

Squeezed-limit bispectrum, Non-Bunch-Davies vacuum, Scale-dependent bias, and Multi-field consistency relation

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“Pre-Planckian Inflation,” University of Minnesota, Minneapolis
October 7, 2011

This talk is based on...

- Squeezed-limit bispectrum
 - *Ganc & Komatsu, JCAP, 12, 009 (2010)*
- Non-Bunch-Davies vacuum
 - *Ganc, PRD 84, 063514 (2011)*
- Scale-dependent bias
 - *Ganc & Komatsu, in preparation*
- Multi-field consistency relation
 - *Sugiyama, Komatsu & Futamase, PRL, 106, 251301 (2011)*

Motivation

- Can we falsify inflation?

Falsifying “inflation”

- We still need inflation to explain the flatness problem!
 - (Homogeneity problem can be explained by a bubble nucleation.)
- However, the observed fluctuations may come from different sources.
- So, what I ask is, “can we rule out inflation as a mechanism for generating the observed fluctuations?”

First Question:

- Can we falsify **single-field** inflation?

An Easy One: Adiabaticity

- Single-field inflation = One degree of freedom.
- Matter and radiation fluctuations originate from a single source.

$$S_{c,\gamma} \equiv \frac{\delta\rho_c}{\rho_c} - \frac{3\delta\rho_\gamma}{4\rho_\gamma} = 0$$

Dark Matter Photon

* A factor of 3/4 comes from the fact that, in thermal equilibrium, $\rho_c \sim (1+z)^3$ and $\rho_\gamma \sim (1+z)^4$. 6

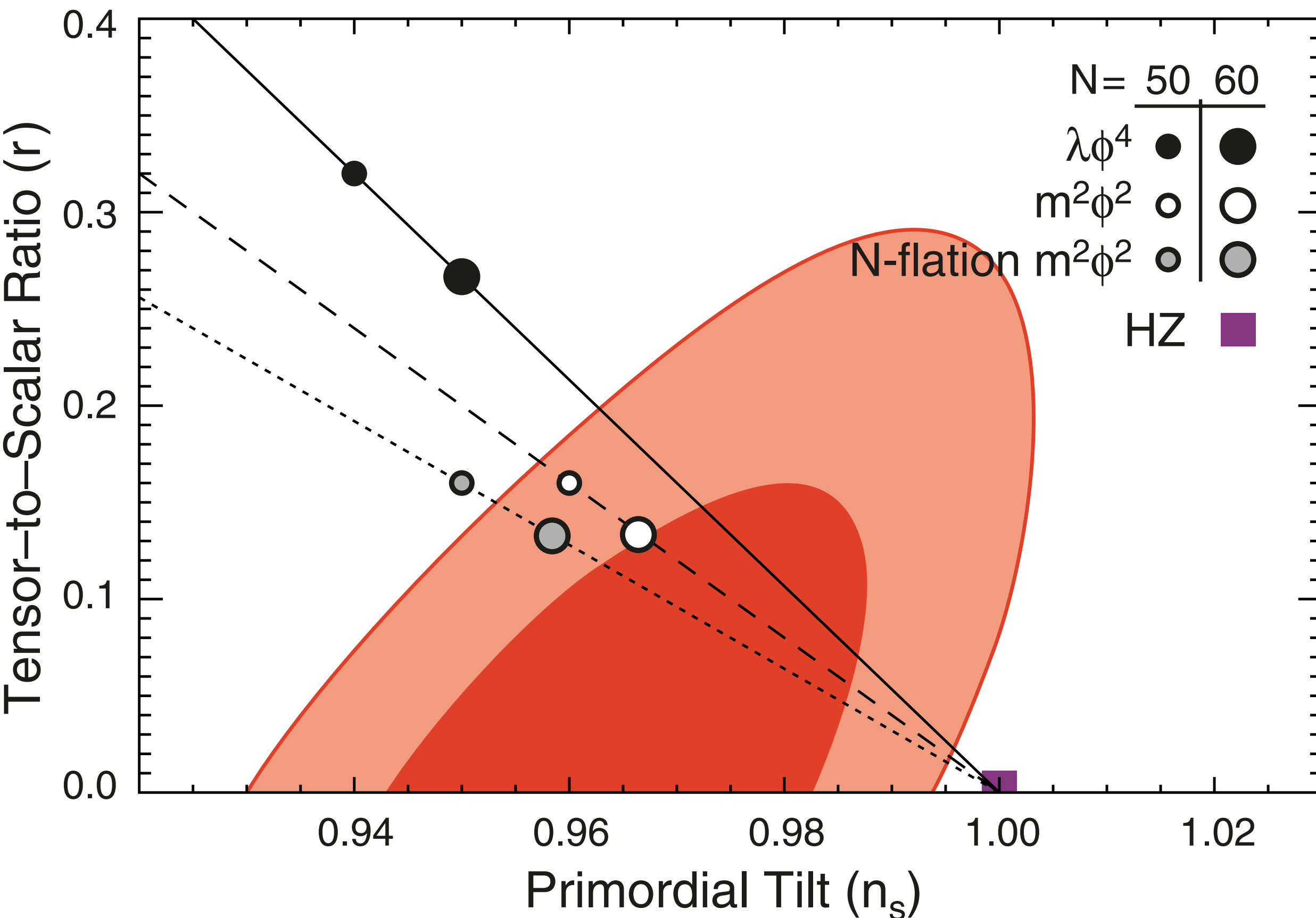
Non-adiabatic Fluctuations

- Detection of non-adiabatic fluctuations immediately rule out single-field inflation models.

The data are consistent with adiabatic fluctuations:

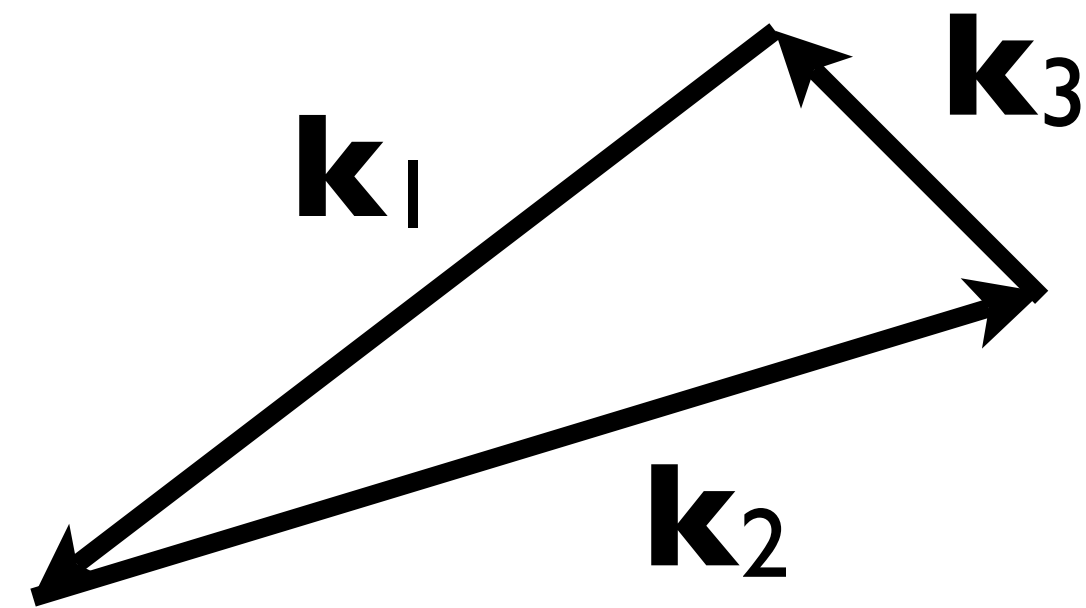
$$\frac{|\delta\rho_c/\rho_c - 3\delta\rho_\gamma/(4\rho_\gamma)|}{\frac{1}{2}[\delta\rho_c/\rho_c + 3\delta\rho_\gamma/(4\rho_\gamma)]} < 0.09 \quad (95\% \text{ CL})$$

Single-field inflation looks good (in 2-point function)



- **$n_s = 0.968 \pm 0.012$ (68%CL; WMAP7+BAO+ H_0)**
- **$r < 0.24$ (95%CL; WMAP7+BAO+ H_0)**

So, let's use 3-point function



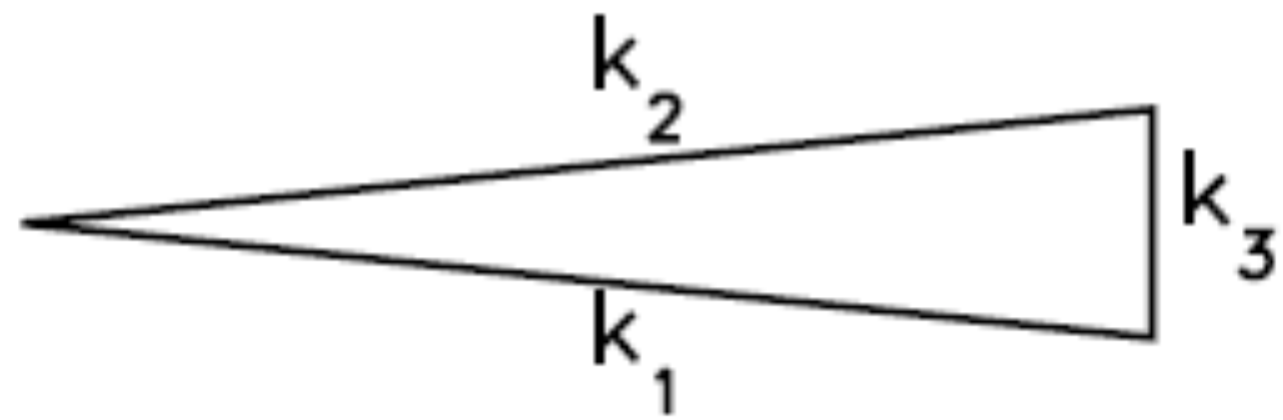
- Three-point function!

- $B_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$

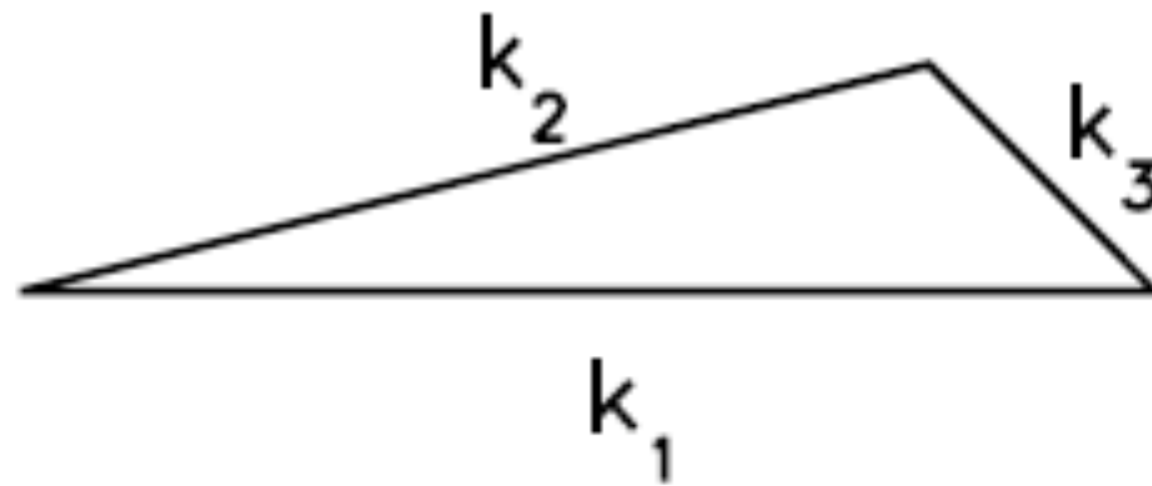
$$= \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (\text{amplitude}) \times (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) b(k_1, k_2, k_3)$$

model-dependent function

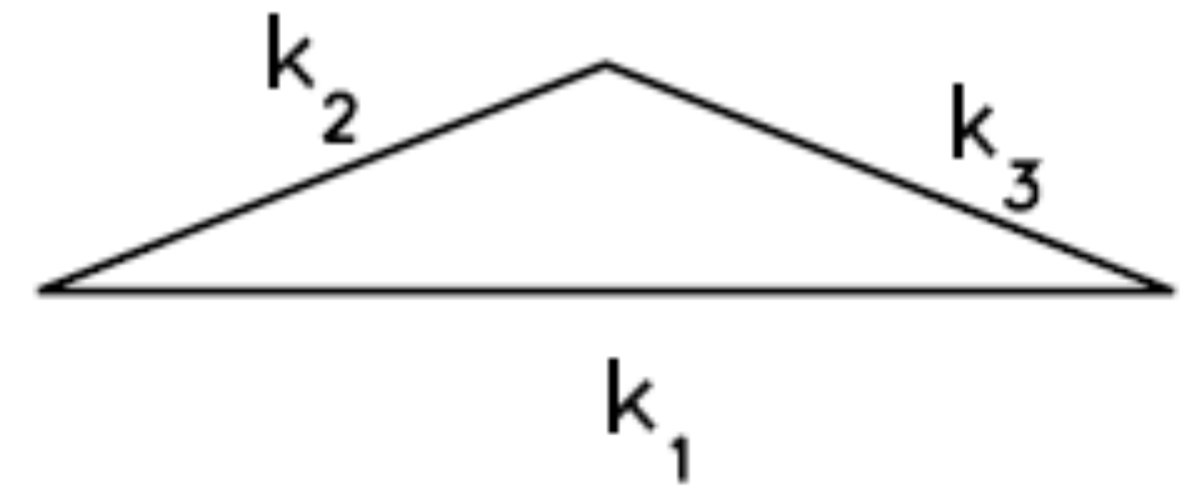
(a) squeezed triangle
($k_1 \simeq k_2 \gg k_3$)



(b) elongated triangle
($k_1 = k_2 + k_3$)

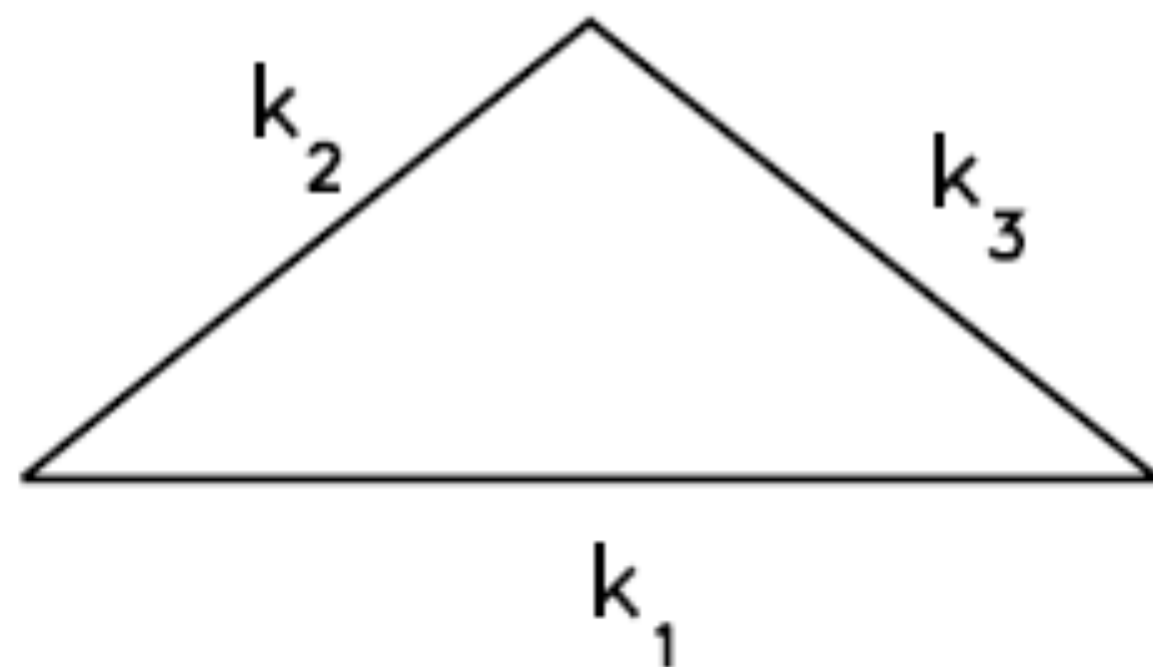


(c) folded triangle
($k_1 = 2k_2 = 2k_3$)

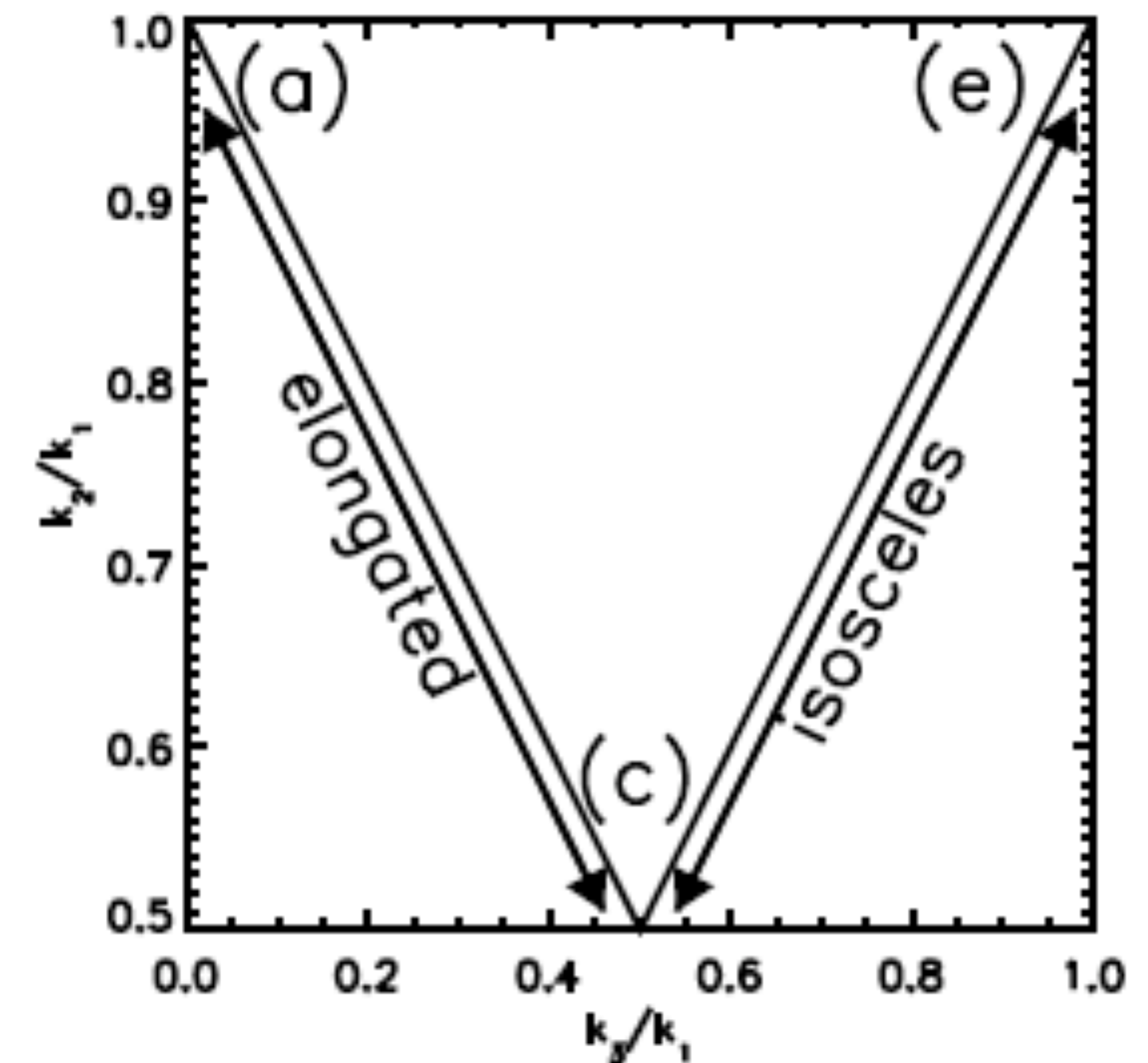
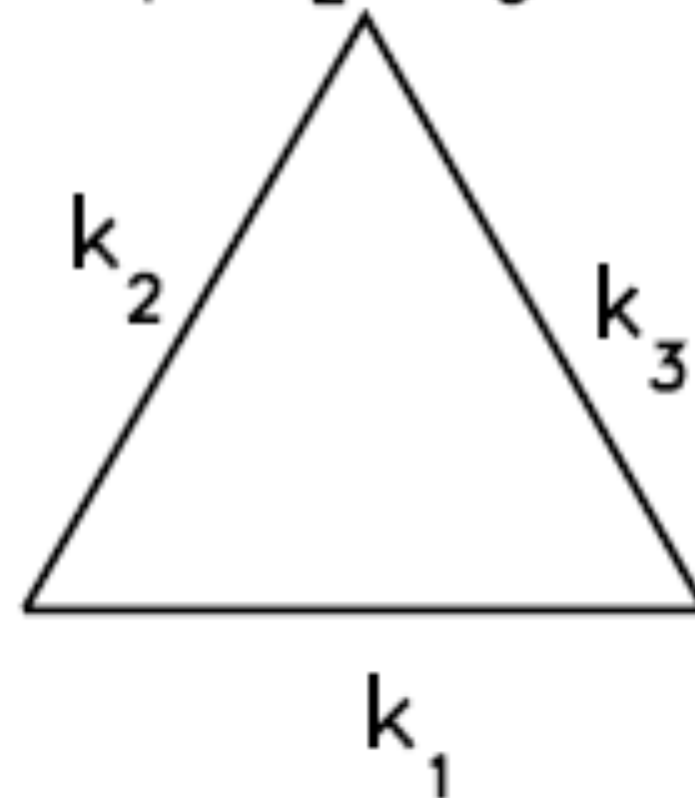


**MOST IMPORTANT, for falsifying
single-field inflation**

(d) isosceles triangle
($k_1 > k_2 = k_3$)



(e) equilateral triangle
($k_1 = k_2 = k_3$)



Single-field Theorem (Consistency Relation)

- For **ANY** single-field models*, the bispectrum in the squeezed limit is given by
- $B_{\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \approx (1-n_s) \times (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \times P_{\zeta}(k_1) P_{\zeta}(k_3)$
- Therefore, all single-field models predict $f_{\text{NL}} \approx (5/12)(1-n_s)$.
- With the current limit $n_s=0.96$, f_{NL} is predicted to be 0.017.

* for which the single field is solely responsible for driving inflation and generating observed fluctuations.

Understanding the Theorem

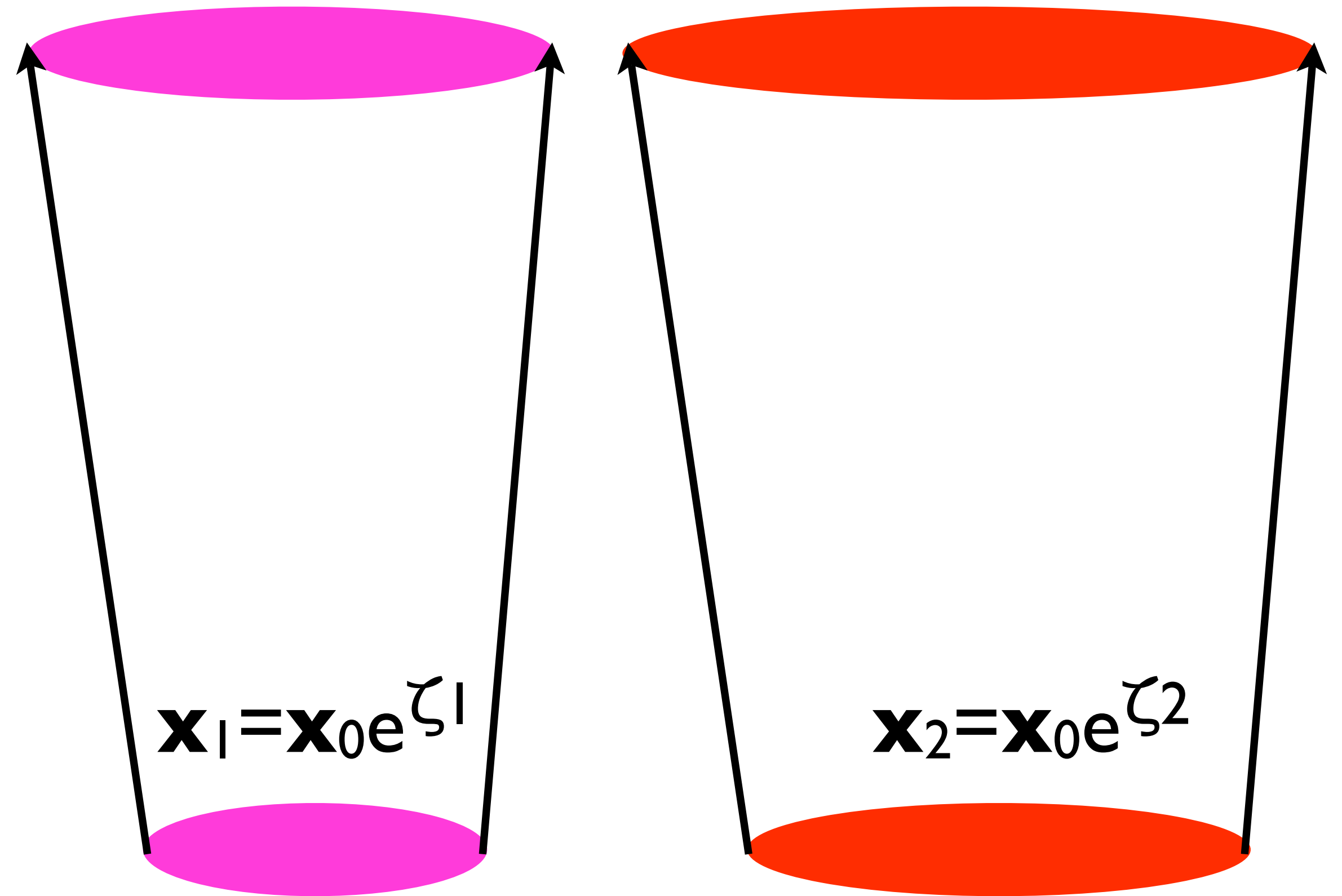
- First, the squeezed triangle correlates one very long-wavelength mode, $k_L (=k_3)$, to two shorter wavelength modes, $k_S (=k_1 \approx k_2)$:
 - $\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \approx \langle (\zeta_{k_S})^2 \zeta_{k_L} \rangle$
- Then, the question is: “why should $(\zeta_{k_S})^2$ ever care about ζ_{k_L} ?”
 - The theorem says, “it doesn’t care, if ζ_k is exactly scale invariant.”

$\zeta_{\mathbf{k}L}$ rescales coordinates

- The long-wavelength curvature perturbation rescales the spatial coordinates (or changes the expansion factor) within a given Hubble patch:

- $ds^2 = -dt^2 + [a(t)]^2 e^{2\zeta} (d\mathbf{x})^2$

Separated by more than H^{-1}

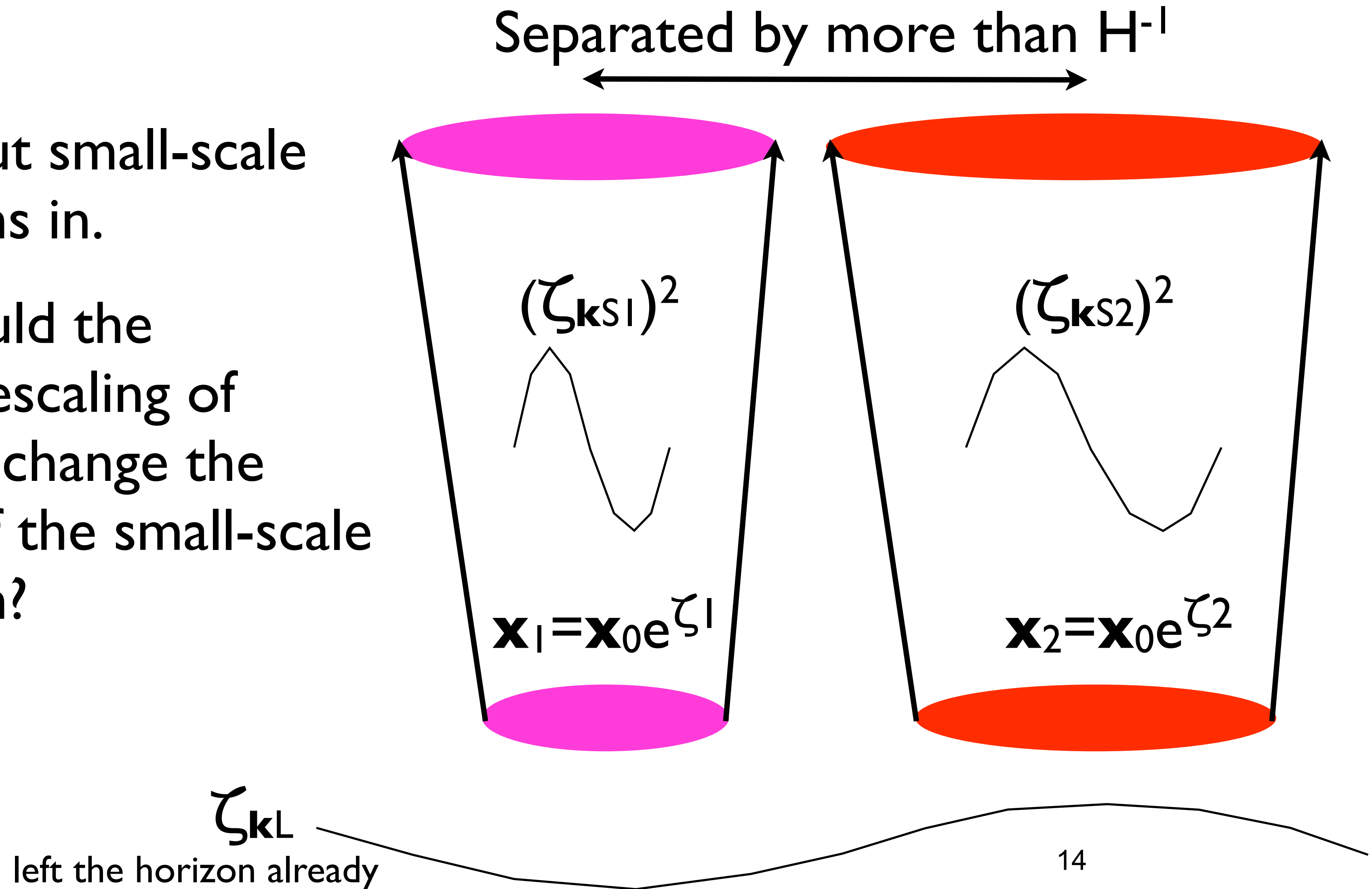


$\zeta_{\mathbf{k}L}$

left the horizon already

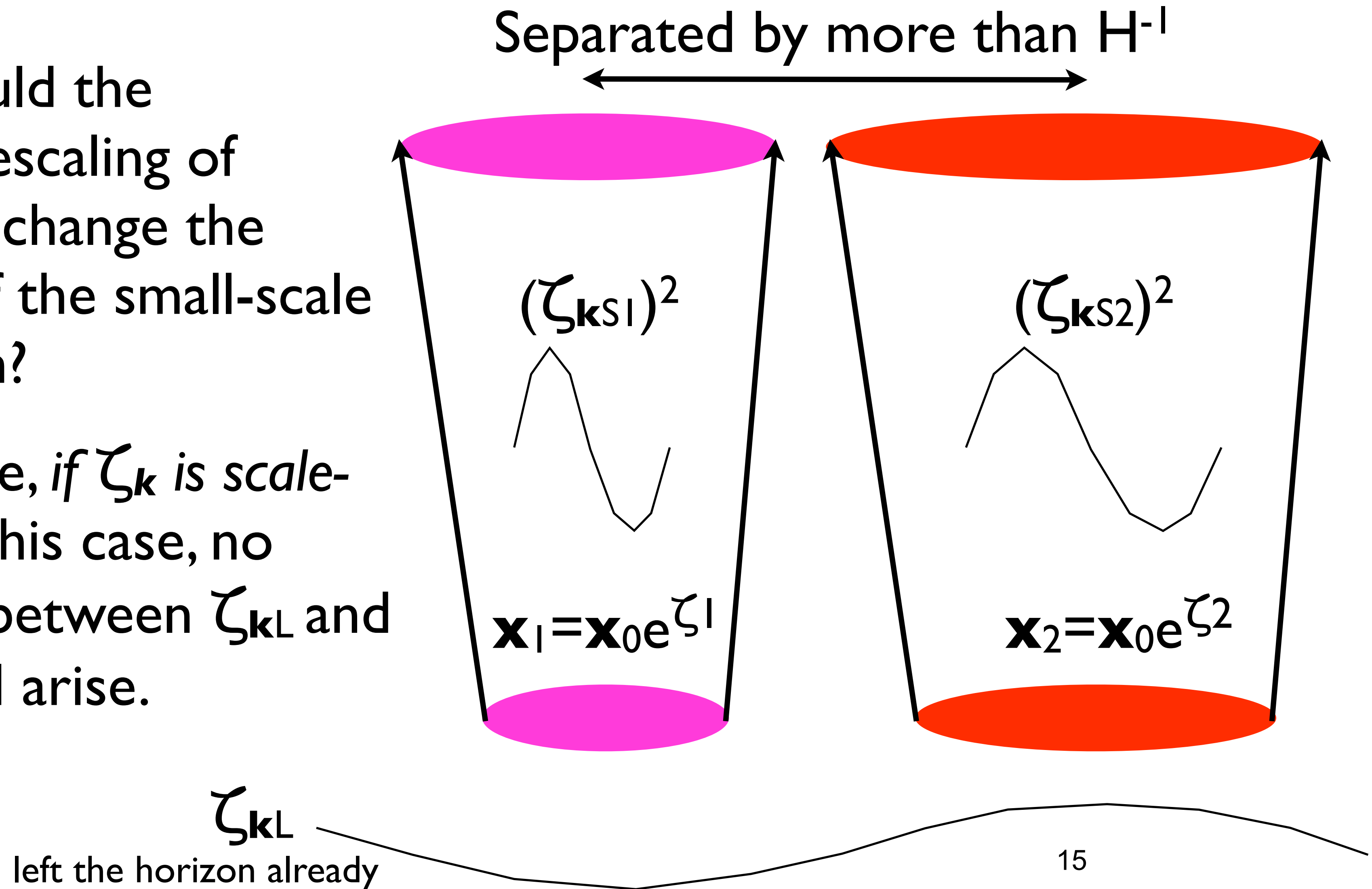
$\zeta_{\mathbf{k}L}$ rescales coordinates

- Now, let's put small-scale perturbations in.
- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?



ζ_{kL} rescales coordinates

- Q. How would the conformal rescaling of coordinates change the amplitude of the small-scale perturbation?
- A. No change, if ζ_k is scale-invariant. In this case, no correlation between ζ_{kL} and $(\zeta_{kS})^2$ would arise.



Real-space Proof

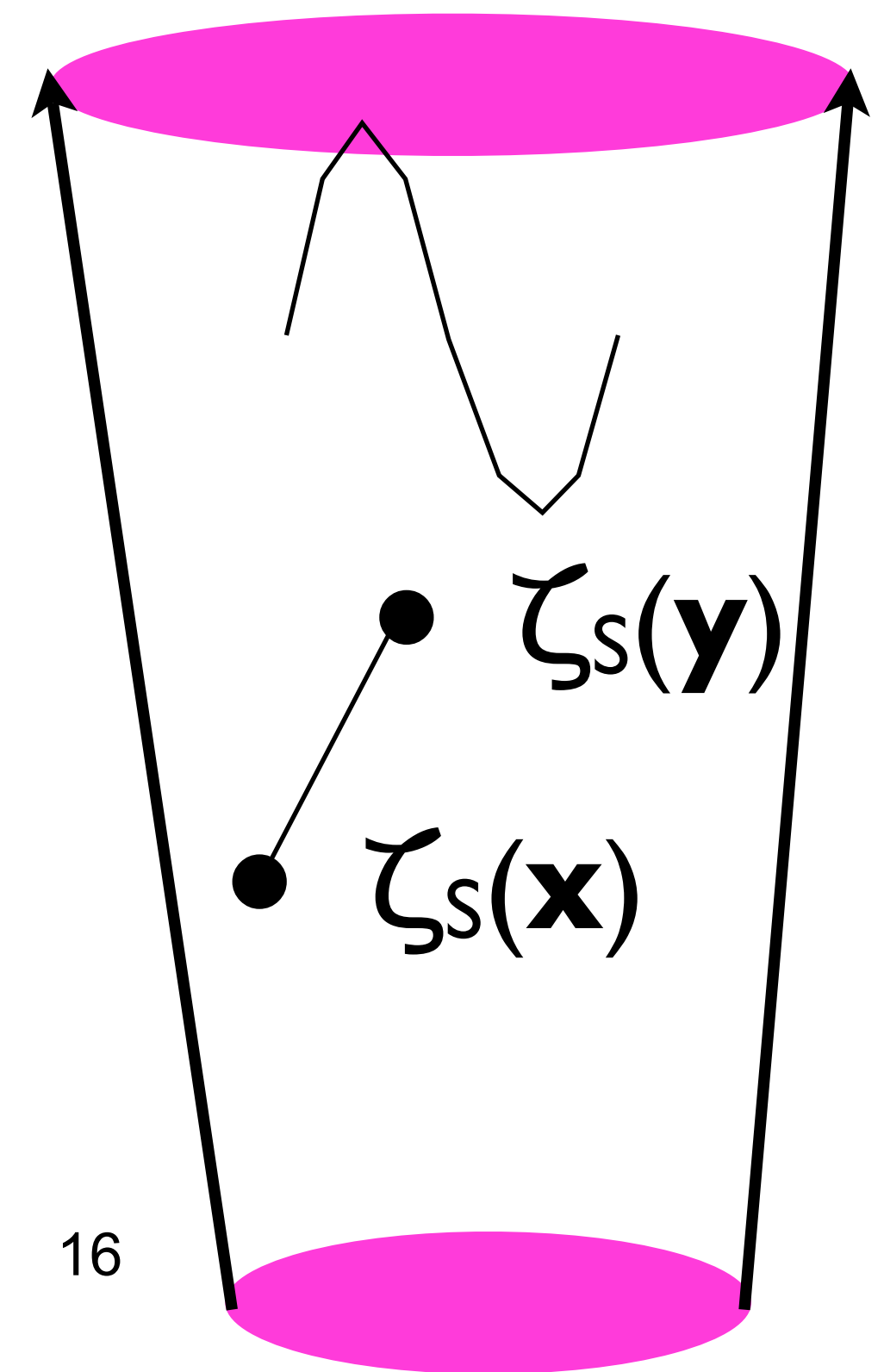
- The 2-point correlation function of short-wavelength modes, $\xi = \langle \zeta_s(\mathbf{x}) \zeta_s(\mathbf{y}) \rangle$, within a given Hubble patch can be written in terms of its vacuum expectation value (in the absence of ζ_L), ξ_0 , as:

- $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d\zeta_L]$

- $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L [d\xi_0(|\mathbf{x}-\mathbf{y}|)/d \ln |\mathbf{x}-\mathbf{y}|]$

- $\xi_{\zeta_L} \approx \xi_0(|\mathbf{x}-\mathbf{y}|) + \zeta_L (1-n_s) \xi_0(|\mathbf{x}-\mathbf{y}|)$

$$\begin{aligned} \text{3-pt func.} &= \langle (\zeta_s)^2 \zeta_L \rangle = \langle \xi_{\zeta_L} \zeta_L \rangle \\ &= (1-n_s) \xi_0(|\mathbf{x}-\mathbf{y}|) \langle \zeta_L^2 \rangle \end{aligned}$$



This is great, but...

- The proof relies on the following Taylor expansion:
 - $\langle \zeta_S(\mathbf{x}) \zeta_S(\mathbf{y}) \rangle_{\zeta_L} = \langle \zeta_S(\mathbf{x}) \zeta_S(\mathbf{y}) \rangle_0 + \zeta_L [d \langle \zeta_S(\mathbf{x}) \zeta_S(\mathbf{y}) \rangle_0 / d \zeta_L]$
- Perhaps it is interesting to show this explicitly using the in-in formalism.
 - Such a calculation would shed light on the limitation of the above Taylor expansion.
 - Indeed it did - we found a non-trivial “counter-example” (more later)

An Idea

- How can we use the in-in formalism to compute the two-point function of short modes, given that there is a long mode, $\langle \zeta_s(\mathbf{x}) \zeta_s(\mathbf{y}) \rangle_{\zeta_L}$?
- Here it is!

$$\langle \zeta_S^2(\bar{t}) \rangle_{\zeta_L} = -i \int_{-(1-i\epsilon)\infty}^{\bar{t}} dt' \langle 0 | [\zeta_S^2(\bar{t}), H_I^{(3)}(t')] | 0 \rangle$$

Long-short Split of H_I

$$\langle \zeta_S^2(\bar{t}) \rangle_{\zeta_L} = -i \int_{-(1-i\epsilon)\infty}^{\bar{t}} dt' \langle 0 | [\zeta_S^2(\bar{t}), H_I^{(3)}(t')] | 0 \rangle$$

- Inserting $\zeta = \zeta_L + \zeta_S$ into the cubic action of a scalar field, and retain terms that have one ζ_L and two ζ_S 's.

$$S_{\text{int}}^{(3)} = \int d^4x \left[\left(\frac{1}{4} \frac{\dot{\phi}_0^4}{H^4} - \frac{1}{16} \frac{\dot{\phi}_0^6}{H^6} \right) a^3 \zeta_L \dot{\zeta}_S^2 + \frac{1}{4} \frac{\dot{\phi}_0^4}{H^4} a \zeta_L (\partial \zeta_S)^2 - \frac{\dot{\phi}_0^4}{2H^4} a^3 \dot{\zeta}_S \partial_i \zeta_S \partial_i \partial^{-2} \dot{\zeta}_L + \right. \\ \left. + \frac{1}{16} \frac{\dot{\phi}_0^6}{H^6} a^3 \partial_i \partial_j \partial^{-2} \dot{\zeta}_S \partial_i \partial_j \partial^{-2} \dot{\zeta}_S \zeta_L + 2 \frac{\dot{\phi}_0^2}{H^2} a^3 \zeta_L \frac{d}{dt} \left[\frac{1}{2} \frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{4} \frac{\dot{\phi}_0^2}{H^2} \right] \dot{\zeta}_S \zeta_S - f(\zeta) \frac{\delta L_0}{\delta \zeta_S} \right],$$

Result

$$\langle \zeta_{S, \mathbf{k}_1} \zeta_{S, \mathbf{k}_2} \rangle_{\zeta_{\mathbf{k}_3}} = \zeta_{L, \mathbf{k}_1 + \mathbf{k}_2} \left[K + \left(\frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{2} \frac{\dot{\phi}_0^2}{H^2} \right) P(k_1) \right]$$

• where

$$K \equiv i u_{k_1}^2(\bar{\eta}) \int_{-\infty(1-i\epsilon)}^{\bar{\eta}} d\eta \left[\frac{1}{2} \frac{\dot{\phi}_0^4}{H^4} a^2 u_{k_1}^{\prime*2}(\eta) + \frac{1}{2} \frac{\dot{\phi}_0^4}{H^4} a^2 k_1^2 u_{k_1}^{*2}(\eta) + \right. \\ \left. + 2 \frac{\dot{\phi}_0^2}{H^2} a^3 \frac{d}{dt} \left(\frac{\ddot{\phi}_0}{\dot{\phi}_0 H} + \frac{1}{2} \frac{\dot{\phi}_0^2}{H^2} \right) u_{k_1}^{\prime*}(\eta) u_{k_1}^*(\eta) \right] + \text{c.c.}$$

Result

- Although this expression looks nothing like $(1-n_s)P(k_L)\zeta_{KL}$, we have verified that it leads to the known consistency relation for (i) slow-roll inflation, and (ii) power-law inflation.
- But, there was a curious case – Alexei Starobinsky’s exact $n_s=1$ model.
 - If the theorem holds, we should get a vanishing bispectrum in the squeezed limit.

Starobinsky's Model

- The famous Mukhanov-Sasaki equation for the mode function is

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{z} \frac{d^2 z}{d\eta^2} \right) u_k = 0$$

where

$$z = \frac{a\dot{\phi}}{H}$$

- The scale-invariance results when $\frac{1}{z} \frac{d^2 z}{d\eta^2} = \frac{2}{\eta^2}$

So, let's write **$z=B/\eta$**

Result

$$\langle \zeta_{S,\mathbf{k}_1} \zeta_{S,\mathbf{k}_2} \rangle_{\zeta_{\mathbf{k}_3}} = \zeta_{L,\mathbf{k}_1+\mathbf{k}_2} 4P(k_1) (k_1 \eta_{\text{start}})^2 e^{-\frac{1}{2} \phi_{\text{end}}^2}$$

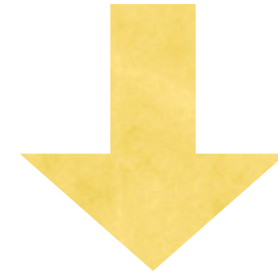
- **It does not vanish!**
- But, it approaches zero when Φ_{end} is large, meaning the duration of inflation is very long.
 - In other words, this is a condition that **the longest wavelength that we observe, \mathbf{k}_3 , is far outside the horizon.**
 - In this limit, the bispectrum approaches zero.

Vacuum State

- What we learned so far:
 - The squeezed-limit bispectrum is proportional to $(1-n_s)P(k_1)P(k_3)$, provided that ζ_{k_3} is far outside the horizon when k_1 crosses the horizon.
 - What if the state that ζ_{k_3} sees is not a Bunch-Davies vacuum, but something else?
 - The exact squeezed limit ($k_3 \rightarrow 0$) should still obey the consistency relation, but perhaps something happens when **k_3/k_1 is small but finite.**

Back to in-in

$$\langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle$$

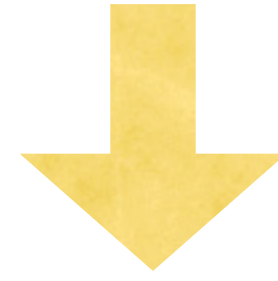


$$B_\zeta(k_1, k_2, k_3) = 2i \frac{\dot{\phi}^4}{H^6} \sum_i \left(\frac{1}{k_i^2} \right) \tilde{u}_{k_1}(\bar{\eta}) \tilde{u}_{k_2}(\bar{\eta}) \tilde{u}_{k_3}(\bar{\eta}) \int_{\eta_0}^{\bar{\eta}} d\eta \frac{1}{\eta^3} u'_{k_1}{}^* u'_{k_2}{}^* u'_{k_3}{}^* + \text{c.c.}$$

- The Bunch-Davies vacuum: $u_k' \sim \eta e^{-ik\eta}$ (positive frequency mode)
- The integral yields $1/(k_1+k_2+k_3) \rightarrow 1/(2k_1)$ in the squeezed limit

Back to in-in

$$\langle \zeta^3(t^*) \rangle = -i \int_{t_0}^{t^*} dt' \langle 0 | [\zeta^3(t^*), H_I(t')] | 0 \rangle$$



$$B_\zeta(k_1, k_2, k_3) = 2i \frac{\dot{\phi}^4}{H^6} \sum_i \left(\frac{1}{k_i^2} \right) \tilde{u}_{k_1}(\bar{\eta}) \tilde{u}_{k_2}(\bar{\eta}) \tilde{u}_{k_3}(\bar{\eta}) \int_{\eta_0}^{\bar{\eta}} d\eta \frac{1}{\eta^3} u'_{k_1}{}^* u'_{k_2}{}^* u'_{k_3}{}^* + \text{c.c.}$$

- Non-Bunch-Davies vacuum: $u_k' \sim \eta(A_k e^{-ik\eta} + B_k e^{+ik\eta})$ **negative frequency mode**
- The integral yields $1/(k_1 - k_2 + k_3)$, peaking in the folded limit
Chen et al. (2007); Holman & Tolley (2008)
- The integral yields $1/(k_1 - k_2 + k_3) \rightarrow 1/(2k_3)$ in the squeezed limit

Enhanced by k_1/k_3 : this can be a big factor!

Agullo & Parker (2011)

How about the consistency relation?

$$B(k_1, k_2, k_3) \xrightarrow{k_3/k_1 \ll 1} P(k_1)P(k_3) \left\{ (1 - n_s) + 4 \frac{\dot{\phi}^2}{H^2} \frac{k_1}{k_3} [1 - \cos(k_3 \eta_0)] \right\}$$

- When k_3 is far outside the horizon at the onset of inflation, η_0 (whatever that means), $k_3 \eta_0 \rightarrow 0$, and thus the above additional term vanishes.

- The consistency relation is restored. *Sounds familiar!*

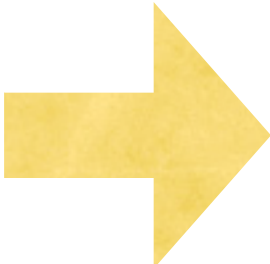
Checking

“Not-so-squeezed Limit”

- Creminelli, D’Amico, Musso & Norena, arXiv:1106.1462 showed that all single-field models have the next-to-leading behavior of the squeezed bispectrum given by

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle \simeq -(2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) P(k_1)P(k_S) \left[\frac{d \ln(k_S^3 P(k_S))}{d \ln k_S} + \mathcal{O}\left(\frac{k_1^2}{k_S^2}\right) \right], \quad k_1 \ll k_S.$$

The non-Bunch-Davies vacuum case seems to violate this: the solution is that, in order for their result to hold, k_3 must be small enough so that k_3 is already **far** outside the horizon.

 We already saw that, in this limit, the non-Bunch-Davies vacuum result reproduces the standard result. But...

Checking “Not-so-squeezed Limit”

$$B(k_1, k_2, k_3) \xrightarrow{k_3/k_1 \ll 1} P(k_1)P(k_3) \left\{ (1 - n_s) + 4 \frac{\dot{\phi}^2}{H^2} \frac{k_1}{k_3} [1 - \cos(k_3 \eta_0)] \right\}$$

- The Taylor expansion of the second term yields $O(k_1 k_3 \eta_0^2)$, which is not the same as $(k_3/k_1)^2$. Hmm...

Anyway, an interesting possibility:

- What if $k_3\eta_0 = O(1)$?
- The squeezed bispectrum receives an enhancement of order $\epsilon k_1/k_3$, which can be sizable.
- Most importantly, **the bispectrum grows faster than the local-form toward $k_1/k_3 \rightarrow 0$!**
 - $B(k_1, k_2, k_3) \sim 1/k_3^3$ [Local Form]
 - $B(k_1, k_2, k_3) \sim 1/k_3^4$ [non-Bunch-Davies]
- This has an observational consequence – particularly a scale-dependent bias.

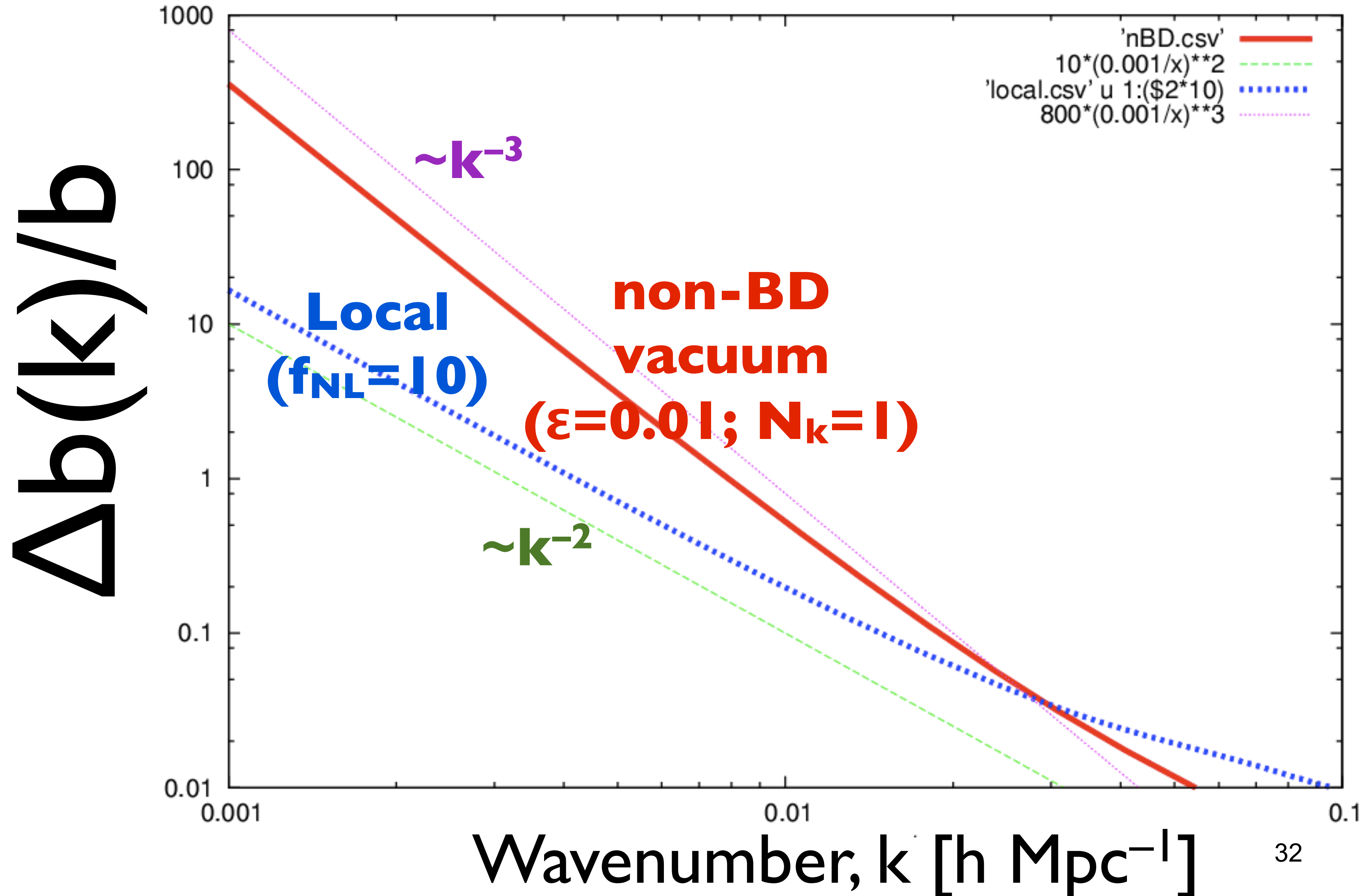
Scale-dependent Bias

$$\frac{\Delta b_h(k, R)}{b_h} = \frac{\delta_c}{D(z)\mathcal{M}_R(k)} \frac{1}{8\pi^2\sigma_R^2} \int_0^\infty dk_1 k_1^2 \mathcal{M}_R(k_1) \times \int_{-1}^1 d\mu \mathcal{M}_R\left(\sqrt{k^2 + k_1^2 + 2kk_1\mu}\right) \frac{\mathbf{B}\left(k_1, \sqrt{k^2 + k_1^2 + 2kk_1\mu}, k\right)}{P_\zeta(k)}$$

- A rule-of-thumb:
 - For $\mathbf{B}(k_1, k_2, k_3) \sim 1/k_3^p$, the scale-dependence of the halo bias is given by $b(k) \sim 1/k^{p-1}$
 - For a local-form ($p=3$), it goes like $b(k) \sim 1/k^2$
 - For a non-Bunch-Davies vacuum ($p=4$), would it go like $b(k) \sim 1/k^3$?

It does!

Ganc & Komatsu (in prep)



CMB?

- The expected contribution to $f_{\text{NL}}^{\text{local}}$ as measured by CMB is typically $f_{\text{NL}}^{\text{local}} < 2(\epsilon/0.01)$.
 - A lot bigger than $(5/12)(1-n_s)$, but still small enough.

How about...

- Falsifying multi-field inflation?

Strategy

- We look at the local-form four-point function (trispectrum).
- Specifically, we look for a consistency relation between the local-form bispectrum and trispectrum that is respected by (almost) all models of multi-field inflation.

- We found one: $\tau_{\text{NL}} > \frac{1}{2} \left(\frac{6}{5} f_{\text{NL}} \right)^2$

provided that 2-loop and higher-order terms are ignored.

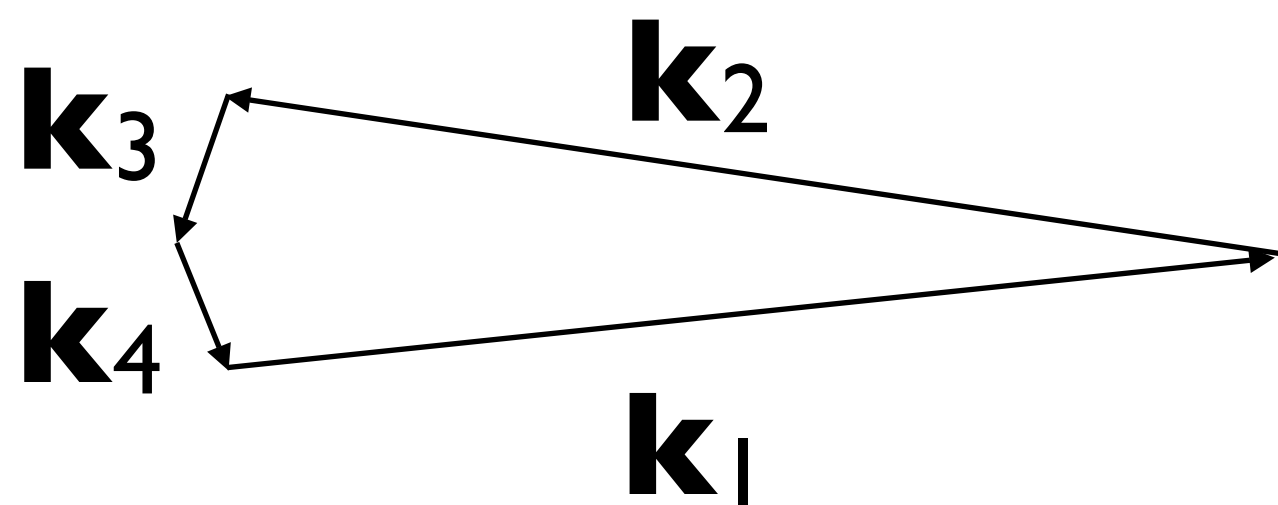
Which Local-form Trispectrum?

- The local-form bispectrum:
 - $B_{\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) f_{\text{NL}} [(6/5) P_{\zeta}(k_1) P_{\zeta}(k_2) + \text{cyc.}]$
- can be produced by a curvature perturbation in position space in the form of:
 - $\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + (3/5) f_{\text{NL}} [\zeta_g(\mathbf{x})]^2$
- This can be extended to higher-order:
 - $\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + (3/5) f_{\text{NL}} [\zeta_g(\mathbf{x})]^2 + (9/25) g_{\text{NL}} [\zeta_g(\mathbf{x})]^3$

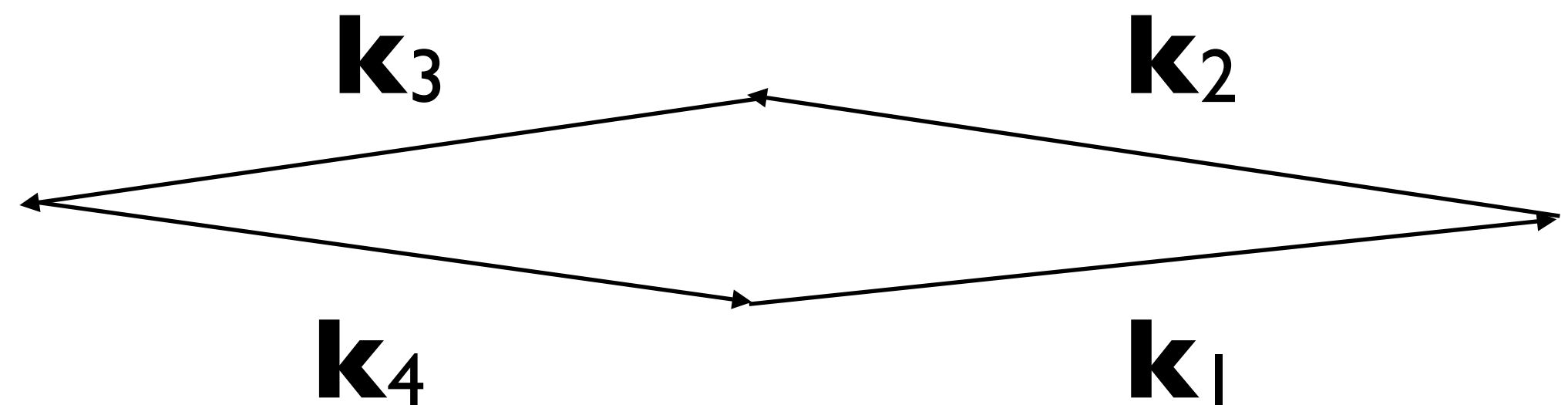
This term (ζ^3) is too small to see, so I will ignore this in this talk.

Two Local-form Shapes

- For $\zeta(\mathbf{x}) = \zeta_g(\mathbf{x}) + (3/5)f_{\text{NL}}[\zeta_g(\mathbf{x})]^2 + (9/25)g_{\text{NL}}[\zeta_g(\mathbf{x})]^3$, we obtain the trispectrum:
 - $T_\zeta(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \{ g_{\text{NL}}[(54/25)P_\zeta(k_1)P_\zeta(k_2)P_\zeta(k_3) + \text{cyc.}] + (f_{\text{NL}})^2[(18/25)P_\zeta(k_1)P_\zeta(k_2)(P_\zeta(|\mathbf{k}_1 + \mathbf{k}_3|) + P_\zeta(|\mathbf{k}_1 + \mathbf{k}_4|)) + \text{cyc.}] \}$



g_{NL}



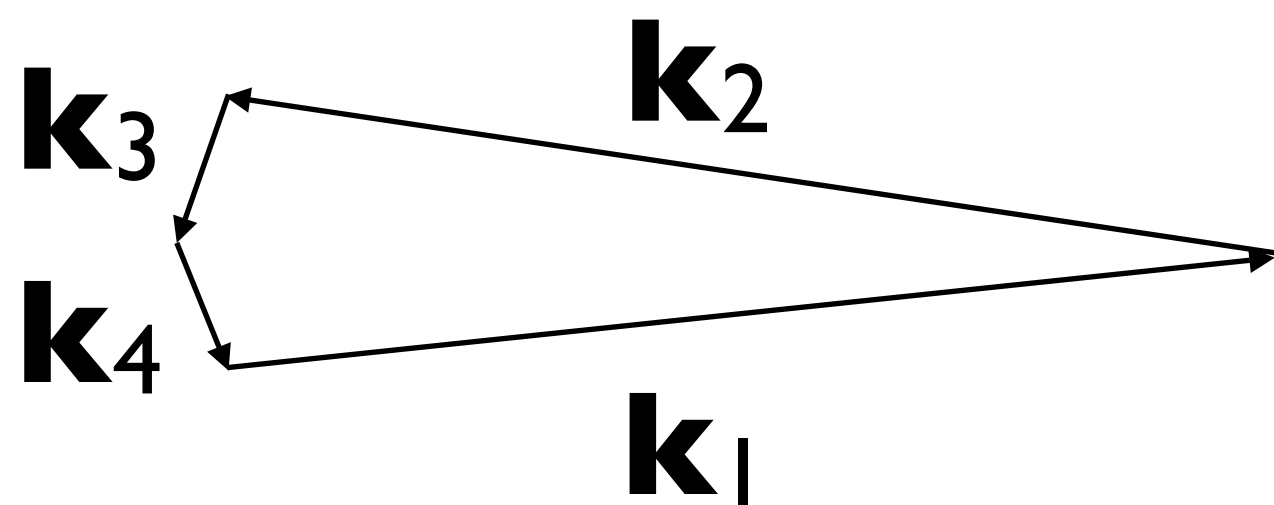
f_{NL}^2

Generalized Trispectrum

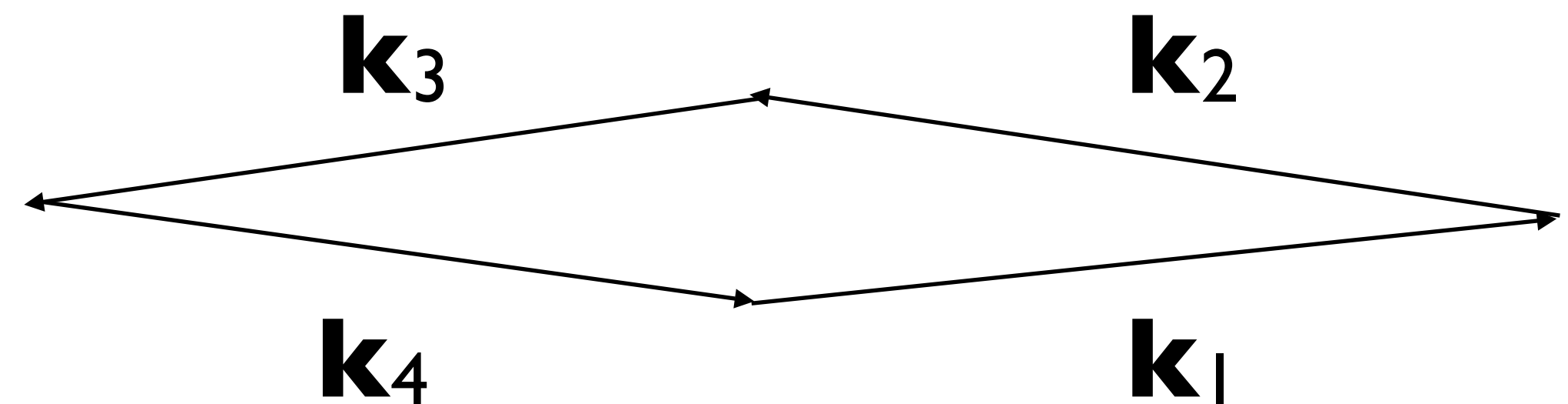
- $T_{\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \{ \mathbf{g}_{NL} [(54/25) P_{\zeta}(k_1) P_{\zeta}(k_2) P_{\zeta}(k_3) + \text{cyc.}] + \mathbf{T}_{NL} [P_{\zeta}(k_1) P_{\zeta}(k_2) (P_{\zeta}(|\mathbf{k}_1 + \mathbf{k}_3|) + P_{\zeta}(|\mathbf{k}_1 + \mathbf{k}_4|)) + \text{cyc.}] \}$

The single-source local form consistency relation,

*$\tau_{NL} = (6/5)(f_{NL})^2$, may not be respected –
additional test of multi-field inflation!*



\mathbf{g}_{NL}

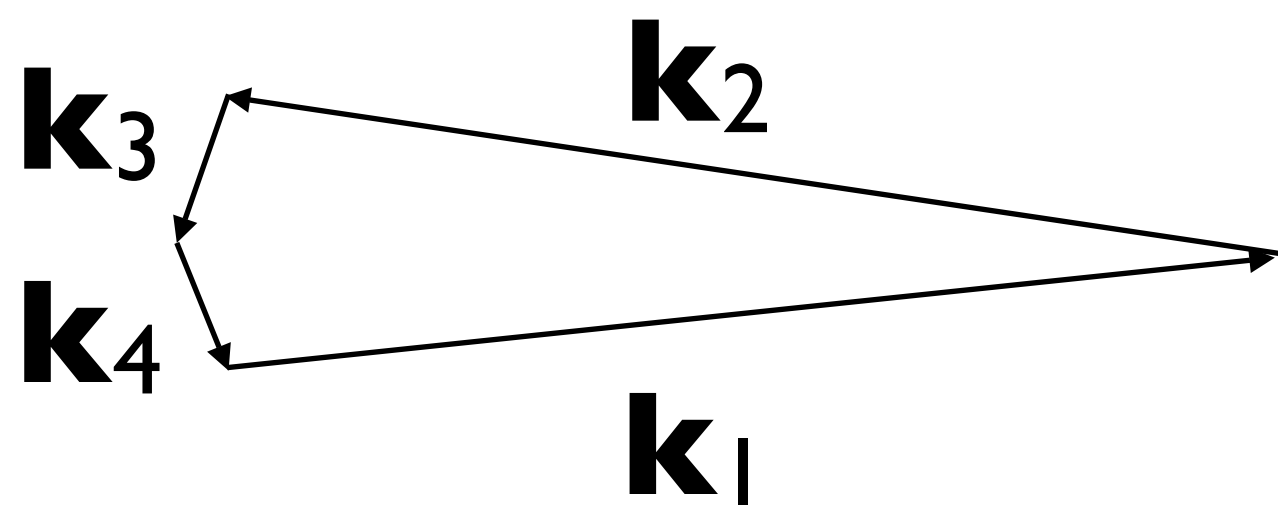


τ_{NL}

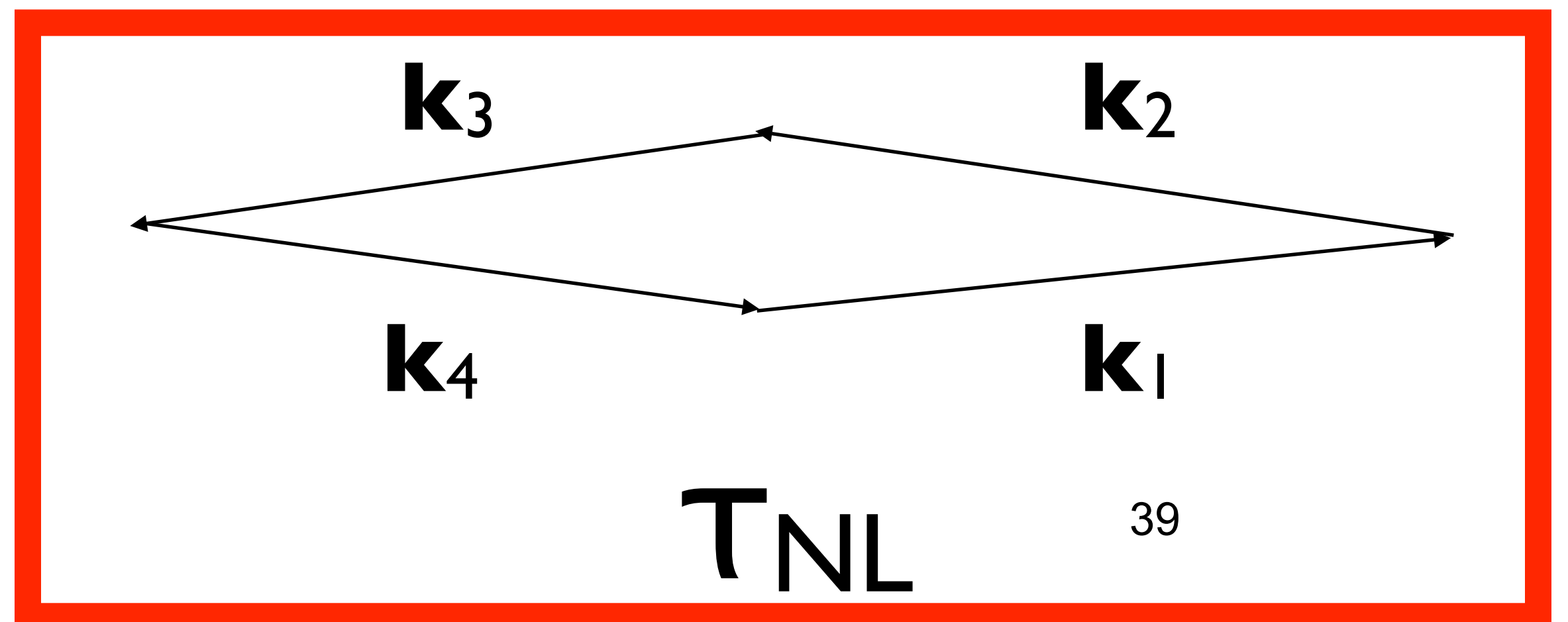
(Slightly) Generalized Trispectrum

- $T_{\zeta}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \{ g_{NL} [(54/25) P_{\zeta}(k_1) P_{\zeta}(k_2) P_{\zeta}(k_3) + \text{cyc.}] + \tau_{NL} [P_{\zeta}(k_1) P_{\zeta}(k_2) (P_{\zeta}(|\mathbf{k}_1 + \mathbf{k}_3|) + P_{\zeta}(|\mathbf{k}_1 + \mathbf{k}_4|)) + \text{cyc.}] \}$

The single-source local form consistency relation, $\tau_{NL} = (6/5)(f_{NL})^2$, may not be respected – additional test of multi-field inflation!



g_{NL}



τ_{NL}

Tree-level Result (Suyama & Yamaguchi)

- Usual δN expansion to the second order

$$\zeta = \sum_I \frac{\partial N}{\partial \phi_I} \delta \phi_I + \frac{1}{2} \sum_{IJ} \frac{\partial^2 N}{\partial \phi_I \partial \phi_J} \delta \phi_I \delta \phi_J + \dots$$

gives:

$$\frac{6}{5} f_{\text{NL}}^{\text{local}} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\text{NL}} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

Now, stare at these.

$$\frac{6}{5} f_{\text{NL}}^{\text{local}} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\text{NL}} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

Change the variable...

$$\frac{6}{5} f_{\text{NL}}^{\text{local}} = \frac{\sum_{IJ} N_{,IJ} N_{,I} N_{,J}}{[\sum_I (N_{,I})^2]^2},$$

$$\tau_{\text{NL}} = \frac{\sum_{IJK} N_{,IJ} N_{,J} N_{,IK} N_{,K}}{[\sum_I (N_{,I})^2]^3} = \frac{\sum_I (\sum_J N_{,IJ} N_{,J})^2}{[\sum_I (N_{,I})^2]^3}$$

$$a_I = \frac{\sum_J N_{,IJ} N_{,J}}{[\sum_J (N_{,J})^2]^{3/2}}$$

$$b_I = \frac{N_{,I}}{[\sum_J (N_{,J})^2]^{1/2}}$$

$$(6/5) f_{\text{NL}} = \sum_I a_I b_I$$

$$\tau_{\text{NL}} = (\sum_I a_I^2) (\sum_I b_I^2)_{42}$$

Then apply the Cauchy-Schwarz Inequality

$$\left(\sum_I a_I^2 \right) \left(\sum_J b_J^2 \right) \geq \left(\sum_I a_I b_I \right)^2$$

- Implies (Suyama & Yamaguchi 2008)

$$\tau_{\text{NL}} \geq \left(\frac{6 f_{\text{NL}}^{\text{local}}}{5} \right)^2$$

But, this is valid only at the tree level!

Harmless models can violate the tree-level result

- The Suyama-Yamaguchi inequality does not always hold because the Cauchy-Schwarz inequality can be $0=0$. For example:

$$\zeta = \frac{\partial N}{\partial \phi_1} \delta \phi_1 + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_2^2} \delta \phi_2^2$$

In this harmless two-field case, the Cauchy-Schwarz inequality becomes $0=0$ (both f_{NL} and τ_{NL} result from the second term).

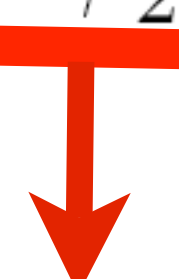
In this case,

$$\tau_{\text{NL}} \sim 10^3 (f_{\text{NL}}^{\text{local}})^{4/3}$$

(Suyama & Takahashi 2008) ⁴⁴

“1 Loop”

$$\zeta = \frac{\partial N}{\partial \phi_1} \delta \phi_1 + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_2^2} \delta \phi_2^2$$



Fourier transform this,
and multiply 3 times

$$\begin{aligned} & \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 s}{(2\pi)^3} \langle \delta \tilde{\phi}_2(\mathbf{k}_1 - \mathbf{p}) \delta \tilde{\phi}_2(\mathbf{p}) \delta \tilde{\phi}_2(\mathbf{k}_2 - \mathbf{q}) \delta \tilde{\phi}_2(\mathbf{q}) \delta \tilde{\phi}_2(\mathbf{k}_3 - \mathbf{s}) \delta \tilde{\phi}_2(\mathbf{s}) \rangle \\ &= \left(\frac{H^2}{2} \right)^3 (2\pi)^3 \delta_D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^3 |\mathbf{k}_1 - \mathbf{p}|^3 |\mathbf{k}_3 + \mathbf{p}|^3} + (\text{permutations}) \\ &\approx \left(\frac{H^2}{2} \right)^3 (2\pi)^3 \delta_D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{8 \ln(k_b L)}{2\pi^2} \left[\frac{1}{k_1^3 k_3^2} + \frac{1}{k_2^3 k_3^2} + \frac{1}{k_1^3 k_2^2} \right] \end{aligned}$$

- $k_b = \min(k_1, k_2, k_3)$

Ignoring details...

- I don't have time to show you the derivation (you can look it up in the paper), but the result is somewhat weaker than the Suyama-Yamaguchi inequality:

$$\tau_{\text{NL}} > \frac{1}{2} \left(\frac{6}{5} f_{\text{NL}} \right)^2$$

Detection of a violation of this relation can potentially falsify inflation as a mechanism for generating cosmological fluctuations.

A Comment

- Even without using the physics argument, the statistics argument can give a bound (*Smith, LoVerde & Zaldarriaga, arXiv:1108.1805*):

$$\tau_{NL} \geq \left(\frac{6}{5} f_{NL} \right)^2 - A$$

where

$$A = \frac{1}{2P(k_L)V_S} = \frac{2\pi^2}{k_L^3 P(k_L)} \frac{1}{4\pi^2} \frac{3}{4\pi} \left(\frac{k_L}{k_S} \right)^3 \approx 2 \times 10^6 \left(\frac{k_L}{k_S} \right)^3 \neq 0$$

The statistics argument does **not** preclude a physical violation of the Suyama-Yamaguchi inequality

Summary

- A more insight into the single-field consistency relation for the squeezed-limit bispectrum using in-in formalism.
- Non-Bunch-Davies vacuum can give an enhanced bispectrum in the $k_3/k_1 \ll 1$ limit, yielding a distinct form of the scale-dependent bias.
- Multi-field consistency relation between the 3-point and 4-point function can be used to rule out multi-field inflation, as well as single-field.