ON BOUNDED AND HARMONIZABLE SOLUTIONS
ON INFINITE ORDER ARMA SYSTEMS

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A. MAKAGON* AND H. SALEHI†

Abstract. The problem of existence and uniqueness of stationary, harmonizable and bounded solutions to infinite ARMA systems is studied. Specifically it is shown that an infinite ARMA system has a harmonizable solution if and only if it has a stationary solution. Under some regularity conditions on the coefficients of the system (which are automatically satisfied for finite ARMA models) it is proved that if the system has a bounded solution then it has a stationary solution. It is our hope that the consideration of infinite order ARMA systems will contribute to better understanding of the analysis of time series with large sample data.

Key Words and Phrases: Infinite ARMA systems; stationary, harmonizable and bounded solutions; existence and uniqueness of solutions; form of solutions.

AMS(MOS) subject classifications. 60, 40

1. Introduction. In this paper the problem of existence and uniqueness of stationary, harmonizable and bounded solutions to ARMA equations of infinite order is studied. The question of existence and uniqueness of stationary solutions to the classical ARMA equations

\[ \sum_{k=0}^{p} \phi_k x_{n-k} = \sum_{k=0}^{q} \theta_k z_{n-k}, \quad n \in \mathbb{Z} \]

(1.1)

where \( (z_n) \) is a discrete parameter white noise, has been extensively studied in literatures. For example, in [2] it is shown that the equation (1.1) admits a stationary solution if and only if \( \frac{\theta}{\phi} \) is bounded, where \( \theta(t) = \sum_{k=0}^{q} \theta_k e^{itk}, \phi(t) = \sum_{k=0}^{p} \phi_k e^{itk}. \)

However, to the best of our knowledge, the problem of existence of harmonizable and possibly bounded non-harmonizable solutions have not been studied. We will deal with these questions for finite order as well as infinite order ARMA equations.

The justification for studying infinite order ARMA systems comes from the infinite order autoregressive representation of a stationary sequence whose spectral density satisfies some regularity conditions, see for example [8] and the references therein. It is our hope that the consideration of infinite order ARMA systems will contribute to better understanding of the analysis of time series with large sample data. One such useful study for infinite order autoregressive models is carried out in [9], where asymptotically efficient selection of the order of the model is sought.

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In Section 2 the general equation

\[ \sum_{k=-\infty}^{+\infty} \phi_k x_{n-k} = y_n, \ n \in \mathbb{Z} \]

where \( \sum |\phi_k| < \infty \) and \((y_n)\) is an arbitrary stationary sequence is considered. The main thrust of this section is to show that the existence of a stationary solution to equation (1.2) is equivalent to the existence of a harmonizable solution to (1.2); and the latter holds if and only if \( \frac{1}{\phi} \) is square integrable with respect to (w.r.t.) the spectral measure of the process \((y_n)\).

In Section 3 the sequence \((y_n)\) occurring in equation (1.2) is of the form \( y_n = \sum_{k=-\infty}^{+\infty} \phi_k z_{n-k}, \ n \in \mathbb{Z} \), with \( \sum |\theta_k|^2 < \infty \), and the search is for a series solution in the form \( \sum_{k=-\infty}^{+\infty} \psi_k z_{n-k}, \ n \in \mathbb{Z} \), where \((z_n)\) is a standard white noise sequence.

Section 4 deals with the existence of bounded solutions to infinite order ARMA equations. Under some analytic conditions which are automatically satisfied for finite order ARMA models, it is shown that every bounded solution to the equation

\[ \sum_{k=-\infty}^{+\infty} \phi_k x_{n-k} = \sum_{k=-\infty}^{+\infty} \theta_k z_{n-k}, \ n \in \mathbb{Z}, \]

where \((z_n)\) is a discrete parameter white noise, is harmonizable. Hence if (1.1) has a bounded solution then it has a stationary solution.

Throughout the paper, \( \mathbb{C}, \mathbb{Z}, \mathbb{N} \) and \( H \) will stand for complex numbers, integers, positive integers and a complex separable Hilbert space with an inner product \((,)\) and the norm \( \| \| \), respectively; \( T = \{z \in \mathbb{C} : |z| = 1\} \). The set \( T \) will be identified with the interval \((-\pi, \pi]\). For \( \varphi \) defined on \((-\pi, \pi]\), \( \tilde{\varphi} \) will stand for its replica on \( T \), i.e., \( \varphi(t) = \tilde{\varphi}(e^{it}), \ t \in (-\pi, \pi] \). All integrals will be over \((-\pi, \pi]\) and all sums and sequences will be indexed by the set of integers, unless it is stated otherwise. If \( F \) is a complex Borel measure on \((-\pi, \pi]\), then \( L^2(F) \) will denote the space of all square integrable functions w.r.t. the variation of \( F \). \( dt \) will denote the Lebesgue measure and we will abbreviate \( L^2(dt) \) by \( L^2 \). As usual \( 1_A \) will denote the indicator of a set \( A \) and \( \delta_{nm} \) will stand for the Kronecker symbol. \( \ell^2 \) will be the class of all complex square summable sequences indexed by \( \mathbb{Z} \). Any countably additive set function \( Z \) defined on the Borel \( \sigma \)-algebra \( \mathcal{B} \) of \((-\pi, \pi]\) with values in a Hilbert space \( H \) will be called an \( (H \text{-valued}) \) measure. An \( H \text{-valued} \) measure \( Z \) is orthogonally scattered (o.s.) if for any disjoint \( \Delta_1, \Delta_2 \in \mathcal{B}, \ Z(\Delta_1) \) is orthogonal to \( Z(\Delta_2) \). A measure \( W \) is absolutely continuous w.r.t. \( Z \) \((W \ll Z)\) if \( Z(\Delta) = 0 \) implies \( W(\Delta) = 0 \), \( \Delta \in \mathcal{B} \).

Let \( Z \) be an \( H \text{-valued} \) measure. Following the notion of integrability introduced in [7], a complex valued function \( f \) is said to be \( Z \) integrable if

(i) \( f \) is \((Z, x)\) integrable for each \( x \in H \), and
(ii) for each $\Delta \in \mathcal{B}$ there is an element of $H$, denoted by $\int_{\Delta} f \, dZ$, such that

$$\left( \int_{\Delta} f \, dZ, x \right) = \int_{\Delta} f \, d(Z, x), \quad x \in H,$$

where $(Z, x)$ denotes the complex valued measure $\Delta \rightarrow (Z(\Delta), x), \quad x \in H$. Below we state couple of basic properties of this integral. Justifications are included in [7].

(A) A function $f$ is $Z$ integrable (in the sense given above) if and only if it is integrable w.r.t. $Z$ in the sense of Dunford–Schwartz ([3], IV.10.7), see [7], Thm. 2.4.

(B1) If $f$ is $Z$ integrable, then the set function defined on $\mathcal{B}$ by $v(\Delta) = \int_{\Delta} f \, dZ$ is a measure and the semivariation $|||v|||(E)$ of $v$ on $E$ satisfies

$$|||v|||(E) = \sup_{\|x\| \leq 1} \int_{E} |f| \, dV(Z, x),$$

where $V(Z, x)$ denotes the total variation measure of the scalar measure $(Z, x)$, (see [7], Thm. 2.2).

(B2) If $f$ is measurable and $\sup_{\|x\| \leq 1} \int_{-\pi}^{\pi} |f| \, dV(Z, x) < \infty$, then $f$ is $Z$ integrable and the conclusion of (B1) holds. Indeed, for every $\Delta \in \mathcal{B}, x \in H$,

$$\left| \int_{\Delta} f \, d(Z, x) \right| \leq \int_{\Delta} |f| \, dV(Z, x) \leq C \|x\|,$$

with $C = \sup_{\|x\| \leq 1} \int |f| \, dV(Z, x)$. Therefore there exists an element $\nu(\Delta)$ in $H$ such that

$$(\nu(\Delta), x) = \int_{\Delta} f \, d(Z, x), \quad x \in H.$$

(C) The space of all $Z$ integrable functions becomes a Banach space (if two functions which are equal $Z$ almost everywhere are identified) under the norm

$$\|f\|_{L^1(Z)} = ||| \int f \, dZ ||| (-\pi, \pi) = \sup_{\|x\| \leq 1} \int |f| \, dV(Z, x) \mathcal{D},$$

and the simple functions are dense in $L^1(Z)$. ([1], p. 75).

1.3 Lemma. Let $Z$ be an $H$-valued measure.

(A) If $Z$ is orthogonally scattered, then

$$\mu(\Delta) = \|Z(\Delta)\|^2, \quad \Delta \in \mathcal{B},$$
is a positive measure and $L^1(Z) = L^2(\mu)$.

(B) If $W$ is an $H$-valued measure and $\varphi$ is a bounded Borel measurable function such that for all $\Delta \in \mathcal{B}$

$$W(\Delta) = \int_{\Delta} \varphi(t) \, Z(dt)$$

then $\frac{1}{\varphi} \in L^1(W)$ and for all $\Delta \in \mathcal{B}$

$$\int_{\Delta} \frac{1}{\varphi} \, dW = Z(\Delta \cap D^c)$$

where $D = \{t : \varphi(t) = 0\}$.

Proof. (A) If $f = \sum a_k 1_{\Delta_k}$ is a step function and $Z$ is orthogonally scattered then

$$\|f\|_{L^1(Z)}^2 = \sup_{\pi = \{D_1, \ldots, D_n\}} \left\| \sum \varepsilon_k \int_{D_k} f \, dZ \right\|^2 =$$

$$= \sup_{\pi} \sum_k \left\| \int_{D_k} \varepsilon_k f \, dZ \right\|^2 =$$

$$= \sup_{\pi} \sum_k \int_{D_k} |f|^2 d\mu = \int |f|^2 d\mu, \text{ where } \varepsilon_k = \pm 1.$$ 

Since step functions are dense in $L^1(Z)$ and $L^2(\mu)$, part (A) is proved.

To prove (B) let $\psi = \frac{1}{\varphi} 1_{D^c}$, where $D = \{t : \varphi(t) = 0\}$. Then

$$\int |\psi| dV(W, x) = \int |\psi| |\varphi| \, dV(Z, x) = V(Z, x)(D^c) \leq C\|x\|,$$

([2], IV 10.4). Therefore $\psi \in L^1(W)$, and because $D$ is a null set of $W$, $\frac{1}{\varphi} \in L^1(W)$.

Hence for all $x \in H$ and $\Delta \in \mathcal{B}$,

$$\int_{\Delta} \frac{1}{\varphi} \, d(W, x) = \int_{\Delta \cap D^c} \frac{1}{\varphi} \, d(W, x) = (Z(\Delta \cap D^c), x). \quad \square$$

1.4 Definition. A sequence $x = (x_n) \subset H$ is called harmonizable if there exists an $H$-valued measure $Z_x$ such that for all $n \in \mathbb{Z}$

$$x_n = \int e^{-int} Z_x(dt).$$

The measure $Z_x$ is uniquely determined by the sequence $(x_n)$ and is called the random spectral measure of $(x_n)$. For a harmonizable sequence $x = (x_n)$ and $A \subset \mathbb{Z}$ we will denote $M_x(A) = \overline{sp}\{x_n : n \in A\}, \quad M_x = M_x(\mathbb{Z})$. If $Z$ is an $H$-valued measure
and $\Delta \in \mathcal{B}$ then we define $M(Z, \Delta) = \overline{sp}\{Z(\Delta') : \Delta' \subset \Delta, \Delta' \in \mathcal{B}\}, \quad M(Z) = M(Z, (-\pi, \pi])$. For the random spectral measure $Z_x$ of a sequence $(x_n)$ we have $M_x = M(Z_x)$. A harmonizable sequence $(x_n)$ is stationary iff its random spectral measure is orthogonally scattered. If it is so, then $F_x(\Delta) = \|Z_x(\Delta)\|^2$, $\Delta \in \mathcal{B}$, is a positive measure satisfying

$$(x_n, x_m) = \int e^{-i(n-m)t} F_x(dt), \quad n, m \in \mathbb{Z}. $$

The measure $F_x$ is called the spectral measure of the sequence $(x_n)$. A harmonizable sequence $(x_n)$ is called regular if $\bigcap_n M_x(-\infty, n] = \{0\}$. A stationary sequence $(x_n)$ is regular if and only if $F_x$ is absolutely continuous w.r.t. the Lebesgue measure and its density satisfies $\int \ln \frac{dF_x}{dt} dt > -\infty$ ([6]).

1.5 Definition. Let $A$ denote the class of all continuous functions $f$ on $(-\pi, \pi]$ such that $\sum_{n=-\infty}^{+\infty} |\hat{f}(n)| < \infty$, where $\hat{f}(n) = \frac{1}{2\pi} \int e^{-int} f(t)dt$, $n \in \mathbb{Z}$.

2. Existence and Uniqueness. In this section we will discuss the existence of a solution to the equation

$$ (2.1) \quad \sum_{k=-\infty}^{+\infty} \phi_k x_{n-k} = y_n, \quad n \in \mathbb{Z}. $$

Unless otherwise is stated, we will assume that:

$$ (2.2) \quad \sum_{k=-\infty}^{+\infty} |\phi_k| < \infty, \quad \phi_k \in \mathbb{C}, \quad k \in \mathbb{Z}, $$

$$ (2.3) \quad \{y_n : n \in \mathbb{Z}\} \text{ is a stationary sequence with the spectral measure } F_y. $$

Let $\phi(t) = \sum \phi_k e^{ikt}$. If $\phi_k$ satisfies (2.2) then $\phi(t)$ is a continuous function belonging to the class $A$ (see Def. (1.5)).

2.4 Theorem (Existence). Assume (2.2) and (2.3). Then the following conditions are equivalent:

1. the equation (2.1) has a harmonizable solution,
2. the equation (2.1) has a stationary solution,
3. the equation (2.1) has a unique stationary solution satisfying $M_x \subset M_y$,
4. the equation (2.1) has a unique harmonizable solution satisfying $Z_x \ll Z_y$,
5. $\frac{1}{\phi} \in L^2(F_y)$. 

5
If (5) is satisfied then the solution in (3) and (4)) is given by the formula

\[ x_n^0 = \int e^{-int} \frac{1}{\phi(t)} Z_y(dt). \]  

(2.5)

Proof. (1) ⇒ (5). Suppose that \( x_n = \int e^{-int} Z_x(dt) \) is a harmonizable solution of (2.1). Then \( \sum \phi_k x_{n-k} = \int \phi(t) e^{-int} Z_x(dt) = \int e^{-int} Z_y(dt) \) for every \( n \in \mathbb{Z} \), and from the uniqueness of the Fourier transform, it follows that

\[ \phi dZ_x = dZ_y. \]

(2.6)

Therefore, from Lem. 1.3, it follows that

\[ \frac{1}{\phi} \in L^1(dZ_y) = L^2(dF_y). \]

(5) ⇒ (4). For every \( \Delta \in \mathcal{B} \), let

\[ Z_0(\Delta) = \int_{\Delta} \frac{1}{\phi} dZ_y. \]

Then \( Z_0 \) is an \( M_y \)-valued measure and

\[ x_n^0 = \int e^{-int} Z_0(dt), \quad n \in \mathbb{Z} \]

(2.7)

is a harmonizable (in fact stationary) sequence. Moreover

\[ \sum \phi_k x_{n-k}^0 = \int \phi(t) e^{-int} Z_0(dt) = \int \phi(t) e^{-int} \frac{1}{\phi(t)} dZ_y(t) = y_n, \quad n \in \mathbb{Z}, \]

so \( (x_n^0) \) is a solution to (2.1). Clearly \( Z_{x^0} = Z_0 \ll Z_y \). Now let \( (x_n) \) be any other harmonizable solution to (2.1) satisfying \( Z_x \ll Z_y \). From (1) ⇒ (5) we conclude that \( \phi dZ_x = dZ_y \) and by Lem. 1.3

\[ Z_x(\Delta \cap D^C) = \int_{\Delta} \frac{1}{\phi} dZ_y = Z_0(\Delta), \quad \Delta \in \mathcal{B}, \]

where \( D = \{ t : \phi(t) = 0 \} \). Notice that if \( \Delta \subset D \), \( \Delta \in \mathcal{B} \), then \( Z_y(\Delta) = 0 \) and \( Z_x(\Delta) = 0 \) since \( Z_x \ll Z_y \). Therefore for every \( \Delta \in \mathcal{B} \)

\[ Z_x(\Delta) = Z_x(\Delta \cup D^C) = Z_0(\Delta), \]

which proves that

\[ x_n = \int e^{-int} Z_x(dt) = x_n^0, \quad n \in \mathbb{Z}. \]

(5) ⇒ (3). Let \( (x_n^0) \) be the stationary solution to (2.1) defined by (2.5). Then \( M_{x^0} = M_y \). Assume \( (x_n) \) is any other stationary solution satisfying \( M_x \subset M_y \). Then from (1) ⇒ (5), \( \phi dZ_x = dZ_y \) and by Lem. 1.3

\[ Z_x(\Delta \cap D^C) = Z_0(\Delta), \quad \Delta \in \mathcal{B}, \]
where \( D = \{ t : \phi(t) = 0 \} \).

This implies that \( M(Z_x, D^c) = M(Z_0, (-\pi, \pi]) = M_x^0 = M_y \). Since \( Z_x \) is orthogonally scattered and by assumption

\[
M(Z_x, (-\pi, \pi]) = M(Z_x, D) \oplus M(Z_x, D^c) \subseteq M_y,
\]

\( M(Z_x, D) = 0 \) and consequently for all \( \Delta \in \mathbb{B} \).

\[
Z_x(\Delta) = Z_x(\Delta \cap D) \oplus Z_x(\Delta \cap D^c) = Z_0(\Delta).
\]

Thus \( x_n = x_n^0 \), \( n \in \mathbb{Z} \). This completes the proof since the implications (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1) and (4) \( \Rightarrow \) (1) are obvious. \( \square \)

In general the equation (2.1) may have many stationary and harmonizable solutions if we do not require that \( Z_x \ll Z_y \) (or \( M_x \subset M_y \), which is equivalent to the previous in the stationary case). The following proposition describes all stationary and harmonizable solutions to (2.1). Note that from (2.1) and Thm. 2.4 it follows that if \( \frac{1}{\phi} \in L^2(F_y) \), then \( M_y = M_x^0 \).

2.8 Theorem (Form of solutions). Suppose that \( \frac{1}{\phi} \in L^2(F_y) \). Then

(1) every harmonizable solution to (2.1) has the form

\[
x_n = x_n^0 + x_n^1,
\]

where \( x_n^0 \) is the stationary sequence given by (2.5) and \( x_n^1 \) is a harmonizable sequence with the random spectral measure concentrated on \( D = \{ t : \phi(t) = 0 \} \),

(2) every stationary solution to (2.1) has the form

\[
x_n = x_n^0 \oplus x_n^1
\]

where \( x_n^0 \) is given by (2.5), \( x_n^1 \) is a stationary sequence with random spectral measure concentrated on \( D \) and \( M_{x^1} \perp M_{x^0} \).

Proof. (1). Let \( x_n \) be a harmonizable solution to (2.1). Then \( x_n^1 = x_n - x_n^0 \), \( n \in \mathbb{Z} \) is a harmonizable sequence and \( \int e^{-int} \phi(t) dZ_{x^1}(t) = 0 \), \( n \in \mathbb{Z} \). Therefore \( Z_{x^1} \) is concentrated on \( D \).

(2) Now suppose that \( (x_n) \) is a stationary solution to (2.1). Then \( x_n^1 = x_n - x_n^0 \), \( n \in \mathbb{Z} \) is harmonizable with the spectral random measure \( Z_{x^1} \) concentrated on \( D \). Moreover,

\[
x_n = \int e^{-int} \frac{1}{\phi(t)} dZ_y + \int e^{-int} dZ_{x^1} = \int_{D^c} e^{-int} \frac{1}{\phi(t)} dZ_y + \int_{D} e^{-int} dZ_{x^1} = \int e^{-int} dZ_x.
\]
Hence

\[ Z_x(\Delta) = \int_{\Delta \cap D^c} \frac{1}{\phi(t)} dZ_y + \int_{\Delta \cap D} dZ_x, \quad \Delta \in \mathcal{B}. \]

Since \((x_n)\) is stationary, \(Z_x\) is orthogonally scattered and

\[ M_{x^1} = \overline{sp}\{Z_x(\Delta \cap D) : \Delta \in \mathcal{B}\} \perp \overline{sp}\{Z_x(\Delta \cap D^c) : \Delta \in \mathcal{B}\} = M_{x^0}. \]

Therefore \((x_n^1)\) is stationary and orthogonal to \((x_n^0)\). \(\square\)

As an easy consequence of Thm. 2.8 we obtain that

2.11 COROLLARY (Uniqueness). The following three conditions are equivalent:

(i) (2.1) has a unique harmonizable solution,

(ii) (2.1) has a unique stationary solution,

(iii) \(\phi(t) \neq 0\) everywhere.

Proof. Implications (iii) \(\Rightarrow\) (i) \(\Rightarrow\) (ii) follow from Thm. 2.8, because \(\phi\) is continuous and \(\phi(t) \neq 0\) everywhere implies that \(\frac{1}{\phi}\) bounded. If \(\phi(t_0) = 0\), then

\[ x_n = x_n^0 + e^{-int_0} v, \quad n \in \mathbb{Z}, \text{ where } v \perp M_y = M_{x^0}, \text{ satisfies (2.1). This proves (ii) \(\Rightarrow\) (iii).} \quad \square \]

Now suppose that the sequence \((y_n)\) in (2.1) has a spectral density (with respect to the Lebesgue measure). Then, as it is shown below, \(y_n = \sum \theta_k z_{n-k}\) where \(\sum |\theta_k|^2 < \infty\) and \((z_n)\) is an orthonormal system. In this case the equation (2.1) takes the form

\[ \sum_{-\infty}^{\infty} \phi_k x_{n-k} = \sum_{-\infty}^{\infty} \theta_k z_{n-k}, \]

\[ \sum |\phi_k| < \infty, \quad \sum |\theta_k|^2 < \infty, \quad (z_n, z_m) = \delta_{nm}. \]

2.13 LEMMA. A stationary sequence \((y_n)\) admits a representation

\[ y_n = \sum \theta_k z_{n-k} \]

with \((z_n)\) being an orthonormal system if and only if \(F_y \ll dt\). Moreover, if it does then

(i) \(\sum |\theta_k|^2 < \infty\) (so \(\theta(t) = \sum \theta_k e^{ikt} \in L^2\)),

(ii) \(\frac{dF_y}{dt} = |\theta(t)|^2 dt. \text{ a.e.,}\)

(iii) \(M_y = M_x\) if and only if \(\theta(t) \neq 0 dt. \text{ a.e.}\)
Proof. Suppose that \( y_n = \sum \theta_k z_{n-k} \). Since \((z_n)\) is an orthonormal system, \( \sum |\theta_k|^2 < \infty \) and \( \vartheta(t) = \sum \theta_k e^{ikt} \) converges in \( L^2 \); hence (i). Consider the \( L^2 \)-valued process \( y_n = e^{-in^\cdot \vartheta(\cdot)} \). Then

\[
(y_n, y_m) = (y_n^0, y_m^0)_{L^2} = \int e^{-i(n-m)^t |\vartheta(t)|^2} dt
\]

Therefore \( F_y \ll dt \) and \( \frac{dF_y}{dt} = |\vartheta|^2 dt \) a.e. (so we have (ii)). To prove sufficiency suppose that \( F_y \ll dt \). Let \( \vartheta = \sqrt{\frac{dF_y}{dt}} \). Then \( \vartheta \in L^2 \), \( \vartheta = \sum \theta_k e^{ikt} \), where \( \theta_k = \frac{1}{2\pi} \int e^{-ikt} \vartheta(t) dt \), and the series above converges in \( L^2 \). Let

\[
y_n^0(\cdot) = e^{i n^\cdot \vartheta(\cdot)}, \quad n \in \mathbb{Z},
\]

\[
z_n^0(\cdot) = e^{i n^\cdot}, \quad n \in \mathbb{Z}.
\]

Then \( (z_n^0) \) is a complete orthonormal system in \( L^2 \) and

\[
(y_n^0, y_m^0)_{L^2} = \int e^{-i(n-m)^t |\vartheta(t)|^2} dt = (y_n, y_m).
\]

Therefore there exists an isometry \( V : M_y \to L^2 \) such that \( V(y_n) = y_n^0, \quad n \in \mathbb{Z} \). Note that

\[
VM_y = M_{y^0} = \overline{sp}\{e^{-in^\cdot \vartheta} : \quad n \in \mathbb{Z}\}
\]

\[
= \{ f \in L^2 : f = f1_{\vartheta \neq 0} dt \ \text{a.e.}\}
\]

Let \( N = L^2 \Theta M_{y^0} \) \((N \text{ could be equal to } \{0\})\) and let \( K = M_y \oplus N \). Define

\[
U : L^2 = M_{y^0} \oplus N \to M_y \oplus N
\]

by \( U = V^{-1} \oplus I \). Then \( U \) is unitary, \( z_n = U z_n^0, \quad n \in \mathbb{Z}, \) is an orthonormal system in \( K \) and

\[
y_n = U y_n^0 = \sum \theta_k z_{n-k}.
\]

To see (iii) notice that if

\[
y_n = \sum \theta_k z_{n-k}.
\]

and \( U : M_x \to L^2 \) is the unitary operator defined by

\[
U(\sum a_k z_k) = \sum a_k e^{i nt}, \quad (a_k) \in \ell^2,
\]

then \( U y_n = e^{-n^\cdot \vartheta(\cdot)} y_n^0 \). Clearly \( M_y = M_x \) if and only if \( L^2 = M_{y^0} = \{ f \in L^2 : f = f1_{\vartheta(t) \neq 0} \} \), which holds if and only if \( \{ t : \vartheta(t) \neq 0 \} \) has the Lebesgue measure zero. \( \square \)

The following two results follow immediate from Thms. 2.4 and 2.8.
2.14. **Theorem.** Suppose that $\sum |\phi_k| < \infty$, $\sum |\theta_k|^2 < \infty$. Let $\phi(t) = \sum \phi_n e^{-int}$, $\theta(t) = \sum \theta_n e^{-int}$. Then the following conditions are equivalent:

(i) (2.12) has a harmonizable solution,

(ii) (2.12) has a stationary solution,

(iii) (2.12) has a unique stationary solution in the space $M_y$, where $y_n = \sum \theta_k z_{n-k}$,

$$n \in \mathbb{Z}$$

(iv) $\frac{\theta}{\phi} \in L^2$ (with convention $\frac{0}{0} = 0$).

Moreover, if (iv) holds then the unique stationary solution in $M_y$ is given by

$$(2.15) \quad x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} Z_z(dt) = \sum \psi_k z_{n-k}, \quad n \in \mathbb{Z},$$

where $Z_z$ is the random spectral measure of $(z_n)$ and $\psi_k = \frac{1}{2\pi} \int e^{-ikt} \frac{\theta(t)}{\phi(t)} dt$,

$k \in \mathbb{Z}$.

Note that in general $M_y$ might be a proper subset of $M_z$ so (2.15) need not be a unique stationary solution in $M_z$.

2.16. **Theorem.** Let $\frac{\theta}{\phi} \in L^2$. Then

(1) every harmonizable solution to (2.12) in $M_z$ has the form

$$x_n = x_n^0 + x_n^1$$

where

(i) $x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ_z$, $n \in \mathbb{Z}$,

(ii) $x_n^1$ is an $M_z$-valued harmonizable sequence with the random spectral measure concentrated on $D = \{t; \phi(t) = 0\}$,

(2) every stationary solution to (2.12) in $M_z$ has the form

$$x_n = x_n^0 \oplus x_n^1, \quad n \in \mathbb{Z}$$

where

(i) $x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ_z$, $n \in \mathbb{Z}$

(ii) $M_{x^0} \perp M_{x^1}$,

(iii) $(x_n^1)$ is an $M_z$-valued stationary sequence with spectral measure concentrated on $D = \{t : \phi(t) = 0\}$.

The next proposition provides a necessary and sufficient conditions for the existence of a unique stationary solution to (2.12) in $M_z$. 
2.17. Theorem. The equation (2.12) has a unique stationary solution in $M_z$ if and only if either one of the following two conditions is satisfied:

(1) $\phi(t) \neq 0$ everywhere, or

(2) $\theta(t) \neq 0$ dt a.e., and $\frac{\theta}{\phi} \in L^2$.

Proof. If (1), then by Cor. 2.11, the equation (2.12) has a unique stationary solution. If (2), then from Lem. 2.13 (iii), $M_y = M_z$, where $(y_n)$ is defined in Prop. 2.14. Therefore by Prop. 2.14 (iii), (2.12) has a unique stationary solution in $M_z$. Conversely, suppose that $\theta(t) = 0$ on a set $\Delta$ of positive Lebesgue measure, $\phi(t_0) = 0$ and suppose that (2.12) has a stationary solution. Then $\frac{\theta}{\phi} \in L^2$ and the sequence

$$x_n^1 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ_z(t) + Z_z(\Delta)e^{-int_0}, n \in \mathbb{Z}$$

is a stationary solution to (2.12) with values in $M_z$ different from $(x_n^0)$. □

3. Series Solution. Suppose that $\sum |\phi_k| < \infty, \sum |\theta_k|^2 < \infty$ and consider the equation

$$\sum \phi_k x_{n-k} = \sum \theta_k z_{n-k}$$

where $(z_n, z_m) = \delta_{nm}$. Let, as before, $\phi(t) = \sum \phi_k e^{ikt}, \theta(t) = \sum \theta_k e^{ikt}$

3.2. Definition. We will say that (3.1) has a series solution if there exists a sequence of scalars $(\varphi_k)^{+\infty}$ such that

$$x_n = \sum_{-\infty}^{+\infty} \varphi_k z_{n-k}, n \in \mathbb{Z},$$

is a solution to (3.1).

It is obvious that if (3.1) has a series solution $(x_n)$, then $\sum |\varphi_k|^2 < \infty$ and $(x_n)$ is stationary. Therefore by Thm. 2.14 and the formula (2.15) therein

the equation (3.1) has a series solution iff $\frac{\theta}{\phi} \in L^2$.

Theorem below discusses the problem of the uniqueness of a series solution.

3.3. Theorem. The equation (3.1) has a unique series solution iff

(1) $\frac{\theta}{\phi} \in L^2$, and

(2) $\phi(t) \neq 0$ dt a.e.
Proof. We have already noticed that (3.1) has a series solution \( \frac{\theta}{\phi} \in L^2 \). Let \( D = \{ t : \phi(t) = 0 \} \). Assume first that (3.1) has a series solution and \( dt(D) \neq 0 \). Then

\[
x_n^0 = \int e^{-int} \left( \frac{\theta(t)}{\phi(t)} \right) dZ_z = \sum_{-\infty}^{+\infty} (\theta/\phi)^\wedge(k)z_{n-k} \quad \text{and}
\]

\[
x_n^1 = \int e^{-int} \left( \frac{\theta(t)}{\phi(t)} \right) + 1_D(t) dZ_z = \sum_{-\infty}^{+\infty} \left( \frac{\theta}{\phi} + 1_D \right)^\wedge(k)z_{n-k}
\]

are two different series solutions to (3.1). Conversely, assume (1) and (2). Suppose that \( (x_n) \) is a series solution to (3.1). Then \( x_n^2 = x_n - x_n^0 \), where \( x_n^0 \) is as above has the form \( x_n^2 = \sum a_kz_{n-k}, n \in \mathbb{Z} \), and by Thm. 28 its spectral measure is concentrated on \( D \). Since \( dt(D) = 0 \), from Lem. 2.13, we conclude that \( x_n^2 = 0 \). \( \square \)

3.4. Remark. If (3.1) has a series solution \( x_n = \sum \varphi_k z_{n-k}, n \in \mathbb{Z} \), then the coefficients \( (\varphi_k) \) satisfy the infinite system of equations

\[
\sum_{-\infty}^{+\infty} \phi_k \varphi_{n-k} = \theta_n, \quad n \in \mathbb{Z}.
\]

(3.5)

In fact, there is one-to-one correspondence between \( \ell^2 \)-solutions of (3.5) and series solutions of (3.1). To see this it is enough to notice that the Fourier transform converts (3.5) into the functional equation

\[
\phi(t)\varphi(t) = \theta(t) \quad dt \text{ a.e.,}
\]

where \( \varphi(t) = \sum \varphi_k e^{ikt}, \varphi \in L^2 \). On the other hand, if \( x_n = \sum \varphi_k z_{n-k} \) is a solution to (3.1) then

\[
\int e^{-int}\phi(t)\varphi(t)dZ_z = \int e^{-int}\theta(t)dZ_z, \quad n \in \mathbb{Z},
\]

and \( \phi(t)\varphi(t) = \theta(t) \) \( dt \) a.e. \( Z_z \). This leads to an alternative proof of Thm. 3.3.

If the coefficients \( \phi_k \) and \( \theta_k \) in (3.1) vanish for negative \( k \) the functions \( \phi \) and \( \theta \) are in the Hardy class \( H^2 \) (\( H^2 \) consists of all functions in \( L^2 \) whose negative Fourier coefficient vanish). Since for every \( f \in H^2 \), \( \log |f| \in L^1 \) (e.g. [5]) and in particular \( f(t) \neq 0 \) \( dt \) a.e., we obtain the following theorem.

3.6. Theorem. Assume that \( \sum_{k=0}^{\infty} |\phi_k| < \infty, \sum_{k=0}^{\infty} |\theta_k|^2 < \infty \). Consider the equation

\[
\sum_{k=0}^{\infty} \phi_k x_{n-k} = \sum_{k=0}^{\infty} \theta_k z_{n-k}, \quad n \in \mathbb{Z},
\]

(3.7)

where \( (z_n, z_m) = \delta_{nm} \).
Then the following conditions are equivalent:

(1) (3.7) has a harmonizable solution,
(2) (3.7) has a stationary solution,
(3) (3.7) has a unique series solution,
(4) (3.7) has a unique regular stationary solution,
(5) (3.7) has a unique stationary solution in $M_z$,
(6) $\frac{\theta}{\phi} \in L^2$.

If (6) holds then the unique regular series solution, whose existence is guaranteed by (4) and (5), is given by

$$x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ_z = \sum_{-\infty}^{+\infty} \psi_k z_{n-k}, \quad n \in \mathbb{Z},$$

where $\psi_n = \frac{1}{2\pi} \int e^{-int} \frac{\theta(t)}{\phi(t)} dt, \quad n \in \mathbb{Z}$.

Moreover, if $\phi(t) \neq 0$ everywhere, then the equation (3.7) has only one harmonizable solution given by (3.8).

Proof. Because $\phi \in H^2$, $\phi(t) \neq 0$ dt a.e. Therefore the equivalence of (1), (2), (3), (5), (6) follows from Thms. 2.14, 2.17 and 3.3. We need to prove only that (6) $\Rightarrow$ (4). Since the spectral density of $(x_n^0)$, $|\frac{\theta}{\phi}|^2$, has integrable logarithm, $(x_n^0)$ is regular. Let $(x_n)$ be any other regular stationary solution to (3.7). Then from Thm. 2.16 it follows that $dF_x = |\frac{\theta}{\phi}|^2 dt + dF_{x1}$, and $dF_{x1}$ is singular w.r.t. $dt$. Since $(x_n)$ is regular, $F_{x1} = 0$. The part "moreover" follows from Cor. 2.11. □

3.9. Remark. The solution (3.8) has the form

$$x_n = \sum_{k=0}^{\infty} \psi_k z_{n-k},$$

iff $\frac{\theta}{\phi} \in H^2$. In this case in view of Rem. 3.4 the coefficients $\psi_k$, $k \geq 0$, can be computed by solving the system of equations

$$\sum_{k=0}^{n} \phi_k \psi_{n-k} = \theta_n, \quad n \geq 0.$$ 

4. Bounded Solutions. In this section we discuss the problem of the existence of bounded solutions to the equation (2.12).
4.1. Definition. Assume that $\sum |\phi_k| < \infty$, $\sum |\theta_k|^2 < \infty$, $(z_k, z_n) = \delta_{kn}$, $n, k \in \mathbb{Z}$. Every norm bounded sequence $x_n \in H$, $n \in \mathbb{Z}$, satisfying the equation

$$
\sum_{-\infty}^{+\infty} \phi_k x_{n-k} = \sum_{-\infty}^{+\infty} \theta_k z_{n-k}, n \in \mathbb{Z}
$$

is called a bounded solutions to (4.2).

As before, let $A$ be the space of all continuous functions $\varphi(t)$, $t \in (-\pi, \pi]$ with absolutely summable Fourier series. It is known ([4], 11.4.17) that $A$ with the norm $\|\varphi\|_A = \sum_{-\infty}^{+\infty} |\varphi_k| < \infty$ is a Banach algebra under pointwise multiplication. Moreover it is easy to see that the relations

$$
\begin{cases}
  x_n = F(e^{-in\cdot}), & n \in \mathbb{Z} \\
  F(\varphi) = \sum_{k=-\infty}^{+\infty} \varphi_k x_{n-k}, & \varphi \in A,
\end{cases}
$$

establishes a one-to-one correspondence between the class of all bounded linear operators from $A$ to $H$ and the class of all bounded $H$-valued sequences $(x_n)$. Here and in the sequel $(\varphi_k)$ will stand for Fourier coefficients of $\varphi$.

4.4. Lemma. Let $Z$ denote the random spectral measure of $(z_n)$. The equation (4.2) has a bounded solution if and only if there exists a bounded linear operator $F : A \to H$ such that

$$
F(\phi \varphi) = \int \theta \varphi dZ
$$

for every trig polynomial $\varphi$ (or equivalently for every $\varphi \in A$).

Proof. Let $(x_n)$ be a bounded solution of equation (4.2) and let $F$ denote the corresponding bounded operator as described above, that is $x_n = F(e^{-in\cdot})$, $n \in \mathbb{Z}$. Then (4.2) takes the form

$$
F(\phi e^{-in\cdot}) = \sum \phi_k F(e^{-i(n-k)\cdot}) = \sum \theta_k \int e^{-i(n-k)t} Z(dt) = \int \theta(t) e^{-int} Z(dt), n \in \mathbb{Z}.
$$

Let $\varphi \in A$. Then $\varphi(t) = \sum_{-\infty}^{+\infty} \varphi_k e^{ikt}$ where the convergence is uniform and in $A$.

Since multiplication in $A$ is continuous and $F$ is bounded $\sum \varphi_k F(\phi e^{ik\cdot}) = F(\phi \varphi)$. On the other hand

$$
\sum \phi_k \int \theta(t) F(t) e^{-int} Z(dt) = \int \theta(t) \varphi(t) Z(dt)
$$

because the Lebesgue dominated theorem applies. Consequently we have

$$
F(\phi \varphi) = \int \theta \varphi dZ, \ \varphi \in A.
$$

The proof of the converse is easy and can be similarly carried out. []

As an immediate consequence of the Lemma above and the Wiener theorem for the algebra $A$ we obtain the following corollary.
4.5. **COROLLARY.** If \( \phi(t) \neq 0 \) everywhere then the equation (4.2) has a unique bounded solution given by \( x_n^0 = \int e^{-int} \frac{\theta}{\phi} dZ, \ n \in \mathbb{Z}. \)

**Proof.** Since \( \phi(t) \) is continuous, \( \frac{\theta}{\phi} \in L^2. \) By Thm. 2.14 \( x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ \) is a stationary, and hence bounded, solution to (4.2). Let \( (y_n) \) be any bounded solution to (4.2). Then \( x_n = y_n - x_n^0 \) satisfies \( \sum_{k=-\infty}^{+\infty} \phi_k x_{n-k} = 0 \) and the operator \( F \) generated by \( (x_n) \) by the formula (4.3) satisfies \( F(\varphi) = 0, \ \varphi \in A. \) Since, by the Wiener theorem ([4], 11.4.17), \( \frac{1}{\phi} \in A, \) we conclude that \( F(\varphi) = 0 \) for all \( \varphi \in A. \) Thus \( F = 0. \)

The problem of describing all bounded solutions to (4.2) for arbitrary \( \phi \in A \) and \( \theta \in L^2 \) seems to be rather difficult. However, if:

\[
(4.6) \quad \text{the functions } \tilde{\phi}(z) = \sum_{k=-\infty}^{+\infty} \phi_k z^k \text{ and } \tilde{\theta}(z) = \sum_{k=\infty}^{\infty} \theta_k z^k \text{ z \in C.}
\]

are analytic in an annulus \( 1 - \varepsilon < |z| < 1 + \varepsilon, \ \varepsilon > 0, \)

then one can prove that every bounded solution to (4.2) is harmonizable, which if combined with previous results, gives a complete description of all bounded solutions to (4.2) under assumptions (4.6). The proof of this statement is divided into three lemmas, which seem to be of independent interest. Note that if \( \tilde{\phi} \) and \( \tilde{\theta} \) satisfy condition (4.6), then by Laurent expansion the functions \( \phi(t) = \tilde{\phi}(e^{it}) \) and \( \theta(t) = \tilde{\theta}(e^{it}) \) belong to class \( A. \)

4.7. **LEMMA.** If \( \phi(t) = \phi_1(t)\phi_2(t), \ \phi_1, \phi_2 \in A \) and the equation \( \sum \phi_k x_{n-k} = \sum \theta_k z_{n-k}, \sum |\theta_k|^2 < \infty, (z_n, z_m) = \delta_{nm} \) has a bounded solution, say \( (x_n) \), then the sequence \( y_n = \sum \phi_{j,k} x_{n-k} \) is a bounded solution to the equation

\[
\sum \phi_{1,k} y_{n-k} = \sum \theta_k z_{n-k}
\]

where \( \phi_j(t) = \sum_{k=-\infty}^{+\infty} \phi_{j,k} e^{ikt} \quad j = 1, 2. \)

**Proof.** Let \( F \) be associated with \( (x_n) \) by the formula (4.3). Since \( (x_n) \) is a bounded solution to \( \sum \phi_k x_{n-k} = \sum \theta_k z_{n-k}, \) by Lem. 4.4 we have

\[
F(\phi_1 \phi_2 \varphi) = \int \varphi \theta dZ
\]

for all \( \varphi \in A. \) Since \( A \) is a Banach algebra, the mapping \( \phi_2 F : A \rightarrow H \) defined by \( (\phi_2 F)(\psi) = F(\phi_2 \psi) \) is bounded and satisfies

\[
(\phi_2 F)(\phi_1 \varphi) = \int \theta \varphi dZ, \quad \varphi \in A.
\]

Therefore, using once again Lem. 4.4, it is easy to see that \( y_n = (\phi_2 F)(e^{-in \cdot}) = \sum \phi_{2,k} x_{n-k}, \ n \in \mathbb{Z}, \) satisfies

\[
\sum \phi_{1,k} y_{n-k} = \sum \theta_k z_{n-k}, \ n \in \mathbb{Z}.
\]
4.8. LEMMA. Assume that \( \tilde{\phi}(z) = \sum_{k=-\infty}^{+\infty} \phi_k z^k \), \( z \in \mathbb{Z} \) is a non-zero analytic function in an annulus \( 1 - \varepsilon < |z| < 1 + \varepsilon, \varepsilon > 0 \). Let \( t_0, \ldots, t_k \) be the zeros of the function \( \phi(t) = \tilde{\phi}(e^{it}), t \in (-\pi, \pi) \). Then a sequence \( (x_n) \) is a bounded solution to the homogeneous equation

\[
(\star) \quad \sum_{k=-\infty}^{+\infty} \phi_k x_{n-k} = 0
\]

if and only if \( x_n = \sum_{j=0}^{k} v_j e^{-int_j}, v_j \in H \).

Proof. First we note that since \( \phi \) is analytic, \( \tilde{\phi}(z) \) has only finitely many zeros in \( 1 - \varepsilon/2 < |z| < 1 + \varepsilon/2 \), unless it is the zero function. Therefore \( \phi(t) = \tilde{\phi}(e^{it}) \) has only finitely many zeros on \((-\pi, \pi] \) and \( \phi(t) \) can be written in the form

\[
\phi(t) = (e^{it} - e^{it_0})^{j_0}(e^{it} - e^{it_1})^{j_1} \cdots (e^{it} - e^{it_k})^{j_k} \phi_0(t),
\]

where \( \phi_0(t) \neq 0 \) on \((-\pi, \pi] \), \( j_r \in \mathbb{N}, 0 \leq r \leq k \).

Let

\[
\phi_0^m(t) = \left[(e^{it} - e^{it_0})^m\right]^{-1} \phi(t), \quad 1 \leq m \leq j_0,
\]

\[
\phi_1^m(t) = \left[(e^{it} - e^{it_0})^{j_0}(e^{it} - e^{it_1})^m\right]^{-1} \phi(t), \quad 1 \leq m \leq j_1,
\]

\[\vdots\]

\[
\phi_k^m(t) = \left[(e^{it} - e^{it_0})^{j_0} \cdots (e^{it} - e^{it_{k-1}})^{j_{k-1}}(e^{it} - e^{it_k})^m\right]^{-1} \phi(t), \quad 1 \leq m \leq j_k,
\]

\[
\varphi_{N,j}(t) = \frac{e^{iNt} - e^{iNt_j}}{e^{it} - e^{it_j}},
\]

\[
\varphi_{-N,j}(t) = \frac{e^{-iNt} - e^{-iNt_j}}{e^{it} - e^{it_j}}.
\]

At the points \( t_0, \ldots, t_k \) the functions are defined by continuity. In particular

\[
\varphi_{N,j}(t_j) = Ne^{-i(N+1)t_j}, \quad \varphi_{N,j}(t_j) = Ne^{-i(N-1)t_j}, \quad N \in \mathbb{N}.
\]

Suppose that \( (x_n) \) is a bounded solution to \( (\star) \) and let the associated \( F \) be given by (4.3). Then from Lems. 4.4 and 4.7 it follows that

\[
(\phi_0^1 F)((e^{it} - e^{it_0})\varphi) = 0
\]

for every polynomial \( \varphi \). Letting \( \varphi = \varphi_{N,0} \) and \( \varphi = \varphi_{-N,0} \) we obtain \( (\phi_0^1 F)(e^{iNt}) = e^{iNt}(\phi_0^1 F)(1), \quad N \in \mathbb{Z} \). Therefore for every trig polynomial \( \varphi \), \( (\phi_0^1 F)(\varphi) = \varphi(t_0)w_0 \), where \( w_0 = (\phi_0^1 F)(1) \). If \( j_0 = 1 \) this shows that \( (\phi_0^{j_0} F)(\varphi) = \varphi(t_0)w_0, \) \( \varphi \in A \). If \( j_0 \geq 2 \), let us assume that \( m \leq j_0 - 1 \) and that

\[
(\phi_0^m F)(\varphi) = \varphi(t_0)w_0.
\]
We will show that \((\phi^m_0 F)(\varphi) = \varphi(t_0)w_0\), for possibly different \(w_0\). From Lem. 4.7
\[
(\phi^m_0 F)((e^{it} - e^{i t_0})\varphi) = \varphi(t_0)w_0.
\]

Letting \(\varphi = \varphi_{N,0}\) and \(\varphi_{-N,0}\) we obtain
\[
(\phi^m_0 F)(e^{iNt} - e^{iNt_0}) = \varphi_N(t_0)w_0 = Ne^{i(N-1)t_0}w_0, \quad N \in \mathbb{N},
\]
and
\[
(\phi^m_0 F)e^{-iNt_0} - e^{-iNt}) = \varphi_{-N}(t_0)w_0 = Ne^{-i(N+1)t_0}w_0, \quad N \in \mathbb{N}.
\]

Since the left sides are bounded as functions of \(N\), \(w_0 = 0\) and consequently
\[
(\phi^m_0 F)(e^{iNt_0}) = e^{iNt_0}(\phi^m_0 F)(1), \quad N \in \mathbb{Z}.
\]

Hence \((\phi^m_0 F)(\varphi) = \varphi(t_0)w_0\), for every trig polynomial \(\varphi\). Repeating the argument we conclude that \((\phi^m_0 F)(\varphi) = \varphi(t_0)w_0\), for every \(\varphi \in \mathbb{A}\) and possibly different \(w_0\). Using Lem. 4.7 once again we obtain
\[
(\phi^1_1 F)[(e^{it} - e^{i t_1})\varphi] = \varphi(t_0)w_0, \quad \varphi \in \mathbb{A}.
\]

Setting \(\varphi = \varphi_{N,1}\) and \(\varphi = \varphi_{-N,1}\) we get,
\[
(\phi^1_1 F)(e^{iNt} - e^{iNt_1}) = \varphi_{N,1}(t_0)w_0 = (e^{i t_0 N} - e^{i t_1 N})\frac{w_0}{e^{it_0} - e^{it_1}}, \quad N \in \mathbb{N},
\]
and
\[
(\phi^1_1 F)(e^{-iNt_1} - e^{-iNt}) = \varphi_{-N,1}(t_0)w_0 = (e^{-iNt_1} - e^{-iNt_0})\frac{w_0}{e^{it_0} - e^{it_1}}, \quad N \in \mathbb{N}.
\]

Hence for all \(N \in \mathbb{Z}\),
\[
(\phi^1_1 F)(e^{iNt_1})) = e^{iNt_1}((\phi^1_1 F)(1)\frac{w_0}{e^{it_0} - e^{it_1}}) + e^{iNt_0}\frac{w_0}{e^{it_0} - e^{it_1}}.
\]

From the linearity we obtain
\[
(\phi^1_1 F)(\varphi) = \varphi(t_1)w_1 + \varphi(t_0)w_0, \quad \varphi \in \mathbb{A}.
\]

where \(w_0\) and \(w_1\) are now the coefficients of \(e^{it_0}\) and \(e^{i t_1}\), respectively. If \(j_1 = 1\), this step is completed. If \(j_1 \geq 2\), let us assume that \(m \leq j_1 - 1\) and that
\[
(\phi^m_1 F)(\varphi) = \varphi(t_1)w_1 + \varphi(t_0)w_0, \quad \varphi \in \mathbb{A}.
\]

Then
\[
(\phi^{m+1}_1 F)((e^{it} - e^{i t_1})\varphi] = \varphi(t_1)w_1 + \varphi(t_0)w_0.
\]

Now setting again \(\varphi = \varphi_{N,1}\) and \(\varphi = \varphi_{-N,1}\) we obtain
\[
(\phi^{m+1}_1 F)(e^{iNt} - e^{i t_1 N}) = Ne^{i(N-1)t_1}w_1 + (e^{iNt_0} - e^{i t_1 N})\frac{w_0}{e^{it_0} - e^{it_1}}, \quad N \in \mathbb{N},
\]
\[
(\phi^{m+1}_1 F)(e^{-iNt_1} - e^{-iNt}) = Ne^{-i(N+1)t_1}w_1 + (e^{-iNt_1} - e^{-iNt})\frac{w_0}{e^{it_0} - e^{it_1}}, \quad N \in \mathbb{N}.
\]

Therefore \( w_1 = 0 \), since the terms not containing \( w_1 \) are bounded in \( N \). Hence

\[
(\phi_1^{m+1} F)(e^{iNt}) = e^{iNt_1} \left( (\phi_1^{m+1} F)(1) - \frac{w_0}{e^{it_0} - e^{it_1}} \right) + e^{iNt_0} \frac{w_0}{e^{it_0} - e^{it_1}}, \quad N \in \mathbb{Z}.
\]

This yields that \((\phi_1^i F)(\varphi) = \varphi(t_0)w_0 + \varphi(t_1)w_1\), \( \varphi \in \mathbb{A} \), for some \( w_0, w_1 \) (which may not be the same as earlier \( w_0, w_1 \)). Repeating the sequence of arguments given above \( k \) times, we obtain

\[
F(\phi_0 \varphi) = \varphi(t_0)w_0 + \cdots + \varphi(t_k)w_k, \quad \varphi \in \mathbb{A},
\]

for some \( w_0, \ldots, w_k \) in \( H \). Setting \( \varphi = \frac{1}{\phi_0} \psi, \psi \in \mathbb{A} \), we obtain

\[
F(\psi) = \sum_{j=0}^{k} \psi(t_j) \frac{w_j}{\phi_0(t_j)}
\]

Therefore \( x_n = F(e^{-int}) = \sum_{j=0}^{k} e^{int_j} v_j \), with \( v_j = \frac{w_j}{\phi_0(t_j)}, \quad j = 1, \ldots, k \). The converse implication is trivial since

\[
\sum_{k=-\infty}^{+\infty} \phi_k \left( \sum_{j=0}^{k} e^{-int_j} e^{-inkt_j} v_j \right) = \sum_{j=0}^{k} e^{-int_j} \phi(t_j) v_j = 0. \quad \square
\]

4.9. LEMMA. Assume (4.6). Then the equation

\[
(4.10) \quad \sum \phi_k x_{n-k} = \sum \theta_k z_{n-k}, \quad (z_n, z_m) = \delta_{nm}
\]

has a bounded solution iff \( \frac{\theta}{\phi} \) is a bounded function on \((-\pi, \pi]\).

Proof. Since \( \tilde{\phi} \) is analytic in some neighborhood of \( T \), \( \phi(t) \) has finitely many zeros, say \( t_0, \ldots, t_k \). We examine the behavior of \( \frac{\theta}{\phi} \) on a neighborhood of each zero of \( \phi \). Specifically we show that \( \frac{\theta}{\phi} \) is bounded on each neighborhood. Since the proofs are the same for all points we will merely give the proof for \( t_0 \). Let \( z_0 = e^{it_0}, \tilde{\phi}(e^{it_0}) = 0 \). Then \( \tilde{\phi}(t) = (e^{it} - e^{it_0})^m \phi_0(t) \), where \( \phi_0(z) = \frac{\tilde{\phi}(z)}{(z - z_0)^m} \) is analytic, \( \tilde{\phi}_0(z_0) \neq 0 \), and \( m \) is the order of \( z_0 \). Let \( \theta(t) = a_0 + (e^{it} - e^{it_0})\theta_1(t) \) with \( \tilde{\theta}_1(z) - \frac{a_0}{z - z_0}, \quad a_0 = \tilde{\theta}(z_0) \). Suppose first that the equation (4.10) has a bounded solution \( (x_n) \). Let \( F \) be associated with \( (x_n) \) by formula (4.3). Then (4.10) can be written as

\[
(\phi_0 F)((e^{it} - e^{it_0})^m \varphi) = \int a_0 \varphi dZ + \int (e^{it} - e^{it_0})\theta_1 \varphi dZ,
\]
\( \varphi \in A \), where \( Z \) is the random spectral measure of the sequence \((z_n)\). First we prove that \( a_0 = 0 \). Let \( F_1(\psi) = ((e^{it} - e^{it_0})^{m-1} \phi_0) F(\psi), \) \( F_2(\psi) = \int \theta_1 \psi dZ, \) \( \psi \in A \). Then \( F_0 = F_1 - F_2 \) is a bounded linear operator from \( A \) to \( H \) which satisfies

\[
F_0((e^{it} - e^{it_0})\varphi) = a_0 \int \varphi dZ, \quad \varphi \in A.
\]

Setting

\[
\varphi_N = \frac{e^{iNt} - e^{it_0}}{e^{it} - e^{it_0}} = e^{i(N-1)t_0} \sum_{k=0}^{N-1} e^{itk} e^{it_0k}
\]

we obtain

\[
F_0(e^{iNt} - e^{it_0}) = a_0 \int \varphi_N dZ, \quad N \in \mathbb{N}.
\]

Since \( F_0 \) is bounded,

\[
||F_0(e^{iNt} - e^{it_0})|| \leq 2||F_0||,
\]

while on the other hand

\[
||a_0 \int \varphi_N dZ||^2 = |a_0|^2 \int \sum_{k=0}^{N-1} e^{itk}(e^{-it_0k})^2 dt = N|a_0|^2.
\]

Therefore \( a_0 = 0 \) and \( \theta(t) = (e^{it} - e^{it_0})\theta_1(t) \), where \( \tilde{\theta}_1(z) \) is analytic in the annulus \( 1 - \epsilon < |z| < 1 + \epsilon \). If \( m = 1 \) this shows that \( \frac{\theta}{\varphi} \) is bounded in a neighborhood of \( t_0 \). Assume that \( m > 1, \; 1 \leq k \leq m - 1 \), and suppose that \( \theta(t) = (e^{it} - e^{it_0})^k \theta_k(t) \), where \( \tilde{\theta}_k(z) \) is analytic in the annulus \( 1 - \epsilon < |z| < 1 + \epsilon \). We show that \( \theta_k(t) = (e^{it} - e^{it_0})\theta_{k+1}(t) \), where \( \tilde{\theta}_{k+1}(z) \) is analytic in \( 1 - \epsilon < |z| < 1 + \epsilon \). Write \( \theta_k(t) = a_k + (e^{it} - e^{it_0})\theta_{k+1}(t) \), where \( \tilde{\theta}_{k+1}(z) = \frac{\theta_k(z) - \tilde{\theta}_k(z_0)}{z - z_0} \). Then (4.10) can be written as

\[
F((e^{it} - e^{it_0})^m \phi_0 \varphi) = \int (e^{it} - e^{it_0})^k a_k \varphi dZ + \int (e^{it} - e^{it_0})^{k+1} \theta_{k+1} \varphi dZ, \quad \varphi \in A.
\]

Let \( F_0(\psi) = (\phi_0(e^{it} - e^{it_0})^{m-k-1} F(\psi) - \int \theta_{k+1} \psi dZ, \quad \psi \in A. \)

Then \( F_0 \) is bounded and

\[
F_0((e^{it} - e^{it_0})^{k+1} \varphi) = a_k \int (e^{it} - e^{it_0})^k \varphi dZ, \quad \varphi \in A.
\]

Setting \( \varphi = (\varphi_N)^{k+1} \), we obtain

\[
F_0((e^{iNt} - e^{iNt_0})^{k+1}) = a_k \int (e^{iNt} - e^{iNtt_0})^k \varphi_N dZ.
\]
Since $F_0$ is bounded, $\|F_0(e^{iNt} - e^{iNt_0})^k\| \leq C_k$ for all $N$, while

$$\|a_k \int (e^{iNt} - e^{iNt_0})^k (e^{i(N-1)t_0} \sum_{k=0}^{N-1} e^{itk} e^{-ikt_0}) dZ\|^2 =$$

$$\|a_k e^{i(N-1)t_0} \sum_{m=0}^{k} \sum_{r=0}^{N-1} e^{it(Nm+r)} \binom{k}{m} (-1)^{m-k} e^{i(Nk-nm-r)t_0} \|^2_{L^2} =$$

$$= |a_k|^2 N \sum_{m=0}^{k} \binom{k}{m}^2 \rightarrow +\infty.$$ 

Hence $a_k = 0$. By repetitions of this argument we obtain that $\theta(t) = (e^{it} - e^{it_0})^m \tilde{\theta}_m(t)$, where $\tilde{\theta}_m(z)$ is analytic on the annulus $1 - \varepsilon < |z| < 1 + \varepsilon$. Thus $\frac{\theta}{\phi}$ is bounded in a neighborhood of $t_0$. The converse implication is a trivial consequence of Thm. 2.14. \]

**4.11 Theorem.** If $\phi$ and $\theta$ satisfy assumptions (4.6) then the following conditions are equivalent:

1. the equation (4.2) has a bounded solution,
2. the equation (4.2) has a stationary solution,
3. $\frac{\theta}{\phi}$ is bounded.

Moreover, if (3) is satisfied then every bounded solution to (4.2) is harmonizable and is given by

$$x_n = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ + \sum_{j=0}^{k} e^{-int_j} v_j,$$

where $Z$ is the random spectral measure of $(z_n)$; $t_0, \ldots, t_k$ are zeros of $\phi(t)$ and $v_0, \ldots, v_k$ are arbitrary vectors in $H$.

**Proof.** The equivalence of (1), (2), (3) follows from Lem. 4.9 and Thm. 2.14. Let $x_n^0 = \int e^{-int} \frac{\theta(t)}{\phi(t)} dZ$, $n \in \mathbb{Z}$, and let $x_n$ be an arbitrary bounded solution to (4.2). Then by Lem. 4.8, $x_n^1 = x_n - x_n^0$ has the form

$$x_n^1 = \sum_{m=0}^{k} v_j e^{-int_j} = \int e^{-int} (\sum_{m=0}^{k} \delta_{t_j})(dt) v_j), \ n \in \mathbb{Z},$$

so it is harmonizable. Therefore $x_n^1$ is harmonizable and has the required form. \]

Since a classical ARMA equation is a special case of the system of equations considered in this paper, the results obtained in Thm. 4.11 and those of the earlier sections also hold for ARMA equations. We will not discuss in details the ramifications of our results to the ARMA case. We merely state just one result.
4.12. Theorem. Consider the ARMA equation

\[ \sum_{k=0}^{p} \phi_k x_{n-k} = \sum_{k=0}^{q} \theta_k z_{n-k}, \quad n \in \mathbb{Z}, \]

where \((z_n)\) is an orthonormal system and \(\phi_0, \theta_0, \phi_p, \theta_q\) are nonzero. The equation (4.13) has a bounded solution iff \(\frac{\theta}{\phi}\) is bounded. When this is so, every bounded solution has the form

\[ x_n = \sum_{k=-\infty}^{+\infty} \psi_k z_{n-k} + \sum_{k=1}^{N} e^{-int} v_k, \quad n \in \mathbb{Z}, \]

where \(\psi_k = \frac{1}{2\pi} \int e^{-ikt} \frac{\theta(t)}{\phi(t)} \, dt, k \in \mathbb{Z}\), \(v_1, \ldots, v_N\) are elements of \(H\) and \(t_1, \ldots, t_N\) are zeros of \(\phi(t)\).

4.15. Remark. The fact that when \(\frac{\theta}{\phi}\) is bounded, every bounded solution to (4.13) has the form (4.14) is well known and can be derived from the fact that in this case every solution to the ARMA equation is given by

\[ x_n = x_n^0 + \sum_{m=1}^{M} \left( \sum_{k=0}^{r_m-1} y_{mk} |n|^{k} \left( \frac{1}{a_m} \right)^n \right), \quad n \in \mathbb{Z} \]

where \(x_n^0 = \int e^{-int} \frac{\theta}{\phi} \, dZ = \sum_{k=-\infty}^{+\infty} \psi_k z_{n-k}, \quad n \in \mathbb{Z}\), \(\psi_k = \frac{1}{2\pi} \int e^{-ikt} \frac{\theta}{\phi} \, dt, \quad k \in \mathbb{Z}\), \(\alpha_1, \ldots, a_M\) are zeros of \(\phi(z) = \sum_{k=0}^{p} \phi_k z^k\) and \(r_1, \ldots, r_M\) their multiplicities (see [2]). However, even in this case, the fact that all solutions to (4.13) are unbounded, if \(\frac{\theta}{\phi}\) is not bounded, seems to be new.

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