EIGENVALUES OF ALMOST SKEW SYMMETRIC MATRICES AND TOURNAMENT MATRICES

By

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Eigenvalues of almost skew symmetric matrices and tournament matrices

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ABSTRACT

A real square matrix C is called almost skew symmetric if $C = S + A$ where S is a rank one real symmetric matrix and A is a real skew symmetric matrix. We shall show that the real eigenvalues of almost skew symmetric matrices satisfy remarkable inequalities. We shall apply these and other inequalities to estimate the spectral radii of the tournament matrices.

§1. Introduction

Let C be a real $n \times n$ matrix. Set

$$C = S + A,\ S = \frac{C + C^T}{2},\ A = \frac{C - C^T}{2}. \tag{1.1}$$

Thus, S and A are the symmetric and the skew (anti) symmetric components of C. We call C almost skew symmetric if S is a rank one matrix. In the terminology of [**M-P**] $C - \frac{I}{2}$ is a pseudo-tournament matrix. In §2 we discuss remarkable properties of the spectrum of almost skew symmetric matrices. Under certain conditions we estimate the largest real eigenvalue of an almost skew symmetric C from below and above when the nonzero eigenvalue of S is positive. Our estimate are quadratic in the entries of A. We also give conditions on A which ensure the existence of real eigenvalues for C in the case where the dimension of C is even. Our results are almost unrelated to the results discussed in [**M-P**]. A tournament matrix is a square $(0,1)$ matrix T such that $T + T^T = J - I$ where J is the all ones matrix. In §3 we note that a tournament matrix is an almost skew symmetric matrix minus $\frac{I}{2}$. We then apply our results to get some estimates on the spectral radius of tournament matrices. Our results can be viewed as a natural extension of the basic inequality due to Kirkland [**Kir**]. In §4 we bring a generalization of Ostrowski's inequality [**Ost**] for the spectral radius of nonnegative matrices which we apply to generalized tournament matrices. (A nonnegative matrix T is called a generalized tournament matrix if $T + T^T = J - I$.) In the last section we discuss the extremal tournament matrices of an even order n which maximize the spectral radius of the tournament matrices and the corresponding values of their spectral radii as a function of n. We were not able to prove or disprove the Brualdi - Li conjecture [**B-L**] on these extremal matrices. We did show that the Brualdi-Li matrix maximizes a function which is closely related to the spectral radius of a tournament matrix.

§2. Perturbation of a rank one symmetric matrix by a skew symmetric matrix

As usual, denote by $M_n(\mathbf{F})$, $S_n(\mathbf{F})$, $A_n(\mathbf{F})$ be the algebra of $n \times n$ matrices, the space of $n \times n$ symmetric matrices and the space of $n \times n$ skew symmetric matrices with the entries in a field $\mathbf{F}$. Let $C \in M_n(\mathbf{R})$ and assume the decomposition (1.1). Let $spec(C) = \{\lambda_1(C), ..., \lambda_n(C)\}$ be the spectrum of C. Then $spec(S), \sqrt{-1}spec(A) \subset \mathbf{R}$. Arrange the eigenvalues of S in the decreasing order:

$$\lambda_1(S) \geq \lambda_2(S) \geq ... \geq \lambda_n(S).$$

As $x^T C x = x^T S x,\ x \in \mathbf{R}^n$ it follows that

$$spec(C) \cap \mathbf{R} \subset [\lambda_n(S), \lambda_1(S)].$$

Thus, the real eigenvalues of C can be viewed as a perturbation of the eigenvalues of S. More precisely, consider the coefficients of the characteristic polynomial of C as polynomials in the entries of S and A. As

$$det(\lambda I - C) = det(\lambda I - (S + A)) = det(\lambda I - A^T) = det(\lambda I - (S - A))$$

we deduce that the coefficients of the characteristic polynomial of C involve only even degree mononomials in the entries of A. That is, the characteristic polynomial of C is a quadratic perturbation of the characteristic polynomial of S by the entries of A. The following theorem exhibits more precisely what happens to the spectrum of a real symmetric matrix S under the perturbation by a real skew symmetric matrix A:

Theorem 2.1. *Let*

$$C(z) = S + zA,\ S \in S_n(\mathbf{R}),\ A \in A_n(\mathbf{R}),\ z \in \mathbf{C}. \tag{2.2}$$

Then for a small enough disk $|z| < r$ the matrix $C(z)$ has n analytic eigenvalues $\lambda_1(z), ...,$ $\lambda_n(z)$ with corresponding n independent analytic eigenvectors. Let $\lambda \in spec(S)$ be an eigenvalue of multiplicity k. Assume that the eigenvalues $\lambda_i(z), ..., \lambda_{i+k-1}(z)$ correspond to λ, i.e. $\lambda_j(0) = \lambda,\ j = i, ..., i + k - 1$. Suppose furthermore that k is odd. Then at least one of the eigenvalues $\lambda_j(z),\ i \leq j \leq i + k - 1$ is real analytic. Furthermore, any real analytic eigenvalue $\lambda_l(z)$ is a function of z^2.

Proof. Let $z = \sqrt{-1}w$. Then $C(\sqrt{-1}w) = S + wA_1,\ A_1 = \sqrt{-1}A$. Thus, A_1 is a hermitian martrix. It then follows that $C(\sqrt{-1}w)$ is hermitian for real w. Rellich's theorem, e.g. [**Kat**], yields that for a small enough disk $|w| < r$ the matrix $C(\sqrt{-1}w)$

has n real analytic eigenvalues $\lambda_1(w), ..., \lambda_n(w)$ and corresponding n independent analytic eigenvectors. Replace $w = -\sqrt{-1}z$ to deduce the corresponding result for $C(z)$.

Assume that $\lambda_j(z)$ is real analytic. As $\lambda_j(\sqrt{-1}w)$ is real analytic we deduce that the power series of $\lambda_j(z)$ contain only even powers of z. Suppose finally that $\lambda \in spec(A)$ is of an odd multiplicity k. Assume that z is real. Then the eigenvalues of $C(z)$ are either real or come as conjugate complex pairs. As k is odd one of the eigenvalues $\lambda_i(z), ..., \lambda_{i+k-1}(z)$ is real analytic. $\diamond$

In the space $\mathbf{R}^n$ we consider the standard inner product $(x,y) = x^T y$ and the l_2 norm $\|x\| = (x,x)^{\frac{1}{2}}$. The following lemma is well known, e.g. [**G-V**, §9.1.1, p'483] but we bring its proof for completeness:

Lemma 2.3. *Let $A \in A_n(\mathbf{R})$. Assume that $e \in \mathbf{R}^n$, $\|e\| = 1$. Let $U = span\{e, Ae, ..., A^{k-1}e\} \subset \mathbf{R}^n$ be a cyclic subspace of A, that is $e, Ae, ..., A^{k-1}e$ are linearly independent and $A^k e \in U$. Assume that $k > 1$. Let $e^1 = e, ..., e^k$ be the orthonormal vectors obtained by the Gram-Schmidt process from the vectors $e, Ae, ..., A^{k-1}e$:*

$$e^1 = e,\ e^2 = \frac{Ae}{\|Ae\|},\ e^j = \frac{Ae^{j-1} - ((e^{j-2})^T Ae^{j-1})e^{j-2}}{\|Ae^{j-1} - ((e^{j-2})^T Ae^{j-1})e^{j-2}\|},\ j = 3, ..., k. \tag{2.4}$$

Then in the basis $e^1, ..., e^k$ the matrix A is represented by a skew symmetric $k \times k$ tridiagonal matrix $\hat{A} = (\hat{a}_{ij})_1^k$ whose lower nontrivial diagonal is given by:

$$\hat{a}_{21} = \alpha_1 = \|Ae\|,\ a_{(j+1)j} = \alpha_j = \|Ae^j + \alpha_{j-1}e^{j-1}\| = (e^{j+1})^T Ae^j, j = 2, ..., k-1. \tag{2.5}$$

Proof. As $x^T Ax = 0$ we deduce that Ae is orthogonal to e. Since $k > 1$ $Ae \neq 0$ hence $e_2 = \frac{Ae}{\|Ae\|}$. Assume by the induction hypothesis that (2.4) holds for $j = 1, ..., m-1$. Let e^m be given by (2.4). As $(e^{m-1})^T Ae^{m-1} = (e^{m-1})^T e^{m-2} = 0$ it follows that $(e^m)^T e^{m-1} = 0$. The formula for e^m yields that $(e^m)^T e^{m-2} = 0$. Consider $(e^m)^T e^j$ for $j < m-2$. By the induction hypothesis $(e^{m-2})^T e^j = 0$. As $Ae^j \in span\{e^1, ..., e^{j+1}\}$ we deduce that $(e^{m-1})^T Ae^j = 0$. Hence, $(e^m)^T e^j = 0$. This shows that $e^1, ..., e^k$ form an orthonormal basis of U which is constructed by the Gramm-Schmidt process. The formula (2.4) yields that A is represented by a tridiagonal matrix $\hat{A}$ in the orthonormal basis $e^1, ..., e^k$. As A is skew symmetric so is $\hat{A}$. The equalities (2.5) follow straight forward from (2.4). $\diamond$

Let $\bar{\mathbf{R}} = \mathbf{R} \cup \{-\infty\}$. We adopt the natural convention $-\infty < x$, $\forall x \in \mathbf{R}$. Let $x, y \in \bar{\mathbf{R}}$. We then let $x \prec y$ $(y \succ x)$ if either $x = y = -\infty$ or $y \in \mathbf{R}$ and $x < y$.

Theorem 2.6. *Let $S \in S_n(\mathbf{R})$, $A \in A_n(\mathbf{R})$. Assume furthermore that S is a rank one matrix with positive eigenvalue $\delta > 0$ and the corresponding eigenvector $Se = \delta e$, $\|e\| = 1$. If $Ae = 0$ then the $spec(S + A)$ is obtained from $spec(A)$ by removing one zero eigenvalue and replacing it by δ. Assume next that $Ae \neq 0$. Let $e^1 = e, ..., e^k$ be the orthonormal basis of the cyclic subspace of $U = span\{e, Ae, ..., A^{k-1}e\} \subset \mathbf{R}^n$ as in Lemma 2.3. Then in the basis $e^1, ..., e^k$ S is represented by the $k \times k$ diagonal matrix $D = (d_{ij})_1^k = diag\{\delta, 0, ..., 0\}$*

and A by the $k \times k$ tridiagonal skew symmetric matrix $\hat{A} = (\hat{a}_{ij})_1^k$ given by (2.5). Furthermore, any real eigenvalue of $D + \hat{A}$ is nonzero. Moreover, all nonzero real eigenvalues of $S + A$ are the eigenvalues of $D + \hat{A}$. Set:

$$q_l(\lambda) = det(\lambda I - (d_{ij} + \hat{a}_{ij})_1^l), \ l = 1, ..., k. \tag{2.7}$$

Denote by ρ_l the maximal real root of $q_l(\lambda)$ for $l = 1, ..., k$. If l is even and $q_l(\lambda)$ does not have a real root we then let $\rho_l = -\infty$. Then

$$\rho_i > \rho_j, \ j = i + 1, ..., k; \ i = 1, 3, \ \rho_j \succ \rho_2, \ j = 3, ..., k. \tag{2.8}$$

Suppose that $q_{2i-1}(\lambda)$, $i = 1, ..., \lfloor \frac{k+1}{2} \rfloor$ have exactly one simple real root ρ_{2i-1}, $i = 1, ..., \lfloor \frac{k+1}{2} \rfloor$. Then

$$\begin{aligned} &\rho_{2i-1} > \rho_{2i+1}, \ i = 1, ..., \lfloor \frac{k-1}{2} \rfloor, \\ &\rho_{2i} \prec \rho_{2i+2}, \ i = 1, ..., \lfloor \frac{k-2}{2} \rfloor, \\ &\rho_{2i-1} > \rho_{2j}, \ i = 1, ..., \lfloor \frac{k+1}{2} \rfloor, \ j = 1, ..., \lfloor \frac{k}{2} \rfloor. \end{aligned} \tag{2.9}$$

Proof. Let $U^\perp \subset \mathbf{R}^n$ be the orthogonal complement of U. As A is skew symmetric we deduce that $AU^\perp \subset U^\perp$. Clearly, $S span(e)^\perp = 0$. Hence $SU \subset U$, $SU^\perp = 0$ and

$$spec(S + A) = spec(S + A|_U) \cup spec(A|_{U^\perp}).$$

Assume first that $Ae = 0$. Then $U = span(e)$ and the above equality yield that the $spec(S + A)$ is obtained from the $spec(A)$ by replacing a zero eigenvalue of A by δ.

We now assume that $Ae \neq 0$, that is, $k = dim(U) > 1$. Suppose that $(S + A)x = 0$, $0 \neq x \in U$. As $0 = x^T(S + A)x = x^T S x$ we deduce that $Sx = 0$. Thus $Ax = 0$. As U is a cyclic subspace of A we deduce that $U = span(x)$ contrary to our assumption. Thus, the matrix $D + A$ does not have a zero eigenvalue. We now show the inequalities (2.8).

Let

$$p_l(\lambda) = det(\lambda I - (d_{ij} + \hat{a}_{ij})_{k-l+1}^k), \ l = 1, ..., k. \tag{2.10}$$

Set $p_0(\lambda) = q_0(\lambda) = 1$. The following recursive formulas are established in a straightforward manner:

$$\begin{aligned} &p_k(\lambda) = q_k(\lambda) = q_i(\lambda) p_{k-i}(\lambda) + q_{i-1}(\lambda) \alpha_i^2 p_{k-i-1}, \ i = 1, ..., k-1, \\ &p_i(\lambda) = \lambda p_{i-1}(\lambda) + \alpha_{k-i+1}^2 p_{i-2}, \ i = 2, ..., k-1. \end{aligned} \tag{2.11}$$

Note that $p_i(\lambda) > 0$, $\lambda > 0$, $i = 1, ..., k-1$. The formulas (2.11) yield

$$q_k(\lambda) = (\lambda - \delta) p_{k-1}(\lambda) + \alpha_1^2 p_{k-2}(\lambda).$$

Hence, $q_k(\lambda) \geq \alpha_1^2 p_{k-2}(\lambda) > 0$ for $\lambda \geq \delta (> 0)$. Therefore, $\rho_k < \rho_1 = \delta$. By considering the matrix $(d_{ij} + \hat{a}_{ij})_1^l$ we deduce that $\rho_l < \rho_1$, $l = 2, ..., k$. Consider ρ_2. If $\rho_2 = -\infty$ we trivially have that $\rho_j \succ \rho_2$, $j = 3, ..., k$. Assume now that $\rho_2 \in \mathbf{R}$. Our arguments yield that $0 < \rho_2 < \delta$. The first set of the equalities of (2.11) imply:

$$q_k(\lambda) = q_2(\lambda) p_{k-2}(\lambda) + (\lambda - \delta)\alpha_2^2 p_{k-3}(\lambda).$$

Thus $q_k(\rho_2) = (\delta - \rho_2)\alpha_2^2 p_{k-3}(\rho_2) < 0$. Hence, q_k has a real root and $\rho_k > \rho_2$. The arguments above show that each q_l, $l = 3, ..., k$ has a real root and $\rho_l > \rho_2$, $l = 3, ..., k$. Assume finally that all q_{2i-1} have simple real roots. We prove the inequalities (2.9) by the induction on k. Suppose (2.9) holds for $k = l$. Let $k = l + 1$. The first set of equalities of (2.11) yield that

$$q_{l+1} = \lambda q_l + \alpha_l^2 q_{l-1}. \tag{2.12}$$

Suppose first that l is even. Then the induction hypothesis yield that $\rho_l < \rho_{l-1}$. It then follows that $q_{l+1}(\lambda) > 0$ for $\lambda \geq \rho_{l-1}$. Hence, $\rho_{l+1} < \rho_{l-1}$. If $\rho_l = -\infty$ we obviously have the inequality $\rho_l < \rho_{l+1}$. If $0 < \rho_l$ then (2.12) yields $q_{l+1}(\rho_l) = \alpha_l^2 q_{l-1}(\rho_l) < 0$. The last inequality follows from the inequality $\rho_l < \rho_{l-1}$ and the assumption that ρ_{l-1} is the only real root of q_{l-1} which is also simple. Therefore $\rho_{l+1} > \rho_l$. Similar arguments apply in the case l is odd. ◇

The two roots of:

$$q_2(\lambda) = \lambda^2 - \delta\lambda + \alpha_1^2 \tag{2.13}$$

are given by

$$\tau_1 = \frac{\delta - \sqrt{\delta^2 - 4\alpha_1^2}}{2}, \ \tau_2 = \frac{\delta + \sqrt{\delta^2 - 4\alpha_1^2}}{2}. \tag{2.14}$$

Thus, if $\delta^2 \geq 4\alpha_1^2$ then $\rho_2 = \tau_2 \geq \frac{\delta}{2}$. Then Theorem 2.6 yields:

$$\rho_i > \rho_2 = \frac{\delta + \sqrt{\delta^2 - 4\alpha_1^2}}{2}, \ i = 3, ...k, \ \delta^2 \geq 4\alpha_1^2. \tag{2.15}$$

In particular, the matrix $S + A$ has a positive real eigenvalue which is not less than ρ_2. This last statement can be deduced from the recent inequality of [**Kir**] for hypertournament matrices. We adopt Kirkland's proof to state a more precise result about the spectrum of almost skew symmetric matrices in the case $\delta^2 \geq 4\alpha_1^2$.

Theorem 2.16. *Let $S \in S_n(\mathbf{R})$, $A \in A_n(\mathbf{R})$. Assume furthermore that S is a rank one matrix with positive eigenvalue $\delta > 0$ and the corresponding eigenvector $Se = \delta e$, $\|e\| = 1$. Set $\alpha_1 = \|Ae\|$. Assume that $\delta^2 \geq 4\alpha_1^2$. Then $S + A$ has one real eigenvalue λ satisfying the inequality:*

$$\lambda \geq \frac{\delta + \sqrt{\delta^2 - 4\alpha_1^2}}{2}. \tag{2.17}$$

Furthermore, all other $n - 1$ eigenvalues of $A + S$ satisfy the inequality:

$$\Re(\lambda) \leq \frac{\delta - \sqrt{\delta^2 - 4\alpha_1^2}}{2}. \tag{2.18}$$

Proof. Using a continuity argument it is enough to consider the case $\delta^2 > 4\alpha_1^2$. Let

$$H = S - \frac{\delta}{2}I,\ M_t = S + A - tI,\ P_t = M_t H + H M_t^T,\ \frac{\delta - \sqrt{\delta^2 - 4\alpha_1^2}}{2} < t < \frac{\delta + \sqrt{\delta^2 - 4\alpha_1^2}}{2}. \tag{2.19}$$

Then $P_t = N_t \oplus \frac{t\delta}{2} I_{n-2}$. A straightforward calculation shows that the 2×2 matrix N_t is positive definite for the values of t prescribed in (2.19). Thus, P_t is positive definite. The result of Ostrowski and Schneider [**O-S**] claims that M_t and H have the same inertia. That is, M_t has exactly one eigenvalue with positive real part and all other $n-1$ eigenvalues have negative real part. As M_t is a real matrix we deduce that the unique eigenvalue with a positive real part must be real. To deduce (2.17) let t tend to $\rho_2 = \tau_2$ from below and to deduce (2.18) let t tende to τ_1 from above. $\diamond$

Theorem 2.20 *For $\delta > 0,\ \alpha_1^2, \alpha_2^2 \geq 0$ Let $\rho_3(\alpha_1^2, \alpha_2^2)$ be the largest real root of*

$$q_3(\lambda) = \lambda^3 - \delta\lambda^2 + \lambda(\alpha_1^2 + \alpha_2^2) - \delta\alpha_2^2. \tag{2.21}$$

For a fixed value $\alpha_1^2 > 0,\ 4\alpha_1^2 \leq \delta^2$, ρ_3 is a strictly increasing function of α_2^2 such that

$$\rho(\alpha_1^2, 0) = \rho_2(\alpha_1^2) = \frac{\delta + \sqrt{\delta^2 - 4\alpha_1^2}}{2},\ \rho_2(\alpha_2^2, \infty) = \delta,\ \delta^2 \geq 4\alpha_1^2 > 0. \tag{2.22}$$

For a fixed value of $\alpha_2 > 0$ ρ_3 is a strictly decreasing function of α_1 on the interval $[0, \frac{\delta}{2}]$ such that $\rho_3(0, \alpha_2^2) = \delta$. For a fixed value $\alpha_2^2 \geq \frac{\delta^2}{3}$ ρ_3 is a strictly decreasing function of α_1^2 such that:

$$\rho_3(0, \alpha_2^2) = \delta,\ \rho_3(\infty, \alpha_2^2) = 0, \tag{2.23}$$

Assume finally that $\alpha_1^2, \alpha_2^2 > 0$ and $4\alpha_1^2 \leq \delta^2$. Then

$$0 < \rho_3 - \rho_2 < \frac{\alpha_2^2(\delta - \rho_2)}{(2\rho_2 - \delta)\rho_2 + \alpha_2^2}. \tag{2.24}$$

Let $S \in S_n(\mathbf{R}),\ A \in A_n(\mathbf{R})$. Assume furthermore that S is a rank one matrix with positive eigenvalue $\delta > 0$ and the corresponding eigenvector $Se = \delta e,\ \|e\| = 1$. Assume that $A^2 e \neq 0$. Set

$$\alpha_1^2 = \|Ae\|^2,\ \alpha_2^2 = \|\frac{A^2 e}{\alpha_1} + Ae\|^2 = \frac{\|A^2 e\|^2}{\|Ae\|^2} - \|Ae\|^2. \tag{2.25}$$

Then $S + A$ has exactly one positive simple eigenvalue λ satisfying

$$\rho_2(\alpha_1^2) < \lambda \leq \rho_3(\alpha_1^2, \alpha_2^2). \tag{2.26}$$

Proof. Using the definitions (2.13) and (2.2) we deduce in a straightforward manner the equality

$$q_3(\lambda) = \lambda q_2(\lambda) + \alpha_2^2(\lambda - \delta), \tag{2.27}$$

which is a special case of (2.12). Then ρ_3 is the maximal real solution of

$$\frac{\lambda q_2(\lambda)}{\delta - \lambda} = \alpha_2^2. \tag{2.28}$$

Note that $q_2(\lambda)$ is an increasing function of λ for $\lambda \geq \frac{\delta}{2}$. Also, for $\delta^2 \geq 4\alpha_1^2 > 0, \alpha_2^2 > 0$ the argument in the proof of Theorem 2.6 yields that

$$\delta > \rho_3(\alpha_1^2, \alpha_2^2) > \rho_2(\alpha_1^2) \geq \frac{\delta}{2},\ 0 < 4\alpha_1^2 \leq \delta^2,\ 0 < \alpha_2^2. \tag{2.29}$$

It then follows that for a fixed α_1, $\delta^2 \geq 4\alpha_1^2 > 0$ ρ_3 is an increasing function in $\alpha_2^2 > 0$. Thus, when $\alpha_2^2 = 0$ we obtain the first equality in (2.23). As $\rho_3 < \delta$ we deduce that if $\alpha_2^2 \to \infty$ then $\rho_3 \to \delta$. This proves the second equality of (2.23).

Fix the value of $\alpha_2 > 0$. Then for $\alpha_1 \in (0, \frac{\delta}{2})$ we know that $\rho_3 > \rho_2 > \frac{\delta}{2}$. Note that the equalities (2.27) and (2.13) directly yield that for a fixed $\lambda > 0$, q_3 is an increasing function of α_1^2. Fix the values of α_1^2, α_2^2. Then q_3 is a strictly increasing function of λ for $\lambda > \frac{\delta}{2}$. These arguments imply straightforward the $\rho_3(\alpha_1, \alpha_2)$ is a strictly decreasing function of $\alpha_1 \in (0, \frac{\delta}{2})$. Assume that $\alpha_2^2 \geq \frac{\delta^2}{3}$. A straightforward calculation shows that $q_3' \geq 0$ for all λ. The previous arguments yield that ρ_3 is a striclty decreasing function of $\alpha_1 > 0$. The equalities (2.23) are straightforward.

We now prove (2.24). Assume that $0 < \alpha_1 \leq \frac{\delta}{2}$, $0 < \alpha_2$. Then the inequality $\rho_2 < \rho_3$ is a consequence of (2.8). Clearly

$$q_2(\rho_3) = q_2'(\rho_2)(\rho_3 - \rho_2) + (\rho_3 - \rho_2)^2 > q_2'(\rho_2)(\rho_3 - \rho_2).$$

Since ρ_3 satisfies the equation (2.28) from the above inequality and the fact $q_2'(\rho_2) \geq 0$ we deduce:

$$\alpha_2^2 > \frac{\rho_3 q_2'(\rho_2)(\rho_3 - \rho_2)}{\delta - \rho_3} \geq \frac{\rho_2 q_2'(\rho_2)(\rho_3 - \rho_2)}{\delta - \rho_3}.$$

The inequality (2.24) is equivalent to the observation that α_2^2 is greater than the last term in the above inequality. $\diamond$

Theorem 2.30. *Let $S \in S_n(\mathbf{R})$, $A \in A_n(\mathbf{R})$. Assume furthermore that S is a rank one matrix with positive eigenvalue $\delta > 0$ and the corresponding eigenvector $Se = \delta e$, $\|e\| = 1$. Assume that that $Ae \neq 0$. Let*

$$e^1 = e, ..., e^k, D, \hat{A}, q_1, ..., q_k, \rho_1, ..., \rho_k$$

be defined as in Theorem 2.6. Assume that $\alpha_1 = \|Ae\| \leq \frac{\delta}{2}$. Then all $\rho_1, ..., \rho_k$ are positive numbers which satisfy the strict inequalities:

$$\begin{aligned} &\rho_{2i-1} > \rho_{2i+1},\ i = 1, ..., \lfloor\frac{k-1}{2}\rfloor, \\ &\rho_{2i} < \rho_{2i+2},\ i = 1, ..., \lfloor\frac{k-2}{2}\rfloor, \\ &\rho_{2i-1} > \rho_{2j},\ i = 1, ..., \lfloor\frac{k+1}{2}\rfloor,\ j = 1, ..., \lfloor\frac{k}{2}\rfloor. \end{aligned} \tag{2.31}$$

Proof. We prove (2.31) by the induction on k. For $k = 1, 2, 3$ we showed the validity of (2.31). Assume that (2.31) holds for $k = l \geq 3$. Let $k = l + 1$. Suppose first that l is even. The induction hypothesis yields that $\rho_l < \rho_{l-1}$. Use (2.12) to deduce that $q_{l+1}(\lambda) > 0$ for $\lambda \geq \rho_{l-1}$. Hence $\rho_{l+1} < \rho_{l-1}$. Combine Theorem 2.16, (2.12) and the induction hypothesis $\rho_{l-1} > \rho_l > \rho_2$ to deduce that $q_{l+1}(\rho_l) = \alpha_l^2 q_{l-1}(\rho_l) < 0$. Therefore $\rho_{l+1} > \rho_l$.Similar arguments apply in the case l is odd. $\diamond$

Note that under the assumptions of Theorem 2.30 ρ_k is the maximal real eigenvalue of the matrix $S + A$.

§3. Tournament matrices

A *tournament matrix* is a square $(0, 1)$ matrix T such that $T + T^T = J - I$ where J is the all ones matrix. These matrices have many interesting properties, see [**Moo**] and the references therein. The spectral properties of these matrices were studied by quite a few numbers of authors, see for example [**M-P**] and [**Kir**] and the references therein. We first note that any tournament matrix is an almost skew symmetric matrix minus $\frac{I}{2}$

$$T = C - \frac{I}{2},\ C = \frac{J}{2} + A,\ A = \frac{T - T^T}{2},\ J = uu^T,\ u = (1, ..., 1)^T. \tag{3.1}$$

To estimate the spectral radius $\rho(T)$ of T it is enough to estimate the maximal real eigenvalue of an almost skew symmetrtic matrix C. In the notation of Section 2

$$\delta = \frac{n}{2},\ \epsilon = \frac{u}{\sqrt{n}} = \frac{(1, ..., 1)^T}{\sqrt{n}}. \tag{3.2}$$

Set

$$b = (b_1, ..., b_n)^T = Tu - T^T u \in \mathbf{Z}^n. \tag{3.3}$$

That is b_i is the number of wins minus the number of losses of the player number i. We call b the balance vector. Use (3.1), (2.4) and (2.25) to deduce:

$$\alpha_1^2 = \frac{b^T b}{4n},\ \alpha_2^2 = -\frac{b^T A^2 b}{b^T b} - \frac{b^T b}{4n}. \tag{3.4}$$

We assume that $\alpha_1 = \alpha_2 = 0$ if $b = 0$. Theorems 2.6 and 2.30 yield:

Theorem 3.5 *Let T be an $n \times n$ tournament matrix. Assume* $(3.1)-(3.5)$. *Let $\rho_3(\alpha_1^2, \alpha_2^2)$ be the biggest real root of* (2.21). *Then*

$$\rho(T) \leq \rho_3(\alpha_1^2, \alpha_2^2) - \frac{1}{2}. \tag{3.6}$$

Suppose furthermore that $4b^Tb \le n^3$. *Let* $\rho_2(\alpha_1^2)$ *be the maximal root of* (2.13). *Then*

$$\rho(T) \ge \rho_2(\alpha_1^2) - \frac{1}{2} \tag{3.7}$$

and equality holds iff either $b = 0$ *or* $b \ne 0, \alpha_2 = 0$.

The inequality (3.7) is due to Kirkland [**Kir**]. He also gives the condition for equality stated in a different form.

The difficulty in using the inequality (3.6) is due to the fact that ρ_3 is a solution to the cubic equation. We now bring an upper bound on the spectral tournament matrices using Ostrowski's theorem. Let $B = (b_{ij})_1^n$ be a nonegative matrix. Set

$$r_i(B) = \sum_{j=1}^n b_{ij}, c_i(B) = r_i(B^T),\ \hat{r}_i(B) = r_i(B) - b_{ii},\ \hat{c}_i(B) = \hat{r}_i(B^T)\ i = 1, ..., n. \tag{3.8}$$

Ostrowski's theorem [**Ost**], see also [**Bru**] for generalization of Ostrowski's theorem, claims that *spec*(B) - the spectrum of B is contained in the following complex disks:

$$|z - b_{ii}| \le \hat{r}_i(B)^t \hat{c}_i(B)^{1-t},\ i = 1, ..., n$$

for any $0 \le t \le 1$. In particular:

$$\rho(B) \le \mu_t(B),\ \mu_t(B) = \max_{1 \le i \le n} b_{ii} + \hat{r}_i(B)^t \hat{c}_i(B)^{1-t},\ 0 \le t \le 1. \tag{3.9}$$

Corollary 3.10 *Let* T *be an* $n \times n$ *tournament matrix. Then*

$$\rho(T) \le \mu_{\frac{1}{2}}(T). \tag{3.11}$$

Assume that n is even. Let ρ_n^* be the maximal spectral radius of all $n \times n$ tournament matrices. By abusing the notation we let $\rho^* = \rho_n^*$. A conjecture due to Brualdi-Li [**B-L**] gives a "natural" candidade for the tournament matrix T^*, $\rho(T^*) = \rho^*$.

$$\begin{aligned} &Q = (Q_{ij})_1^2,\ Q_{11} = Q_{22} = V,\ Q_{12} = I + V^T,\ Q_{21} = V^T, \\ &V = (v_{ij})_1^{\frac{n}{2}},\ v_{ij} = 1\ for\ 1 \le i < j \le \frac{n}{2},\ v_{ij} = 0\ for\ 1 \le j \le i \le \frac{n}{2}. \end{aligned} \tag{3.12}$$

In particular, Q is almost regular. Recall that a tournament matrix T is called almost regular if $|r_i(T) - \frac{n-1}{2}| = \frac{1}{2},\ i = 1, ..., n$. A weaker version of Brualdi-Li conjecture is that any extremal matrix T^* is almost regular. We now study these conjectures.

Theorem 3.13 *Let* T *be a tournament matrix of an even order* n. *Then*

$$\rho(T) \le \sqrt{\frac{n}{2}(\frac{n}{2} - 1)}. \tag{3.14}$$

Assume furthermore that T satisfies the condition

$$|r_i(T) - \frac{n-1}{2}| > \frac{1}{2},\ i = 1, ..., n. \tag{3.15}$$

Then

$$\rho(T) \le \sqrt{(\frac{n}{2}+1)(\frac{n}{2}-2)}. \tag{3.16}$$

In particular, any tournament matrix which satisfies (3.15) *satisfies the inequality*

$$\rho(T) < \frac{n-1}{2} - \frac{1}{n+\sqrt{n^2-4}}. \tag{3.17}$$

Proof. Note that for any tournament matrix of an even order n we have the inequality $r_i(T)c_i(T) \le \frac{n}{2}(\frac{n}{2}-1)$. Hence, (3.11) yields (3.14). The assumption (3.15) implies that $r_i(T)c_i(T) \le (\frac{n}{2}+1)(\frac{n}{2}-2), i = 1, ..., n$. Thus, (3.11) implies (3.16). A straightforward calculation shows that the right-hand side of (3.16) is strictly less than the righ-hand side of (3.17). $\diamond$

The following lemma is due to Kirkland:

Lemma 3.18. *Let $T = (T_{ij})_1^{2m}$, $T_{ij} \in M_m(\mathbf{R})$, $i, j = 1, 2$ be an almost regular tournament matrix of an even order. Assume furthermore that the row sum of each of the first m rows of T is m and $T_{11} = T_{22} = V$ where V is given by* (3.12). *Then T is equal to the Brualdi-Li matrix Q given by* (3.12).

Proof. As T is an almost regular tourmanent matrix with $r_i(T) = m,\ i = 1, ..., m$, we deduce that $c_{m+i}(T) = 2m-1-r_{m+i}(T) = m,\ i = 1, ..., m$. Clearly, $r_i(V) = c_{m-i+1}(V) = m-i,\ i = 1, ..., m$. The equalities $r_i(T) = c_{m+i}(T) = m,\ i = 1, ..., m$, yield

$$r_i(T_{12}) = m - r_i(V) = i,\ c_i(T_{12}) = m - c_i(V) = m - i + 1,\ i = 1, ..., m. \tag{3.19}$$

We claim that the above equations have a unique $0-1$ matrix solution $T_{12} = I + V^T$. For $m = 1, 2$ the claim is trivial. Assume that the claim holds for $m \le k-1$. Let $m = k$. As T_{12} is a $0-1$ matrix it follows that the first column and the last row of T_{12} consists of all ones. The equality $r_1(T_{12}) = 1$ yields that the only nonzero entry on the first row of T_{12} is the entry $(1, 1)$. As $c_k(T_{12}) = 1$ we deduce that the only nonzero entry on the last column of T_{12} is the entry (k, k). Delete the first and the last rows and columns of T_{12} and use the induction hypothesis to show that the equalities (3.19) have a unique $0-1$ matrix solution $T_{12} = I + V^T$. As T is a tournament we deduce that $T_{21} = V^T$. $\diamond$

Theorem 3.20. *Let T be an almost regular tournament matrix of an even order n. Assume* (3.1) $-$ (3.4). *Then $\alpha_1 = \frac{1}{2}$. In particular, Kirkland's inequality holds* [**Kir**]:

$$\rho(T) \ge \rho_2(\frac{1}{4}) - \frac{1}{2} = \frac{n-1}{2} - \frac{1}{n+\sqrt{n^2-4}}. \tag{3.21}$$

Furthermore

$$\alpha_2^2 \le \frac{n^2-4}{12}. \tag{3.22}$$

Equality holds iff T is permutationally similar to the Brualdi-Li matrix given by (3.12). In particular, any almost regular tournament matrix T of an even order satisfies the inequality

$$\rho(T) \le \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2} \tag{3.23}$$

which implies

$$\rho(T) < \frac{n-1}{2} - \frac{3}{8n}. \tag{3.24}$$

Proof. Let $n = 2m$. Then the balance vector b has m entries equal to 1 and m entries equal to -1. Use the first part of (3.4) to deduce the equality $\alpha_1^2 = \frac{1}{4}$. According to Theorem 2.20 we have that $\rho(\frac{1}{4}, \alpha_2^2)$ is an increasing function function of α_2^2. (Here $\delta = \frac{n}{2}$.) Use (3.4) to deduce that the maximal value of α_2^2 is achieved for the almost regular tournament matrices for which $\|Ab\|$ is maximal. Since $Jb = 0$ we get:

$$\begin{aligned} \|Ab\|^2 = \|(A+\frac{J}{2})b\|^2 = \|(T+\frac{I}{2})b\|^2 = b^T T^T T b + \frac{b^T b}{4} + \frac{b^T(T+T^T)b}{2} = \\ b^T T^T T b + \frac{n}{4} + \frac{b^T(J-I)b}{2} = b^T T^T T b - \frac{n}{4}. \end{aligned} \tag{3.25}$$

Thus, it is enough to consider almost regular tournament matrices which maximize $\|Tb\|^2$. Rename the indices $1, ..., 2m$ so that the first m coordinates of b are equal to 1 and the last m coordinates of b are equal to -1. Let $T = (t_{ij})_1^n = (T_{ij})_1^2$, $T_{ij} \in M_m(\mathbf{R})$. Assume that s_i, t_i are the $i-th$ row sum of T_{11}, T_{22} respectively, for $i = 1, ...m$. It then follows that

$$b^T T^T T b = \sum_1^m (2s_i - m)^2 + (2t_i - (m-1))^2,$$

$$s_i, t_i \in \mathbf{Z},\ 0 \le s_i, t_i \le m-1,\ \sum_1^m s_i = \sum_1^m t_i = \frac{m(m-1)}{2}.$$

We next observe that if $x \le y$ are two integers then $x^2 + y^2 < (x-2)^2 + (y+2)^2$. We now maximize the expression

$$\sum_1^m (2s_i - m)^2 + (2t_i - (m-1))^2. \tag{3.26}$$

on all tournament matrices of an order $n = 2m$. Let $T = (t_{ij})_1^n$. Assume that $t_{ij} = 1$, $1 \le i \ne j \le m$. Then we may change the value of (3.26) by letting $t_{ij} = 0$. This is equivalent to replacing s_i, s_j by $s_i - 1$ and $s_j + 1$ and leaving all other s_k and t_k unchanged. Assume that T maximizes (3.26). We claim that after renaming the first m vertices if necessary we have the equality: $T_{11} = V$. In particular $s_i = m - i,\ i = 1, ..., m$. Assume that we renamed the indices so that $s_1 \geq s_2 \geq ... \geq s_m$. Suppose to the contrary that $s_m > 0$, that is, $t_{mj} = 1, 1 \leq j < m$. Then replace s_m, s_j by $s_m - 1, s_j + 1$ to condradict the optimality of T. Thus, the last row of T_{11} is a zero row and the first $m-1$ entries of the last column of T_{11} are equal to zero. It then follows that $s_i - 1,\ i = 1, ..., m-1$ are the $i-th$ row sum of $(t_{ij})_1^{m-1}$. It now follows that $s_{m-1} - 1 = 0$. Thus, the last row of the matrix $(t_{ij})_1^{m-1}$ is equal to zero. Continuing in the same manner we deduce that $T_{11} = V$. Assume that we have renamed the indices $m+1, ..., 2m$ so that $t_1 \geq ... \geq t_m$. Our arguments yield that $T_{22} = V$. In particular, Q maximize the expression (3.26).

Assume now that T is an almost regular matrix with $r_i(T) = m,\ i = 1, ..., m$, which maximizes (3.26). By renaming the indices in the sets $\{1, ..., m\}$ and $\{m+1, ..., 2m\}$ if necessary we deduce from the arguments above that $T_{11} = T_{22} = V$. Lemma 3.18 yields that $T = Q$.

In what follows we assume that T is an almost regular tournament matrix. Using the standard formula

$$\sum_1^k i^2 = \frac{(2k+1)(k+1)k}{6}$$

we deduce that

$$\|Tb\|^2 \leq \frac{(2m^2+1)m}{3}.$$

¿From (3.25), (3.4) and the above inequality we obtain the inequality (3.22) and the equality holds iff T is permutationally similar to the Brualdi-Li matrix. Theorem 2.20 yields the inequality (3.23). To deduce (3.24) let $r_n = \frac{n}{2} - \frac{3}{8n}$. A straightforward calculation shows that

$$\frac{r_n q_2(r_n)}{\delta - r_n} > \frac{n^2-4}{12},\ \delta = \frac{n}{2},\ \alpha_1^2 = \frac{1}{4}.$$

The arguments of the proof of Theorem 2.20 yield that

$$\rho_3(\frac{1}{4}, \frac{n^2-4}{12}) < \frac{n}{2} - \frac{3}{8n}. \tag{3.27}$$

§4. A generalization of Ostrowski's inequality and applications

In this section we let $h = (1, ..., 1)^T$.

Theorem 4.1 *Let B be an $n\times$ nonnegative irreducible matrix matrix. Set*

$$Bu = \rho(B)u,\ B^T v = \rho(B)v,$$
$$u = (u_1, ..., u_n)^T,\ v = (v_1, ..., v_n)^T,\ 0 < u_i, v_i,\ i = 1, ..., n,\ \sum_1^n u_i v_i = 1. \tag{4.2}$$

Let $0 \le s \le \frac{1}{n}$, $0 \le t \le 1$ *be given. Suppose that*

$$s \le u_i v_i,\ i = 1, ..., n. \tag{4.3}$$

Then

$$log\rho(B) \le s(\sum_1^n tlogr_i(B) + (1-t)logc_i(B)) + (1-ns)log(\max_{1\le i\le n} tlogr_i(B) + (1-t)logc_i(B)). \tag{4.4}$$

Assume that $0 < t < 1$ *and* B *is fully indecomposable. (*PBQ *is irreducible for any permutation matrices* P, Q*.) Then the equality sign holds iff* $\frac{B}{r_1(B)}$ *is doubly stochastic.*

Proof. The characterization of Friedland-Karlin [**F-K**] claims that

$$log\rho(B) = \min_{x=(x_1,\dots,x_n),0<x_i,i=1,\dots,n} \sum_1^n u_i v_i log \frac{(Bx)_i}{x_i}. \tag{4.5}$$

By chosing $x = h$ we deduce that

$$log\rho(B) \le \sum_1^n u_i v_i logr_i(B).$$

Replace B by B^T to obtain

$$log\rho(B) \le \sum_1^n u_i v_i logc_i(B).$$

Multiply the first inequality by t the second inequality by $1-t$ and use (4.3) to get (4.4). Suppose that B is fully indecomposable. Then the minimum in (4.5) is achieved only for the eigenvector $x = \alpha u, \alpha > 0$. (Contrary to the claim in [**F-K**], if B is irreducible but not fully indecomposable then the equality sign in (4.5) can be achieved for vectors x which may not be the eigenvector of B. See [**Fri**].) As $0 < t < 1$, the equality sign in (4.4) implies that h is the left and the right eigenvector of B . That is $\frac{B}{r_1(B)}$ is a doubly stochastic matrix. In that case we obviuosly have the equality sign in (4.4).$\diamond$

Remark 4.6 We view (4.4) as a generalization of (3.9). Indeed, for any nonnegative irreducible matrix B (4.4) yields the inequality

$$\rho(B) \le \nu_t(B),\ \nu_t(B) = \max_{1\le i\le n} r_i(B)^t c_i(B)^{1-t},\ 0 \le t \le 1. \tag{4.7}$$

Use a continuity argument to deduce that the above inequality holds for any nonegative B. It is not hard to show that (3.9) implies (4.7). Of course, in the case B has a zero diagonal (3.9) is equivalent to (4.7).

Remark 4.8 Clearly, if B is reducible we might have the equality sign in the inequality $\rho(B) \leq \nu_{\frac{1}{2}}(B)$ without the requirement that $\frac{A}{r_1(B)}$ is a doubly stochastic matrix. This also may be the case for an irreducible B which is not completely irreducible. Indeed, assume that C is a nonnegative cyclic matrix of even order - $n = 2m$. Suppose that all odd row sums of C and all even column sums of C are equal to $\alpha > 0$ and all even row sums and odd column row sums are equal to $\beta > 0$. As $C^n = \nu_{\frac{1}{2}}(B)^n I$ we deduce that we have the equality sign in $\rho(B) = \nu_{\frac{1}{2}}(B)$. Clearly, for $\alpha \neq \beta$ the matrix $\frac{B}{\alpha}$ is not doubly stochastic.

A generalized tournament matrix T is a nonnegative square matrix satisfying

$$T + T^T = J - I \qquad (4.9).$$

See [**M-P**]. We now apply (4.4) to give an upper bound for the spectral radius of generalized tournament matrices. According to (3.1) any generalized tournament matrix is an almost skew symmetric matrix minus $\frac{I}{2}$. Thus, we can use the inequalities (3.6) – (3.7) for $\rho(T)$. As we pointed out before there is an intrinsic difficulty in applying (3.6) as $\rho_3(\alpha_1^2, \alpha_2^2)$ is a solution of a cubic equation. In what follows we shall deduce upper bounds for $\rho(T)$ using (4.4) provided that $\rho(T)$ is sufficiently big. This condition can be verified by using (3.7).

Let T be a generalized tournament matrix and assume that

$$Tu = \rho(T)u, \ u = (u_1, ..., u_n)^T, \ u_i \geq 0, \ i = 1, ..., n, \ u^T u = 1, \ \alpha = \frac{\sum_1^n u_i}{n}. \qquad (4.10)$$

Combine (4.9) and (4.10) to deduce

$$T^T u = Ju - (\rho(T) + 1)u = n\alpha h - (\rho(T) + 1)u. \qquad (4.11)$$

Multiply (4.11) by u^T to deduce:

$$n^2\alpha^2 = 2\rho(T) + 1. \qquad (4.12)$$

It then follows

$$\epsilon = (\epsilon_1, ..., \epsilon_n)^T = u - \alpha h, \ \epsilon^T \epsilon = u^T u - 2u^T \alpha h + \alpha^2 h^T h = 1 - n\alpha^2 = 1 - \frac{2\rho(T) + 1}{n}. \qquad (4.13)$$

Theorem 4.14. *Let T be an $n \times n$ generalized tournament matrix. Assume that*

$$\rho(T) > \frac{n-2}{2} + \frac{1}{2(n+1)}. \qquad (4.15)$$

Then T is an irreducible matrix. Furthermore

$$log\rho(T) \leq \left(\frac{\sqrt{2\rho(T)+1}}{n} - \sqrt{1 - \frac{2\rho(T)+1}{n}}\right)^2 \frac{\sum_1^n log r_i(T) c_i(T)}{2} + \left(1 - n\left(\frac{\sqrt{2\rho(T)+1}}{n} - \sqrt{1 - \frac{2\rho(T)+1}{n}}\right)^2\right) log\nu_{\frac{1}{2}}(T). \qquad (4.16)$$

Proof. In view of (4.12) and (4.13) we deduce that (4.15) is equivalent to the inequality $\alpha^2 > \epsilon^T\epsilon$. By the definition $\alpha = \frac{u^T h}{n} > 0$. Hence, the inequality (4.15) yields that $0 < u_i,\ i = 1, ..., n$. Let

$$T^T w = \rho(T)w,\ w = (w_1, ..., w_n)^T,\ 0 \le w_i,\ i = 1, ..., n,\ w^T w = 1. \tag{4.17}$$

Using the arguments for the eigenvector u we deduce:

$$\alpha = \frac{w^T h}{n},\ \delta = (\delta_1, ..., \delta_n)^T = w - \alpha h,\ \delta^T\delta = 1 - \frac{2\rho(T)+1}{n}. \tag{4.18}$$

Thus, the assumption (4.15) yields that w is a vector with strictly positive coordinates. It is well known that if a nonnegative matrix with positive spectral radius has stricitly positive left and right eigenvectors then T is a direct sum of irreducible matrices. The assumption (4.9) implies that this sum reduces only to one component, i.e. T is irreducible. We next note that $0 < w^T u \le (w^T w u^T u)^{\frac{1}{2}} = 1$. Also, the vector v satisfying (4.2) is given by:

$$v = \frac{1}{w^T u} w. \tag{4.19}$$

Therefore

$$u_i v_i \ge (\alpha - (\epsilon^T\epsilon)^{\frac{1}{2}})(\alpha - (\delta^T\delta)^{\frac{1}{2}}) = (\frac{\sqrt{2\rho(T)+1}}{n} - \sqrt{1 - \frac{2\rho(T)+1}{n}})^2 = s,\ i = 1, ..., n. \tag{4.20}$$

Use (4.4) with $t = \frac{1}{2}$ and the above s to deduce (4.16). $\diamond$

Theorem 4.21 *Let T be an $n \times n$ generalized tournament matrix. Assume that*

$$\sqrt{2\rho(T)+1} = \sqrt{n}(1 - \frac{\tau^2}{8n}),\ 0 \le \tau \le 1. \tag{4.22}$$

Then

$$log\rho(T) \le (1-\tau)log(\prod_1^n r_i(T)c_i(T))^{\frac{1}{2n}} + \tau log\nu_{\frac{1}{2}}(T). \tag{4.23}$$

Proof. The condition (4.22) implies:

$$\frac{2\rho(T)+1}{n} = 1 - \frac{\tau^2}{4n} + \frac{\tau^4}{64n^2}. \tag{4.24}$$

Hence

$$\sqrt{1 - \frac{2\rho(T)+1}{n}} \le \frac{\tau}{2\sqrt{n}}. \tag{4.25}$$

Therefore:

$$(\frac{\sqrt{2\rho(T)+1}}{n} - \sqrt{1 - \frac{2\rho(T)+1}{n}})^2 \ge \frac{1}{n}(1 - \frac{\tau}{2} - \frac{\tau^2}{8n})^2 \ge \frac{1-\tau}{n}. \tag{4.26}$$

Use the above inequality in (4.16) to obtain (4.23). ◇

For an $n \times n$ generalized tournament matrix T let:

$$\bar{r}_i(T) = r_i(T) - \frac{n-1}{2},\ \bar{c}_i(T) = c_i(T) - \frac{n-1}{2} = -\bar{r}_i(T),$$
$$|\bar{r}_i(T)| = |\bar{c}_i(T)| \leq \frac{n-1}{2},\ i = 1, ..., n. \tag{4.27}$$

Then:

$$r_i(T)c_i(T) = (\frac{n-1}{2})^2 - \bar{r}_i(T)^2 = (\frac{n-1}{2})^2 - \bar{c}_i(T)^2. \tag{4.28}$$

Set:

$$\omega(T) = (\prod_1^n r_i(T)c_i(T))^{\frac{1}{2n}}. \tag{4.29}$$

Combine the standard inequality $log(1-x) \leq -x,\ x \leq 1$ with (4.28) to deduce

$$log\omega(T) \leq log\frac{n-1}{2} - \frac{2}{n(n-1)^2}\sum_1^n \bar{r}_i(T)^2. \tag{4.30}$$

Corollary 4.31. *Let the assumptions of Theorem* 4.21 *hold. Then*

$$log\frac{2\rho(T)}{n-1} \leq -(1-\tau)\frac{2}{n(n-1)^2}\sum_1^n \bar{r}_i(T)^2. \tag{4.31}$$

We stress again that the inequalities (4.16), (4.23) and (4.31) should be viewed as a refined version of (3.11). (Note that for generalized tournament matrices T $\mu_{\frac{1}{2}}(T) = \nu_{\frac{1}{2}}(T)$.) To apply these inequalities one should first use the lower bound (3.7) which in particular results in an upper bound for τ.

§5. Remarks and open problems

Inequalities (3.17) and (3.21) yield that a matrix T which satisfy (3.15) also satisfy $\rho(T) < \rho^*$, i.e. T is not maximal. The inequality (3.21) implies

$$\frac{n-1}{2} - \frac{1}{2n-1} < \rho_n^*. \tag{5.1}$$

The inequality (3.14) imply

$$\rho_n^* < \frac{n-1}{2} - \frac{1}{4n}. \tag{5.2}$$

Note that for almost regular matrices (3.24) gives a better upper bound than (5.2). Brualdi-Li conjecture implies in particular that the inequality (3.23) holds for all tournament matrices of an even order n. We conjecture that for any tournament matrix of an even order n we have the inequality:

$$\rho_3(\alpha_1^2, \alpha_2^2) \leq \rho_3(\frac{1}{4}, \frac{n^2-4}{12}). \tag{5.3}$$

Here α_1^2, α_2^2 are defined by (3.4). Furthermore, the equality sign holds iff T is permutationally similar to Brualdi-Li matrix.

The inequalities (5.1) – (5.2) imply that

$$\rho_n^* = \frac{n-1}{2} - \frac{K_n}{n}. \tag{5.4}$$

Roughly speaking $\frac{1}{4} \leq K_n \leq \frac{1}{2}$. Conjecture (5.3) yields the inequality $\frac{3}{8} \leq K_n$. It would be interesting to find the asymptotic distribution of the spectral radius of Q_n given by (3.12). A straightforward computation shows that

$$\rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2} = \frac{n-1}{2} - \frac{3}{8n} + O(\frac{1}{n^2}). \tag{5.5}$$

Thus, it is possible to estimate the asymptotics of $\rho(Q_n)$ by finding the corresponding value of α_3^2 and evaluating the asymptotics of $\rho_4(\alpha_1^2, \alpha_2^2, \alpha_3^2)$ for the matrix Q_n. We decided to skip this computation here. We conjecture that

$$lim_{n\to\infty} K_n = \frac{3}{8}. \tag{5.6}$$

However, $\rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2} > \rho(T)$ for any almost regular tournament matrix.

Lemma 5.7 *Let $n = 2m$ and assume that T is a generalized tournament matrix such that $r_i(T) = m,\ i = 1, ..., m,\ r_i(T) = m - 1,\ i = m + 1, ..., 2m$. Suppose furthermore that $T = (T_{ij})_1^2,\ T_{ij} \in M_m(\mathbf{R}),\ T_{11} = T_{22} = V$ where V is given by (3.12) . Then $\rho(T) \leq \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}$. Furthermore, $\rho(T) = \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}$ iff the following conditon hold:*

$$\begin{aligned}&(T - T^T)(w, -w)^T = \tau(v, -v)^T,\\ &v = (1, ..., 1), w = (w_1, ..., w_m) \in \mathbf{Z}^m, w_i = m + 1 - 2i,\ i = 1, ..., m,\ \tau = -\frac{2(m^2-1)}{3}.\end{aligned} \tag{5.8}$$

There is no almost regular tournament matrix which satisfies $\rho(T) = \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}$.

Proof. Let $2A = T - T^T,\ u = (v, v)^T$ and consider the cyclic subspace V generated by $u, Au, ..., A^{n-1}u$. As in the proof of Theorem (3.20), we deduce that for any generalized tournament matrix which satisfies the conditions of the theorem one has the the equalities $\alpha_1^2 = \frac{1}{4},\ \alpha_2^2 = \frac{n^2-4}{12}$. Theorem 2.30 implies the inequality $\rho(T) \leq \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}$ and the equality sign holds iff V is three dimensional. Clearly, $u, b = (v, -v)^T \in V$. It is straightforward to show that Tb and hence Ab in view of (3.25) is spanned by $u, b, x = (w, -w)^T$. Furthermore, $u^T w = b^T w = 0$. Thus V is three dimensional iff Ax is a linear combination of u, b, w. As $x^T Ax = u^T Ax = 0$ it follows that V is three dimensional iff $2Ax = \tau b$. Moreover, $\tau = -\frac{x^T 2Ab}{b^T b}$. A straighforward calculation shows that $\tau = -\frac{2(m^2-1)}{3}$.

Assume that T is an almost regular tournament matrix which satisfies $\rho(T) = \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}$. ¿From the proof of Theorem 2.6 we obtain that $spec(T) \supset spec(F - \frac{I}{2})$, $F \in A_{n-3}(\mathbf{R})$. As $n-3$ is odd we deduce that $0 \in spec(F)$. That is $-\frac{1}{2} \in spec(T)$. This is impossible, since all the eigenvalues of T are algebraic integers. $\diamond$

We close the paper by giving some numerical data. For $n = 4, 6, 8, 10, 12$ we computed the quantities

$$\rho_2(\frac{1}{4}) - \frac{1}{2} < \rho(Q_n) < \rho_3(\frac{1}{4}, \frac{n^2-4}{12}) - \frac{1}{2}, \; \delta = \frac{n}{2}$$

using MATLAB. The matrix Q_n is given by (3.12). Here are the results:

$$\begin{aligned}
n = 4: &\quad 1.3660, \; 1.3953, \; 1.3969; \\
n = 6: &\quad 2.4142, \; 2.4340, \; 2.4350; \\
n = 8: &\quad 3.4365, \; 3.4513, \; 3.4521; \\
n = 10: &\quad 4.4495, \; 4.4614, \; 4.4620; \\
n = 12: &\quad 5.4580, \; 5.4680, \; 5.4684.
\end{aligned}$$

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