# A NOTE ON NONABELIAN VORTICES

By

**Yisong Yang**

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# A NOTE ON NONABELIAN VORTICES

YISONG YANG†

**Abstract.** This note is a case study of a general $SU(2)$ gauge vortex model where the two Higgs multiplets are in the adjoint representation of the gauge group and the Higgs potential energy density contains cross interaction terms. The vacuum decay of finite energy solutions is proved and the existence of a nontrivial solution is established. Moreover, for a special choice of the parameter region, the occurrence of a nonlinear desingularization phenomenon is observed.

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**1. Introduction.** Gauge theories offer the greatest promise to describe the elementary forces in nature. The Weinberg-Salam model and quantum chromodynamics are the two existing gauge theories of real phenomenological importance. These theories can be formulated in terms of Feynman path integrals and hence, if everything were known about classical field configurations, then in principle all questions concerning the quantum theory could be answered. This belief motivates present active research in classical gauge theories and a rich spectrum of stable regular classical solutions have been found: instantons, monopoles, and vortices are among those solutions having relevant physical implications and elegant topological features.

Nonabelian vortices arise in spontaneously broken gauge theories in two dimensions and play an important role in grand unified theories, especially in the context of cosmology [10]. Unlike instantons and monopoles, such vortices cannot be found in closed forms in any parameter region and their properties may only be investigated by numerical simulations and mathematical analysis. Extensive studies have been carried out towards an understanding of abelian vortices [8, 9, 13, 14] but not much has been done for nonabelian vortices as far as mathematical rigorousness is concerned.

The additional difficulty in the study of nonabelian vortex models with the Higgs fields in the adjoint representation of the gauge group origins from the need of introducing at least two Higgs multiplets [4, 7, 11, 12, 15] to achieve a maximal breakdown of symmetry which means that the vacuum is invariant only under the unit matrix in the adjoint representation, and consequently, the most general Higgs potential energy density may contain cross interaction terms of the Higgs fields [4, 5, 12]. The main purpose of this note is to study the asymptotic behaviour of finite energy vortices in a general $SU(2)$ theory.

†Institute for Mathematics and its Applications, University of Minnesota, Room 514 Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455

**2. The nonabelian vortex model.** Assume that the gauge group $G = SU(2)$. As usual, denote by $\sigma^a (a = 1, 2, 3)$ the Pauli matrices. Then $T^a = \sigma^a/2i$ $(a = 1, 2, 3)$ are the generators of $SU(2)$ and the relation

$$[T^a, T^b] = \epsilon^{abc} T^c$$

holds. Here the structure constants $\epsilon^{abc}$ are totally antisymmetric with respect to interchange of indices and $\epsilon^{123} = 1$.

Consider a general $SU(2)$ vortex model defined on the $(2+1)$-dimensional Minkowski space-time by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu} F^{a\mu\nu}\right) + \frac{1}{2}(D_\mu \phi^a D^\mu \phi^a) + \frac{1}{2}(D_\mu \psi^a D^\mu \psi^a) - \lambda V(\phi, \psi) \tag{2.1}$$

where $\lambda > 0$ and the two Higgs fields $\phi, \psi$ are in the adjoint representation of $SU(2)$ with

$$A_\mu = A^a_\mu T^a \,, \quad \phi = \phi^a T^a \,, \quad \psi = \psi^a T^a,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu],$$

$$D_\mu \phi = \partial_\mu \phi + e[A_\mu, \phi],$$

$e$ is the fundamental charge, $V$ is the Higgs potential density:

$$\begin{aligned} V(\phi, \psi) = &-\frac{1}{2}\,\mu_1^2 |\phi|^2 - \frac{1}{2}\,\mu_2^2 |\psi|^2 + \frac{1}{4}\,\lambda_1 |\phi|^4 + \frac{1}{4}\,\lambda_2 |\psi|^4 \\ &+ \frac{1}{2}\,\beta(\phi^a \psi^a)^2 + \frac{1}{2}\,\gamma |\phi|^2\, |\psi|^2. \end{aligned} \tag{2.2}$$

In the expression (2.2), $|\phi|^2 = \phi^a \phi^a, |\psi|^2 = \psi^a \psi^a$, and the real parameters $\mu_i, \lambda_i, \beta, \gamma$ satisfy the inequality [4]

$$\gamma, \beta > 0, \quad \gamma + \beta < \min\left\{\lambda_1 \left(\frac{\mu_2}{\mu_1}\right)^2, \lambda_2 \left(\frac{\mu_1}{\mu_2}\right)^2\right\}, \lambda_1, \lambda_2 > 0, \tag{2.3}$$

in order to get the maximal symmetry breaking and to ensure that the energy is bounded from below.

The ground states (vacuum solutions) are determined through [4]:

$$\phi^a_0 \psi^a_0 = 0,$$

$$\overline{\phi}_0 \equiv |\phi_0| = \left(\frac{\mu_1^2 \lambda_2 - \mu_2^2 \gamma}{\lambda_1 \lambda_2 - \gamma^2}\right)^{\frac{1}{2}},$$

$$\overline{\psi}_0 \equiv |\psi_0| = \left(\frac{\mu_2^2 \lambda_1 - \mu_1^2 \gamma}{\lambda_1 \lambda_2 - \gamma^2}\right)^{\frac{1}{2}}.$$

Choose the temporal gauge $A_0 = 0$ and consider the static case. The energy density is given in the form:

$$\mathcal{E} = \frac{1}{4} F^a_{ij}F^a_{ij} + \frac{1}{2} D_i\phi^a D_i\phi^a + \frac{1}{2} D_i\psi^a D_i\psi^a + \lambda\widetilde{V}(\phi, \psi), \tag{2.4}$$

where

$$\widetilde{V} = V + \frac{1}{4} \mu_1^2\overline{\phi}_0^2 + \frac{1}{4} \mu_2^2\overline{\psi}_0^2.$$

The equations of motion of the energy (2.4) or the Lagrangian (2.1) are

$$\begin{cases} D_jD_j\phi = \lambda[(\lambda_1|\phi|^2 - \mu_1^2)\phi + \gamma|\psi|^2\phi + \beta(\phi^a\psi^a)\psi], \\ D_jD_j\phi = \lambda[(\lambda_2|\psi|^2 - \mu_2^2)\psi + \gamma|\phi|^2\psi + \beta(\phi^a\psi^a)\phi], \\ D_jF_{jk} = [\phi, D_k\phi] + [\psi, D_k\psi]. \end{cases} \tag{2.5}$$

In order to establish the asymptotic decay of finite energy solutions of Eqs. (2.5), we need the following well-known lemmas:

LEMMA 2.1. *For $p \geq 2$, there is an embedding $W^{1,2}(\mathbf{R}^2) \to L^p(\mathbf{R}^2)$.*

LEMMA 2.2. *For $u \in W^{1,p}(\mathbf{R}^n)$ $(p > n)$, we have $u(x) \to 0$ as $|x| \to \infty$.*

**3. Pointwise bounds of the Higgs fields.** Due to the complicated cross interaction of the Higgs fields in Eqs. (2.5), we are unable to prove that the magnitudes of $\phi, \psi$ are bounded from above by their vacuum expectation values $\overline{\phi}_0, \overline{\psi}_0$ at this moment. However, the following weaker result holds which is already sufficient for us to obtain the desired asymptotic decay of the finite energy solutions of the vortex model:

LEMMA 3.1. *For a finite energy solution of Eqs. (2.5), we have*

$$|\phi|^2 \leq \mu_1^2/\lambda_1, \quad |\psi|^2 \leq \mu_2^2/\lambda_2.$$

*Proof.* Let $\Omega \subset \mathbf{R}^2$ be an arbitrary bounded domain. Let $\chi$ be an $su(2)$-valued functions so that $\chi \in W_0^{1,2}(\Omega)$. Then, from Eq. (2.5a),

$$\int_\Omega dx \left\{ D_i\phi^a D_i\chi^a + \lambda[(\lambda_1|\phi|^2 - \mu_1^2)\phi^a\chi^a + \gamma|\psi|^2\phi^a\chi^a + \beta(\phi^a\psi^a)\psi^a\chi^a] \right\} = 0. \tag{3.1}$$

Suppose otherwise that

$$\Omega_\rho^+ = \{x \in \Omega_\rho | \ |\phi|^2 > \mu_1^2/\lambda_1\} \neq \emptyset \quad \text{for } \rho \geq \text{some } \rho_0 > 0,$$

where $\Omega_\rho = \{x \in \mathbf{R}^2 | \ |x| < \rho\}$.

Let $\eta \in C_0^\infty(\mathbf{R}^1)$ be a real-valued function with the properties

$$0 \le \eta \le 1, \quad \eta(s) = 1 \quad \text{for } |s| \le 1, \quad \eta(s) = 0 \quad \text{for } |s| \ge 2.$$

Set $\eta_\rho(x) = \eta(|x|/\rho), x \in \mathbf{R}^2$, and

$$\chi_\rho(x) = \eta_\rho(x)(|\phi(x)| - \overline{\phi}_0)^+ \frac{\phi(x)}{|\phi(x)|}$$

where for $\theta \in \mathbf{R}^1, \quad \theta^+ = \max\{\theta, 0\}$.

Define $f = \phi/|\phi|$ on $(\mathbf{R}^2)^+$. Then $|f| = 1$ and on $\Omega_{2\rho}^+$:

$$\begin{aligned} &D_j\chi_\rho = (\partial_j\eta_\rho)(|\phi| - \overline{\phi}_0)f + [(\partial_j|\phi|)f + (|\phi| - \overline{\phi}_0)D_jf]\eta_\rho, \\ &D_j\phi = (\partial_j|\phi|)f + |\phi|D_jf, \\ &(D_jf)^af^a = 0. \end{aligned}$$

Substituting these results into (3.1) we get, after a simple algebraic manipulation,

$$\begin{aligned} (3.2) \quad &\int_{\Omega_{2\rho}^+} dx \left\{ \eta_\rho|\nabla|\phi|\,|^2 + (\nabla|\phi| \cdot \nabla\eta_\rho)(|\phi| - \overline{\phi}_0) + \eta_\rho|\phi|(|\phi| - \overline{\phi}_0)(D_jf)^a(D_jf)^a + \right. \\ &\left. + \lambda\eta_\rho\left( |\phi|(\lambda_1|\phi|^2 - \mu_1^2)(|\phi| - \overline{\phi}_0) + \gamma|\psi|^2(|\phi| - \overline{\phi}_0)|\phi| + \beta(\phi^a\psi^a)^2\frac{1}{|\phi|}\,(|\phi| - \overline{\phi}_0)\right)\right\} = 0. \end{aligned}$$

Since $\overline{\phi}_0 < \mu_1^2/\lambda_1$, therefore $(|\phi| - \overline{\phi}_0) \le \phi_0^{-1}(|\phi|^2 - \overline{\phi}_0^2)$ on $\Omega_{2\rho}^+$. From the Schwarz inequality,

$$(3.3) \quad \left| \int_{\Omega_{2\rho}^+} (|\phi| - \overline{\phi}_0)\nabla|\phi| \cdot \nabla\eta_\rho dx \right| \le \overline{\phi}_0^{-1} \|(|\phi|^2 - \overline{\phi}_0^2)\|_{L^2(\Omega_{2\rho}^+)} \|(\nabla|\phi| \cdot \nabla\eta_\rho)\|_{L^2(\Omega_{2\rho}^+)}.$$

Consider the following quadratic form in $X, Y \in \mathbf{R}^1$:

$$Q(X,Y) = \left(\frac{1}{2}\lambda_1X^2 - \mu_1^2X + \frac{1}{2}\mu_1^2\overline{\phi}_0^2\right) + \left(\frac{1}{2}\lambda_2Y^2 - \mu_2^2Y + \frac{1}{2}\mu_2^2\overline{\psi}_0^2\right) + \gamma XY.$$

The condition (2.3) implies the inequality

$$(3.4) \qquad C_1[(X - \overline{\phi}_0^2)^2 + (Y - \overline{\psi}_0^2)^2] \le Q(X,Y) \le C_2[(X - \overline{\phi}_0^2)^2 + (Y - \overline{\psi}_0^2)^2]$$

where $C_1, C_2 > 0$ are constants.

It can be verified that

$$\phi^a \partial_j \phi^a = \phi^a (D_j \phi)^a. \tag{3.5}$$

As an immediate consequence of (3.5), the inequality

$$\big|\partial_j |\phi|\big| \leq |D_j \phi| \tag{3.6}$$

holds.

Finally, using (3.4) and (3.6) in (3.3) and substituting the resulting bound into (3.2) we get

(3.7)

$$\begin{aligned}\int_{\Omega^+_{2\rho}} dx \Big\{ & |\nabla |\phi|\,|^2 + |\phi|(|\phi| - \overline{\phi}_0)(D_j f)^a (D_j f)^a + \lambda \Big( |\phi|(\lambda_1 |\phi|^2 - \mu_1^2)(|\phi| - \overline{\phi}_0) \\ & + \gamma |\psi|^2 (|\phi| - \overline{\phi}_0)|\phi| + \beta(\phi^a \psi^a)^2 \frac{1}{|\phi|}(|\phi| - \overline{\phi}_0) \Big) \Big\} \eta_\rho \\ \leq & \frac{1}{\rho} C(E(A, \phi, \psi)),\end{aligned}$$

where $C$ is constant depending only on the total energy $E = \int \mathcal{E} dx$. On taking the limit $\rho \to \infty$ in (3.7) we find $(\mathbf{R}^2)^+ = \emptyset$. This contradiction proves $|\phi|^2 \leq \mu_1^2/\lambda_1$. Similarly, one shows $|\psi|^2 \leq \mu_2^2/\lambda_2$.

**4. Decay of $|\phi| - \overline{\phi}_0$, $|\psi| - \overline{\psi}_0$, and $\phi^a \psi^a$.** In order to simplify the notation, we set $g_l = D_l \phi$ , $h_l = D_l \psi$.

If $\chi$ is an $su(2)$-valued function, then

$$(D_j D_k - D_k D_j)\chi = [D_j, D_k]\chi = [F_{jk}, \chi]. \tag{4.1}$$

Using (2.5) and (4.1) we have

$$\begin{aligned} D_j D_j g_l &= [D_j F_{jl}, \phi] + 2[F_{jl}, g_j] + D_l D_j D_j \phi \\ &= \Big[[\phi, g_l], \phi\Big] + \Big[[\psi, h_l], \phi\Big] + 2[F_{jl}, g_j] \\ &\quad + (\lambda_1 |\phi|^2 - \mu_1^2) g_l + 2(\phi^a g_l^a)\phi + \gamma |\psi|^2 g_l \\ &\quad + 2\gamma(\psi^a h_l^a)\phi + \beta(\phi^a \psi^a) h_l + \beta(\psi^a g_l^a)\psi + \beta(\phi^a h_l^a)\psi. \end{aligned} \tag{4.2}$$

Let $\eta_\rho$ be the cutoff function introduced in Section 3.

Multiplying both sides of (4.2) by $\eta_\rho^2 g_l$ and integrating by parts, we obtain

$$\text{the left-hand-side} = \int_{\mathbf{R}^2} dx \Big\{ \eta_\rho^2 g_l^a (D_j D_j g_l)^a \Big\}$$

$$= -\int_{\mathbf{R}^2} dx \Big\{ \eta_\rho^2 (D_j g_l)^a (D_j g_l)^a \Big\} - \frac{2}{\rho} \int_{\mathbf{R}^2} dx \Big\{ (\partial_j \eta) g_l^a (\eta_\rho D_j g_l)^a \Big\},$$

$$|\text{the right-hand-side}| \leq C_1 + C_2 \int_{\mathbf{R}^2} dx \Big\{ |F_{jl}|\ |g_j|\ |g_l| \Big\} \eta_\rho^2.$$

where $C_1, C_2 > 0$ are constants depending on $E(A, \phi, \psi)$.

In virtue of the above and the inequalities

$$\frac{1}{2} \int_{\mathbf{R}^2} dx \Big\{ |F_{jl}|\ |g_j|\ |g_l|\ \eta_\rho^2 \Big\} \leq \int_{\mathbf{R}^2} dx \Big\{ |F_{12}|\ |g_1|^{\frac{1}{2}}\ |\eta_\rho g_2|^{\frac{3}{2}} \Big\}$$

$$\leq \|F_{12}\|_{L^2(\mathbf{R}^2)} \|g_1\|^{\frac{1}{2}}_{L^2(\mathbf{R}^2)} \|\eta_\rho g_2\|^{\frac{3}{2}}_{L^6(\mathbf{R}^2)},$$

$$\|\eta_\rho g_l\|_{L^6(\mathbf{R}^2)} \leq C \|\eta_\rho g_l\|_{W^{1,2}(\mathbf{R}^2)} \quad \text{(cf. Lemma 2.1)}$$

$$\leq C_1 \|\eta_\rho \nabla |g_l|\ \|_{L^2(\mathbf{R}^2)} + C_2 \|g_l\|_{L^2(\mathbf{R}^2)}$$

where $C_1, C_2 > 0$ are constants independent of $\rho \geq 1$, and substituting $g_l$ into (3.6), we find from a simple interpolation inequality:

$$\|\eta_\rho D_j g_l\|_{L^2(\mathbf{R}^2)} \leq C_3$$

where $C_3 > 0$ is a constant independent of $\rho \geq 1$. This proves $D_j g_l \in L^2(\mathbf{R}^2)$.

Applying (3.6) to $g_l$ again we reach $|g_l| \in W^{1,2}(\mathbf{R}^2)$. From Lemma 2.1, we conclude that $|g_l| \in L^p(\mathbf{R}^2)$ $(p \geq 2)$.

On the other hand, by (3.5) and Lemma 3.1, one finds

$$|\partial_j w| \leq 2\mu_1 |g_j| / \sqrt{\lambda_1} \tag{4.3}$$

where $w = |\phi|^2 - \overline{\phi}_0^2$. Hence $w \in W^{1,2}(\mathbf{R}^2)$ because $w \in L^2(\mathbf{R}^2)$ in virtue of (3.4) and the structure of $\widetilde{V}$. So $w \in L^p(\mathbf{R}^2)$ $(p \geq 2)$. But (4.3) says that $w \in W^{1,p}(\mathbf{R}^2)$ $(p \geq 2)$. This proves $w = |\phi|^2 - \overline{\phi}_0^2 \to 0$ as $|x| \to \infty$ (Lemma 2.2).

Similarly one shows that $|\psi|^2 - \overline{\psi}_0^2 \to 0$ as $|x| \to \infty$.

To show $(\phi^a \psi^a) \to 0$ as $|x| \to \infty$, we first notice that, since $\beta > 0$, $(\phi^a \psi^a) \in L^2(\mathbf{R}^2)$.

From the identity

$$\partial_l (\phi^a \psi^a) = (g_l^a \psi^a) + (\phi^a h_l^a), \tag{4.4}$$

we see that $\partial(\phi^a \psi^a) \in L^p(\mathbf{R}^2)$ $(p \geq 2)$ since we shown that $|g_l|$, $|h_l| \in L^p(\mathbf{R}^2)$ $(p \geq 2)$. In particular, $(\phi^a \psi^a) \in W^{1,2}(\mathbf{R}^2)$, hence $(\phi^a \psi^a) \in L^p(\mathbf{R}^2)$ $(p \geq 2)$. Finally, on applying Lemma 2.2, the desired decay of $(\phi^a \psi^a)$ follows.

**5. Decay of $F_{jk}, D_l\phi$, and $D_l\psi$.** Let $\chi_1, \chi_2$ be two $su(2)$-valued functions. Then it can be verified that

$$D_k[\chi_1, \chi_2] = [D_k\chi_1, \chi_2] + [\chi_1, D_k\chi_2]. \tag{5.1}$$

An application of (5.1) to (2.5c) gives:

$$D_1D_1F_{12} = [g_1, g_2] + [h_1, h_2] + [\phi, D_1g_2] + [\psi, D_1h_2]. \tag{5.2}$$

Multiply both sides of (5.2) by $\eta_\rho^2 F_{12}$. An integration by parts yields:

$$\|\eta_\rho D_1 F_{12}\|_{L^2(\mathbf{R}^2)} \le C,$$

where $C > 0$ depends on the quantities $\|F_{12}\|_{L^2(\mathbf{R}^2)}, \|g_j\|_{L^4(\mathbf{R}^2)}, \|h_j\|_{L^4(\mathbf{R}^2)}, \|D_kg_l\|_{L^2(\mathbf{R}^2)}$, and $\|D_kh_l\|_{L^2(\mathbf{R}^2)}$.

Letting $\rho \to \infty$ we get $D_1F_{12} \in L^2(\mathbf{R}^2)$.

Similarly, $D_2F_{12} \in L^2(\mathbf{R}^2)$.

Now, applying (3.6) to $F_{12}$ we reach $|F_{12}| \in W^{1,2}(\mathbf{R}^2)$. In particular, $F_{12} \in L^p(\mathbf{R}^2)$ $(p \ge 2)$.

On the other hand, in virtue of (5.2), $D_1D_1F_{12} \in L^2(\mathbf{R}^2)$. In a similar way, we find $D_2D_1F_{12} \in L^2(\mathbf{R}^2)$. Applying (3.6) again to $D_1F_{12}$ we get $|D_1F_{12}| \in W^{1,2}(\mathbf{R}^2)$ and hence $D_1F_{12} \in L^p(\mathbf{R}^2)(p \ge 2)$. By the same reason, $D_2F_{12} \in L^p(\mathbf{R}^2)$. Thus, (3.6) applied to $F_{12}$ implies $\nabla|F_{12}| \in L^p(\mathbf{R}^2)$. This proves $|F_{12}| \in W^{1,p}(\mathbf{R}^2)(p \ge 2)$. Using Lemma 2.2, the decay $F_{12} \to 0$ as $|x| \to \infty$ follows.

Our next goal is to show $D_kD_jg_l \in L^2(\mathbf{R}^2)$.

First assume that $\chi$ is a smooth $su(2)$-valued function with compact support. An integration by parts gives us:

$$\int_{\mathbf{R}^2} \left\{ (D_kD_k\chi)^a(D_jD_j\chi)^a \right\} dx = \int_{\mathbf{R}^2} \left\{ (D_kD_j\chi)^a(D_kD_j\chi)^a \right\} dx \tag{5.3}$$
$$+ \int_{\mathbf{R}^2} \left\{ [F_{jk}, \chi]^a(D_kD_j\chi)^a \right\} dx + \int_{\mathbf{R}^2} \left\{ (D_k\chi)^a[F_{jk}, D_j\chi]^a \right\} dx.$$

Define $\chi = \eta_\rho g_l$. Then

$$D_kD_k\chi = \eta_\rho(D_kD_kg_l) + (\nabla^2\eta_\rho)g_l + 2(\partial_k\eta_\rho)(D_kg_l). \tag{5.4}$$

On inserting (5.4) into (5.3), noting $F_{jk}, g_l \in L^p(\mathbf{R}^2)(p \ge 2), D_kg_l \in L^2(\mathbf{R}^2)$ (see Section 3), using (4.2) and the Hölder inequality, we obtain

$$\int_{\mathbf{R}^2} dx \left\{ (D_kD_j\eta_\rho g_l)^a(D_kD_j\eta_\rho g_l)^a \right\} \le C_1 + C_2 \int_{\mathbf{R}^2} \left\{ |D_k\eta_\rho g_l|\ |F_{jk}|\ |D_j\eta_\rho g_l| \right\} dx, \tag{5.5}$$

where $C_1, C_2 > 0$ are constants independent of $\rho \geq 1$.

In (5.5), if we use

$$D_k\eta_\rho g_l = (\partial_k\eta_\rho)g_l + \eta_\rho(D_k g_l),$$
$$D_k D_j\eta_\rho g_l = (\partial_k\partial_j\eta_\rho)(D_k g_l) + D_k\eta_\rho(D_j g_l),$$

then it becomes

$$\int_{\mathbf{R}^2} dx \left\{ [D_k(\eta_\rho D_j g_l)]^a [D_k(\eta_\rho D_j g_l)]^a \right\} \leq C_3 + C_4 \|D_1 g_l\|_{L^2(\mathbf{R}^2)} \|F_{12}\|_{L^4(\mathbf{R}^2)} \|\eta_\rho D_2 g_l\|_{L^4(\mathbf{R}^2)}, \tag{5.6}$$

with $C_3, C_4$ independent of $\rho \geq 1$.

From Lemma 2.1, one has, on using (3.6),

$$\begin{aligned} \|\eta_\rho D_2 g_l\|_{L^4(\mathbf{R}^2)} &\leq C\|\eta_\rho D_2 g_l\|_{W^{1,2}(\mathbf{R}^2)} \\ &\leq C_5 + C_6 \|(\nabla|\eta_\rho D_2 g_l|)\|_{L^2(\mathbf{R}^2)} \\ &\leq C_5 + C_6 \sum_{j=1}^{2} \|D_j(\eta_\rho D_2 g_l)\|_{L^2(\mathbf{R}^2)}. \end{aligned} \tag{5.7}$$

We may now substitute (5.7) into (5.6) to get

$$\sup_{j,k,l} \|D_k(\eta_\rho D_j g_l)\|_{L^2(\mathbf{R}^2)} \leq C_7$$

where $C_7$ is independent of $\rho \geq 1$. Therefore we have shown that $D_k D_j g_l \in L^2(\mathbf{R}^2)$.

Finally, applying (3.6) to $D_j g_l$ we reach $|D_j g_l| \in W^{1,2}(\mathbf{R}^2)$. So $D_j g_l \in L^p(\mathbf{R}^2)$ $(p \geq 2)$. This proves $g_l \in W^{1,p}(\mathbf{R}^2)(p \geq 2)$ and hence $g_l \to 0$ as $|x| \to \infty$.

By the same argument one can prove $h_l \to 0$ as $|x| \to \infty$.

**6. Existence of nontrivial solutions.** Since in our vortex model, the symmetry breaking is maximal, the residual symmetry group $H$ is the center of the gauge group $SU(2)$. In this case the homotopy group $\pi_1(SU(2)/H) = \pi_1(\mathbf{RP}^3) = \mathbf{Z}_2$ and therefore the model *may* possess one nontrivial topologically stable solution. In order to show the existence of such a solution, in this section we take the radial-ansätz [4]:

$$\begin{cases} \phi = \dfrac{f(\rho)}{\rho}(x^1T^1 + x^2T^2), \ \psi = \dfrac{g(\rho)}{\rho}(-x^2T^1 + x^1T^2), \\ A_j = \epsilon^{jk}x^k\dfrac{a(\rho)}{\rho^2}T^3, \quad A_0 = 0 \end{cases} \tag{6.1}$$

where $f(\rho)$, $g(\rho)$, $a(\rho)$ are real-valued functions, $\epsilon^{jk} = k - j$, $j, k = 1, 2$, and $\rho^2 = (x^1)^2 + (x^2)^2$. Hence Eqs. (2.5) are reduced to a system of ordinary differential equations:

$$(6.2) \qquad \begin{cases} f'' + \dfrac{1}{\rho} f' - \dfrac{1}{\rho^2} f(1+ea)^2 + \lambda(\mu_1^2 f - \lambda_1 f^3 - \gamma f g^2) = 0, \\ g'' + \dfrac{1}{\rho} g' - \dfrac{1}{\rho^2} g(1+ea)^2 + \lambda(\mu_2^2 g - \lambda_2 g^3 - \gamma g f^2) = 0, \ \rho > 0, \\ a'' - \dfrac{1}{\rho} a' - e(1+ea)(f^2+g^2) = 0. \end{cases}$$

The total energy is now given by:

$$(6.3) \quad E(a,f,g) = \pi \int_0^\infty d\rho \left\{ \frac{1}{\rho}(a')^2 + \rho(f')^2 + \rho(g')^2 + \frac{1}{\rho}(1+ea)^2(f^2+g^2) + \rho\lambda P(f,g) \right\}$$

where $P(f,g) = Q(f^2, g^2)$ and $Q$ is defined as in Section 3. It can be checked that (6.2) are the equations of motion of the energy (6.3). Consequently, using (3.4) and the method of [13, 16], one easily shows that Eqs. (6.2) have a finite energy solution $(a, f, g)$ satisfying:

$$\lim_{\rho \to 0} f(\rho) = \lim_{\rho \to 0} g(\rho) = \lim_{\rho \to 0} a(\rho) = 0,$$
$$\lim_{\rho \to \infty} f(\rho) = \overline{\phi}_0, \ \lim_{\rho \to \infty} g(\rho) = \overline{\psi}_0, \ \lim_{\rho \to \infty} a(\rho) = -e^{-1},$$

$a(\rho)$ is monotonically decreasing and $f(\rho), g(\rho) \geq 0$. Such a solution gives us a topologically nontrivial smooth solution of Eqs. (2.5) via the substitution (6.1). Moreover, the exponential decay estimates:

$$0 \leq 1 + ea(\rho) \leq C(\epsilon) \exp(-\sigma[1-\epsilon]\rho)$$

for arbitrary $\epsilon \in (0,1)$ can be established. Here $\sigma = e(\overline{\phi}_0^2 + \overline{\psi}_0^2)^{\frac{1}{2}}$ is the vector boson mass. Despite some effort, we were unable to get suitable exponential decay estimates for the Higgs fields.

Finally, let us make a remark concerning the behaviour of solutions in the $\lambda \to \infty$ limit.

In the abelian case, the static vortex model is recognized as the Ginzburg-Landau theory [6] for low temperature superconductivity. For large values of the parameter $\lambda$ the theory describes type II superconductivity which is characterized by the occurrence of Abrikosov's mixed states [1]. Although the magnetic field in a superconductor cannot be determined analytically due to the complexity of the nonlinear Ginzburg-Landau equations and in general numerical methods have to be used, the heuristic argument of Abrikosov shows that, if $\lambda$ is very large, the region in which the order parameter differs from the superconducting vacua is small compared with the distances at which the interaction between vortex lines takes place and the vortex centers will play the role of singularities in solutions and their detailed structure will not be of importance. In view of this, the magnetic field for large $\lambda$ can be determined approximately by the solutions of a linear elliptic differential equation with a singular nonhomogeneous source term which is a sum of the Dirac distributions concentrated at the centers of vortices. This interesting phenomenon is called the ***nonlinear desingularization*** [3] and has rigorously been justified recently under the standard radial-ansätz [2].

It may be observed that the above phenomenon occurs also in the nonabelian vortex model under discussion.

In fact, under the ansätz (6.1), the components of the field strength are:

$$F_{0\mu} = 0, \qquad F_{12} = -\frac{a'(\rho)}{\rho}\, T^3.$$

Define the "magnetic" field strength $B$ by

$$B = F^3_{12}.$$

From Eqs. (6.2), we have

$$\Delta B_\lambda - \sigma B_\lambda = T_\lambda$$

where

$$T_\lambda = -\frac{2e}{\rho}(1 + ea_\lambda)(f_\lambda f'_\lambda + g_\lambda g'_\lambda) + \frac{a'_\lambda}{\rho}\left(\sigma - e^2(f^2_\lambda + g^2_\lambda)\right),$$

and the subscript denotes the dependence of the fields on the coupling parameter $\lambda > 0$.

The vortex is centered at $\rho = 0$. It may be shown that, as $\lambda \to \infty$, $T_\lambda \to$ the singular source function $-(2\pi/e)\sigma\delta(x)$ and $B_\lambda \to$ the Green's function $G(x) : \Delta G - \sigma G = -(2\pi/e)\sigma\delta(x)$ ($G$ is bounded in $\mathbf{R}^2$) in a suitable sense as expected.

To see this, let us consider for simplicity the parameter range $\mu_1 = \mu_2$. The condition (2.3) allows us to take a further ansätz in (6.1):

$$g = \xi f \qquad \text{with } \xi = \left(\frac{\lambda_1 - \gamma}{\lambda_2 - \gamma}\right)^{\frac{1}{2}}.$$

Then, up to some constant factors, Eqs. (6.2) and the energy (6.3) are reduced to those of the abelian Higgs vortex model studied in [2]. Therefore $T_\lambda \to -(2\pi/e)\sigma\delta(x)$ in the sense of distribution (say) and $B_\lambda \to G(x)$ in $W^{1,p}(\mathbf{R}^2)(1 \le p < 2)$ as $\lambda \to \infty$.

It seems highly desirable to prove the occurrence of the nonlinear desingularization phenomenon for the more complicated case $\mu_1 \neq \mu_2$.

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