Curvature estimates and applications for steady and expanding Ricci solitons

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Abstract

The interplay between curvature, geometry and topology of a manifold has long been a main theme in Riemannian geometry. In this thesis, we attempt to reveal the geometry of Ricci soliton from the curvature point of view. Ricci soliton is of great significance since it is a self similar solution to the Ricci flow and appears in the singularity analysis of the flow. It is also a natural generalization of the Einstein manifold.

Firstly, we study the curvature decay of noncompact steady gradient Ricci soliton. One conjectural behavior of steady soliton is that if the curvature ever decays, then the decay rate is either linear or exponential. We shall focus on the steady soliton whose decay rate is faster than the former case. Under some conditions on the potential function, we refine an estimate by Munteanu-Sung-Wang and establish a sharp exponential curvature decay for steady gradient soliton with curvature decay rate faster than linear rate. In dimension four, we show that the curvature tensor is bounded from above by the scalar curvature if the scalar curvature goes to zero at infinity, improving a result by Cao-Cui.

Secondly, we continue our investigation on steady soliton with fast curvature decay in complex geometric setting. We classify Kähler steady gradient Ricci soliton with nonnegative Ricci curvature and integrable scalar curvature. It is shown that such a soliton is isometric to a quotient of a product of the cigar soliton and some Kähler Ricci flat manifold. As an application, we classify Kähler steady gradient Ricci soliton with nonnegative Ricci curvature and scalar curvature decay rate faster than linear rate.

Thirdly, we study the scalar curvature of noncompact expanding Ricci soliton. We derive a sharp lower bound for the scalar curvature of noncompact expanding gradient Ricci soliton provided that the scalar curvature is non-negative and the potential function is proper. We then give a sufficient condition for the scalar curvature of expanding gradient soliton being nonnegative. In parallel to our study on steady soliton with fast curvature decay, we prove that a three dimensional noncompact expanding gradient Ricci soliton with scalar curvature decaying faster than quadratic rate must be flat.

Contents

Chapter 1

Introduction

Ricci flow was introduced by Hamilton [56] to study three manifolds with positive Ricci curvature. Let M^n be an n dimensional smooth manifold. Ricci flow deforms the metric in the direction of its Ricci curvature:

$$\begin{aligned} \frac{\partial g(t)}{\partial t} &= -2\mathrm{Ric}(g(t)) \\ g(0) &= g, \end{aligned} \tag{1.1}$$

where $g(t)$ is a family of Riemannian metrics on M and $\mathrm{Ric}(g(t))$ is the Ricci curvature of the metric $g(t)$, $t \geq 0$. The flow (1.1) is a second order nonlinear parabolic system. Like the heat flow which tends to smooth out the initial datum, Ricci flow perturbs the initial metric g to a nicer and more informative metric $g(t)$ while keeping the topology and smooth structure of the manifold M unchanged. In other words, it helps to extract different topological information by constructing special and restrictive metric on the manifold.

Definition 1. Let (M^n, g) be a Riemannian manifold and X be a smooth vector field on M, the triple (M, g, X) is said to be a Ricci soliton if there is a constant λ such that the following equation is satisfied

$$\mathrm{Ric} + \frac{1}{2} L_X g = \lambda g, \tag{1.2}$$

where Ric and L_X denote the Ricci curvature and Lie derivative with respect to X

respectively. A Ricci soliton is called shrinking (steady, expanding) if $\lambda > 0$ $(= 0, < 0)$. The soliton is said to be complete if (M, g) is complete as a Riemannian manifold. It is said to be gradient if X can be chosen such that $X = \nabla f$ for some smooth function f on M. In this case, (1.2) can be written as

$$\mathrm{Ric} + \nabla^2 f = \lambda g. \tag{1.3}$$

To motivate the definition of Ricci soliton, we consider some special solutions to the Ricci flow. A smooth family of Riemannian metrics $g(t)$ on M is said to be a self similar solution to the Ricci flow (1.1) if it satisfies (1.1) and can be written as

$$g(t) = \sigma(t)\varphi(t)^* g, \tag{1.4}$$

for some t independent metric g, some positive function $\sigma(t)$ and a one parameter family of diffeomorphisms $\varphi(t)$ of M. It is natural to ask which metric g supports a self similar solution to the Ricci flow, i.e. there exist function $\sigma(t)$ and diffeomorphisms $\varphi(t)$ making $g(t)$ defined by (1.4) a solution to the flow (1.1). We shall see that such a metric g satisfies a Ricci soliton equation. Indeed, given any self similar solution to the Ricci flow $g(t)$, we take the derivative of $g(t)$ with respect to t and get

$$\begin{aligned}
\frac{\partial g(t)}{\partial t} =& -2\mathrm{Ric}(g(t)) \\
=& -2\mathrm{Ric}(\sigma(t)\varphi(t)^* g) \\
=& -2\mathrm{Ric}(\varphi(t)^* g) \\
=& \varphi(t)^*(-2\mathrm{Ric}(g)) \\
=& \sigma'(t)\varphi(t)^* g + \sigma(t)\varphi(t)^*(L_{X(t)} g),
\end{aligned} \tag{1.5}$$

where $X(t) := \frac{\partial}{\partial s}|_{s=t}\Big(\varphi(s) \circ \varphi(t)^{-1}\Big)$ is a t dependent vector field and $L_{X(t)}$ is the Lie derivative. We also used the scaling invariance $\mathrm{Ric}(cg) = \mathrm{Ric}(g)$ and diffeomorphism invariance $\mathrm{Ric}(\varphi^* g) = \varphi^* \mathrm{Ric}(g)$ of the Ricci tensor in the above equations. It implies that

$$\varphi(t)^*\{2\mathrm{Ric}(g) + \sigma(t) L_{X(t)} g + \sigma'(t) g\} = 0.$$

Since $\varphi(t)$ are diffeomorphisms, we may simply write

$$\mathrm{Ric}(g) + \frac{1}{2} L_{\sigma(t)X(t)} g = -\frac{\sigma'(t)}{2} g.$$

Hence fix each t, we have the metric g in (1.4) must satisfy a Ricci soliton equation. It motivates the definition of Ricci soliton.

Conversely given a Ricci soliton, we consider $\sigma(t) := 1 - 2\lambda t$ and $\varphi(t)$ be the solution of the following non-autonomous ODE

$$\begin{aligned} \frac{d\varphi(t)}{dt} =& \frac{X(\varphi(t))}{\sigma(t)} \\ \varphi(0) =& \ \mathrm{id}\ , \end{aligned} \tag{1.6}$$

provided that the solution exists (it holds if the soliton is gradient, see [99] and Theorem 6). Then $g(t) = (1 - 2\lambda t)\varphi(t)^* g$ is a self similar solution to the Ricci flow (1.1).

We shall abbreviate shrinking, steady and expanding solitons as shrinker, steadier and expander respectively. Hereinafter, Rm and S denote the curvature tensor and the scalar curvature of the Riemannian manifold (M, g) respectively. Fix a point p_0 in M, for any x in M, r, $r(x)$ and $d(x, p_0)$ will be used interchangeably and refer to the distance between x and p_0. $B_R(p)$ is the geodesic ball centered at p with radius R. For any smooth function ω and tensor T, we define the weighted Laplacian of T w.r.t. ω by

$$\Delta_\omega T := \Delta T - \nabla_{\nabla \omega} T,$$

where Δ is the Laplace-Beltrami operator. We also adopt the Einstein summation convention in this thesis, i.e. any repeated indices are interpreted as summation over that indices.

The Ricci flow has been one of the most important flows in geometric analysis. Due to its nonlinear nature, the flow may develop different types of singularities. A gradient Ricci soliton, as a self similar solution to the Ricci flow, often arises as a rescaled limit of the flow near its singularities (see [18],[50], [1] and ref. therein). The study of Ricci soliton is therefore crucial for understanding the singularity formation of the Ricci flow [59]. Indeed, it played a significant role in the resolution of Poincaré conjecture by Perelman [87], [88] and [89]. Please consult the excellent survey paper [18] by Cao for

more details in this direction.

Ricci soliton can also be viewed as a generalization of the Einstein metric $\mathrm{Ric} = \lambda g$. Let (M, g) be a Riemannian manifold and f be any smooth function on M. The triple $(M, g, e^{-f} dv_g)$ is called a smooth metric measure space, where dv_g is the volume element induced by the metric g. Bakry and Emery first introduced the Bakry Emery Ricci curvature $\mathrm{Ric}_f := \mathrm{Ric} + \nabla^2 f$ in [3]. The Bakry Emery curvature is closely related to the diffusion processes, logarithmic Sobolev inequalities and isoperimetric inequalities (see [73], [23] and [96]). It is one of the most fundamental geometric quantities in the theory of smooth metric measure spaces. Like the Ricci curvature, the Bakry Emery curvature satisfies a Bochner formula

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f(\nabla u, \nabla u), \tag{1.7}$$

for any smooth function u. Therefore the Bakry Emery Ricci curvature also plays an important role in the analysis of harmonic functions, spectral geometry and connectedness at infinity of the smooth metric measure spaces (see [77], [78] and [79]). Since $L_{\nabla f} g = 2\nabla^2 f$, the gradient Ricci solitons equation (1.3) can be rewritten as

$$\mathrm{Ric}_f = \lambda g, \tag{1.8}$$

which is a natural extension of the Einstein metric.

1.1 Examples of Ricci solitons

In this section, we try to list a number of Ricci solitons in the literature, we follow closely the survey paper by Cao [18], the monographs [38] and [36]. Since the construction of Ricci soliton is an active research area in differential geometry, I apologize that the list here is far from being exhaustive and some recent examples are not included.

1.1.1 Shrinking Ricci solitons

Example 1. Gaussian shrinking soliton: the flat metric on $\mathbb{R}^n$ together with the potential function f given by $\frac{\lambda}{2}|x|^2$, where $\lambda > 0$ give a shrinking soliton structure on the Euclidean space. By choosing λ to be 0 or negative, we also have steady and expanding

solitons structure respectively.

Example 2. Einstein manifolds: quotients of spheres $\mathbb{S}^n/\Gamma$, product of two spheres of same dimension $\mathbb{S}^n \times \mathbb{S}^n$ and the complex projective spaces $\mathbb{CP}^n$ are examples of Einstein manifolds with positive Ricci curvature. By choosing f to be constant function, they are endorsed with shrinking soliton structure.

Example 3. Cylinders: $\mathbb{R}^{n-k} \times \mathbb{S}^k, k \geq 2$. Let $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{S}^k$, the potential function is given by $f(x, y) := \frac{k-1}{2}|x|^2$ and $\lambda = k - 1$, note that these examples have positive constant scalar curvature.

Example 4. Non trivial compact shrinkers: non-Einstein compact gradient Kähler Ricci shrinker on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ with positive Ricci curvature was constructed independently by Koiso [65] and Cao [15]. Later Wang-Zhu [95] constructed non-Einstein compact gradient Kähler Ricci shrinker on $\mathbb{CP}^2\#2(-\mathbb{CP}^2)$.

Example 5. F.I.K. shrinker: An $U(n)$ invariant complete shrinking Kähler gradient Ricci soliton on twisted complex line bundle over $\mathbb{CP}^{n-1}, n \geq 2$ was constructed by Feldman, Ilmanen and Knopf [52]. The Ricci curvature of the manifold is negative in some directions. Following the approach by Cao [15], they first constructed an incomplete $U(n)$ invariant shrinking soliton metric on $\mathbb{C}^n\backslash\{0\}$ and hence on $\mathbb{C}^n\backslash\{0\}/\mathbb{Z}_k, 0 < k < n$, then a complete soliton metric was obtained by gluing a $\mathbb{CP}^{n-1}$ to $\mathbb{C}^n\backslash\{0\}/\mathbb{Z}_k$ at the origin.

1.1.2 Steady Ricci solitons

Example 6. Einstein manifolds: n dimensional torus $\mathbb{T}^n$ and the Eguchi-Hanson metric on the cotangent bundle of $\mathbb{S}^2$ (see [14] and [51]). By choosing f as constant, these manifolds become steady gradient Ricci solitons.

Example 7. Hamilton's cigar soliton Σ: Hamilton [58] constructed a two dimensional complete gradient steady soliton on $\mathbb{R}^2$. It is rotationally symmetric with positive sectional curvature. In the standard coordinate of $\mathbb{R}^2$, its metric can be written as follows (see [18])

$$g_\Sigma = \frac{4(dx^2 + dy^2)}{1 + x^2 + y^2}.$$

The potential function f is given by $f(x,y) = -\log(1+x^2+y^2)$. The scalar curvature S decays like e^{-r} as $r \to \infty$.

Example 8. Bryant soliton: Bryant [12] constructed an n dimensional complete gradient steady soliton on $\mathbb{R}^n$, where $n \geq 3$. It is rotationally symmetric with positive sectional curvature. The scalar curvature S decays like r^{-1} as $r \to \infty$.

Example 9. Cao steady soliton on $\mathbb{C}^n$: Cao [15] constructed a complete $U(n)$ invariant steady gradient Kähler Ricci soliton with positive sectional curvature on $\mathbb{C}^n$, $n \geq 2$. It has positive sectional curvature and its scalar curvature S decays like r^{-1} as $r \to \infty$.

Example 10. Ivey soliton: Ivey [62] generalized Bryant's method to construct Ricci steadier using doubly warped products. Given any Einstein manifold (E, g_E) with positive Ricci curvature, there is a complete gradient steadier on $\mathbb{R}^n \times E, n \geq 2$ and the metric can be written as

$$g = dr^2 + w_1^2(r) g_{\mathbb{S}^{n-1}} + w_2^2(r) g_E$$

for some functions $w_i(r)$, $i = 1, 2$. Unlike Bryant soliton, this soliton may have nontrivial topology and its sectional curvature may change sign.

Example 11. Buzano-Dancer-Wang soliton: Buzano, Dancer and Wang [13] modified the construction in [41] and gave non-Kähler steady gradient Ricci soliton with non negative Ricci curvature on $\mathbb{R}^2 \times E_1 \times \cdots \times E_k$, where E_i are Einstein manifolds with positive Ricci curvature.

1.1.3 Expanding Ricci solitons

Example 12. Einstein manifolds: quotients of the hyperbolic spaces $\mathbb{H}^n/\Gamma$.

Example 13. Cylinders: $\mathbb{R}^{n-k} \times \mathbb{H}^k$. Let $(x,y) \in \mathbb{R}^{n-k} \times \mathbb{H}^k$, the potential function is given by $f(x,y) := \frac{-(k-1)}{2}|x|^2$ and $\lambda = -(k-1)$, note that these examples have negative constant scalar curvature.

Example 14. Bryant expanding soliton [12]: Same ODE method as in Example 8 was adopted to construct complete rotationally symmetric expanding gradient soliton with positive sectional curvature on $\mathbb{R}^n$, $n \geq 3$. Moreover the sectional curvature decays like

r^{-2} near infinity. Similar argument also gave negatively curved rotationally symmetric gradient expander on $\mathbb{R}^n$, where $n \geq 3$.

Example 15. Cao expanding soliton on $\mathbb{C}^n$: Using some ODE argument, Cao [16] gave a one parameter family of complete $U(n)$ invariant expanding Kähler gradient Ricci solitons on $\mathbb{C}^n$, where $n \geq 1$. The sectional curvatures of these solitons have signs, either positive or negative, depending on the parameter. When $n = 1$, the curvature decays exponentially in r^2 near infinity. For $n \geq 2$, the scalar curvature decays like r^{-2} near infinity.

Example 16. Nil: Baird and Danielo [2] constructed a 3 dimensional non-gradient expander with constant scalar curvature $-\frac{1}{2}$ on the 3 dimensional Heisenberg group (set of all 3×3 upper triangular matrices with all diagonal entries are 1). It is diffeomorphic to $\mathbb{R}^3$ and in the standard coordinate, the metric g is given by

$$g = g_e + x^2 dy \otimes dy + x dy \otimes dz + x dz \otimes dy,$$

where g_e is the Euclidean metric on $\mathbb{R}^3$. The soliton vector field X is given by

$$X = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2z\frac{\partial}{\partial z}.$$

They give the soliton structure with $\lambda = -\frac{3}{2}$. Its Ricci curvature has both positive and negative eigenvalues.

Example 17. F.I.K. expander: A similar method as in Example 5 was used by Feldman, Ilmanen and Knopf [52] to construct an $U(n)$ invariant complete noncompact expanding Kähler Ricci soliton on twisted complex line bundle over $\mathbb{CP}^{n-1}, n \geq 2$.

1.2 Basic properties and some known classification results

We are going to cover some equations satisfied by Ricci solitons, some known classification results of solitons and estimates on the potential functions. It was proven by Hamilton [59] that $|\nabla f|^2 + S - 2\lambda f$ is always a constant for gradient Ricci soliton.

Lemma 1. *[59] Let (M, g, f) be a complete gradient Ricci soliton. Then there exists a*

constant C such that

$$|\nabla f|^2 + S = 2\lambda f + C,$$

where λ is the constant in (1.3).

The following identities are well known and relate different geometric quantities to the potential function of the soliton. Their proofs can be found in [36], [90] and [49].

$$\Delta f + S = n\lambda, \tag{1.9}$$

$$\nabla S = 2\mathrm{Ric}(\nabla f) \text{ and} \tag{1.10}$$

$$\Delta_f S = 2\lambda S - 2|\mathrm{Ric}|^2. \tag{1.11}$$

In an orthonormal frame, we have

$$R_{kj,i} - R_{ki,j} = R_{ijkl} f_l, \tag{1.12}$$

$$\Delta_f R_{ij} = 2\lambda R_{ij} - 2R_{iklj}R_{kl} \text{ and} \tag{1.13}$$

$$\Delta_f R_{ijkl} = 2\lambda R_{ijkl} + 2(R_{ipjq}R_{kplq} - R_{ipjq}R_{lpkq} + R_{ipkq}R_{jplq} - R_{iplq}R_{jpkq}), \tag{1.14}$$

where $\Delta_f = \Delta - \nabla_{\nabla f}$.

We shall list some classification results of gradient Ricci solitons in low dimensions.

Theorem 1. *[59] Any 2 dimensional complete shrinking gradient Ricci soliton is isometric to $\mathbb{R}^2$ or a quotient of the sphere $\mathbb{S}^2$.*

Theorem 2. *[36] A 2 dimensional complete steady gradient Ricci soliton is isometric to the cigar soliton Σ or a quotient of $\mathbb{R}^2$.*

Please see [6] and ref. therein for more classification results in dimension 2. The classification of 3 dimensional gradient shrinker is completed and due to the efforts of numerous mathematicians. Please see [83] for more details.

Theorem 3. (*[61], [59], [88], [20], [86], [29], [85]*) *Let (M, g, f) be a 3 dimensional complete shrinking gradient Ricci soliton. Then M is isometric to $\mathbb{R}^3$ or a quotient of $\mathbb{S}^3$ or a quotient of $\mathbb{R} \times \mathbb{S}^2$.*

Nonetheless, the classifications of 3 dimensional steady and expanding solitons in full generality are still unclear. One significant result for gradient steadier was done by Brendle [8] (see also [45]), he confirmed a conjecture by Perelman [87] and classified non-collapsed 3 dimensional gradient steadier.

A Riemannian manifold (M^n, g) is said to be non-collapsed if for some $\kappa > 0$, for any $r > 0$ and $x \in M$ satisfying $|\mathrm{Rm}(y)| \leq r^{-2}$ for any $y \in B_r(x)$, then

$$\frac{V(B_r(x))}{r^n} \geq \kappa,$$

where $V(B_r(x))$ is the volume of the geodesic ball $B_r(x)$.

Theorem 4. *[8] A 3 dimensional complete non-collapsed and non-flat steady gradient Ricci soliton is isometric to the Bryant soliton.*

It was proved by Hamilton [59] and Ivey [61] that any compact non-shrinking Ricci solitons must be Einstein. In view of Example 4, we see that compact shrinking Ricci solitons may not be Einstein in general.

Theorem 5. *([59], [61]) Let (M^n, g, X) be a compact steady or expanding Ricci soliton (not necessarily gradient) with dimension $n \geq 2$. Then M is Einstein and X is a Killing vector field (i.e. $L_X g = 0$).*

There are also various classification results for solitons under different conditions like conformal flatness, curvature sign and decay conditions, please see [9], [18], [19], [34], [35], [42], [45], [47], [76], [82] and ref. therein for more results in higher dimensions.

Theorem 6. *[99] Let (M^n, g, f) be a complete gradient Ricci soliton with dimension $n \geq 2$. Then the flow of the vector field ∇f exists for all time $t \in \mathbb{R}$.*

Consequently, taking $X = \nabla f$, we have the solution φ_t in (1.6) exists on $(-\infty, \frac{1}{2\lambda})$ $\big($ resp. $\mathbb{R}, (\frac{1}{2\lambda}, \infty)\big)$ if $\lambda > 0$ $\big($ resp. $= 0, < 0\big)$. Hence a complete gradient Ricci soliton always generates a self similar solution to the Ricci flow.

Chen [29] showed that any complete ancient solution to the Ricci flow must have non-negative scalar curvature. As a result, gradient shrinker and steadier have non-negative scalar curvature. Zhang [99] provided an elliptic proof for the result. Moreover, he [99] gave a lower bound for the scalar curvature in the expanding case (which is not sharp).

Later, Pigola-Rimoldi-Setti [91] and Zhang [98] independently sharpened the bound for the scalar curvature of expander.

Theorem 7. (*[29], [99], [91], [98]*) *Let (M^n, g, f) be a complete gradient Ricci soliton with dimension $n \geq 2$. Then the following are true:*

1. *If it is shrinking or steady, then $S \geq 0$.*

2. *If it is expanding, then $S \geq n\lambda$.*

Moreover, in each of the above cases, the inequality is strict on M unless M is Einstein.

The following theorems are concerned with the growth of the potential function for noncompact gradient Ricci soliton.

Theorem 8. *[22] Let (M^n, g, f) be a complete noncompact shrinking gradient Ricci soliton with dimension $n \geq 2$ and $\lambda = \frac{1}{2}$. Then there exist positive constants c_0 and r_0, both depending only on n, such that for all x with $d(x, p_1) \geq r_0$*

$$\frac{1}{4}\Big(d(x,p_1) - c_0\Big)^2 \leq f(x) \leq \frac{1}{4}\Big(d(x,p_1) + c_0\Big)^2, \tag{1.15}$$

where p_1 is a minimum point of f on M, which always exists.

Theorem 9. (*[19], [23]*) *Let (M^n, g, f) be a complete noncompact steady gradient Ricci soliton with dimension $n \geq 2$ and the scaling convention* (2.6)*. Suppose that the scalar curvature S attains its maximum at a point $p \in M$. If in addition either one of the following conditions holds:*

1. *$Ric > 0$ on M;*

2. *$Ric \geq 0$ on M and $\limsup_{x\to\infty} S < \max_M S$,*

then there exist positive constants $\alpha \in (0, 1)$ and C such that

$$-r(x) - C \leq f(x) \leq -\alpha r(x) + C \text{ on } M. \tag{1.16}$$

Theorem 10. *[38] Let (M^n, g, f) be a complete noncompact expanding gradient Ricci soliton with dimension $n \geq 2$ and $\lambda = -\frac{1}{2}$. If for some $\delta > 0$, $Ric \geq (\delta - \frac{1}{2})g$ outside*

some compact subset of M, then there exists a positive constant C such that

$$-\frac{1}{4}r^2(x) - Cr(x) - C \leq f(x) \leq -\frac{\delta}{2}r^2(x) + Cr(x) + C \text{ on } M. \tag{1.17}$$

Remark 1. Upon scaling the soliton metric g by a constant if necessary, the λ in (1.3) can be chosen to be either $\frac{1}{2}$, 0 or $-\frac{1}{2}$.

Perelman [87] proved a Laplacian comparison theorem for gradient Ricci soliton (see also [99]).

Theorem 11. *[87] Let (M^n, g, f) be an n dimensional complete gradient Ricci soliton with $n \geq 2$. Then on $M \setminus B_1(p_0)$*

$$\Delta_f r \leq -\lambda r + n - 1 + |\nabla f|(p_0) + \frac{2}{3}\max_{B_1(p_0)} Ric_+, \tag{1.18}$$

where $Ric_+(x) := \sup_{v \in T_xM, |v|=1} \max\{Ric(v,v), 0\}$ and r is the distance function from the based point p_0.

Wei-Wylie [96] proved a more general Laplacian comparison theorem under Ric_f lower bound and some conditions on f.

1.3 Curvature estimates for steady Ricci solitons

In Chapter 2, we study the curvature of steady Ricci soliton. Recently, there has been lots of researches concerning the curvature estimate of gradient Ricci soliton, for example, [39], [84], [80], [21], [42], [83], [76], [49], [69], [31], [68] and [37]. Chow-Freedman-Shin-Zhang [37] showed that any four dimensional complete gradient steady Ricci soliton arising as a blow up limit of finite time singularity of Ricci flow on closed manifold must have bounded curvature. For any four dimensional complete gradient shrinking Ricci soliton arising from similar way, they proved that the Riemann curvature Rm of such shrinking Ricci soliton has at most quadratic growth. In [76], Munteanu-Sung-Wang studied the solvability of weighted Poisson equation for certain class of smooth metric measure space. As an application, they proved the following:

Theorem 12. *[76] Let (M^n, g, f) be an n dimensional complete non Ricci flat gradient steady Ricci soliton with $n \geq 2$. Suppose the potential function f is bounded above by a*

constant and $\lim_{r\to\infty} r|Rm| = 0$*, then there exists a positive constant* C *such that*

$$|Rm|(x) \leq C(1 + r(x))^{3(n+1)} e^{-r(x)}, \tag{1.19}$$

where $r = r(x)$ *is the distance of* x *from a fixed point* $p_0 \in M$*.*

It is not known whether the decay rate on the right hand side of (1.19) is sharp or not. In Chapter 2, we shall sharpen the upper bound under a weaker curvature decay condition. Instead of applying the Green's function estimate in [76], we adopt the method based on the maximum principle introduced by Deruelle [49] in order to study the curvature properties of expanding gradient Ricci solitons. Here is the main result of Chapter 2:

Theorem 13. *Let* (M^n, g, f) *be an* n *dimensional complete non Ricci flat gradient steady Ricci soliton with* $n \geq 2$*. Suppose the potential function* f *is bounded above by a constant and the curvature tensor* Rm *satisfies* $\limsup_{r\to\infty} r|Rm| < \frac{1}{5}$*. Then there exists a positive constant* C *such that*

$$|Rm|(x) \leq Ce^{-r(x)} \text{ on } M, \tag{1.20}$$

where $r = r(x)$ *is the distance of* x *from a fixed point* $p_0 \in M$*.*

Remark 2. The decay rate is sharp on $\Sigma \times \mathbb{T}^{n-2}$, where Σ and $\mathbb{T}^{n-2}$ denote the Hamilton's cigar soliton and $n-2$ dimensional flat torus respectively (see Example 7 in Section 2).

Remark 3. The constant $\frac{1}{5}$ in Theorem 13 is technical. For the n dimensional Bryant soliton, $\lim_{r\to\infty} r|Rm| = \sqrt{\frac{(n-1)}{2(n-2)}} > \frac{1}{5}$ (see [8] and [9]). It is unclear at this point whether the constant $\frac{1}{5}$ is optimal.

Remark 4. In view of Lemma 3, the conclusion (1.20) is still true under a seemingly weaker condition on f, namely, $\overline{\lim}_{r\to\infty} r^{-1} f < 1$, for the sake of simplicity, we state the theorem in the above way. Ric ≥ 0 and $S \to 0$ at infinity are sufficient for f to be bounded above by a constant (see [23]).

Remark 5. The quantity $r|Rm|$ is not scaling invariant, we are using the scaling convention such that (2.6) holds.

In view of Theorem 12 and 13, one possible conjecture concerning the curvature decay of gradient steady Ricci soliton is that, the decay rate is either linear or exponential. It would be very useful for the classification of steady soliton if one can establish this dichotomy. There are a number of classification results under certain curvature decay conditions. For exponential curvature decay, Deng-Zhu [45] generalized a result of [76] and showed that any non-flat complete gradient steady Ricci soliton with non-negative sectional curvature is a quotient of $\mathbb{R}^{n-2} \times \Sigma$ if rS is small near infinity, where Σ and S denote the cigar soliton and scalar curvature respectively (see also [46] and [24]). For linear curvature decay, Brendle [8] solved a conjecture of Perelman [87] and proved that any three dimensional non-flat and non-collapsed gradient steady Ricci soliton is the Bryant soliton. Deng-Zhu [43], [45] improved the result of Brendle [8] and proved that any non-flat three dimensional gradient steady soliton with linear curvature decay is the Bryant soliton. See [9] and [44] for more classification results of higher dimensional steady Ricci soliton with curvature decay conditions.

It is also worth mentioning that very lately Brendle [10] confirmed a conjecture by Perelman [87] and showed that any three dimensional complete non-compact, non-collapsed ancient solution to the Ricci flow with bounded positive sectional curvature is eternal and isometric to the Bryant soliton (see also [70], [4] and [11]).

Munteanu-Wang [80] proved that a complete four dimensional gradient shrinking Ricci soliton must have bounded Riemann curvature Rm if its scalar curvature S is bounded (see [21] for the estimate of steady soliton by Cao-Cui). Furthermore, $|Rm|$ is controlled by S pointwise, more precisely, there exists a constant $c > 0$ such that

$$|Rm| \leq cS \text{ on } M. \tag{1.21}$$

In dimension three, (1.21) is a direct consequence of the Hamilton-Ivey estimate which implies that three dimensional complete gradient shrinking and steady Ricci solitons have non-negative sectional curvature ([61], [59] and [29]). However in dimension four, the sectional curvature of shrinking and steady Ricci solitons may change sign (see [18]).

Recently, Conlon-Deruelle-Sun [40] classified all non-flat gradient Kähler shrinking Ricci soliton of complex dimension two with $\lim_{r\to\infty} S = 0$ and showed that it is isometric to the Feldman-Ilmanen-Knopf shrinking Ricci soliton [52] (see [40] for more results

on gradient Kähler shrinking and expanding Ricci solitons).

One special feature observed by Munteanu-Wang [80] is that in four dimensional gradient Ricci soliton, the Riemann curvature Rm can be bounded by Ric and ∇Ric:

$$|Rm| \leq A_0\Big(|\mathrm{Ric}| + \frac{|\nabla \mathrm{Ric}|}{|\nabla f|}\Big), \tag{1.22}$$

for some universal positive constant A_0. Please see Lemma 2 for the precise statement and proof of (1.22).

We will show that estimate (1.21) is also true for four dimensional non-trivial gradient steady Ricci soliton with scalar curvature $S \to 0$ as $r \to \infty$. Under the additional assumption that S has at most polynomial decay, Cao-Cui [21] proved that for any $a \in (0, \frac{1}{2})$, there is a positive constant C such that

$$|Rm| \leq CS^a.$$

We raise the exponent a to 1:

Theorem 14. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. There exists a positive constant c such that*

$$|Rm| \leq cS \text{ on } M. \tag{1.23}$$

As an application, we show that the Riemann curvature Rm of a four dimensional complete nontrivial gradient steady Ricci soliton decays exponentially if the potential f is bounded from above and $\overline{\lim}_{r\to\infty} rS$ is sufficiently small.

Theorem 15. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. Suppose the potential function f is bounded from above by a constant and $\overline{\lim_{r\to\infty}} rS < \frac{1}{5A_0^2}$, where A_0 is the constant in (1.22). Then there is a constant $C > 0$ such that*

$$|Rm|(x) \leq Ce^{-r(x)} \text{ on } M.$$

1.4 Kähler steady Ricci solitons with nonnegative Ricci curvature and integrable scalar curvature

Other than the pointwise curvature decay, it is natural to study solitons with curvature decay in the integral sense. In Chapter 3, we prove a classification result for steady soliton under a integrability condition of the curvature. In [46], Deruelle proved that any complete non-flat gradient steady Ricci soliton with non-negative sectional curvature and scalar curvature $S \in L^1(M, g)$ is isometric to a quotient of $\Sigma \times \mathbb{R}^{m-2}$, where Σ denotes the Hamilton's cigar soliton. Later Catino-Mastrolia-Monticelli [24] weakened the integrability condition of S to

$$\liminf_{r\to\infty} \frac{1}{r} \int_{B_r(p_0)} S = 0, \tag{1.24}$$

for some $p_0 \in M$ (see also [76] by Munteanu-Sung-Wang for a different proof). Since $S \geq 0$, It is clear that the above condition is true independent on the base point p_0, i.e. (1.24) holds for some p_0 in M if and only if it holds for all p_0 in M.

Theorem 16. *[46], [24], [76] Let (M, g, f) be a real m dimensional non-flat complete gradient steady Ricci soliton with non-negative sectional curvature. Suppose in addition that the scalar curvature S satisfies (1.24), then the universal cover of (M, g) is isometric to $\Sigma \times \mathbb{R}^{m-2}$, where Σ denotes the Hamilton's cigar soliton.*

Remark 6. It is not difficult to see from the proof of Theorem 16 that Any non-flat gradient steady Kähler Ricci soliton with non-negative bisectional curvature and S satisfying (1.24) is a quotient of $\Sigma \times \mathbb{C}^{n-1}$.

It was shown by Hamilton [59], Ivey [61] and Chen [29] that any real 3 dimensional complete gradient shrinking or steady Ricci soliton must have non-negative sectional curvature. However, this significant feature doesn't hold true for higher dimensions, Feldman, Ilmanen and Knopf [52] constructed some shrinkers with Ricci curvature being negative in some directions (see also [15] by Cao who constructed a steadier on the anticanonical line bundle on $\mathbb{CP}^n$ which doesn't have non-negative bisectional curvature). It is natural to ask whether we can classify steady Ricci soliton under weaker curvature condition. Deng and Zhu [42] showed that any complete Ricci non-negative

gradient steady Kähler Ricci soliton with average of scalar curvature over large ball decaying faster than linear rate must be Ricci flat. It would be interesting to know more about the Kähler steadier with non-negative Ricci curvature. In [46], Deruelle proved the following local splitting theorem:

Theorem 17. *[46] Let (M, g, f) be a real m dimensional complete gradient steady Ricci soliton with $Ric \geq 0$ and $S > 0$. Suppose the following conditions are satisfied:*

1. *S is integrable, i.e. $S \in L^1(M, g)$;*
2. *$|Rm| \to 0$ as $r \to \infty$;*
3. *$|\nabla f|^2 S \geq 2Ric(\nabla f, \nabla f)$,*

then $M \setminus A$ is locally isometric to $\Sigma \times \mathbb{R}^{m-2}$, where $A := \{\nabla f = 0\}$ and Σ is the Hamilton's cigar soliton.

Remark 7. Condition 3 in the above theorem is automatic if (M, g, f) is a gradient Kähler Ricci soliton with $\mathrm{Ric} \geq 0$.

We shall generalize Theorem 16 and Theorem 17 under the Kähler condition.

Theorem 18. *Let (M, g, f) be a complex n dimensional complete non Ricci flat gradient steady Kähler Ricci soliton with $Ric \geq 0$ and $n \geq 2$. Suppose the scalar curvature S satisfies (1.24), i.e.*

$$\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S = 0,$$

then it is isometric to a quotient of $\Sigma \times \mathbb{C}^{n-1-k} \times N^k$, where Σ and N denote the Hamilton's cigar soliton and some simply connected compact Kähler Ricci flat manifold of complex dimension k respectively.

The result is no longer true if one allows $\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S > 0$. Indeed, let Σ_2 be the positively curved $U(2)$ invariant soliton on $\mathbb{C}^2$ constructed by Cao [15] and $\mathbb{T}^{n-2}$ be any flat Tori of complex dimension $n-2$. $\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S > 0$ for $\Sigma_2 \times \mathbb{T}^{n-2}$ but its universal cover is not isometric to $\Sigma \times \mathbb{C}^{n-1-k} \times N^k$.

One difficulty we encounter is that in real dimension $m \geq 4$, the strong maximum principle for the Ricci tensor of Hamilton [57], Cao [17] and the splitting theorem of

soliton by Guan, Lu and Xu [55] are not available in the absence of non-negative sectional or bisectional curvature condition. Moreover, the classical Cheeger Gromoll splitting theorem ([28] and [67]) cannot be applied directly as the soliton under consideration has no line. Thanks to the observation by Deruelle in [46], in order to split the manifold, one suffices to show that ∇f is an eigenvector of the Ricci tensor. Motivated by the arguments in [24] and [76], we will prove this by an integration by part argument.

In view of Theorem 16, one may ask when $\mathbb{C}^{n-1-k} \times N^k$ is flat, i.e. $k = 0$. Under the assumptions of the previous theorem, we give a necessary and sufficient condition for the flatness of $\mathbb{C}^{n-1-k} \times N^k$.

Corollary 1. *Let (M^n, g, f) be a complex n dimensional complete non Ricci flat gradient steady Kähler Ricci soliton with $Ric \geq 0$ and $n \geq 2$. Suppose that*

$$\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S = 0.$$

For n=2, it is isometric to a quotient of $\Sigma \times \mathbb{C}$. For $n \geq 3$, M is isometric to a quotient of $\Sigma \times \mathbb{C}^{n-1}$ if and only if $|Rm| \to 0$ as $r \to \infty$.

The integrability condition (1.24) is closely related to the volume growth of the manifold. Indeed, it was shown in [46] (see also [24]) that for a complete gradient steady Ricci soliton with Ric ≥ 0 and scaling convention (2.6), the scalar curvature S must satisfy

$$\frac{1}{V(B_r(p))} \int_{B_r(p)} S \leq \frac{m}{r}, \tag{1.25}$$

for all $r > 0$ and $p \in M^m$. With the above inequality, Catino, Mastrolia and Monticelli [24] showed that any non-flat complete gradient steady Ricci soliton with non-negative sectional curvature and subquadratic volume growth is a quotient of $\Sigma \times \mathbb{R}^{m-2}$. Motivated by their result, we prove an analog in the Kähler case with Ric ≥ 0 using Theorem 18.

Corollary 2. *Let (M^n, g, f) be a complex n dimensional complete non Ricci flat gradient steady Kähler Ricci soliton with $Ric \geq 0$. Suppose the volume of geodesic ball is of subquadratic growth, i.e. $V(B_r) = o(r^2)$, then the universal covering space of M is isometric to $\Sigma \times \mathbb{C}^{n-1-k} \times N^k$, where N is a simply connected compact Kähler Ricci flat manifold of complex dimension k.*

Recently, there have been lots of researches about the classification of Ricci solitons according to the decay rate of the scalar curvature. For example, Brendle [8] showed that any real 3 dimensional complete non-flat and non-collapsed gradient steady Ricci soliton is the Bryant soliton (see also [9]). Deng and Zhu [43],[45] later generalized Brendle's result and classified real 3 dimensional complete gradient Ricci steadier under $S \leq Cr^{-1}$. Munteanu, Sung and Wang [76] proved that any real m dimensional non-flat gradient steadier with non-negative sectional curvature and decay rate of the scalar curvature faster than linear rate is isometric to a quotient of $\Sigma \times \mathbb{R}^{m-2}$. Lately, Deng and Zhu [45] generalized the result in [76]:

Theorem 19. *[45] Let (M, g, f) be a real m dimensional complete non-flat gradient steady Ricci soliton with non-negative sectional curvature and the scaling convention (2.6). There exists a constant $\varepsilon = \varepsilon(m) > 0$ depending only on m such that if S satisfies*

$$rS \leq \varepsilon$$

near infinity, then the universal covering space of M is isometric to $\Sigma \times \mathbb{R}^{m-2}$.

Using a result by Catino, Mastrolia and Monticelli [24] and Corollary 2, we can have a sharp dimension free bound for the ε in Theorem 19.

Theorem 20. *Let (M, g, f) be a real m dimensional complete non-flat gradient steady Ricci soliton with non-negative sectional curvature and the scaling convention (2.6). In addition, we assume that*

$$\limsup_{r\to\infty} rS < 1.$$

Then M is isometric to a quotient of $\Sigma \times \mathbb{R}^{m-2}$ and $\limsup_{r\to\infty} rS = 0$.

Theorem 21. *Let (M, g, f) be a complex n dimensional complete non-Ricci flat gradient steady Kähler Ricci soliton with non-negative Ricci curvature and the scaling convention (2.6). In addition, we assume that*

$$\limsup_{r\to\infty} rS < 1.$$

Then M is isometric to a quotient of $\Sigma \times \mathbb{C}^{n-1-k} \times N^k$ and $\limsup_{r\to\infty} rS = 0$, where N is a simply connected compact Kähler Ricci flat manifold.

If one allows $\limsup_{r\to\infty} rS \leq 1$, then both Theorems 20 and 21 will not be true. The counter example for the real case is the 3 dimensional Bryant soliton and for the Kähler case is the positively curved $U(2)$ invariant example constructed by Cao on $\mathbb{C}^2$ [15], both satisfy $\lim_{r\to\infty} rS = 1$ but they are not the quotient of $\Sigma \times \mathbb{R}$ or $\Sigma \times \mathbb{C}$. Higher dimensional counter examples can be obtained by taking product with flat torus of suitable dimensions.

1.5 Curvature estimates and gap theorems for expanding Ricci solitons

In Chapter 4, we study the geometry of expanding solitons with nonnegative scalar curvature. Chen [29] showed that any complete ancient solutions to the Ricci flow have non-negative scalar curvature. Consequently, any complete shrinking and steady gradient Ricci solitons have non-negative scalar curvature (see also [99] for a different proof). By the strong minimum principle, the scalar curvature of gradient shrinker and steadier must be positive unless the manifolds are Ricci flat (and hence is flat in the former case). Chow-Lu-Yang [39] applied an estimate of the potential function for gradient shrinker by Cao-Zhou [22] and obtained a sharp positive lower bound for the scalar curvature of noncompact and nonflat shrinker.

Theorem 22. *[39] Let (M^n, g, f) be an n dimensional complete noncompact and nonflat shrinking gradient Ricci soliton. Then there exists a positive constant c such that*

$$S \geq \frac{c}{r^2} \tag{1.26}$$

outside a compact set of M.

Remark 8. As pointed out in [39], the estimate (1.26) is sharp on the shrinker constructed by Feldman-Ilmanen-Knopf [52].

For the case of steady soliton, Chow-Lu-Yang [39] also provided a sharp lower estimate for the scalar curvature under some conditions on the potential function (see also [76]).

Theorem 23. *[39] Let (M^n, g, f) be an n dimensional complete noncompact and non*

Ricci flat steady gradient Ricci soliton with the scaling convention $|\nabla f|^2 + S = 1$. *Suppose* $f \leq 0$ *and* $\lim_{x\to\infty} f(x) = -\infty$. *Then*

$$S \geq \frac{1}{\sqrt{\frac{n}{2}} + 2} e^f \text{ on } M. \tag{1.27}$$

Remark 9. Using some localization and cut off function arguments, Munteanu-Sung-Wang [76] showed $S \geq Ce^f$ without assuming $\lim_{r\to\infty} f = -\infty$. However the constant C depends not only on the dimension n, but also the metric g. The decay rate in (1.27) is sharp on cigar soliton (see [39], [58] and [36]).

Unlike shrinker and steadier, expander may have negative scalar curvature. Pigola-Rimoldi-Setti [91] and S. J. Zhang [98] independently improved an estimate in [99] and showed the scalar curvature S of an n dimensional complete expanding gradient Ricci soliton satisfies

$$S \geq -\frac{n}{2}, \tag{1.28}$$

with equality holds somewhere if and only if the expander is Einstein with scalar curvature $-\frac{n}{2}$ (see [91] and [98]).

Roughly speaking, the inequalities (1.26) and (1.27) in the above theorems provide some quantitative estimates to measure the gap between the solitons and the flat or Ricci-flat spaces. It is natural to ask whether such a gap exists or not if the scalar curvature of the expander is nonnegative. We provide an affirmative answer to the above question if the potential function f is proper. No upper boundedness of the scalar curvature is assumed.

Theorem 24. *Let* (M^n, g, f) *be an* n *dimensional complete non-compact gradient expanding Ricci soliton with* $n \geq 2$. *Assume that* $\lim_{r\to\infty} f = -\infty$ *and* (M, g) *is non-flat and has non-negative scalar curvature* S. *Then there exists a positive constant* C *such that*

$$S \geq Cv^{1-\frac{n}{2}} e^{-v} \text{ on } M, \tag{1.29}$$

where $v := \frac{n}{2} - f$.

Remark 10. The estimate (1.29) is sharp when $n = 2$ on 2 dimensional positively curved gradient expander and when $n = 2m$, $m \geq 2$ on the Kähler gradient expander

constructed by Feldman-Ilmanen-Knopf (see [52] and [94]). By (1.28) and (4.2), $f \leq \frac{n}{2}$, hence f is proper (preimages of compact subsets are compact) if and only if $\lim_{r\to\infty} f = -\infty$. One sufficient condition for the properness of f is that for some $\varepsilon > 0$,

$$\mathrm{Ric} \geq (\varepsilon - \frac{1}{2})g$$

outside a compact subset of M (see Theorem 10).

Remark 11. Under the additional assumption that $\lim_{r\to\infty} r^2|\mathrm{Ric}| = 0$, Deruelle [49] also gave a lower bound (possibly zero) of $v^{\frac{n}{2}-1}e^v S$ which depends on $\liminf_{r\to\infty} v^{\frac{n}{2}-1}e^v S$. The limit $\lim_{r\to\infty} v^{\frac{n}{2}-1}e^{v-\frac{n}{2}} S$ is also related to the notion of the scalar curvature at infinity in [48].

For expander with nonpositive Ricci curvature and proper potential function, we prove an upper bound for the scalar curvature.

Theorem 25. *Let (M^n, g, f) be an n dimensional complete non-compact gradient expanding Ricci soliton with $n \geq 2$. Suppose that $\lim_{r\to\infty} f = -\infty$ and (M, g) is non-flat and has non-positive Ricci curvature. Then there is a positive constant C such that*

$$S \leq -Cv^{1-\frac{n}{2}}e^{-v}, \tag{1.30}$$

where $v := \frac{n}{2} - f$.

Remark 12. The upper estimate is sharp on two dimensional negatively curved gradient expanders with curvature decaying to zero at infinity.

In view of Theorem 24, it is interesting to see when the scalar curvature of an expander is nonnegative. The lower bound of the scalar curvature is also important since it is related to the connectedness at infinity of the expander (see [78] and [32]). It is known that there doesn't exist any compact expander with nonnegative scalar curvature. Indeed, any compact expander must be Einstein with scalar curvature $-\frac{n}{2}$ (see [36]). Motivated by various studies of solitons with integrable curvature (see [91], [46], [24], [76] and [26]), we prove an analog in the noncompact case.

Theorem 26. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature S by S_-,*

i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. *If* S_- *is integrable, that is,*

$$\int_M S_- dv_g < \infty, \tag{1.31}$$

then either $S \geq 0$ *everywhere or* M *is isometric to an Einstein manifold with scalar curvature* $-\frac{n}{2}$ *and finite volume, where* dv_g *is the volume element with respect to the metric* g.

Remark 13. The curvature of negatively curved Cao's Kähler expander in complex dimension one decays exponentially in r^2 (see [16] and [42]). Hence Theorem 26 doesn't hold in real dimension two.

One immediate consequence of the above theorem is that any complete noncompact gradient expander with nonnegative scalar curvature outside compact subset must have nonnegative scalar curvature everywhere. Using the asymptotic curvature estimates in [49], we also have:

Corollary 3. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with dimension* $n \geq 3$. *Suppose in addition that*

$$\lim_{x\to\infty} r^2 |Ric| = 0,$$

where r *is the distance function from a fixed point. Then* M *has nonnegative scalar curvature.*

Using Theorem 26, we give another sufficient condition for nonnegativity of the scalar curvature.

Theorem 27. *Let* (M^n, g, f) *be a complete noncompact gradient expanding Ricci soliton with dimension* $n \geq 3$ *and bounded scalar curvature* S. *Suppose the following conditions are satisfied:*

1. $\lim_{x\to\infty} f = -\infty$;
2. $\liminf_{x\to\infty} r^2 S \geq 0$.

Then the scalar curvature $S \geq 0$ *on* M.

Remark 14. It can be seen from the proof of Theorem 27 that $S \geq 0$ holds under a weaker assumption on S, namely $\limsup_{x\to\infty} v^{-1}S < 1$ instead of S being bounded, where $v = \frac{n}{2} - f$. Under the conditions of Corollary 3 or Theorem 27, it follows from Theorem 24 that the scalar curvature S satisfies (1.29) provided that M is not flat. In view of the negatively curved Bryant's expander and Cao's Kähler expander, we know that the quadratic factor r^2 in Corollary 3 and Theorem 27 is sharp (see [16], [12], [36] and [42]). We also see from the F.I.K. expander [52] that analogous result is not true if one replaces the scalar curvature S by the Ricci curvature Ric.

We then study the curvature estimates for expanders of low dimensions, namely dimensions 3 and 4. Motivated by the estimates of the curvature tensor of shrinker and steadier in [80], [21], [81] and [83], we prove some analogs for 3 dimensional gradient expander.

Theorem 28. *Let (M^3, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Then the curvature tensor Rm is bounded if the scalar curvature S is bounded. Moreover, the following hold:*

1. *If S is nonnegative and bounded, then*

$$|Rm| \leq c\sqrt{S} \text{ on } M,$$

   for some positive constant c;

2. *If $S \to 0$ as $x \to \infty$, then $\lim_{x\to\infty} |Rm| = 0$.*

As an application, we show that f is proper if the scalar curvature S decays at infinity in dimension three.

Corollary 4. *Let (M^3, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose that $\lim_{x\to\infty} S = 0$. Then $\lim_{x\to\infty} f = -\infty$, furthermore*

$$\lim_{x\to\infty} \frac{4f(x)}{r^2(x)} = -1. \tag{1.32}$$

Remark 15. In view of Theorem 36 and Proposition 8, we see that (1.32) is also true in higher dimensions if in addition |Rm| is bounded on M.

Munteanu-Wang [80] showed that any 4 dimensional complete gradient shrinker must have bounded curvature if the scalar curvature is bounded (see also [21] for the estimates in steady soliton). We prove that it is also true for gradient expanding soliton if in addition the potential function $f \to -\infty$ as $r \to \infty$.

Theorem 29. *Let (M^4, g, f) be a 4 dimensional complete non-compact gradient expanding Ricci soliton with bounded scalar curvature. Suppose that $\lim_{r\to\infty} f = -\infty$. Then the curvature tensor Rm is bounded.*

It was proved by Deng-Zhu [42] and Deruelle [49] that any gradient expander with non-negative Ricci curvature must have bounded scalar curvature. Together with Remark 10, we have the following corollary.

Corollary 5. *Let (M^4, g, f) be a 4 dimensional complete noncompact gradient expanding Ricci soliton with $Ric \geq 0$. Then it has bounded curvature tensor.*

Using Shi's estimate [38], Theorem 36 and Lemma 2, we establish the equivalence of the properness of f and the curvature decay at infinity.

Corollary 6. *Let (M^4, g, f) be a 4 dimensional complete noncompact gradient expanding Ricci soliton with $\lim_{x\to\infty} S(x) = 0$. Then $\lim_{x\to\infty} f(x) = -\infty$ if and only if $\lim_{x\to\infty} |Rm|(x) = 0$.*

As an application of the previous curvature estimates, we study the gap theorem of expanding soliton. Gap phenomenon of manifold has long been an interesting problem in geometry. A special case of the problem is concerned with the flatness of noncompact manifold, more precisely, for a non-compact manifold with curvature decaying sufficiently fast at infinity, is it necessarily flat? The tangent cone at infinity of gradient expander with $\lim_{x\to\infty} r^2|\mathrm{Rm}| = 0$ was proven to be flat by Chen [30] and Chen-Deruelle [32]. The expander itself is also flat if in addition $\mathrm{Ric} \geq 0$ (see [5] and [32]). Deng and Zhu [42] proved that a Kähler gradient expander with complex dimension $n \geq 2$, $\mathrm{Ric} \geq 0$ and $\lim_{r\to\infty} r^2 S = 0$ is isometric to $\mathbb{C}^n$ (see [75], [54] and [5] for more results on the gap theorems of general Riemannian manifolds). In either cases, Ricci curvature is assumed to be non-negative. Using the positive mass theorem, asymptotic curvature estimates in [49], Theorems 26 and 28(2), we are able to prove a gap theorem for 3 dimensional gradient expander by only imposing condition on the scalar curvature.

Theorem 30. *Let (M^3, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose the scalar curvature S satisfies*

$$\lim_{x\to\infty} r^2(x)S(x) = 0.$$

Then M is isometric to $\mathbb{R}^3$.

Remark 16. Analogous result is not true in higher even dimensions. The non-flat Kähler gradient expander constructed by Feldman-Ilmanen-Knopf [52] has nonnegative scalar curvature which decays exponentially in r.

Remark 17. Positive mass theorem was also used by Ma [74] to prove the flatness of 3 dimensional Ricci pinched gradient expander. Using the asymptotic curvature estimates [49] and volume comparison theorem, Deruelle showed that any 4 dimensional Ricci pinched gradient expander is flat (see [74] and [49] for more results in higher dimensions under additional assumptions). Again $\mathrm{Ric} \geq 0$ was assumed in both cases.

Recently the positive mass theorem for asymptotically flat manifold with dimension $n \geq 9$ has been proven by Lohkamp [71], [72], and later also by Schoen-Yau [93]. Hence similar argument for Theorem 30 gives the following gap theorem in higher odd dimensions without sign condition on the Ricci curvature.

Theorem 31. *Let (M^{2m+1}, g, f) be a complete noncompact gradient expanding Ricci soliton with odd dimension $2m+1$ and $m \geq 2$. If $\lim_{x\to\infty} r^2|Rm| = 0$, then M is isometric to $\mathbb{R}^{2m+1}$.*

Remark 18. The theorem is not true in even dimensions. The non-flat Feldman-Ilmanen-Knopf Kähler expander [52] has flat asymptotic cone and satisfies $\lim_{x\to\infty} r^2|\mathrm{Rm}| = 0$. Nonetheless, if in addition either one of the following conditions holds:

1. M is smoothly asymptotic to the cone $(C(\mathbb{S}^{n-1}(1)), dt^2 + t^2 g_{\mathbb{S}^{n-1}(1)})$ in sense of [49];
2. $\liminf_{R\to\infty} \frac{Vol_g(B_R(p_0))}{R^n} > \frac{\omega_{n-1}}{2n}$, where ω_{n-1} is the volume of $\mathbb{S}^{n-1}(1)$ in $\mathbb{R}^n$ with respect to the Euclidean metric;
3. M is simply connected at infinity;

4. $\mathrm{Ric} \geq \left(\delta - \frac{1}{2}\right) g$ on M for some $\delta > 0$,

then M^n is isometric to $\mathbb{R}^n$, where $n = 2m$ and $m \geq 2$. As mentioned in Remark 14, the quadratic factor r^2 in Theorems 30 and 31 is optimal.

Chapter 2

Curvature estimates for steady Ricci solitons

The results in this chapter are based on [25].

2.1 Preliminaries

Let (M^n, g) be a connected smooth n dimensional Riemannian manifold and f be a smooth function on M. (M^n, g, f) is said to be a gradient steady Ricci soliton with potential function f if

$$\mathrm{Ric} + \nabla^2 f = 0. \tag{2.1}$$

(M, g) is assumed to be complete as a Riemannian manifold throughout this thesis. Ricci soliton is related to the self similar solution to the Ricci flow (solution which evolves by time dependent scaling and reparametrization of a fixed metric, see [36]). We consider the flow of the vector field ∇f and denote it by ϕ_t, ϕ_0 is the identity map. By a result of Zhang [99], ϕ_t exists for all time $t \in \mathbb{R}$. Let $g(t) := \phi_t^* g$, then $g(t)$ is a solution to the Ricci flow (1.1) with initial datum $g(0) = g$.

For any smooth function ω, the weighted Laplacian with respect to ω is defined to be the operator $\Delta_\omega := \Delta - \nabla_{\nabla\omega}$. The following equations are known for gradient steady

Ricci soliton (see [36] or [90] for derivation of these formulas):

$$S + \Delta f = 0, \tag{2.2}$$

$$\nabla S = 2\mathrm{Ric}(\nabla f), \tag{2.3}$$

$$\Delta_f S = -2|\mathrm{Ric}|^2 \text{ and} \tag{2.4}$$

$$\Delta_f R_{ij} = -2R_{iklj}R_{kl}. \tag{2.5}$$

It was proved by Hamilton [59] that $S + |\nabla f|^2 = C$ for some constant C. When the metric is not Ricci flat, S and hence C are positive on M (see the discussion after (2.10)). We can normalize C by scaling the metric and get:

$$S + |\nabla f|^2 = 1 \text{ and} \tag{2.6}$$

$$\Delta_f f = -1. \tag{2.7}$$

We define a function v in the following way:

$$v := -f. \tag{2.8}$$

From (2.6) and (2.7), we have

$$\Delta_f v = 1 \text{ and} \tag{2.9}$$

$$S + |\nabla v|^2 = 1. \tag{2.10}$$

Chen [29] showed that any complete ancient solution to the Ricci flow must have non-negative scalar curvature. As a result, the scalar curvature S of a complete gradient steady Ricci soliton is non-negative (see also [99]). Moreover by strong minimum principle, $S = 0$ somewhere if and only if the steady soliton is Ricci flat. It is also known that any compact steady Ricci soliton must be Ricci flat (see [36]). Hence any complete non Ricci flat gradient steady Ricci soliton must be non-compact and have positive scalar curvature.

The following lemma is due to Munteanu-Wang [80] and stated in a slightly different form in [80] and [21]. For the sake of completeness, we sketch the proof here.

Lemma 2. *[80] Let (M^4, g, f) be a four dimensional gradient Ricci soliton. There exists a universal positive constant A_0 such that if $\nabla f \neq 0$ at $q \in M$, then at q*

$$|Rm| \leq A_0\Big(|Ric| + \frac{|\nabla Ric|}{|\nabla f|}\Big). \tag{2.11}$$

Proof. This proof was presented by Munteanu during one of his talks on [80]. Whenever $\nabla f \neq 0$, we consider an orthonormal frame $\{e_i\}_{i=1}^4$ with $e_4 = \frac{\nabla f}{|\nabla f|}$. For any gradient Ricci soliton, by Ricci identity and (1.8)

$$R_{kl,i} - R_{ki,l} = R_{ilkj} f_j.$$

Hence for any $1 \leq i, j, k \leq 4$,

$$|R_{ijk4}| \leq \frac{c|\nabla \mathrm{Ric}|}{|\nabla f|}.$$

Since the dimension is four, we have by direct computation,

$$R_{1221} = \frac{1}{2}(R_{11} + R_{22} - R_{33} - R_{44}) + R_{3443}.$$

From the above identity, we see that Lemma 2 holds. □

2.2 Estimates on the potential function f

We first provide some equivalent criteria for the properness of the potential function f which will be used in later sections. Please see [23], [19] and Theorem 9 for more results on the linear growth of f.

Lemma 3. *Let (M^n, g, f) be an n dimensional complete non Ricci flat steady gradient Ricci soliton with scalar curvature $S \to 0$ as $r \to \infty$. Then the following are equivalent:*

(i) $\overline{\lim_{r\to\infty}} r^{-1} f < 1$;
(ii) $\lim_{r\to\infty} f = -\infty$;
(iii) $\forall\ \alpha \in (0,1)$, *there exists constant* $D > 0$ *such that*

$$r + D \geq -f \geq \alpha r - D \text{ on } M,$$

in particular $\lim_{r\to\infty} r^{-1}f = -1$.

Remark 19. By (2.6), we always have $-1 \leq \overline{\lim_{r\to\infty}} r^{-1}f \leq 1$.

Proof. $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious.
$(i) \Rightarrow (iii)$: The upper bound of $-f$ follows from (2.6). We now consider the lower bound. Since the scalar curvature S decays at infinity, for any $\alpha \in (0,1)$ there exists a compact subset K of M such that

$$|\nabla f| \geq \alpha \text{ on } M \setminus K.$$

We consider the flow of the vector field $\frac{\nabla f}{|\nabla f|^2}$ and denote it by ψ_t, ψ_0 is the identity map. Let $q \in M \setminus K$ and for small non-negative t,

$$d(\psi_t(q), q) \leq \int_0^t \frac{1}{|\nabla f|(\psi_s(q))} ds \leq \frac{t}{\alpha}. \tag{2.12}$$

By short time existence of ordinary differential equation, $\psi_t(q)$ exists as long as it is in $M \setminus K$. Therefore we can define T as follows

$$T := \sup\{a, \psi_t(q) \in M \setminus K \text{ for all } t \in [0,a]\}. \tag{2.13}$$

Since K is compact, T is positive. We claim that T is finite. We first assume the claim and prove the lemma. Under $T < \infty$, we know that $\psi_T(q)$ exists and lies in K, and by (3.3)

$$\begin{aligned} -f(q) &= T - f(\psi_T(q)) \\ &\geq \alpha(d(q,p_0) - d(\psi_T(q), p_0)) - f(\psi_T(q)) \\ &\geq \alpha d(q,p_0) - \alpha \sup_K d(p_0,\cdot) - \sup_K |f|. \end{aligned}$$

The above inequality also holds for $q \in K$. Thus we showed (iii) is true. It remains to justify the claim, i.e. $T < \infty$. We argue by contradiction, suppose $T = \infty$. By (3.3)

and (2.6), for all $t \in [0, \infty)$,

$$\begin{aligned} d(\psi_t(q), q) &\leq \int_0^t \frac{1}{|\nabla f|(\psi_s(q))} ds \\ &= t + \int_0^t \frac{1 - |\nabla f|}{|\nabla f|} ds \\ &= t + \int_0^t \frac{S}{|\nabla f|(1 + |\nabla f|)} ds \\ &\leq t + \frac{1}{\alpha} \int_0^t S(\psi_s(q)) ds. \end{aligned}$$

We have for all $t \geq 0$

$$d(\psi_t(q), q) \leq t + \frac{1}{\alpha} \int_0^t S(\psi_s(q)) ds. \tag{2.14}$$

We shall write $d(q, p_0)$ as $r(q)$, $r(\psi_t(q))$ as r_t, where p_0 is a fixed reference point. By triangular inequality and (2.6),

$$\begin{aligned} f(\psi_t(q)) - f(q) &= t \\ &\leq d(\psi_t(q), q) \\ &\leq r_t + r_0. \end{aligned} \tag{2.15}$$

Hence by (2.14)

$$\begin{aligned} r_t - r_0 &\leq d(\psi_t(q), q) \\ &\leq f(\psi_t(q)) - f(q) + \frac{1}{\alpha} \int_0^t S(\psi_s(q)) ds \\ &\leq f(\psi_t(q)) - f(q) + \frac{1}{\alpha} \int_0^{r_t + r_0} S(\psi_s(q)) ds. \end{aligned}$$

We also used (2.15) in the last inequality. Dividing the inequality by r_t, we deduce that

$$1 \leq \frac{f(\psi_t(q))}{r_t} + \frac{r_0 - f(q)}{r_t} + \frac{1}{\alpha r_t} \int_0^{r_t + r_0} S(\psi_s(q)) ds. \tag{2.16}$$

Since $f(\psi_t(q)) - f(q) = t$, we see that $r_t \to \infty$ as $t \to \infty$. Using L Hospital's Rule and

$\lim_{r\to\infty} S = 0$,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{r_t}\int_0^{r_t+r_0} S(\psi_s(q))ds &= \lim_{x\to\infty}\frac{1}{x}\int_0^{x+r_0} S(\psi_s(q))ds \\ &= \lim_{x\to\infty} S(\psi_{x+r_0}(q)) \\ &= 0.\end{aligned}$$

Now taking $\limsup\, t\to\infty$ in (2.16), we have $1 \le \overline{\lim_{t\to\infty}} r_t^{-1} f(\psi_t(q))$ which contradicts with (i). We conclude that $T < \infty$ and finish the proof. □

We next show that $f + r$ is bounded from above and below provided that f is proper and scalar curvature S decays sufficiently fast.

Lemma 4. *Let (M^n, g, f) be an n dimensional non Ricci flat complete steady gradient Ricci soliton. Suppose that $\lim_{r\to\infty} f = -\infty$ and for some positive constant C*

$$S \le Ce^f \text{ on } M.$$

Then there exists positive constant c_0 such that

$$r - c_0 \le -f \le r + c_0 \text{ on } M.$$

Proof. The upper bound of $-f$ again follows from (2.6). For the lower bound, there is a small t_0 such that on $\{f \le t_0\}$,

$$Ce^f \le \frac{1}{2} \text{ and} \tag{2.17}$$

$$\begin{aligned}|\nabla f|^2 &= 1 - S \\ &\ge 1 - Ce^f \\ &\ge \frac{1}{2}.\end{aligned}$$

Let $z \in \{f < t_0\}$ and $t := f(z)$. We again denote the flow of the vector field $\frac{\nabla f}{|\nabla f|^2}$ by ψ_s, ψ_0 is the identity map. By short time existence of ordinary differential equation,

$\psi_s(z)$ exists as long as it is in $\{f \le t_0\}$. The following are true:

$$f(\psi_s(z)) - f(z) = s \text{ and}$$

$$\begin{aligned}
d(\psi_{t_0-t}(z), z) &\le \int_0^{t_0-t} |\dot{\psi}_\tau(z)| d\tau \\
&= \int_0^{t_0-t} \frac{1}{|\nabla f|} d\tau \\
&= \int_0^{t_0-t} \frac{1}{\sqrt{1-S}} d\tau \\
&\le \int_0^{t_0-t} \frac{1}{\sqrt{1-Ce^{\tau+t}}} d\tau \\
&= t_0 - t + \int_0^{t_0-t} \frac{1-\sqrt{1-Ce^{\tau+t}}}{\sqrt{1-Ce^{\tau+t}}} d\tau \\
&= f(\psi_{t_0-t}(z)) - f(z) \\
&\quad + \int_0^{t_0-t} \frac{Ce^{\tau+t}}{(1+\sqrt{1-Ce^{\tau+t}})\sqrt{1-Ce^{\tau+t}}} d\tau \\
&\le f(\psi_{t_0-t}(z)) - f(z) + \int_0^{t_0-t} \sqrt{2} Ce^{\tau+t} d\tau \\
&\le f(\psi_{t_0-t}(z)) - f(z) + \sqrt{2} Ce^{t_0}.
\end{aligned}$$

From the above inequality and $f(\psi_{t_0-t}(z)) = t_0$, we have

$$\begin{aligned}
-f(z) &\ge r(z) - r(\psi_{t_0-t}(z)) - f(\psi_{t_0-t}(z)) - \sqrt{2} Ce^{t_0} \\
&\ge r(z) - \sup_K r(\cdot) - \sup_K |f| - \sqrt{2} Ce^{t_0},
\end{aligned}$$

where $K := \{f \ge t_0\}$ is a compact set. Result follows. □

2.3 Proof of theorem 13

Motivated by the study of certain weighted elliptic equation and the choice of barrier functions by Deruelle [49] and Munteanu-Sung-Wang [76], we estimate the decay rate of subsolution u of $\Delta_f u \ge -Au^2$, where $A \ge 0$ is a constant.

Lemma 5. *Let (M^n, g, f) be an n dimensional non Ricci flat complete steady gradient*

Ricci soliton. Suppose that $\lim_{r\to\infty} f = -\infty$ *and* u *is a non-negative subsolution of the following differential inequality*

$$\Delta_f u \geq -Au^2, \tag{2.18}$$

where A *is a non-negative constant. If in addition* $\overline{\lim_{r\to\infty}} ru =: B < \infty$ *and* $AB < 1$*, then there exists positive constant* C *such that*

$$u \leq C(v^2+1)e^{-v} \text{ on } M, \tag{2.19}$$

where $v := -f$*.*

Proof. Since $v = -f \to \infty$, $v > 0$ near infinity. Pick any $\alpha \in (AB, 1)$, we will show that $v^\alpha u$ and $v^{3\alpha}e^{-v}$ are subharmonic and superharmonic functions respectively with respect to the operator $\Delta_{f+2\alpha\ln v}$, then we will apply the maximum principle to $v^\alpha u - bv^{3\alpha}e^{-v}$ for some well chosen constant b. For $v^\alpha u$, we compute directly using (2.9) and (2.18)

$$\begin{aligned}
\Delta_f(v^\alpha u) &= v^\alpha \Delta_f u + 2\langle \nabla v^\alpha, \nabla(v^\alpha u v^{-\alpha})\rangle + u\Delta_f v^\alpha \\
&= v^\alpha \Delta_f u + 2\alpha\langle \nabla \ln v, \nabla(v^\alpha u)\rangle - 2\alpha^2 |\nabla v|^2 v^{\alpha-2}u + u\Delta_f v^\alpha \\
&\geq -Av^\alpha u^2 + 2\alpha\langle \nabla \ln v, \nabla(v^\alpha u)\rangle - 2\alpha^2|\nabla v|^2 v^{\alpha-2}u \\
&\quad +\alpha v^{\alpha-1}u + \alpha(\alpha-1)|\nabla v|^2 v^{\alpha-2}u \\
&= \alpha v^{\alpha-1}u - Av^\alpha u^2 - \alpha(\alpha+1)v^{\alpha-2}|\nabla v|^2 u \\
&\quad +2\alpha\langle \nabla \ln v, \nabla(v^\alpha u)\rangle.
\end{aligned}$$

Hence we get the following inequality

$$\begin{aligned}
\Delta_{f+2\alpha\ln v}(v^\alpha u) &\geq v^{\alpha-1}u\big[\alpha - Auv - \alpha(\alpha+1)\frac{|\nabla v|^2}{v}\big] \\
&= v^{\alpha-1}u\big[\alpha - AB + A(B-uv) - \alpha(\alpha+1)\frac{|\nabla v|^2}{v}\big].
\end{aligned}$$

By (2.6) and (2.10), $|\nabla f| = |\nabla v| \leq 1$, we see that $v \leq r + D$ for some constant D and thus

$$\overline{\lim_{r\to\infty}} vu \leq \overline{\lim_{r\to\infty}} ru = B.$$

Using $\lim_{r\to\infty} v = \infty$, we have

$$\begin{aligned}\Delta_{f+2\alpha\ln v}(v^\alpha u) &\geq v^{\alpha-1}u\frac{(\alpha-AB)}{4}\\ &\geq 0\end{aligned}$$

outside some compact subset of M. For the function $v^{3\alpha}e^{-v}$, by (2.10)

$$\begin{aligned}\Delta_f e^{-v} &= -e^{-v}\Delta_f v + e^{-v}|\nabla v|^2\\ &= e^{-v}(|\nabla v|^2 - 1)\\ &= -Se^{-v}.\end{aligned} \tag{2.20}$$

Direct computation also yields

$$\begin{aligned}\Delta_f v^{3\alpha} &= 3\alpha v^{3\alpha-1}\Delta_f v + 3\alpha(3\alpha-1)v^{3\alpha-2}|\nabla v|^2\\ &= 3\alpha v^{3\alpha-1} + 3\alpha(3\alpha-1)v^{3\alpha-2}|\nabla v|^2.\end{aligned}$$

Using (2.10) and $\lim\limits_{r\to\infty} v = \infty$,

$$\begin{aligned}\Delta_f(v^{3\alpha}e^{-v}) &= v^{3\alpha}\Delta_f e^{-v} + e^{-v}\Delta_f v^{3\alpha} + 2\langle\nabla v^{3\alpha}, \nabla e^{-v}\rangle\\ &= -Sv^{3\alpha}e^{-v} + 3\alpha(3\alpha-1)v^{3\alpha-2}e^{-v}|\nabla v|^2\\ &\quad +3\alpha v^{3\alpha-1}e^{-v} - 6\alpha v^{3\alpha-1}e^{-v}|\nabla v|^2\\ &= -Sv^{3\alpha}e^{-v} + 3\alpha(3\alpha-1)v^{3\alpha-2}e^{-v}|\nabla v|^2\\ &\quad -3\alpha v^{3\alpha-1}e^{-v} + 6\alpha v^{3\alpha-1}e^{-v}S\\ &= -3\alpha v^{3\alpha-1}e^{-v} + 3\alpha(3\alpha-1)v^{3\alpha-2}e^{-v}|\nabla v|^2\\ &\quad +Sv^{3\alpha}e^{-v}(-1+6\alpha v^{-1})\\ &= v^{3\alpha-1}e^{-v}\Big[-3\alpha + 3\alpha(3\alpha-1)\frac{|\nabla v|^2}{v}\Big]\\ &\quad +Sv^{3\alpha}e^{-v}(-1+6\alpha v^{-1})\\ &= v^{3\alpha-1}e^{-v}(-3\alpha+o(1)) + Sv^{3\alpha}e^{-v}(-1+o(1))\\ &\leq v^{3\alpha-1}e^{-v}(-3\alpha+o(1)).\end{aligned}$$

$$\begin{aligned} -2\alpha\langle\nabla \ln v, \nabla(v^{3\alpha}e^{-v})\rangle &= -6\alpha^2 v^{3\alpha-2}e^{-v}|\nabla v|^2 + 2\alpha v^{3\alpha-1}e^{-v}|\nabla v|^2 \\ &= -6\alpha^2 v^{3\alpha-2}e^{-v}|\nabla v|^2 + 2\alpha v^{3\alpha-1}e^{-v} \\ &\quad -2\alpha v^{3\alpha-1}e^{-v}S \\ &\leq 2\alpha v^{3\alpha-1}e^{-v}. \end{aligned}$$

We see that near infinity

$$\begin{aligned} \Delta_{f+2\alpha\ln v}(v^{3\alpha}e^{-v}) &\leq v^{3\alpha-1}e^{-v}(-\alpha + o(1)) \\ &< 0. \end{aligned}$$

Hence $v^{3\alpha}e^{-v}$ is superharmonic with respect to $\Delta_{f+2\alpha\ln v}$. To proceed, we choose a large R_0 such that on $M \setminus B_{R_0}(p_0)$, $v > 1$,

$$\Delta_{f+2\alpha\ln v}(v^{\alpha}u) \geq 0 \text{ and} \tag{2.21}$$

$$\Delta_{f+2\alpha\ln v}(v^{3\alpha}e^{-v}) < 0. \tag{2.22}$$

Pick a large $b > 0$ such that on $\partial B_{R_0}(p_0)$

$$v^{\alpha}u - bv^{3\alpha}e^{-v} < 0. \tag{2.23}$$

Let $Q := v^{\alpha}u - bv^{3\alpha}e^{-v}$. Since $\alpha < 1$ and $\overline{\lim_{r\to\infty}} vu \leq B$, we have $\lim_{r\to\infty} Q = 0$. Moreover by (2.21) and (2.22), on $M \setminus B_{R_0}(p_0)$

$$\Delta_{f+2\alpha\ln v}Q > 0.$$

By the maximum principle, we know that $Q \leq 0$ on $M \setminus B_{R_0}(p_0)$. Result follows. □

With the preliminary estimate in Lemma 5, we are able to remove the quadratic factor v^2 in (2.19) and thus improve the decay estimate of u.

Proposition 1. *Let (M^n, g, f) be an n dimensional non Ricci flat complete steady gradient Ricci soliton. Suppose that $\lim_{r\to\infty} f = -\infty$ and u is a non-negative subsolution*

of the following differential inequality

$$\Delta_f u \geq -Au^2, \tag{2.24}$$

where A is a non-negative constant. If in addition $\overline{\lim_{r\to\infty}} ru =: B < \infty$ and $AB < 1$, then u satisfies

$$u \leq Ce^{-v} \text{ on } M,$$

for some positive constant C, where $v := -f$.

Remark 20. By considering $v + \sigma$ for some large σ, one can show by similar arguments that both Lemma 5 and Proposition 1 are still true if the condition $\lim_{r\to\infty} f = -\infty$ is replaced by the boundedness of f from above by a constant.

Remark 21. The condition $AB < 1$ may look artificial but is critical indeed. In three dimensional Bryant steady soliton, $S^2 \geq 2|\mathrm{Ric}|^2$ since the sectional curvature is non-negative (see [76]). Hence the scalar curvature $S \geq 0$ satisfies

$$\Delta_f S \geq -S^2.$$

However, $\lim_{r\to\infty} rS = \lim_{r\to\infty} -r^{-1}f = 1$ (see [8]). $AB = 1$ and S doesn't decay exponentially.

For non-trivial gradient steady soliton with $\mathrm{Ric} \geq 0$ and $\lim_{r\to\infty} S = 0$, we know that $S^2 \geq |\mathrm{Ric}|^2$ and $\lim_{r\to\infty} f = -\infty$ (see [23]). We have an immediate consequence of Proposition 1 and Lemma 4:

Corollary 7. *Let (M^n, g, f) be an n dimensional complete non Ricci flat gradient steady Ricci soliton with $\mathrm{Ric} \geq 0$. Suppose that $\overline{\lim_{r\to\infty}} rS < \frac{1}{2}$, then there exists a constant $C > 0$ such that*

$$S \leq Ce^{-r} \text{ on } M.$$

Proof of Proposition 1. Firstly, we prove that $ue^{-v^{-1}}$ is subharmonic with respect to the operator $\Delta_{f-2v^{-1}}$ near infinity.

$$\begin{aligned}
\Delta_f e^{-v^{-1}} &= v^{-2}e^{-v^{-1}}\Delta_f v + |\nabla v|^2\Big(-2v^{-3}e^{-v^{-1}} + v^{-4}e^{-v^{-1}}\Big) \\
&= v^{-2}e^{-v^{-1}} + e^{-v^{-1}}|\nabla v|^2\Big(-2v^{-3} + v^{-4}\Big).
\end{aligned}$$

$$\begin{aligned}
\Delta_f(ue^{-v^{-1}}) &= e^{-v^{-1}}\Delta_f u + 2\langle\nabla u, \nabla e^{-v^{-1}}\rangle + u\Delta_f e^{-v^{-1}} \\
&= e^{-v^{-1}}\Delta_f u + 2v^{-2}e^{-v^{-1}}\langle\nabla(e^{-v^{-1}}ue^{v^{-1}}), \nabla v\rangle \\
&\quad + u\Delta_f e^{-v^{-1}} \\
&= e^{-v^{-1}}\Delta_f u + 2v^{-2}\langle\nabla(e^{-v^{-1}}u), \nabla v\rangle \\
&\quad - 2v^{-4}ue^{-v^{-1}}|\nabla v|^2 + u\Delta_f e^{-v^{-1}} \\
&\geq -Ae^{-v^{-1}}u^2 - ue^{-v^{-1}}|\nabla v|^2\Big(2v^{-3} + v^{-4}\Big) \\
&\quad + v^{-2}ue^{-v^{-1}} + 2v^{-2}\langle\nabla(e^{-v^{-1}}u), \nabla v\rangle.
\end{aligned}$$

Hence

$$\Delta_{f-2v^{-1}}(ue^{-v^{-1}}) \geq v^{-2}ue^{-v^{-1}}\Big[1 - Av^2u - |\nabla v|^2\big(2v^{-1} + v^{-2}\big)\Big]. \tag{2.25}$$

By Lemma 5, $v^2u \to 0$ as $r \to \infty$. Therefore it is not difficult to see that $\Delta_{f-2v^{-1}}(ue^{-v^{-1}}) \geq 0$ outside a compact subset of M.

Secondly, we want to show that $e^{-v}e^{-4v^{-1}}$ is superharmonic with respect to the operator $\Delta_{f-2v^{-1}}$ near infinity. Using (2.20) in Lemma 5

$$\Delta_f e^{-v} = -Se^{-v}.$$

$$\begin{aligned}
\Delta_f e^{-4v^{-1}} &= 4v^{-2}e^{-4v^{-1}}\Delta_f v + 8|\nabla v|^2e^{-4v^{-1}}\big(-v^{-3} + 2v^{-4}\big) \\
&= 4v^{-2}e^{-4v^{-1}} + 8|\nabla v|^2e^{-4v^{-1}}\big(-v^{-3} + 2v^{-4}\big).
\end{aligned}$$

By $\lim_{r\to\infty} v = \infty$, outside a compact subset of M,

$$\begin{aligned}
\Delta_f(e^{-v}e^{-4v^{-1}}) &= e^{-4v^{-1}}\Delta_f e^{-v} + e^{-v}\Delta_f e^{-4v^{-1}} + 2\langle\nabla e^{-v}, \nabla e^{-4v^{-1}}\rangle \\
&= e^{-4v^{-1}}\Delta_f e^{-v} + e^{-v}\Delta_f e^{-4v^{-1}} - 8e^{-v}e^{-4v^{-1}}v^{-2}|\nabla v|^2 \\
&= e^{-4v^{-1}}\Delta_f e^{-v} + \Big[e^{-v}\Delta_f e^{-4v^{-1}} - 8e^{-v}e^{-4v^{-1}}v^{-2}\Big] \\
&\quad +8e^{-v}e^{-4v^{-1}}v^{-2}S \\
&= e^{-4v^{-1}}\Delta_f e^{-v} + 8e^{-v}e^{-4v^{-1}}v^{-2}S - 4e^{-v}e^{-4v^{-1}}v^{-2} \\
&\quad +8|\nabla v|^2 e^{-v}e^{-4v^{-1}}\big(-v^{-3} + 2v^{-4}\big) \\
&= -Se^{-v}e^{-4v^{-1}} + 8e^{-v}e^{-4v^{-1}}v^{-2}S - 4e^{-v}e^{-4v^{-1}}v^{-2} \\
&\quad +8|\nabla v|^2 e^{-v}e^{-4v^{-1}}\big(-v^{-3} + 2v^{-4}\big) \\
&= e^{-v}e^{-4v^{-1}}v^{-2}\Big[-4 + 8|\nabla v|^2\big(-v^{-1} + 2v^{-2}\big)\Big] \\
&\quad +Se^{-v}e^{-4v^{-1}}(-1 + 8v^{-2}) \\
&\leq e^{-v}e^{-4v^{-1}}v^{-2}(-4 + o(1)).
\end{aligned}$$

$$\begin{aligned}
-\langle\nabla(-2v^{-1}), \nabla(e^{-v}e^{-4v^{-1}})\rangle &= -2v^{-2}\langle\nabla v, \nabla(e^{-v}e^{-4v^{-1}})\rangle \\
&= 2e^{-v}e^{-4v^{-1}}v^{-2}|\nabla v|^2 \\
&\quad -8e^{-v}e^{-4v^{-1}}v^{-4}|\nabla v|^2 \\
&\leq 2e^{-v}e^{-4v^{-1}}v^{-2}.
\end{aligned}$$

$$\begin{aligned}
\Delta_{f-2v^{-1}}(e^{-v}e^{-4v^{-1}}) &= \Delta_f(e^{-v}e^{-4v^{-1}}) - \langle\nabla(-2v^{-1}), \nabla(e^{-v}e^{-4v^{-1}})\rangle \\
&\leq e^{-v}e^{-4v^{-1}}v^{-2}(-2 + o(1)) \\
&< 0.
\end{aligned}$$

We showed that $e^{-v}e^{-4v^{-1}}$ is superharmonic with respect to $\Delta_{f-2v^{-1}}$ near infinity.

To finish the proof, we choose a large R_0 such that on $M \setminus B_{R_0}(p_0)$, $v > 1$,

$$\Delta_{f-2v^{-1}}(ue^{-v^{-1}}) \geq 0 \text{ and} \tag{2.26}$$

$$\Delta_{f-2v^{-1}}(e^{-v}e^{-4v^{-1}}) < 0. \tag{2.27}$$

Let $b > 0$ be a large number such that on $\partial B_{R_0}(p_0)$

$$ue^{-v^{-1}} - be^{-v}e^{-4v^{-1}} < 0. \tag{2.28}$$

Let $Q := ue^{-v^{-1}} - be^{-v}e^{-4v^{-1}}$. Obviously, we have $\lim_{r\to\infty} Q = 0$. Moreover by (2.26) and (2.27), on $M \setminus B_{R_0}(p_0)$

$$\Delta_{f-2v^{-1}}Q > 0.$$

By the maximum principle, $Q \leq 0$ on $M \setminus B_{R_0}(p_0)$ which implies that

$$u \leq be^{-v}e^{-3v^{-1}} \leq be^{-v}.$$

Choosing a larger b if necessary, the above inequality holds globally on M. □

We are about to prove the main theorem. For the convenience of reader, we repeat the main theorem here:

Theorem. *Let (M, g, f) be an n dimensional non Ricci flat complete steady gradient Ricci soliton with $n \geq 2$. Suppose the potential function f is bounded from above by a constant and $\overline{\lim_{r\to\infty}} r|Rm| < \frac{1}{5}$. Then there exists a positive constant C such that*

$$|Rm|(x) \leq Ce^{-r(x)} \text{ on } M. \tag{2.29}$$

Proof. The potential function f is bounded above by a constant and $|Rm| \to 0$ as $r \to \infty$, hence by Lemma 3, $\lim_{r\to\infty} f = -\infty$ (one may also apply the inequality $S \geq Ce^f$ in [76]). From the evolution equation of Rm of a solution to the Ricci flow ([38] c.f. P.119 Lemma 2.51), we have in any orthonormal frame,

$$\Delta_f R_{abcd} = 2(R_{apbq}R_{cpdq} - R_{apbq}R_{dpcq} + R_{apcq}R_{bpdq} - R_{apdq}R_{bpcq}).$$

By first Bianchi identity,

$$R_{cpdq} - R_{dpcq} = R_{cdpq} \text{ and}$$

$$R_{apbq} = R_{aqbp} + R_{abpq}.$$

Therefore we have

$$\begin{aligned} R_{apbq}R_{cpdq} - R_{apbq}R_{dpcq} &= R_{apbq}R_{cdpq} \\ &= R_{aqbp}R_{cdpq} + R_{abpq}R_{cdpq} \\ &= -R_{apbq}R_{cdpq} + R_{abpq}R_{cdpq}. \end{aligned}$$

We conclude that

$$\begin{aligned} 2R_{apbq}R_{cpdq} - 2R_{apbq}R_{dpcq} &= 2R_{apbq}R_{cdpq} \\ &= R_{abpq}R_{cdpq}. \end{aligned}$$

The equation for $\Delta_f Rm$ becomes

$$\Delta_f R_{abcd} = R_{abpq}R_{cdpq} + 2(R_{apcq}R_{bpdq} - R_{apdq}R_{bpcq}).$$

The above formula can also be derived using the evolution equation in [7]. Using Kato's inequality and $2|Rm|\Delta_f|Rm| + 2|\nabla|Rm||^2 = 2|\nabla Rm|^2 + 2\langle Rm, \Delta_f Rm\rangle$, we see that

$$\Delta_f|Rm| \geq -5|Rm|^2 \tag{2.30}$$

holds in the weak sense and on the set of points where $|Rm|$ is non-zero. The soliton is not Ricci flat and $S > 0$, we may take $u = |Rm| > 0$. By Proposition 1 and $\overline{\lim_{r\to\infty}} r|Rm| < 5^{-1}$,

$$\begin{aligned} |Rm| &\leq Ce^{-v} \\ &= Ce^{f}, \end{aligned}$$

which implies that $S \leq Ce^f$. We apply Lemma 4 and conclude that $|Rm| \leq Ce^{-r+c_0}$

on M.

□

2.4 Proofs of theorem 14 and 15

We recall the following estimate for the Ricci curvature of four dimensional gradient steady Ricci soliton by Cao-Cui [21]:

Theorem 32. *[21] Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. Then Rm is bounded and $\forall\ a \in (0, \frac{1}{2})$, there exists a positive constant C such that*

$$|Ric| \leq CS^a \text{ on } M. \tag{2.31}$$

We first prove a slightly better estimate for the Ricci curvature.

Lemma 6. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. There exists a positive constant A_1 such that*

$$|Ric| \leq A_1 S \text{ on } M. \tag{2.32}$$

Remark 22. The constant A_1 depends on A_0 in (2.11) and $\sup_{B_{R_0}(p_0)} \frac{|\mathrm{Ric}|}{S}$ for some large ball $B_{R_0}(p_0)$.

Proof. We claim that for some large $R_0 > 0$, on $M \setminus B_{R_0}(p_0)$

$$\Delta_f(|\mathrm{Ric}| + |\mathrm{Ric}|^2) \geq -(6A_0 + 20A_0^2)|\mathrm{Ric}|^2, \tag{2.33}$$

where A_0 is the universal constant in (2.11). We first assume the claim and prove the lemma. There exists a large constant $C > 0$ such that the following inequalities are true

$$|\mathrm{Ric}| + |\mathrm{Ric}|^2 - CS < 0 \text{ on } \partial B_{R_0}(p_0), \tag{2.34}$$

$$C \geq 3A_0 + 10A_0^2 + 1. \tag{2.35}$$

Using (2.33), (2.35) and (2.4),

$$\begin{aligned}\Delta_f(|\mathrm{Ric}| + |\mathrm{Ric}|^2 - CS) &= \Delta_f(|\mathrm{Ric}| + |\mathrm{Ric}|^2) + 2C|Ric|^2 \\ &\geq (2C - 6A_0 - 20A_0^2)|\mathrm{Ric}|^2 \\ &\geq 2|\mathrm{Ric}|^2 \\ &> 0.\end{aligned}$$

By Theorem 32, we have $\lim_{r\to\infty}(|\mathrm{Ric}| + |\mathrm{Ric}|^2 - CS) = 0$. Using the maximum principle with boundary condition (2.34), we see that on $M \setminus B_{R_0}(p_0)$, $|\mathrm{Ric}| + |\mathrm{Ric}|^2 - CS \leq 0$. Choosing a bigger constant C if necessary,

$$\begin{aligned}|\mathrm{Ric}| &\leq |\mathrm{Ric}| + |\mathrm{Ric}|^2 \\ &\leq CS\end{aligned}$$

holds globally on M. To finish the argument, we need to show the inequality (2.33) is true. By (2.5) and (2.11)

$$\begin{aligned}\Delta_f|\mathrm{Ric}|^2 &= 2|\nabla\mathrm{Ric}|^2 + 2\langle\mathrm{Ric}, \Delta_f\mathrm{Ric}\rangle \\ &= 2|\nabla\mathrm{Ric}|^2 - 4R_{ij}R_{iklj}R_{kl} \\ &\geq 2|\nabla\mathrm{Ric}|^2 - 4|Rm||\mathrm{Ric}|^2 \\ &\geq 2|\nabla\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - 4A_0|\mathrm{Ric}|^2\frac{|\nabla\mathrm{Ric}|}{|\nabla f|}.\end{aligned}$$

Using Kato's inequality, we have

$$\Delta_f|\mathrm{Ric}| \geq -2A_0|\mathrm{Ric}|^2 - 2A_0|\mathrm{Ric}|\frac{|\nabla\mathrm{Ric}|}{|\nabla f|}. \tag{2.36}$$

We can further simplify the inequality for $|Ric|^2$,

$$\begin{aligned}
\Delta_f|\mathrm{Ric}|^2 &\geq |\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 + \Big(|\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^2\frac{|\nabla \mathrm{Ric}|}{|\nabla f|}\Big) \\
&= |\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&\quad + \Big(|\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^2\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} + \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2}\Big) \\
&= |\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&\quad + \Big(|\nabla \mathrm{Ric}| - \frac{2A_0|\mathrm{Ric}|^2}{|\nabla f|}\Big)^2 \\
&\geq |\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2}. \qquad (2.37)
\end{aligned}$$

Hence by (2.36)

$$\begin{aligned}
\Delta_f(|\mathrm{Ric}| + |\mathrm{Ric}|^2) &\geq -2A_0|\mathrm{Ric}|^2 - 2A_0|\mathrm{Ric}|\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} \\
&\quad + |\nabla \mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&= \Big(|\nabla \mathrm{Ric}|^2 - 2A_0|\mathrm{Ric}|\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} + \frac{A_0^2|\mathrm{Ric}|^2}{|\nabla f|^2}\Big) \\
&\quad - \frac{A_0^2|\mathrm{Ric}|^2}{|\nabla f|^2} - 2A_0|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 \\
&\quad - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&= \Big(|\nabla \mathrm{Ric}| - \frac{A_0|\mathrm{Ric}|}{|\nabla f|}\Big)^2 - \frac{A_0^2|\mathrm{Ric}|^2}{|\nabla f|^2} \\
&\quad - 2A_0|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&\geq -\frac{A_0^2|\mathrm{Ric}|^2}{|\nabla f|^2} - 2A_0|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 \\
&\quad - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2}.
\end{aligned}$$

By Theorem 32, (2.6) and $\lim_{r\to\infty} S = 0$, there exists a large positive R_0 such that on

$M \setminus B_{R_0}(p_0)$

$$|\mathrm{Ric}| \le 1 \text{ and}$$

$$|\nabla f| \ge \frac{1}{2}.$$

Using the above two inequalities, we have

$$\begin{aligned}
\Delta_f(|\mathrm{Ric}| + |\mathrm{Ric}|^2) &\ge -\frac{A_0^2|\mathrm{Ric}|^2}{|\nabla f|^2} - 2A_0|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 \\
&\quad -\frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&\ge -4A_0^2|\mathrm{Ric}|^2 - 2A_0|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^2 \\
&\quad -16A_0^2|\mathrm{Ric}|^2 \\
&= -(6A_0 + 20A_0^2)|\mathrm{Ric}|^2.
\end{aligned}$$

We are done with the proof of the lemma. □

We can apply the Ricci curvature estimate to bound the curvature tensor Rm. To prepare for the proof, we start with the following computational lemma.

Lemma 7. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. There exists a large constant $R_0 > 1$ such that for all $\lambda \ge 0$, on $M \setminus B_{R_0}(p_0)$*

$$\Delta_{f-2\ln S}(S^{-2}|Ric|^2) \ge \frac{1}{32A_0^2}\Big(S^{-1}|Rm| + \lambda S^{-2}|Ric|^2\Big)^2 - c\lambda^2 - c, \tag{2.38}$$

where c is a positive constant depending only on A_0 and A_1 in (2.11) and (2.32) respectively.

Proof.

$$\Delta_f(S^{-2}|\mathrm{Ric}|^2) = S^{-2}\Delta_f|Ric|^2 + 2\langle\nabla S^{-2}, \nabla|\mathrm{Ric}|^2\rangle + |\mathrm{Ric}|^2\Delta_f S^{-2}.$$

By (2.4)

$$\begin{aligned}
\Delta_f S^{-2} &= -2S^{-3}\Delta_f S + 6S^{-4}|\nabla S|^2 \\
&= 4S^{-3}|\mathrm{Ric}|^2 + 6S^{-2}|\nabla \ln S|^2.
\end{aligned}$$

$$\begin{aligned}\langle \nabla S^{-2}, \nabla |\mathrm{Ric}|^2 \rangle &= -2S^{-3}\langle \nabla S, \nabla (S^{-2}|\mathrm{Ric}|^2 S^2)\rangle \\ &= -2\langle \nabla \ln S, \nabla (S^{-2}|\mathrm{Ric}|^2)\rangle - 4|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2.\end{aligned}$$

By Kato's inequality,

$$\begin{aligned}\langle \nabla S^{-2}, \nabla |\mathrm{Ric}|^2 \rangle &= -2S^{-3}\langle \nabla S, 2|\mathrm{Ric}|\nabla |\mathrm{Ric}|\rangle \\ &\geq -4S^{-2}|\nabla \ln S||\mathrm{Ric}||\nabla \mathrm{Ric}| \\ &\geq -\frac{1}{2}S^{-2}|\nabla \mathrm{Ric}|^2 - 8|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2.\end{aligned}$$

$$\begin{aligned}\Delta_f (S^{-2}|\mathrm{Ric}|^2) &\geq S^{-2}\Delta_f |Ric|^2 - 2\langle \nabla \ln S, \nabla (S^{-2}|\mathrm{Ric}|^2)\rangle \\ &\quad -12|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2 - \frac{1}{2}S^{-2}|\nabla \mathrm{Ric}|^2 + |\mathrm{Ric}|^2 \Delta_f S^{-2} \\ &= S^{-2}\Delta_f |Ric|^2 - 2\langle \nabla \ln S, \nabla (S^{-2}|\mathrm{Ric}|^2)\rangle \\ &\quad -12|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2 - \frac{1}{2}S^{-2}|\nabla \mathrm{Ric}|^2 \\ &\quad +4S^{-3}|\mathrm{Ric}|^4 + 6|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2 \\ &\geq S^{-2}\Delta_f |Ric|^2 - 2\langle \nabla \ln S, \nabla (S^{-2}|\mathrm{Ric}|^2)\rangle \\ &\quad -6|\nabla \ln S|^2 S^{-2}|\mathrm{Ric}|^2 - \frac{1}{2}S^{-2}|\nabla \mathrm{Ric}|^2.\end{aligned}$$

By Lemma 6 and (2.3), $|\nabla S| = 2|\mathrm{Ric}(\nabla f)| \leq 2|\mathrm{Ric}||\nabla f| \leq 2A_1 S$, where A_1 is the constant in (2.32). We now consider a large $R_0 > 0$ such that on $M \setminus B_{R_0}(p_0)$, $|\mathrm{Ric}| \leq 1$

and $|\nabla f| \geq \frac{1}{2}$. Hence by (2.37) in Lemma 6, on $M \setminus B_{R_0}(p_0)$,

$$\begin{aligned}
\Delta_{f-2\ln S}(\frac{|\mathrm{Ric}|^2}{S^2}) &\geq \frac{|\nabla \mathrm{Ric}|^2}{2S^2} - 4A_0\frac{|\mathrm{Ric}|^3}{S^2} \\
&\quad -\frac{4A_0^2|\mathrm{Ric}|^4}{S^2|\nabla f|^2} - 6|\nabla \ln S|^2\frac{|\mathrm{Ric}|^2}{S^2} \\
&\geq \frac{|\nabla \mathrm{Ric}|^2}{2S^2} - (4A_0 + 16A_0^2 + 6|\nabla \ln S|^2)\frac{|\mathrm{Ric}|^2}{S^2} \\
&\geq \frac{|\nabla \mathrm{Ric}|^2}{2S^2} - (4A_0 + 16A_0^2 + 24A_1^2)\frac{|\mathrm{Ric}|^2}{S^2} \\
&\geq \frac{|\nabla \mathrm{Ric}|^2}{2S^2} - (4A_0 + 16A_0^2 + 24A_1^2)A_1^2.
\end{aligned}$$

Using (2.11), $|Rm|^2 \leq 2A_0^2(|\mathrm{Ric}|^2 + \frac{|\nabla \mathrm{Ric}|^2}{|\nabla f|^2}) \leq 2A_0^2(|\mathrm{Ric}|^2 + 4|\nabla \mathrm{Ric}|^2)$. We apply the inequality $2^{-1}(a+b)^2 - b^2 \leq a^2$ to conclude that for all $\lambda \geq 0$,

$$\begin{aligned}
\Delta_{f-2\ln S}(S^{-2}|\mathrm{Ric}|^2) &\geq \frac{1}{16A_0^2}S^{-2}|Rm|^2 - \frac{1}{8}S^{-2}|\mathrm{Ric}|^2 \\
&\quad -(4A_0 + 16A_0^2 + 24A_1^2)A_1^2 \\
&\geq \frac{1}{32A_0^2}\Big(S^{-1}|Rm| + \lambda S^{-2}|\mathrm{Ric}|^2\Big)^2 - \frac{1}{8}S^{-2}|\mathrm{Ric}|^2 \\
&\quad -\frac{\lambda^2}{16A_0^2}S^{-4}|\mathrm{Ric}|^4 - (4A_0 + 16A_0^2 + 24A_1^2)A_1^2 \\
&\geq \frac{1}{32A_0^2}\Big(S^{-1}|Rm| + \lambda S^{-2}|\mathrm{Ric}|^2\Big)^2 - \frac{\lambda^2}{16A_0^2}A_1^4 \\
&\quad -\frac{1}{8}A_1^2 - (4A_0 + 16A_0^2 + 24A_1^2)A_1^2 \\
&\geq \frac{1}{32A_0^2}\Big(S^{-1}|Rm| + \lambda S^{-2}|\mathrm{Ric}|^2\Big)^2 - c\lambda^2 - c,
\end{aligned}$$

where c only depends on A_0 and A_1. This ends the proof of the lemma. □

Let us recall the statement of Theorem 14:

Theorem. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. There exists a positive constant c such that*

$$|Rm| \leq cS \text{ on } M. \tag{2.39}$$

Remark 23. The constant c in (2.39) again depends on A_0 in (2.11), A_1 in (2.32) and $\sup_{B_{R_0}(p_0)} S^{-1}|Rm|$ for some large ball $B_{R_0}(p_0)$.

Remark 24. No assumption on the potential function f is made in the above theorem. In particular, f is not known to be proper in this setting. To construct a cut off function for the maximum principle argument, we use the distance function r together with an appropriate Laplacian comparison theorem in [96] (see also Theorem 11).

Proof. The argument is essentially due to Munteanu-Wang [80] and Cao-Cui [21]. We know that by (2.30)

$$\Delta_f |Rm| \geq -5|Rm|^2.$$

$$\begin{aligned}
\Delta_f(S^{-1}|Rm|) &= S^{-1}\Delta_f|Rm| + 2\langle \nabla S^{-1}, \nabla(SS^{-1}|Rm|)\rangle \\
&\quad + |Rm|\Delta_f S^{-1} \\
&= S^{-1}\Delta_f|Rm| - 2\langle \nabla \ln S, \nabla(S^{-1}|Rm|)\rangle \\
&\quad -2|\nabla \ln S|^2 S^{-1}|Rm| + |Rm|\Delta_f S^{-1} \\
&\geq -5S^{-1}|Rm|^2 - 2\langle \nabla \ln S, \nabla(S^{-1}|Rm|)\rangle \\
&\quad + |Rm|(2S^{-2}|\mathrm{Ric}|^2 + 2S^{-1}|\nabla \ln S|^2) \\
&\quad -2|\nabla \ln S|^2 S^{-1}|Rm| \\
&\geq -5S^{-1}|Rm|^2 - 2\langle \nabla \ln S, \nabla(S^{-1}|Rm|)\rangle.
\end{aligned}$$

Hence

$$\Delta_{f-2\ln S}(S^{-1}|Rm|) \geq -5S^{-1}|Rm|^2. \tag{2.40}$$

We apply Lemma 7 and take $\lambda = 192A_0^2$, on $M \setminus B_{R_0}(p_0)$

$$\begin{aligned}
\Delta_{f-2\ln S}\Big(\frac{|Rm|}{S} + \lambda\frac{|\mathrm{Ric}|^2}{S^2}\Big) &\geq \big(\frac{\lambda}{32A_0^2} - 5S\big)\Big(\frac{|Rm|}{S} + \lambda\frac{|\mathrm{Ric}|^2}{S^2}\Big)^2 \\
&\quad -c\lambda^3 - c\lambda \\
&= \big(6 - 5S\big)\Big(\frac{|Rm|}{S} + \lambda\frac{|\mathrm{Ric}|^2}{S^2}\Big)^2 \\
&\quad -c\lambda^3 - c\lambda \\
&\geq \Big(\frac{|Rm|}{S} + \lambda\frac{|\mathrm{Ric}|^2}{S^2}\Big)^2 - c,
\end{aligned} \tag{2.41}$$

where c only depends on A_0 and A_1. Let $W := \left(S^{-1}|Rm|+\lambda S^{-2}|\mathrm{Ric}|^2\right)$ and $G := \phi^2 W$, where ϕ is a non-negative cut off function. By (2.41), on the set where ϕ is positive,

$$\begin{aligned}
\Delta_{f-2\ln S}G &= \phi^2\Delta_{f-2\ln S}W + 4\phi\langle\nabla\phi, \nabla(\phi^2 W\phi^{-2})\rangle + W\Delta_{f-2\ln S}\phi^2 \\
&= \phi^2\Delta_{f-2\ln S}W + 4\langle\nabla\ln\phi, \nabla G\rangle - 8|\nabla\phi|^2 W \\
&\quad + W\Delta_{f-2\ln S}\phi^2 \\
&= \phi^2\Delta_{f-2\ln S}W + 4\langle\nabla\ln\phi, \nabla G\rangle \\
&\quad + (2\phi\Delta_f\phi + 4\phi\langle\nabla\ln S, \nabla\phi\rangle - 6|\nabla\phi|^2)W \\
&\geq \phi^2 W^2 - c\phi^2 + 4\langle\nabla\ln\phi, \nabla G\rangle \\
&\quad + (2\phi\Delta_f\phi - 4\phi^2|\nabla\ln S|^2 - 7|\nabla\phi|^2)W.
\end{aligned}$$

By Lemma 6 and (2.3), $|\nabla S| = |2\mathrm{Ric}(\nabla f)| \leq 2A_1 S$, we deduce that

$$\begin{aligned}
\phi^2\Delta_{f-2\ln S}G &\geq G^2 - c\phi^4 + 4\phi\langle\nabla\phi, \nabla G\rangle \\
&\quad + (2\phi\Delta_f\phi - 16\phi^2 A_1^2 - 7|\nabla\phi|^2)G.
\end{aligned} \tag{2.42}$$

Let $R \geq R_0 \geq 1$ and $\psi : [0,\infty) \to \mathbb{R}$ be a smooth real valued function satisfying the following: $0 \leq \psi \leq 1$, $\psi' \leq 0$,

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 2 \leq t \end{cases}$$

and

$$|\psi''(t)| + |\psi'(t)| \leq c \text{ for all } t \geq 0.$$

We take $\phi(x) := \psi(\frac{r(x)}{R})$, then

$$|\nabla\phi| = \frac{|\psi'|}{R} \leq \frac{c}{R}. \tag{2.43}$$

Using the Laplacian comparison theorem for smooth metric measure space with $\mathrm{Ric}_f \geq 0$ and $|\nabla f| \leq 1$ (Theorem 1.1a in [96], see also Theorem 11), we have

$$\Delta_f r \leq \frac{3}{r} + 1.$$

$$\begin{aligned}\Delta_f\phi &= \frac{\psi'}{R}\Delta_f r + \frac{\psi''}{R^2}|\nabla r|^2 \\ &\geq -\frac{c}{R}\big(\frac{3}{R}+1\big) - \frac{c}{R^2} \\ &\geq -\frac{c_1}{R}.\end{aligned}$$

$$\begin{aligned}2\phi\Delta_f\phi - 16\phi^2 A_1^2 - 7|\nabla\phi|^2 &\geq -\frac{c_2}{R} - 16A_1^2 \\ &\geq -c,\end{aligned} \tag{2.44}$$

where c is a constant depending on A_1. Suppose G attains its maximum at q. If $q \in \overline{B_{R_0}(p_0)}$, then

$$G \leq G(q) \leq W(q) \leq \sup_{\overline{B_{R_0}(p_0)}} W,$$

right hand side is obviously independent of R. If $q \in M \setminus \overline{B_{R_0}(p_0)}$, then by (2.42) and (2.44), we have

$$\begin{aligned}0 &\geq \phi^2(q)\Delta_{f-2\ln S}G(q) \\ &\geq G^2(q) - c - cG(q).\end{aligned}$$

From this we see that there exists a positive constant C depending only on A_0 and A_1 such that

$$G \leq G(q) \leq C.$$

Result then follows by letting $R \to \infty$. □

The constant c in Theorem 14 may depend on the local geometry of (M, g) and provide no information on the size of $r|Rm|$ near infinity if one simply assumes $\overline{\lim}_{r\to\infty} rS$ is small. To prove Theorem 15 by Theorem 13 and 14, we look for a universal positive constant in front of the scalar curvature S and then estimate the size of $\overline{\lim}_{r\to\infty} r|Rm|$. To have a better control on $\overline{\lim}_{r\to\infty} r|Rm|$, we need to refine our previous estimate on the Ricci curvature:

Proposition 2. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient*

steady Ricci soliton with $\lim_{r\to\infty} S = 0$. *Suppose that* $\lim_{r\to\infty} f = -\infty$ *and* $\overline{\lim_{r\to\infty}} rS < \infty$, *then there exists a positive constant* C *such that*

$$|Ric| \leq A_0 S + C v^{-\frac{3}{2}} \tag{2.45}$$

outside some compact subset of M, *where* $v := -f$. *In particular,*

$$\overline{\lim_{r\to\infty}} r|Ric| \leq A_0 \overline{\lim_{r\to\infty}} rS. \tag{2.46}$$

Proof. Since $\overline{\lim_{r\to\infty}} rS < \infty$, we have by Lemma 3 and Theorem 14 there exists a constant C_0 such that outside a compact set,

$$|\mathrm{Ric}| \leq c|Rm| \leq CS \leq \frac{C_0}{v}. \tag{2.47}$$

By viewing steady soliton as a solution to the Ricci flow and Shi's derivative estimate [38],

$$|\nabla \mathrm{Ric}| \leq c|\nabla Rm| \leq \frac{C_1}{v^{\frac{3}{2}}}. \tag{2.48}$$

Let R_0 be a large positive number such that on $M \setminus B_{R_0}(p_0)$, $|\nabla f| \geq \frac{1}{2}$, $v > 1$, (2.47) and (2.48) hold, moreover, v satisfies

$$\frac{1}{2} - \frac{15}{4} v^{-1} \geq 0.$$

We then choose a big positive constant C such that $C \geq 4A_0 C_0 C_1 + 1$ and

$$|\mathrm{Ric}| - A_0 S - C v^{-\frac{3}{2}} < 0 \text{ on } \partial B_{R_0}(p_0).$$

By (2.4) in Section 2, (2.36) in Lemma 6, (2.47) and (2.48), we have

$$\begin{aligned}
\Delta_f(|\mathrm{Ric}| - A_0 S - Cv^{-\frac{3}{2}}) &\geq -2A_0|\mathrm{Ric}|^2 - 2A_0|\mathrm{Ric}|\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} + 2A_0|\mathrm{Ric}|^2 \\
&\quad + \frac{3C}{2}v^{-\frac{5}{2}}\Delta_f v - \frac{15C}{4}|\nabla v|^2 v^{-\frac{7}{2}} \\
&\geq -2A_0|\mathrm{Ric}|\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} + \frac{3C}{2}v^{-\frac{5}{2}} - \frac{15C}{4}v^{-\frac{7}{2}} \\
&\geq -4A_0C_0C_1v^{-\frac{5}{2}} + \frac{3C}{2}v^{-\frac{5}{2}} - \frac{15C}{4}v^{-\frac{7}{2}} \\
&= v^{-\frac{5}{2}}\Big(\frac{3C}{2} - 4A_0C_0C_1 - \frac{15C}{4}v^{-1}\Big) \\
&\geq v^{-\frac{5}{2}}\Big(C - 4A_0C_0C_1\Big) \\
&\geq v^{-\frac{5}{2}} \\
&> 0.
\end{aligned}$$

Since $\lim_{r\to\infty}(|\mathrm{Ric}| - A_0 S - Cv^{-\frac{3}{2}}) = 0$, by the maximum principle,

$$|\mathrm{Ric}| - A_0 S - Cv^{-\frac{3}{2}} \leq 0 \text{ on } M \setminus B_{R_0}(p_0).$$

□

For the reader's convenience, we recall the statement of Theorem 15:

Theorem. *Let (M^4, g, f) be a four dimensional complete non Ricci flat gradient steady Ricci soliton with $\lim_{r\to\infty} S = 0$. Suppose the potential function f is bounded from above by a constant and $\overline{\lim_{r\to\infty}} rS < \frac{1}{5A_0^2}$. Then there is a constant $C > 0$ such that*

$$|Rm| \leq Ce^{-r} \text{ on } M.$$

Proof. Since $S \to 0$ as $r \to \infty$ and f is bounded from above, we know by Lemma 3 that $f \to -\infty$ as $r \to \infty$. By Proposition 2, Shi's estimate (2.48) and Lemma 2,

$$|Rm| \leq A_0^2 S + CA_0 v^{-\frac{3}{2}} + 2C_1A_0v^{-\frac{3}{2}}.$$

It implies that

$$\begin{aligned} \overline{\lim_{r\to\infty}} r|Rm| &\leq A_0^2 \overline{\lim_{r\to\infty}} rS \\ &< \frac{1}{5}. \end{aligned}$$

Hence by Theorem 13, $|Rm|$ decays exponentially.

□

Chapter 3

Kähler steady Ricci solitons with nonnegative Ricci curvature and integrable scalar curvature

The materials in this chapter are based on [26].

3.1 Preliminaries

Let (M, J) be a complex manifold and g be a Riemannian metric on M. (M, g, J) is Kähler if $g(JX, JY) = g(X, Y)$ for any tangent vectors X and Y, and $\nabla J = 0$, where ∇ is the Levi Civita connection of g. A Kähler manifold (M, g, J) is a gradient steady Kähler Ricci soliton if M satisfies (2.1) for some smooth function f on M (see [36]). Let $\beta \in \mathbb{R}$ and h be any function on M, $h = o(r^\beta)$ means that $\lim_{r\to\infty} r^{-\beta} h = 0$.

One of the simplest examples of steady Kähler Ricci soliton is the cigar soliton (Σ, g_Σ, f) (see Example 7). With the standard complex structure on $\mathbb{C}$, (Σ, g_Σ, f) is a complete gradient steady Kähler Ricci soliton. It is also the unique (up to scaling and isometry) real 2 dimensional non-flat complete gradient steady Ricci soliton (see [36], [6] and ref. therein).

A normalized geodesic $\gamma : \mathbb{R} \to M$ is called a line if for all real numbers a and b with $a \leq b$, $\gamma\,|_{[a,b]}$ is distance minimizing. We need the following splitting theorems by

Cheeger and Gromoll [28] (see also [67]).

Theorem 33. *[28] Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature. If M contains a line, then M splits isometrically as $\mathbb{R} \times M_1$ for some complete Riemannian manifold M_1 with nonnegative Ricci curvature.*

Theorem 34. *[28] Let (M, g) be a compact Riemannian manifold with nonnegative Ricci curvature. Then the universal covering space $\widetilde{M}$ of M splits isometrically as $\mathbb{R}^k \times M_0$ for some compact Riemannian manifold M_0 and integer $k \geq 0$.*

It was shown by Chen [29] that any complete ancient solution to the Ricci flow must have nonnegative scalar curvature. Using strong maximum principle, we see that any complete gradient steady Ricci soliton must have positive scalar curvature $S > 0$ unless it is Ricci flat (see also [99]). It is also known that any compact steady Ricci soliton is Ricci flat [36] and hence any non Ricci flat complete gradient steady Ricci soliton is non-compact. We shall adopt the scaling convention (2.6) throughout this thesis.

It was shown in [19] and [23] (see also [46] and Theorem 9) that for a complete gradient steady Ricci soliton with $\mathrm{Ric} > 0$ and S attaining maximum (or $\mathrm{Ric} \geq 0$ with $\limsup_{r\to\infty} S < \max_M S$), then there exist $a \in (0, 1)$ and $D > 0$ such that

$$r + D \geq -f \geq ar - D \text{ on } M.$$

We first prove a similar bound for f under different conditions which suffice for the arguments in later sections. Similar estimate was also obtained independently by Deng and Zhu [45] without non Ricci flat condition, instead $\mathrm{Ric} \geq 0$ on M and $-f$ being equivalent to r are assumed.

Proposition 3. *Let (M, g, f) be a real m dimensional complete non-Ricci flat gradient steady Ricci soliton with $Ric \geq 0$ outside some compact subset of M. Further suppose that $S \to 0$ as $r \to \infty$. Then for all $\alpha \in (0, 1)$, there exists $D > 0$ such that*

$$r + D \geq -f \geq \alpha r - D \text{ on } M, \tag{3.1}$$

where r is the distance function from a fixed reference point $p_0 \in M$. In particular, $\lim_{r\to\infty} \frac{-f}{r} = 1$.

Proof. The upper bound of $-f$ follows from (2.6) and $|\nabla f| \leq 1$. For the lower bound, let δ be a small positive constant to be chosen. Since $|\nabla f|^2 + S \equiv 1$ and $S \to 0$ at infinity, there is a compact subset K of M such that $p_0 \in K$ and on $M \setminus K$, Ric ≥ 0 and

$$|\nabla f| \geq \frac{1}{1+\delta}. \tag{3.2}$$

Let ψ_t be the flow of $\frac{\nabla f}{|\nabla f|^2}$ with ψ_0 be the identity map. Let $q \in M \setminus K$ and for small $t \geq 0$, by (3.2),

$$d(\psi_t(q), q) \leq \int_0^t \frac{1}{|\nabla f|(\psi_s(q))} ds \leq (1+\delta)t. \tag{3.3}$$

By short time existence of O.D.E., $\psi_t(q)$ exists as long as it is in $M \setminus K$. Therefore we can define T as follows

$$T := \sup\{a : \psi_t(q) \in M \setminus K \text{ for all } t \in [0, a]\}. \tag{3.4}$$

Obviously, $T = T(q) > 0$ by compactness of K. For $0 \leq t < T$

$$f(\psi_t(q)) - f(q) = \int_0^t \langle \nabla f, \dot{\psi}_s(q) \rangle ds = \int_0^t 1 ds = t. \tag{3.5}$$

We first show that $T < \infty$. Suppose not, then $T = \infty$ and by (3.5), there is a sequence of $t_k \to \infty$ such that $\psi_{t_k}(q) \to \infty$ as $k \to \infty$. But by (2.3)

$$\begin{aligned} S(\psi_{t_k}(q)) - S(q) &= \int_0^{t_k} \langle \nabla S, \dot{\psi}_s(q) \rangle ds \\ &= \int_0^{t_k} \langle \nabla S, \frac{\nabla f(\psi_s(q))}{|\nabla f|^2} \rangle ds \\ &= \int_0^{t_k} \frac{2\mathrm{Ric}(\nabla f, \nabla f)}{|\nabla f|^2} ds \\ &\geq 0. \end{aligned}$$

Hence $S(\psi_{t_k}(q)) \geq S(q) > 0$ and $\lim_{k\to\infty} S(\psi_{t_k}(q)) \neq 0$, contradicting to our assumption that $S = o(1)$. We proved that $T < \infty$ and $\psi_T(q) \in K$. By (3.3), $d(\psi_T(q), q) \leq (1+\delta)T$.

$$r(q) = d(p_0, q) \leq d(\psi_T(q), q) + d(\psi_T(q), p_0) \leq (1+\delta)T + \mathrm{diam}K,$$

where diam$K$ is the diameter of the subset K. We have

$$\begin{aligned} -f(q) &= T - f(\psi_T(q)) \\ &\geq T - \sup_K |f| \\ &\geq \frac{1}{1+\delta} r(q) - \frac{\mathrm{diam}K}{1+\delta} - \sup_K |f|. \end{aligned}$$

(3.1) follows by choosing $\delta > 0$ small enough such that $\frac{1}{1+\delta} \geq \alpha$. $-r^{-1}f \to 1$ as $r \to \infty$ is now a consequence of (3.1). □

3.2 Proof of theorem 18

To start with, we recall a result on the kernel of the Ricci tensor of steady soliton satisfying (1.24). It was proved in [76] in the real case with non-negative sectional curvature. However, the argument also works well in the Kähler case with non-negative Ricci curvature. For the sake of completeness, we include the proof of the result here.

Proposition 4. *[76] Let (M, g, f) be a complex n dimensional complete non Ricci flat gradient steady Kähler Ricci soliton with $Ric \geq 0$. Suppose that*

$$\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S = 0.$$

Then $S^2 \equiv 2|Ric|^2$ and the null space E of the Ricci tensor is a subbundle of the tangent bundle TM with real rank $2n-2$.

Proof. The argument is essentially due to [76]. Let $\lambda_i, i = 1, 2, \ldots, 2n$ be the eigenvalues of the Ricci tensor. By J invariance of Ric, we may assume $\lambda_i = \lambda_{n+i}, i = 1, 2, \ldots, n$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Hence

$$\begin{aligned} S - 2\lambda_i &= S - 2\lambda_{n+i} \\ &= \big(\sum_{j\neq i}^{2n} \lambda_j\big) - \lambda_i \\ &= \big(\sum_{j\neq i}^{2n} \lambda_j\big) - \lambda_{n+i} \\ &= \sum_{j\neq i,n+i}^{2n} \lambda_j \geq 0. \end{aligned}$$

From this, we know that

$$2|\mathrm{Ric}|^2 = \sum_{j=1}^{2n} 2\lambda_j^2 \leq \sum_{j=1}^{2n} \lambda_j S = S^2,$$

with equality holds at a point p iff $\lambda_n = \lambda_{2n} = \frac{S}{2}$ at p iff the dimension of the null space of Ric at p is $2n-2$. We are going to show $2|\mathrm{Ric}|^2 \equiv S^2$ on M. Let φ be a non-negative cut off function which $\equiv 1$ on $B_R(p_0)$, $\equiv 0$ outside $B_{2R}(p_0)$ and $|\nabla\varphi| \leq \frac{c}{R}$. We know that by the contracted second Bianchi identity $2\mathrm{div}(\mathrm{Ric}) = \nabla S$,

$$\begin{aligned} 0 &\leq \int_M \varphi^2(S^2 - 2|\mathrm{Ric}|^2) \\ &= \int_M \varphi^2(-S\Delta f + 2R_{ij}f_{ij}) \\ &= \int_M \varphi^2(\langle\nabla S, \nabla f\rangle - 2R_{ij,j}f_i) \\ &\quad + \int_M 2\varphi S\langle\nabla\varphi, \nabla f\rangle - \int_M 4\varphi R_{ij}f_i\varphi_j \\ &\leq \frac{c}{R}\int_{B_{2R}(p_0)} S. \end{aligned}$$

By condition (1.24), one can pick a sequence of $R_k \to \infty$ such that R.H.S. goes to zero as $k \to \infty$, we show that $S^2 = 2|\mathrm{Ric}|^2$ everywhere. It is not difficult to see from the previous argument that Ric only has two distinct eigenvalues, one is 0 with multiplicity $2n-2$, another one is $\frac{S}{2}$ with multiplicity 2, result follows.

□

Since we do not impose any condition on the sign of bisectional curvature, the non-triviality of the kernel of the Ricci tensor doesn't suffice for the splitting. Motivated by the local splitting result in [46] (see Theorem 17), we show that ∇f is always an eigenvector of Ric which eventually leads to the splitting of M.

Proposition 5. *Let (M, g, f) be a complex n dimensional complete gradient steady Kähler Ricci soliton with $Ric \geq 0$. Suppose that*

$$\liminf_{r\to\infty} \frac{1}{r} \int_{B_r} S = 0,$$

then $|\nabla f|^2 S = 2Ric(\nabla f, \nabla f)$ on M. In particular if M is not Ricci flat, then it is isometric to a quotient of $\Sigma \times N$, where Σ and N denote the cigar soliton and some simply connected complete Kähler Ricci flat manifold respectively.

Proof. We are done if g is Ricci flat, so we can assume Ric is not identically zero. Since $\mathrm{Ric} \geq 0$ and the curvature tensor is J invariant, we have $|\nabla f|^2 S \geq 2\mathrm{Ric}(\nabla f, \nabla f)$. Let $Q := \sqrt{f^2 + 1} \geq 1$. Then $\nabla Q = Q^{-1} f \nabla f$. Let $\phi \in C_c^\infty(M)$ be any smooth compactly supported function on M.

$$\begin{aligned} 0 &\leq \int_M \phi^2 Q^{-1}(|\nabla f|^2 S - 2\mathrm{Ric}(\nabla f, \nabla f)) \\ &= \int_M \phi^2 Q^{-1} f_i f_i S - \int_M 2\phi^2 Q^{-1} R_{ij} f_i f_j \\ &=: (I) + (II) \end{aligned}$$

Using integration by part, we have

$$\begin{aligned} (I) &= -\int_M 2\phi\phi_i Q^{-1} f f_i S + \int_M \phi^2 Q^{-3} f f_i f f_i S \\ &\quad - \int_M \phi^2 Q^{-1} f f_{ii} S - \int_M \phi^2 Q^{-1} f f_i S_i \\ &= -\int_M 2\phi\phi_i Q^{-1} f f_i S + \int_M \phi^2 Q^{-3} f^2 f_i f_i S \\ &\quad + \int_M \phi^2 Q^{-1} f S^2 - \int_M \phi^2 Q^{-1} f f_i S_i, \end{aligned}$$

where we use the fact that (2.2) $\Delta f + S = 0$. Similarly, using (2.3) $\nabla S = 2\mathrm{Ric}(\nabla f)$ and the contracted second Bianchi identity $2\mathrm{divRic} = \nabla S$, we see that

$$\begin{aligned}
(II) &= \int_M 4\phi\phi_i Q^{-1} f f_j R_{ij} - \int_M 2\phi^2 Q^{-3} f f_i f f_j R_{ij} \\
&\quad + \int_M 2\phi^2 Q^{-1} f R_{ij,i} f_j + \int_M 2\phi^2 Q^{-1} f R_{ij} f_{ji} \\
&= \int_M 2\phi\phi_i Q^{-1} f S_i - \int_M 2\phi^2 Q^{-3} f^2 f_i f_j R_{ij} \\
&\quad + \int_M \phi^2 Q^{-1} f f_j S_j - \int_M 2\phi^2 Q^{-1} f |\mathrm{Ric}|^2 \\
&= \int_M 2\phi\phi_i Q^{-1} f S_i - \int_M 2\phi^2 Q^{-3} f^2 f_i f_j R_{ij} \\
&\quad + \int_M \phi^2 Q^{-1} f f_j S_j - \int_M \phi^2 Q^{-1} f S^2,
\end{aligned}$$

we also use the identity $S^2 = 2|\mathrm{Ric}|^2$ (see [76] and Proposition 4). Hence, we have

$$\begin{aligned}
&\int_M \phi^2 Q^{-1}(|\nabla f|^2 S - 2\mathrm{Ric}(\nabla f, \nabla f)) \\
= &- \int_M 2\phi\phi_i Q^{-1} f f_i S + \int_M 2\phi\phi_i Q^{-1} f S_i \\
&+ \int_M \phi^2 Q^{-3} f^2 (f_i f_i S - 2 f_i f_j R_{ij}) \\
= &- \int_M 2\phi\phi_i Q^{-1} f f_i S + \int_M 2\phi\phi_i Q^{-1} f S_i \\
&+ \int_M \phi^2 Q^{-3} f^2 (|\nabla f|^2 S - 2\mathrm{Ric}(\nabla f, \nabla f)).
\end{aligned}$$

Since $Q^{-1} - Q^{-3} f^2 = Q^{-3}$, we know that

$$\begin{aligned}
&\int_M \phi^2 Q^{-3}(|\nabla f|^2 S - 2\mathrm{Ric}(\nabla f, \nabla f)) \\
= &- \int_M 2\phi\phi_i Q^{-1} f f_i S + \int_M 2\phi\phi_i Q^{-1} f S_i.
\end{aligned}$$

Now we take $0 \leq \phi \leq 1$ be a cut off function $\equiv 1$ on B_R, vanishes outside B_{2R} and

$|\nabla\phi| \leq \frac{c}{R}$.

$$\begin{aligned} |\int_M 2\phi\phi_i Q^{-1} f f_i S| &\leq \int_{B_{2R}\setminus B_R} \frac{2c}{R} Q^{-1}|f|S \\ &\leq \frac{2c}{R}\int_{B_{2R}\setminus B_R} S. \end{aligned}$$

Since $\mathrm{Ric} \geq 0$, $|\nabla S| \leq 2|\mathrm{Ric}| \leq cS$.

$$\begin{aligned} |\int_M 2\phi\phi_i Q^{-1} f S_i| &\leq \int_{B_{2R}\setminus B_R} \frac{2c}{R} Q^{-1}|f||\nabla S| \\ &\leq \frac{c_1}{R}\int_{B_{2R}\setminus B_R} S. \end{aligned}$$

All in all, there is a positive constant c_2 independent of R such that

$$\begin{aligned} 0 &\leq \int_{M\cap B_R} Q^{-3}(|\nabla f|^2 S - 2\mathrm{Ric}(\nabla f, \nabla f)) \\ &\leq \frac{c_2}{R}\int_{B_{2R}\setminus B_R} S. \end{aligned}$$

Using the condition $\liminf_{r\to\infty} \frac{1}{r}\int_{B_r} S = 0$, we may pick a sequence of $R_k \to \infty$ such that R.H.S. goes to zero as $k \to \infty$, we conclude that $|\nabla f|^2 S = 2\mathrm{Ric}(\nabla f, \nabla f)$ on M. We now proceed to prove the splitting of M. By $\mathrm{Ric} \geq 0$ and J invariance of Ric, we have for any tangent vector v with $|v|_g = 1$,

$$2\mathrm{Ric}(v, v) = \mathrm{Ric}(v, v) + \mathrm{Ric}(Jv, Jv) \leq S.$$

Hence whenever $\nabla f \neq 0$, $2\mathrm{Ric}(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}) = S$, we deduce that ∇f is an eigenvector with eigenvalue equal to $\frac{S}{2}$ and thus ∇f is always perpendicular to the nullspace of Ric. Let E be the nullspace of Ric, it is a smooth subbundle of the tangent bundle of real rank $2n-2$ ([76] and Proposition 4). Suppose at p, $\nabla f \neq 0$, the tangent space at p decomposes orthogonally as $T_pM = E_p \oplus_\perp span\{\nabla f, J\nabla f\}$. Let X be a smooth section of E defined locally near p and Y be any smooth vector field defined near p, then JX

is also a smooth section of E. At p

$$\begin{aligned}\langle \nabla_Y X, \nabla f\rangle &= Y\langle X, \nabla f\rangle - \langle X, \nabla_Y \nabla f\rangle \\ &= \mathrm{Ric}(X, Y) \\ &= 0.\end{aligned}$$

Similarly, $\nabla_Y JX \perp \nabla f$, thus $\nabla_Y X(p)$ is in E_p. If $\nabla f = 0$ at p, by real analyticity of g (see [66] and ref. therein), $\{\nabla f = 0\} = \{S = 1\}$ has no interior point in M (indeed if p is an interior point, then by (2.4) $0 = \Delta S(p) - \langle \nabla f, \nabla S\rangle(p) = -2|\mathrm{Ric}|^2(p)$, which is absurd). We may find a sequence $p_k \to p$ with $\nabla f(p_k) \neq 0$,

$$\mathrm{Ric}(\nabla_Y X)(p) = \lim_{k\to\infty} \mathrm{Ric}(\nabla_Y X)(p_k) = 0.$$

From this, we conclude that E is invariant under parallel translation. By de Rham splitting theorem (see [64]) and the classification of real 2 dimensional complete gradient steady Ricci solitons (see [36], [6] and ref. therein), the universal cover of M splits like $\Sigma \times N$ for some Kähler Ricci flat N. □

Proof of Theorem 18. By Proposition 5, the universal covering space of M splits isometrically as $\Sigma \times N$ for some simply connected complete Kähler Ricci flat N. By de Rham decomposition theorem (see [64]), N is isometric to $\mathbb{C}^{n-1-k} \times N_1$ with N_1 being a product of irreducible Kähler Ricci flat manifolds. It remains to show N_1 is compact. By Proposition 6 (will be proved in the coming section), $\mathbb{R}^{2n-2-2k} \times N_1 \cong_{\text{isom}} \mathbb{R}^l \times Q$, for some simply connected compact Ricci flat manifold Q. Since both N_1 and Q have no line, we conclude that $2n-2-2k = l$ and N_1 is diffeomorphic to the compact Q. □

3.3 Geometry of $\Sigma \times N/\sim$

In this section, we study the geometry of the quotient manifold $M = \Sigma \times N/\sim$ with scalar curvature S satisfying (1.24), where Σ and N denote the cigar soliton and some real $m-2$ dimensional simply connected complete (not necessarily Kähler) Ricci flat manifold respectively. The main goal of this section is to prove the following:

Proposition 6. *Let $M^m = \Sigma \times N/\sim$, for some simply connected complete Ricci flat manifold N. Suppose that on M*

$$\liminf_{r\to\infty} \frac{1}{r}\int_{B_r} S = 0,$$

then there exist positive constants C and $\alpha \in (0,1)$ such that

$$C^{-1}e^{-\frac{r}{\alpha}} \leq S \leq Ce^{-\alpha r} \text{ on } M.$$

Moreover, N is isometric to $\mathbb{R}^{m-2-k} \times Q^k$, where Q is some simply connected compact Ricci flat manifold.

Remark 25. It can be seen from the above proposition that M must have bounded curvature and S is integrable. Using an estimate in [46], the level sets of f (function constructed in Lemma 8) have uniformly bounded diameter, hence α in the above proposition can indeed be chosen to be 1. Alternatively, $\alpha = 1$ also follows from the curvature estimates in [25] and [76].

To prepare for the proof of Proposition 6, we recall some basic properties of Σ (see [36]). Let $\tilde{r}$ and $\tilde{f}$ be the distance function of Σ from its origin and the potential function respectively. In the geodesic polar coordinate, the metric is given by

$$g_\Sigma = d\tilde{r}^2 + 4\tanh^2(\frac{\tilde{r}}{2})d\theta^2.$$

We also have $\tilde{f} = \tilde{f}(\tilde{r}) = -2\log\cosh\frac{\tilde{r}}{2}$ and the scalar curvature

$$S_\Sigma = \frac{1}{\cosh^2(\frac{\tilde{r}}{2})} = e^{\tilde{f}} > 0. \tag{3.6}$$

Let $\rho : \Sigma \times N \to M$ and $\pi : \Sigma \times N \to \Sigma$ be the Riemannian covering and the projection into the first factor respectively. $\tilde{r}\circ\pi$ and $\tilde{f}\circ\pi$ are functions defined on $\Sigma \times N$. By abuse of notation, we shall not distinguish $\tilde{r}\circ\pi$ from $\tilde{r}$, $\tilde{f}\circ\pi$ from $\tilde{f}$, namely for all $(a,b) \in \Sigma\times N$, $\tilde{r}\circ\pi(a,b)$ and $\tilde{f}\circ\pi(a,b)$ will be written as $\tilde{r}(a,b)$ and $\tilde{f}(a,b)$ respectively.

Lemma 8. *Let M be the manifold as in Proposition 6. There is a smooth function f on M such that $f\circ\rho = \tilde{f}$. With this f, (M,g,f) is a complete gradient steady Ricci*

soliton.

Proof. Let (a, b) and $(c, d) \in \Sigma \times N$ such that $\rho(a, b) = \rho(c, d)$. Since N is Ricci flat,

$$S_\Sigma(a) = S_{\Sigma\times N}(a, b) = S_{\Sigma\times N}(c, d) = S_\Sigma(c).$$

By (3.6), we conclude that $\tilde{r}(a, b) = \tilde{r}(a) = \tilde{r}(c) = \tilde{r}(c, d)$ and $\tilde{f}(a, b) = \tilde{f}(c, d)$. $\tilde{f}$ respects the quotient map ρ and thus induces a map $f : M \to \mathbb{R}$ such that $f \circ \rho = \tilde{f}$. (M, g, f) is a gradient steady Ricci soliton then follows from the facts that ρ is a local isometry and $\tilde{f}$ is a potential function for the steady soliton $\Sigma \times N$. □

Lemma 9. *Let f be the function as in Lemma 8. The level sets $\Sigma_t := \{f = t\}$ are connected compact embedded hypersurfaces in M for all $t < 0$.*

Proof. Note that $0 = \max_M f = \max_{\Sigma\times N} \tilde{f}$. M has $\mathrm{Ric} \geq 0$ and thus both f and $\tilde{f}$ are concave functions with

$$\{f = 0\} = \{\nabla f = 0\} \text{ and } \{\tilde{f} = 0\} = \{\nabla \tilde{f} = 0\}.$$

For $t < 0$, $\Sigma_t := \{f = t\}$ are embedded hypersurfaces and complete w.r.t. the induced metric from (M, g). Since $\rho^{-1}(\Sigma_t) = \{\tilde{f} = t\}$ is diffeomorphic to $\mathbb{S}^1 \times N$, Σ_t is connected for all $t < 0$. Let ψ_t be the flow of $\frac{\nabla f}{|\nabla f|^2}$ with ψ_0 be the identity map. Using the level set flow ψ_t, we see that $\Sigma_t = \{f = t\}$ are diffeomorphic to each other for all $t < 0$. Moreover, $\psi_t(\Sigma_{-2}) = \Sigma_{t-2}$, for $t \in [0, 1]$. Therefore, it suffices to show that Σ_{-2} is compact. Assume by contradiction that Σ_{-2} is not compact. On $\{\tilde{f} \leq -1\} \subseteq \Sigma \times N$, by (2.6) and (3.6)

$$|\nabla \tilde{f}|^2 = 1 - e^{\tilde{f}} \geq 1 - e^{-1}.$$

Using $\rho_* \nabla \tilde{f} = \nabla f$, there exists $\delta > 0$ such that on $\{f \leq -1\}$

$$|\nabla f| \geq \delta. \tag{3.7}$$

Let $v \in T\Sigma_{-2}$, then

$$\begin{aligned}\frac{\partial}{\partial t}\psi_t^* g(v,v) &= \psi_t^*(L_{\frac{\nabla f}{|\nabla f|^2}} g)(v,v) \\ &= 2\nabla^2 f(\psi_{t*}v, \psi_{t*}v)|\nabla f|^{-2} \\ &= -2\mathrm{Ric}(\psi_{t*}v, \psi_{t*}v)|\nabla f|^{-2} \\ &\leq 0.\end{aligned}$$

Since the Ricci curvature of M is bounded, there is a constant $C > 0$ such that for $t \in [0,1]$

$$Cg_0 \leq g_t \leq g_0 \tag{3.8}$$

where $g_t := \psi_t^* g$ on Σ_{-2}. Let $B_R^t(q)$ be the intrinsic ball of (Σ_t, g) with radius R centered at q. It is not difficult to see that for $t < 0$

$$B_R^t(q) \subseteq \Sigma_t \cap B_R(q), \text{ for } q \in \Sigma_t, \tag{3.9}$$

where $B_R(q)$ is the geodesic ball in the ambient manifold (M, g). Fix any q_0 in Σ_{-2}. Let $r_0 = r(q_0) := d(q_0, p_0)$, $p_0 \in M$ is a fixed reference point. Hence by (3.7) for $t \in [0,1]$

$$d(\psi_t(q_0), q_0) \leq \int_0^t \frac{1}{|\nabla f|(\psi_s(q_0))} ds \leq \frac{t}{\delta}. \tag{3.10}$$

Next we show that for large $R > 0$,

$$B_{R-\frac{1}{\delta}}(\psi_t(q_0)) \subseteq B_{R+r_0}(p_0). \tag{3.11}$$

For all $z \in L.H.S.$

$$\begin{aligned}d(z,p_0) &\leq d(z, \psi_t(q_0)) + d(\psi_t(q_0), q_0) + d(q_0, p_0) \\ &< R - \frac{1}{\delta} + \frac{t}{\delta} + r_0 \\ &\leq R + r_0,\end{aligned}$$

we proved the inclusion (3.11). To proceed, we also need the following inclusion: for t

$\in [0,1]$

$$\psi_t(B^{-2}_{R-\frac{1}{\delta}}(q_0)) \subseteq B^{t-2}_{R-\frac{1}{\delta}}(\psi_t(q_0)), \tag{3.12}$$

where $B^t_R(q)$ is defined before (3.9). Let $z \in B^{-2}_{R-\frac{1}{\delta}}(q_0)$ and $\alpha \subseteq \Sigma_{-2}$ be an intrinsic minimizing geodesic w.r.t (Σ_{-2}, g) joining z and q_0. The length of $\psi_t \circ \alpha$ is given by

$$\begin{aligned} l_g(\psi_t \circ \alpha) &= \int |d\psi_t(\dot{\alpha})|_g \\ &= \int |\dot{\alpha}|_{\psi_t^* g} \\ &\leq \int |\dot{\alpha}|_g, \end{aligned}$$

where we use (3.8) in the last inequality and (3.12) follows. From (3.6), we see that there is a positive constant C_0 such that on $\{-2 \leq f\}$,

$$S \geq C_0. \tag{3.13}$$

We are going to derive a contradiction using the weak integrability condition (1.24)

of S. By (3.13), coarea formula, (3.11), (3.9), (3.12),

$$\begin{aligned}
\int_{B_{R+r_0}(p_0)} S &\geq \int_{B_{R+r_0}(p_0)\cap\{-2\leq f\leq -1\}} S \\
&\geq C_0 \int_{B_{R+r_0}(p_0)\cap\{-2\leq f\leq -1\}} \\
&= C_0 \int_0^1 \int_{B_{R+r_0}(p_0)\cap\Sigma_{t-2}} \frac{1}{|\nabla f|} d\sigma_t dt \\
&\geq C_0 \int_0^1 \int_{B_{R-\frac{1}{\delta}}(\psi_t(q_0))\cap\Sigma_{t-2}} d\sigma_t dt \\
&\geq C_0 \int_0^1 \int_{B^{t-2}_{R-\frac{1}{\delta}}(\psi_t(q_0))} d\sigma_t dt \\
&\geq C_0 \int_0^1 \int_{\psi_t(B^{-2}_{R-\frac{1}{\delta}}(q_0))} d\sigma_t dt \\
&= C_0 \int_0^1 \int_{B^{-2}_{R-\frac{1}{\delta}}(q_0)} \psi_t^* d\sigma_t dt \\
&\geq C_0 C_1 \int_0^1 \int_{B^{-2}_{R-\frac{1}{\delta}}(q_0)} d\sigma_0 dt \\
&= C_0 C_1 A(B^{-2}_{R-\frac{1}{\delta}}(q_0)),
\end{aligned}$$

where we also use (3.8) in the last inequality, $d\sigma_t$ and $A(B^{-2}_{R-\frac{1}{\delta}}(q_0))$ denote the volume element of Σ_{t-2} and the volume of the intrinsic geodesic ball $B^{-2}_{R-\frac{1}{\delta}}(q_0)$ in (Σ_{-2}, g) respectively. One can check readily that the induced metric on $\{\tilde{f} = -2\}$ is given by

$$4(1 - e^{-2})d\theta^2 + g_N,$$

where g_N is the metric on N. $\{\tilde{f} = -2\}$ is obviously Ricci flat. $\rho^{-1}(\Sigma_{-2}) = \{\tilde{f} = -2\}$ and thus Σ_{-2} is covered by $\{\tilde{f} = -2\}$. Hence Σ_{-2} is also Ricci flat (in particular $\mathrm{Ric} \geq 0$). If Σ_{-2} is noncompact, then by Yau's lower volume estimate on noncompact manifolds with $\mathrm{Ric} \geq 0$ (see [97] and [67]), there exists positive constant C_2 independent on all large R such that

$$A(B^{-2}_{R-\frac{1}{\delta}}(q_0)) \geq C_2(R - \frac{1}{\delta}).$$

From this we see that for all large R

$$\begin{aligned}\frac{1}{R+r_0}\int_{B_{R+r_0}(p_0)} S &\geq C_0C_1\frac{A(B^{-2}_{R-\frac{1}{\delta}}(q_0))}{R+r_0}\\ &\geq C_0C_1C_2\frac{R-\frac{1}{\delta}}{R+r_0},\end{aligned}$$

contradicting to the weak integrability condition (1.24) of S. We proved that Σ_{-2} and hence Σ_t are compact as long as $t<0$. □

Lemma 10. *Let f be the function as in Lemma 8. $\{f\geq -A\}$ is compact subset of M for all $A>0$*

Proof. Suppose it is not true for some A, then by the completeness of M, $\{f\geq -A\}$ is unbounded and there exists a sequence $x_k\to\infty$ with $f(x_k)\geq -A$. Pick a sequence y_k with $f(y_k)\to-\infty$, let γ_k be a normalized minimizing geodesic joining x_k to y_k, then $\gamma_k\cap\Sigma_{-A-1}\neq\phi$. By Lemma 9, Σ_{-A-1} is compact and it implies that after passing to a subsequence, γ_k converges to a line γ_∞. By Cheeger Gromoll splitting theorem (see [28], [67] and Theorem 33), M splits isometrically as $M_1\times\mathbb{R}$ for some complete manifold M_1. Let $(\alpha,\beta)\in M_1\times\mathbb{R}$, then $S_{M_1\times\mathbb{R}}(\alpha,\beta)=S_{M_1}(\alpha)>0$. Moreover, one have for all $R>0$

$$B_R(\alpha,\beta)\supseteq B^{M_1}_{\frac{R}{\sqrt{2}}}(\alpha)\times B^{\mathbb{R}}_{\frac{R}{\sqrt{2}}}(\beta).$$

Then

$$\begin{aligned}\int_{B_R(\alpha,\beta)} S &\geq \int_{B^{M_1}_{\frac{R}{\sqrt{2}}}(\alpha)\times B^{\mathbb{R}}_{\frac{R}{\sqrt{2}}}(\beta)} S_{M_1}\\ &\geq \sqrt{2}R\int_{B^{M_1}_{\frac{R}{\sqrt{2}}}(\alpha)} S_{M_1}\\ &\geq \sqrt{2}R\int_{B^{M_1}_{\frac{1}{\sqrt{2}}}(\alpha)} S_{M_1},\end{aligned}$$

again contradicting to the integrability condition (1.24) of S. □

Lemma 11. *S decays exponentially.*

Proof. We first show that $S \to 0$ at infinity. Let $x_k \in M \to \infty$ as $k \to \infty$, $(a_k, b_k) \in \Sigma \times N$ such that $\rho(a_k, b_k) = x_k$. Then $\tilde{r}_k := \tilde{r}(a_k, b_k) \to \infty$, where $\tilde{r}(a_k, b_k)$ is understood as $\tilde{r} \circ \pi(a_k, b_k) = \tilde{r}(a_k)$ as in the discussion right before Lemma 8. Otherwise, it is bounded for some subsequence k_j, then $f(x_{k_j}) = \tilde{f}(a_{k_j}, b_{k_j}) = -2 \log \cosh(\frac{\tilde{r}_{k_j}}{2})$ is bounded, by Lemma 10, x_{k_j} has convergent subsequence, which is impossible. Hence $\tilde{r}_k := \tilde{r}(a_k, b_k) \to \infty$. Then by (3.6), $S(x_k) = 1/\cosh^2(\frac{\tilde{r}_k}{2}) \to 0$ as $k \to \infty$. We deduce that $\lim_{r \to \infty} S = 0$. By Proposition 3 (see also [23]), there exist $\alpha \in (0, 1)$ and $D > 0$ such that,

$$r(x) + D \geq -f(x) \geq \alpha r(x) - D \text{ on } M \tag{3.14}$$

and

$$\tilde{r}(a) + D \geq -\tilde{f}(a) \geq \alpha \tilde{r}(a) - D \text{ on } \Sigma. \tag{3.15}$$

By the above two inequalities, we have for $x \in M$ and $(a, b) \in \Sigma \times N$ with $\rho(a, b) = x$,

$$\alpha r(x) - 2D \leq \tilde{r}(a) \leq \frac{r(x) + 2D}{\alpha}$$

and hence for some positive constants C_1 and C_2

$$\begin{aligned} S(x) = \frac{1}{\cosh^2(\frac{\tilde{r}(a)}{2})} &\leq 4e^{-\tilde{r}(a)} \\ &\leq C_1 e^{-\alpha r(x)}, \end{aligned}$$

similarly for the lower bound,

$$\begin{aligned} S(x) &\geq e^{-\tilde{r}(a)} \\ &\geq C_2 e^{-\frac{r(x)}{\alpha}}. \end{aligned}$$

□

To finish the proof of Proposition 6, it remains to show N is isometric to $\mathbb{R}^l \times Q$, for some compact simply connected Ricci flat manifold Q.

Proof of Proposition 6. Fix any $t < 0$, the induced metric on $\{\tilde{f} = t\} \subseteq \Sigma \times N$ is equal

to

$$4(1-e^t)d\theta^2 + g_N.$$

We see that $\{\tilde{f} = t\}$ is isometric to $\mathbb{S}^1 \times N$ and universally covered by $\mathbb{R} \times N$. By de Rham decomposition theorem (see [63]), N is isometric to $\mathbb{R}^q \times \widetilde{N}$, where $q \geq 0$ and $\widetilde{N} = N_1 \times N_2 \times \cdots \times N_l$ is a product of irreducible simply connected Ricci flat manifolds N_i with $\dim_{\mathbb{R}} N_i \geq 2$ $\forall i$. $\widetilde{N}$ has no line otherwise N_i splits for some i, contradicting to its irreducibility.

Since ρ is a Riemannian covering map, Σ_t is covered by $\{\tilde{f} = t\}$ and is compact Ricci flat. We have by Cheeger Gromoll splitting theorem (see [28], [67] and Theorem 34) and the uniqueness of universal Riemannian covering space that

$$\mathbb{R} \times N \cong_{\text{isom}} \mathbb{R}^{q+1} \times \widetilde{N} \cong_{\text{isom}} \mathbb{R}^k \times Q, \tag{3.16}$$

for some simply connected compact Ricci flat Q. Both $\widetilde{N}$ and Q do not have a line, we must have $q+1 = k$ and $\widetilde{N}$ is diffeomorphic to the compact Q. We are done with the proof of the proposition. □

3.4 Proof of Theorems 20 and 21

In this section, we will show Theorems 20 and 21. They essentially follow from the volume estimate on large geodesic balls:

Proposition 7. *Let (M, g, f) be a real m dimensional non Ricci flat complete gradient steady Ricci soliton with $Ric \geq 0$. Suppose there is a finite positive constant l such that*

$$\limsup_{r\to\infty} rS \leq l. \tag{3.17}$$

Then for all $p_0 \in M$ and $\varepsilon > 0$, there exists positive constant C such that for all large R,

$$V(B_R(p_0)) \leq CR^{l+1+\varepsilon}.$$

In particular if $l < 1$, then M has subquadratic volume growth.

Proof. By Proposition 3 (see also [23] and Theorem 9), there are $\alpha \in (0,1)$ and $D > 0$

such that

$$r + D \geq -f \geq \alpha r - D. \tag{3.18}$$

Hence f attains maximum, adding a constant if necessary, we may assume $\max_M f = 0$. Since f is concave, we have

$$\{\nabla f = 0\} = \{f = 0\}.$$

By (3.18), $\{-f = t\}$ are compact embedded hypersurfaces and diffeomorphic to each other for all $t > 0$. Let δ be a small positive number to be chosen later. By $S \to 0$ and (2.6), for all large r,

$$|\nabla f|^2 \geq (1+\delta)^{-1}. \tag{3.19}$$

Let $n := -\frac{\nabla f}{|\nabla f|}$ be the normal of $\{-f = t\}$, the second fundamental form of $\{-f = t\}$ w.r.t. n is given by $\frac{\mathrm{Ric}}{|\nabla f|}$. We consider the flow of $-\frac{\nabla f}{|\nabla f|^2}$ and denote it by ϕ_s with $\phi_0 = id$, then

$$\phi_s(\{-f = t\}) = \{-f = t + s\}.$$

Let $A(t)$ be the area of the level set $\{-f = t\}$. By the first variation formula, (3.19), (3.18) and (3.17) (see [46]), for all large t,

$$\begin{aligned} A'(t) &= \int_{\{-f=t\}} \frac{S - \mathrm{Ric}(n,n)}{|\nabla f|^2} \\ &\leq \int_{\{-f=t\}} \frac{S}{|\nabla f|^2} \\ &\leq (1+\delta) \int_{\{-f=t\}} S \\ &\leq \frac{(1+\delta)^2 l}{t} A(t), \end{aligned}$$

we used $\mathrm{Ric} \geq 0$ in the first inequality. The above differential inequality implies that there is a t_1 such that for $t \geq t_1$

$$A(t) \leq \frac{A(t_1)}{t_1^{(1+\delta)^2 l}} t^{(1+\delta)^2 l}. \tag{3.20}$$

Integrate the above inequality w.r.t t, together with (3.19) and (3.18), we see that for

all large R,

$$V(B_R(p_0)) \leq CR^{(1+\delta)^2 l+1}.$$

Result then follows by choosing $\delta > 0$ small enough such that $(1+\delta)^2 l < l + \varepsilon$. □

Proof of Theorems 20 and 21. By Proposition 7, M has subquadratic volume growth. Theorem 20 then follows from [24]. Theorem 21 is now a consequence of Corollary 2. By (1.25), we know that S satisfies (1.24). By Proposition 6, S decays exponentially and thus $\limsup_{r\to\infty} rS = 0$. □

Chapter 4

Curvature estimates and gap theorems for expanding Ricci solitons

The proofs in this chapter are based on [27].

4.1 Preliminaries

Let (M^n, g, f) be an n dimensional gradient expanding Ricci soliton, i.e.

$$\mathrm{Ric} + \nabla^2 f = -\frac{1}{2}g.$$

It generates a self similar solution to the Ricci flow. Indeed, let ψ_t be the flow of the vector field $\frac{\nabla f}{1+t}$ with ψ_0 being the identity map. We define $g(t) := (1+t)\psi_t^* g$, then $g(t)$ is a solution to the Ricci flow for $t \in (-1, \infty)$ with $g(0) = g$. The following equations for gradient expanders are known (see [36], [90] and [49])

$$S + \Delta f = -\frac{n}{2}, \tag{4.1}$$

$$S + |\nabla f|^2 = -f, \tag{4.2}$$

$$\Delta_f f = f - \frac{n}{2}, \tag{4.3}$$

$$\Delta_f S = -S - 2|\mathrm{Ric}|^2, \tag{4.4}$$

$$\Delta_f R_{ij} = -R_{ij} - 2R_{iklj}R_{kl} \tag{4.5}$$

and

$$\Delta_f \mathrm{Rm} = -\mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}, \tag{4.6}$$

where S is the scalar curvature and $\Delta_f := \Delta - \nabla_{\nabla f}$. We define a function v in the following way:

$$v := \frac{n}{2} - f. \tag{4.7}$$

From (4.2) and (4.3), we have

$$\Delta_f v = v \text{ and} \tag{4.8}$$

$$S + |\nabla v|^2 = v - \frac{n}{2}. \tag{4.9}$$

By (1.28) and (4.9), we have $|\nabla v|^2 \leq v$ and $|\nabla\sqrt{v+\delta}| \leq \frac{1}{2}$ for any $\delta > 0$. Integrating the inequality along minimizing geodesics and letting $\delta \to 0$, we get

$$|\nabla f| = |\nabla v| \leq \sqrt{v} \leq \frac{1}{2}r + \sqrt{v(p_0)}, \tag{4.10}$$

where p_0 is a fixed point in M. Suppose that S is now non-negative. Then we have by (4.2) that $f \leq 0$. Using strong minimum principle [53] and (4.4), we conclude that $S > 0$ unless $S \equiv 0$ on M (i.e. flat, see [91]). Moreover v satisfies

$$v \geq \frac{n}{2}. \tag{4.11}$$

In general the scalar curvature may be negative, the lower bound of v (4.11) is no longer available. However, it is known that v is positive under some mild conditions (see [49]).

Lemma 12. *Let (M^n, g, f) be a complete noncompact expanding gradient Ricci soliton. If $\mathrm{Ric} \neq -\frac{g}{2}$ somewhere on M, then $v > 0$ on M.*

Proof. From (1.28) and (4.9), we see that $v \geq |\nabla v|^2 \geq 0$. If v vanishes somewhere, then by the strong minimum principle [53] and (4.8), $v \equiv 0$. Hence $\mathrm{Ric} = -\frac{g}{2}$ on M, which is absurd. □

If $\lim_{r\to\infty} f = -\infty$, then $\mathrm{Ric} \neq -\frac{g}{2}$ somewhere and hence $v > 0$. The following

lemmas are immediate consequences of the computations by Deruelle [49]. We include the calculations for the sake of completeness.

Lemma 13. *Let (M^n, g, f) be an n dimensional complete gradient expanding Ricci soliton and $p \in M$. If $v(p) > 0$, then at p*

$$\Delta_{f+2\ln v}(vS) = -2v|Ric|^2 - 2|\nabla \ln v|^2 vS. \tag{4.12}$$

Proof. Using (4.4) and (4.8),

$$\begin{aligned}
\Delta_f(vS) &= v\Delta_f S + S\Delta_f v + 2\langle \nabla v, \nabla S\rangle \\
&= v(-S - 2|\mathrm{Ric}|^2) + Sv + 2\langle \nabla v, \nabla(vSv^{-1})\rangle \\
&= -2v|\mathrm{Ric}|^2 + 2\langle \nabla \ln v, \nabla(vS)\rangle - 2|\nabla \ln v|^2 vS.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 14. *Under the same assumption in Lemma 13, then at p*

$$\begin{aligned}
\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}) &= e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}\Big\{ \frac{1}{2\sqrt{v}} - (S+\frac{n}{2})\frac{1}{v^{\frac{3}{2}}} + \frac{1}{v^{\frac{3}{2}}}(\frac{n}{2}-2) \\
&\quad -(S+\frac{n}{2})\frac{1}{v^{\frac{5}{2}}}(\frac{n}{2}-2) - S + (\frac{n}{2}-2)(\frac{n}{2}+1)\frac{1}{v} \\
&\quad -(S+\frac{n}{2})(\frac{n}{2}-2)(\frac{n}{2}+1)\frac{1}{v^2} - (2S+n)(\frac{n}{2}-1)\frac{1}{v} \\
&\quad +\frac{1}{v^{\frac{3}{2}}}\big[\frac{7}{4} + \frac{1}{4\sqrt{v}}\big] - \frac{1}{v^{\frac{5}{2}}}(S+\frac{n}{2})\big[\frac{7}{4} + \frac{1}{4\sqrt{v}}\big]\Big\}.
\end{aligned} \tag{4.13}$$

Consequently if M is noncompact and $v \to \infty$ as $r \to \infty$, then near infinity

$$\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}) = e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}\Big\{\big(\frac{1}{2} + o(1)\big)\frac{1}{\sqrt{v}} + S\big(-1 + o(1)\big)\Big\}. \tag{4.14}$$

Proof.

$$\begin{aligned}
\Delta_f v^{2-\frac{n}{2}} &= (2-\frac{n}{2})v^{1-\frac{n}{2}}\Delta_f v + (\frac{n}{2}-2)(\frac{n}{2}-1)v^{-\frac{n}{2}}|\nabla v|^2 \\
&= (2-\frac{n}{2})v^{2-\frac{n}{2}} + (\frac{n}{2}-2)(\frac{n}{2}-1)v^{2-\frac{n}{2}}|\nabla \ln v|^2.
\end{aligned}$$

By (4.9),

$$\begin{aligned}\Delta_f e^{-v} &= -e^{-v}\Delta_f v + e^{-v}|\nabla v|^2 \\ &= e^{-v}(|\nabla v|^2 - v) \\ &= -(S+\frac{n}{2})e^{-v}.\end{aligned}$$

$$\begin{aligned}2\langle\nabla(v^{2-\frac{n}{2}}),\nabla e^{-v}\rangle &= 2(2-\frac{n}{2})v^{1-\frac{n}{2}}|\nabla v|^2(-e^{-v}) \\ &= 2(\frac{n}{2}-2)v^{2-\frac{n}{2}}e^{-v} - (2S+n)(\frac{n}{2}-2)v^{1-\frac{n}{2}}e^{-v}.\end{aligned}$$

$$\begin{aligned}\Delta_f(v^{2-\frac{n}{2}}e^{-v}) &= e^{-v}\Delta_f v^{2-\frac{n}{2}} + v^{2-\frac{n}{2}}\Delta_f e^{-v} + 2\langle\nabla(v^{2-\frac{n}{2}}),\nabla e^{-v}\rangle \\ &= (2-\frac{n}{2})v^{2-\frac{n}{2}}e^{-v} + (\frac{n}{2}-2)(\frac{n}{2}-1)v^{2-\frac{n}{2}}e^{-v}|\nabla\ln v|^2 \\ &\quad -(S+\frac{n}{2})v^{2-\frac{n}{2}}e^{-v} + 2(\frac{n}{2}-2)v^{2-\frac{n}{2}}e^{-v} \\ &\quad -(2S+n)(\frac{n}{2}-2)v^{1-\frac{n}{2}}e^{-v} \\ &= v^{2-\frac{n}{2}}e^{-v}\Big[-2-S+(\frac{n}{2}-2)(\frac{n}{2}-1)|\nabla\ln v|^2 \\ &\qquad -(2S+n)(\frac{n}{2}-2)v^{-1}\Big].\end{aligned}$$

On the other hand, by (4.9)

$$\begin{aligned}-2\langle\nabla\ln v,\nabla(v^{2-\frac{n}{2}}e^{-v})\rangle &= 2(\frac{n}{2}-2)v^{1-\frac{n}{2}}e^{-v}\langle\frac{\nabla v}{v},\nabla v\rangle \\ &\quad +2v^{2-\frac{n}{2}}e^{-v}\langle\frac{\nabla v}{v},\nabla v\rangle \\ &= 2(\frac{n}{2}-2)v^{2-\frac{n}{2}}e^{-v}|\nabla\ln v|^2 \\ &\quad +2v^{2-\frac{n}{2}}e^{-v}\frac{(v-S-\frac{n}{2})}{v} \\ &= v^{2-\frac{n}{2}}e^{-v}\Big[2-(2S+n)v^{-1} \\ &\qquad +2(\frac{n}{2}-2)|\nabla\ln v|^2\Big]\end{aligned}$$

Hence

$$\begin{aligned}\Delta_{f+2\ln v}(v^{2-\frac{n}{2}}e^{-v}) &= v^{2-\frac{n}{2}}e^{-v}\Big[-S+(\frac{n}{2}-2)(\frac{n}{2}+1)|\nabla \ln v|^2 \\ &\quad -(2S+n)(\frac{n}{2}-1)v^{-1}\Big]. \qquad (4.15)\end{aligned}$$

$$\begin{aligned}\Delta_f e^{\frac{1}{\sqrt{v}}} &= -\frac{1}{2}v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}\Delta_f v + e^{\frac{1}{\sqrt{v}}}\big[\frac{3}{4}v^{-\frac{5}{2}}+\frac{1}{4}v^{-3}\big]|\nabla v|^2 \\ &= -\frac{1}{2}v^{-\frac{1}{2}}e^{\frac{1}{\sqrt{v}}} + e^{\frac{1}{\sqrt{v}}}\big[\frac{3}{4}v^{-\frac{5}{2}}+\frac{1}{4}v^{-3}\big](v-S-\frac{n}{2}) \\ &= -\frac{1}{2}v^{-\frac{1}{2}}e^{\frac{1}{\sqrt{v}}} + v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}\big[\frac{3}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big] \\ &\quad -v^{-\frac{5}{2}}e^{\frac{1}{\sqrt{v}}}(S+\frac{n}{2})\big[\frac{3}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big].\end{aligned}$$

$$\begin{aligned}\Delta_{f+2\ln v}e^{\frac{1}{\sqrt{v}}} &= \Delta_f e^{\frac{1}{\sqrt{v}}} - 2\langle\frac{\nabla v}{v}, \nabla e^{\frac{1}{\sqrt{v}}}\rangle \\ &= \Delta_f e^{\frac{1}{\sqrt{v}}} + \frac{(v-S-\frac{n}{2})}{v^{\frac{5}{2}}}e^{\frac{1}{\sqrt{v}}} \\ &= -\frac{1}{2}v^{-\frac{1}{2}}e^{\frac{1}{\sqrt{v}}} + v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big] \\ &\quad -v^{-\frac{5}{2}}e^{\frac{1}{\sqrt{v}}}(S+\frac{n}{2})\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big].\end{aligned}$$

$$\begin{aligned}2\langle\nabla e^{\frac{1}{\sqrt{v}}}, \nabla(v^{2-\frac{n}{2}}e^{-v})\rangle &= -v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}\langle\nabla v, \nabla v\rangle(2-\frac{n}{2})v^{1-\frac{n}{2}}e^{-v} \\ &\quad -v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}\langle\nabla v, \nabla v\rangle v^{2-\frac{n}{2}}(-e^{-v}) \\ &= v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}(v-S-\frac{n}{2})(\frac{n}{2}-2)v^{1-\frac{n}{2}}e^{-v} \\ &\quad +v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}(v-S-\frac{n}{2})v^{2-\frac{n}{2}}e^{-v} \\ &= v^{-\frac{1}{2}}e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v} - (S+\frac{n}{2})v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v} \\ &\quad +v^{-\frac{3}{2}}e^{\frac{1}{\sqrt{v}}}(\frac{n}{2}-2)v^{2-\frac{n}{2}}e^{-v} \\ &\quad -(S+\frac{n}{2})v^{-\frac{5}{2}}e^{\frac{1}{\sqrt{v}}}(\frac{n}{2}-2)v^{2-\frac{n}{2}}e^{-v}.\end{aligned}$$

Using (4.15), we have

$$\begin{aligned}
\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}) &= e^{\frac{1}{\sqrt{v}}}\Delta_{f+2\ln v}(v^{2-\frac{n}{2}}e^{-v}) + v^{2-\frac{n}{2}}e^{-v}\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}}) \\
&\quad +2\langle\nabla e^{\frac{1}{\sqrt{v}}}, \nabla(v^{2-\frac{n}{2}}e^{-v})\rangle \\
&= e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}\Big\{\frac{1}{2\sqrt{v}} - (S+\frac{n}{2})v^{-\frac{3}{2}} + v^{-\frac{3}{2}}(\frac{n}{2}-2) \\
&\quad -(S+\frac{n}{2})v^{-\frac{5}{2}}(\frac{n}{2}-2) - S + (\frac{n}{2}-2)(\frac{n}{2}+1)|\nabla\ln v|^2 \\
&\quad -(2S+n)(\frac{n}{2}-1)v^{-1} + v^{-\frac{3}{2}}\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big] \\
&\quad -v^{-\frac{5}{2}}(S+\frac{n}{2})\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big]\Big\} \\
&= e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}\Big\{\frac{1}{2\sqrt{v}} - (S+\frac{n}{2})v^{-\frac{3}{2}} + v^{-\frac{3}{2}}(\frac{n}{2}-2) \\
&\quad -(S+\frac{n}{2})v^{-\frac{5}{2}}(\frac{n}{2}-2) - S + (\frac{n}{2}-2)(\frac{n}{2}+1)v^{-1} \\
&\quad -(S+\frac{n}{2})(\frac{n}{2}-2)(\frac{n}{2}+1)v^{-2} - (2S+n)(\frac{n}{2}-1)v^{-1} \\
&\quad +v^{-\frac{3}{2}}\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big] - v^{-\frac{5}{2}}(S+\frac{n}{2})\big[\frac{7}{4}+\frac{1}{4}v^{-\frac{1}{2}}\big]\Big\}.
\end{aligned}$$

We showed (4.13). We can then separate the terms with S from those without S in R.H.S of (4.13) to get (4.14). □

4.2 Proof of Theorem 24

With all the computations in the previous section, we are going to prove Theorem 24:

Theorem. *Let (M^n, g, f) be an n dimensional complete non-compact gradient expanding Ricci soliton with $n \geq 2$. Suppose that $\lim_{r\to\infty} f = -\infty$ and (M, g) is not flat and has non-negative scalar curvature S. Then there exists a positive constant C such that*

$$S \geq Cv^{1-\frac{n}{2}}e^{-v} \text{ on } M,$$

where $v := \frac{n}{2} - f$.

Proof. Since the scalar curvature $S \geq 0$ and $v \geq \frac{n}{2}$, we have by (4.12)

$$\Delta_{f+2\ln v}(vS) \leq 0. \tag{4.16}$$

Moreover by $\lim_{x\to\infty} v(x) = \infty$ and (4.14), we see that outside a compact subset of M,

$$\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}) \;=\; e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}\Big\{\big(\frac{1}{2}+o(1)\big)\frac{1}{\sqrt{v}}+S\big(-1+o(1)\big)\Big\}.$$

From the above equation,

$$\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}) \geq e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}(\frac{1}{4\sqrt{v}}-2S). \tag{4.17}$$

Since $v = \frac{n}{2} - f \to \infty$ as $r \to \infty$, by taking a larger compact set if necessary, we may assume that

$$\frac{1}{8} > e^{\frac{1}{\sqrt{v}}}v^{\frac{3}{2}-\frac{n}{2}}e^{-v} \tag{4.18}$$

near infinity. Let R_0 be a large positive number such that (4.16), (4.17) and (4.18) hold on $M \setminus B_{R_0}(p_0)$. Hence by $S > 0$ and (4.11), there exists a constant $b \in (0,1)$ such that

$$vS > be^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v} \text{ on } \partial B_{R_0}(p_0). \tag{4.19}$$

Let $Q := vS - be^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}$. It is not difficult to see that

$$\liminf_{r\to\infty} Q \geq 0.$$

Fix any y in $M \setminus \overline{B_{R_0}(p_0)}$ and any $\varepsilon > 0$, there exists a large positive $T > R_0$ such that $y \in B_T(p_0)$ and

$$Q \geq -\varepsilon \text{ on } \partial B_T(p_0). \tag{4.20}$$

Let $\Omega := B_T(p_0) \setminus \overline{B_{R_0}(p_0)}$ and $z \in \overline{\Omega}$ such that Q attains its minimum over $\overline{\Omega}$ at z, i.e.

$$Q(z) = \min_{\overline{\Omega}} Q.$$

If $z \in \partial\Omega$, then by (4.19) and (4.20), we have

$$Q(y) \geq Q(z) \geq -\varepsilon.$$

If $z \in \Omega$, we have by (4.16) and (4.17) that at z

$$\begin{aligned} 0 &\leq \Delta_{f+2\ln v} Q \\ &= \Delta_{f+2\ln v}(vS) - b\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}) \\ &\leq -b\Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}) \\ &\leq 2be^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}(S - \frac{1}{8\sqrt{v}}). \end{aligned}$$

Together with (4.18), we know that at z

$$\begin{aligned} vS &\geq \frac{\sqrt{v}}{8} \\ &\geq e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v} \\ &\geq be^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}, \end{aligned}$$

i.e. $0 \leq Q(z) \leq Q(y)$. Hence in any cases, $Q(y) \geq -\varepsilon$. Result then follows by letting $\varepsilon \to 0$ and choosing a smaller b to make $Q \geq 0$ on the entire M. □

4.3 Proof of Theorem 25

To prepare for the maximum principle argument, we first show that the scalar curvature is negative in a gradient expander with nonpositive Ricci curvature.

Lemma 15. *Let (M^n, g, f) be a complete noncompact and nonflat expanding gradient Ricci soliton with $Ric \leq 0$. Then $S < 0$ on M.*

Proof. Using $\mathrm{Ric} \leq 0$, we have $|Ric|^2 \leq S^2$ and by (4.4)

$$\begin{aligned} \Delta_f S &= -S - 2|\mathrm{Ric}|^2 \\ &\geq -(1+2S)S. \end{aligned}$$

We argue by contradiction. If $S(z) = 0$ for some z in M, then S attains its interior maximum at z. By the strong maximum principle [53], $S \equiv 0$ and hence by (4.4) $\mathrm{Ric} \equiv 0$. From (1.8), $-2\nabla^2 f = g$. g is flat by a result in [91], which is absurd. □

With Lemma 15, we can finish the proof of Theorem 25:

Theorem. *Let (M^n, g, f) be an n dimensional complete non-compact gradient expanding Ricci soliton with $n \geq 2$. Suppose that $\lim_{r\to\infty} f = -\infty$ and (M, g) is not flat and has non-positive Ricci curvature. Then there is a positive constant C such that*

$$S \leq -Cv^{1-\frac{n}{2}}e^{-v},$$

where $v := \frac{n}{2} - f$.

Proof. Using the discussion after the proof of Lemma 12, we have $v > 0$ on M. By (4.9),

$$\begin{aligned} |\nabla \ln v|^2 v &= \frac{|\nabla v|^2}{v} \\ &= 1 - \frac{\frac{n}{2} + S}{v}. \end{aligned}$$

Using $\mathrm{Ric} \leq 0$ and (4.12), we see that $|\mathrm{Ric}|^2 \leq S^2$ and

$$\begin{aligned} \Delta_{f+2\ln v}(vS) &\geq -2vS^2 - 2|\nabla \ln v|^2 vS \\ &= -2vS^2 - 2S + \frac{nS + 2S^2}{v} \\ &= S\Big(-2 + \frac{n + 2S}{v} - 2vS\Big). \end{aligned} \tag{4.21}$$

From $\lim_{r\to\infty} v = \infty$ and (4.14), we see that

$$\begin{aligned} \Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}) &= e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}\Big\{\big(\frac{1}{2} + o(1)\big)\frac{1}{\sqrt{v}} + S\big(-1 + o(1)\big)\Big\} \\ &\geq 0 \end{aligned} \tag{4.22}$$

near infinity. We now consider a large R_0 such that on $M \setminus B_{R_0}(p_0)$, $v > 1$ and (4.22) are true, moreover the following hold:

$$\frac{-1 + \frac{n}{2v}}{1 - \frac{1}{v^2}} \leq -\frac{1}{2} \tag{4.23}$$

and

$$\frac{1}{2} \geq e^{\frac{1}{\sqrt{v}}}v^{2-\frac{n}{2}}e^{-v}. \tag{4.24}$$

For such R_0, there exists a positive constant $\alpha \in (0,1)$ such that

$$vS + \alpha e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v} < 0 \text{ on } \partial B_{R_0}(p_0). \tag{4.25}$$

We define $Q := vS + \alpha e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}$. It is easy to see that

$$\limsup_{r\to\infty} Q \leq 0.$$

For any y in $M \setminus \overline{B_{R_0}(p_0)}$ and any $\varepsilon > 0$, there is a positive $T > R_0$ such that $y \in B_T(p_0)$ and

$$Q \leq \varepsilon \text{ on } \partial B_T(p_0). \tag{4.26}$$

Let $\Omega := B_T(p_0) \setminus \overline{B_{R_0}(p_0)}$ and $z \in \overline{\Omega}$ be a point where Q attains its maximum over $\overline{\Omega}$. If $z \in \partial\Omega$, then by the boundary conditions (4.25) and (4.26),

$$Q(y) \leq Q(z) \leq \varepsilon.$$

If $z \in \Omega$, we have by (4.21) and (4.22) that at z

$$\begin{aligned} 0 &\geq \Delta_{f+2\ln v} Q \\ &= \Delta_{f+2\ln v}(vS) + \alpha \Delta_{f+2\ln v}(e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}) \\ &\geq \Delta_{f+2\ln v}(vS) \\ &\geq S\Big(-2 + \frac{n+2S}{v} - 2vS\Big). \end{aligned}$$

Together with $S < 0$, (4.23) and (4.24), at z

$$\begin{aligned} vS &\leq \frac{-1+\frac{n}{2v}}{1-\frac{1}{v^2}} \\ &\leq -\frac{1}{2} \\ &\leq -e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v} \\ &\leq -\alpha e^{\frac{1}{\sqrt{v}}} v^{2-\frac{n}{2}} e^{-v}. \end{aligned}$$

Hence $Q(z) \leq 0$ and result follows by letting $\varepsilon \to 0$. □

4.4 Proof of Theorem 26

Before moving to the proof of Theorem 26, we recall the statement of the theorem:

Theorem. *Let (M^n, g, f) be a complete noncompact gradient expanding Ricci soliton with dimension $n \geq 3$. We denote the negative part of the scalar curvature S by S_-, i.e. $S_- := (-S)_+ := \max\{-S, 0\}$. If S_- is integrable, that is,*

$$\int_M S_- dv_g < \infty, \tag{4.27}$$

then either $S \geq 0$ everywhere or M is isometric to an Einstein manifold with scalar curvature $-\frac{n}{2}$ and finite volume.

Proof. W.L.O.G., we may assume $S < 0$ somewhere and show that M is Einstein. We are going to prove that M has constant scalar curvature $-\frac{n}{2}$. By (4.4) and Cauchy Schwarz inequality,

$$\begin{aligned} \Delta_f S &= -S - 2|\mathrm{Ric}|^2 \\ &\leq -S - \frac{2}{n}S^2. \end{aligned}$$

Let $\delta \in (0, \frac{1}{4})$, $\varepsilon \in (0, 1)$ and ϕ be any nonnegative compactly supported function, we multiply the above inequality by $(-S)_+(S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 \geq 0$ and integrate the inequality over M,

$$\begin{aligned} (I)+(II) &:= \int_M (-S)_+(S^2+\varepsilon)^{\delta-\frac{1}{2}}\Delta S\phi^2 - \int_M (-S)_+(S^2+\varepsilon)^{\delta-\frac{1}{2}}\langle\nabla f, \nabla S\rangle\phi^2 \\ &\leq \int_M (-S)_+(-S-\frac{2}{n}S^2)(S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 \\ &= \int_{\{S<0\}} (-S)(-S-\frac{2}{n}S^2)(S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 \\ &= \int_{\{S<0\}} \frac{2}{n}S^2(S+\frac{n}{2})(S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2. \end{aligned}$$

By [53], the weak derivative of $(-S)_+$ is given by

$$\begin{aligned} \nabla(-S)_+ &= \chi_{\{S<0\}}\nabla(-S) \\ &= -\chi_{\{S<0\}}\nabla S, \end{aligned}$$

where $\chi_{\{S<0\}}$ is the characteristic function of the set $\{S<0\}$ which is 1 on $\{S<0\}$ and vanishes elsewhere. Using integration by part, we see that

$$\begin{aligned}
(I) &= \int_M (-S)_+ (S^2+\varepsilon)^{\delta-\frac{1}{2}} \Delta S \phi^2 \\
&= -\int_M \chi_{\{S<0\}} \langle -\nabla S, \nabla S\rangle (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 - \int_M \chi_{\{S<0\}}(-S)\phi^2 \langle \nabla (S^2+\varepsilon)^{\delta-\frac{1}{2}}, \nabla S\rangle \\
&\quad -\int_M \chi_{\{S<0\}}(-S) 2\phi \langle \nabla\phi, \nabla S\rangle (S^2+\varepsilon)^{\delta-\frac{1}{2}} \\
&\geq \int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 + (2\delta-1)\int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{3}{2}} S^2\phi^2 \\
&\quad -\int_{\{S<0\}} 2\phi |\nabla\phi||\nabla S||S| (S^2+\varepsilon)^{\delta-\frac{1}{2}} \\
&\geq 2\delta \int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 - \delta \int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 \\
&\quad -\frac{1}{\delta}\int_{\{S<0\}} S^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}} |\nabla\phi|^2 \\
&= \delta \int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 - \frac{1}{\delta}\int_{\{S<0\}} S^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}} |\nabla\phi|^2.
\end{aligned}$$

By (4.1) and integration by part again, we have

$$\begin{aligned}
(II) &= -\int_M (-S)_+ (S^2+\varepsilon)^{\delta-\frac{1}{2}} \langle \nabla f, \nabla S\rangle \phi^2 \\
&= \frac{1}{2\delta+1}\int_M \langle \nabla f, \nabla ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}}\rangle \phi^2 \\
&= -\frac{1}{2\delta+1}\int_M \Delta f ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}}\phi^2 - \frac{2}{2\delta+1}\int_M \phi\langle\nabla\phi,\nabla f\rangle ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}} \\
&= \frac{1}{2\delta+1}\int_M (S+\frac{n}{2}) ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}}\phi^2 - \frac{2}{2\delta+1}\int_M \phi\langle\nabla\phi,\nabla f\rangle ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}}.
\end{aligned}$$

All in all, we have

$$\begin{aligned}
0 &\leq \delta \int_{\{S<0\}} |\nabla S|^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 \\
&\leq \frac{1}{\delta}\int_{\{S<0\}} S^2 (S^2+\varepsilon)^{\delta-\frac{1}{2}} |\nabla\phi|^2 + \frac{2}{2\delta+1}\int_M \phi\langle\nabla\phi,\nabla f\rangle ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}} \\
&\quad + \int_{\{S<0\}} \frac{2}{n} S^2 (S+\frac{n}{2}) (S^2+\varepsilon)^{\delta-\frac{1}{2}}\phi^2 - \frac{1}{2\delta+1}\int_M (S+\frac{n}{2}) ((-S)_+^2+\varepsilon)^{\delta+\frac{1}{2}}\phi^2.
\end{aligned}$$

Using $(-S)_+ = -\chi_{\{S<0\}}S$, ϕ is compactly supported and Dominated Convergence theorem, we may let $\varepsilon \to 0$ and get

$$\begin{aligned} 0 &\leq \frac{1}{\delta}\int_{\{S<0\}}(S^2)^{\delta+\frac{1}{2}}|\nabla\phi|^2 + \frac{2}{2\delta+1}\int_{\{S<0\}}\phi\langle\nabla\phi,\nabla f\rangle(S^2)^{\delta+\frac{1}{2}} \\ &\quad + \int_{\{S<0\}}\frac{2}{n}(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}}\phi^2 - \frac{1}{2\delta+1}\int_{\{S<0\}}(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}}\phi^2 \\ &= \frac{1}{\delta}\int_{\{S<0\}}(S^2)^{\delta+\frac{1}{2}}|\nabla\phi|^2 + \frac{2}{2\delta+1}\int_{\{S<0\}}\phi\langle\nabla\phi,\nabla f\rangle(S^2)^{\delta+\frac{1}{2}} \\ &\quad + (\frac{2}{n}-\frac{1}{2\delta+1})\int_{\{S<0\}}(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}}\phi^2. \end{aligned}$$

Since $n \geq 3$ and $\delta < \frac{1}{4}$, we have $\frac{2}{n} - \frac{1}{2\delta+1} < 0$ and by (1.28)

$$\begin{aligned} 0 &\leq (\frac{1}{2\delta+1}-\frac{2}{n})\int_{\{S<0\}}(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}}\phi^2 \\ &\leq \frac{1}{\delta}\int_{\{S<0\}}(S^2)^{\delta+\frac{1}{2}}|\nabla\phi|^2 + \frac{2}{2\delta+1}\int_{\{S<0\}}\phi\langle\nabla\phi,\nabla f\rangle(S^2)^{\delta+\frac{1}{2}}. \end{aligned} \tag{4.28}$$

Let $R \geq 1$ and $\psi : [0,\infty) \to \mathbb{R}$ be a smooth real valued function satisfying the following: $0 \leq \psi \leq 1$, $\psi' \leq 0$,

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 2 \leq t \end{cases}$$

and

$$|\psi'(t)| \leq c \text{ for all } t \geq 0.$$

We take the cut off function $\phi(x) := \psi(\frac{r(x)}{R})$, then

$$\phi = \begin{cases} 1 & \text{on } B_R(p_0) \\ 0 & \text{on } M \setminus B_{2R}(p_0) \end{cases}$$

and

$$|\nabla\phi| = \frac{|\psi'|}{R} \leq \frac{c}{R}\chi_{B_{2R}\setminus B_R}, \tag{4.29}$$

where $\chi_{B_{2R}\setminus B_R}$ is the characteristic function of the set $B_{2R}(p_0) \setminus B_R(p_0)$ which is 1 on

$B_{2R}(p_0) \setminus B_R(p_0)$ and vanishes elsewhere. We have

$$\begin{aligned}\frac{1}{\delta}\int_{\{S<0\}}(S^2)^{\delta+\frac{1}{2}}|\nabla\phi|^2 &\leq \frac{c^2n^{2\delta}}{\delta 4^\delta R^2}\int_{\{S<0\}\setminus B_R}|S| \\ &= \frac{c^2n^{2\delta}}{\delta 4^\delta R^2}\int_{M\setminus B_R}(-S)_+.\end{aligned}$$

Since $|\nabla f| \leq C(r+1)$ (see (4.10)),

$$\begin{aligned}\frac{2}{2\delta+1}\int_{\{S<0\}}\phi\langle\nabla\phi,\nabla f\rangle(S^2)^{\delta+\frac{1}{2}} &\leq \frac{2cn^{2\delta}}{(2\delta+1)4^\delta}\int_{\{S<0\}\setminus B_R}|S| \\ &= \frac{2cn^{2\delta}}{(2\delta+1)4^\delta}\int_{M\setminus B_R}(-S)_+.\end{aligned}$$

By (4.27), we let $R\to\infty$ and conclude by (4.28) that

$$\int_{\{S<0\}}(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}} = 0.$$

Hence we may apply (1.28) again to get

$$(S+\frac{n}{2})(S^2)^{\delta+\frac{1}{2}} \equiv 0 \text{ on } \{S<0\}.$$

From this we see that $S=-\frac{n}{2}$ wherever $\{S<0\}$. Since we assume that $\{S<0\}$ is nonempty, we know by connectedness that $S\equiv-\frac{n}{2}$. M is Einstein since

$$\begin{aligned}0 &= \Delta_f S \\ &= -S-2|\mathrm{Ric}|^2 \\ &= -\frac{2S}{n}\big(S+\frac{n}{2}\big)-2|\mathrm{Ric}-\frac{S}{n}g|^2.\end{aligned}$$

Finiteness of volume is a now consequence of (4.27). □

4.5 Proof of the Theorems 28 and 30

In this section, we study the geometry of three dimensional gradient expander. For the convenience of reader, we recall the statement of Theorem 28(1).

Theorem. *Let (M, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton with bounded non-negative scalar curvature. Then the curvature tensor Rm is bounded, moreover*

$$|Rm| \leq c\sqrt{S} \text{ on } M,$$

for some constant $c > 0$.

Proof. Throughout this proof, we use c to denote an absolute constant, its value may be different from line by line. W.L.O.G, we may assume M is not flat, or else we have nothing to prove. By the strong minimum principle and (4.4), S is positive on M. Since the Weyl tensor is zero in dimension three, it suffices to bound the Ricci tensor. By (4.5)

$$\begin{aligned} \Delta_f |\mathrm{Ric}|^2 &= 2|\nabla \mathrm{Ric}|^2 + 2\langle \mathrm{Ric}, \Delta_f \mathrm{Ric}\rangle \\ &= 2|\nabla \mathrm{Ric}|^2 - 2|\mathrm{Ric}|^2 - 4R_{ij}R_{iklj}R_{kl} \\ &\geq 2|\nabla \mathrm{Ric}|^2 - 2|\mathrm{Ric}|^2 - c|\mathrm{Ric}|^3, \end{aligned} \tag{4.30}$$

for some absolute constant c. Using (4.4),

$$\begin{aligned} \Delta_f S^{-1} &= -S^{-2}\Delta_f S + 2S^{-1}|\nabla \ln S|^2 \\ &= -S^{-2}(-S - 2|\mathrm{Ric}|^2) + 2S^{-1}|\nabla \ln S|^2 \\ &= S^{-1} + 2S^{-2}|\mathrm{Ric}|^2 + 2S^{-1}|\nabla \ln S|^2. \end{aligned} \tag{4.31}$$

Hence by Kato's inequality, (4.30) and (4.31),

$$\begin{aligned} \Delta_f(S^{-1}|\mathrm{Ric}|^2) &= S^{-1}\Delta_f|\mathrm{Ric}|^2 + 2\langle \nabla S^{-1}, \nabla |\mathrm{Ric}|^2\rangle + |\mathrm{Ric}|^2\Delta_f S^{-1} \\ &\geq 2S^{-1}|\nabla \mathrm{Ric}|^2 - 2S^{-1}|\mathrm{Ric}|^2 - cS^{-1}|\mathrm{Ric}|^3 \\ &\quad -4S^{-1}|\nabla \ln S||\mathrm{Ric}||\nabla \mathrm{Ric}| + S^{-1}|\mathrm{Ric}|^2 \\ &\quad +2S^{-2}|\mathrm{Ric}|^4 + 2S^{-1}|\nabla \ln S|^2|\mathrm{Ric}|^2. \end{aligned} \tag{4.32}$$

By completing square, we see that

$$\begin{aligned} 2S^{-1}|\nabla \mathrm{Ric}|^2 - 4S^{-1}|\nabla \ln S||\mathrm{Ric}||\nabla \mathrm{Ric}| &= 2S^{-1}\Big(|\nabla \mathrm{Ric}| - |\nabla \ln S||\mathrm{Ric}|\Big)^2 \\ &\quad -2S^{-1}|\nabla \ln S|^2|\mathrm{Ric}|^2 \\ &\geq -2S^{-1}|\nabla \ln S|^2|\mathrm{Ric}|^2. \end{aligned}$$

$$\Delta_f(S^{-1}|\mathrm{Ric}|^2) \geq 2S^{-2}|\mathrm{Ric}|^4 - cS^{-1}|\mathrm{Ric}|^3 - S^{-1}|\mathrm{Ric}|^2.$$

Let $u := S^{-1}|\mathrm{Ric}|^2$, the above differential inequality can be rewritten as

$$\begin{aligned} \Delta_f u &\geq 2u^2 - cS^{\frac{1}{2}}u^{\frac{3}{2}} - u \\ &\geq u^2 - c(1+S)u \\ &\geq u^2 - c_0 u, \end{aligned} \tag{4.33}$$

where c_0 is a positive constant depending on the global upper bound of the scalar curvature. Let $R \geq 1$ and $\psi : [0,\infty) \to \mathbb{R}$ be a smooth real valued function satisfying the following: $0 \leq \psi \leq 1$, $\psi' \leq 0$,

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & 2 \leq t \end{cases}$$

and

$$|\psi''(t)| + |\psi'(t)| \leq c \text{ for all } t \geq 0.$$

We consider a function $\phi(x) := \psi(\frac{r(x)}{R})$, then

$$\phi = \begin{cases} 1 & \text{on } B_R(p_0) \\ 0 & \text{on } M \setminus B_{2R}(p_0) \end{cases}$$

and

$$|\nabla \phi| = \frac{|\psi'|}{R} \leq \frac{c}{R}. \tag{4.34}$$

Using the Laplacian comparison theorem (see Theorem 3.1 in [96] and Theorem 11), there exists a positive constant β (independent on R and $x \in M \setminus B_1(p_0)$) such that on $M \setminus B_1(p_0)$

$$\Delta_f r(x) \leq \frac{1}{2} r(x) + \beta. \tag{4.35}$$

Hence

$$\begin{aligned} \Delta_f \phi &= \frac{\psi'}{R} \Delta_f r + \frac{\psi''}{R^2} |\nabla r|^2 \\ &\geq -\frac{c}{R}\big(R + \beta\big) - \frac{c}{R^2} \\ &\geq -c - \frac{c(1+\beta)}{R}. \end{aligned} \tag{4.36}$$

Let G be $\phi^2 u$, we have by (4.33), (4.34) and (4.36)

$$\begin{aligned} \phi^2 \Delta_f G &= \phi^4 \Delta_f u + 4\phi^3 \langle \nabla \phi, \nabla(\phi^2 u \phi^{-2}) \rangle + (2\phi \Delta_f \phi + 2|\nabla \phi|^2) G \\ &\geq G^2 - c_0 G + 4\phi \langle \nabla \phi, \nabla G \rangle + (2\phi \Delta_f \phi - 6|\nabla \phi|^2) G \\ &\geq G^2 - \big(c_0 + c + \frac{c(1+\beta)}{R}\big) G + 4\phi \langle \nabla \phi, \nabla G \rangle. \end{aligned} \tag{4.37}$$

Suppose G attains its maximum at q. If $q \in \overline{B_1(p_0)}$, then

$$G \leq G(q) \leq \sup_{\overline{B_1(p_0)}} u,$$

R.H.S. is independent on $R \geq 1$. If $q \in M \setminus \overline{B_1(p_0)}$, then by the maximum principle, we have by (4.37),

$$0 \geq G^2(q) - \big(c_0 + c + \frac{c(1+\beta)}{R}\big) G(q).$$

In either cases, we get the following bound for G:

$$G \leq \sup_{\overline{B_1(p_0)}} u + c_0 + c + \frac{c(1+\beta)}{R},$$

result then follows by letting $R \to \infty$. □

For general gradient expander, the scalar curvature may be negative. Nonetheless, Rm is still bounded if the scalar curvature is bounded.

Theorem 35. *Let (M^3, g, f) be a 3 dimensional complete noncompact gradient expanding Ricci soliton. Then there exist positive constant C_1 and C_2 such that on M*

$$|Rm| \leq C_1 S + C_2. \tag{4.38}$$

In particular the curvature tensor is bounded If in addition the scalar curvature S is bounded.

Proof. Since in dimension 3, the curvature tensor is controlled by the Ricci tensor, we shall estimate the Ricci tensor. Using the computation (4.30) in the proof of Theorem 28(1) and Kato's inequality, we have

$$\Delta_f |\mathrm{Ric}| \geq -|\mathrm{Ric}| - c|\mathrm{Ric}|^2$$

for some absolute constant $c > 0$ wherever $|\mathrm{Ric}| > 0$. Hence by (4.4) and (1.28)

$$\begin{aligned} \Delta_f(|\mathrm{Ric}| - AS) &\geq (2A - c)|\mathrm{Ric}|^2 - |\mathrm{Ric}| + AS \\ &\geq |\mathrm{Ric}|^2 - |\mathrm{Ric}| + AS \\ &\geq |\mathrm{Ric}|^2 - |\mathrm{Ric}| - \frac{3A}{2}, \end{aligned} \tag{4.39}$$

for all large positive constant A. Let $u := |\mathrm{Ric}| - AS$ and $G := \phi^2 u$. For any $R \geq 1$, we consider the cutoff function $\phi(x) = \psi(\frac{r(x)}{R})$ as in (4.34), (4.35) and (4.36). Using (1.28) and (4.39), we compute directly as in (4.37) on the set where G is positive to get

$$\begin{aligned} \phi^2 \Delta_f G &= \phi^4 \Delta_f u + 4\phi\langle\nabla\phi, \nabla G\rangle + (2\phi\Delta_f\phi - 6|\nabla\phi|^2)G \\ &\geq \phi^4 \Delta_f u - (c + \frac{c(1+\beta)}{R})G + 4\phi\langle\nabla\phi, \nabla G\rangle \\ &\geq \phi^4 \Delta_f u - (c + \frac{c(1+\beta)}{R})\phi^2|\mathrm{Ric}| - \frac{3Ac}{2}(1 + \frac{(1+\beta)}{R}) + 4\phi\langle\nabla\phi, \nabla G\rangle \\ &\geq \phi^4|\mathrm{Ric}|^2 - (1 + c + \frac{c(1+\beta)}{R})\phi^2|\mathrm{Ric}| - \frac{3A}{2}(1 + c + \frac{c(1+\beta)}{R}) \\ &\quad + 4\phi\langle\nabla\phi, \nabla G\rangle, \end{aligned} \tag{4.40}$$

where β is the positive constant in (4.35). Suppose G attains its maximum at q. If

$q \in \overline{B_1(p_0)}$, then

$$G \le G(q) \le \sup_{\overline{B_1(p_0)}} (|\mathrm{Ric}| + A|S|),$$

R.H.S. is independent on $R \ge 1$. If $q \in M \setminus \overline{B_1(p_0)}$, we may assume $G(q) > 0$. Hence $|\mathrm{Ric}|(q) > 0$ and u is smooth near q. By the maximum principle, (1.28) and (4.40), we have

$$\begin{aligned} G &\le G(q) \\ &= \phi^2(q)|\mathrm{Ric}|(q) - \phi^2(q)AS(q) \\ &\le \phi^2(q)|\mathrm{Ric}|(q) + \frac{3A}{2} \\ &\le 1 + c + \frac{c(1+\beta)}{R} + \sqrt{\frac{3A}{2}\Big(1 + c + \frac{c(1+\beta)}{R}\Big)} + \frac{3A}{2}. \end{aligned}$$

Results follows by letting $R \to \infty$. □

To prove Theorem 28(2), we need to control the change in distance along the flow of ∇f. Recall that it is a result of Zhang [99] that the flow of ∇f exists for all time $t \in \mathbb{R}$.

Lemma 16. *Let (M^n, g, f) be an n dimensional complete gradient expanding Ricci soliton and ϕ_s be the flow of ∇f with ϕ_0 being the identity map. Then for any $x \in M$ and $s \ge 0$,*

$$r(\phi_s(x)) \le e^{\frac{s}{2}}(r(x) + 2\sqrt{v(p_0)}), \tag{4.41}$$

where $v = \frac{n}{2} - f$ and p_0 is the base point of the distance function r.

Proof. It follows from (4.10) that

$$|\nabla f| \le \frac{1}{2}r + \sqrt{v(p_0)}. \tag{4.42}$$

For simplicity, we denote $r(\phi_s(x))$ by r_s, $\sqrt{v(p_0)}$ by c_1. By triangle inequality and

(4.42),

$$\begin{aligned} r_s &\leq r_0 + \int_0^s |\dot{\phi}_\tau(x)| d\tau \\ &= r_0 + \int_0^s |\nabla f(\phi_\tau(x))| d\tau \\ &\leq r_0 + \int_0^s \frac{1}{2} r_\tau + c_1 d\tau. \end{aligned} \tag{4.43}$$

Let $w(s) := r_0 + \int_0^s \frac{1}{2} r_\tau + c_1 d\tau + 2c_1$, the above inequality can be rewritten as

$$w' \leq \frac{w}{2}.$$

Integrating the inequality w.r.t. s, we have by (4.43)

$$\begin{aligned} r_s &\leq w(s) \\ &\leq e^{\frac{s}{2}}(r_0 + 2c_1) \\ &= e^{\frac{s}{2}}(r(x) + 2\sqrt{v(p_0)}). \end{aligned}$$

□

Theorem 28(2) is now a consequence of Theorem 35 and the following theorem. The proof is motivated by the arguments in [45] and [81].

Theorem 36. *Let (M^n, g, f) be an n dimensional complete noncompact gradient expanding Ricci soliton with bounded curvature. If $\lim_{x\to\infty} S(x) = 0$, then*

$$\lim_{x\to\infty} |Ric|(x) = 0.$$

Proof. We argue by contradiction. Suppose the claim is not true. Then there exist a sequence $y_k \in M \to \infty$ as $k \to \infty$ and $\varepsilon_0 \in (0, 1)$ such that for all k

$$P_k := |\mathrm{Ric}(g)|(y_k) \geq \varepsilon_0. \tag{4.44}$$

Since $|\mathrm{Rm}|$ is assumed to be bounded on M, we see that the derivatives of Rm of any orders are also bounded by the Shi's derivative estimate (see [38]). For any nonnegative

integer l, we may let $L_l := \sup_M |\nabla^l \mathrm{Rm}(g)| < \infty$. Let $g(t)$ be the Ricci flow generated by the soliton, i.e. $g(t) = (1+t)\psi_t^* g$ for all $t > -1$, where ψ_t is the flow of the non-autonomous vector field $\frac{\nabla f}{1+t}$ with ψ_0 being the identity map. We are going to rescale $g(t)$ near y_k. We consider the flow $g_k(t) := P_k g(\frac{t}{P_k}) = (P_k + t)\psi^*_{\frac{t}{P_k}} g$. Then all $t \in [-\frac{\varepsilon_0}{10}, \frac{\varepsilon_0}{10}]$ and $z \in M$

$$\begin{aligned} |\mathrm{Rm}(g_k(t))|_{g_k(t)}(z) &= \frac{1}{P_k} |\mathrm{Rm}((1+\frac{t}{P_k})\psi^*_{\frac{t}{P_k}} g)|_{(1+\frac{t}{P_k})\psi^*_{\frac{t}{P_k}} g}(z) \\ &= \frac{1}{P_k + t} |\mathrm{Rm}(g)|_g(\psi_{\frac{t}{P_k}}(z)) \\ &\leq \frac{10}{9\varepsilon_0} L_0. \end{aligned}$$

Similarly for all the derivatives of the curvature Rm,

$$\begin{aligned} |\nabla^l_{g_k(t)} \mathrm{Rm}(g_k(t))|_{g_k(t)}(z) &= (P_k + t) |\nabla^l_{g_k(t)} \mathrm{Rm}(\psi^*_{\frac{t}{P_k}} g)|_{g_k(t)}(z) \\ &= (P_k + t) |\nabla^l_{\psi^*_{\frac{t}{P_k}} g} \mathrm{Rm}(\psi^*_{\frac{t}{P_k}} g)|_{g_k(t)}(z) \\ &= \frac{1}{(P_k + t)^{\frac{l+2}{2}}} |\nabla^l_{\psi^*_{\frac{t}{P_k}} g} \mathrm{Rm}(\psi^*_{\frac{t}{P_k}} g)|_{\psi^*_{\frac{t}{P_k}} g}(z) \\ &= \frac{1}{(P_k + t)^{\frac{l+2}{2}}} |\nabla^l_g \mathrm{Rm}(g)|_g(\psi_{\frac{t}{P_k}}(z)) \\ &\leq \Big(\frac{10}{9\varepsilon_0}\Big)^{\frac{l+2}{2}} L_l. \end{aligned}$$

For all $k \in \mathbb{N}$, let $\iota_k : \mathbb{R}^n \to T_{y_k} M$ (with metric $g_k(0)$) be any linear isometry such that $\iota_k(0) = 0$ and $F_k := \exp^{g_k(0)}_{y_k} \circ \iota_k$, where $\exp^{g_k(0)}_{y_k}$ denotes the exponential map w.r.t. $g_k(0)$ at y_k. By Rauch comparison theorem, F_k is a local diffeomorphism on $\{x \in \mathbb{R}^n : |x| < \sqrt{\frac{9\pi^2\varepsilon_0}{10L_0}}\}$. By Hamilton compactness theorem (see [60], [36] and the proof of Lemma 4.4 in [45]), there exist positive δ (depending only on ε_0, L_0 and n) and subsequence k_j such that as $j \to \infty$

$$\big(B_\delta(0), F^*_{k_j} g_{k_j}(t)\big) \to \big(B_\delta(0), h_\infty(t)\big) \tag{4.45}$$

in C^∞_{loc} sense on $B_\delta(0) \times (-\frac{\varepsilon_0}{10}, \frac{\varepsilon_0}{10})$, where $B_\delta(0) := \{x \in \mathbb{R}^n : |x| < \delta\}$ and $h_\infty(t)$ is a solution to the Ricci flow. Moreover, by the local smooth convergence of the metric, we

have

$$\begin{aligned}|\mathrm{Ric}(h_\infty(0))|_{h_\infty(0)}(0) &= \lim_{j\to\infty}|\mathrm{Ric}(g_{k_j}(0))|_{g_{k_j}(0)}(y_{k_j})\\ &= \lim_{j\to\infty}\frac{1}{P_{k_j}}|\mathrm{Ric}(g)|_g(y_{k_j})\\ &= \lim_{j\to\infty}1\\ &= 1.\end{aligned}$$

Hence $h_\infty(0)$ is not Ricci flat. We claim that $h_\infty(t)$ is scalar flat for all $t\in(-\frac{\varepsilon_0}{10},0]$. We first assume the claim and prove the theorem. By the evolution equation of the scalar curvature along the Ricci flow (see [38])

$$2|\mathrm{Ric}(h_\infty(t))|^2_{h_\infty(t)} = \frac{\partial}{\partial t}S_{h_\infty(t)} - \Delta_{h_\infty(t)}S_{h_\infty(t)} = 0,$$

which is impossible. It remains to justify our claim, i.e. $h_\infty(t)$ is scalar flat. For all $(z,t)\in B_\delta(0)\times(-\frac{\varepsilon_0}{10},0]$, the scalar curvature with respect to the metric $h_{\infty(t)}$ satisfies

$$\begin{aligned}S_{h_{\infty(t)}}(z) &= \lim_{j\to\infty}S_{F^*_{k_j}g_{k_j}(t)}(z)\\ &= \lim_{j\to\infty}\frac{1}{(P_{k_j}+t)}S_{\psi^*_{\frac{t}{P_{k_j}}}g}(F_{k_j}(z)) \qquad (4.46)\\ &= \lim_{j\to\infty}\frac{1}{(P_{k_j}+t)}S_g(\psi_{\frac{t}{P_{k_j}}}\circ F_{k_j}(z)).\end{aligned}$$

By the assumption $\lim_{x\to\infty}S(x)=0$, we are done if $\lim_{j\to\infty}\psi_{\frac{t}{P_{k_j}}}(F_{k_j}(z))=\infty$. Since $r_k := d(y_k,p_0)\to\infty$ as $k\to\infty$, there is a $N_0\in\mathbb{N}$ such that for all $j\geq N_0$,

$$r_{k_j} > \sqrt{\frac{10}{9}}\Big[\sqrt{r_{k_j}}+2\sqrt{v(p_0)}\Big]+\frac{\delta}{\sqrt{\varepsilon_0}}. \qquad (4.47)$$

We are going to show that for all $j\geq N_0$, $z\in B_\delta(0)$ and $t\in(-\frac{\varepsilon_0}{10},0]$

$$d(\psi_{\frac{t}{P_{k_j}}}(F_{k_j}(z)),p_0)\geq\sqrt{r_{k_j}}. \qquad (4.48)$$

Assume by contradiction that it is not true. Then $d(\psi_{\frac{t}{P_{k_j}}}(F_{k_j}(z)),p_0)<\sqrt{r_{k_j}}$ for some

$j \geq N_0$, $z \in B_\delta(0)$ and $t \in (-\frac{\varepsilon_0}{10}, 0]$. We consider the flow ϕ_s of ∇f with ϕ_0 being the identity map, it is related to ψ_t in the following way

$$\psi_t = \phi_{\ln(1+t)} \text{ for all } t > -1.$$

By Lemma 16 and $\phi_{-\ln(1+\frac{t}{P_{k_j}})} \circ \psi_{\frac{t}{P_{k_j}}}(F_{k_j}(z)) = F_{k_j}(z)$,

$$\begin{aligned} d(F_{k_j}(z), p_0) &\leq \frac{1}{\left(1+\frac{t}{P_{k_j}}\right)^{\frac{1}{2}}} \Big[d(\psi_{\frac{t}{P_{k_j}}}(F_{k_j}(z)), p_0) + 2\sqrt{v(p_0)} \Big] \\ &\leq \sqrt{\frac{10}{9}} \Big[\sqrt{r_{k_j}} + 2\sqrt{v(p_0)} \Big], \end{aligned} \tag{4.49}$$

we used $\frac{t}{P_k} \geq -\frac{\varepsilon_0}{10P_k} \geq -\frac{1}{10}$ in the last inequality. By the definition of $F_k := \exp_{y_k}^{g_k(0)} \circ \iota_k$,

$$d_{g_{k_j}(0)}(F_{k_j}(z), y_{k_j}) = \sqrt{P_{k_j}} d_g(F_{k_j}(z), y_{k_j}) \leq \delta.$$

Furthermore by triangle inequality and (4.49),

$$\begin{aligned} r_{k_j} - \frac{\delta}{\sqrt{P_{k_j}}} &\leq d_g(F_{k_j}(z), p_0) \\ &\leq \sqrt{\frac{10}{9}} \Big[\sqrt{r_{k_j}} + 2\sqrt{v(p_0)} \Big], \end{aligned}$$

which is absurd by (4.44) and (4.47). We proved that (4.48) holds. $S_{h_\infty(t)} \equiv 0$ now follows from (4.44), (4.46) and (4.48).

□

Proof of Theorem 28(2): By Theorem 35, M has bounded curvature. We then apply Theorem 36 to conclude that $\mathrm{Ric} \to 0$ as $x \to \infty$. Since the Weyl tensor is zero in dimension 3, we have $\lim_{x\infty} |\mathrm{Rm}| = 0$. □

Corollary 4 is a consequence of the following proposition and Theorem 28(2). The proposition is known and we give a proof for the sake of completeness.

Proposition 8. *Let (M^n, g, f) be an n dimensional complete noncompact gradient expanding Ricci soliton. If $\liminf_{x\to\infty} Ric \geq 0$, then*

$$\lim_{x\to\infty} \frac{4v(x)}{r^2(x)} = 1,$$

where $v = \frac{n}{2} - f$.

Proof. It can be seen from (4.10) that $\overline{\lim}_{x\to\infty} \frac{v(x)}{r^2(x)} \leq \frac{1}{4}$. Since $\liminf_{x\to\infty} \mathrm{Ric} \geq 0$, for any small positive number ε, $\exists\ R_0 \gg 1$ such that on $M \setminus B_{R_0}(p_o)$

$$\begin{aligned} \nabla^2 v &= -\nabla^2 f \\ &= \mathrm{Ric} + \frac{1}{2} g \\ &\geq (\frac{1}{2} - \varepsilon) g. \end{aligned}$$

For any x in $M \setminus B_{R_0}(p_o)$, consider a normalized minimizing geodesic γ joining p_0 to x. We integrate the above inequality along the geodesic to get

$$\langle \nabla v, \dot{\gamma} \rangle(\gamma(t)) - \langle \nabla v, \dot{\gamma} \rangle(\gamma(R_0)) \geq (\frac{1}{2} - \varepsilon)(t - R_0),$$

where $t \geq R_0$. By integrating the inequality w.r.t. t, we see that

$$\begin{aligned} v(x) - v(\gamma(R_0)) &\geq (\frac{1}{4} - \frac{\varepsilon}{2})(r(x) - R_0)^2 + \langle \nabla v, \dot{\gamma} \rangle(\gamma(R_0))(r(x) - R_0) \\ &\geq (\frac{1}{4} - \frac{\varepsilon}{2})(r(x) - R_0)^2 - \sup_{\overline{B_{R_0}(p_o)}} |\nabla v| (r(x) - R_0). \end{aligned}$$

Hence it is clear that $\underline{\lim}_{x\to\infty} \frac{v(x)}{r^2(x)} \geq \frac{1}{4} - \frac{\varepsilon}{2}$. We let $\varepsilon \to 0$ and conclude that $\lim_{x\to\infty} \frac{4v}{r^2} = 1$. □

Lemma 17. *Let (M^3, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. If $\lim_{x\to\infty} r^2(x) S(x) = 0$, then $\lim_{x\to\infty} r^2(x) |Rm|(x) = 0$.*

Proof. By Theorem 28(2), $|\mathrm{Rm}| = o(1)$. Using local Shi's estimate (see Lemma 2.6 in

[49]), there exists a positive constant C such that for all $p \in M$ and $R \geq 1$

$$|\nabla \mathrm{Rm}|(p) \leq C \sup_{B_R(p)} |\mathrm{Rm}| \left[1 + \sup_{B_R(p)} |\mathrm{Rm}| + \frac{\sup_{B_R(p) \setminus B_{\frac{R}{2}}(p)} |\nabla f|}{R} \right]^{\frac{1}{2}}. \tag{4.50}$$

Hence $|\nabla \mathrm{Rm}| = o(1)$. By Corollary 4, (4.2) and (4.10),

$$C^{-1} r \leq |\nabla f| \leq Cr \tag{4.51}$$

near infinity for some constant $C > 0$. Let $\nu := \frac{\nabla f}{|\nabla f|}$. In dimension three, the Weyl tensor vanishes and (see also [49])

$$R_{ij} = R_{\nu ij \nu} + \frac{S}{2}(g_{ij} - g_{i\nu} g_{j\nu}) - R_{\nu\nu} g_{ij} + R_{j\nu} g_{i\nu} + R_{i\nu} g_{j\nu}. \tag{4.52}$$

To proceed, we need the following identity for gradient Ricci solitons which follows from Ricci identity and (1.8):

$$R_{ij,k} - R_{ik,j} = R_{kjil} f_l. \tag{4.53}$$

By (4.51), we see that

$$\begin{aligned} R_{ijk\nu} &= \frac{R_{kj,i} - R_{ki,j}}{|\nabla f|} \\ &= o(r^{-1}). \end{aligned} \tag{4.54}$$

By (4.52), It is now clear that $|\mathrm{Rm}| \leq c|\mathrm{Ric}| = o(r^{-1})$. By (4.50) and (4.54) again, we have $|\nabla \mathrm{Rm}| = o(r^{-1})$ and $R_{ijk\nu} = o(r^{-2})$. $|\mathrm{Rm}| = o(r^{-2})$ then follows from (4.52). □

Before we move on, let us recall some basic definitions:

Definition 2. ([92], [93]) An n dimensional complete Riemannian manifold M is asymptotically flat if there exist $R_0 > 0$, compact set $K \subseteq M$ and diffeomophism $\psi : M \setminus K \to \mathbb{R}^n \setminus \{|x| \leq R_0\}$ such that in this coordinate

$$|x|^p |g_{ij} - \delta_{ij}| + |x|^{1+p} |\partial g_{ij}| + |x|^{2+p} |\partial^2 g_{ij}| \leq C \text{ as } x \to \infty,$$

for some $p > \frac{n-2}{2}$ and $C > 0$, where ∂ denotes the partial derivative. We also require

the scalar curvature $S = O(|x|^{-q})$ for some $q > n$.

Let X be any smooth $n-1$ dimensional closed manifold with Riemannian metric g_X, $C(X)$ is defined to be the cone over X, i.e. $\{(t,\omega) : t > 0, \omega \in X\}$. g_C and ∇_C denote the metric $dt^2 + t^2 g_X$ on $C(X)$ and its Riemannian connection respectively. $\overline{B(o,R)} \subseteq C(X)$ is the set given by $\{(t,\omega) : R \geq t > 0, \omega \in X\}$. With the above preparations, we are going to prove Theorem 30:

Theorem. *Let (M, g, f) be a 3 dimensional complete non-compact gradient expanding Ricci soliton. Suppose that*

$$\lim_{x\to\infty} r^2(x)S(x) = 0.$$

Then M is isometric to $\mathbb{R}^3$.

Proof. By Theorem 1.3 in [49] and Lemma 17, the curvature tensor Rm satisfies

$$|\nabla^k \mathrm{Rm}|_g = O(v^{1+\frac{k-3}{2}} e^{-v}) \text{ as } r \to \infty, \text{ for any nonnegative integer } k. \tag{4.55}$$

Furthermore, M is smoothly asymptotic to a three dimensional cone at exponential rate [49], i.e. there exist $R > 0$, compact set K in M, smooth closed surface X and diffeomorphism $\phi : M \setminus K \to C(X) \setminus \overline{B(o,R)}$ such that for any nonnegative integer k

$$|\nabla_C^k \big[(\phi^{-1})^* g - g_C\big]|_{g_C}(t,\omega) = O(t^{k-3} e^{-\frac{t^2}{4}}) \text{ as } t \to \infty; \tag{4.56}$$

$$\frac{n}{2} - v \circ \phi^{-1}(t,\omega) = f \circ \phi^{-1}(t,\omega) = -\frac{t^2}{4} + c_0, \tag{4.57}$$

where c_0 is some constant. Hence the scalar curvature S is integrable and nonnegative by Theorem 26. Moreover, $(C(X), g_C)$ is Ricci flat [49] which implies that X has constant Gauss curvature equal to 1. By (4.57) or the proof of Theorem 3.2 in [49], X is diffeomorphic to the level sets $\{-f = s\}$ for all large s (see also [32]). Since $S \geq 0$, using the results of Chen-Deruelle [32] and Munteanu-Wang [78], we see that M is connected at infinity. Hence X is connected.

Case 1 M is orientable.

The level sets of f at infinity are orientable closed surfaces in M. Hence X is also orientable and isometric to $\mathbb{S}^2(1)$. As a result, $C(X)$ is isometric to $\mathbb{R}^3 \setminus \{0\}$. By (4.55) and (4.56), (M, g) is asymptotically flat and ∂g_{ij} decay exponentially in $t = |x|$ with

$x \in \mathbb{R}^3$, in the standard coordinate of $\mathbb{R}^3 \setminus \{0\}$, where ∂ denotes the partial derivative. The A.D.M. mass m vanishes since

$$m := \frac{1}{32\pi} \lim_{t\to\infty} \int_{|x|=t} \sum_{i,j=1}^{3} \left(\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j}\right) \frac{x_j}{t} d\sigma_t = 0, \tag{4.58}$$

where $d\sigma_t$ is the volume element of the Euclidean sphere $\{|x| = t\}$ w.r.t. the Euclidean metric. By the rigidity case of the positive mass theorem, M is isometric to $\mathbb{R}^3$ (see [92], [71], [72], [93] and ref.therein).

Case 2 M is non-orientable.

We prove that it is impossible. Suppose on the contrary that M is non-orientable. Then we consider $\pi : N \to M$, the orientable double cover of M. With $\pi^* g$ and $\pi^* f$, N is endowed with the structure of a complete expanding Ricci soliton. Moreover, $S_N \geq 0$ and $\lim_{y\to\infty} r_N^2(y) S_N(y) = 0$. By Case 1, N and thus M are flat. $\nabla_g^2 f = -\frac{1}{2} g$ on M and M is diffeomorphic to $\mathbb{R}^3$, contradicting to the non-orientability of M. □

4.6 Proof of Theorem 27

We start with an estimate on the potential function f under some growth conditions on f and S.

Lemma 18. *Let (M^n, g, f) be an n dimensional complete noncompact gradient expanding Ricci soliton. If f is proper, i.e. $\lim_{x\to\infty} f = -\infty$, and*

$$\alpha := \limsup_{x\to\infty} \frac{S}{v} < 1, \tag{4.59}$$

where $v = \frac{n}{2} - f$, then for any $\delta \in (0, 1-\alpha)$ there exists a positive constant C such that

$$\frac{\delta r^2}{4} - Cr - C \leq -f \leq \frac{r^2}{4} + Cr + C \text{ on } M. \tag{4.60}$$

Remark 26. It can be seen from (1.28) and (4.9) that when f is proper, α is a real number in $[0, 1]$.

Proof. The upper bound of (4.60) follows from (4.10) without any conditions on f and S. For the lower bound, fix any $\delta \in (0, 1-\alpha)$. We consider the flow G_t of the vector

field $\frac{\nabla v}{|\nabla v|^2}$ with G_0 be the identity map, where $v = \frac{n}{2} - f$. The flow makes sense since from (4.9) and (4.59),

$$\begin{aligned} |\nabla v|^2 &= v - \frac{n}{2} - S \\ &= v\Big(1 - \frac{n}{2v} - \frac{S}{v}\Big) \\ &\geq \delta v \\ &> 1, \end{aligned} \tag{4.61}$$

near infinity. Let ρ be a large constant such that the above inequality holds on $\{v \geq \rho\}$ and $\{v = \rho\}$ is nonempty. It is not difficult to see that for all $q \in \{v \geq \rho\}$, there are $t \geq 0$ and $z \in \{v = \rho\}$ such that $G_t(z) = q$ and

$$\begin{aligned} v(q) - v(z) &= \int_0^t \langle \nabla v, \frac{\nabla v}{|\nabla v|^2} \rangle (G_\tau(z)) d\tau \\ &= t. \end{aligned}$$

Moreover, we have

$$\begin{aligned} d(q,z) &= d(G_t(z), z) \\ &\leq \int_0^t \frac{1}{|\nabla v|(G_\tau(z))} d\tau \\ &= \int_0^t \frac{1}{\sqrt{v(G_\tau(z)) - \frac{n}{2} - S}} d\tau \\ &\leq \int_0^t \frac{1}{\sqrt{\delta(\tau + \rho)}} d\tau \\ &\leq \frac{2}{\sqrt{\delta}} \sqrt{v(q)}. \end{aligned}$$

We used (4.61) in the second last inequality. By triangular inequality,

$$\begin{aligned} r(q) &\leq d(q,z) + \sup_{\{v=\rho\}} d(\cdot, p_0) \\ &\leq \frac{2}{\sqrt{\delta}} \sqrt{v(q)} + K_0, \end{aligned}$$

where $K_0 = \sup_{\{v=\rho\}} d(\cdot, p_0)$. Hence near infinity,

$$v(q) \geq \frac{\delta r^2(q)}{4} - \frac{\delta K_0 r(q)}{2} + \frac{\delta K_0^2}{4}. \tag{4.62}$$

□

Using the above lemmas, We prove a lower bound for the scalar curvature.

Proposition 9. *Let (M^n, g, f) be a complete noncompact expanding gradient Ricci soliton with proper potential function f and dimension $n \geq 2$. If*

$$\liminf_{x\to\infty} vS \geq 0, \tag{4.63}$$

then there exists a positive constant C such that

$$0 \leq S + Cv^{1-\frac{n}{2}} e^{-v} \text{ on } M, \tag{4.64}$$

where $v = \frac{n}{2} - f$.

Proof. Let u be $-S$. By (4.63),

$$\limsup_{x\to\infty} vu \leq 0. \tag{4.65}$$

From (4.4), we see that

$$\Delta_f u \geq -u. \tag{4.66}$$

(4.64) then follows from (4.65), (4.66) and the proof of Lemma 2.9 in [49]. We include the details for the convenience of readers. The calculations were essentially done by Deruelle [49]. By the properness of f, $\lim_{x\to\infty} v = \infty$ and $v \geq 1$ outside some compact subset of M. We claim that for any constant $a \geq 4$, there exist positive constants R_0 and b such that

$$\Delta_{f+2\ln v - 2av^{-1}}(e^{-\frac{a}{v}} uv) \geq 0 \text{ on } \{u \geq 0\} \setminus B_{R_0}(p_0) \text{ and} \tag{4.67}$$

$$\Delta_{f+2\ln v - 2av^{-1}}(e^{-\frac{b}{v}} v^{2-\frac{n}{2}} e^{-v}) < 0 \text{ on } \{u \geq 0\} \setminus B_{R_0}(p_0). \tag{4.68}$$

We first assume the above inequalities and prove (4.64). There exists a positive constant β such that

$$e^{-\frac{a}{v}}uv - \beta e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v} < 0 \text{ on } \partial B_{R_0}(p_0). \tag{4.69}$$

Let $Q := e^{-\frac{a}{v}}uv - \beta e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}$. From $\lim_{x\to\infty} v = \infty$ and (4.65), it is evident that $\limsup_{x\to\infty} Q \leq 0$. For any $y \in M \setminus \overline{B_{R_0}(p_0)}$ and $\varepsilon > 0$, there exists $T > R_0$ such that $y \in B_T(p_0)$ and

$$Q < \varepsilon \text{ on } \partial B_T(p_0). \tag{4.70}$$

We consider the open set $\Omega := B_T(p_0) \setminus \overline{B_{R_0}(p_0)}$. Suppose Q attains its maximum over $\overline{\Omega}$ at $z \in \overline{\Omega}$. If $z \in \partial\Omega$, then by (4.69) and (4.70), $Q(y) \leq Q(z) \leq \varepsilon$. If $z \in \Omega$, we show that $Q(z) \leq 0$. Suppose on the contrary that $Q(z) > 0$. Then we know that $u(z) \geq 0$ and by (4.67) and (4.68) at z,

$$\begin{aligned} 0 &\geq \Delta_{f+2\ln v-2av^{-1}}Q \\ &= \Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{a}{v}}uv) - \beta\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}) \\ &\geq -\beta\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}) \\ &> 0, \end{aligned}$$

which is impossible. Hence, $Q(z) \leq 0$. In either cases, $Q(y) \leq Q(z) \leq \varepsilon$. We then obtain (4.64) by letting $\varepsilon \to 0$. It remains to prove (4.67) and (4.68). For (4.67), by (4.8) and (4.66), we have

$$\begin{aligned} \Delta_f(uv) &= v\Delta_f u + u\Delta_f v + 2\langle\nabla u, \nabla v\rangle \\ &\geq -uv + uv + 2\langle\nabla\ln v, \nabla(vu)\rangle - 2|\nabla\ln v|^2uv; \end{aligned}$$

$$\Delta_{f+2\ln v}(uv) \geq -2|\nabla\ln v|^2uv. \tag{4.71}$$

For any $a > 0$, we compute directly using (4.8) as in Lemma 2.9 of [49] to get

$$\Delta_{f+2\ln v}e^{-\frac{a}{v}} = \frac{a}{v}e^{-\frac{a}{v}} + \big(-\frac{4a}{v^3} + \frac{a^2}{v^4}\big)|\nabla v|^2e^{-\frac{a}{v}}. \tag{4.72}$$

A straightforward calculation using (4.71) and (4.72) yields

$$\begin{aligned}
\Delta_{f+2\ln v}(e^{-\frac{a}{v}}uv) &= e^{-\frac{a}{v}}\Delta_{f+2\ln v}(uv) + uv\Delta_{f+2\ln v}e^{-\frac{a}{v}} + 2\langle\nabla e^{-\frac{a}{v}}, \nabla(uv)\rangle \\
&\geq -2|\nabla \ln v|^2 e^{-\frac{a}{v}}uv + \frac{a}{v}e^{-\frac{a}{v}}uv + \big(-\frac{4a}{v^3} + \frac{a^2}{v^4}\big)|\nabla v|^2 e^{-\frac{a}{v}}uv \\
&\quad +2\langle\nabla e^{-\frac{a}{v}}, \nabla(uv)\rangle \\
&= \big(\frac{a}{v} - 2|\nabla \ln v|^2\big)e^{-\frac{a}{v}}uv + \big(-\frac{4a}{v^3} + \frac{a^2}{v^4}\big)|\nabla v|^2 e^{-\frac{a}{v}}uv \\
&\quad -2a\langle\nabla v^{-1}, \nabla(e^{-\frac{a}{v}}uv)\rangle - \frac{2a^2}{v^4}|\nabla v|^2 e^{-\frac{a}{v}}uv \\
&= \big(\frac{a}{v} - 2|\nabla \ln v|^2\big)e^{-\frac{a}{v}}uv - \big(\frac{4a}{v^3} + \frac{a^2}{v^4}\big)|\nabla v|^2 e^{-\frac{a}{v}}uv \\
&\quad -2a\langle\nabla v^{-1}, \nabla(e^{-\frac{a}{v}}uv)\rangle.
\end{aligned}$$

Hence we have

$$\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{a}{v}}uv) \geq \big(\frac{a}{v} - 2|\nabla \ln v|^2\big)e^{-\frac{a}{v}}uv - \big(\frac{4a}{v^3} + \frac{a^2}{v^4}\big)|\nabla v|^2 e^{-\frac{a}{v}}uv.$$

On the set where u is nonnegative and for $a \geq 4$, we may simplify the above inequality by $|\nabla v|^2 \leq v$

$$\begin{aligned}
\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{a}{v}}uv) &\geq \frac{a-2}{v}e^{-\frac{a}{v}}uv - \big(\frac{4a}{v^2} + \frac{a^2}{v^3}\big)e^{-\frac{a}{v}}uv \\
&\geq \frac{1}{v}e^{-\frac{a}{v}}uv \\
&\geq 0
\end{aligned}$$

near infinity. We proved the inequality (4.67). For (4.68), by $|\nabla v|^2 \leq v$, (4.9), (1.28) and previous computation (4.15),

$$\begin{aligned}
\Delta_{f+2\ln v}(v^{2-\frac{n}{2}}e^{-v}) &= v^{2-\frac{n}{2}}e^{-v}\Big[-S + (\frac{n}{2}-2)(\frac{n}{2}+1)|\nabla \ln v|^2 \\
&\qquad -(2S+n)(\frac{n}{2}-1)v^{-1}\Big]. \\
&\leq v^{2-\frac{n}{2}}e^{-v}\Big[-S + |\frac{n}{2}-2|(\frac{n}{2}+1)v^{-1}\Big].
\end{aligned}$$

$$\begin{aligned} 2a\langle \nabla v^{-1}, \nabla(v^{2-\frac{n}{2}}e^{-v})\rangle &= a(n-4)\frac{|\nabla \ln v|^2}{v}v^{2-\frac{n}{2}}e^{-v} + 2a|\nabla \ln v|^2 v^{2-\frac{n}{2}}e^{-v} \\ &\le v^{2-\frac{n}{2}}e^{-v}\Big[\frac{a|n-4|}{v^2} + \frac{2a}{v}\Big]. \end{aligned}$$

Hence

$$\Delta_{f+2\ln v - 2av^{-1}}(v^{2-\frac{n}{2}}e^{-v}) \le v^{2-\frac{n}{2}}e^{-v}\Big[-S + \frac{C_0}{v}\Big],$$

where $C_0 = C_0(a,n)$ is a positive constant depending on a and n. Using (4.63), we have $vS \ge -1$ near infinity. We can rewrite the above inequality as

$$\Delta_{f+2\ln v - 2av^{-1}}(v^{2-\frac{n}{2}}e^{-v}) \le C_1 v^{1-\frac{n}{2}}e^{-v}, \tag{4.73}$$

where $C_1 = C_1(a,n)$ is a positive constant depending on a and n. For any $b > 0$,

$$2a\langle \nabla v^{-1}, \nabla e^{-\frac{b}{v}}\rangle = -2ab|\nabla \ln v|^2 v^{-2}e^{-\frac{b}{v}}. \tag{4.74}$$

By (4.72) and $|\nabla v|^2 \le v$,

$$\begin{aligned} \Delta_{f+2\ln v - 2av^{-1}}e^{-\frac{b}{v}} &= e^{-\frac{b}{v}}\Big[\frac{b}{v} + \Big(-\frac{4b}{v^3} + \frac{b^2}{v^4}\Big)|\nabla v|^2 - 2ab|\nabla \ln v|^2 v^{-2}\Big] \\ &\le e^{-\frac{b}{v}}\Big[\frac{b}{v} + \frac{b^2}{v^4}|\nabla v|^2\Big] \\ &\le e^{-\frac{b}{v}}\Big[\frac{b}{v} + \frac{b^2}{v^3}\Big]. \end{aligned} \tag{4.75}$$

It follows from (4.9) that

$$\begin{aligned} 2\langle \nabla(v^{2-\frac{n}{2}}e^{-v}), \nabla e^{-\frac{b}{v}}\rangle &= 2bv^{-2}e^{-\frac{b}{v}}|\nabla v|^2(2-\frac{n}{2})v^{1-\frac{n}{2}}e^{-v} \\ &\quad + 2bv^{-2}e^{-\frac{b}{v}}|\nabla v|^2(-e^{-v})v^{2-\frac{n}{2}} \\ &= 2bv^{-2}e^{-\frac{b}{v}}|\nabla v|^2(2-\frac{n}{2})v^{1-\frac{n}{2}}e^{-v} \\ &\quad -\frac{2b}{v}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}} + \frac{b(2S+n)}{v^2}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}} \\ &\le -\frac{2b}{v}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}} + \frac{b(2S+n+|n-4|)}{v^2}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}}. \end{aligned} \tag{4.76}$$

Using (4.73), (4.75) and (4.76), on the set where u is nonnegative (i.e. $S \leq 0$),

$$\begin{aligned}
\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}) &= e^{-\frac{b}{v}}\Delta_{f+2\ln v-2av^{-1}}(v^{2-\frac{n}{2}}e^{-v}) \\
&\quad +v^{2-\frac{n}{2}}e^{-v}\Delta_{f+2\ln v-2av^{-1}}e^{-\frac{b}{v}} \\
&\quad +2\langle\nabla(v^{2-\frac{n}{2}}e^{-v}),\nabla e^{-\frac{b}{v}}\rangle \\
&\leq \frac{C_1}{v}e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}+\Big[\frac{b}{v}+\frac{b^2}{v^3}\Big]e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v} \\
&\quad -\frac{2b}{v}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}}+\frac{b(2S+n+|n-4|)}{v^2}e^{-\frac{b}{v}}e^{-v}v^{2-\frac{n}{2}} \\
&\leq e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}\Big[\frac{C_1-b}{v}+\frac{C_2}{v^2}\Big],
\end{aligned}$$

where $C_2 = C_2(b,n)$ is positive and depends on b and n. We choose $b = C_1 + 2$, then

$$\begin{aligned}
\Delta_{f+2\ln v-2av^{-1}}(e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}) &\leq e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v}\Big[-\frac{2}{v}+\frac{C_2}{v^2}\Big] \\
&\leq -\frac{1}{v}e^{-\frac{b}{v}}v^{2-\frac{n}{2}}e^{-v} \\
&< 0
\end{aligned}$$

near infinity. We showed (4.68) and completed the proof of the proposition. □

With the above preparation, we give a proof for Theorem 27.

Proof of Theorem 27: From (4.10), we see that (4.63) holds. By Proposition 9, the negative part of the scalar curvature S_- satisfies

$$0 \leq S_- \leq Cv^{1-\frac{n}{2}}e^{-v} \text{ on } M,$$

for some constant $C > 0$. Using Lemma 18, we have $v \sim r^2$. Hence there exists $\delta > 0$ such that

$$S_- \leq Ce^{-\delta r^2}. \tag{4.77}$$

By (4.10) and Theorem 1.1 in [79], there exists a positive constant C such that for all $R > 0$

$$Vol_g(B_R(p_0)) \leq Ce^{\sqrt{n-1}R},$$

where $Vol_g(B_R(p_0))$ is the volume of the geodesic ball $B_R(p_0)$. It implies that

$$\int_M S_- < \infty.$$

The nonnegativity of the scalar curvature is a consequence of $\liminf_{x\to\infty} r^2 S \geq 0$ and Theorem 26. □

4.7 Proof of theorem 29

It was observed by Munteanu-Wang [80] that in four dimensional gradient Ricci soliton, the Riemann curvature Rm can be bounded by Ric and ∇Ric:

Lemma 19. *[80] Let (M^4, g, f) be a four dimensional gradient Ricci soliton. There exists a universal positive constant A_0 such that if $\nabla f \neq 0$ at $q \in M$, then at q*

$$|Rm| \leq A_0\Big(|Ric| + \frac{|\nabla Ric|}{|\nabla f|}\Big). \tag{4.78}$$

Please see [80], [25] and Lemma 2 for a proof of Lemma 19. To prove that the curvature tensor is bounded, we first give estimate on the Ricci tensor.

Theorem 37. *Let (M^4, g, f) be a 4 dimensional complete noncompact expanding gradient Ricci soliton. Suppose that it has bounded scalar curvature and $f \to -\infty$ as $x \to \infty$. Then the Ricci tensor Ric is bounded.*

Proof. Let $L := \sup_M |S| < \infty$. If $L = 0$, then by (4.4) M is Ricci flat and we are done with the proof. We may assume $L > 0$ and introduce the function $F(S) := (S+3L)^{-a} > 0$, where $a > 0$. We see that

$$\frac{1}{4^a L^a} \leq F \leq \frac{1}{2^a L^a}, \tag{4.79}$$

and

$$\nabla F = -a(S+3L)^{-a-1}\nabla S. \tag{4.80}$$

By (4.4)

$$\begin{aligned} \Delta_f F &= -a(S+3L)^{-a-1}\Delta_f S + a(a+1)(S+3L)^{-a-2}|\nabla S|^2 \\ &= a(S+3L)^{-a-1}(S+2|\mathrm{Ric}|^2) + a(a+1)(S+3L)^{-a-2}|\nabla S|^2. \end{aligned} \tag{4.81}$$

Whenever $\nabla f \neq 0$, we compute directly using (4.5) and get

$$\begin{aligned}
\Delta_f|\mathrm{Ric}|^2 &= 2|\nabla \mathrm{Ric}|^2 + 2\langle \mathrm{Ric}, \Delta_f \mathrm{Ric}\rangle \\
&\geq 2|\nabla \mathrm{Ric}|^2 - 2|\mathrm{Ric}|^2 - 4|\mathrm{Ric}|^2|\mathrm{Rm}| \\
&\geq 2|\nabla \mathrm{Ric}|^2 - 2|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - 4A_0|\mathrm{Ric}|^2\frac{|\nabla \mathrm{Ric}|}{|\nabla f|} \qquad (4.82) \\
&\geq |\nabla \mathrm{Ric}|^2 - 2|\mathrm{Ric}|^2 - 4A_0|\mathrm{Ric}|^3 - \frac{4A_0^2|\mathrm{Ric}|^4}{|\nabla f|^2}.
\end{aligned}$$

It is not difficult to see from Cauchy Schwarz inequality, (4.81), (4.82) and (4.79) that

$$\begin{aligned}
\Delta_f(F|\mathrm{Ric}|^2) &= F\Delta_f|\mathrm{Ric}|^2 + 2\langle \nabla F, \nabla|\mathrm{Ric}|^2\rangle + |\mathrm{Ric}|^2\Delta_f F \\
&\geq F\Delta_f|\mathrm{Ric}|^2 - 4|F'||\nabla S||\mathrm{Ric}||\nabla \mathrm{Ric}| + |\mathrm{Ric}|^2\Delta_f F \\
&\geq F\Delta_f|\mathrm{Ric}|^2 - F|\nabla \mathrm{Ric}|^2 - 4\frac{(F')^2}{F}|\nabla S|^2|\mathrm{Ric}|^2 \\
&\quad + |\mathrm{Ric}|^2\Delta_f F \\
&\geq F\Delta_f|\mathrm{Ric}|^2 - F|\nabla \mathrm{Ric}|^2 - 4\frac{(F')^2}{F}|\nabla S|^2|\mathrm{Ric}|^2 \\
&\quad + a(S+3L)^{-a-1}(S+2|\mathrm{Ric}|^2)|\mathrm{Ric}|^2 \\
&\quad + a(a+1)(S+3L)^{-a-2}|\nabla S|^2|\mathrm{Ric}|^2 \\
&= F\Delta_f|\mathrm{Ric}|^2 - F|\nabla \mathrm{Ric}|^2 - 4a^2(S+3L)^{-a-2}|\nabla S|^2|\mathrm{Ric}|^2 \\
&\quad + a(S+3L)^{-a-1}(S+2|\mathrm{Ric}|^2)|\mathrm{Ric}|^2 \\
&\quad + a(a+1)(S+3L)^{-a-2}|\nabla S|^2|\mathrm{Ric}|^2 \\
&\geq -\big(2+\frac{a}{2}\big)F|\mathrm{Ric}|^2 - 4A_0F|\mathrm{Ric}|^3 - \frac{4A_0^2F|\mathrm{Ric}|^4}{|\nabla f|^2} \\
&\quad + 2a(S+3L)^{a-1}F^2|\mathrm{Ric}|^4 \\
&\quad + a(1-3a)(S+3L)^{-a-2}|\nabla S|^2|\mathrm{Ric}|^2.
\end{aligned}$$

We may take $a = \frac{1}{3}$ and consider the function $w := F|\mathrm{Ric}|^2$, then the above inequalities can be rewritten as

$$\Delta_f w \geq \Big(\frac{2}{3}F^2 - \frac{4A_0^2}{F|\nabla f|^2}\Big)w^2 - 4A_0F^{-\frac{1}{2}}w^{\frac{3}{2}} - \frac{13}{6}w.$$

By (4.2), $|\nabla f|^2 = -f - S \geq -f - L \to \infty$. Using (4.79), we have outside a compact

set,

$$\begin{aligned}\Delta_f w &\geq \Big(\frac{1}{2^{\frac{1}{3}}3L^{\frac{2}{3}}} - \frac{4^{\frac{4}{3}}A_0^2 L^{\frac{1}{3}}}{|\nabla f|^2}\Big)w^2 - 4A_0F^{-\frac{1}{2}}w^{\frac{3}{2}} - \frac{13}{6}w \\ &\geq \frac{1}{6L^{\frac{2}{3}}}w^2 - 4A_0F^{-\frac{1}{2}}w^{\frac{3}{2}} - \frac{13}{6}w.\end{aligned}$$

It is easy to see that the differential inequality of w resembles the one of u in (4.33). Hence a similar cut off function argument as in the proof of Theorem 28(1) and the Laplacian comparison theorem (4.35) (see [96] and Theorem 11) yields the boundedness of w. □

Theorem 29 is now a consequence of the following theorem. The proof of the theorem is essentially due to Munteanu-Wang [80] and Cao-Cui [21]. We shall only give a sketch of the proof.

Theorem 38. *Let (M^4, g, f) be a 4 dimensional complete noncompact expanding gradient Ricci soliton. Suppose that it has bounded Ricci curvature and $f \to -\infty$ as $x \to \infty$. Then the curvature tensor Rm is bounded*

Proof. Throughout this proof, we use c_0 to denote some constants depending on the global upper bound of $|\mathrm{Ric}|$ and A_0 in Lemma 19, its value may be different from line by line. Since $|\nabla f|^2 = -f - S \to \infty$ as $x \to \infty$, we have by Lemma 19

$$\begin{aligned}|\nabla \mathrm{Ric}|^2 &\geq \frac{1}{2A_0^2}|\mathrm{Rm}|^2 - |\mathrm{Ric}|^2 \\ &\geq \frac{1}{2A_0^2}|\mathrm{Rm}|^2 - c_0\end{aligned}$$

outside a compact subset of M. Moreover by (4.82)

$$\begin{aligned}\Delta_f|\mathrm{Ric}|^2 &\geq |\nabla \mathrm{Ric}|^2 - c_0 \\ &\geq \frac{1}{2A_0^2}|\mathrm{Rm}|^2 - c_0 \\ &\geq \frac{1}{4A_0^2}(|\mathrm{Rm}| + \lambda|\mathrm{Ric}|^2)^2 - \frac{\lambda^2}{2A_0^2}|\mathrm{Ric}|^4 - c_0 \\ &\geq \frac{1}{4A_0^2}(|\mathrm{Rm}| + \lambda|\mathrm{Ric}|^2)^2 - \frac{\lambda^2}{2A_0^2}c_0 - c_0,\end{aligned} \tag{4.83}$$

where λ is any non-negative constant. We apply (4.6) and get

$$\begin{aligned}\Delta_f|\mathrm{Rm}|^2 &= 2|\nabla\mathrm{Rm}|^2 + 2\langle \mathrm{Rm}, \Delta_f \mathrm{Rm}\rangle \\ &\geq 2|\nabla\mathrm{Rm}|^2 - 2|\mathrm{Rm}|^2 - c|\mathrm{Rm}|^3,\end{aligned}$$

where c is an absolute constant. By Kato's inequality,

$$\Delta_f|\mathrm{Rm}| \geq -|\mathrm{Rm}| - c|\mathrm{Rm}|^2$$

on the set where $|\mathrm{Rm}| > 0$. Using (4.83), we see that for all $\lambda \geq 0$,

$$\Delta_f\big(|\mathrm{Rm}| + \lambda|\mathrm{Ric}|^2\big) \geq \big(\frac{\lambda}{4A_0^2} - c\big)\big(|\mathrm{Rm}| + \lambda|\mathrm{Ric}|^2\big)^2 - |\mathrm{Rm}| - \frac{\lambda^3}{2A_0^2}c_0 - c_0\lambda.$$

Let $W := |\mathrm{Rm}| + \lambda|\mathrm{Ric}|^2$, we may rewrite the above inequality as

$$\begin{aligned}\Delta_f W &\geq \big(\frac{\lambda}{4A_0^2} - c\big)W^2 - W - \frac{\lambda^3}{2A_0^2}c_0 - c_0\lambda \\ &\geq W^2 - W - \frac{\lambda^3}{2A_0^2}c_0 - c_0\lambda\end{aligned}$$

for all sufficiently large λ. Similar argument as in the proofs of Theorems 28(1) and 37 gives the boundedness of W. □

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