More on Monopoles, Vortices and Confinement

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Abstract

(1) Brief review: monopoles and confinement
(2) Behavior of strongly-coupled nonabelian monopoles
(3) Nonabelian monopoles from vortex moduli: DUAL GROUP

Work with Marmorini, Yokoi, Vinci, Eto, Ohashi, Nitta, ...

I. Confinement in QCD: nonabelian superconductor?

- Confinement in QCD $\sim$ dual superconductor ('t Hooft, Mandelstam):

$$\langle \chi_{n_m, n_e} \rangle \neq 0, \quad (n_m, n_e) \in (Z_N, Z_N)$$

$\Rightarrow$ particle with (n'_m, n'_e) confined ($SU(N)$ theory) if

$$n'_m \, n_e - n_m \, n'_e \neq 0, \quad Mod \, [N]$$

- What is $\chi_{1,0}$ or $\chi_{2,0}$ in QCD?
- No hint that $SU(3) \to U(1)^2$ in Nature
- No hint of weakly-coupled monopoles in QCD
- Hint that confinement is related to dynamical symmetry breaking (lattice)
- Interesting hints about quantum mechanical behavior of nonabelian monopoles and confinement ($\mathcal{N} = 1, 2$ Susy Gauge Theories)

Semiclassical "nonabelian monopoles"

Basic results (Goddard-Olive-Nuyts, Bais, '77, E.Weinberg '80)

$$G \xrightarrow{\langle\phi\rangle\neq 0} H$$

$$\phi \sim U \cdot \langle\phi\rangle \cdot U^{-1} \sim \Pi_2(G/H) = \Pi_1(H); \quad (\text{if } \ \Pi_1(G) = \emptyset),$$

$$A_i^a \sim U \cdot \partial_i U^\dagger \to F_{ij} \sim \epsilon_{ijk} \frac{r_k}{r^3} (\beta \cdot \mathbf{H}), \qquad H_i \in \text{Cartan S.A. of} \quad H$$

Topological (Dirac) Quantization $\Longrightarrow$

$$2\,\alpha \cdot \beta \subset \mathbf{Z}, \qquad cfr. \quad 2\,g_e\,g_m = n$$

$$\beta_i = \text{weight vectors of } \tilde{H} \quad (= \text{dual of } H).$$

$$\alpha \Leftrightarrow \alpha^* \equiv \frac{\alpha}{(\alpha \cdot \alpha)}$$

Dual Groups

$SU(N)/\mathbb{Z}_N$	$\Leftrightarrow$	$SU(N)$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

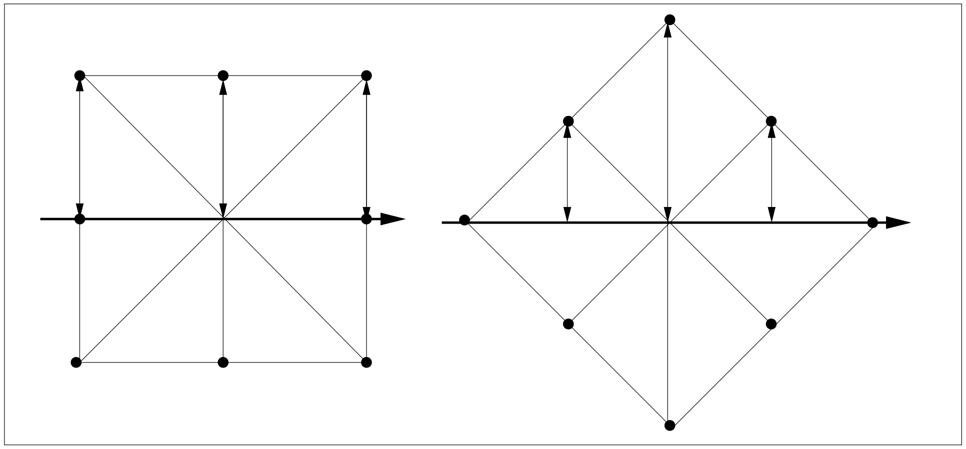

Figure 1: Root vectors in $SO(5)$ and $USp(4)$

Simple Example:

$$SU(3) \xrightarrow{\langle\phi\rangle} \frac{SU(2)\times U(1)}{\mathbb{Z}_2}, \qquad \langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

$$S^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad S^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad S^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \phi(\mathbf{r}) &= \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3\,v\hat{S}\cdot\hat{r}\phi(r), \\ \vec{A}(\mathbf{r}) &= \hat{S}\wedge\hat{r}A(r), \end{aligned}$$

$\phi(r)$ and $A(r)$ are BPS- 't Hooft's fnc with

$$\phi(\infty) = 1, \qquad \phi(0) = 0, \qquad A(\infty) = -1/r.$$

$\Rightarrow$ two *degenerate* $SU(3)$ solutions: Topology: $\Pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}$

GNO Duality in more general settings

- $SU(N+1) \to \frac{SU(N)\times U(1)}{\mathbb{Z}_N} \Rightarrow N$ degenerate solutions
- $\underline{N}$ of $\widetilde{S}U(N) \sim SU(N)$
- Actually, part of continuously degenerate set:

  Multiplicity $= 1, N, \infty$?
- $USp(2N+2) \to USp(2N) \times U(1)$

  $\Rightarrow 2N+1$ monopoles of $\tilde{H} = SO(2N+1)$!
- $SO(2N+3) \to SO(2N+1) \times U(1)$

  $\Rightarrow 2N$ degenerate monopoles of $USp(2N)$
- Is GNOW duality valid quantum mechanically?

Homotopy Groups in Sytems $G \to H$

- Exact sequence

$$\cdots \to \pi_n(G) \to \pi_n(G/H) \xrightarrow{f} \pi_{n-1}(H) \to \pi_{n-1}(G) \to \cdots .$$

- $\pi_2(G) = \emptyset$ for any Lie Group: short exact sequence

$$0 \to \pi_2(G/H) \xrightarrow{f} \pi_1(H) \to \pi_1(G) \to 0.$$

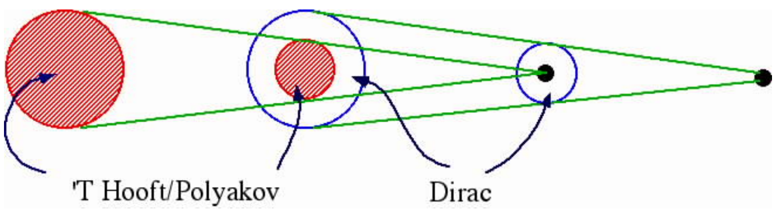

Figure 2:

Regular (BPS) monopoles $\sim \pi_2(G/H)$; $\pi_1(G) \sim \frac{\pi_1(H)}{\pi_2(G/H)}$

Regular monopoles $\sim$ kernel of mapping $\pi_1(H) \to \pi_1(G)$ (Coleman)

Quantum mechanical nonabelian monopoles

Quantum mechanical nonabelian monopoles

- Dynamics matters, *e.g.,* H can break itself dynamically (Pure $\mathcal{N}=2$, $SU(3)$): a Q.M. concept;

- "Colored dyons" do not exist (no charge fractionalization) (Abouelsaood, Coleman, ...) *i.e.* For $SU(3)/SU(2)\times U(1)$, no $(\underline{2},\underline{1}^*)$ monopoles. Global topological obstruction in $T^i(\hat{x})\sim u(\hat{x})\,T^i\,u(\hat{x})^\dagger$.

- Some monopole zero modes non-normalizable (Coleman-Nelson, Dorey-Fraser-Hollowood-Kneipp): standard quantization procedure not straightforward (*cfr* Jackiw-Rebbi)

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- NA monopoles are multiplets of $\tilde{H}$: (WANT TO SHOW): $\tilde{H}$ well-defined when H is strongly coupled (!?);

- $\mathcal{N}=2$ SQCD: monopoles carrying charges $(\underline{2}^*,\underline{1}^*)$!

Jackiw-Rebbi

- Dirac equation in the background of a 't Hooft-Polyakov monopole admits normalizable fermion zeromode (s).

$$[H, \mathbf{L} + \mathbf{s} + \frac{1}{2}\tau] = 0, \qquad H = \gamma_0 \gamma_i \mathcal{D}^i.$$

- For $\underline{2}$ fermion, $\psi_0^\dagger |M\rangle$ is again a boson! ($\psi_{adj}^\dagger |M\rangle$ is a fermion)
- (Flavor-) degenerate monopoles (Also, Gauntlett, Blum)
- Generalization to $SU(N)$, $USp(2N)$, $SO(N)^*$ (*subtle)
- $SU(N)$ with N_f: $\psi^{i\,\dagger}|M\rangle$, $\psi^{i\,\dagger}\,\psi^{j\,\dagger}|M\rangle$, etc., in antisymmetric tensor representations of $SU(N_f)$
- $USp(2N)$ with N_f: 2^{N_f} monopoles in spinor repres of $SO(N_f)$

Softly broken $\mathcal{N}=2$ susy gauge theories

- Abelian confining vacua (dual superconductor):
- NA confining vacua (nonabelian superconductor); weakly coupled monopoles;
- NA confining vacua: Strongly coupled NA monopoles and dyons

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Abelian confining vacua

- 't Hooft-Mandelstam picture
- Dynamical abelianization
- Abelian monopoles are singlets of $SU(N_f)$ (except for $SU(2)$)

Nonabelian, weakly coupled confining vacua

- $r-$ vacua with magnetic $SU(r) \times U(1)^{N-r}$ gauge symmetry;
- "Dual quarks" in $(\underline{r}, \underline{1})$: light GNO monopoles (Bolognesi, Konishi);
- They are in $\underline{N_f}$ of $SU(N_f)$
- The quantum $r-$ vacua only for $r < \frac{n_f}{2}$ (*cfr.* $r_{clas} \leq min[N_f, N_c]$)

$$b_0^{(dual)} \propto -2\, r + n_f > 0, \qquad b_0 \propto -2\, n_c + n_f \leq 0.$$

- $r = r_{clas}, \quad r_{clas} < \frac{N_f}{2}$;

  $r = N_f - r_{clas}, \quad r_{clas} > \frac{N_f}{2}$ (Seiberg dual, but is also GNOW dual)
- Light nonabelian monopoles needs flavor (*cfr.* $\mathcal{N} = 2$ YM)

Abelian or nonabelian monopoles

- Nonabelian monopoles of r vacua vs abelian monopoles

$$A_{i_1\,i_2\,\dots i_r} \sim \epsilon_{\alpha_1\alpha_2\dots\alpha_r}\, M_{i_1}^{\alpha_1}\, M_{i_2}^{\alpha_2}\, \dots M_{i_r}^{\alpha_r}.$$

- $SU(r)$ is IR free: $A_{i_1\,i_2\,\dots i_r}$ breaks up into M_i^{α}'s (*cfr.* $r = N_f/2$: see below)

- $U(1)$ monopole in r-antisymmetric flavor repr. (Jackiw-Rebbi) $\Rightarrow$ accidental symmetry $SU({}_{N_f}C_r)$

- Huge number of Nambu-Goldstone bosons

- $SU(2)$: No paradox, apparently because N_f, $\binom{N_f}{r}$, etc. are *small* numbers!

- SCFT Vacua in N_f flavored $USp(2N)$: Abelian monopoles in spinor repr. with multip. 2^{N_f-1};

- They disintegrate into fundamentals of $SU(r) \subset USp(2N)$, $r = 0, 1, \dots, \frac{N_f}{2}$, and in the fundamental repr of flavor!

Strongly coupled NA monopoles (and dyons)

- Example: $r = \frac{N_f}{2}$ of $SU(N)$ theory. Nontrivial SCFT.
- Nonabelian Argyres-Douglas vacua
- *e.g.*, $r = 2$ vacua of $SU(3)$, $N_f = 4$ theory (Auzzi, Grena, Konishi '03):
- *e.g.*, $USp(4)$, $N_f = 4$ theory (Auzzi, Grena '04):
- Actually, all confining vacua of $USp(2N)$ with $m_i = 0$ are SCFT

  no massless GNO monopoles of the dual groups $SO(2r + 1)$.
- How do monopoles behave?

Argyres-Douglas Vacua

- Pure $\mathcal{N}=2$, $SU(3)$ Yang-Mills theory

$$W = A_\mu, \lambda; \qquad \Phi = \phi, \psi$$

- $U = \langle \mathrm{Tr}\Phi^2 \rangle = 0$, $V = \langle \mathrm{Tr}\Phi^2 \rangle = \pm\Lambda^3$

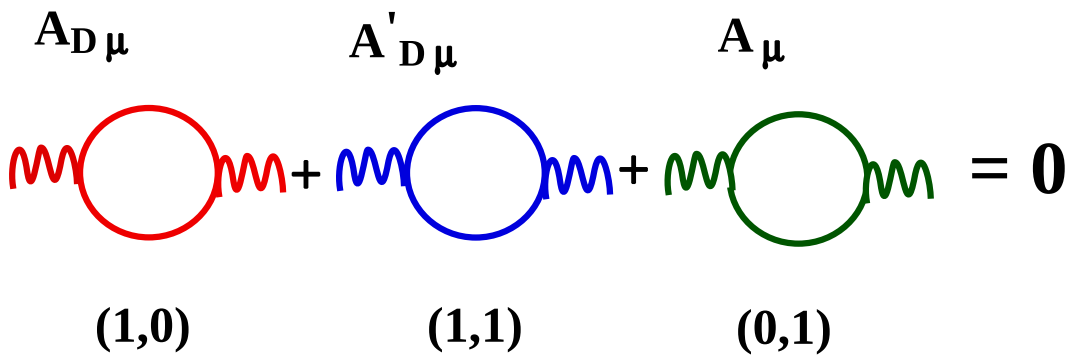
- $SU(3) \to U(1) \times U(1)$; the second $U(1)$ is a IR free; the first is strongly coupled with

$$\tau_{11} \sim \tau^* = e^{2\pi i/3}$$

- A light monopole, a dyon and an electron cancel in β function

$$\sum_i (n_{e\,i} + n_{m\,i}\,\tau^*)^2 = 0.$$

- Many other examples in $\mathcal{N}=2$, $\mathcal{N}=1$ (Bilal-Ferrari, Gorky-Vainshtein-Yung)



A_{D μ}
A'_{D μ}
A_{μ}
+
+
= 0
(1,0)
(1,1)
(0,1)

“$r = 2$” Vacua of $\mathcal{N} = 2$, $SU(3)$, $N_f = 4$

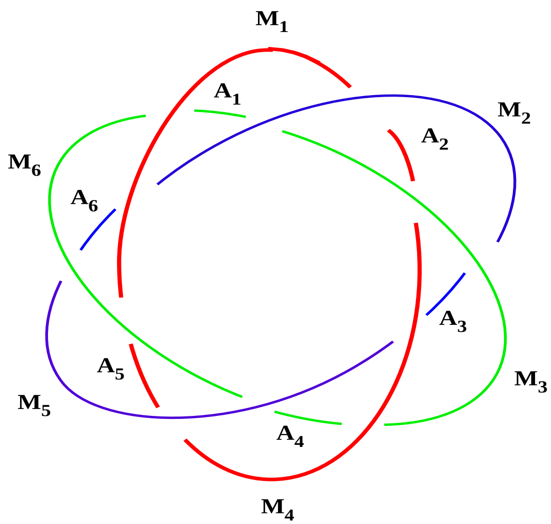
- $U = \langle \mathrm{Tr}\Phi^2 \rangle = \pm \Lambda^2, \quad V = \langle \mathrm{Tr}\Phi^2 \rangle = 0$
- $G_{eff} = SU(2) \times U(1);\ G_f = SU(4) \times U(1)$
- Low energy effective deg. freedom $=$

  4 monopole doublets, 1 dyon doublet, 1 electric doublet
- Nonlocal cancellation of beta function ($\tau^* = \frac{-1+i}{2}$).
- $\mu \neq 0$, Symmetry breaking pattern inferred from $\mu \gg \Lambda$:

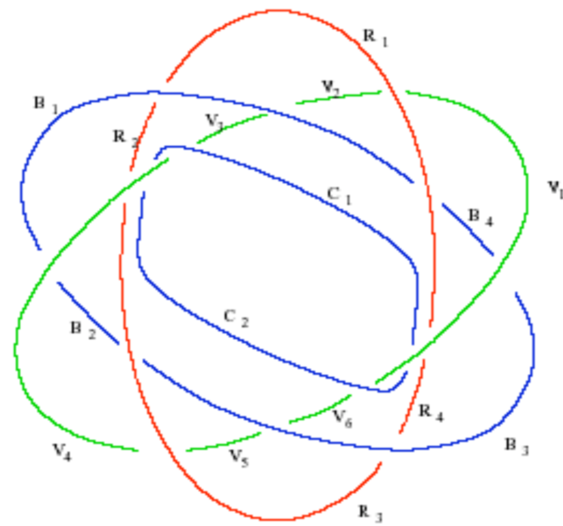
$$G_F = U(4) \times U(1) \to U(2) \times U(2) \quad \to \quad \langle \epsilon_{\alpha\beta} M_\alpha^i \, M_\beta^j \rangle \neq 0. \tag{1}$$

- A subtlety: Is baryon number of the massless monopoles zero!?

  Yes (Baryon number fractionalization - quenching Argyres-Plesser-Seiberg, Carlino-Konishi-Terao)



M_1
A_1
M_2
A_2
M_6
A_6
A_3
M_3
A_5
M_5
A_4
M_4



R_1
B_1
V_2
V_3
R_2
V_1
C_1
B_4
C_2
B_2
V_6
R_4
V_4
V_5
B_3
R_3

SCFT in General $SU(N)$ Theory

- SCFT with $r = \frac{N_f}{2}$;
- On $N = 1$ perturbation $\mu \mathrm{Tr}\, \Phi^2$, the symmetry breaking pattern known at $\mu \gg \Lambda$:

$$G_F = U(N_f) \times U(1) \to U(\frac{N_f}{2}) \times U(\frac{N_f}{2}).$$

- Need the condensate

$$\langle \epsilon_{\alpha_1 \alpha_2 \ldots \alpha_{N_f/2}} M_{i_1}^{\alpha_1} M_{i_2}^{\alpha_2} \ldots M_{i_{N_f/2}}^{\alpha_{N_f/2}} \rangle \neq 0. \qquad (2)$$

- Strong $SU(\frac{N_f}{2})$ interactions fundamental for confinement and DSB.

Another Example Auzzi, Grena

- $USp(4)$, $N_f = 4 \quad G_f = SO(8)$
- Nontrivial SCFT at $U = \langle \mathrm{Tr}\Phi^2 \rangle = \pm\Lambda^2$, $V = \langle \mathrm{Tr}\Phi^2 \rangle = 0$
- Low energy effective deg. freedom ($SU(2) \times U(1) \subset USp(4)$)

Particles	Charge
$M, \tilde{M}$	$(\pm 1, 1, 0, 0)^4$
$D, \tilde{D}$	$(\pm 2, -2, \pm 1, 0)$
$E, \tilde{E}$	$(0, 2, \pm 1, 0)$
$C, \tilde{C}$	$(\pm 2, 0, \pm 1, 0)$

- Breaking $G_F = SO(8) \to U(4)$ known ($\mu \gg \Lambda$)
- $\mu \neq 0 \to \langle M^i_\alpha \tilde{M}^j_\alpha \rangle = \mathrm{const.}\, \delta_{i\,j}$:

Confining vacua of N_f flavored $USp(2N)$

- All confining vacua are of SCFT type (deformed SCFT)
- No local effective Lagrangians
- No local set of fields representing $SO(2N_f)$ linearly
- Abelian monopoles disintegrate in nonabelian monopoles in $\underline{r}$ of $SU(r) \subset USp(2N)$, $r = 0, 1, \ldots, \frac{N_f}{2}$;
- Confinement due to
$$\langle M_\alpha^i \, \tilde{M}_\alpha^j \rangle = \text{const.} \, \delta_{i\,j} \; : \tag{3}$$
- Strongly interacting monopoles
- Chiral $SO(2N_f) \to U(N_f)$ symmetry breaking induced by the same order parameters

Strongly-interacting monopoles and confinement: Moral

- Confinement and symmetry breaking due to condensation of composite, nonabelian monopoles (1), (2), (3).

- Mechanism basically different from confinement in weakly-coupled nonabelian (r vacua) or abelian dual Meissner effects. (where $\langle M \rangle \neq 0$)

III. Nonabelian monopoles from monopole-vortex complex

- Nonabelian monopoles from $G \xrightarrow{v_1} H$: H nonabelian;
- Nonabelian monopoles as a multiplet of the dual group $(\tilde{H})$;
- Dynamics of H (or $\tilde{H}$) matters
- $\tilde{H}$ in Higgs phase $\Rightarrow$ no multiplet structure;
- $\tilde{H}$ in confinement phase $\Rightarrow$ monopole multiplet BUT H in Higgs phase!
- Consider

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \emptyset, \qquad v_1 \gg v_2.$$

- v_2 acts as IR cutoff

Nonabelian Vortices

Hanany-Tong, Auzzi-Bolognesi-Evslin-Konishi-Yung, Shifman-Yung, Gorsky-Shifman-Yung, Eto-Isozumi-Nitta-Ohashi-Sakai

$$SU(3) \xrightarrow{v_1} \frac{SU(2)\times U(1)}{\mathbb{Z}_2} \xrightarrow{v_2} 0, \qquad v_1 \gg v_2,$$

- HE theory has Monopoles; LE (monopoles heavy) has Vortices

  N=2, $4 \leq n_f \leq 5$,

  Bare quark mass m, adj mass μ, such that $\xi = \sqrt{\mu\, m} \ll m$

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}, \; < q^{kA} >=< \bar{\tilde{q}}^{kA} >= \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Exact $SU(2)_{C+F}$ invariance crucial

Two model systems for non-abelian vortices

- $\mathcal{N}=2$, $SU(N+1)\to U(N)$ (BPS only for $v_1 \gg v_2$):

$$\mathcal{W}=\mu\,\Phi^2+Q_i\,(\sqrt{2}\Phi+m_i)\,\tilde{Q}_i, \qquad v_1=m, \quad v_2=\sqrt{\mu\,m}=\xi;$$

Set $\Phi=\langle\Phi\rangle$; $q=\tilde{q}^\dagger$; and $q\to\frac{1}{2}q$:

$$\int d^4x\left[-\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2-\frac{1}{4g_1^2}\left(F^8_{\mu\nu}\right)^2+\left|\nabla_\mu q^A\right|^2-\frac{g_2^2}{8}\left(\bar{q}_A\tau^a q^A\right)^2-\frac{g_1^2}{24}\left(\bar{q}_A q^A-2\xi\right)^2\right]$$

- $U(N)$ ($\mathcal{N}=2$) with Fayet-Iliopoulos in the $U(1)$ part ($\tilde{Q}=0$). BPS if $c=g^2/4$. (q is $N\times N_F$ color-flavor matrix)

$$\mathcal{L}=\mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}-\mathcal{D}_\mu\,q\,\mathcal{D}^\mu q^\dagger-\lambda\left(c\,\mathbf{1}_N-q\,q^\dagger\right)^2\right]$$

- Equivalent for $m_i=m$ and $v_1\gg v_2$ (related by an $SU_R(2)$); distinct for $m_i\neq m_j$. $T_i\neq T_j$ in the first; $T_i=T_j$ in the second.

Nonabelian Bogomolnyi Equations

$$\int d^4x \left[-\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{4g_1^2}\left(F^8_{\mu\nu}\right)^2 + \left|\nabla_\mu q^A\right|^2 - \frac{g_2^2}{8}\left(\bar{q}_A\tau^a q^A\right)^2 - \frac{g_1^2}{24}\left(\bar{q}_A q^A - 2\xi\right)^2 \right],$$

$$\begin{aligned} T \;\; &= \;\; \int d^2x \left[\sum_{a=1}^{3} \left(\frac{1}{2g_2} F^{(a)}_{ij} \pm \frac{g_2}{4} \left(\bar{q}_A \tau^a q^A \right) \epsilon_{ij} \right] ^2 \\ &+ \;\; \left[\frac{1}{2g_1} F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}} \left(|q^A|^2 - 2\xi \right) \epsilon_{ij} \right]^2 + \frac{1}{2} \left| \nabla_i \, q^A \pm i\epsilon_{ij} \nabla_j \, q^A \right|^2 \pm \frac{\xi}{2\sqrt{3}} \tilde{F}^{(8)} \right) \end{aligned}$$

- Exact Symmetry

  $SU(2)_{C+F}$, broken only (to a $U(1)$) by the vortex $\Rightarrow$ **zeromodes** (moduli) of

$$SU(2)/U(1) = S^2 = \mathbf{CP}^1,$$

- $SU(N) \times U(1) \xrightarrow{\langle q \rangle \neq 0} \emptyset$: Vortex zeromodes (vortex moduli):

$$\mathcal{M}_{1,N} \simeq \frac{SU(N)}{SU(N-1) \times U(1)} = \mathbf{CP}^{N-1},$$

- Vortex moduli $\mathbf{CP}^{N-1}$ precisely corresponds to the moduli of an N-state system in quantum mechanics (heavy monopoles in $\underline{N}$ of (dual) $SU(N)$)

- Appropriate N_f $(= 4, 5$ for $SU(2))$ required; otherwise, H breaks dynamically. $\Rightarrow$ abelian ANO vortices.

- Semilocal NA string

Vortex of Generic Orientation (Zero Modes)

$$q^{kA} = U \begin{pmatrix} e^{i\varphi}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = e^{\frac{i}{2}\varphi(1+n^a\tau^a)}\, U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1},$$

$$\mathbf{A}_i(x) = U[-\frac{\tau^3}{2}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)]]U^{-1} = -\frac{1}{2}\,n^a\tau^a\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_3(r)],$$

$$A_i^8(x) = -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2}\,[1-f_8(r)]$$

where

$$U \in SU(2)_{C+F}, \qquad U\,\tau^3\,U^\dagger = n^a\,\tau^a,$$

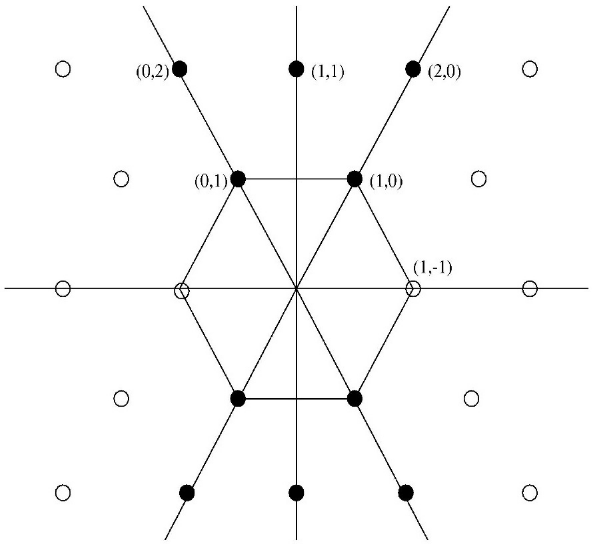
$$n^a = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha), \qquad U = e^{-i\beta\,\tau_3/2}\,e^{-i\alpha\,\tau_2/2}.$$

The tension

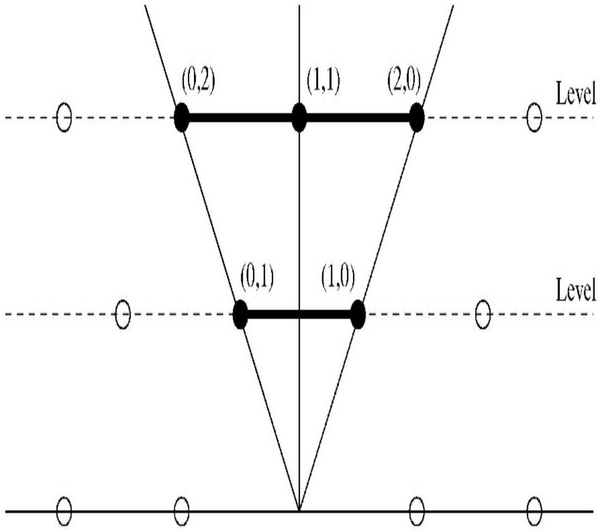
$$T = 2\pi\xi$$

independent of U.

Reduction of Vortex Spectrum



Lattice of (n, k) vortices in the theory $SU(3) \to U(1)^2$.



Reduced lattice of $\mathbb{Z}$ vortices $SU(3) \to SU(2) \times U(1)$

Method of moduli matrix

Eto-Isozumi-Nitta-Ohashi-Sakai

For the vortices of $U(N)$, $N_f = N$ (also for monopoles and walls):

- BPS equation ($H = q$ is an $N \times N$ color-flavor matrix)

$$(\mathcal{D}_1 + i\mathcal{D}_2)\; H = 0, \quad F_{12} + \frac{g^2}{2}\left(c\,\mathbf{1}_N - H\,H^\dagger\right) = 0.$$

- The solution ($z = x_1 + i\,x_2$):

$$H = S^{-1}(z,\bar{z})\,H_0(z), \quad W_1 + iW_2 = -2\,i\,S^{-1}(z,\bar{z})\,\bar{\partial}_z S(z,\bar{z}),$$

where $H_0(z)$, $N \times N$, holomorphic, $\det H_0(z) \sim z^k$.

$H_0(z)$: all the information on the moduli parameters.

- $U(2)$ with $N_f = 2$ ($a_0 = 1/b_0$; $CP^1 \sim SU(2)/U(1)$), $H_0(z) \sim V(z)\,H_0(z)$

$$H_0^{(1,0)} \simeq \begin{pmatrix} z - z_0 & 0 \\ -b_0 & 1 \end{pmatrix}; \qquad H_0^{(0,1)} \simeq \begin{pmatrix} 1 & -a_0 \\ 0 & z - z_0 \end{pmatrix}, \tag{4}$$

- Master equation

$$\partial_z \left(\Omega^{-1}\, \partial_{\bar z}\Omega\right) = \frac{g^2}{4} \left(c\, \mathbf{1} - \Omega^{-1}\, H_0\, H_0^\dagger \right), \qquad \Omega \equiv S\, S^\dagger$$

- Vortex moduli $\mathcal{M}_{N,k}$ (Kähler quotient) from D brane construction: $(\mathbf{Z}_{k\times k}, \mathbf{\Psi}_{N\times k})$

$$\mathcal{M}_{N,k} \simeq \{[\mathbf{Z}^\dagger, \mathbf{Z}] + \mathbf{\Psi}^\dagger \mathbf{\Psi} \propto \mathbf{1}_k\}/U(k),$$

$$\mathbf{Z} \to U\, \mathbf{Z}\, U^{-1} \quad \mathbf{\Psi} \to \mathbf{\Psi}\, U^{-1}; \qquad dim[\mathcal{M}_{N,k}] = 2\, k\, N.$$

Tong

- $H_0(z) \Leftrightarrow (\mathbf{Z}, \mathbf{\Psi})$ Isozumi-Nitta-Ohashi-Sakai

$$H_0(z)\, \mathbf{\Phi}(z) = \mathbf{J}(z)\, P(z), \qquad P(z) = \det H_0(z);$$

$$z\, \mathbf{\Phi}(z) = \mathbf{\Phi}(z)\, \mathbf{Z} + \mathbf{\Psi}\, P(z).$$

Axially symmetric (co-axial) $k = 2$ vortices

- $\mathcal{M}_{2,2} \sim CP^2$ (Hashimoto-Tong '05) or $\frac{CP^2}{Z_2}$ (Auzzi-Shifman-Yung '05) ?

- H_0 relates the explicit vortex construction and Tong's result

$$H_0(z) = \begin{pmatrix} 1 & -az-b \\ 0 & z^2 \end{pmatrix}, \quad \begin{pmatrix} z-\phi & -\varphi \\ -\tilde{\varphi} & z+\phi \end{pmatrix}, \quad \begin{pmatrix} z^2 & 0 \\ -a'z-b' & 1 \end{pmatrix},$$

$$\phi^2 + \varphi\tilde{\varphi} = 0, \qquad \phi = X_3 = \frac{b}{a}, \quad \varphi = X_1 - i\,X_2 = -\frac{b^2}{a}, \quad \varphi = X_1 + i\,X_2 = \frac{1}{a};$$

then

$$(-a, b, 1) \sim (a', 1, b') \sim (1, x, y), \qquad \leftarrow \qquad \phi = x\,y, \quad \varphi = x^2, \quad \tilde{\varphi} = -y^2,$$

$$(z_1, z_2, z_3) \sim (\lambda^2 z_1, \lambda z_2, \lambda z_3)$$

- $\mathcal{M}_{2,2} = WCP^2 \sim \frac{CP^2}{Z_2} \quad (z_1 = \tilde{z}_1^2)$ (Auzzi-Shifman-Yung);

  $\mathcal{M}_{N,2} = WGr_{N+1,2\,(1,\ldots,1,0)}$ Eto-Konishi-Marmorini-Nitta-Ohashi-Vinci-Yokoi

Transformation property of $k = 2$ vortices

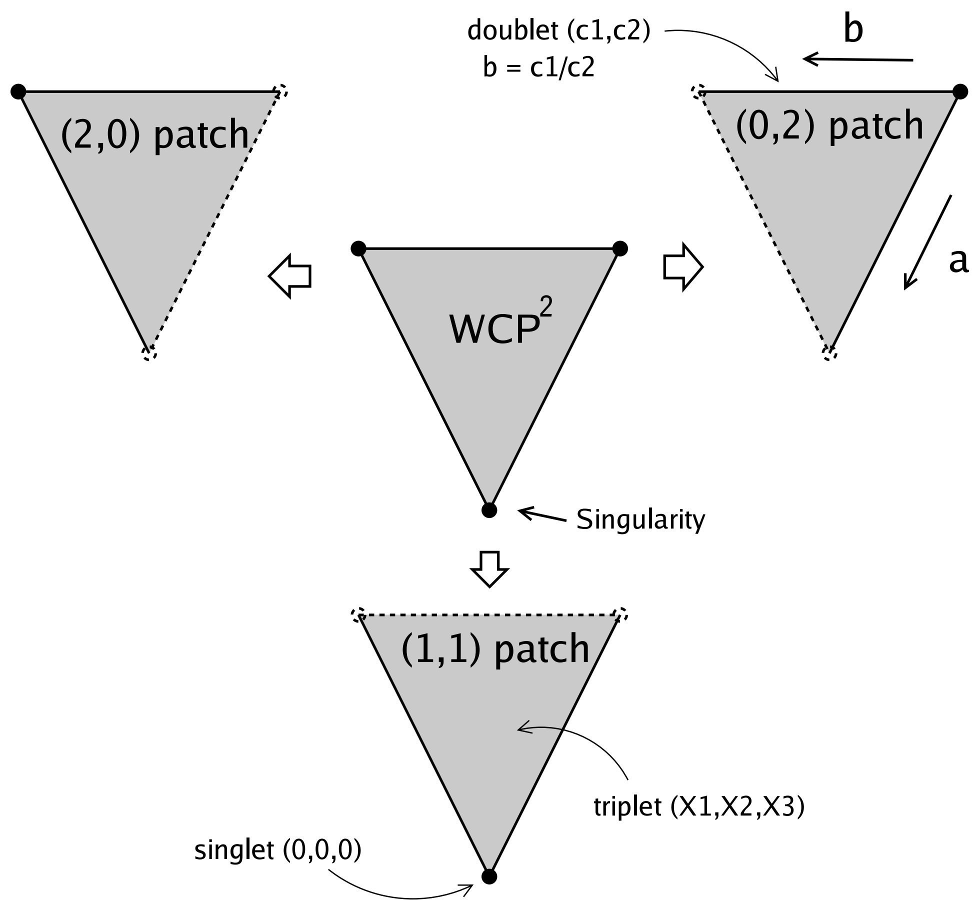


Figure 3:

Nonabelian monopoles from monopole-vortex complex: 2

- Consider systems

$$G \stackrel{\langle\phi_1\rangle\neq 0}{\longrightarrow} H \stackrel{\langle\phi_2\rangle\neq 0}{\longrightarrow} \emptyset, \qquad \langle\phi_1\rangle \gg \langle\phi_2\rangle.$$

- Assume exact unbroken H_{C+F};

- ($k=1$) Vortices carry nonabelian flux (e.g. CP^{N-1} for $G = SU(N+1)$, $H = SU(N) \times U(1)/Z_N$;

- Monopoles in $\pi_2(G/H);$ vortices in $\pi_1(H)$;

- $\pi_2(G) = \emptyset$: monopoles not topologically stable: confined by vortices with $F_v(H) = F_m(H)$ (Fig.)

- The energy of the whole system invariant under H_{C+F} rotations (Fig.)

$$\mathcal{M}^{mono}_{N,1} \Leftrightarrow \mathcal{M}^{vor}_{N,1} \tag{5}$$

NA monopoles form $\tilde{H}$ multiplet

- H_0 for $U(2)$ with $N_f = 2$ (Eq.(4))

$$H_0 \to U\, H_0\, U^{-1} \sim H_0', \qquad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{6}$$

$$a_0 \to \frac{\alpha a_0 + \beta}{\alpha^* + \beta^* a_0}.$$

- OK with

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \frac{a_1}{a_2} = a_0.$$

- Precisely the $SU(2)$ transformation of a two-state quantum mechanical system

$$a_1\,|1\rangle + a_2\,|2\rangle.$$

- Together with Eq.(5), in the system with a hierarchical breaking, this proves that the massive non-abelian monopoles transform as a $\tilde{H}$ multiplet.

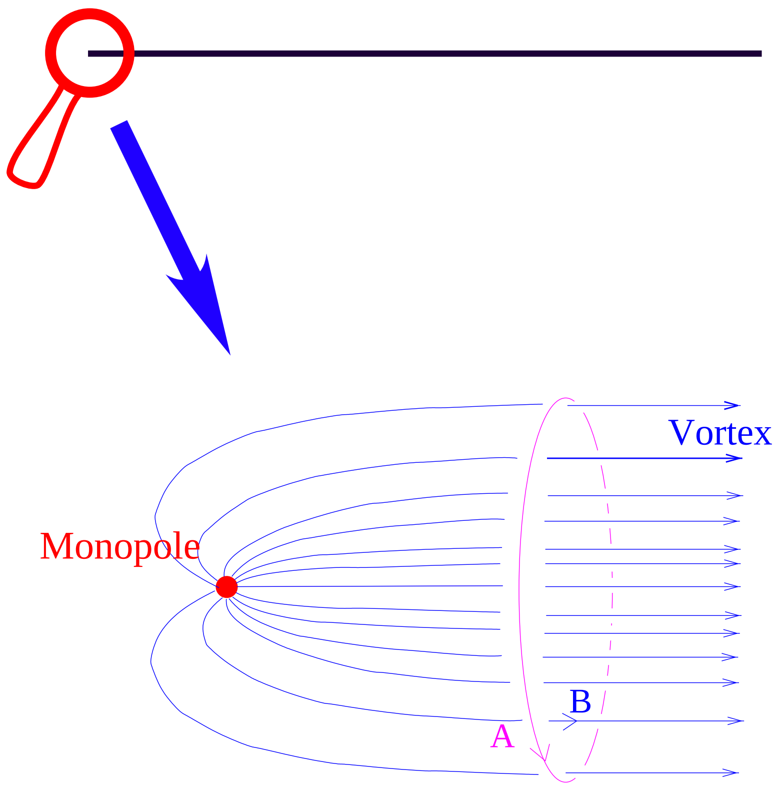


Figure 4:

Dual group transformation

- The energy of the whole system invariant under $SU(N)_{C+F}$ (exact symmetry broken only by the composite solitons)

- Monopoles appear (in the approximation $v_1 \gg v_2$) to transform under $SU(N)_C$ only, but actually needs flavor transformation (through regularization);

- The whole system is not BPS (no topological stability); can be stabilized dynamically;

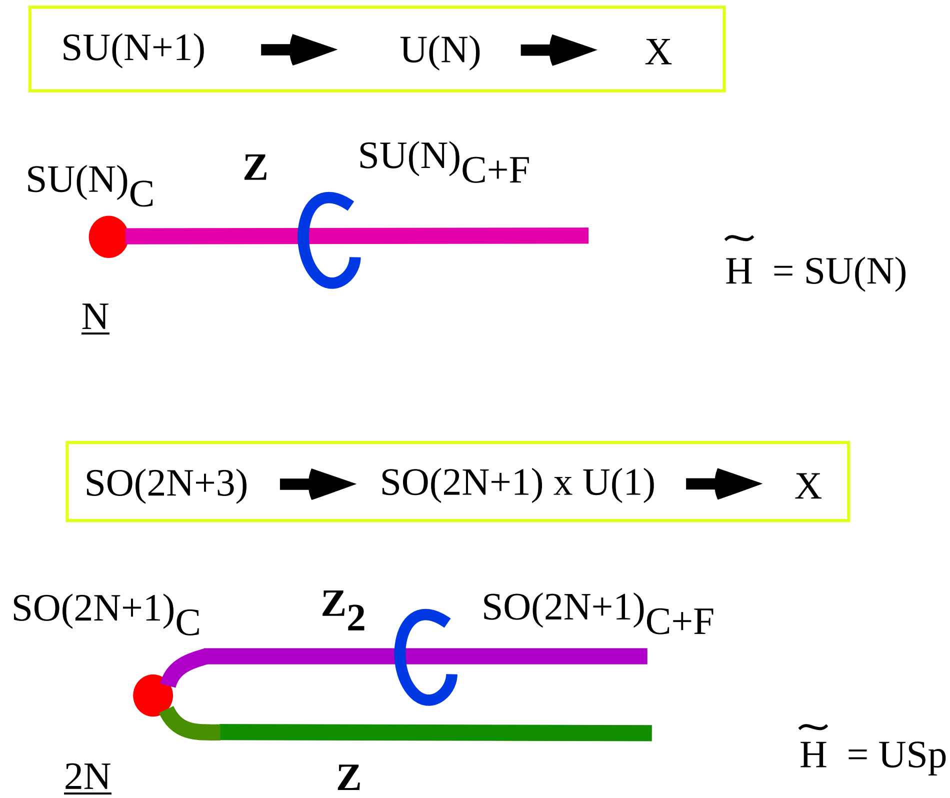
- $SO(5) \to U(2)$; GNOW monopoles belong to $\underline{3}$ of $S\tilde{U}(2) \sim SO(3)$.

- $\pi_1(SO(5)) = Z_2$. $\therefore$ Regular monopoles of $SO(5) \to U(2)$ cannot exist in the full theory $SO(5) \to U(2) \to \emptyset$. Confined by $k = 2$ vortces of low energy system which transform as a triplet of $SU(2)$!

$$SO(5) \to SU(2) \times U(1) \to \emptyset$$

$$\langle \phi \rangle = \begin{pmatrix} 0 & i\,v & 0 & 0 & 0 \\ -i\,v & 0 & 0 & 0 & 0 \\ 0 & 0 & i\,v & 0 & 0 \\ 0 & -i\,v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Monopole 1 (in 1234)) in $SO(4) \sim SU(2) \times SU(2) \to U(1) \times SU(2)$;
- Monopole 2 (in 125) in $SO(3) \to U(1)$;
- Monopole 3 (in 345) in $SO(3) \to U(1)$;
- Degenerate but not related by the unbroken $SU(2)$ (nontrivial GNO)
- $S\tilde{U}(2) = SO(3) \sim SU(2)_{C+F}$



SU(N+1) → U(N) → X
SU(N)$_C$
Z
SU(N)$_{C+F}$
N
$\tilde{H}$ = SU(N)
SO(2N+3) → SO(2N+1) x U(1) → X
SO(2N+1)$_C$
Z_2
SO(2N+1)$_{C+F}$
2N
Z
$\tilde{H}$ = USp

Vortex dynamics

(Hanany-Tong, Auzzi-Bolognesi-Evslin-Konishi-Yung, Shifman-Yung)

- Vortex orientation n^a as a dynamical variable $n^a(t, z)$

- Vortex dynamics $\sim$ a 2 D sigma model, e.g., in $\mathcal{N} = 2$ $SU(2) \times U(1)$ model above:

$$S = \beta \int d^2x \, \frac{1}{2} \, (\partial n^a)^2 + fermions \, :$$

  $\mathcal{N} = 2$ susy $O(3)$ sigma model

- $U(2) \to U(1)^2$ when $m_1 \neq m_2$: light 't Hooft-Polyakov (abelian) monopoles, and W bosons; Stable (BPS) V-M-V system (monopoles "confined" by two vortices); (Hanany-Tong, Auzzi,Bolognesi,Evslin, TiTech)

- Vortex (2D sigma model) dynamics $\sim$ 4D gauge dynamics (Dorey);

- $\mathcal{N} = 1^*$ $SU(2)$ model; nonsupersymmetric $O(3)$ sigma model (Markov-Marshakov-Yung, Gorsky-Shifman-Yung, ...)

Conclusion: Part II

Confining vacua due to condensation of

- Weakly-coupled abelian monopoles;
- Weakly-coupled nonabelian monopoles; or
- Strongly-coupled nonabelian monopoles and dyons.

The last class (nonabelian Argyres-Douglas vacua) characterized by the facts:

- Close to nontrivial SCFT (relatively nonlocal dyons);
- Confinement caused by strongly-coupled (nonabelian) monopoles;
- Symmetry breaking induced by the same composites monopole condensate;
- Best candidate for QCD ? (Abelian monopoles $M_i^j \sim \psi_L^\dagger \psi_R^{c\,\dagger} \, |M\rangle$ would not do)

Conclusion: Part III

1. Quantum behavior of nonabelian monopoles through the study of vortices;

2. Moduli matrices: powerful tool to study the properties of vortices (monopoles, domain walls, etc. etc.);

3. Dual group crucially involves the massless flavors (H unbroken dynamically; $\tilde{H}$ related to H_{C+F}

4. Nonabelian monopoles transform as a multiplet of the dual (GNOW) group, but $\tilde{H} \sim H_{C+F}$, if acting on the electric variables.