{\sigma} \chi_{j L} \left( \epsilon^{\mu \nu \rho \sigma} \bar{\psi}_{\mu} \gamma_{\nu} \psi_{\rho} - i \bar{\psi}^{\mu} \gamma^{5} \gamma_{\sigma} \psi_{\mu} \right) \\
+ \left( \frac{1}{8} G^{ijkl} - \frac{i}{16} G^{ij} G_{mn} G^{mkkl} - \frac{1}{16} G^{ijkl} \right) \bar{\chi}_{i R} \chi_{j L} \chi_{k L} \chi_{l R} + \text{h.c.}
\]

(B.11)

and

\[
e^{−1}L_{F,\text{int}}^{G} = \frac{1}{4} e^{G/2} (\tilde{f}_{ab})^{ij} G^{jk} \bar{\lambda}_{a L} \lambda_{b R} - \frac{g}{2} G^{i} (T_{a})_{ij} \phi_{j} \bar{\psi}_{\mu R} \gamma^{\mu} \lambda_{a R} \\
- ig\sqrt{2} G^{ij} (T_{a})_{jk} \phi_{k} \bar{\lambda}_{a L} \chi_{i R} - \frac{i}{2\sqrt{2}} g (\text{Re} \ f_{ab})^{-1} (f_{bc})^{k} G^{i} (T_{a})_{ij} \phi_{j} \bar{\chi}_{k R} \lambda_{c L} \\
+ \frac{3}{32} [ (\text{Re} \ f_{ab}) \bar{\lambda}_{a R} \gamma_{\mu} \lambda_{b R}]^{2} + \frac{i}{16} (\text{Re} \ f_{ab}) \bar{\lambda}_{a R} \gamma^{\mu} \sigma^{\rho \sigma} \psi_{\mu} \bar{\psi}_{\rho} \gamma_{\sigma} \lambda_{b} \\
- \frac{1}{32} G_{kl} (f_{ab})^{ij} (\tilde{f}_{cd})^{k} \bar{\lambda}_{a R} \lambda_{b L} \bar{\lambda}_{c L} \lambda_{d R} \\
+ \frac{1}{4\sqrt{2}} (f_{ab})^{ij} \left( \bar{\chi}_{i R} \sigma^{\mu \nu} \lambda_{a L} \bar{\psi}_{\nu R} \gamma_{\mu} \lambda_{b R} + \frac{i}{2} \bar{\psi}_{\mu L} \gamma^{\mu} \chi_{i L} \bar{\lambda}_{a L} \lambda_{b L} \right) \\
+ \frac{1}{16} \bar{\chi}_{i R} \gamma^{\mu} \chi_{j R} \lambda_{d L} \gamma_{\mu} \lambda_{c L} \left[ G^{ij} (\text{Re} \ f_{cd}) + \frac{1}{2} \text{Re} \left( f_{ab}^{-1} (f_{ac})^{i} (f_{bd})^{j} \right) \right] \\
- \frac{1}{16} \bar{\chi}_{i R} \gamma^{\mu} \chi_{j L} \lambda_{c R} \lambda_{d L} \left( 2G^{ij} G_{kl} (f_{cd})^{i j} - 2 (f_{cd})^{i j} \right) + \frac{1}{2} \text{Re} \ f_{ab}^{-1} (f_{ac})^{i} (f_{bd})^{j} \right) \\
+ \frac{1}{128} \bar{\chi}_{i R} \sigma^{\mu \nu} \chi_{j L} \lambda_{c R} \sigma^{\mu \nu} \lambda_{d L} \text{Re} \left( f_{ab}^{-1} (f_{ac})^{i} (f_{bd})^{j} \right) + \text{h.c.}
\]

(B.12)

The transformation laws of local supersymmetry can be found in [192, 156].
Appendix C

Slow-Roll Inflation

We recall some equations for single-field slow-roll inflation. For a general inflationary potential \( V(\phi) \), we find the following Klein-Gordon equation of motion

\[
\ddot{\phi} = -3H\dot{\phi} - V'(\phi),
\] (C.1)

where the evolution of the scalar field is driven by the potential gradient term \( V' = dV/d\phi \). In order to treat the slow-roll approximation, we introduce the following slow-roll parameters

\[
\epsilon \equiv \frac{1}{2} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv \left( \frac{V''}{V} \right),
\] (C.2)

where \( \epsilon, |\eta| \ll 1 \).

Next, we introduce the expressions of cosmological observables in terms of the slow-roll parameters. The tensor-to-scalar ratio for single scalar field is given by

\[
r \simeq 16\epsilon,
\] (C.3)

and the scalar power spectrum is expressed as

\[
n_s - 1 \simeq -6\epsilon + 2\eta.
\] (C.4)

To solve the flatness and horizon problems, we require the total number of inflationary \( e \)-folds to be \( N_\ast \simeq 50 - 60 \) before the end of inflation. The number of \( e \)-folds before the
end of inflation is given by

$$N_* = \int_{t_*}^{t} H dt \approx \int_{\phi_*}^{\phi} \frac{1}{\sqrt{2\epsilon}} \ d\phi. \quad \text{(C.5)}$$

We find the following expressions for the slow-roll parameters (C.2) in the $\alpha$-Starobinsky scalar model (7.12):

$$\epsilon = \frac{4}{3\alpha} \left( 1 - e^{\sqrt{\frac{2}{3\alpha}} x} \right)^{-2}, \quad \text{(C.6)}$$

and

$$\eta = \frac{4}{3\alpha} \left( \frac{2 - e^{\sqrt{\frac{2}{3\alpha}} x}}{1 - e^{\sqrt{\frac{2}{3\alpha}} x}} \right)^2. \quad \text{(C.7)}$$

Combining (C.6) with (C.5), we obtain

$$N_* = -\frac{3\alpha}{4} \left( 1 - e^{\sqrt{\frac{2}{3\alpha}} x} \right) - \frac{\sqrt{3\alpha}}{2\sqrt{2}} x, \quad \text{(C.8)}$$

and solving it for $x$, we find

$$x = \frac{-4\sqrt{6}N_* - 3\sqrt{6}\alpha - 3\sqrt{6}\alpha W_{-1}(-e^{-1-\frac{4N_*}{3\alpha}})}{6\sqrt{\alpha}}, \quad \text{(C.9)}$$

where $W_k(z)$ is the Lambert $W$ function with $k$ an integer, which is defined as the inverse function of $f(W) = We^W$. Using the expressions (C.3, C.4), the slow-roll parameters (C.6, C.7) and expression (C.9), we find

$$r = \frac{64}{3\alpha} \left( 1 + W_{-1}(-e^{-1-\frac{4N_*}{3\alpha}}) \right)^{-2}, \quad \text{(C.10)}$$

and

$$n_s = 1 - \frac{8}{3\alpha} \left( \frac{1 - W_{-1}(-e^{-1-\frac{4N_*}{3\alpha}})}{1 + W_{-1}(-e^{-1-\frac{4N_*}{3\alpha}})} \right). \quad \text{(C.11)}$$

If we expand (C.10) and (C.11) for large $N_*/\alpha$, we recover (7.2).
Appendix D

Field Shifts in the Minima

As noted in Section 8.1.3, the shifts in the minima of the Polonyi field and the inflaton depend on the modular weight of the Polonyi superpotential. Introducing a modular weight $\delta$ for the Polonyi superpotential (8.8):

$$\mu(Z + b) \rightarrow \mu(Z + b) \left( T + \frac{1}{2} \right)^{\delta}, \quad (D.1)$$

we find the following shifted VEVs in the untwisted case:

$$\langle T \rangle \simeq \frac{1}{2} + \left( \frac{2\alpha - 1 - \frac{2\delta}{3}}{\alpha c^2} \right) \Delta^2, \quad (D.2)$$

$$\langle \phi \rangle \simeq \left( \frac{\sqrt{3}}{c} - \frac{\delta}{\sqrt{3}\alpha c} \right) \Delta, \quad (D.3)$$

$$\langle Z \rangle \simeq \frac{\sqrt{\alpha}}{6\sqrt{3}} \Lambda_Z^2, \quad (D.4)$$

$$b \simeq \frac{1}{\sqrt{3}\alpha} - \frac{1}{2\sqrt{3}\alpha^{3/2}c^2} \left( 1 + 3\alpha(\alpha - 1) + \frac{\delta(\delta + 3\alpha(2 - 6\alpha + \delta))}{9\alpha} \right) \Delta^2, \quad (D.5)$$
and in the twisted case:

\[
\langle T \rangle \approx \frac{1}{2} + \left( \frac{2\alpha}{c^2} - \frac{2\delta}{3c^2} \right) \Delta^2,
\]

\[
\langle \phi \rangle \approx \left( \frac{\sqrt{3\alpha}}{c} - \frac{\delta}{\sqrt{3\alpha}c} \right) \Delta,
\]

\[
\langle Z \rangle \approx \frac{1}{2\sqrt{3}} \Lambda^2_Z,
\]

\[
b \approx \frac{1}{\sqrt{3}} - \left( \frac{(3\alpha - \delta)^2}{6\sqrt{3}c^2} \right) \Delta^2.
\]

These results reduce to those given in the text when \( \delta = 0 \).
Appendix E

Analytical approximations

As stated in the main text, the power spectrum and reheating constraints summarized in Fig. 9.1 have been obtained numerically. In this appendix we provide analytical approximations to the relevant inflationary quantities.

The end of inflation corresponds to the end of the epoch of accelerated expansion, i.e., \( \ddot{a} = 0 \) or \( \epsilon_H = 1 \), where \( \epsilon_H = -\dot{H}/H^2 \) is the first Hubble flow function. In terms of the potential slow-roll parameters (2.46, 2.47), it can be shown that the end of inflation occurs approximately when

\[
\epsilon \simeq (1 + \sqrt{1 - \eta/2})^2. \tag{E.1}
\]

This expression can be used to obtain the following closed-form estimates for the value of the inflaton field at the end of inflation for \( \alpha \)-Starobinsky models,

\[
\frac{\varphi_{\text{end}}}{M_P} \simeq \frac{\sqrt{3\alpha}}{2} \ln \left[ \frac{2(6\alpha + 3\sqrt{3\alpha} - 2)}{12\alpha - 1} \right], \tag{E.2}
\]

and for T-models,

\[
\frac{\varphi_{\text{end}}}{M_P} \simeq \frac{\sqrt{3\alpha}}{2} \ln \left[ \frac{4 - 6\sqrt{\alpha(5 + 4\alpha)}}{1 - 12\alpha} \right] + \sqrt{\frac{75}{5 + 68\alpha + 16\sqrt{\alpha(5 + 4\alpha)}}}. \tag{E.3}
\]
As expected, for $\alpha = 1$, we recover Eqs. (9.5) and (9.6). Compared to the exact values, the analytic approximations have errors of 2% (2%, 4%) for $\alpha = 1$ (0.1, 10) in the case of $\alpha$-Starobinsky models, and of 5% (3%, 5%) for $\alpha = 1$ (0.1, 10) for T-models.

The value of the inflaton field at the moment when the pivot scale crosses the horizon can be estimated by integrating Eq. (2.48). In the case of $\alpha$-Starobinsky models, \[ \frac{\varphi_*}{M_P} \simeq \sqrt{\frac{3\alpha}{2}} \left[ 1 + \frac{3\alpha}{4N_* - 3\alpha} \right] \times \ln \left( \frac{4N_*}{3\alpha} + e^{\sqrt{\frac{2}{3} \frac{\varphi_{\text{end}}}{M_P}} - \sqrt{\frac{2}{3} \frac{\varphi_{\text{end}}}{M_P}}} \right), \] (E.4)

and for T-models, \[ \frac{\varphi_*}{M_P} \simeq \sqrt{\frac{3\alpha}{2}} \cosh^{-1} \left[ \frac{4N_*}{3\alpha} + \cosh \left( \sqrt{\frac{2}{3} \frac{\varphi_{\text{end}}}{M_P}} \right) \right]. \] (E.5)

For $40 < N_* < 60$ the relative errors are at most 0.3% (0.3%, 3%) for $\alpha = 1$ (0.1, 10) in the $\alpha$-Starobinsky case, and 0.5% (0.4%, 0.7%) for $\alpha = 1$ (0.1, 10) in the case of T-model inflation.

The logarithm of the so-called reheating parameter [144], \[ \ln R_{\text{rad}} \equiv \ln \left[ \frac{a_{\text{end}}}{a_{\text{rad}}} \left( \frac{\rho_{\text{end}}}{\rho_{\text{rad}}} \right)^{1/4} \right] \] (E.6)

may be estimated by noting that the energy density of the relativistic inflaton decay products, assuming a constant decay rate $\Gamma_\varphi$, can be written as [137] \[ \rho_{\text{rad}} = \rho_{\text{end}} \left( \frac{a_{\text{end}}}{a_{\text{rad}}} \right)^4 \int_0^{v_{\text{end}}} \left( \frac{a(u)}{a_{\text{end}}} \right) e^{-u} du, \] (E.8)

where $v \equiv \Gamma_\varphi (t - t_{\text{end}})$. Approximating the equation-of-state parameter as $w \simeq 0$ during
reheating, we can further write

\[
\frac{a(t)}{a_{\text{end}}} \simeq \left( \sqrt{\frac{3}{4} \rho_{\text{end}} \frac{t - t_{\text{end}}}{M_P}} \right)^{\frac{2}{3}} = \left( \frac{3H_{\text{end}} v}{2\Gamma_\varphi} \right)^{\frac{2}{3}}. \tag{E.9}
\]

Substitution of (E.9) into (E.8) and subsequently into (E.6) results in the following simple approximation for the reheating parameter,

\[
\ln R_{\text{rad}} \simeq \frac{1}{6} \ln \left( \frac{\Gamma_\varphi}{H_{\text{end}}} \right). \tag{E.10}
\]

This result allows us to write simple analytical expressions for the number of \( e \)-folds after horizon crossing as functions of the effective Yukawa coupling responsible for reheating. As an example for \( \alpha = 1 \), substitution of (E.2), (E.4) and (E.10) into (2.83) gives

\[
N_\ast \simeq 57.68 - \frac{1}{2} \ln N_\ast + \frac{1}{3} \ln y - \frac{1}{12} \ln g_{\text{reh}}, \tag{E.11}
\]

for \( \alpha\)-Starobinsky models at the pivot scale \( k_\ast = 0.05 \text{ Mpc}^{-1} \), and for T-models

\[
N_\ast \simeq 57.82 - \frac{1}{2} \ln N_\ast + \frac{1}{3} \ln y - \frac{1}{12} \ln g_{\text{reh}}. \tag{E.12}
\]

In the range of values shown in the left panels of Fig. 9.1, the maximum differences of these approximations from the full numerical results are 0.2% (0.1%) for the \( \alpha \)-Starobinsky models (T-models).

For other analyses of reheating in attractor models, see [303, 309].
Appendix F

Computing the CMB observables

In order to compute accurately the inflationary observables, in particular the scalar tilt $n_s$, we have integrated the linear equations for the curvature fluctuation numerically. To calculate the gauge-invariant Mukhanov-Sasaki variable $Q$,\(^\dagger\) we integrate the equation of motion \[119\],
\[
\ddot{Q} + 3H\dot{Q} + \left[ \frac{k^2}{a^2} + 3\dot{\varphi}^2 - \frac{\dot{\varphi}^4}{2H^2} + 2\frac{\dot{\varphi}V_{\varphi}}{H} + V_{\varphi\varphi} \right] Q = 0,
\]
with the Bunch-Davies initial condition $Q_{k_0} = e^{-ik\tau/a\sqrt{2k}}$, where $d\tau = dt/a$ is the conformal time. The corresponding metric fluctuation and its power spectrum are in turn given by
\[
\mathcal{R} = \frac{H}{|\dot{\varphi}|} Q, \quad (F.2)
\]
\[
\langle \mathcal{R}(k)\mathcal{R}^\ast(k') \rangle = \frac{2\pi^2}{k^3} P_\mathcal{R} \delta(k - k'). \quad (F.3)
\]
The scalar tilt is then computed using its definition,
\[
n_s = 1 + \frac{d\ln P_\mathcal{R}}{d\ln k}, \quad (F.4)
\]
\(^\dagger\)In the Newtonian gauge, $Q = \delta\varphi + \frac{\Psi}{H}$, where $\delta\varphi$ and $\Psi$ denote the field and the metric perturbations, respectively.
and the tensor-to-scalar-ratio is
\[ r = \frac{P_T}{P_R}, \] (F.5)
where in the case of the tensor spectrum we take the horizon-crossing value \( P_T = 2H^2/\pi^2 \).

Comparing the numerical results obtained by the procedure above with the slow-roll approximations (2.87) and (2.88) we find a discrepancy \( \gtrsim 1 \) e-fold for \( N_s = N_s(n_s) \), see the dashed line in Fig. F.1. This difference can be reduced if, instead of the potential slow-roll parameters \( \epsilon_V \) one uses the Hubble slow-roll parameters,
\[ \epsilon_H = -\frac{\dot{H}}{H}, \quad \eta_H = 2\epsilon_H - \frac{\dot{\epsilon}_H}{2\epsilon_H H}, \] (F.6)
see the dotted line in Fig. F.1.

This difference remains even when the higher-order slow-roll corrections are included. Ultimately, it is due to the fact that curvature modes do not immediately freeze upon leaving the horizon, which corresponds to the condition \( k = aH \). Hence there is always a shift between the approximate horizon-crossing value, used in our semi-analytical estimates, and the final “freeze-out” values used in our full numerical results, in particular in Fig. 9.1.
Figure F.1: The scalar tilt $n_s$ as a function of the number of e-folds after horizon crossing, $N_*$, for the $\alpha$-Starobinsky model with $\alpha = 1$. The continuous blue line is the numerical solution of Eqs. (F.1)-(F.4). The dotted grey line is the slow-roll approximation (2.87) with the Hubble parameters $\epsilon_H, \eta_H$ defined in (F.6). The dashed black line is the slow-roll approximation (2.87) calculated using the potential parameters $\epsilon, \eta$ defined in (2.46, 2.47).