# Supply Chain Pricing, Risk-Return Analysis, and Online Resource Allocation

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# Dedication

To my father Shengli, my mother Xiuqin, and Tianbai.

# Abstract

This dissertation studies a few models in two categories of operations management. The first part of the dissertation focuses on supply chain management related topics. We consider a supply chain model with one supplier and one retailer who acts as a newsvendor. The first model in this dissertation focuses on the supplier and the retailer's optimal policies in a multi-period newsvendor model. We derive the optimal pricing and ordering policies for demand with Increasing Generalized Failure Rate (IGFR) property and obtain comparative statics for the optimal prices. We discover that under certain conditions of the demand distribution, the supplier's optimal prices are increasing in time. Moreover, the price increments are increasing in the backorder cost and the optimal prices are increasing in the backorder cost as well. We also perform a distribution-free analysis of the multi-period newsvendor model and provide the structure of the worst-case distribution.

In addition to the pricing and ordering decisions, we also analyze the risk-return trade-off in single-period newsvendor models using the mean-variance approach. We discuss the classic newsvendor model which uses the wholesale-price contract and two variations of the model, a spot market model and a revenue-sharing contract model. We derive the risk-return curve for the retailer and the corresponding distribution in closed-form for a two-point distribution and a three-point distribution in the classic model. When the demand follows a multi-point distribution or a continuous distribution, we provide a linear program to compute the risk-return curves and show the curves' upper bounds. An approximation algorithm is introduced to efficiently calculate the risk-return curve in the continuous distribution models.

Introducing some variation to the basic model, we consider a supply chain setting with a spot market where unsatisfied demand can purchase from the supplier at the market price. The supplier's decisions are the wholesale price and the buffer inventory for the spot market. We derive the supplier's optimal decisions and study the supplier's risk-return trade-off under uniform and exponential distributions. Another problem that we consider is the risk-return analysis under a revenue-sharing model. We derive the supplier's optimal pricing policy and characterize the effect of $\phi$ on both the supplier and the retailer's decisions and risks. Numerical experiments are conducted to demonstrate the results.

The second part of this thesis concerns resource allocation in an online setting, specifically, the online matching problems. Online matching problems are used as the backstage algorithm by search engines to match advertisements with each search. We focus on the online matching problem with concave return functions and a random permutation model. In this dissertation, we introduce two online learning algorithms to solve the associated matching problem. The main idea is to utilize the observed data in the allocation process and project it into the future. We begin with the one-time learning algorithm that only uses the data to compute an allocation rule once. This algorithm achieves near-optimal performance when input data satisfy certain conditions. To further improve the performance, we introduce a dynamic learning algorithm which updates the allocation rule at a geometric pace, at time $\epsilon n, 2\epsilon n, 4\epsilon n$ and so on. This algorithm achieves near-optimal performance with fewer restrictions on the input data conditions. We compare the performance of the one-time learning algorithm, the dynamic learning algorithm, and the greedy algorithm in numerical experiments.

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# Chapter 1

# Introduction

This dissertation consists of two parts. The first half of the dissertation focuses on supply chain management related problems. Supply chain refers to a network of resources and organizations that moves goods or services from the suppliers to the end customer. A supply chain is a very complex and dynamic system which involves material flow, information flow and financial flow. The Council of Supply Chain Management Professionals (CSCMP) defines supply chain management as "the planning and management of all activities involved in sourcing and procurement, conversion, and all logistics management activities, plus coordination and collaboration with channel partners, which can be suppliers, intermediaries, third-party service providers, and customers". Supply chain management is a crucial part of a business's success and customer satisfaction. There are countless examples of companies that went out of business because of poor management of the supply chain. Thus, supply chain management has drawn much attention from researchers and scholars in the past few decades and there are still challenges left unsolved.

Pricing and ordering are integral parts of supply chain management. In some sense, the goal of supply chain management is to match supply with demand. This is done with the help of proper pricing strategy, correct ordering decision, and efficient inventory management. For any company to stay competitive and financially stable, reducing operational costs (such as purchasing cost, inventory cost, transportation cost) and risk (such as stockout risk, defective products, delivery failure) at the same time increasing profit is the never-ending pursuit. Thus, it is crucial to find a suitable pricing and ordering strategy based on the company's supply chain characteristics, and this is the focus of Chapter 2 of this thesis.

In Chapter 2, we consider a two-echelon supply chain model with one supplier and one retailer who acts as a newsvendor. The newsvendor model is one of the most classical models in inventory management. The newsvendor model's order quantity is based on the tradeoff between the overage cost and the shortage cost. One common underlying assumption of the newsvendor model is a fixed purchase price. In practice, however, suppliers also try to maximize their revenue. The model proposed in this dissertation looks at a setting where the supplier of a multi-period newsvendor strategically sets the prices to maximize its revenue. In this setting, the supplier knows the newsvendor's ordering strategy and cost parameters, and actively proposes a set of prices for all periods, then the newsvendor makes the ordering decision accordingly. In our model, the pricing and ordering decisions for all periods must be made before the start of the selling horizon. The newsvendor's inventory can be carried to the next period if there is overage, and demand can be backlogged if there is shortage. There are several applications of such a model. One example is in the fresh produce retail industry. In the fresh produce retail industry, due to the long lead time of production, the order quantity and prices must be planned ahead of the growing season. Moreover, fresh produce has short shelf lives and high holding cost, making it necessary to have small portion deliveries throughout the selling season.

Chapter 2 focuses on the optimal pricing policy of the supplier and the optimal ordering policy of the retailer in such a multi-period supply chain model. We first show the existence of an optimal pricing policy, then propose a procedure for computing the optimal prices. We show that the optimal prices are decreasing in time under our model and explain the intuition behind our result. We also adopt the approach used in Lariviere and Porteus (2001) to obtain comparative statics for the optimal prices. For the supplier's profit function, our analysis shows that it is unimodal when the demand distributions have increasing generalized failure rate. For the supplier's optimal prices, we discover that under certain conditions of the demand distribution, the price increments are increasing. Moreover, the price increments are increasing in the backorder cost. Therefore, the optimal price is increasing in the backorder cost. Numerical experiments confirm the monotonicity results of the supplier's optimal prices and the revenue functions of the supplier and the retailer.

In most of the studies of the newsvendor models, demand information are assumed as public information. However, in the real world, it is not possible to obtain the actual demand distribution function. In view of that, people are interested in the characteristics of the worst-case distribution function and the corresponding robust ordering policy. For example, Scarf (1958) proposed a single-period newsvendor model under worst-case distribution where only the mean $\mu$ and variance $\sigma^2$ are known. Gallego and Moon (1993) later provided a simplified proof to that result and extended the distribution-free model to other cases. In Section 2.5, we extend the model in Scarf (1958) to a multi-period newsvendor model. In particular, we present a distribution-free analysis of the multi-period newsvendor model and provide the structure of the worst-case distribution.

In addition to pricing and ordering decisions, another important aspect in supply chain management is the profit vs. risk relationship between different players, particularly, the way that the supply chain's profit and risk are split between the supplier and the retailer. In Chapter 3 of this dissertation, we perform a risk-return analysis for a single-period supply chain model. The concept of risk and return analysis is to study the trade-off between the expected revenue and the potential variability in the revenue. The approach adopted for this analysis in Chapter 3 is the mean-variance approach. We discuss the classic newsvendor model which uses the wholesale-price contract and two variations of the model, a spot market model and a revenue-sharing model. In the classic newsvendor model, the retailer pays the supplier according to the wholesale-price contract. The supplier receives a fixed payment; thus, does not face any uncertainty in revenue. Given this, the focus of the risk analysis for the classic model is the retailer's problem. We first look at the retailer's minimum risk given an expected return. Under our model setting, the only available demand information is the mean and number of points in the distribution. By fixing different values of the expected profit, we are able to find the risk-return curve (efficient frontier) for the retailer. Moreover, we can compute the exact distribution function that achieves the optimal risk-return curve when the demand distribution is a two-point distribution and a three-point distribution. When the demand follows a multi-point distribution, we provide a linear program to compute the risk-return curve. We show the upper bound of the retailer's risk-return curve and the smallest point in the optimal distribution. We then generalize this analysis to the continuous case in Section 3.2.4. We model the problem using an infinite program and propose an approximation algorithm to efficiently calculate the risk-return curve. At the end of this section, numerical results of a continuous distribution risk-return curve solved using the approximation algorithm are demonstrated. After analyzing the supply chain's profit variation, we look at how the supply chain's profit is split between the supplier and the retailer. In particular, we derive lower bounds of the supplier's portion of the total supply chain's profit. We show that this lower bound is tight by providing distributions that could achieve the lower bound. The lower bound shows that the supplier is always guaranteed to receive a certain portion of the supply chain profit regardless of the demand distribution.

In Section 3.4, we consider a supply chain structure with a spot market. The retailer who is a newsvendor purchases from the supplier based on the wholesale price given. The retailer sells the inventory to the end customer. When the retailer's order quantity cannot meet all the demand, the unsatisfied demand will turn to the spot market to purchase the item. The supplier can build up buffer inventory to sell on the spot market at the market price. Therefore, the supplier's decisions are the price and buffer inventory for the spot market. In this section, we investigate the supplier's optimal decisions. In the spot market, the supplier is directly facing the random demand. So, we analyze the supplier's risk-return trade-off in this case. Furthermore, numerical examples with a uniform distribution and an exponential distribution demonstrate the risk-return curves.

Section 3.5 focuses on a different setup. We adopt a revenue-sharing contract instead of the wholesale-price contract, see Cachon and Lariviere (2005). Specifically, the retailer pays the supplier a wholesale price upon purchase of the inventory, plus a percentage $(1 - \phi)$ of the revenue received in the selling period. The shared revenue includes sales revenue and salvage value of the unsold items. We study the supplier and the retailer's optimal decisions and risk-return trade-offs. We also characterize the effect of $\phi$ on the optimal decisions, profit expectation, and variance functions. Section 3.5.1 shows the supplier and the retailer's optimal decisions and risk-return functions in closed-form for uniform and exponential distributions.

The second part of this thesis concerns resource allocation in an online setting. Specifically, we consider an important class of online resource allocation problem — the online matching problems, and propose efficient algorithms for this class of problems. The online matching problem usually considers a bipartite matching. In a bipartite matching, there is a bipartite graph. We need to find a matching between the two sides in graph $G$ such that no edges in the matching have common endpoints. When all the vertices in both sides are given upfront, we call this problem an offline problem. In the online matching problems, there is a bipartite graph with a part of the vertices known and a part of vertices arriving one at a time. When a vertex arrives, it needs to be matched to an available vertex from the other side. The objective is to maximize the size of the matching, see Karp et al. (1990) and Mehta (2012). Online matching problems are widely used in practice because they are the backstage algorithm used by search engines (e.g., Google and Bing) to match advertisements with each search. In that setting (known as the Adwords problem), there are $m$ advertisers (which we also call the bidders). A sequence of $n$ keywords are searched during a fixed time horizon. Based on the relevance of the keyword, the $i$th bidder would bid a certain amount $b_{ij}$ to show his advertisement on the result page of the $j$th keyword. The search engine's decision is to allocate each keyword to one of the $m$ bidders. Note that each allocation decision can only depend on the information earlier in the arrival sequence but not on any future data.

In Chapter 4, we study the online matching problem with concave return functions. Most prior research uses linear or piece-wise linear return functions. However, as pointed out in Devanur and Jain (2012), there are several practical motivations for considering a concave function of the matched bids. Among them are convex penalty costs for under-delivery in search engine-advertiser contracts, the concavity of the click-through rate in the number of allocated bids observed in empirical data and fairness considerations. In each of these situations mentioned above, the objective can be expressed as a concave function.

In Chapter 4, we introduce two online learning algorithms to discern the demand and solve the associated Adwords problem. The main idea is to utilize the observed data in the allocation process. When the input data arrives, the search engine could use the past data and project it into the future. Section 4.2 focuses on a one-time learning algorithm that only uses the data to compute an optimal allocation rule once. By selecting the first $\epsilon n$ amount of data as our learning sample and projecting the demand arriving pattern into the future, this algorithm achieves near-optimal performance when input data satisfy certain conditions. In Section 4.3, we introduce a dynamic learning algorithm which achieves near-optimal performance with less restrictions on the input data conditions. In the dynamic learning algorithm, the allocation rule is updated at a geometric pace, at time $\epsilon n, 2\epsilon n, 4\epsilon n, \dots$ and so on. This choice of the updating points balances the tradeoff between exploration and exploitation. Thus, the dynamic learning algorithm makes better use of the input data without slowing the allocation process down too much. We compare the performance of the one-time learning algorithm, the dynamic learning algorithm, and the greedy algorithm in numerical experiments in the last part of Chapter 4.

Finally, we note that the materials in the dissertation have resulted in two published papers. More specifically, part of Chapter 2 is from the paper *Optimal Pricing for Selling to a Static Multi-Period Newsvendor* Chen et al. (2017) and the majority of Chapter 4 is from the paper *A Dynamic Learning Algorithm for Online Matching Problems with Concave Returns* Chen and Wang (2015).

# Chapter 2

# Optimal Pricing in a Multi-Period Newsvendor Model

## 2.1 Introduction and Literature Review

The newsvendor model is probably the most classical model in inventory and supply chain management. In a newsvendor model, a retailer decides the order quantity of a product when facing uncertain demand, trading off the possibility of overstocking with understocking. There have been extensive studies on newsvendor models in the literature. For some recent reviews, we refer the readers to Khouja (1999) and Qin et al. (2011).

In practice, the newsvendor (retailer) is often supplied by a supplier who has pricing power over the product. The traditional newsvendor model assumes that the purchase price of the product is fixed. However, the supplier may utilize its pricing power to decide a price that maximizes his profit. Such a model is first studied in Lariviere and Porteus (2001), in which the authors consider a supplier who sells to a newsvendor and derives the optimal price for the supplier. In this chapter, we extend the model in Lariviere and Porteus (2001) to a multi-period setting. We consider a supplier who sells a product to a multi-period newsvendor. Inventory holding cost and backorder cost are incurred if there are excess or a shortage of inventory at the end of each period. The supplier posts a price for each period in the beginning of the time horizon, and the retailer must commit to an ordering quantity for each period before the selling horizon starts.

There are many practical situations where such a multi-period model is relevant. One example is in the fresh produce industry. One major characteristic of the fresh produce industry is the long lead time of the production due to the growing cycle of the produce. Therefore, the retailer's order quantities must be planned ahead of the growing season despite demand uncertainty so that the supplier can deliver the desired quantity during the selling season. Moreover, due to the short shelf lives of fresh produce and the high holding costs, it is not possible to have all products delivered at once and stored during the selling season. Indeed, it is common practice in the fresh produce industry for the retailers to commit to a sequence of ordering quantities before the selling season starts. According to i2, a supply chain results company, Dole Asia (a division of Dole Food Company) plans the plant according to the customer's demand Bossers (2007); and Tesco, the UK's largest retailer, signs three- to five-year contract with its fresh produce suppliers, allowing them to plan further ahead Knowles (2016). In addition, the supplier sometimes has the pricing power over the product due to the scarcity of supply (e.g., see Richards et al. (2011)). In such situations, the problem for the supplier is one that sells to a multi-period newsvendor as described in our model.

In this chapter, we solve the problem of optimal pricing for selling to a multi-period newsvendor. Particularly, we provide a simple solution procedure for the optimal prices as well as a closed-form solution to the optimal ordering quantities under the optimal prices. We find that the optimal prices are decreasing in time under our model. We also adopt the analysis in Lariviere and Porteus (2001) to obtain comparative statics for the optimal prices.

In the remainder of this section, we review how this model relate to and differ from the literature. First of all, this model is related to the inventory theory. Various inventory models have been studied and many results have been obtained, see, e.g., Nahmias (2011), Porteus (2002) and Snyder and Shen (2011). However, instead of the typical continuous review or periodic review models, we consider a case where all the inventory decisions must be made at the beginning (which is equivalent to having a lead time that is longer than the planning horizon). Therefore, our decision model invokes only one shot of inventory decisions for the retailer rather than decisions that depend on the state of the system.

This work is also related to the revenue management literature (see Talluri and van Ryzin (2004) for a comprehensive review). In revenue management, the seller decides the optimal prices for its products to maximize the revenue, which is the case for the supplier in our model. However, in our model, the demand function (i.e., the purchase quantity of the retailer) is resulting from solving a multi-period newsvendor model. Therefore, our problem is a very specific form of pricing problem.

There has also been a wealth of literature of supply chain contracts in which the supplier and retailer sign a contract in terms of cost-sharing, buy-back, etc, to coordinate the supply chain and to increase the overall efficiency Cachon (2003), Lariviere (1999), Tsay (1999). Although a coordinated supply chain performs better in many aspects, in practice, an uncoordinated supply chain is still widespread due to its simple structure. Our work will be focused on uncoordinated supply chains.

Finally, as we mentioned in the introduction, one application of this model is in the fresh produce industry. There has been much work that is related to the fresh produce supply chain. We refer the readers to Ahumada and Villalobos (2009) for a thorough review. However, in most of those studies, the supply chain is modeled by a dynamic system in which the lead time is relatively short compared to the selling season. For example, Chao et al. (2018, 2015), Zhang et al. (2016) propose approximation algorithms for perishable inventory systems under different settings. However, those assumptions may not be realistic in practice because of the long growing cycle of the produce and the limited spot market Maruyama and Hirogaki (2007). In contrast, we consider the case where the supplier and the retailer only make a one-time decision about the price and the order quantity ahead of the selling season.

## 2.2 Model

In this section, we first state the model assumptions and event setups. Then, the mathematical model in the retailer's problem and the supplier's problem are presented.

### 2.2.1 Model Assumptions

We consider a supplier who provides the supply of a certain product to a retailer over a horizon of $T$ periods. Before the start of the time horizon, the supplier decides the selling price in all time periods $p_t$, for $t = 1, 2, \ldots, T$. After observing the supplier's prices, the retailer decides the order quantities $Q_t$ for each period. In each period $t$, the retailer faces a random demand $D_t$ with $F_t(\cdot)$ being the cumulative distribution function. We do not assume that the demands in each period are independent. The retailer incurs a per unit holding cost $h$ to carry inventory from one period to the next, and a per unit backorder cost $b$ for each unit of unsatisfied demand. In addition, the selling price of the retailer is $r$. In our model, we assume the holding cost, the backorder cost, and the selling price are constant throughout the entire horizon. However, our model can be easily extended to the case where they differ across different periods.

In our model, before the start of the first period, the supplier decides a sequence of prices $\mathbf{p} = \{p_t\}_{t=1}^T$. Then the retailer decides $\mathbf{Q} = \{Q_t\}_{t=1}^T$ based on $\mathbf{p}$. The initial inventory of the retailer is $I_0 = 0$. Then, in each period $t$, the following sequence of events take place.

1. The retailer receives order quantity $Q_t$ and pays the supplier $p_t Q_t$.
2. Demand $D_t$ realizes, and the inventory at the end of period $t$ is $I_t = I_{t-1} + Q_t - D_t$.
3. If $t \leq T - 1$, then the profit in period $t$ is $R_t = rD_t - hI_t^+ - bI_t^-$, where $I_t^+ = \max\{I_t, 0\}$ represents the positive part of $I_t$, and $I_t^- = \max\{-I_t, 0\}$ represents the negative part of $I_t$. In the last period $T$, the profit is $R_T = r\min\{D_T, I_{T-1} + Q_T\} - hI_T^+ - bI_T^-$.

Here, we can view the backorder cost as the per period loss of goodwill cost when the seller is unable to satisfy the demand (regardless of whether the demand can be satisfied eventually). The total profit of the retailer is $R(\mathbf{Q}) = \sum_{t=1}^{T} (R_t - p_t Q_t)$, and the total revenue for the supplier is $\sum_{t=1}^{T} p_t Q_t$.

### 2.2.2 Retailer's Problem

Having observed the supplier's prices $p_t$, the retailer decides how many units to purchase in each period before demand arrives. The retailer's problem can be modeled as follows:

$$\begin{aligned} \max \quad & R(\mathbf{Q}) = \mathbb{E}\left[r \min\left\{\textstyle\sum_{t=1}^{T} Q_t, \sum_{t=1}^{T} D_t\right\}\right] - \textstyle\sum_{t=1}^{T} \mathbb{E}\left[hI_t^+ + bI_t^- + p_t Q_t\right] \\ \text{s.t.} \quad & I_0 = 0,\ I_t = I_{t-1} + Q_t - D_t, \qquad \forall t = 1, \ldots, T \\ & Q_t \geq 0, \qquad \forall t = 1, \ldots, T. \end{aligned} \tag{2.1}$$

Here, the first part of the objective function is the expected revenue, and the second part of the objective function is the expected cost. We can rewrite the objective function as follows:

$$r\mathbb{E}\left[\textstyle\sum_{t=1}^{T} D_t\right] - r\mathbb{E}\left[\textstyle\sum_{t=1}^{T} D_t - \sum_{t=1}^{T} Q_t\right]^+ - \textstyle\sum_{t=1}^{T} \mathbb{E}\left[hI_t^+ + bI_t^- + p_t Q_t\right].$$

Note that the first term is a constant. Therefore, we can write (2.1) as a cost minimization problem.

$$\begin{aligned} \min \quad & r\mathbb{E}\left[I_T^-\right] + \textstyle\sum_{t=1}^{T} \mathbb{E}\left[hI_t^+ + bI_t^- + p_t Q_t\right] \\ \text{s.t.} \quad & I_0 = 0,\ I_t = I_{t-1} + Q_t - D_t, && \forall t = 1, \ldots, T \\ & Q_t \geq 0, && \forall t = 1, \ldots, T. \end{aligned}$$

Then, using the inventory dynamics $I_t = \sum_{i=1}^{t}(Q_i - D_i)$ to replace the $I_t$'s in the objective function, we can rewrite the cost minimization problem as

$$\begin{aligned} \min \quad & C(\mathbf{Q}) = r\mathbb{E}\left[\textstyle\sum_{i=1}^{T}(Q_i - D_i)\right]^- \\ & \quad + \textstyle\sum_{t=1}^{T} \mathbb{E}\left[h\left(\sum_{i=1}^{t}(Q_i - D_i)\right)^+ + b\left(\sum_{i=1}^{t}(Q_i - D_i)\right)^- + p_t Q_t\right] \\ \text{s.t.} \quad & Q_t \geq 0, \quad \forall t = 1, \ldots, T. \end{aligned} \tag{2.2}$$

In this chapter, we assume that $D_i$ follows a continuous distribution with support on $[0, +\infty)$. Therefore, $C(\mathbf{Q})$ is a strictly convex function in $(Q_1, \ldots, Q_T)$ on $Q_i \geq 0$.

Thus, for any given $\mathbf{p}$, there is a unique optimal solution $\mathbf{Q}(\mathbf{p})$ for the retailer. In the following, we use $\mathbf{Q}^*(\mathbf{p})$ to denote the optimal purchasing strategy of the retailer under price $\mathbf{p}$.

### 2.2.3 Supplier's Problem

In our model, we assume the supplier knows the retailer's ordering strategy and cost parameters. The supplier's problem is to find the optimal pricing policy to maximize the revenue under the retailer's ordering strategy. The supplier's problem can be written as follows:

$$\begin{array}{ll} \max & S(\mathbf{p}) = \sum_{t=1}^{T} p_t Q_t^*(\mathbf{p}) \\ \text{s.t.} & p_t \geq 0, \quad \forall t = 1, \ldots, T. \end{array} \tag{2.3}$$

Having stated the model assumptions and formulations, we focus on the supplier and the retailer's decisions and revenues in the next section.

## 2.3 Analysis

In this section, we analyze the model and provide a procedure to solve for the optimal pricing policy. In addition, we show some properties of the optimal pricing policy. Then, there will be numerical results to demonstrate the revenue improvement from using the proposed pricing policy in Section 2.4.

### 2.3.1 Optimal Policy

To begin with, we study the optimal policy of the supplier and the retailer. We define the following notation. Let $\tilde{F}_t(\cdot)$ denote the CDF of the cumulative demand up to time $t$, i.e., $\sum_{i=1}^{t} D_i$. We have the following theorem.

**Theorem 1.** *Let* $p_t = \sum_{i=t}^{T-1} \Delta_i^* + p_T^*$ *for* $t = 1, \ldots, T-1$ *and* $p_T = p_T^*$*, where* $\Delta_t^*$ *is a maximizer for* $\Delta_t \tilde{F}_t^{-1}(\frac{b-\Delta_t}{h+b})$*, and* $p_T^*$ *is a maximizer for* $p_T \tilde{F}_T^{-1}(\frac{b-p_T+r}{h+b+r})$*. Then* $\mathbf{p} = (p_1, \ldots, p_T)$ *is an optimal pricing policy, and it satisfies* $p_1 \geq p_2 \geq \cdots \geq p_T$ *with* *strict inequalities hold if $b > 0$. Furthermore, the corresponding order quantities $\mathbf{Q}^*(\mathbf{p})$ are given by*

$$\begin{aligned}
&Q_1^* = \tilde{F}_1^{-1}(\tfrac{b-p_1+p_2}{h+b});\\
&Q_2^* = \tilde{F}_2^{-1}(\tfrac{b-p_2+p_3}{h+b}) - \tilde{F}_1^{-1}(\tfrac{b-p_1+p_2}{h+b});\\
&\dots\\
&Q_k^* = \tilde{F}_k^{-1}(\tfrac{b-p_k+p_{k+1}}{h+b}) - \tilde{F}_{k-1}^{-1}(\tfrac{b-p_{k-1}+p_k}{h+b});\\
&\dots\\
&Q_T^* = \tilde{F}_T^{-1}(\tfrac{b-p_T+r}{h+b+r}) - \tilde{F}_{T-1}^{-1}(\tfrac{b-p_{T-1}+p_T}{h+b}).
\end{aligned} \tag{2.4}$$

**Proof**. First, we prove that there exists an optimal $\mathbf{p}$ such that $\mathbf{Q}^*(\mathbf{p})$ has values given in (2.4). Given a set of $\mathbf{p}$, the KKT conditions of (2.2) are:

$$\begin{aligned}
\frac{\partial C}{\partial Q_t}(\mathbf{Q}^*) \quad &= \quad \sum_{i=t}^{T} \left\{ h\tilde{F}_i\left(\sum_{k=1}^{i} Q_k^*(\mathbf{p})\right) - b\left(1 - \tilde{F}_i\left(\sum_{k=1}^{i} Q_k^*(\mathbf{p})\right)\right)\right\} + p_t \\
&\quad -r\left(1 - \tilde{F}_T\left(\sum_{k=1}^{T} Q_k^*(\mathbf{p})\right)\right) \geq 0, \quad \forall t = 1, \dots, T \qquad (2.5)\\
\frac{\partial C}{\partial Q_t}(\mathbf{Q}^*) \quad &\cdot \quad Q_t^*(\mathbf{p}) = 0, \quad Q_t^*(\mathbf{p}) \geq 0, \quad \forall t = 1, \dots, T \qquad (2.6)
\end{aligned}$$

Let $\mathbf{p}$ be an optimal solution to problem (2.3), and $\mathbf{Q}^*(\mathbf{p})$ be the corresponding optimal solution to (2.2) under $\mathbf{p}$. Thus, $\mathbf{p}$ and $\mathbf{Q}^*(\mathbf{p})$ satisfy the KKT conditions (2.5) to (2.6). If $\partial C(\mathbf{Q}^*)/\partial Q_t = 0$ holds for all $t$, then solving the KKT conditions yields (2.4). Otherwise, we construct a new set of prices $\bar{\mathbf{p}} = (\bar{p}_1, \dots, \bar{p}_T)$ as follows:

$$\bar{p}_t = \begin{cases} p_t, & \text{if } \frac{\partial C}{\partial Q_t}(\mathbf{Q}^*) = 0,\\ \sum_{i=t}^{T}\left\{b - (h+b)\tilde{F}_i\left(\sum_{k=1}^{i} Q_k^*(\mathbf{p})\right)\right\} + r\left(1 - \tilde{F}_T\left(\sum_{k=1}^{T} Q_k^*(\mathbf{p})\right)\right), & \text{otherwise.}\end{cases}$$

We observe that $\mathbf{Q}^*(\mathbf{p})$ satisfies condition (2.5) with $\bar{C}$ replacing $C$. Here, we use $\bar{C}$ to denote the retailer's cost function in (2.2) where $p_t$'s are replaced by $\bar{p}_t$'s. Thus, $\mathbf{Q}^*(\mathbf{p})$ is also an optimal solution under price $\bar{\mathbf{p}}$.

In addition, the supplier's revenues under the two prices are the same, i.e., $S(\bar{\mathbf{p}}) = S(\mathbf{p})$. This is because $\bar{p}_t \neq p_t$ only when $Q_t(\mathbf{p}) = 0$. Therefore, we can conclude that there exists a $\mathbf{p}$ such that $\mathbf{Q}^*(\mathbf{p})$ satisfies (2.4).

Next, substituting (2.4) into (2.3), the objective function of (2.3) can be converted into

$$S(\mathbf{p}) = \sum_{t=1}^{T-1}(p_t - p_{t+1})\tilde{F}_t^{-1}\left(\frac{b - p_t + p_{t+1}}{h+b}\right) + p_T\tilde{F}_T^{-1}\left(\frac{b - p_T + r}{h+b+r}\right). \tag{2.7}$$

This objective function can be regarded as a function of variables $(p_t - p_{t+1})$'s and $p_T$. Instead of maximizing the original objective function, now we can maximize each $(p_t - p_{t+1})\tilde{F}_t^{-1}(\frac{b-p_t+p_{t+1}}{h+b})$ term separately, and thus we have the result for $\mathbf{p}$. Moreover, since all the demands are positive, $\tilde{F}_t^{-1}(\frac{b-p_t+p_{t+1}}{h+b})$'s must be positive as well. To maximize each term, we need $p_t - p_{t+1} \geq 0$. Therefore, we have $p_1 \geq p_2 \geq \cdots \geq p_T$. Furthermore, if $b > 0$, then there exists $p_t - p_{t+1}$ such that $(p_t - p_{t+1})\tilde{F}_t^{-1}(\frac{b-p_t+p_{t+1}}{h+b}) > 0$. Therefore, the optimal solution must satisfy that $p_t > p_{t+1}$. □

Theorem 1 indicates that the supplier should set the prices in the starting period the highest, and decrease the prices as time proceeds. Now we provide some explanation for this result. Intuitively, in our model, lowering the price in early periods will simply shift demand from later periods to early periods without increasing the total order quantity. This is because the retailer can expect leftovers from the excessive orderings in the early periods and reduce the ordering quantities in later periods. Meanwhile, lowering the prices in later periods could attract more demand in total, because it will shift the critical ordering point in the last period. Thus, lowering the prices in later periods is more appealing than lowering price in early periods for the supplier.

To further see this mathematically, we consider a two-period model and argue that the optimal prices must satisfy $p_1^* \geq p_2^*$. If $p_1^* < p_2^*$, then by increasing $p_1^*$ to $p_2^*$, the order quantity in the first period will decrease while the order quantity in the second period will increase. Meanwhile, a key observation is that the total order quantity only depends on $p_2^*$ according to the optimality condition (2.4). Thus, the total order quantity stays the same during the process. As a result, more units will be sold at the higher price $p_2^*$ instead of at the lower price $p_1^*$. Therefore, $p_1^* < p_2^*$ cannot be optimal. This argument can be applied similarly to the multi-period model and show iteratively that $p_t^* \geq p_{t+1}^*$ for $t = 1, \ldots, T-1$.

### 2.3.2 Unimodality of Supplier's Profit Function

As shown in Theorem 1, the profit optimization problem for the supplier can be written as

$$\max \sum_{t=1}^{T-1} \Delta_t \tilde{F}_t^{-1}\left(\frac{b-\Delta_t}{h+b}\right) + p_T \tilde{F}_T^{-1}\left(\frac{b-p_T+r}{h+b+r}\right). \tag{2.8}$$

Let $\tilde{Q}_t = \tilde{F}_t^{-1}((b-\Delta_t)/(h+b))$ and $\tilde{Q}_T = \tilde{F}_T^{-1}((b-p_T+r)/(h+b+r))$. Then, each term in (2.8) can be rewritten as $W_t(\tilde{Q}_t) = \tilde{Q}_t[b-(h+b)\tilde{F}_t(\tilde{Q}_t)]$ for $t = 1, \ldots, T-1$, and $W_T(\tilde{Q}_T) = \tilde{Q}_T[b+r-(h+b+r)\tilde{F}_T(\tilde{Q}_T)]$. In the following, we give conditions for this problem to be unimodal and thus it has a unique optimal solution. We shall use the notion of increasing generalized failure rate, which is defined by Lariviere and Porteus (2001) as below.

**Definition 1** (Increasing Generalized Failure Rate)**.** *A distribution with cumulative distribution function $F$ and density function $f$ is said to have increasing generalized failure rate (IGFR) if $g(x) = xf(x)/(1-F(x))$ is weakly increasing for all $x$ such that $F(x) < 1$.*

Now we are ready to state the theorem for the unimodality of $W_t(\tilde{Q}_t)$.

**Theorem 2.** *Suppose $\tilde{F}_t(\cdot)$ is IGFR. Then $W_t(\tilde{Q}_t)$ is unimodal on $[0, \infty)$.*

**Proof**. In the proof, we show the case for $t < T$. The case for $t = T$ can be proved in a similar way. The first order derivative of $W_t(\tilde{Q}_t)$ is $W_t'(\tilde{Q}_t) = b - (h+b)\tilde{F}_t(\tilde{Q}_t) - (h+b)\tilde{Q}_t\tilde{f}_t(\tilde{Q}_t)$. Setting $W_t'(\tilde{Q}_t) = 0$, we have

$$\frac{(h+b)\tilde{Q}_t\tilde{f}_t(\tilde{Q}_t)}{b-(h+b)\tilde{F}_t(\tilde{Q}_t)} = 1. \tag{2.9}$$

We argue that equation (2.9) has at most one solution. To see this, we rewrite it as follows:

$$\frac{(h+b)\tilde{Q}_t\tilde{f}_t(\tilde{Q}_t)}{1-\tilde{F}_t(\tilde{Q}_t)} \frac{1-\tilde{F}_t(\tilde{Q}_t)}{b-(h+b)\tilde{F}_t(\tilde{Q}_t)} = 1.$$

Since $\tilde{F}_t(\cdot)$ is IGFR, the first term is increasing in $\tilde{Q}_t$. The second term can be written as

$$\frac{1-\tilde{F}_t(\tilde{Q}_t)}{b-(h+b)\tilde{F}_t(\tilde{Q}_t)} = \frac{1}{h+b}\left(1+\frac{h}{b-(h+b)\tilde{F}_t(\tilde{Q}_t)}\right),$$

which is also increasing in $\tilde{Q}_t$ on $\tilde{F}_t(\tilde{Q}_t) < b/(b+h)$ (it is obvious that $W_t'(\tilde{Q}_t) = 0$ does not have a solution on $\tilde{F}_t(\tilde{Q}_t) \geq b/(b+h)$). Hence, the left-hand side of equation (2.9) is increasing in $\tilde{Q}_t$. Therefore, there exists at most one solution to the first order condition of $W_t(\tilde{Q}_t)$. Thus, $W_t(\tilde{Q}_t)$ is unimodal on $[0, +\infty)$. □

Recall that $\tilde{F}_t(\cdot)$ is the CDF of the cumulative demand up to time $t$. By Banciu and Mirchandani (2013), many classes of distributions would make $\tilde{F}_t(\cdot)$ satisfy the IGFR property (e.g., when $D_t$'s are i.i.d. normal distributions, standard uniform distributions or exponential distributions). Therefore, the condition in Theorem 2 is mild.

### 2.3.3 Determinants of Supplier's Price

In this section, we analyze how the demand distributions and cost parameters affect supplier's optimal prices. We first define some notation. Let $\bar{F}_t$, $\hat{F}_t$ denote the cumulative distribution function up to period $t$ for two sequences of demands $\bar{\mathbf{D}}$ and $\hat{\mathbf{D}}$, and $\bar{f}_t$ and $\hat{f}_t$ denote the corresponding p.d.f. Let $\bar{p}_t^*$ and $\hat{p}_t^*$ denote the optimal prices that maximize supplier's profit under demand $\bar{\mathbf{D}}$ and $\hat{\mathbf{D}}$ respectively. Define $\bar{\Delta}_t^* = \bar{p}_t^* - \bar{p}_{t+1}^*$ and $\hat{\Delta}_t^* = \hat{p}_t^* - \hat{p}_{t+1}^*$. We have the following theorem:

**Theorem 3.** *Suppose both $\bar{F}_t$ and $\hat{F}_t$ are IGFR. If $N(\xi) = \bar{F}_t^{-1}\left(\hat{F}_t(\xi)\right)/\xi$ is increasing in $\xi$ for all $\xi$ on $[0, +\infty)$, then $\bar{\Delta}_t^* \leq \hat{\Delta}_t^*$.*

In Lariviere and Porteus (2001), the authors called that $\bar{F}_t$ is greater than $\hat{F}_t$ in the star order (symbolically, $\bar{F}_t \geq_* \hat{F}_t$) if $N(\xi)$ is increasing in $\xi$. Therefore, Theorem 3 shows that if the cumulative demand function at time $t$ increases in the star order, then $\Delta_t^* = p_t^* - p_{t+1}^*$ decreases. Particularly, if all other parameters stay the same, then the prices in periods 1 to $t$ will be higher while the rest of the prices stay the same.

**Proof**. Let $\bar{Q}_t^*$ and $\hat{Q}_t^*$ denote the optimal total order quantity up to time $t$ under demand $\bar{\mathbf{D}}$ and $\hat{\mathbf{D}}$, respectively. From Theorem 1, we have $\bar{\Delta}_t^* = b - (h+b)\bar{F}_t(\bar{Q}_t^*)$ and $\hat{\Delta}_t^* = b - (h+b)\hat{F}_t(\hat{Q}_t^*)$. In the following, we show that $\bar{F}_t(\bar{Q}_t^*) \geq \hat{F}_t(\hat{Q}_t^*)$. By the optimality condition of $\bar{Q}_t^*$ and $\hat{Q}_t^*$, we have

$$b-(h+b)\bar{F}_t(\bar{Q}_t^*)-(h+b)\bar{Q}_t^*\bar{f}_t(\bar{Q}_t^*) = \bar{W}_t'(\bar{Q}_t^*) = 0 = \hat{W}_t'(\hat{Q}_t^*) = b-(h+b)\hat{F}_t(\hat{Q}_t^*)-(h+b)\hat{Q}_t^*\hat{f}_t(\hat{Q}_t^*),$$

where $\bar{W}_t(\bar{Q}_t) = \bar{Q}_t[b - (h+b)\bar{F}_t(\bar{Q}_t)]$ and $\hat{W}_t(\hat{Q}_t) = \hat{Q}_t[b - (h+b)\hat{F}_t(\hat{Q}_t)]$. Thus, we must have

$$\bar{F}_t(\bar{Q}_t^*) + \bar{Q}_t^* \bar{f}_t(\bar{Q}_t^*) = \hat{F}_t(\hat{Q}_t^*) + \hat{Q}_t^* \hat{f}_t(\hat{Q}_t^*).$$

Let $\bar{\alpha}^* = \bar{F}_t(\bar{Q}_t^*)$ and $\hat{\alpha}^* = \hat{F}_t(\hat{Q}_t^*)$. The above equation can be written as

$$\bar{\alpha}^* + \bar{F}_t^{-1}(\bar{\alpha}^*)\bar{f}_t(\bar{F}_t^{-1}(\bar{\alpha}^*)) = \hat{\alpha}^* + \hat{F}_t^{-1}(\hat{\alpha}^*)\hat{f}_t(\hat{F}_t^{-1}(\hat{\alpha}^*)). \tag{2.10}$$

Note that when $N(\xi)$ is increasing, we have

$$N'(\xi) = -\frac{1}{\xi^2}\bar{F}_t^{-1}\left(\hat{F}_t(\xi)\right) + \frac{1}{\xi}\frac{\hat{f}_t(\xi)}{\bar{f}_t\left(\bar{F}_t^{-1}\left(\hat{F}_t(\xi)\right)\right)} \geq 0.$$

Let $\alpha = \hat{F}_t(\xi)$, or $\xi = \hat{F}_t^{-1}(\alpha)$. Then the above inequality can be simplified to

$$\bar{F}_t^{-1}(\alpha)\bar{f}_t\left(\bar{F}_t^{-1}(\alpha)\right) \leq \hat{F}_t^{-1}(\alpha)\hat{f}_t\left(\hat{F}_t^{-1}(\alpha)\right). \tag{2.11}$$

Using $\hat{\alpha}^*$ to replace $\alpha$ in inequality (2.11), then combine it with equation (2.10), we have

$$\begin{aligned} \bar{\alpha}^* + \bar{F}_t^{-1}(\bar{\alpha}^*)\bar{f}_t\left(\bar{F}_t^{-1}(\bar{\alpha}^*)\right) &= \hat{\alpha}^* + \hat{F}_t^{-1}(\hat{\alpha}^*)\hat{f}_t\left(\hat{F}_t^{-1}(\hat{\alpha}^*)\right) \\ &\geq \hat{\alpha}^* + \bar{F}_t^{-1}(\hat{\alpha}^*)\bar{f}_t\left(\bar{F}_t^{-1}(\hat{\alpha}^*)\right). \end{aligned} \tag{2.12}$$

Now, we claim that $H(x) = x + \bar{F}_t^{-1}(x)\bar{f}_t(\bar{F}_t^{-1}(x))$ is increasing in $x$ when $x$ is between $\hat{\alpha}^*$ and $\bar{\alpha}^*$. Consider $1 - H(x)$, we have

$$\begin{aligned} 1 - H(x) &= 1 - x - \bar{F}_t^{-1}(x)\bar{f}_t(\bar{F}_t^{-1}(x)) \\ &= (1-x)\left(1 - \frac{\bar{F}_t^{-1}(x)\bar{f}_t(\bar{F}_t^{-1}(x))}{1-x}\right). \end{aligned}$$

The first term is decreasing in $x$, and we will show that the second term is positive between $\hat{\alpha}^*$ and $\bar{\alpha}^*$ and is also decreasing in $x$. To see the second term is decreasing, we note that $\frac{\bar{F}_t^{-1}(x)\bar{f}_t(\bar{F}_t^{-1}(x))}{1-x}$ is the generalized failure rate of $\bar{F}_t$ at $\bar{F}_t^{-1}(x)$, which is increasing in $x$ by the IGFR assumption. To see that it is positive, we first note that by the optimality condition $\bar{W}_t'(\bar{Q}_t^*) = 0$, we have

$$\left(1 - \bar{F}_t(\bar{Q}_t^*)\right)\left(1 - \frac{\bar{Q}_t^*\bar{f}_t(\bar{Q}_t^*)}{1 - \bar{F}_t(\bar{Q}_t^*)}\right) = \frac{h}{h+b}.$$

Therefore, we have

$$1-\frac{\bar{F}_t^{-1}(\bar{\alpha}^*)\bar{f}_t\left(\bar{F}_t^{-1}(\bar{\alpha}^*)\right)}{1-\bar{\alpha}^*}\geq 0.$$

Also, from inequality (2.11), we have,

$$\frac{\bar{F}_t^{-1}(\hat{\alpha}^*)\bar{f}_t\left(\bar{F}_t^{-1}(\hat{\alpha}^*)\right)}{1-\hat{\alpha}^*}\leq\frac{\hat{F}_t^{-1}(\hat{\alpha}^*)\hat{f}_t\left(\hat{F}_t^{-1}(\hat{\alpha}^*)\right)}{1-\hat{\alpha}^*}.$$

And because of the optimality condition $\hat{W}_t'(\hat{Q}_t^*)=0$, we have

$$\left(1-\hat{F}_t(\hat{Q}_t^*)\right)\left(1-\frac{\hat{Q}_t^*\hat{f}_t(\hat{Q}_t^*)}{1-\hat{F}_t(\hat{Q}_t^*)}\right)=\frac{h}{h+b},$$

which implies that

$$\frac{\hat{F}_t^{-1}(\hat{\alpha}^*)\hat{f}_t\left(\hat{F}_t^{-1}(\hat{\alpha}^*)\right)}{1-\hat{\alpha}^*}\leq 1.$$

As a result,

$$\left(1-\frac{\bar{F}_t^{-1}(x)\bar{f}_t(\bar{F}_t^{-1}(x))}{1-x}\right)\geq 0$$

at both $\bar{\alpha}^*$ and $\hat{\alpha}^*$. As proved earlier, it is decreasing in $x$; thus, it must be positive between $\bar{\alpha}^*$ and $\hat{\alpha}^*$. Therefore, $1-H(x)$ is decreasing in $x$ and $H(x)$ is increasing in $x$ on the interval between $\bar{\alpha}^*$ and $\hat{\alpha}^*$. Thus, combining (2.12), we conclude that $\bar{\alpha}^*\geq\hat{\alpha}^*$, which is equivalent to $\bar{F}_t(\bar{Q}_t^*)\geq\hat{F}_t(\hat{Q}_t^*)$. The rest of the theorem follows. □

Next, we study the effect of the backorder cost on the optimal prices. We have the following theorem:

**Theorem 4.** *If $\tilde{F}_t$ ($\tilde{F}_T$, resp.) is IGFR, then $\Delta_t^*$ ($p_T^*$, resp.) is increasing in $b$.*

**Proof**. By Theorem 2, if $\tilde{F}_t$ is IGFR, then $W_t(\tilde{Q}_t)$ is unimodal in $\tilde{Q}_t$. Since $\tilde{Q}_t$ is decreasing in $\Delta_t$, $\Delta_t\tilde{F}_t^{-1}\left(\frac{b-\Delta_t}{h+b}\right)$ is also unimodal. Thus, $\Delta_t^*$ must satisfy the first order condition

$$\frac{\tilde{F}_t^{-1}\left(\frac{b-\Delta_t^*}{h+b}\right)\tilde{f}_t\left(\tilde{F}_t^{-1}\left(\frac{b-\Delta_t^*}{h+b}\right)\right)}{1-\frac{b-\Delta_t^*}{h+b}}-\frac{\Delta_t^*}{h+\Delta_t^*}=0. \tag{2.13}$$

If $b$ increases, then $\frac{b-\Delta_t^*}{h+b}$ increases. Since $\tilde{F}_t$ is IGFR, the first term in (2.13) increases in $b$. Thus, the optimal $\Delta_t$ must also increase (because the derivative at $\Delta_t^*$ is greater than 0). Thus, $\Delta_t^*$ is increasing in $b$.

Similarly, if $\tilde{F}_T$ is IGFR, then $p_T^*$ must satisfy

$$\frac{\tilde{F}_T^{-1}\left(\frac{b+r-p_T^*}{h+b+r}\right)\tilde{f}_T\left(\tilde{F}_T^{-1}\left(\frac{b+r-p_T^*}{h+b+r}\right)\right)}{1-\frac{b+r-p_T^*}{h+b+r}}-\frac{p_T^*}{h+p_T^*}=0.$$

By the same arguments, $p_T^*$ increases in $b$. Thus, the theorem holds. $\square$

**Corollary 1.** *If $\tilde{F}_t$ is IGFR for all $t$, then $p_t^*$ is increasing in $b$ for all $t$.*

Next, we study a property of the optimal prices under some conditions on the distribution. In Theorem 1, we already showed that the optimal prices are decreasing in $t$. In the following, we study the relationship between the price differences.

**Theorem 5.** *Suppose $\tilde{F}_t$ and $\tilde{F}_{t+1}$ are IGFR. If $\tilde{F}_{t+1} \leq_* \tilde{F}_t$, then $\Delta_{t+1}^* \geq \Delta_t^*$.*

**Proof**. By (2.11), if $\tilde{F}_{t+1} \leq_* \tilde{F}_t$, then for any $\alpha$, $\tilde{F}_{t+1}^{-1}(\alpha)\tilde{f}_{t+1}(\tilde{F}_{t+1}^{-1}(\alpha)) \geq \tilde{F}_t^{-1}(\alpha)\tilde{f}_t(\tilde{F}_t^{-1}(\alpha))$. Therefore, by (2.13)

$$\frac{\tilde{F}_{t+1}^{-1}\left(\frac{b-\Delta_t^*}{h+b}\right)\tilde{f}_{t+1}\left(\tilde{F}_{t+1}^{-1}\left(\frac{b-\Delta_t^*}{h+b}\right)\right)}{1-\frac{b-\Delta_t^*}{h+b}}-\frac{\Delta_t^*}{h+\Delta_t^*}\geq 0.$$

Thus, by the same argument as in the proof of Theorem 4, we have $\Delta_{t+1}^* \geq \Delta_t^*$. $\square$

**Corollary 2.** *If $\tilde{F}_t$ is IGFR for all $t$ and $\tilde{F}_t$'s are ranked in reverse star order, i.e., $\tilde{F}_t \geq_* \tilde{F}_{t+1}$ for all $t$, then $\Delta_{t+1}^* \geq \Delta_t^*$ for all $t$.*

### 2.3.4 Constant Pricing Model

In this subsection, we study a special case of the multi-period model. We are interested in the retailer's ordering behavior when the supplier's prices are constant. Suppose the retailer faces i.i.d. demand in each period. The price offered by the supplier is the same through the entire horizon. The retailer still incurs holding costs or backorder costs for excessive inventory or shortage. Before stating the next theorem, we define a property for demand distributions.

**Definition 2** (Convex Sum Property)**.** *A distribution $X$ has convex sum property at $\eta$ if the following inequalities hold at $\eta$*

$$2F_1^{-1}(\eta) < F_2^{-1}(\eta) \quad \textit{and} \quad 2F_t^{-1}(\eta) < F_{t-1}^{-1}(\eta) + F_{t+1}^{-1}(\eta), \quad \forall t \geq 2. \tag{2.14}$$

*where $F_t^{-1}(\cdot)$ is the inverse CDF of the sum of $t$ i.i.d. $X$. We say a distribution has concave sum property if $-X$ has convex sum property.*

We have the following results for normal distribution.

**Lemma 1.** *Normal distribution has convex sum property at $\eta$ for all $\eta \in [0, 0.5]$, and concave sum property for all $\eta \in [0.5, 1]$.*

**Proof**. Let $X \sim N(\mu, \sigma^2)$. Then, $F_t^{-1}(\cdot)$ is the inverse CDF of $N(t\mu, t\sigma^2)$, i.e.,

$$F_t^{-1}(\eta) = t\mu + \Phi^{-1}(\eta)\sqrt{t}\sigma,$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. If $\eta \in [0, 0.5]$, then we have

$$2F_1^{-1}(\eta) = 2\mu + 2\Phi^{-1}(\eta)\sigma < 2\mu + \sqrt{2}\Phi^{-1}(\eta)\sigma = F_2^{-1}(\eta),$$

and

$$\begin{aligned}
2F_t^{-1}(\eta) &= 2\sqrt{t}\Phi^{-1}(\eta)\sigma + 2t\mu \\
&< \sqrt{t-1}\Phi^{-1}(\eta)\sigma + (t-1)\mu + \sqrt{t+1}\Phi^{-1}(\eta)\sigma + (t+1)\mu \\
&= F_{t+1}^{-1}(\eta) + F_{t-1}^{-1}(\eta).
\end{aligned}$$

Thus, normal distribution has convex sum property for $\eta \in [0, 0.5]$. By symmetry, normal distribution has concave sum property for $\eta \in [0.5, 1]$. □

Now we are ready to state the theorem.

**Theorem 6.** *Suppose the demand distributions have concave sum property at $\frac{b}{b+h}$. Under constant pricing, the retailer's optimal order quantities are decreasing in $t$, $\forall t = 1, \ldots, T-1$, i.e.*

$$Q_1^* > Q_2^* > \cdots > Q_{T-1}^*.$$

*Similarly, suppose the demand distributions have convex sum property at* $\frac{b}{b+h}$. *Under constant pricing, the retailer's optimal order quantities are increasing in t,* $\forall t = 1, \ldots, T-1$, *i.e.*

$$Q_1^* < Q_2^* < \cdots < Q_{T-1}^*.$$

**Proof**. When all $Q_i$s are positive, we can compute $Q_i$s from the first order optimality condition,

$$Q_t^* = F_t^{-1}(\tfrac{b}{h+b}) - F_{t-1}^{-1}(\tfrac{b}{h+b}) \quad \text{and} \quad Q_T^* = F_T^{-1}(\tfrac{b+r-p}{h+b+r}) - F_{T-1}^{-1}(\tfrac{b}{h+b}). \tag{2.15}$$

Now we will show $Q_1^* > Q_2^*$. By definition of the concave sum property, we have

$$2F_1^{-1}(\frac{b}{h+b}) > F_2^{-1}(\frac{b}{h+b}).$$

Therefore, we have showed $Q_1^* > Q_2^*$.

For $t \geq 2$, again by the definition of the concave sum property, we have

$$2F_t^{-1}(\frac{b}{h+b}) > F_{t-1}^{-1}(\frac{b}{h+b}) + F_{t+1}^{-1}(\frac{b}{h+b}),$$

which is equivalent to $Q_t^* > Q_{t+1}^*$, for $2 < t < T-2$. Therefore, the first part of the theorem is proved. The proof for the second part is very similar to that of the first part with inequality signs in the opposite direction. □

Theorem 6 suggests that under constant pricing, the optimal order quantities are decreasing if the demand distribution has concave sum property at certain points. This result is somewhat surprising at first because if the demand distribution and prices are identical in each period, it is natural to think that the order quantities should be the same throughout the horizon. However, when the distribution has concave sum property, the cumulated demand becomes less variable as time proceeds. For a fixed critical ratio, the newsvendor should order more aggressively in the earlier period, while more conservative in the later periods. The excessive inventory from the earlier periods can be used as a buffer for the uncertain demand. When the holding cost is very high, carrying inventory becomes costly. To maximize the expected profit for the items, the retailer should order less in the early periods to avoid holding costs. In an extreme case, when the backorder cost is zero, the retailer should backorder all the demand in the first $T-1$ periods and wait until the last period to satisfy all the demand.

The proof gives another interesting result. From (2.15) we can see that when parameters $h, b$ are fixed, retailer's order quantities $Q_t, \forall t = 1, \ldots, T-1$ does not depend on $p$. So the supplier's price only affects the last period's order quantity. This is due to the exogenous demand. As long as the demand distribution stays the same, the optimal order quantity should not be affected by price $p$. Intuitively, retailer's problem is to balance the overage cost and shortage cost in each period. When $h$ and $b$ are fixed, increasing $p$ only increases the overage cost in the last period, so the retailer would prefer to order less in the final period.

When the demand at the retailer's end is normally distributed with mean $\mu$, and standard deviation $\sigma$, the following result will hold.

**Corollary 3.** *When the demand in each period is normally distributed, then under constant pricing, if $b > h$, then the retailer's optimal order quantities are decreasing in $t$, $\forall t = 1, \ldots, T$, i.e. $Q_1^* > Q_2^* > \cdots > Q_{T-1}^* > \mu$; if $b < h$, then the retailer's optimal order quantities are increasing in $t$, $\forall t = 1, \ldots, T$, i.e. $Q_1^* < Q_2^* < \cdots < Q_{T-1}^* < \mu$.*

*In addition, if $\sqrt{T}\Phi^{-1}(\frac{b+r-p}{h+b+r}) > \sqrt{T-1}\Phi^{-1}(\frac{b}{h+b})$, then we have $Q_T^* > \mu$. Otherwise, $Q_T^* < \mu$.*

**Proof**. We will show the proof of case 1 where $b > h$. To get the proof for case 2, one can just change the inequality signs in the proof of case 1. It is shown in Lemma 1 that normal distribution has convex sum property on $\eta \in [0.5, 1]$. Thus, Theorem 6 applies here. Now, we will show $Q_{T-1}^* > \mu$, which is equivalent to

$$\Phi^{-1}(\frac{b}{h+b})\sigma_{T-1} + \mu_{T-1} - \Phi^{-1}(\frac{b}{h+b})\sigma_{T-2} - \mu_{T-2} > \mu.$$

Since $b > h$, $\eta = b/(h+b) \in [0.5, 1]$. Rearranging terms, we will see this holds for all $T \geq 2$.

Lastly, we will show the conditions for $Q_T^*$. From (2.15) we have,

$$Q_T^* = \sigma_T \Phi^{-1}(\tfrac{b+r-p}{h+b+r}) - \sigma_{T-1}\Phi^{-1}(\tfrac{b}{h+b}) + \mu.$$

Therefore, if $\sqrt{T}\Phi^{-1}(\frac{b+r-p}{h+b+r}) > \sqrt{T-1}\Phi^{-1}(\frac{b}{h+b})$, then $Q_T^* > \mu$. Otherwise, $Q_T^* < \mu$. $\square$

When $b > h$, it is more beneficial for the retailer to overstock than understock. To avoid understock, the retailer should place larger orders in the earlier period and gradually increase the order quantity over time. Note that inventory from the earlier periods has more opportunities to be sold, so we also take into consideration the number of selling periods ahead when computing the order quantity. Consider an extreme case where $h = 0$ and $b$ is very large. In this case, the retailer incurs no holding cost for inventory. It is obvious that the optimal solution is to place a large order for the entire planning horizon in the first period and deplete the inventory gradually.

When $b < h$, it is more beneficial for the retailer to understock than overstock. To avoid overstock, retailer should place smaller orders in the earlier period and gradually increase the order quantities as demand realizes overtime. An extreme case is when $b = 0$ and $h$ is very large. In this case, the retailer incurs no backorder cost for not satisfying demand. The optimal strategy in this case is to wait for the last period to place order for all the backordered demand.

In the final period, unsatisfied demand remains unsatisfied but incurs backorder costs. The leftover inventory has no value and incurs extra disposal costs equal to $h$. When the backorder cost and expected sales is low, the understock cost is low. As a result, the retailer tends to be conservative when ordering, resulting in a order quantity smaller than the mean. When the backorder cost and expected sales is high, the understock cost is high. The retailer tends to order more to reduce the probability of stockout. Thus, the order quantity is greater than the mean.

## 2.4 Numerical Experiment

In this section, we conduct some numerical experiments to illustrate our results. We consider a five-period model in our experiment and set $h = 1, b = 1, r = 20$, and $F_i(\cdot) \sim \Gamma(0.5, 30)$ as the base problem. According to Banciu and Mirchandani (2013), Gamma distributions are IGFR. Also, by Shaked and Shanthikumar (2007) and van

Zwet (1964), Gamma distributions $X$ and $Y$ with the same scale parameter and shape parameters $\theta$ and $\gamma$, respectively, satisfy that if $\theta \leq \gamma$, then $X \geq_* Y$. Therefore, the test problems satisfy the conditions in Theorems 4 and 5.

In the numerical experiments, we calculate the optimal price curves as we vary $b$ or $h$ while keeping the other parameters as in the base problem. The results are shown in Figures 2.1 and 2.2. From Figure 2.1, we can see that the optimal prices are decreasing in time as Theorem 1 suggests, and as $b$ increases, the price path lies higher on the plot, which is consistent with Theorem 4. Similarly, in Figure 2.2, we see that the prices are decreasing in time in all cases, and as $h$ increases, the price path lies higher on the plot. Also, both figures show that as $t$ increases, the differences between consecutive prices are increasing, i.e., $p_t - p_{t-1} \leq p_{t+1} - p_t$, which is consistent with Theorem 5.

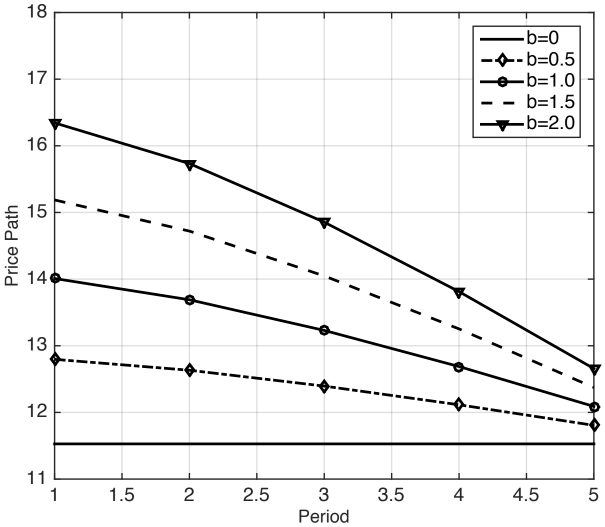

Figure 2.1: Price path for various $b$ with $h = 1$

In the next two figures, we keep the parameters fixed as the base problem and change $h$ or $b$ to see the effect of the parameter on the supplier's and retailer's revenue. Figure 2.3 and 2.4 shows the plot of supplier's and retailer's revenue for various $b$ and $h$. The plot on the left shows that as $b$ increases, the retailer's revenue decreases, but is always positive. The supplier's revenue is increasing in $b$. Figure 2.4 shows that both supplier's

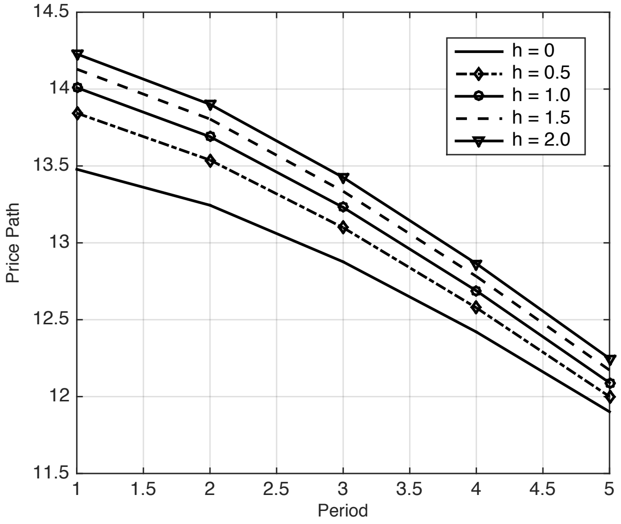

Figure 2.2: Price path for various $h$ with $b = 1$

and retailer's revenue is decreasing in $h$ for $h > 0$.

Next, we compare the case when the supplier uses the optimal pricing policy versus a constant pricing policy. Particularly, we still choose $h = 1, b = 1, r = 20$, and $F_i(\cdot) \sim \Gamma(0.5, 30)$ to be the base problem, and vary parameter $b$ or $h$ in the experiment. Under each parameter, we compute the optimal revenue by using the optimal pricing policy and the best constant pricing policy, and study the gap between them. The best constant pricing policy is computed by enumerating a range of prices and computing the corresponding $Q_t^*$ using the KKT conditions (2.5) to (2.6) and choosing the one that achieves the highest revenue. The price that gives the largest $p(\sum_{t=1}^{T} Q_t)$ is the optimal constant price. The results are shown in Tables 2.1 and 2.2. In the tables, we use $R^*$ to denote the revenue under optimal prices, $R_c^*$ to denote the revenue under optimal constant prices, and $p_c^*$ to denote the optimal constant price. The percentage revenue differences $(1 - R_c^*/R^*) \times 100\%$ are shown in the parentheses next to the constant pricing revenue.

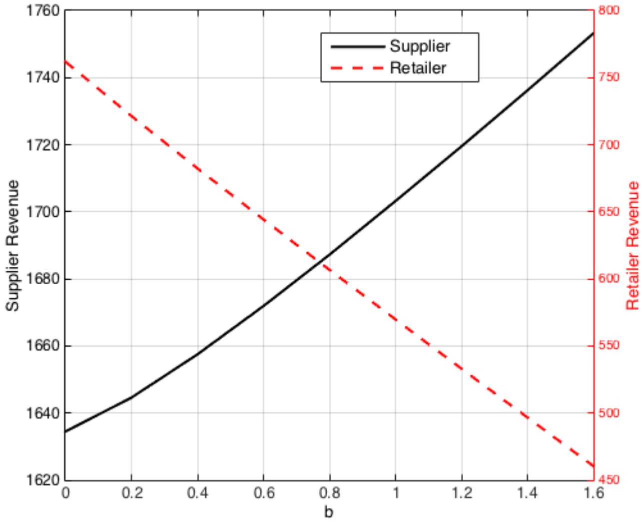

Figure 2.3: Supplier and retailer's revenue for various $b$

| $b$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $R^*$ | 636.08 | 663.12 | 698.26 | 736.94 | 777.61 |
| $R_c^*$ | 636.08 (0.00%) | 652.54 (1.60%) | 669.00 (4.19%) | 694.10 (5.81%) | 723.01 (7.02%) |
| $p_c^*$ | 11.52 | 11.81 | 12.09 | 13.24 | 13.78 |

Table 2.1: Revenue difference between optimal pricing policy and constant pricing policy for various $b$

From Tables 2.1 and 2.2, we can see that using the optimal pricing policy could lead to significant revenue improvement over the best constant pricing policy. Particularly, the difference is larger when $b$ is large, while the difference is relatively stable in $h$.

## 2.5 Distribution-Free Multi-Period Newsvendor Model

The model in the previous section requires a known demand distribution. However, in most cases, the distribution information is very limited. Sometimes, only the mean and variance are accessible. In this section, we study the optimal ordering problem in a multi-period newsvendor model under a worst-case demand distribution. In particular, we derive a second-order cone programming (SOCP) formulation to calculate the optimal

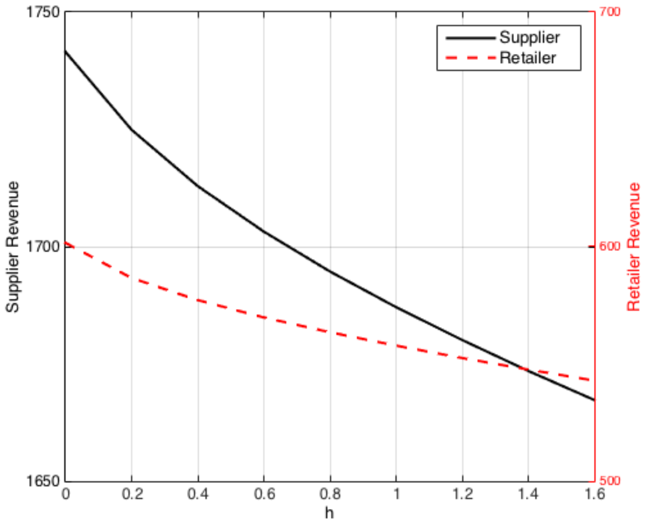

Figure 2.4: Supplier and retailer's revenue for various $h$

| $h$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $R^*$ | 751.81 | 718.37 | 698.26 | 682.47 | 668.92 |
| $R_c^*$ | 744.89 (0.92%) | 689.90 (3.96%) | 669.00 (4.19%) | 658.10 (3.57%) | 647.77 (3.16%) |
| $p_c^*$ | 13.11 | 12.56 | 12.09 | 12.17 | 12.25 |

Table 2.2: Revenue difference between optimal pricing policy and constant pricing policy for various $h$

ordering quantity under this model and obtain some characteristics for the worst-case distribution.

Before we proceed, we briefly review the related literature about this problem. In Scarf (1958), Scarf addressed the single-period newsvendor problem with demand information of only the mean and variance. Later, Gallego and Moon (1993) reviewed his contribution and extended the results to other modified newsvendor models. We state the Scarf's Rule for the single-period newsvendor model from Scarf (1958).

**Proposition 1. (Scarf's Rule)** *Given $b + r > p$, in the distribution-free model, the* *retailer's optimal order quantity is*

$$Q^s = \mu + \frac{\sigma}{2}\left(\sqrt{\frac{b-p+r}{h+p}} - \sqrt{\frac{h+p}{b-p+r}}\right). \tag{2.16}$$

There exists a two-point distribution that gives $Q^s$ as the optimal order quantity.

This distribution-free model can be extended to multi-period problem with a total of $T$ periods. Let $\mathbf{x} = (x_1, \ldots, x_T)$ be the random variable for demand in each period. Suppose the demands are unknown except that the marginal mean and variance in period $t$ are $\mu_t$ and $\sigma_t$, respectively. In the following, we denote this set of distributions for the demand by $\mathcal{H}$. The retailer's problem can be written as

$$\begin{aligned}\min_{Q_i \geq 0} \sup_{F \in \mathcal{H}} \quad & \{\sum_{t=1}^{T}((T+1-t)h + p_t)Q_t + (h+b)\sum_{t=1}^{T-1}\mathbb{E}(\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} Q_1)^+ \\ & +(h+b+r)\mathbb{E}(\sum_{i=1}^{T} x_i - \sum_{i=1}^{T} Q_i)^+\}\end{aligned} \tag{2.17}$$

Let $F(\mathbf{x})$ be the CDF of the joint distribution of the demands for period 1 to $T$. Then, given $Q_i$s, the worst-case distribution can be computed via the following problem,

$$\begin{array}{ll} \max_{F(\mathbf{x})} & (h+b)\sum_{t=1}^{T-1}\int_{\mathbb{R}^T} C_t(\mathbf{Q},\mathbf{x})dF(\mathbf{x}) + (h+b+r)\int_{\mathbb{R}^T} C_T(\mathbf{Q},\mathbf{x})dF(\mathbf{x}) \\ \text{s.t.} & \int_{\mathbb{R}^T} x_t dF(\mathbf{x}) = \mu_t, \quad \forall t = 1, \ldots, T \\ & \int_{\mathbb{R}^T} x_t^2 dF(\mathbf{x}) = \mu_t^2 + \sigma_t^2, \quad \forall t = 1, \ldots, T \\ & \int_{\mathbb{R}^T} dF(\mathbf{x}) = 1, \\ & dF(\mathbf{x}) \geq 0, \end{array} \tag{2.18}$$

where $C_t(\mathbf{Q},\mathbf{x}) = (\sum_{i=1}^{t} x_i - \sum_{i=1}^{t} Q_i)^+$.

We use dual variables $\boldsymbol{y}, \boldsymbol{z}, \lambda$ for the equality constraints in (2.18) and obtain its dual problem,

$$\begin{array}{ll} \min_{\boldsymbol{y},\boldsymbol{z},\lambda} & \sum_{t=1}^{T}\mu_t y_t + \sum_{t=1}^{T}(\mu_t^2+\sigma_t^2)z_t + \lambda \\ \text{s.t.} & \sum_{t=1}^{T}(x_t y_t + x_t^2 z_t) + \lambda \geq (h+b)\sum_{t=1}^{T-1} C_t(\mathbf{Q},\mathbf{x}) + (h+b+r)C_T(\mathbf{Q},\mathbf{x}), \forall \mathbf{x}. \end{array} \tag{2.19}$$

Let $\Omega = \{0,1\}^n$ denote the set of all possible $0,1$ combination vectors with $n$ dimensions, and $\omega_i$ represent the $i$th element in the vector $\boldsymbol{\omega} \in \Omega$. The total number of vectors in $\Omega$ is $2^n$. The constraints in the dual problem (2.19) can be expanded into $2^T$ constraints as follows:

$$\sum_{t=1}^{T}(x_t y_t + x_t^2 z_t) + \lambda - (h+b)\sum_{t=1}^{T-1}\omega_t \sum_{i=1}^{t}(x_i - Q_i) - (h+b+r)\omega_T \sum_{i=1}^{T}(x_i - Q_i) \geq 0,\ \forall \mathbf{x}, \boldsymbol{\omega}. \tag{2.20}$$

Note that each constraint is quadratic in $\mathbf{x}$. By choosing $\mathbf{x}$ that minimizes the quadratic function, we can further transform the constraints to the following:

$$\begin{aligned} -\sum_{t=1}^{T}\frac{[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T - y_t]^2}{4z_t} + \lambda & \\ +\sum_{t=1}^{T} Q_t\left[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T\right] &\geq 0, \quad \forall \boldsymbol{\omega} \qquad (2.21) \\ z_t &\geq 0. \end{aligned}$$

The new set of constraints are convex sets. By replacing the fractions with new variables, we can further write (2.21) in an equivalent second-order cone program (SOCP).

$$\begin{array}{ll} \min_{\mathbf{y},\mathbf{z},\lambda,\mathbf{s}} & \sum_{t=1}^{T}\mu_t y_t + \sum_{t=1}^{T}(\mu_t^2 + \sigma_t^2) z_t + \lambda \\ \text{s.t.} & \sum_{t=1}^{T} s_{\omega t} \leq 4\lambda + 4\sum_{t=1}^{T} Q_t\left[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T\right], \quad \forall \boldsymbol{\omega} \\ & 4[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T - y_t]^2 + (z_t - s_{\omega t})^2 \leq (z_t + s_{\omega t})^2, \quad \forall s_{\omega t} \\ & z_t \geq 0. \end{array} \tag{2.22}$$

Together with the outer optimization problem in (2.17), the multi-period distribution-free problem can be written as:

$$\begin{array}{ll} \min_{\mathbf{Q},\mathbf{y},\mathbf{z},\lambda,\mathbf{s}} & \sum_{t=1}^{T}\mu_t y_t + \sum_{t=1}^{T}(\mu_t^2 + \sigma_t^2) z_t + \lambda + \sum_{t=1}^{T}(\sum_{i=1}^{T-1} h + p_t) Q_t \\ \text{s.t.} & \sum_{t=1}^{T} s_{\omega t} \leq 4\lambda + 4\sum_{t=1}^{T} Q_t\left[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T\right], \quad \forall \boldsymbol{\omega} \\ & 4[(h+b)\sum_{i=t}^{T-1}\omega_i + (h+b+r)\omega_T - y_t]^2 + (z_t - s_{\omega t})^2 \leq (z_t + s_{\omega t})^2, \forall s_{\omega t} \\ & z_t \geq 0,\ Q_t \geq 0. \end{array} \tag{2.23}$$

From the solution of (2.22), we can compute the worst-case distribution function, and (2.23) gives the optimal order quantities under the worst-case distribution. The next theorem on the worst-case distribution function hold for problems with $T$ periods.

**Theorem 7.** *In a T-period distribution-free model, the worst-case demand distribution has at most* $T+1$ *points.*

**Proof**. First, by the complementarity conditions, if the optimal solution to (2.18) has a strictly positive mass at $\mathbf{x}$, then it must be that the corresponding constraint in (2.19) holds as equality for that $\mathbf{x}$ at an optimal solution $(\mathbf{y}^*, \mathbf{z}^*, \lambda^*)$. In the following, we show that for any dual optimal solution $(\mathbf{y}^*, \mathbf{z}^*, \lambda^*)$, there exist at most $T+1$ $\mathbf{x}$s such that constraints in (2.19) hold as equality.

To show this, it suffices to show that for any given dual optimal solution $(\mathbf{y}^*, \mathbf{z}^*, \lambda^*)$, there exist at most $T+1$ $\boldsymbol{\omega} \in \Omega$ such that (2.21) holds as equality. This is because, first, $z_t^* > 0$ at optimal for all $t$. Otherwise (2.19) cannot hold for all $\mathbf{x}$. Then, given a $\boldsymbol{\omega} \in \Omega$, the left hand side of (2.20) is a strictly convex function of $\mathbf{x}$. So, a unique minimizer exists. Thus, for each $\boldsymbol{\omega}$, there is only one possible $\mathbf{x}$ such that (2.20) hold as equality. Therefore, the number of $\mathbf{x}$s that make the constraints in (2.19) hold as equality is no more than the number of $\boldsymbol{\omega}$s that makes (2.21) hold as equality.

Finally, we show that there are at most $T+1$ $\boldsymbol{\omega} \in \Omega$ such that (2.21) holds as equality. In the following, we use $(\boldsymbol{\omega})$ to denote the value of the left hand side of (2.21) for that $\boldsymbol{\omega}$. We will show that if $(\boldsymbol{\omega}_1) = (\boldsymbol{\omega}_2) = 0$, it must be $\boldsymbol{\omega}_1 \leq \boldsymbol{\omega}_2$ or $\boldsymbol{\omega}_2 \leq \boldsymbol{\omega}_1$, where "$\leq$" means component-wise. Apparently, if this is true, then there will be at most $T+1$ $\boldsymbol{\omega}$s which satisfy $(\boldsymbol{\omega}) = 0$.

Consider $(\boldsymbol{\omega}_1) = (\boldsymbol{\omega}_2) = 0$ and $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$. Let $\bar{\boldsymbol{\omega}} = \boldsymbol{\omega}_1 \vee \boldsymbol{\omega}_2$ and $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2$ where $\vee$ and $\wedge$ mean component-wise maximum and minimum. If neither $\boldsymbol{\omega}_1 \leq \boldsymbol{\omega}_2$ nor $\boldsymbol{\omega}_2 \leq \boldsymbol{\omega}_1$ hold, then $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \bar{\boldsymbol{\omega}}, \underline{\boldsymbol{\omega}}\}$ are four distinct vectors. We first argue that we must have $(\bar{\boldsymbol{\omega}}) = (\underline{\boldsymbol{\omega}}) = 0$, too. This is because the left hand side of (2.21) is submodular in $\boldsymbol{\omega}$ when $z_t^* > 0$ (it is easy to verify that all cross-partial derivatives for $\boldsymbol{\omega}$ are less than 0). Therefore, the minimizers of the left hand sides of (2.21) must form a sublattice Topkis (1978). Note that when $(\boldsymbol{\omega}_1) = (\boldsymbol{\omega}_2) = 0$, they must be minimizers of the left hand side of (2.21) since it cannot be less than zero on $[0,1]^n$. Therefore, if $(\boldsymbol{\omega}_1) = (\boldsymbol{\omega}_2) = 0$, $\bar{\boldsymbol{\omega}}$

and $\underline{\boldsymbol{\omega}}$ must also be minimizers, i.e. $(\bar{\boldsymbol{\omega}}) = (\underline{\boldsymbol{\omega}}) = 0$. However, we have

$$\begin{aligned} & (\bar{\boldsymbol{\omega}}) - (\boldsymbol{\omega}_1) - [(\boldsymbol{\omega}_2) - (\underline{\boldsymbol{\omega}})] \\ = \quad & -\sum_{t=1}^{T} \frac{2[(h+b)\sum_{i=t}^{T-1}(\omega_{1i} - \underline{\omega}_i) + (h+b+r)(\omega_{1T} - \underline{\omega}_T)]}{4z_t^*}[(h+b)\sum_{i=t}^{T-1}\bar{\omega}_i \\ & -\omega_{1i} + (h+b+r)(\bar{\omega}_T - \omega_{1T})] \\ < \quad & 0, \end{aligned}$$

where the first equality is because $\bar{\omega}_i - \omega_{1i} = \omega_{2i} - \underline{\omega}_i$ for all $i$, and the last inequality is because $z_t^* > 0$ and $\bar{\omega}_i \geq \omega_{1i}$ for all $i$ while $\bar{\omega}_i > \omega_{1i}$ for at least one $i$ (since $\bar{\boldsymbol{\omega}} \neq \boldsymbol{\omega_1}$). This contradicts with $(\bar{\boldsymbol{\omega}}) = (\underline{\boldsymbol{\omega}}) = 0$. Thus, if $(\boldsymbol{\omega}_1) = (\boldsymbol{\omega}_2) = 0$, it must be either $\boldsymbol{\omega}_1 \leq \boldsymbol{\omega}_2$ or $\boldsymbol{\omega}_2 \leq \boldsymbol{\omega}_1$. Therefore, the theorem is proved. $\square$

Therefore, Theorem 7 combined with formulation (2.23) presents an extension of Scarf's result to the multi-period case. Particularly, one can view Scarf's result as a special case of our reslut when $T = 1$.

## 2.6 Conclusion

In this chapter, we studied the optimal pricing problem for selling to a multi-period newsvendor. We derived a procedure for solving the optimal prices in the problem, as well as some comparative statics. We also find that the optimal pricing sequence is decreasing in time under our model. In addition, in the constant pricing model, the retailer's optimal order quantity should be decreasing over time. The last part of this chapter focuses on a distribution-free multi-period newsvendor model and shows a property of the worst-case demand distribution.

One important future research direction is to consider the amount of time each item has been sitting in the inventory. Another way to describe this is the time before expiration. Although one motivation of our model is from the fresh produce industry in which shelf life is important, we have not explicitly taken that into account in our model. One way to incorporate inventory shelf life is to track the shelf life of each unit, and expire the unit after a certain day. However, this will lead to much more complicated models and solutions because each unit needs to be tracked individually in every period instead of the total inventory level. Rather than using $I_t$ to denote the inventory level in period $t$, the inventory level should be characterized as $I_t^n$, where $n$ represents the number of periods this inventory is in the system. It would be interesting to study how the optimal pricing and ordering strategy would change in this case.

# Chapter 3

# Supply Chain Risk and Return Analysis

## 3.1 Introduction and Literature Review

This chapter focuses on the risk analysis in a supplier-retailer relationship. Risk-return analysis has long been applied to study inventory control and pricing policies. For example, Bouakiz and Sobel (1992), Buzacott et al. (2011) performed a risk analysis for different types of inventory control models. He and Zhang (2008), Lau and Lau (1999) studied risk analysis for supply chain contracts, pricing and return policies. There are also papers on supply chain channel coordination Choi et al. (2008). For risk analysis for the newsvendor, Chen and Federgruen (2010) used the mean-variance approach to analyze the classic newsvendor model without considering the backorder cost. Eeckhoudt et al. (1995) studied the effect of risk aversion for a newsvendor model without backorder cost. Lau (1980) used the mean-standard deviation approach to study the newsvendor model. Two papers that considered backorder costs in the risk analysis and provided some useful results for the risk-averse newsvendor model are Wang and Webster (2009), Wu et al. (2009). In Wang and Webster (2009), the objective function used is a loss aversion utility function. Their main result states that a loss-averse newsvendor will order less than a risk neutral newsvendor if he faces low shortage cost, and order more if he faces high shortage cost. In Wu et al. (2009), the authors adopt the mean-variance tradeoff approach. They derived the profit function variance and studied its properties. In addition, their numerical tests showed that mean-variance analysis leads to a lower optimal order quantity and a lower optimal value. Ozler et al. (2009) studied a single-period multi-product newsvendor model using value-at-risk approach. They provided exact distribution function for the two-product newsvendor problem and an approximation method for $N$-product case. Rubio-Herrero et al. (2015) considered a single-period single product newsvendor who has the decision of order quantity and selling price. They used a mean-variance approach and their objective is to maximize the expected revenue minus variance with a scaled parameter. They proved concavity of the objective function and existence of a unique optimal solution.

The classic newsvendor model adopts a wholesale-price contract. In this model, the retailer purchases inventory from the supplier at the wholesale price $p$ then sells it to the mass consumer at the retail price $r$. There are no other transactions between the supplier and the retailer. The wholesale-price contract is common and easy to understand. We refer the readers to Cachon (2003) for a brief introduction and do not further elaborate here. One thing to point out here is that the supplier in a wholesale-price contract does not face any uncertainty. Thus, the supplier's revenue is generally low compared with the retailer's expected profit. In the second part of this chapter, we include a spot market in the classic newsvendor model. A spot market is a place where the consumer can purchase the item if it is out of stock at the retailer's side. There is existing literature on the supply chain spot market purchasing strategy (see Haksöz and Seshadri (2007)), most of which discuss how retailers should strategically mix fixed-term contracts with spot market purchases to maximize profit or minimize purchase cost. Seifert et al. (2004) derived the optimal ordering strategy to purchase from the regular contract and the spot market. The paper also discusses the case where the retailer could both buy and sell on the spot market. Golovachkina (2003) studied the buyer-seller coordination in a spot market. This paper assumes the spot market price is stochastic and there is an infinite supply. They derived the optimal strategy for the seller and showed how the market can be coordinated using contracts. We focus specifically on the supplier's problem. We assume the supplier provides inventory to both the retailer and the spot market. The supplier can choose the wholesale price, but the spot market price is the retail price. We are also interested in the supply chain risk sharing when there exists a spot market.

Another model we study in this chapter is the revenue-sharing contract. According to Lariviere (1999), Cachon and Lariviere (2005), and Yao et al. (2008), price-only contracts cannot coordinate the supply chain. The optimal price depends on the demand distribution's coefficient of variation. In a revenue-sharing contract, the retailer pays the supplier a wholesale price upon purchase, plus a percentage $(1 - \phi)$ of the revenue. Cachon and Lariviere (2005) studied the revenue-sharing contracts and showed how revenue-sharing contracts coordinate the supply chain. The authors also discussed some limitations of the revenue-sharing contracts which restrained its popularity. Some literature compares the revenue-sharing contract with other types of contracts. A work that is closely related to ours is Pasternack (2002). The author compared the classic wholesale-price contract newsvendor model with a revenue-sharing contract model. Two types of contracts are presented to the retailer. The retailer chooses the best contract(s) to use. This work assumes that the wholesale price is fixed but the supplier could control the price in the revenue-sharing contract. The optimal ordering strategy of which contract(s) to choose under certain conditions and the optimal order quantities are presented. It also investigates how to achieve supply chain coordination using contracts. Our work differs from this work in the following aspects. First, our model does not involve choosing between contracts. The retailer is presented with the revenue-sharing contract only. Second, we focus on the supplier's optimal pricing strategy as well as the retailer's ordering strategy in the presence of $\phi$. Third, we study how $\phi$ affects the supplier and the retailer's profit expectation and variance and look at the risk-return trade-off. A work that compares revenue-sharing contract with buyback contracts is Zhang et al. (2015). This paper argues that buyback contracts and revenue-sharing contracts are not equivalent when the supplier's loss aversion is taken into consideration. Furthermore, the critical ratio affects the supplier's choice between contracts and the supplier's expected profit. Researchers also used other approaches to study the revenue-sharing contract. Qin and Yang (2008) studied the revenue-sharing contract using a Stackelberg game approach and showed that the party that keeps more than half of the revenue should act as the leader of the game. In Yao et al. (2008), the manufacturer is the Stackelberg leader and sells to two competing retailers using a revenue-sharing contract. This paper studied the impact of demand variability on pricing and ordering decisions.

In this chapter, we focus on the single-period newsvendor models. To measure the return, we use the expected profit. To measure the risk, we use the profit variance. We first study the classic wholesale-price contract. Under this contract, the supplier does not face any uncertainty. The supply chain's risk is all on the newsvendor's (retailer's) side. We study what the risk-return profile looks like for the retailer and how the return of the supply chain is split between the supplier and the retailer. We start our analysis with a simple two-point distribution. Then we extend the analysis to a three-point distribution model. The retailer's risk-return curves and the optimal distributions can be characterized in closed-form expressions. From there, we generalize the analysis to the multi-point distributions and the continuous distributions. Moreover, we derive the lower bounds of the supplier's portion of the supply chain's total profit in the multi-point distribution model. The lower bounds suggest that the supplier is guaranteed to gain a proportion of the supply chain profit regardless of the demand distribution. The continuous model requires solving a sequence of infinite programs. To overcome this challenge, we provide an approximation algorithm that solves the risk-return curve efficiently.

## 3.2 Supply Chain Risk Analysis

In this section, we study the relationship between the retailer's expected profit and the profit variance in a supply chain. We consider a single-period newsvendor model. In this model, there is a retailer facing random demand $D$. The retailer uses the newsvendor ordering policy to place an order with the supplier before the beginning of the selling period. The supplier charges a price $p$ for each unit of good sold, and the pricing decision is made by maximizing his own revenue. The market price for the item is $r$.

### 3.2.1 Two-Point Distribution Risk-Return Analysis

We first study the case when the random demand $D$ follows a two-point distribution with $P(D = d_1) = x_1$ and $P(D = d_2) = x_2$, where $d_1 \leq d_2$ and $x_1 + x_2 = 1, x_1, x_2 \geq 0$. We assume the mean of $D$ is $\mu$. In the following, we compute the minimal profit variance the retailer needs to incur (across different demand distributions) if he wants to achieve an expected profit of $t$.

It is easy to see that under such a two-point distribution, there are two possible order quantities $Q^*$ for the retailer and the corresponding scenarios are described as follows:

1. $Q^* = d_1$: In this case, the optimal price for the supplier is $p^* = r$, and the supplier's revenue is $rd_1$. The expected value and the variance of the retailer's profit are both 0;

2. $Q^* = d_2$: In this case, the optimal price for the supplier is $p^* = (1 - x_1)r$, and the supplier's revenue is $(1-x_1)rd_2$. The retailer's expected profit under this case is $r(d_1x_1 + d_2x_2) - p^*d_2 = rx_1d_1$. The variance of the profit under this case is $r^2(x_1d_1^2 + x_2d_2^2 - 2x_2d_2\mu + x_2^2d_2^2 - x_1^2d_1^2)$.

Without loss of generality, we normalize $r = 1$ in the subsequent discussions. In the following, we find the two-point distribution that gives the minimum profit variance for the retailer given the expected profit is $t$. This problem can be written as follows

$$\begin{aligned}
\min_{x_1,d_1,d_2} \quad & x_1d_1^2+(1-x_1)d_2^2-2(1-x_1)d_2\mu+(1-x_1)^2d_2^2-x_1^2d_1^2 \\
\text{s.t.} \quad & x_1d_1+(1-x_1)d_2=\mu; \\
& x_1d_1=t; \\
& d_1\le(1-x_1)d_2; \\
& 0\le x_1\le 1; \\
& d_2\ge d_1\ge 0.
\end{aligned} \tag{3.1}$$

In (3.1), the equality constraints are the demand expectation constraint and the retailer's profit expectation constraint. The first inequality constraint requires that the supplier's revenue from charging price $r$ be less than that from charging price $(1-x_1)r$. That is, the supplier will prefer case 2.

Now, we analyze the optimal solution to (3.1). First, we argue that if $\mu<2t$, then the problem is infeasible. In other words, if the mean of the demand is $\mu$, then it is not possible for the retailer to achieve an expected profit of more than $\mu/2$. To see this, we note that by the second and third constraints, $(1-x_1)d_2\ge d_1\ge x_1d_1=t$. Therefore, by the first constraint, $\mu=x_1d_1+(1-x_1)d_2\ge 2t$. The next theorem gives the optimal solution to (3.1) when $\mu\ge 2t$.

**Theorem 8.** *Suppose $\mu\ge 2t$. The optimal solution to (3.1) is given by*

$$d_1^*=\mu-t,\ x_1^*=\frac{t}{\mu-t},\ d_2^*=\frac{(\mu-t)^2}{\mu-2t},\ x_2^*=\frac{\mu-2t}{\mu-t}.$$

*The optimal value is $\frac{t^3}{\mu-2t}$.*

**Proof**. Using the first equality constraint, we can reduce one degree of freedom by replacing $d_2$ with

$$d_2=\frac{\mu-x_1d_1}{1-x_1}.$$

Then we can use the second equality constraint to replace $d_1$ by $t/x_1$. The optimization problem becomes

$$\begin{aligned}
\min \quad & \tfrac{t^2}{x_1}+\tfrac{(\mu-t)^2}{1-x_1}-\mu^2 \\
\text{s.t.} \quad & \tfrac{t}{x_1}\le\mu-t; \\
& 0\le x_1\le 1.
\end{aligned}$$

Note that the objective function is convex in $x$ on $[0, 1]$ and achieves minimum at $x_1 = t/\mu$. However, the first constraint requires that $x_1 \geq t/(\mu - t)$. Therefore, the optimal solution is $x_1 = t/(\mu - t)$, and all the other parts of the theorem follow. $\square$

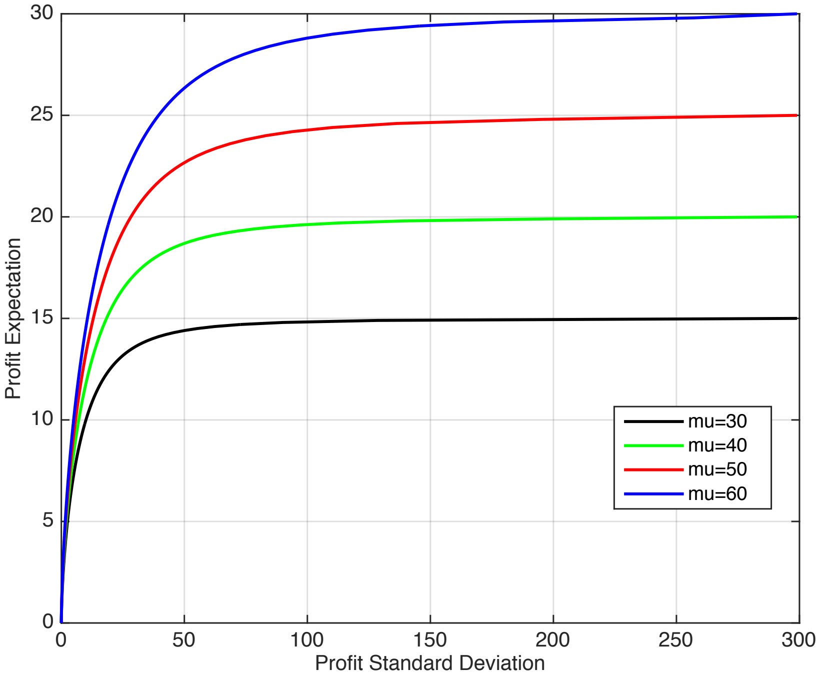

Figure 3.1: Risk-return curve for various demand mean under two-point distribution

The risk-return curves for different values of demand mean $\mu$ are illustrated in Figure 3.1. In Figure 3.1, the region below each curve is a risk-return profile that can be possibly attained under certain demand distribution, while the region above the curve can't be attained by any demand distribution.

### 3.2.2 Three-Point Distribution Risk Return Analysis

Next, let's consider the case with a three-point distribution. Let $D$ denote a three-point random variable, with $P(D = d_1) = x_1, P(D = d_2) = x_2, P(D = d_3) = x_3$. We fix the mean of the demand $D$ to $\mu$. For the retailer, there are three possible optimal order quantities:

1. $Q^* = d_1$: In this case, the optimal price for the supplier is $p^* = r$, and the supplier's revenue is $rd_1$. The expected value and the variance of the retailer's profit are both 0;

2. $Q^* = d_2$: In this case, the optimal price for the supplier is $p^* = (1 - x_1)r$, and the supplier's revenue is $(1 - x_1)rd_2$. The retailer's expected profit under this case is $r(d_1x_1 + d_2(x_2 + x_3)) - p^*d_2 = rx_1d_1$. The variance of the profit is

$$\begin{aligned}\text{Var} &= r^2\left(x_1d_1^2 + (x_2 + x_3)d_2^2 - 2(x_2 + x_3)d_2(x_1d_1 + (x_2 + x_3)d_2) + d_2^2(x_2 + x_3)^2 - (x_1d_1)^2\right) \\ &= r^2x_1(1 - x_1)(d_2 - d_1)^2.\end{aligned}$$

3. $Q^* = d_3$: In this case, the optimal price for the supplier is $p^* = (1 - x_1 - x_2)r$, and the supplier's revenue is $(1 - x_1 - x_2)rd_3$. The retailer's expected profit under this case is $r(d_1x_1 + d_2x_2 + d_3x_3) - p^*d_3 = r(x_1d_1 + x_2d_2)$. The variance of the profit is

$$\begin{aligned}\text{Var} &= r^2\left(x_1d_1^2 + x_2d_2^2 + x_3d_3^2 - 2x_3d_3\mu + (x_3d_3)^2 - (x_1d_1 + x_2d_2)^2\right) \\ &= r^2(x_1d_1^2 + x_2d_2^2 + x_3d_3^2 - \mu^2).\end{aligned}$$

In the first case, the retailer's profit is zero. Thus, it is not optimal for the retailer. We will focus our attention to the optimal distribution in the second and third case.

In the second case, when the supplier chooses price $p^* = (1 - x_1)r$, and the retailer chooses order quantity $Q^* = d_2$, the variance minimization problem is (assuming $r = 1$

without loss of generality)

$$\begin{array}{ll}\min\limits_{x_1,x_2,d_1,d_2,d_3} & x_1d_1^2+(1-x_1)d_2^2-2(1-x_1)d_2(x_1d_1+(1-x_1)d_2)+d_2^2(1-x_1)^2-(x_1d_1)^2\\ \text{s.t.} & x_1d_1+x_2d_2+(1-x_1-x_2)d_3=\mu;\\ & x_1d_1=t;\\ & d_1\le(1-x_1)d_2;\\ & (1-x_1-x_2)d_3\le(1-x_1)d_2;\\ & 0\le x_1,x_2\le 1;\\ & d_3\ge d_2\ge d_1\ge 0.\end{array}$$

Note that the variance in this case is bounded below,

$$\begin{aligned}\text{Var} &= x_1d_1^2+(x_2+x_3)d_2^2-2(x_2+x_3)d_2(x_1d_1+(x_2+x_3)d_2)+d_2^2(x_2+x_3)^2-(x_1d_1)^2\\ &\ge x_1d_1^2+(x_2+x_3)d_2^2-2(x_2+x_3)d_2\mu+d_2^2(x_2+x_3)^2-(x_1d_1)^2.\end{aligned}$$

Equality is only achieved when $d_3$ approaches $d_2$. When this happens, this problem becomes very similar to the two-point problem. For ease of reference, we restate the problem here,

$$\begin{array}{ll}\min\limits_{x_1,x_2,d_1,d_2,d_3} & x_1d_1^2+(1-x_1)d_2^2-2(1-x_1)d_2\mu+d_2^2(1-x_1)^2-t^2.\\ \text{s.t.} & x_1d_1+x_2d_2+(1-x_1-x_2)d_3=\mu;\\ & x_1d_1=t;\\ & d_1\le(1-x_1)d_2;\\ & (1-x_1-x_2)d_3\le(1-x_1)d_2;\\ & 0\le x_1,x_2\le 1;\\ & d_3\ge d_2\ge d_1\ge 0.\end{array}$$

The objective function and most of the constraints are the same except demand expectation constraint and the inequality on $x_3d_3$. Therefore, based on the solution from the two-point problem, we can create an optimal solution to the three-point problem. Consider the following distribution

$$d_1^*=\mu-t,\ x_1^*=\frac{t}{\mu-t},\ d_2^*=\frac{(\mu-t)^2}{\mu-2t},\ x_2^*=\frac{\mu-2t}{\mu-t}-\epsilon,\ d_3^*=\frac{(\mu-t)^2}{\mu-2t}+\epsilon,\ x_3^*=\epsilon.$$

The risk-return curve in this case is also $\frac{t^3}{\mu-2t}$.

Here, we have another result that is similar to the two-point distribution case. If the mean of the demand is $\mu$, then the retailer's expected profit must be no more than $\mu/2$. To see this, we note that by the two inequality constraints, $x_2d_2+x_3d_3 \geq (x_2+x_3)d_2 \geq d_1 \geq x_1d_1 = t$. Therefore, by the first constraint, $\mu = x_1d_1 + x_2d_2 + x_3d_3 \geq 2t$.

In the third case, the risk-return problem is

$$\begin{aligned}
\min_{x_1,x_2,x_3,d_1,d_2,d_3} \quad & x_1d_1^2 + x_2d_2^2 + x_3d_3^2 - \mu^2 \\
\text{s.t.} \quad & x_1d_1 + x_2d_2 + x_3d_3 = \mu; \\
& x_1d_1 + x_2d_2 = t; \\
& d_1 \leq x_3d_3; \\
& (1-x_1)d_2 \leq x_3d_3; \\
& 0 \leq x_1, x_2 \leq 1; \\
& d_3 \geq d_2 \geq d_1 \geq 0.
\end{aligned}$$

Solving this model should give us the risk-return curve when the supplier uses a price that makes the retailer order $d_3$. We have the following results from a detailed analysis of this model.

**Theorem 9.** *Suppose $\mu \geq \frac{3}{2}t$. The optimal solution to (3.2) is given by*

$$\begin{aligned}
d_1^* &= \mu - t, & & x_1^*, \\
d_2^* &= \tfrac{t - x_1^*d_1^*}{x_2^*}, & x_2^* &= \frac{(1-x_1^*)(t - x_1^*(\mu - t))}{\mu - t}, \\
d_3^* &= \tfrac{\mu - t}{x_3^*}, & x_3^* &= 1 - x_1^* - x_2^*,
\end{aligned}$$

*where $x_1^*$ is*

$$x_1^* = \frac{1}{2}\left(\frac{t}{\mu-t} - \sqrt{4 - \frac{\mu}{\mu-t}} + \sqrt{3 + \frac{t^2}{(\mu-t)^2} + \frac{t(-5 + 2\sqrt{4 - \frac{\mu}{\mu-t}})}{\mu-t}}\right).$$

*These distributions achieve the retailer's risk-return curve when demand follows a three-point distribution.*

**Proof.** Using the first two equality constraints and dropping the constant $-\mu^2$ at the end, the model can be simplified to

$$\begin{aligned}
\min_{x_1,x_2,d_1} \quad & F = x_1 d_1^2 + \frac{(t-x_1d_1)^2}{x_2} + \frac{(\mu-t)^2}{1-x_1-x_2} \\
\text{s.t.} \quad & d_1 \leq \mu - t; \\
& (1-x_1)(t-x_1d_1) \leq x_2(\mu-t); \\
& \frac{t-x_1d_1}{x_2} \geq d_1; && (3.2) \\
& \frac{\mu-t}{1-x_1-x_2} \geq \frac{t-x_1d_1}{x_2}; && (3.3) \\
& 0 \leq x_1, x_2 \leq 1; \\
& d_1 \geq 0.
\end{aligned}$$

Constraints (3.2) and (3.3) come from $d_3 \geq d_2 \geq d_1$. The objective function is convex in $x_1, d_1$ and $x_2$ on $0 \leq x_1, x_2 \leq 1, d_1 \geq 0$. The partial derivative with respect to $d_1$ is

$$\frac{\partial F}{\partial d_1} = 2x_1(d_1 - \frac{t-x_1d_1}{x_2}) = 2x_1(d_1 - d_2) \leq 0.$$

Therefore, the objective function is decreasing in $d_1$. Now, we need to find the upper bound of $d_1$. We focus on the constrains and write them as the bounds of $d_1$.

$$\begin{aligned}
d_1 &\leq \mu - t; \\
d_1 &\leq \frac{t}{x_1+x_2}; \\
d_1 &\geq \frac{t}{x_1} - \frac{x_2(\mu-t)}{x_1(1-x_1)}; \\
d_1 &\geq \frac{t}{x_1} - \frac{x_2(\mu-t)}{x_1(1-x_1-x_2)}.
\end{aligned}$$

When $\mu - t \leq t/(x_1+x_2)$, or $x_1 + x_2 \leq t/(\mu-t)$, to minimize the objective function, we must have $d_1^* = \mu - t$. Substituting this into the optimization problem and further simplify the problem, we have,

$$\begin{aligned}
\min_{x_1,x_2} \quad & F = x_1(\mu-t)^2 + \tfrac{(t-x_1(\mu-t))^2}{x_2} + \tfrac{(\mu-t)^2}{1-x_1-x_2} \\
\text{s.t.} \quad & (1-x_1)(t-x_1(\mu-t)) \leq x_2(\mu-t); \\
& x_1 + x_2 \leq \tfrac{t}{\mu-t}; \\
& 0 \leq x_1, x_2 \leq 1.
\end{aligned}$$

The objective function is convex in $x_1, x_2$ on the region $x_1, x_2 \in [0,1]$. Take the partial derivative of the objective function with respect to $x_2$, we have

$$\frac{\partial F}{\partial x_2} = \frac{(\mu - t)^2}{(1 - x_1 - x_2)^2} - \frac{(t - x_1(\mu - t))^2}{x_2^2} = d_3^2 - d_2^2 \geq 0.$$

Therefore, the objective function is increasing in $x_2$. To minimize the variance, we need to find the lower bound of $x_2$. The two constraints in the formulation above gives

$$\begin{aligned} \frac{(1-x_1)(t-x_1(\mu-t))}{\mu-t} \leq \quad & x_2 \quad \leq \frac{t - x_1(\mu - t)}{\mu - t}, \\ 0 \leq \quad & x_2 \quad \leq 1. \end{aligned}$$

We can conclude that $\frac{(1-x_1)(t-x_1(\mu-t))}{\mu-t} \geq 0$ from the result of previous analysis. Thus, we have $x_2^* = \frac{(1-x_1)(t-x_1(\mu-t))}{\mu-t}$. Substituting this into the optimization problem, we have

$$\begin{aligned} \min_{x_1} \quad & x_1(\mu - t)^2 + \tfrac{(\mu-t)(t-x_1(\mu-t))}{1-x_1} + \tfrac{(\mu-t)^2}{1-x_1-\frac{(1-x_1)(t-x_1(\mu-t))}{\mu-t}} \\ \text{s.t.} \quad & 0 \leq x_1 \leq 1. \end{aligned}$$

In addition to the $0 \leq x_1 \leq 1$ feasibility constraint, we could also derive bounds of $x_1$ based on the feasibility constraint from $x_2$. From $x_2^* \geq 0$, we derive $x_1 \leq \frac{t}{\mu-t}$. From $x_1 + x_2 \leq 1$, we derive $x_1 \geq \frac{-\mu+2t}{\mu-t}$.

In fact, when $\mu/2 < t \leq 2\mu/3$, the bound on $x_1$ is

$$\frac{-\mu + 2t}{\mu - t} \leq x_1 \leq 1.$$

When $t \leq \mu/2$, the bound on $x_1$ becomes

$$0 \leq x_1 \leq \frac{t}{\mu - t}.$$

By setting the derivative of the objective function to zero, we can find the extreme points of the function. We use $\tilde{x}_1$ here to denote these extreme point.

$$\tilde{x}_{1,1} = \frac{1}{2}\left(\frac{t}{\mu - t} - \sqrt{4 - \frac{\mu}{\mu - t}} - \sqrt{3 + \frac{t^2}{(\mu - t)^2} + \frac{t(-5 + 2\sqrt{4 - \frac{\mu}{\mu - t}})}{\mu - t}}\right),$$

$$\tilde{x}_{1,2} = \frac{1}{2}\left(\frac{t}{\mu - t} - \sqrt{4 - \frac{\mu}{\mu - t}} + \sqrt{3 + \frac{t^2}{(\mu - t)^2} + \frac{t(-5 + 2\sqrt{4 - \frac{\mu}{\mu - t}})}{\mu - t}}\right),$$

$$\tilde{x}_{1,3} = \frac{1}{2}\left(\frac{t}{\mu - t} + \sqrt{4 - \frac{\mu}{\mu - t}} - \sqrt{3 + \frac{t^2}{(\mu - t)^2} - \frac{t(5 + 2\sqrt{4 - \frac{\mu}{\mu - t}})}{\mu - t}}\right),$$

$$\tilde{x}_{1,4} = \frac{1}{2}\left(\frac{t}{\mu - t} + \sqrt{4 - \frac{\mu}{\mu - t}} + \sqrt{3 + \frac{t^2}{(\mu - t)^2} - \frac{t(5 + 2\sqrt{4 - \frac{\mu}{\mu - t}})}{\mu - t}}\right).$$

Now, we need to verify whether or not this extreme point $\tilde{x}_1$ is in the feasible range that we obtained previously. We conclude the following, $\tilde{x}_{1,1}$ is on the range $[-\sqrt{3}, 0)$, which is out of our feasible region for $x_1$. $\tilde{x}_{1,2}$ is on the range $[0, 1)$ and is increasing in $t$. It is a potential minimizer. $\tilde{x}_{1,3}$ is on the range $[0, 1]$; however, it is greater than $t/(\mu - t)$. $\tilde{x}_{1,4}$ is on the range $[1, \sqrt{3}]$, which is also out of the feasible region for $x_1$.

Therefore, the candidate point that minimizes the objective function are $\tilde{x}_{1,2}$ and two bounds. By taking the second order derivative of the objective function, we conclude that this function is convex on the feasible region. Therefore, $\tilde{x}_{1,2}$ is the minimizer on the feasible region.

Given $x_1^* = \tilde{x}_{1,2}$, the distribution that achieves the risk-return curves can be computed through the following relationships,

$$\begin{aligned} d_1^* &= \mu - t, & x_1^* &= \tilde{x}_{1,2}, \\ d_2^* &= \tfrac{t - x_1^* d_1^*}{x_2^*}, & x_2^* &= \frac{(1 - x_1^*)(t - x_1^*(\mu - t))}{\mu - t}, \\ d_3^* &= \tfrac{\mu - t}{x_3^*}, & x_3^* &= 1 - x_1^* - x_2^*. \end{aligned}$$

Let's consider another possible scenario of $d_1$. When $\mu - t > t/(x_1 + x_2)$, or $x_1 + x_2 > t/(\mu - t)$, to minimize the objective function, we must have $d_1^* = \frac{t}{x_1 + x_2}$. Substituting this into the optimization problem and further simplify the problem, we have,

$$\begin{aligned}\min_{x_1,x_2} \quad & F = x_1(\tfrac{t}{x_1+x_2})^2 + \tfrac{(t-x_1\frac{t}{x_1+x_2})^2}{x_2} + \tfrac{(\mu-t)^2}{1-x_1-x_2} \\ \text{s.t.} \quad & (1-x_1)(t-x_1\tfrac{t}{x_1+x_2}) \leq x_2(\mu-t); \qquad (3.4) \\ & \tfrac{t}{x_1+x_2} \geq \tfrac{t}{x_1} - \tfrac{x_2(\mu-t)}{x_1(1-x_1)}; \qquad (3.5) \\ & x_1+x_2 \geq \tfrac{t}{\mu-t}; \qquad (3.6) \\ & 0 \leq x_1, x_2 \leq 1.\end{aligned}$$

The objective function can be simplified into a function that only contains $x_1 + x_2$,

$$\min F = \quad \frac{t^2}{x_1+x_2} + \frac{(\mu-t)^2}{1-x_1-x_2}.$$

It is convex in $x_1, x_2$ on the region $x_1, x_2 \in [0,1]$. Take the partial derivative of the objective function with respect to $x_2$, we have

$$\frac{\partial F}{\partial x_2} = \frac{(\mu-t)^2}{(1-x_1-x_2)^2} - \frac{t^2}{(x_1+x_2)^2} \geq d_3^2 - \frac{(x_1d_2+x_2d_2)^2}{(x_1+x_2)^2} = d_3^2 - d_2^2 \geq 0.$$

Therefore, the objective function is increasing in $x_2$. To minimize the variance, we need to find the lower bound of $x_2$. Simplify constraints (3.4 - 3.6), we have

$$\begin{aligned} x_2 &\geq \frac{t-x_1\mu}{\mu-t}, \\ x_2 &\geq \frac{t-x_1(\mu-t)}{\mu-t}, \\ x_2 &\geq 0. \end{aligned}$$

When $0 \leq \frac{t-x_1(\mu-t))}{\mu-t} \leq 1$, we have $x_2^* = \frac{(t-x_1(\mu-t))}{\mu-t}$. Substituting this into the optimization problem, the objective function becomes a constant $t(\mu-t) + \frac{(\mu-t)^3}{\mu-2t}$. The optimal distribution must satisfy $x_1^* + x_2^* = \frac{t}{\mu-t}$. However, notice the condition on $x_1 + x_2$ when we first set $d_1^* = \frac{t}{x_1+x_2}$. When $x_1 + x_2 = \frac{t}{\mu-t}$, we have $\mu - t = t/(x_1+x_2)$, which is equivalent to the case when $d_1^* = \mu - t$. Therefore, the only optimal solution for this problem is when $d_1^* = \mu - t$. □

This theorem shows the three-point distribution on the risk-return curve. The function of the risk-return curve is rather complicated. Therefore, we will not show the closed-form here. However, the closed-form is obtained and can be found in Appendix G at the end. The risk-return curves for different values of demand mean $\mu$ are illustrated in Figure 3.2. We use the same base problem as the two-point distribution example. The curves has similar shape as the two-point distribution curves. The region below each curve is a risk-return profile that can be attained under certain demand distribution. The region above the curve cannot be attained by any distribution.

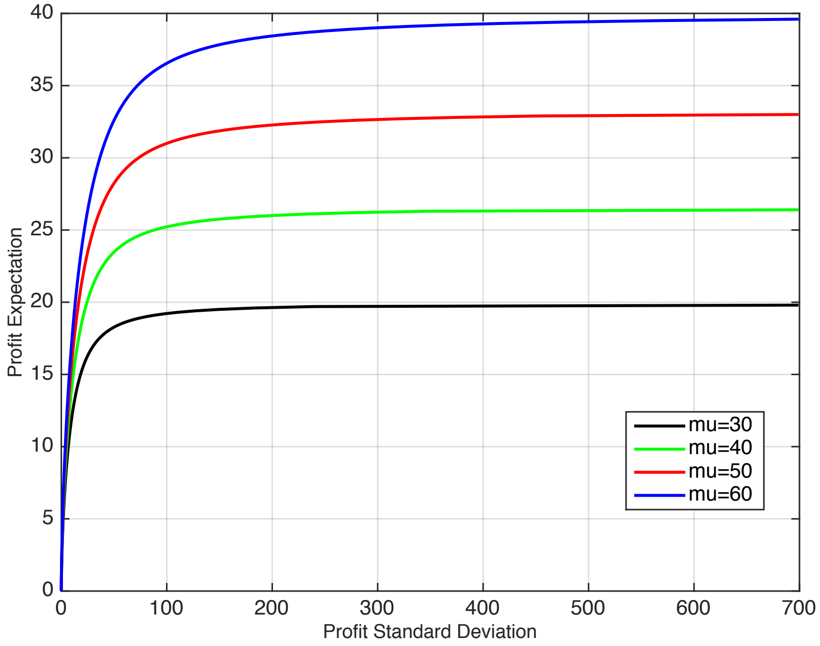

Figure 3.2: Risk-return curves for various demand mean under three-point distributions

We can argue that the retailer's expected profit $t$ cannot be more than $2\mu/3$. From the inequality constraints, we have

$$x_3d_3 \geq d_1 \geq x_1d_1,$$

$$x_3d_3 \geq (x_2 + x_3)d_2 \geq x_2d_2.$$

Thus, we have $2x_3d_3 \geq x_1d_1 + x_2d_2 = t$. By the first equality constraint, $\mu = x_1d_1 +$

$x_2d_2 + x_3d_3 \geq \frac{3}{2}t$. This upper bound of $t$ is larger than the previous case when order quantity is $d_2$. Therefore, to achieve the best risk-return trade-off. The optimal order quantity must be the largest demand, which is $d_3$ in this case.

### 3.2.3 Multi-Point Distribution Risk-Return Analysis

Next, we consider the case in which the demand distribution is a multi-point distribution with $N$ being the number of points in the support of the demand distribution. Similar as previous cases, we can list all the retailer's optimal decisions and profits.

1. $Q^* = d_1$: The optimal price to charge the retailer is $p^* = r$. The supplier's revenue is $rd_1$. The expected value and the variance of the retailer's profit are both 0.

2. $Q^* = d_2$: The optimal price to charge the retailer is $p^* = (1-x_1)r$. The supplier's revenue is $(1-x_1)rd_2$. The retailer's expected profit is $r(x_1d_1+(1-x_1)d_2)-p^*d_2 = rx_1d_1$. The profit variance is

$$\mathrm{Var} = r^2\left(x_1d_1^2 + (1-x_1)d_2^2 - (x_1d_1 + (1-x_1)d_2)^2\right).$$

3. $Q^* = d_3$: The optimal price to charge the retailer is $p^* = (1-x_1-x_2)r$. The supplier's revenue is $(1-x_1-x_2)rd_3$. The retailer's expected profit is $r(x_1d_1 + x_2d_2)$. The profit variance is

$$\mathrm{Var} = r^2\left(x_1d_1^2 + x_2d_2^2 + (1-x_1-x_2)d_3^2 - ((1-x_1-x_2)d_3 + x_1d_1x_2d_2)^2\right).$$

4. $Q^* = d_k$: The optimal price to charge the retailer is $p^* = (1-\sum_{i=1}^{k-1} x_i)r$. The supplier's revenue is $(1-\sum_{i=1}^{k-1} x_i)rd_k$. The retailer's expected profit is $r\sum_{i=1}^{k-1} x_id_i$. The profit variance is

$$\mathrm{Var} = r^2\left(\sum_{i=1}^{k-1} x_id_i^2 + (1-\sum_{i=1}^{k-1} x_i)d_k^2 - (\sum_{i=1}^{k-1} x_id_i + (1-\sum_{i=1}^{k-1} x_i)d_k)^2\right).$$

When the demand distribution follows an $N$-point distribution, we have the following results for the retailer's expected profit.

**Theorem 10.** *Suppose the demand follows an $N$-point distribution with expectation $\mu$. If the supplier's optimal price results in an order quantity of $d_k, k = 1, \ldots, N$, then the retailer's expected profit is bounded above by $\frac{k-1}{k}\mu$. Furthermore, the efficient frontier for the $N$-point distribution is achieved only when $Q^* = d_N$, with an upper bound of $\frac{N-1}{N}\mu$. The smallest demand in the optimal distribution is $d_1^* = \mu - t$.*

**Proof.** First, we will show that the retailer's expected profit is bounded above by $\frac{k-1}{k}\mu$ when the wholesale price results in an order quantity of $d_k$. In order for each of the cases to be feasible, the following constraint must be satisfied,

$$\left(1 - \sum_{i=1}^{k-1} x_i\right) d_k \geq \left(1 - \sum_{i=1}^{j-1} x_i\right) d_j, \quad \forall j = 1, \ldots, N \tag{3.7}$$

They indicate that the supplier's profit from offering price $p^*$ in each case must be higher than offer other prices. From these sets of constraints, we can show that when $k = 2$, or $Q^* = d_2$, we have

$$\sum_{i=2}^{N} x_i d_i \geq \sum_{i=2}^{N} x_i d_k \geq d_1 \geq x_1 d_1 = t.$$

Thus, $\mu = \sum_{i=1}^{N} x_i d_i \geq 2t$.

This result can be generalized to any order quantity $Q = d_k$.

$$(1 - \sum_{i=1}^{k-1} x_i) d_k \geq (1 - \sum_{i=1}^{j-1} x_i) d_j \geq x_j d_j, \qquad \forall j$$

$$\sum_{i=1}^{k-1} x_i d_i = t.$$

Thus, $\mu = \sum_{i=1}^{N} x_i d_i \geq \frac{k}{k-1} t$. The first result in the theorem follows.

Now, we will show how to compute the multi-point distribution that could achieve the risk-return curve. In the following, we will only focus on the case of $Q^* = d_N$.

$$\begin{aligned} \min_{x,d} \quad & \textstyle\sum_{i=1}^{N} x_i d_i^2 - \mu^2 \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{N} x_i d_i = \mu; \\ & \textstyle\sum_{i=1}^{N-1} x_i d_i = t; \\ & x_N d_N \geq \left(1 - \textstyle\sum_{i=1}^{j-1} x_i\right) d_j, \quad \forall j = 1, \ldots, N \\ & 0 \leq x_i \leq 1; \\ & d_k \geq d_{k-1} \geq 0. \end{aligned}$$

Using the equality constraints, we can replace $x_N$ with $1 - \sum_{i=1}^{N-1} x_i$, replace $d_N$ with $(\mu - t)/(1 - \sum_{i=1}^{N-1} x_i)$, and replace $d_{N-1}$ with $(t - \sum_{i=1}^{N-2} x_i d_i)/x_{N-1}$.

We rewrite the problem as

$$\begin{aligned} \min_{x,d} \quad & \textstyle\sum_{i=1}^{N-2} x_i d_i^2 + \frac{(t - \sum_{i=1}^{N-2} x_i d_i)^2}{x_{N-1}} + \frac{(\mu - t)^2}{1 - \sum_{i=1}^{N-1} x_i} \\ \text{s.t.} \quad & x_N d_N \geq \left(1 - \textstyle\sum_{i=1}^{j-1} x_i\right) d_j, \quad \forall j = 1, \ldots, N \\ & 0 \leq x_j \leq 1, \quad \forall j \\ & d_j \geq d_{j-1} \geq 0, \quad \forall j. \end{aligned}$$

The objective function is convex in $x_i$ and $d_i$. Take the partial derivative of the objective function with respect to $d_k$,

$$\frac{\partial F}{\partial d_k} = 2x_k \left( d_k - \frac{(t - \sum_{i=1}^{N-2} x_i d_i)^2}{x_{N-1}} \right) = 2x_1(d_k - d_{N-1}) \leq 0.$$

Thus, the objective function is decreasing in $d_k$ for all $k \leq N-1$. To minimize the objective, the optimal $d_k$ can be obtained from the inequality constraints. From $x_N d_N \geq (1 - \sum_{i=1}^{j-1} x_i) d_j$, we have

$$d_j^* = \frac{x_N d_N}{1 - \sum_{i=1}^{j-1} x_i} = \frac{\mu - t}{1 - \sum_{i=1}^{j-1} x_i}.$$

When $j = 1$, the optimal $d_1 = \mu - t$.

Now, replace all the $d_j^*$, the objective function can be expressed as

$$\min \quad \sum_{i=1}^{N-2} x_i \left(\frac{\mu - t}{1 - \sum_{m=1}^{i-1} x_m}\right)^2 + \frac{\left(t - (\mu - t) \sum_{i=1}^{N-2} \frac{x_i}{1 - \sum_{m=1}^{i-1} x_m}\right)^2}{x_{N-1}} + \frac{(\mu - t)^2}{1 - \sum_{i=1}^{N-1} x_i}.$$

We establish a formulation to calculate the risk-return curve for the retailer and perform a numerical analysis in this case. Suppose the demand distribution is a discrete random variable taking values at $x_i, i = 1, \ldots, N$ with probabilities $w_i, i = 1, \ldots, N$. Also, suppose the mean of the demand is $\mu$. In the following, we would like to compute that given the expected profit of the retailer is $t$, what is the demand distribution that gives the retailer the minimum risk. Again, without loss of generality, we assume the retailer's selling price $r = 1$.

To compute that, it is not hard to see that the optimal ordering quantity for the retailer $Q^*$ must be equal to one of the $x_i$'s. Suppose $Q^* = x_{i^*}$, then one can write the variance minimization problem as

$$\begin{array}{lll} \min_{w_i} & \sum_{i=1}^{i^*-1} x_i^2 w_i & \\ \text{s.t.} & \left(1 - \sum_{i=1}^{i^*-1} w_i\right) x_{i^*} \geq \left(1 - \sum_{i=1}^{j-1} w_i\right) x_j, & \forall i = 1, \ldots, N \\ & \sum_{i=1}^{i^*-1} x_i w_i = t; & \\ & \sum_{i=1}^{N} x_i w_i = \mu; & \\ & \sum_{i=1}^{N} w_i = 1; \quad w_i \geq 0, & \forall i = 1, \ldots, N. \end{array}$$

The above is a linear optimization on $w_i$'s. Solving for all such linear optimizations for $i^* = 1, \ldots, N$ and choose the one that achieves the smallest objective value will give the distribution that gives the smallest profit variance under return $t$.

Next, we show the upper bound of the retailer's expected profit when the mean of the demand is $\mu$.

When the optimal order quantity $Q^* = x_N$, the inequality set is

$$\left(1 - \sum_{i=1}^{N-1} w_i\right) x_N \geq \left(1 - \sum_{i=1}^{j-1} w_i\right) x_j, \quad \forall j = 1, \ldots, N-1.$$

From those constraints, we have the following relationships

$$\begin{aligned} \left(1 - \sum_{i=1}^{N-1} w_i\right) x_N &\geq x_1 \geq x_1 w_1, \\ \left(1 - \sum_{i=1}^{N-1} w_i\right) x_N &\geq (\sum_{i=2}^{N} w_i) x_2 \geq x_2 w_2, \\ \left(1 - \sum_{i=1}^{N-1} w_i\right) x_N &\geq (\sum_{i=j}^{N} w_i) x_j \geq x_j w_j, \\ &\cdots \end{aligned}$$

Summing both sides, we have

$$(N-1)\left(1 - \sum_{i=1}^{N-1} w_i\right) x_N \geq \sum_{i=1}^{N-1} x_i w_i.$$

Thus, we have $(N-1)\left(1 - \sum_{i=1}^{N-1} w_i\right) x_N \geq \sum_{i=1}^{N-1} x_i w_i = t$. Together with the demand expectation equality, the upper bound of the expected profit $t$ can be obtained,

$$\mu = \sum_{i=1}^{N} x_i w_i = t + x_N w_N \geq t + \frac{t}{N-1} t = \frac{N}{N-1} t.$$

Therefore, the retailer's expected profit must be no more than $\frac{N-1}{N}\mu$ when the demand follows an N point distribution. Furthermore, when the optimal order quantity is $Q^* = x_{i^*}$, the retailer's expected profit must be no more than $\frac{i^*-1}{i^*}\mu$. This proof is similar to the one in the three-point distribution case, so we will not elaborate here. □

Theorem 10 provides an upper bound on the retailer's expected profit in this supply chain setting. This result is consistent with what we have seen in the previous sections. Consider the three-point distribution result. When $N = 3$, this theorem shows

$$(1 - \sum_{i=1}^{2} x_i) d_3 \geq (1 - \sum_{i=1}^{j-1} x_i) d_j \geq x_j d_j, \quad \forall j. \tag{3.8}$$

Thus, $\mu = \sum_{i=1}^{N} x_i d_i \geq \frac{3}{2} t$. Another interesting insight from this theorem is that the more points in a distribution, the higher the retailer's expected profit. When the demand mean is $\mu$, the more points in the distribution, the less the variability. When demand variability is smaller, the retailer expects to make more profit.

### 3.2.4 Risk-Return of Continuous Distribution Functions

In this section, we will consider the risk-return model with a continuous demand $D$. The demand has probability distribution function $f(x)$ on range $[0, M]$. Let $\mathbb{E}(X) = \mu$ denote the demand expectation and $F(x)$ denote the cumulative distribution function. The supplier charges price $p$ for each item and the retailer select the order quantity using the newsvendor model. For each order quantity $Q$, the supplier receives revenue of $pQ(p)$. He chooses a price that maximizes the revenue.

We derive the expression of the retailer's expected profit and the profit variance. To make sure the retailer's optimal order quantity is $Q^*$, the supplier's optimal price should be $p^* = r\left(1 - F(Q^*)\right)$. The supplier's revenue is $p^*Q^* = r\left(1 - F(Q^*)\right)Q^*$.
The retailer's expected profit is $r\left(\int_0^{Q^*} xf(x)dx + \int_{Q^*}^{M} Q^*f(x)dx\right) - Q^*r\left(1 - F(Q^*)\right) = r\int_0^{Q^*} xf(x)dx$.
The retailer's profit variance is

$$\begin{aligned}
\text{Var} &= r^2\int_0^{Q^*}(x^2 - 2x(1-F(Q^*))Q^*)f(x)dx + r^2Q^{*2}F(Q^*)(1-F(Q^*)) - \left(r\int_0^{Q^*} xf(x)d_x\right)^2 \\
&= r^2\left(\int_0^{Q^*} x^2f(x)dx + (1-F(Q^*))Q^{*2} - \left((1-F(Q^*))Q^* + \int_0^{Q^*} xf(x)d_x\right)^2\right)
\end{aligned}$$

Without loss of generality, we normalize $r = 1$. The problem of minimizing profit variance under a fixed expected profit $t$ can be modeled as

$$\begin{aligned}
\min_{F(x)} \quad & \int_0^{Q^*} x^2f(x)dx + (1-F(Q^*))Q^{*2} - ((1-F(Q^*))Q^* + t)^2 \\
\text{s.t.} \quad & (1-F(Q^*))Q^* \geq (1-F(x))x, \quad \forall x \in [0, M] && \text{(3.9a)} \\
& \int_0^{Q^*} xdF(x) = t; && \text{(3.9b)} \\
& \int_0^{M} xdF(x) = \mu; && \text{(3.9c)} \\
& \int_0^{M} dF(x) = 1; && \text{(3.9d)} \\
& dF(x) \geq 0. && \text{(3.9e)}
\end{aligned}$$

The decision variable here is the probability distribution function $F(x)$. We are interested in the distribution function that gives the minimum variance. This is a infinite program since the distribution is continuous and the number of decision variables is infinite. When $Q^*$ and $F(Q^*)$ are fixed, the objective function is linear in $f(x)$.

Let's denote the smallest point that has a non-zero probability with $d_{\min}$ and the largest point that has a non-zero probability with $d_{\max}$. We have the following results.

**Theorem 11.** *Suppose the demand follows a continuous distribution with expectation* $\mu$ *on range* $[0, M]$*. The distribution only has non-zero probabilities on range* $[\mu - t, Q^*]$*, which is to say,* $d_{\min} = \mu - t$ *and* $d_{\max} = Q^*$*. In addition, the retailer's expected profit is bounded above by* $\mu$*.*

**Proof**. From Constraint (3.9b, 3.9c), we have

$$\mu = \int_0^M x dF(x) = t + \int_{Q^*}^M x dF(x) \geq t.$$

For a fixed $\mu$, the maximum expected profit $t$ is obtained when $\int_{Q^*}^M x dF(x)$ is at the minimum. Thus, it requires $\int_{Q^*+\epsilon}^M dF(x) = 0$, for any $\epsilon > 0$, which means any points larger than $Q^*$ will have a probability equal to zero. Therefore, we have $Q^* = d_{\max}$.

Constraint (3.9a) requires $(1 - F(Q^*))Q^* \geq \int_x^M dF(x)x$, for all $x \in [0, M]$. When $x = d_{\min}$, this inequality gives $(1 - F(Q^*))Q^* = \mu - t \geq d_{\min}$. Since the objective function is increasing in $x$, $d_{\min} = \mu - t$ at optimal. Therefore, the continuous distribution only has non-zero probabilities on the range $[\mu - t, Q^*]$ at optimal. □

As we can see from this theorem, the continuous case is an extension of the multi-point distribution case. The smaller point with non-zero probability in the continuous case, or $d_{\min}$ corresponds to $d_1$ in the multi-point case. We can think of the continuous case as a multi-point case with $N$ approaching infinity. The upper bound of the retailer's expected profit in the multi-point case approaches 1. Thus, the retailer's expected profit is bounded above by $\mu$ in the continuous case.

The continuous distribution case requires solving an infinite program, which is a hard problem. In the next section, we present an algorithm to efficiently approximate the risk-return curve by reducing the problem dimension. We also show some numerical results from our approximation algorithm.

### 3.2.5 Approximation Algorithm and Numerical Results

Due to the fact that the closed-form solution is not obtainable under a continuous distribution, we will introduce an algorithm to approximate the continuous case. The idea of the algorithm is to use a discrete distribution with fine granularity to approximate the continuous distribution. Then we solve for the optimal multi-point distribution using the results obtained from the previous section. To start with, we define the notation here. Let $N$ denote the number of points used in the discrete distribution, and $D = \{d_1, \dots, d_N\}$ be the support of the demand distribution.

The minimum standard deviation (risk) is the square root of current_best. This algorithm will also produce the optimal probability distribution and the optimal order quantity under each $t$. In Step 2, we did not solve the variance minimization problem directly. Instead, we enumerated $\alpha$ and use it as a constant in the optimization problem. Introducing the $\alpha$ term simplifies our problem to an LP. Without fixing $\alpha$ first, the objective function is non-convex, making it difficult for commercial solvers to solve. Although we've increased the number of optimization problems to be solved, each iteration takes very little time.

**Algorithm 1. $N$-Point Distribution Risk-Return Approximation Algorithm**

```
1. Select $N$ and fix $t$. Set current_best to a large number $\mathcal{M}$.

2. For $k = 1$ to $N$
   Let $Q = d_k$.
   Let $\underline{\alpha} = \min \sum_{i=1}^{k-1} x_i$ and $\bar{\alpha} = \max \sum_{i=1}^{k-1} x_i$.
   For $\alpha = \underline{\alpha}$ to $\bar{\alpha}$
   Let temp_value $= \min_{x_i} \sum_{i=1}^{k-1} d_i^2 x_i + (1-\alpha)Q^2 - (t+\alpha Q)^2$ from the following linear program.

         $\min_{x_i}$   $\sum_{i=1}^{k-1} d_i^2 x_i + (1-\alpha)Q^2 - (t+\alpha Q)^2$
         $s.t.$         $\left(1 - \sum_{i=1}^{k-1} x_i\right) d_k \geq \left(1 - \sum_{i=1}^{j-1} x_i\right) d_j, \quad \forall i = 1, \ldots, N$
                        $\sum_{i=1}^{k-1} d_i x_i = t;$
                        $\sum_{i=1}^{N} d_i x_i = \mu;$
                        $\sum_{i=1}^{k-1} x_i = \alpha;$
                        $\sum_{i=1}^{N} x_i = 1;$
                        $x_i \geq 0, \quad \forall i = 1, \ldots, N.$

       If temp_value $<$ current_best
   current_best $=$ temp_value
   current_Q $= Q$
   current_$\alpha = \alpha$
   distribution$= [x_1, \ldots, x_N]$

3. Output $t$, current_best, current_Q, and distribution.
```

Using this algorithm, we study the risk-return curve of continuous distributions with support $[1, 1000]$. We select $N = 1000$ and integers $\{1, 2, \ldots, 1000\}$ be the support. In Step 2, the CVX package in MATLAB is used to solve the linear optimization problems. After the computation, the retailer's risk-return curve (efficient frontier) is plotted using the plot function in MATLAB. Figure 3.3 shows the risk-return curves for continuous distributions with mean equal 30 and 40. As shown on the plot, the shape of the curves is very similar to what we saw in the two- and three-point distribution cases. Their upper bounds are $\mu$ if the demand upper bound is large enough.

After we've computed the continuous distribution risk-return curve, we want to compare the continuous cases with other multi-point cases to see how much of improvement are there by using a continuous distribution instead of a multi-point case. Figure 3.4 shows the risk-return curves of the continuous distribution, two-point distribution and three-point distribution when the demand mean is 40. This plot shows that there is a significant increase in the upper-bound when we increase from a two-point to a three-point distribution, $(\frac{2}{3} - \frac{1}{2})\mu$ to be exact. However, this effect diminishes dramatically as the number of points increases. As we use more points in the numerical tests, the improvement $(\frac{n}{n+1} - \frac{n-1}{n})\mu$ approaches zero; however, the amount of resource (computation time and memory usage) used increases fast. What's even more interesting is that even in the continuous case, only a few points has significant probability in the optimal distribution. That is to say, using only a few points to model the demand distribution could get results that are close to the continuous case.

## 3.3 Supply Chain Profit Allocation Analysis

In this section, we study the profit allocation problem between the supplier and the retailer in a supply chain. Consider a supplier who sells a single product to a retailer facing a newsvendor problem. The retailer faces an uncertain demand $D$ with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$, and uses the newsvendor ordering policy to decide on the optimal order quantity. The supplier charges a price $p$ for each unit of product sold, and incurs a per unit production cost $c$. The market price

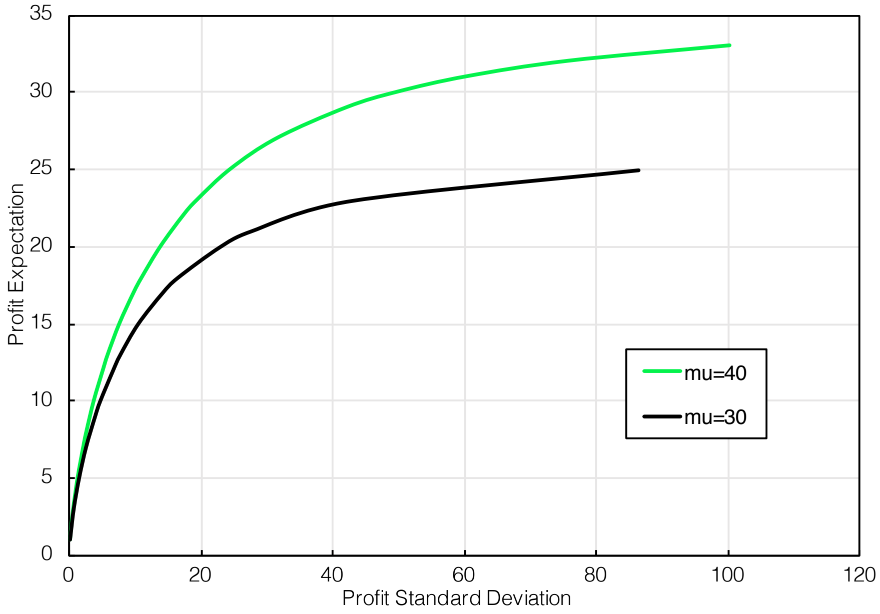

Figure 3.3: Risk-return curve for various demand means under 1000-point distributions

that the retailer sells at is $r$.

Given a price $p$, according to the solution to the newsvendor problem, the retailer will choose order quantity $Q(p) = F^{-1}(\frac{r-p}{r})$. Based on this, the supplier chooses a price $p$ to maximize his profit. That is, the supplier chooses $p$ to maximize $\Pi = (p-c)Q(p)$. In the following, let $p^*$ denote the solution that maximizes $\Pi$ and let $\Pi^* = (p^*-c)Q(p^*)$ and $\mathcal{R}^* = r\mathbb{E}[\min(D, Q(p^*))] - p^*Q(p^*)$ be the corresponding profit of the supplier and the retailer respectively. We further define $\gamma = \frac{\Pi^*}{\Pi^*+\mathcal{R}^*}$ to be the profit division ratio between the supplier and the retailer. The main result in this section is the following theorem regarding lower bounds on $\gamma$.

**Theorem 12.** *Let* $\gamma = \frac{\Pi^*}{\Pi^*+\mathcal{R}^*}$ *be the profit division ratio.*

1. *If the demand distribution is an* $n$*-point distribution, then* $\gamma \geq 1/n$*.*
2. *If the demand distribution has support* $[d_{\min}, d_{\max}]$*, then* $\gamma \geq \mathcal{O}\left(\frac{1}{\log \frac{d_{\max}}{d_{\min}}}\right)$*.*

*Furthermore, the above lower bounds are tight.*

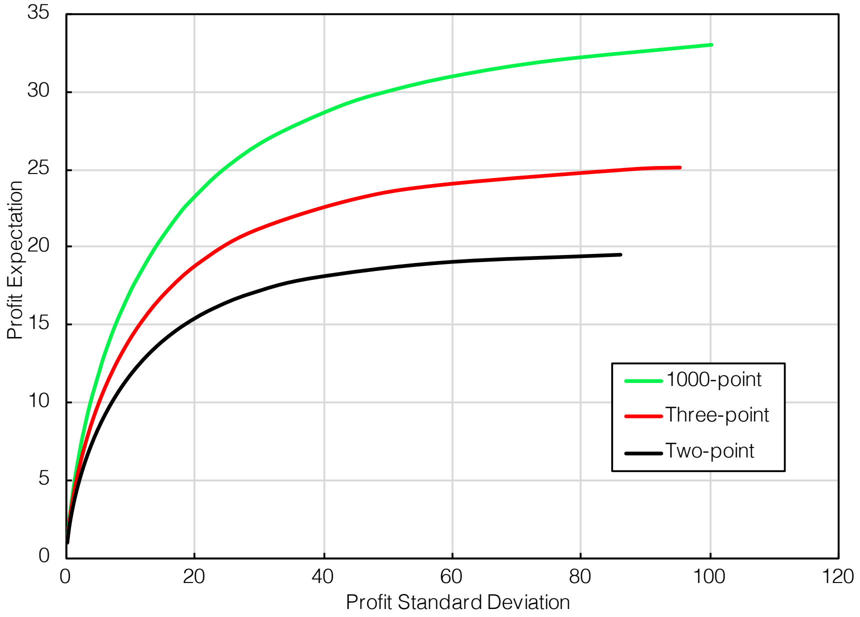

Figure 3.4: Risk-return curve for various distributions with $\mu = 40$

**Proof**. We start with the first part of the theorem. Suppose the demand follows an $n$-point distribution with support $d_i, i = 1, \ldots, n$, where $d_i < d_{i+1}$ for all $i$. Let $w_i = P(D = d_i)$ be the probability for each outcome $d_i$. In the following analyses, we will find a set of $w_i$s that minimizes the profit division ratio of the supplier $\gamma$.

First, we note that for any given price, there exists an optimal ordering strategy of the retailer that takes value in the set $\{d_i\}_{i=1}^{n}$.[1] In the following, we consider the case when the retailer's optimal ordering quantity is $Q^* = d_k$. In this case, the highest price the supplier can charge (in order for $d_k$ to be an optimal order quantity for the retailer) is $p = r(1 - \sum_{i=1}^{k-1} w_i)$. Note that the supply chain's expected profit is divided between the supplier and the retailer. When fixing the retailer's expected profit to $t$, minimizing $\gamma$ is equivalent to minimizing supplier's profit $\Pi$. We consider the following optimization problem to find the demand distribution that gives the supplier

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[1] When the retailer's optimal ordering strategy is not unique, we assume he will choose the largest ordering quantity, which maximizes the social welfare.

the minimal profit division ratio.

$$\gamma_k = \min_{w_i} \quad \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + r\sum_{i=1}^{k-1} w_i d_i} \tag{3.10a}$$

$$\text{s.t.} \quad [r(1-\sum_{i=1}^{k-1} w_i) - c]d_k \geq [r(1-\sum_{i=1}^{j-1} w_i) - c]d_j, \quad \forall j \neq k, \tag{3.10b}$$

$$\sum_{i=1}^{n} w_i = 1; \quad 0 \leq w_i \leq 1, \quad \forall i = 1, \ldots, n. \tag{3.10c}$$

Now we provide some explanations for the above formulation. Given the retailer's optimal order quantity $d_k$, the profit of the supplier and the retailer are $[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k$ and $r\sum_{i=1}^{k-1} w_i d_i$ respectively. Thus, the objective function is the profit division ratio of the supplier. For Constraint (3.10b), it ensures that the supplier's profit when using price $p = r(1-\sum_{i=1}^{k-1} w_i)$ is the highest compared to other prices, which guarantees that the supplier will choose $p = r(1-\sum_{i=1}^{k-1} w_i)$, and thus, the retailer will choose $d_k$. Constraints (3.10c) are feasibility constraints for a probability density function.

By simple algebra, we can obtain the following inequalities from Constraint (3.10b)

$$c \leq \frac{rd_k \sum_{i=k}^{n} w_i - rd_j \sum_{i=j}^{n} w_i}{d_k - d_j}, \quad \forall j < k. \tag{3.11}$$

With these inequalities, for any $j < k$, we have

$$\begin{aligned} \left[ r\left(1 - \sum_{i=1}^{k-1} w_i\right) - c \right] d_k &\geq \left( r\sum_{i=k}^{n} w_i - \frac{rd_k \sum_{i=k}^{n} w_i - rd_j \sum_{i=j}^{n} w_i}{d_k - d_j} \right) d_k \\ &= \frac{rd_k d_j}{d_k - d_j} \sum_{i=j}^{k-1} w_i \geq r w_j d_j. \end{aligned} \tag{3.12}$$

Thus,

$$\begin{aligned} \gamma_k &= \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + r\sum_{i=1}^{k-1} w_i d_i} \\ &\geq \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + (k-1)[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k} = \frac{1}{k}, \end{aligned}$$

where the inequality follows from (3.12). Therefore, if the supplier chooses a price $p$ such that the retailer's optimal ordering quantity is $d_k$, then he will receive at least $1/k$

of the total supply chain's profit. Since the maximum value for $k$ is $n$, the lower bound of the profit devision ratio for an $n$-point distribution is $1/n$.

Next, we show that this bound is tight. Fix any $0 < \epsilon < 1$, we consider a case with $c = 0$ and the following demand distribution:

$$d_i = \begin{cases} \epsilon^{1/i}, & i = 1, \ldots, n-1 \\ 1, & i = n, \end{cases} \quad \text{with} \quad w_i = \begin{cases} \epsilon^{(i-1)/i} - \epsilon^{i/(i+1)}, & i = 1, \ldots, n-2 \\ \epsilon^{(n-2)/(n-1)} - \epsilon, & i = n-1 \\ \epsilon, & i = n. \end{cases}$$

Under this distribution, the supplier's expected profit for charging price $p = r(1 - \sum_{i=1}^{j-1} w_i)$ is

$$\Pi = r \sum_{i=j}^{n} w_i d_j = r\epsilon, \quad \forall j = 1, \ldots, n. \tag{3.13}$$

Therefore, the supplier's expected profit is $r\epsilon$ for all $j$. Thus, $p = rw_n$ is an optimal price, and the retailer's order quantity is $Q = d_n$. The profit division ratio in this case is

$$\gamma = \frac{w_n d_n}{w_n d_n + \sum_{i=1}^{n-1} w_i d_i} = \frac{\epsilon}{\epsilon + (n-1)\epsilon - \sum_{i=1}^{n-2} \epsilon^{(i^2+i+1)/(i^2+i)} - \epsilon^{n/(n-1)}}.$$

When $\epsilon$ approaches 0, $\gamma$ approaches $1/n$. Thus, the lower bound of $1/n$ is tight.

Next, we prove the second part of the theorem. Consider a demand distribution with support $d_i, i = 1, \ldots, n$ with $d_i < d_{i+1}$ for all $i$. So we have $d_{\min} = d_1$ and $d_{\max} = d_n$. Again, let $w_i = P(D = d_i)$ be the probability that demand is $d_i$. We note that although our analysis is done using discrete distribution, it can be extended to continuous distribution in a straightforward way.

Without loss of generality, we assume $d_{\min} = 1$. To prove the lower bound of $\gamma$, similar to part 1 of the proof, we consider the case where the retailer's optimal order quantity is $d_k$. In this case, the highest price the supplier can charge (in order for $d_k$ to be an optimal order quantity for the retailer) is $p = r(1 - \sum_{i=1}^{k-1} w_i)$. In the following, we define a new demand distribution. Let $N = \lceil \log_2 d_k \rceil$, and

$$\tilde{d}_i = \begin{cases} 2^{i-1}, & i = 1, \dots, N \\ d_k, & i = N+1 \end{cases} \quad \text{with} \quad \tilde{w}_i = \begin{cases} \sum_{l:\tilde{d}_i \le d_l < \tilde{d}_{i+1}} w_l, & i = 1, \dots, N \\ \sum_{l:d_l \ge \tilde{d}_i} w_l, & i = N+1. \end{cases}$$

Using the definition for $\tilde{d}_i$ and $\tilde{w}_i$, the profit division ratio satisfies

$$\begin{aligned} \gamma_k &= \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + r\sum_{i=1}^{k-1} w_i d_i} \\ &\ge \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + 2r\sum_{i=1}^{N} \tilde{w}_i \tilde{d}_i}. \end{aligned}$$

Let $j' = \lfloor \log_2 d_j \rfloor + 1$. We can obtain the following relationship between $[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k$ and $r\sum_{i=1}^{N} \tilde{w}_i \tilde{d}_i$,

$$\begin{aligned} \left[ r\left(1 - \sum_{i=1}^{k-1} w_i\right) - c \right] d_k &\ge \left( r\sum_{i=k}^{n} w_i - \frac{r d_k \sum_{i=k}^{n} w_i - r d_j \sum_{i=j}^{n} w_i}{d_k - d_j} \right) d_k \\ &= \frac{r d_k d_j}{d_k - d_j} \sum_{i=j}^{k-1} w_i \ge r \sum_{i=j}^{k-1} w_i d_j \ge \frac{1}{2} r \tilde{w}_{j'} \tilde{d}_{j'}, \end{aligned}$$

where the inequality follows from inequalities (3.11) and the last inequality is because $d_j \ge \frac{1}{2}\tilde{d}_{j'}$ and $\sum_{i=j}^{k-1} w_i \ge \tilde{w}_{j'}$. Therefore,

$$\gamma_k \ge \frac{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k}{[r(1-\sum_{i=1}^{k-1} w_i) - c]d_k + 2r\sum_{i=1}^{N} \tilde{w}_i \tilde{d}_i} \ge \frac{1}{4N+1} = \mathcal{O}\left(\frac{1}{\log d_{\max}}\right).$$

Hence, for any general distribution with support $[d_{\min}, d_{\max}]$, we have $\gamma \ge \mathcal{O}\left(\frac{1}{\log \frac{d_{\max}}{d_{\min}}}\right)$.

Next, we show that this bound is tight. We consider a case with $c = 0$ and the following demand distribution:

$$d_i = 2^{i-1}, \quad i = 1, \dots, n \quad \text{with} \quad w_i = \begin{cases} 2^{-i}, & i = 1, \dots, n-1 \\ 2^{1-n}, & i = n. \end{cases}$$

Under this distribution, the profit division ratio is

$$\gamma = \frac{\sum_{i=k}^{n} w_i d_k}{\sum_{i=k}^{n} w_i d_k + \sum_{i=1}^{k-1} w_i d_i} = \frac{1}{1 + \sum_{i=1}^{k-1} w_i d_i} = \frac{1}{1 + \sum_{i=1}^{k-1} 2^{-1}} = \frac{2}{k+1}.$$

The second equality is because $\sum_{i=k}^{n} w_i d_k = 1$ regardless of $k$. When $k = n$, $\gamma = \frac{2}{n+1} = \mathcal{O}\left(\frac{1}{\log \frac{d_{\max}}{d_{\min}}}\right)$. Thus, the lower bound is tight. $\square$

This theorem suggests that in a supplier-retailer relationship, the supplier is guaranteed to obtain a certain proportion of the supply chain profit. In particular, if the distribution has only few points or the range is narrow, the supplier is guaranteed to obtain a higher proportion.

## 3.4 Pricing Policy and Risk-Return Analysis in a Spot Market

In this section, we consider a newsvendor model with a slight variation. The model is the same as the original model except that the supplier can produce more than the retailer's order quantity and sell the excessive produces on the spot market when the retailer's order quantity cannot satisfy the market demand. The sequence of events that take place are as follows:

Before the start of the first period:

1. The supplier decides the price $p$. Then the retailer decides $Q$ based on $p$ and the demand distribution $F$. The retailer's problem is the original newsvendor's problem without holding and backorder cost. Thus, the optimal order quantity of the retailer is $Q = F^{-1}(\frac{r-p}{p})$.
2. The supplier decides how many units $u$ to stock-up as buffer inventory. The buffer inventory can be sold by the supplier directly to the customer on a spot market at the retail price. For each unit of inventory, the supplier incurs a production cost $c$.

During the selling period:

1. The retailer receives order quantity $Q$ and pays the supplier $pQ$.

2. Demand $D$ realizes. The retailer sells the amount $\min(D, Q)$ at price $r$.

3. If the retailer's order quantity $Q < D$, the supplier will sell $\min(u, D - Q)$ on the spot market at price $r$ to satisfy the demand. The supplier does not incur any backorder cost or holding cost. The supplier's total expected profit is

$$\Pi_S = pQ + r\mathbb{E}[\min(u, (D - Q)^+)] - c(Q + u). \tag{3.14}$$

In this model, the supplier can use two leverages to maximize his profit: the price and buffer inventory. We have the following theorem for the optimal decisions of the supplier.

**Theorem 13.** *Consider a supply chain setting with a spot market where the supplier could produce buffer inventory $u$ to meet unsatisfied demand. The supplier's optimal price and buffer inventory are in the following form,*

1. *The optimal buffer inventory for a given price $p$ is*

$$u^* = F^{-1}(\frac{r - c}{r}) - F^{-1}(\frac{r - p}{r}),$$

   *where $c$ is the supplier's production cost, $r$ is the market price, and $F(\cdot)$ is the CDF of the demand distribution.*

2. *The optimal price is $p^* = r$, and under this price, the optimal buffer inventory is $u^* = F^{-1}(\frac{r-c}{r})$.*

**Proof.** In this model, the decision process can be broken down into two stages. In the first stage, the supplier decides the optimal price $p^*$. In the second stage, he selects the optimal buffer inventory $u^*$ based on the retailer's order decision $Q^*$. To find the optimal decisions, we analyze the two decisions separately in a reverse order. We first study the optimal buffer inventory when the price $p$ is already determined. Under price $p$, the retailer orders $Q^* = F^{-1}(\frac{r-p}{r})$. The supplier then chooses the buffer inventory to maximizes the expected profit,

$$\begin{array}{ll} \max_u & \Pi_S = r\int_{Q^*}^{Q^*+u}(D - Q^*)f(D)dD + r\int_{Q^*+u}^{\infty} uf(D)dD - cu \\ \text{s.t.} & u \geq 0. \end{array}$$

Here, we left out the term $pQ^* - cQ^*$ because it is not relevant in the decision of $u$. Based on the first order condition, we have

$$\begin{aligned}\frac{\partial \Pi_S}{\partial u} &= ruf(u+Q^*) - ruf(Q^*+u) + r(1-F(Q^*+u)) - c \\ &= r(1-F(Q^*+u)) - c = 0.\end{aligned}$$

Therefore, the optimal buffer inventory under price $p$ is $u^* = F^{-1}(\frac{r-c}{r}) - F^{-1}(\frac{r-p}{r})$.

Next, we solve for the optimal price $p^*$ and maximize the supplier's total expected profit. Substituting $u^*$ into (3.14),

$$\begin{aligned}\max_p \quad & \Pi_S = pQ^* + r\int_{Q^*}^{Q^*+u}(D-Q^*)f(D)dD + r\int_{Q^*+u}^{\infty} uf(D)dD - c(Q^*+u) \\ \text{s.t.} \quad & p \geq 0.\end{aligned}$$

To find the optimal price $p$, let's take the first order derivative of $\Pi_S$, but this time, $u$ and $Q^*$ are also functions of $p$.

$$\begin{aligned}\frac{\partial \Pi_S}{\partial p} &= F^{-1}(\frac{r-p}{r}) + p\frac{\partial F^{-1}(\frac{r-p}{r})}{\partial p} - r\int_{Q^*}^{Q^*+u^*} f(D)dD\frac{\partial Q^*}{\partial p} + r(1-F(Q^*+u^*))\frac{\partial u}{\partial p} \\ & \quad -c\frac{\partial u}{\partial p} - c\frac{\partial F^{-1}(\frac{r-p}{r})}{\partial p} \\ &= F^{-1}(\frac{r-p}{r}) + p\frac{\partial F^{-1}(\frac{r-p}{r})}{\partial p} - r(F(Q^*+u^*) - F(Q^*))\frac{\partial F^{-1}(\frac{r-p}{r})}{\partial p} - c\frac{\partial F^{-1}(\frac{r-p}{r})}{\partial p} \\ &= F^{-1}(\frac{r-p}{r}).\end{aligned}$$

From the first order condition, the optimal price $p^*$ must satisfy $p^* = r$ and the optimal buffer inventory is $u^* = F^{-1}(\frac{r-c}{r})$. □

Theorem 13 shows a very interesting result. It indicates that when the supplier maximizes the expected profit, his optimal price is the market price, which leaves the retailer's optimal order quantity to be 0. Then, the supplier chooses the buffer inventory in a similar style as a newsvendor and faces the market demand directly. In other words, to maximize the expected profit, the supplier should take over the retailer's role. From the discussion in Section 3.2 we know that the retailer makes a profit by willing to take the risk before the demand realizes, and the more risk he is willing to take, the more profit he makes. In the original model, the supplier does not take any risk but can still take a fixed amount of profit. However, when the supplier is given the power to fully engage in the market, he is willing to sacrifice the “zero-risk” profit for the “high-risk” profit. For the supplier to obtain the maximum profit, a necessary thing to do is take all the market risk. Thus, the supplier’s decision is to choose a price to minimize the retailer’s order quantity and sell directly to the market demand.

Another observation from this theorem is that when the supplier’s profit is maximized, the two-echelon supply chain structure becomes centralized. One can think of it as downstream supply chain integration by the supplier. This is a common practice in supply chain management, and Theorem 13 provides some insights from the profit aspect on why suppliers prefer to integrate the retailers downstream.

As we mentioned, the supplier now faces uncertainty by having a buffer inventory before the demand realizes. Therefore, we are interested in the amount of risk the supplier takes. When the supplier uses price $p$ and buffer inventory $u$, his profit function can be expressed in the following three expressions.

1. $D \leq Q^*$. Profit $= pQ^* - c(Q^* + u)$.
2. $D \in (Q^*, Q^* + u]$. Profit $= pQ^* - c(Q^* + u) + r(D - Q^*)$.
3. $D > Q^* + u$. Profit $= pQ^* - c(Q^* + u) + ru$.

The supplier’s profit variance is

$$\begin{aligned}
\text{Var} &= \int_0^{Q^*} (pQ^* - c(Q^* + u))^2 f(D)dD + \int_{Q^*}^{Q^*+u} (pQ^* - c(Q^* + u) + r(D - Q^*))^2 f(D)dD \\
&\quad + \int_{Q^*+u}^{\infty} (pQ^* - c(Q^* + u) + ru)^2 f(D)dD - (\mathbb{E}[\text{profit}])^2 \\
&= -2rQ(pQ^* - c(Q^* + u))(F(Q^* + u) - F(Q^*)) + (pQ^* - c(Q^* + u))^2 \\
&\quad +(r^2u^2 + 2ru(pQ^* - c(Q^* + u)))(1 - F(Q^* + u)) \\
&\quad + \int_{Q^*}^{Q^*+u} (r^2(D - Q^*)^2 + 2rD(pQ^* - c(Q^* + u)))f(D)dD \\
&\quad -(pQ^* - c(Q^* + u) - rQ(F(Q^* + u) - F(Q^*)) + ru(1 - F(Q^* + u)) \\
&\quad +r \int_{Q^*}^{Q^*+u} Df(D)dD)^2.
\end{aligned}$$

From the proof of Theorem 13, we know that $F(Q^*+u)=\frac{r-c}{r}$ and $F(Q^*)=\frac{r-p}{r}$. Use these to simplify the profit variance function and obtain

$$\begin{aligned}
\mathrm{Var} &= cru^2-(pQ^*-c(Q^*+u))^2+\int_{Q^*}^{Q^*+u}\left(r^2(D-Q^*)^2+2rD(pQ^*-c(Q^*+u))\right)f(D)dD \\
&\quad -\left(r\int_{Q^*}^{Q^*+u}Df(D)dD\right)^2 \\
&= -(pQ^*-c(Q^*+u))^2+cru^2+ru^2(r-c)-r^2\int_{Q}^{Q+u}2(D-Q^*)F(D)dD \\
&\quad +2(pQ^*-c(Q^*+u))^2+2ru(pQ^*-c(Q^*+u))-2r(pQ^*-c(Q^*+u))\int_{Q}^{Q+u}F(D)dD \\
&\quad -\left(ru+pQ^*-c(Q^*+u)-r\int_{Q^*}^{Q^*+u}F(D)dD\right)^2 \\
&= -2r^2\int_{Q^*}^{Q^*+u}DF(D)dD+2r^2(Q^*+u)\int_{Q^*}^{Q^*+u}F(D)dD-\left(r\int_{Q^*}^{Q^*+u}F(D)dD\right)^2.
\end{aligned}$$

The last two equalities follow from integration by parts.

To see how the supplier's risk behaves as a function of $p$, we select a few common distributions to compute the profit variance. In the Uniform distribution with range $[a,b]$, the CDF is $F(x)=\frac{x-a}{b-a}$. The retailer's order quantity is $Q^*=\frac{(r-p)(b-a)}{r}+a$ and the supplier's buffer inventory satisfies $u+Q^*=\frac{(r-c)(b-a)}{r}+a$. The supplier's profit variance as a function of $p$ is

$$\mathrm{Var}(p) = \frac{(b-a)^2(p-c)^2(4r(2c+p)-3(p+c)^2)}{12r^2}.$$

The supplier's profit expectation as a function of $p$ is

$$\begin{aligned}
\mathbb{E}[\text{Profit}](p) &= pQ^*+\frac{ru^2}{2(b-a)}-\frac{ru(b-Q^*-u)}{b-a}-c(Q^*+u) \\
&= \frac{(p-c)(b(2r-c-p)+a(c+p+4r))}{2r}.
\end{aligned}$$

Now, let's look at some numerical experiments with a uniform distribution on $[0,100]$. We set $c=1/50, r=1$. For each value of price $p\in[c,r]$, we evaluate the supplier's profit standard deviation and profit expectation. Figure 3.5 shows the profit risk-return curve and the relationship between price and profit standard deviation on the right-hand side $y$-axis. As we can see from Figure 3.5, profit standard deviation and profit expectation both increase with the price. In addition, the profit standard deviation has an upper bound which comes from the variance of the uniform distribution. This upper bound is achieved when $p = r$, which is

$$\mathrm{Var} = \frac{(b-a)^2}{12}\frac{(r-c)^3(r+3c)}{r^2}.$$

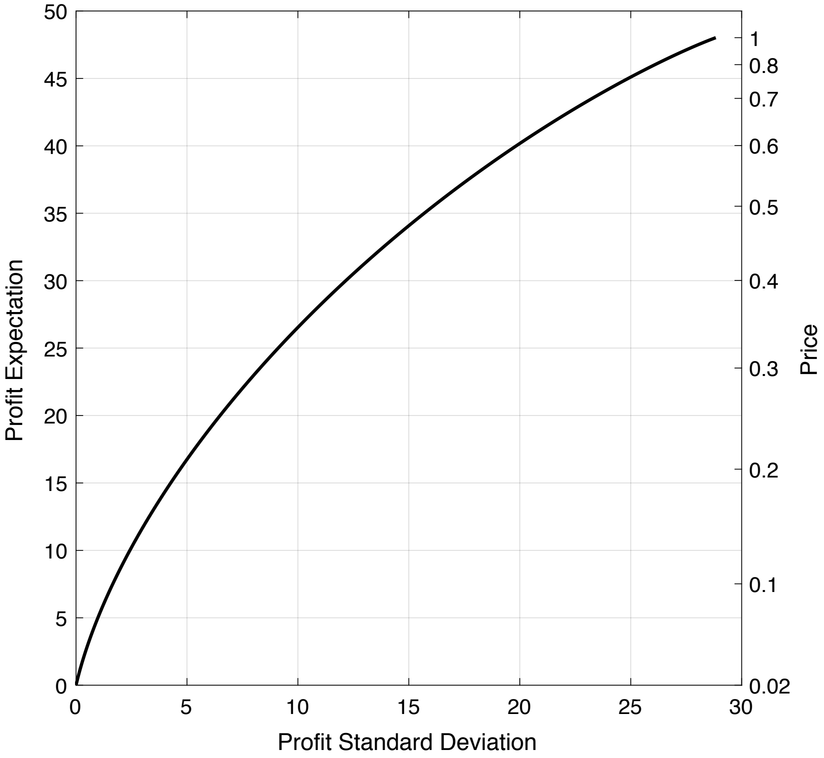

Figure 3.5: Supplier's profit standard deviation and expectation under a uniform distribution

Another continuous distribution we consider is the exponential distribution. The retailer's order quantity is $Q^* = \frac{\ln(r/p)}{\lambda}$, and the supplier's optimal buffer inventory satisfies $u + Q^* = \frac{\ln(r/c)}{\lambda}$. The supplier's profit variance as a function of $p$ is

$$\begin{aligned}
\mathrm{Var}(p) &= \frac{-e^{-2(Q^*+u)\lambda} r^2 (1 - 2e^{u\lambda} + e^{2u\lambda} - 2e^{(Q^*+2u)\lambda} + 2e^{(Q^*+u)\lambda}(1 + u\lambda))}{\lambda^2} \\
&= \frac{(p-c)(c-p+2r) - 2cr\ln(p/c)}{\lambda^2}.
\end{aligned}$$

The supplier's profit expectation as a function of $p$ is

$$\begin{aligned}
\mathbb{E}[\mathrm{Profit}](p) &= pQ^* - c(Q^* + u) + \frac{r}{\lambda} e^{-(Q^*+u)\lambda} \left( e^{u\lambda} - 1 \right) \\
&= \frac{p - c + p\ln(r/p) - c\ln(r/c)}{\lambda}.
\end{aligned}$$

We show a numerical example with an exponential distribution with parameter $\lambda = 1/50$. Again, we set $c = 1/40, r = 1$. The supplier's price range from $c$ to $r$. We plot the supplier's profit standard deviation and expectation as functions of the price. Figure 3.6 shows the profit standard deviation and expectation under the exponential distribution with $\lambda = 1/50$. Again, we can see that the profit standard deviation and expectation are increasing in price and there exists an upper bound for the profit standard deviation which is achieved when $p = r$,

$$\mathrm{Var} = \frac{r^2 - c^2 - 2cr\ln(r/c)}{\lambda^2}.$$

Comparing the uniform distribution and exponential distribution that has the same mean, we can see that when the price is $p = r = 1$, the uniform distribution achieves a slightly higher expected profit, at the cost of a much higher profit standard deviation.

## 3.5 Optimal Policies in the Revenue-Sharing Contract

In our original model, the supplier charges a price for each unit he sells to the retailer. The supplier keeps his profit and is not responsible for any unsold units of the retailer. The retailer takes all the risks in the market and makes a profit from taking the risks.

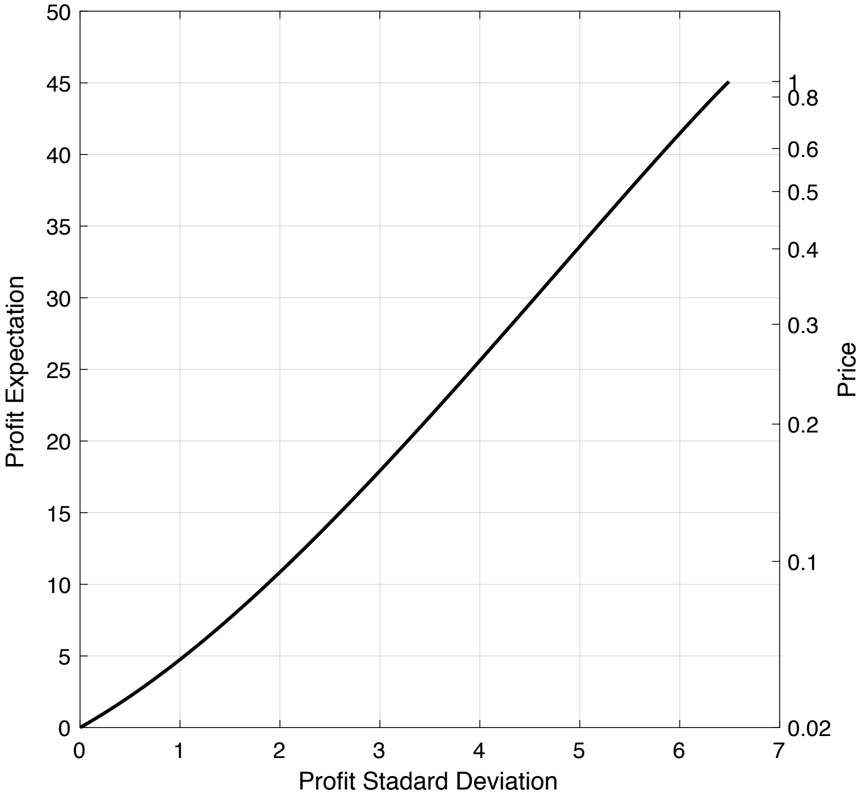

Figure 3.6: Supplier's profit standard deviation and expectation under an exponential distribution

Under such a model, members of the supply chain are only concerned about their own performance. This usually results in low supply chain profit overall Cachon (2003). To facilitate collaboration and maximize the total supply chain profit, members of a supply chain could participate in contracts that award others a portion of the firms' profit. One type of contract that achieves this goal is the revenue-sharing contract. In a revenue-sharing contract, the retailer takes only a portion $\phi$ of the profit and shares $(1 - \phi)$ of his revenue with the supplier in addition to the purchase price. We assume the inventory left at the end of the period has a salvage value of $v$ per unit. To avoid arbitrage opportunity, the salvage value should not exceed the purchase price, i.e., $v \leq p$. This salvage revenue is also shared between the supplier and the retailer. The retailer's profit function is

$$\begin{aligned}\Pi_R(p,Q,\phi) &= \phi r\min(Q,D)+\phi v(Q-\min(Q,D))-pQ\\ &= \phi(r-v)\min(Q,D)-(p-\phi v)Q.\end{aligned}$$

Thus, the total profit of the supplier is

$$\Pi_S(p,\phi)=pQ+(1-\phi)((r-v)\min(Q,D)+vQ).$$

Revenue-sharing contracts induce the supplier to choose a price that will optimize the overall supply chain revenue by aligning the supply chain profit with each firms' objective. Such type of contract can coordinate the supply chain and arbitrarily allocate the profit. Cachon and Lariviere (2005) studied the strengths and limitations of revenue-sharing contracts through analysis. Mortimer (2008) provided evidence that the introduction of revenue-sharing contracts increased the total profit in the video rental industry by 7%.

In a revenue sharing contract between the supplier and the retailer, the following events take place in order.

1. The supplier decides the sale price to the retailer ($p$). The supplier and retailer also agree on the portion of profit that is kept by the retailer ($\phi$). In other words, the supplier receives $r(1-\phi)$ for each unit sold and $v(1-\phi)$ for each unsold unit.

2. The retailer chooses the optimal order quantity $Q$ based on the price $p$ to maximize the expected profit function.

3. At the beginning of the selling period, the retailer receives order quantity $Q$ and pays the supplier $pQ$.

4. Demand $D$ realizes. The retailer sells the amount $\min(D,Q)$ at price $r$. If there are any unsold units by the end of the period, the retailer receives $v$ per unit as salvage revenue. The retailer pays the supplier $1-\phi$ of his revenue. Therefore, the retailer's actual profit is

$$\Pi_R=\phi(r-v)\min(Q,D)-(p-\phi v)Q.$$

5. The supplier's total profit is

$$\Pi_S = pQ + (1-\phi)((r-v)\min(Q,D) + vQ).$$

In this section, we first study the supplier's optimal strategies and the retailer's optimal response. For a revenue-sharing contract, we have the following theorem for the supplier and the retailer.

**Theorem 14.** *Let $f(\cdot)$ and $F(\cdot)$ denote the PDF and the CDF of the demand distribution on $[d_{\min}, d_{\max}]$, respectively. In a revenue-sharing contract with $\phi$ being the revenue kept by the retailer, the supplier's optimal price $p^*$ falls under one of the following cases,*

1. *When $\phi$ and $F(\cdot)$ satisfies $\phi D_{\max} \leq \frac{v}{r-v}\min\left(\frac{\partial F^{-1}(\xi)}{\partial \xi}\right)$, the optimal price is $p^* = \phi v$.*
2. *When $\phi$ and $F(\cdot)$ satisfies $\phi D_{\min} \geq \frac{r}{r-v}\max\left(\frac{\partial F^{-1}(\xi)}{\partial \xi}\right)$, the optimal price is $p^* = \phi r$.*
3. *Otherwise, the optimal price $p^*$ must satisfy*

$$p^* = \phi^2(r-v)f(F^{-1}(\xi))F^{-1}(\xi), \tag{3.15}$$

*where $\xi = \frac{\phi r - p}{\phi(r-v)}$.*

*The retailer's optimal order quantity under this price $p^*$ is $Q^* = F^{-1}(\frac{\phi r - p^*}{\phi(r-v)})$.*

**Proof.** We first show how to obtain the retailer's optimal order quantity for a given price $p$. Although in the revenue-sharing contract model, a term $\phi$ and a salvage value $v$ are introduced, the retailer's problem is still a newsvendor problem. We can think of $\phi(r-v)$ as the market price and $p - \phi v$ as the purchase cost. Therefore, under a certain price $p$, the retailer's optimal order quantity is $Q^* = F^{-1}(\frac{\phi(r-v)-(p-\phi v)}{\phi(r-v)})$, where $\phi r - p \geq 0$.

Next, we show the proof of the first part of the theorem. To arrive at the supplier's optimal price, we start from the supplier's profit maximization problem. The following problem is used to find the optimal price $p^*$ that maximizes the supplier's expected profit,

$$\begin{array}{ll} \max_p & \mathbb{E}\Pi_S = pQ^*(p) + (1-\phi)(r-v)\left(\int_0^{Q^*} Df(D)dD + \int_{Q^*}^{\infty} Q^*(p)f(D)dD\right) + (1-\phi)vQ^* \\ \text{s.t.} & p \leq \phi r, \\ & p \geq 0. \end{array}$$

The objective function simplifies to

$$\begin{aligned} \Pi_S &= pQ^* + (1-\phi)vQ^* + (1-\phi)(r-v)\left(Q^* - \int_0^{Q^*} F(D)dD\right) \\ &= pQ^* + (1-\phi)rQ^* - (1-\phi)(r-v)\int_0^{Q^*} F(D)dD. \end{aligned}$$

Take the derivative of $\Pi_S$ with respect to $p$,

$$\begin{aligned} \frac{\partial \Pi_S}{\partial p} &= Q^* + p\frac{dQ^*}{dp} + (1-\phi)r\frac{dQ^*}{dp} - (1-\phi)(r-v)F(Q^*)\frac{dQ^*}{dp} \\ &= F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right) + \frac{p}{\phi}\frac{dQ^*}{dp} \\ &= F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right) - \frac{p}{\phi^2(r-v)}\frac{\partial F^{-1}(\xi)}{\partial \xi}, \end{aligned}$$

where $\xi = \frac{\phi r - p}{\phi(r-v)}$.

In case 1, when $\phi D_{\max} \leq \frac{v}{r-v}\frac{\partial F^{-1}(\xi)}{\partial \xi}$ holds,

$$\begin{aligned} \frac{\partial \Pi_S}{\partial p} &= F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right) - \frac{p}{\phi^2(r-v)}\frac{\partial F^{-1}(\xi)}{\partial \xi} \\ &\leq D_{\max} - \frac{v}{\phi(r-v)}\min(\frac{\partial F^{-1}(\xi)}{\partial \xi}) \\ &\leq 0. \end{aligned}$$

The first inequality follows from $D_{\max} \geq F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right)$ and $p \geq \phi v$. Therefore, the supplier's expected profit is decreasing in $p$. The optimal price is the lower bound of $p$ which is $p^* = \phi v$.

In case 2, when $\phi D_{\min} \geq \frac{r}{r-v}\frac{\partial F^{-1}(\xi)}{\partial \xi}$ holds,

$$\begin{aligned}\frac{\partial \Pi_S}{\partial p} &= F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right) - \frac{p}{\phi^2(r-v)}\frac{\partial F^{-1}(\xi)}{\partial \xi} \\ &\geq D_{\min} - \frac{r}{\phi(r-v)}\max(\frac{\partial F^{-1}(\xi)}{\partial \xi}) \\ &\geq 0.\end{aligned}$$

Therefore, the supplier's expected profit is increasing in $p$. The optimal price is the upper bound of $p$ which is $p^* = \phi r$.

In case 3, the supplier's expected profit is non-monotone in $p$. The optimal price must satisfy the first order condition.

$$\frac{\partial \Pi_S}{\partial p} = F^{-1}\left(\frac{\phi r - p}{\phi(r-v)}\right) - \frac{p}{\phi^2(r-v)}\frac{\partial F^{-1}(\xi)}{\partial \xi} = 0.$$

Thus, the optimal price must satisfy,

$$p^* = \phi^2(r-v)f(F^{-1}(\xi))F^{-1}(\xi).$$

□

When a specific demand distribution and a set of parameters are given, the optimal price and order quantity can be calculated using Theorem 14. The revenue-sharing parameter $\phi$ also plays a part in the supplier's and the retailer's decision. In the first two cases, the supplier's prices are increasing in $\phi$. This result is consistent with what we see in practice. When $\phi$ is high, the retailer keeps more profit to himself and the supplier receives less from the retailer. To maximize the supplier's revenue, it is optimal to increase the wholesale price. In the third case, the effect of $\phi$ on the supplier's decision depends on the form of the demand distribution. We will look at some specific cases later in this section.

In the following part, we study the supplier and the retailer's risk-return trade-off. We will use $Q^*$ and $p^*$ to denote the supplier and retailer's optimal decisions solved from this theorem. Given that the retailer places the optimal order quantity $Q^*(p^*)$

based on the supplier's optimal price, the supplier's profit variance is

$$\begin{aligned}\mathrm{Var}_S &= \int_0^{Q^*} (p^*Q^* + (1-\phi)((r-v)D + vQ^*))^2 f(D)dD + \int_{Q^*}^{\infty} (p^*Q^* + (1-\phi)rQ^*)^2 f(D)dD \\ &\quad -(\mathbb{E}\Pi_S)^2.\end{aligned}$$

Under the optimal price $p^*$, the retailer's profit expectation is

$$\begin{aligned}\mathbb{E}\Pi_R &= \phi(r-v)\left(\int_0^{Q^*} Df(D)dD + Q^*(1-F(Q^*))\right) - (p^* - \phi v)Q^* \\ &= \phi(r-v)\int_0^{Q^*} Df(D)dD.\end{aligned}$$

The retailer's profit variance is

$$\begin{aligned}\mathrm{Var}_R &= \int_0^{Q^*} (\phi(r-v)D - (p^* - \phi v)Q^*)^2 f(D)dD + (\phi(r-v)Q^* - (p^* - \phi v)Q^*)^2(1-F(Q^*)) \\ &\quad -(\mathbb{E}\Pi_R)^2.\end{aligned}$$

The next corollary discusses the effect of $\phi$ on the supplier and the retailer's decision and profit expectation and variance.

**Corollary 4.** *In a revenue-sharing contract, when $\phi$ is in the range that satisfies*

$$\phi D_{\max} \leq \frac{v}{r-v}\frac{\partial F^{-1}(\xi)}{\partial \xi} \tag{3.16}$$

*or*

$$\phi D_{\min} \geq \frac{r}{r-v}\frac{\partial F^{-1}(\xi)}{\partial \xi} \tag{3.17}$$

*the optimal price is increasing in $\phi$.*

*When $\phi$ is outside of the range, the supplier's optimal price $p^*$ is increasing in $\phi$ when the demand distribution satisfies*

$$\begin{aligned}&\tfrac{\partial F^{-1}(\xi)}{\partial \xi}\left(F^{-1}(\xi)\tfrac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right) \geq \max\left\{-\tfrac{2\phi(r-v)f(F^{-1}(\xi))F^{-1}(\xi)}{p^*}, -\tfrac{1}{\phi}\right\}, \\ &\text{or} \\ &\tfrac{\partial F^{-1}(\xi)}{\partial \xi}\left(F^{-1}(\xi)\tfrac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right) \leq \min\left\{-\tfrac{2\phi(r-v)f(F^{-1}(\xi))F^{-1}(\xi)}{p^*}, -\tfrac{1}{\phi}\right\}.\end{aligned} \tag{3.18}$$

*The retailer's optimal order quantity is decreasing in $\phi$ when demand distribution and price satisfy $p^* - \phi\frac{dp^*}{d\phi} \leq 0$, and increasing in $\phi$ when $p^* - \phi\frac{dp^*}{d\phi} \geq 0$.*

**Proof.** To start with, we first look at how price $p^*$ changes with $\phi$. When $\phi$ is within the range in (3.16) and (3.17), the optimal price is $\phi v$ and $\phi r$, respectively. It is clear that the optimal price is increasing in $\phi$.

For $\phi$ that is outside of this range, we take the derivative of condition (3.15). To simplify notation, we use $RHS$ to denote the right-hand side of (3.15). Remember that in (3.15), $\xi$ is also a function of $p^*$. From (3.15) we have

$$\frac{dp^*}{d\phi} = \frac{\partial RHS}{\partial p^*}\frac{dp^*}{d\phi} + \frac{\partial RHS}{\partial \phi}$$

Therefore, we have

$$\frac{dp^*}{d\phi} = \frac{\partial RHS}{\partial \phi}\frac{1}{1-\frac{\partial RHS}{\partial p}}.$$

For each term in the equation above, we have

$$\begin{aligned}\frac{\partial RHS}{\partial \phi} &= 2\phi(r-v)f(F^{-1}(\xi))F^{-1}(\xi) + \phi^2(r-v)\frac{\partial F^{-1}(\xi)}{\partial \xi}\frac{d\xi}{d\phi}\left(F^{-1}(\xi)\frac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right)\\ &= 2\phi(r-v)f(F^{-1}(\xi))F^{-1}(\xi) + p^*\frac{\partial F^{-1}(\xi)}{\partial \xi}\left(F^{-1}(\xi)\frac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right).\end{aligned}$$

$$\begin{aligned}\frac{\partial RHS}{\partial p} &= \phi^2(r-v)\frac{\partial F^{-1}(\xi)}{\partial \xi}\frac{d\xi(p)}{dp}\left(F^{-1}(\xi)\frac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right)\\ &= -\phi\frac{\partial F^{-1}(\xi)}{\partial \xi}\left(F^{-1}(\xi)\frac{\partial f(F^{-1}(\xi))}{\partial F^{-1}(\xi)} + f(F^{-1}(\xi))\right).\end{aligned}$$

When conditions (3.18) in Corollary 4 are satisfied, $\frac{\partial RHS}{\partial \phi}\frac{1}{1-\frac{\partial RHS}{\partial p}} \geq 0$. Therefore, $\frac{dp^*}{d\phi} \geq 0$ and $p^*$ is increasing in $\phi$.

Next, we study the retailer's optimal order quantity. The partial derivative of $Q^*$ with respect to $\phi$ is

$$\frac{dQ^*}{d\phi} = \frac{\partial F^{-1}(\xi)}{\partial \xi}\left(-\frac{\phi r - p^*}{\phi^2(r-v)} + \frac{r - \frac{dp^*}{d\phi}}{\phi(r-v)}\right) = \frac{\partial F^{-1}(\xi)}{\partial \xi}\frac{p^* - \phi\frac{dp^*}{d\phi}}{\phi^2(r-v)}.$$

This derivative is nonpositive when $p^* - \phi\frac{dp^*}{d\phi} \leq 0$, meaning the optimal order quantity is decreasing in $\phi$. Otherwise, the optimal order quantity is increasing in $\phi$. □

Next, we extend our analysis to study how the retailer's expected profit changes with $\phi$.

$$\frac{d\mathbb{E}\Pi_R}{d\phi} = (r-v)\left(\int_0^{Q^*} Df(D)dD + \phi Q^* f(Q^*)\frac{dQ^*}{d\phi}\right).$$

The effect of $\phi$ on the retailer's expected profit can be derived from the partial derivative of the optimal order quantity $Q^*$. When $\frac{\partial Q^*}{\partial \phi} \leq \frac{\int_0^{Q^*} Df(D)dD}{\phi Q^* f(Q^*)}$, the retailer's expected profit is increasing in $\phi$. Otherwise, it is decreasing in $\phi$.

For the supplier's expected profit, we have

$$\begin{aligned}\frac{d\mathbb{E}\Pi_S}{d\phi} &= (-r+v)\int_0^{Q^*} Df(D)dD - p^*(Q^* - \phi\frac{dQ^*}{d\phi}) + \frac{Q^*}{\phi}(\frac{dp^*}{d\phi} + \phi(r-v)(1-\phi)f(Q^*)\frac{dQ^*}{d\phi}) \\ &= -\frac{d\mathbb{E}\Pi_R}{d\phi} + \frac{1}{\phi^2}\left(\phi^2(r-v)Q^*f(Q^*)\frac{dQ^*}{d\phi} + \phi Q^*\frac{dp^*}{d\phi} - p^*(Q^* - \phi\frac{dQ^*}{d\phi})\right).\end{aligned}$$

The expression depends on $\frac{d\mathbb{E}\Pi_R}{d\phi}$, $\frac{dQ^*}{d\phi}$, and $\frac{dp^*}{d\phi}$. To see some concrete results on how $\phi$ influences each party's optimal decision and profit, we study some specific groups of distributions in the next section.

### 3.5.1 Revenue-Sharing Contracts Under Common Distributions

In this section, we consider some common distributions and analyze the effect of $\phi$ on the supplier and the retailer's decisions. Specifically, we look at the uniform distribution and the exponential distribution.

#### Uniform Distribution

Assume the demand follows a uniform distribution with parameters $[a,b]$. Then, condition (3.15) becomes

$$\begin{aligned}p^* &= \frac{\phi^2(r-v)}{b-a}\left(a + \frac{(\phi r - p^*)(b-a)}{\phi(r-v)}\right) \\ &= \frac{\phi^2(r-v)a}{b-a} + \phi(\phi r - p^*).\end{aligned}$$

Solving for $p^*$ we have

$$p^* = \frac{\phi^2(br - av)}{(1+\phi)(b-a)}.$$

The optimal order quantity is

$$Q^* = a + \frac{(\phi r - p^*)(b-a)}{\phi(r-v)} = \frac{br - av}{(r-v)(1+\phi)}.$$

Here, we can see the optimal price is increasing in $\phi$ and the optimal order quantity is decreasing in $\phi$. When $\phi = 1$, the retailer keeps all the profit. In order to maximize the profit, the supplier will charge the highest price, resulting in a smaller order quantity from the retailer. When $\phi = 0$, the supplier takes the entire supply chain's profit. To maximize the profit, the supplier should choose the lowest price (0 in this case), resulting in a larger optimal order quantity. This is because reducing the retailer's purchase price encourages the retailer to increase the order quantity under the same demand distribution and market price. This will increase the expected number of units sold, in turn, improving the retailer's profit. Since all the profit is shared with the supplier, the supplier's profit is maximized.

Figure 3.7 shows a numerical example of the optimal price and order quantity as a function of $\phi$. In this example, we use a model with uniform distribution on $[0, 100]$. The market price is $r = 1$. The retailer receives a salvage value of $v = 0.2$ per unit from the unsold inventory. The supplier's optimal price is increasing in $\phi$, which is shown in black and the retailer's optimal order quantity is decreasing in $\phi$, which is shown in red.

Next, we look at the profit expectation and variances. The retailer's expected profit under a uniform distribution is

$$\begin{aligned}
\mathbb{E}\Pi_R &= \phi(r-v)\int_0^{Q^*} Df(D)dD \\
&= \frac{\phi(r-v)}{b-a}\frac{Q^{*2}}{2} \\
&= \frac{\phi(br-av)^2}{2(b-a)(r-v)(1+\phi)^2}.
\end{aligned}$$

The retailer's expected profit is increasing in $\phi$ and bounded above by $\frac{(br-av)^2}{8(b-a)(r-v)}$.

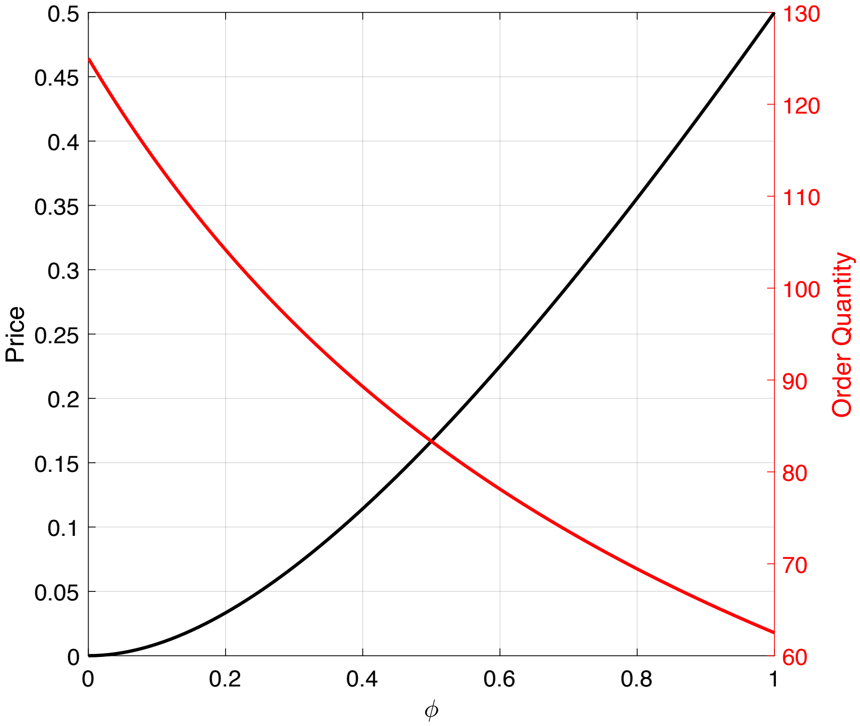

Figure 3.7: The optimal price and order quantity for various $\phi$s when $D \sim \text{Unif}[0, 100]$

The retailer's profit variance under a uniform distribution is

$$\begin{aligned}\text{Var}_R &= \frac{1}{b-a}\int_0^{Q^*} (\phi(r-v)D - (p^* - \phi v)Q^*)^2 dD + (\phi r - p^*)^2 Q^{*2}\frac{p - \phi v}{\phi(r-v)} - (\mathbb{E}\Pi_R)^2 \\ &= \frac{Q^{*2}(-p + r\phi)^2(p^* - v\phi)}{(r-v)\phi} + \frac{Q^{*3}(3p^{*2} - 3p^*(r+v)\phi + (r^2 + rv + v^2)\phi^2)}{3(-a+b)} - (\mathbb{E}\Pi_R)^2.\end{aligned}$$

Figure 3.8 shows a numerical example of the retailer's risk-return curve. The $x$-axis is the retailer's profit standard deviation under different values of $\phi$ and the left $y$-axis is the profit expectation under different $\phi$s. The corresponding $\phi$ can be found on the right $y$-axis. From this figure, we can see that the retailer's profit standard deviation is also increasing in $\phi$. Having a larger $\phi$ guarantees the retailer a higher expected profit, but also at a higher risk.

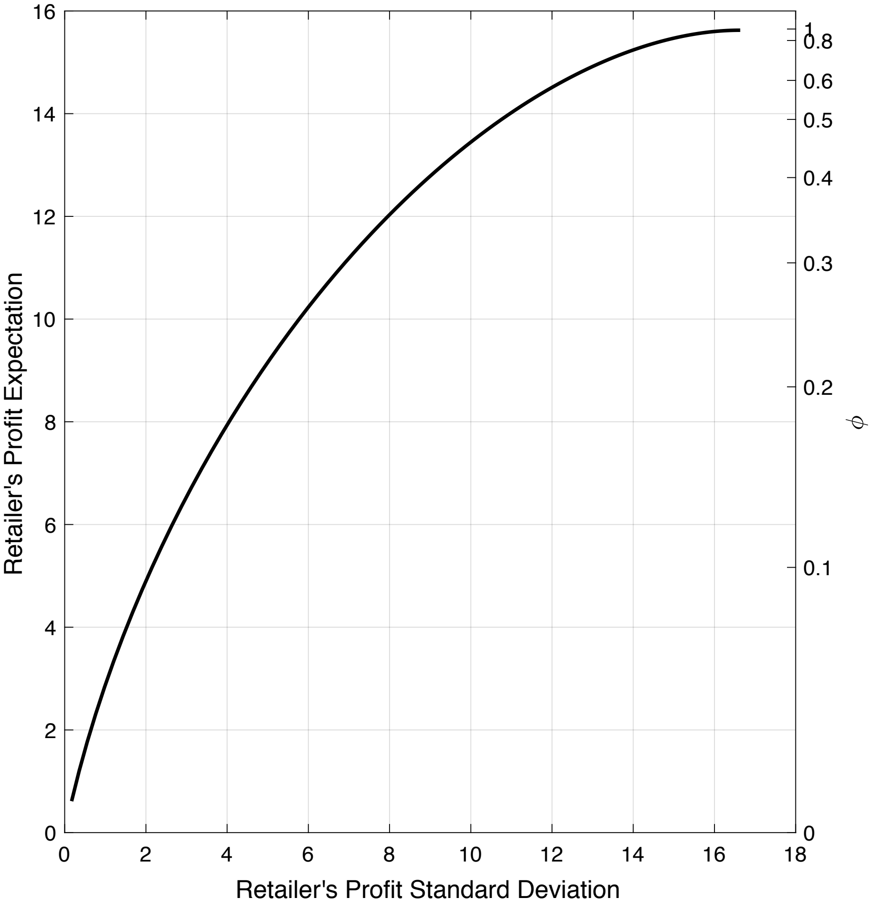

Figure 3.8: Retailer's risk-return curve for various $\phi$s when $D \sim \text{Unif}[0,100]$

The focus of the following part is on the supplier's risk-return trade-off. The supplier's expected profit is

$$\begin{aligned}\mathbb{E}\Pi_S &= p^*Q^* + (1-\phi)(r-v)\left(\int_0^{Q^*}\frac{D}{b-a}dD + Q^*\frac{p^*-\phi v}{\phi(r-v)}\right) + (1-\phi)vQ^* \\ &= Q^*\frac{2p^*(b-a)+Q^*(r-v)(1-\phi)\phi}{2\phi(b-a)} \\ &= \frac{(br-av)^2}{2(b-a)(r-v)(1+\phi)}.\end{aligned}$$

The supplier's expected profit is decreasing in $\phi$ and bounded below by $\frac{(br-av)^2}{4(b-a)(r-v)}$. The supplier's profit variance is rather complicated once we substitute in the formula of the optimal order quantity and price, as well as the supplier's expected profit. Although we do not show the full closed-form expression here, keep in mind that such an expression exists and can be computed quickly.

### Exponential Distribution

Assume the demand follows an exponential distribution with parameter $\lambda$. When $\phi$ is close to zero, the supplier's expected profit is decreasing in $p$ and the optimal price is $\phi v$. For $\phi$s that are outside of the range, when the supplier's price is $p$, the retailer's optimal order quantity is $Q^* = \frac{\ln \frac{\phi(r-v)}{p-\phi v}}{\lambda}$. In this case, condition (3.15) becomes

$$\begin{aligned} p^* &= \phi^2(r-v)\frac{\lambda(p^*-\phi v)}{\phi(r-v)}\frac{\ln\frac{\phi(r-v)}{p^*-\phi v}}{\lambda} \\ &= \phi(p^*-\phi v)\ln\frac{\phi(r-v)}{p^*-\phi v}. \end{aligned}$$

In this equation, the $\lambda$ term disappears during simplification, meaning that the supplier's optimal price that maximizes the expected profit does not rely on the distribution parameter. However, the retailer needs to know the demand expectation in order to find the optimal order quantity. Solving this equation could give us the supplier's optimal price for each $\phi$. However, this function does not have a closed-form solution. To study the monotonicity of $p^*$ with respect to $\phi$, we take the partial derivative of $p$ with respect to $\phi$. Let $RHS$ denote the right-hand side of the equation above.

$$\begin{aligned} \frac{dp}{d\phi} &= \frac{\partial RHS}{\partial p}\frac{dp}{d\phi} + \frac{dRHS}{d\phi} \\ &= \left(\phi\ln\frac{\phi(r-v)}{p^*-\phi v} - \phi\right)\frac{dp}{d\phi} + p + (p-2\phi v)\ln\frac{\phi(r-v)}{p^*-\phi v}. \end{aligned}$$

We have

$$\frac{dp}{d\phi} = \frac{p + (p-2\phi v)\ln\frac{\phi(r-v)}{p^*-\phi v}}{1+\phi-\phi\ln\frac{\phi(r-v)}{p^*-\phi v}}.$$

According to the model discussion, we can think of $\phi(r-v)$ as the market price and $p-\phi v$ as the purchase cost. The retailer's purchase cost should be lower than the market price. Thus, $\frac{\phi(r-v)}{p^*-\phi v}$ is greater than 1. When $\ln \frac{\phi(r-v)}{p^*-\phi v} \leq \min\left(\frac{1+\phi}{\phi}, \frac{p}{-p+2\phi v}\right)$, or when $\ln \frac{\phi(r-v)}{p^*-\phi v} \geq \max\left(\frac{1+\phi}{\phi}, \frac{p}{-p+2\phi v}\right)$, $\frac{dp}{d\phi}$ is greater than 0, i.e., $p$ is increasing in $\phi$.

Let's look at a numerical example with $\lambda = 0.02$. The cost parameters are still the same with the other examples, so we have $r = 1, v = 0.2$. After checking the range of $\phi$ in Theorem 14, it turns out that the optimal $p^*$ solved from condition (3.15) is outside of the feasible range $[\phi v, \phi r]$. In fact, the first order partial derivative of the supplier's profit function is strictly decreasing in $p$. Thus, the optimal price is $\phi v$. Therefore, the optimal price is increasing in $\phi$. The corresponding optimal ordering strategy is to match the maximum demand. When the demand follows an exponential distribution, the model parameters $r, v$ and $\phi$ must satisfy the following relationship to have a $p^*$ inside of the range $[\phi v, \phi r]$,

$$\phi > \frac{1}{\text{ProductLog}(\frac{r-v}{ve})}.$$

The retailer's expected profit under an exponential distribution is given below,

$$\begin{aligned} \mathbb{E}\Pi_R &= \phi(r-v)\int_0^{Q^*} D\lambda e^{-\lambda D} dD \\ &= \frac{1 - e^{-\lambda Q^*}(1+\lambda Q^*)}{\lambda}. \end{aligned}$$

From this, we can see that the retailer's expected profit is increasing in $Q^*$.

## 3.6 Conclusion

In this chapter, we focused on the risk-return analysis for several single-period newsvendor models. The first model we consider is a classic newsvendor model without holding cost or salvage values. We show the closed-form solution of the two-point (Section 3.2.1) and the three-point distribution functions (Section 3.2.2) that achieve the maximum expected profit under a given variance or risk. Next, we generalize the analysis to the multi-point distributions in Section 3.2.3. We present the linear programs that are used to find the risk-return curves. We characterize some properties of the risk-return curves as well as the optimal distribution. For the continuous distributions, we provide the model for solving the risk-return curve which is an infinite program. An approximation algorithm is presented in Section 3.2.4 to efficiently solve for the risk-return curves. This section also studies the supply chain profit allocation between the supplier and the retailer. We obtain lower bounds on the proportion of profits gained by the supplier and the distributions that achieve the lower bounds.

The second model we consider is a spot market model. The retailer still purchases from the supplier using the classic newsvendor model before the period starts; however, in this model, there is a spot market where the supplier could sell directly to the unsatisfied demand. We derive the supplier's optimal pricing and buffer inventory and show the supplier's risk-return curve when demand follows a uniform distribution and an exponential distribution.

In the first two models, the supplier and the retailer uses a wholesale-price contract. In the third model, a revenue-sharing contract is adopted. There is a salvage value for each unsold item for the retailer. Both the retailer's sales revenue and salvage revenue are shared with the supplier. In this model, we obtain the optimality condition for the supplier's price. We also characterize how $\phi$ influences the supplier and the retailer's optimal decisions and their profit expectation and variation in closed-form under uniform and exponential demand distributions.

# Chapter 4

# Online Learning Algorithms for Matching Problems

## 4.1 Introduction and Literature Review

In this chapter, we consider a type of online resource allocation problem — online matching problems. Online matching problems are considered as fundamental problems in online optimization theory and have important applications in the online advertisement allocation problems. In the problem we study, there is an underlying weighted bipartite graph $G = (I, J, E)$ with weights $b_{ij}$ for each edge $(i, j) \in E$. The vertices in $J$ arrive sequentially in some order, and whenever a vertex $j \in J$ arrives, the set of weights $b_{ij}$ is revealed for all $i \in I$, $(i, j) \in E$. The decision maker then has to match $j$ to one of its neighbors $i$, and a value of $b_{ij}$ will be obtained from this matching. In our problem, the decision maker's gain from each vertex $i$ is a function of the total matched value to this vertex, and his goal is to maximize the total gain from all vertices. Mathematically, the problem can be formulated as follows (assume $|I| = m$, $|J| = n$, and let $b_{ij} = 0$ for

$(i,j) \notin E$):

$$\begin{array}{lll} \text{maximize}_{\boldsymbol{x}} & \sum_{i=1}^{m} M_i\left(\sum_{j=1}^{n} b_{ij}x_{ij}\right) & \\ \text{s.t.} & \sum_{i=1}^{m} x_{ij} \leq 1, & \forall j \\ & x_{ij} \geq 0, & \forall i,j, \end{array} \tag{4.1}$$

where $x_{ij}$ denotes the fraction of vertex $j$ that is matched to vertex $i$.[1] In (4.1), the coefficient $\boldsymbol{b}_j = \{b_{ij}\}_{i=1}^{m}$ is revealed only when vertex $j$ arrives, and an irrevocable decision $\boldsymbol{x}_j = \{x_{ij}\}_{i=1}^{m}$ has to be made before observing the next input. For each $i$, $M_i(\cdot)$ is a nondecreasing concave function with $M_i(0) = 0$. In this chapter, we assume that $M_i(\cdot)$s are continuously differentiable.

As mentioned earlier, online matching problems have a very important application in the online advertisement allocation problem, which we will later refer to as the Adwords problem. In the Adwords problem, there are $m$ advertisers (which we also call the bidders). A sequence of $n$ keywords are searched during a fixed time horizon. Based on the relevance of the keyword, the $i$th bidder would bid a certain amount $b_{ij}$ to show his advertisement on the result page of the $j$th keyword. The search engine's decision is to allocate each keyword to one of the $m$ bidders (we only consider a single allocation in this paper). Note that each allocation decision can only depend on the information earlier in the arrival sequence but not on any future data. As pointed out in Devanur and Jain (2012), there are several practical motivations for considering a concave function of the matched bids in the Adwords problem. Among them are convex penalty costs for under-delivery in search engine-advertiser contracts, the concavity of the click-through rate in the number of allocated bids observed in empirical data and fairness considerations. In each of the situations mentioned above, one can write the objective as a concave function. We refer the readers to Devanur and Jain (2012) for a more thorough review of the motivations for this problem. It is worth noting that there is a special case of this problem where $M_i(x) = \min\{x, B_i\}$. In this case, one can view that the bidder has a budget $B_i$ and the revenue from each bidder is bounded by $B_i$.

One important question when studying online algorithms is the assumptions on the

---

[1] We allow fractional allocations in our model. However, our proposed algorithms output integer solutions. Thus all our results hold if one confines to integer solutions.

input data. In this work, we adopt a *random permutation model.* More precisely, we assume:

1. The total number of arrivals $n = |J|$ is known a priori.
2. The weights $\{b_{ij}\}$ can be adversarially chosen. However, the order that $j$ arrives is uniformly distributed over all the permutations.

The random permutation model has been adopted in much recent literature in the study of online matching problems, see, e.g., Agrawal et al. (2014), Devanur and Hayes (2009), Feldman et al. (2010), etc. It is equivalent to saying that a set of $\mathcal{B} = \{\tilde{\boldsymbol{b}}_1, \tilde{\boldsymbol{b}}_2, ..., \tilde{\boldsymbol{b}}_n\}$ is arbitrarily chosen beforehand (unknown to the decision maker). Then the arrivals $\boldsymbol{b}_1, \boldsymbol{b}_2, ..., \boldsymbol{b}_n$ are drawn randomly without replacement from $\mathcal{B}$. The random permutation model is an intermediate path between using a worst-case analysis and assuming each input data is drawn independently and identically distributed (i.i.d.) from a certain distribution. On one hand, compared to the worst-case analysis (see, e.g., Buchbinder et al. (2007), Devanur and Jain (2012), Feldman et al. (2009), Mehta et al. (2005)), the random permutation model is practically reasonable yet much less conservative. On the other hand, the random permutation model is much less restrictive than assuming the inputs are drawn i.i.d. from a certain distribution (Devanur (2011)). Also, the assumption of the knowledge of $n$ is necessary for any online algorithm to achieve near-optimal performance (see Devanur and Hayes (2009)). Therefore, for large problems with relatively stationary inputs, the random permutation model is a good approximation and the study of such models is of practical interest. Next we define the performance measure of an algorithm under the random permutation model:

**Definition 3** ($c$-competitiveness)**.** *Let OPT be the optimal value for the offline problem (4.1). An online algorithm A is called c-competitive in the random permutation model if the expected value of the online solutions by using A is at least c times the optimal value of (4.1), that is*

$$\mathbb{E}_{\sigma}\left[\sum_{i=1}^{m} M_i\left(\sum_{j=1}^{n} b_{ij}x_{ij}(\sigma, A)\right)\right] \geq cOPT,$$

*where the expectation is taken over uniformly random permutations* $\sigma$ *of* $1, ..., n$*, and* $x_{ij}(\sigma, A)$ *is the* $ij$*th decision made by algorithm* $A$ *when the inputs arrive in order* $\sigma$*.*

In Devanur and Jain (2012), the authors propose an algorithm for the online matching problem with concave returns that has a constant competitive ratio under the *worst-case model* (the constant depends on the forms of each $M_i(\cdot)$). They also show that a constant competitive ratio is the *best* possible result under that model. In this chapter, we propose an algorithm under the random permutation model, which achieves *near-optimal performance* under some conditions on the input.

Our main result is stated as follows:

**Theorem 15.** *Fix* $\epsilon \in (0, 1/2)$*. There exists an algorithm (Algorithm DLA) that is* $1-\epsilon$ *competitive for the online matching problem with concave returns* $M_i(\cdot)$*s under the random permutation model if*

$$n \geq \Omega\left(\max\left\{\frac{\log(m/\epsilon)}{\epsilon\bar{b}^2}, \frac{m^2\log(m^2n/\epsilon)F(M,\eta)}{\epsilon^3\bar{b}}\right\}\right), \tag{4.2}$$

*where* $\bar{b} = \frac{1}{n}\min_i\{\sum_{j=1}^n b_{ij}\}$*,* $\eta = \frac{\min_{i,j}\{b_{ij}|b_{ij}>0\}}{\max_{i,j} b_{ij}}$*, and* $F(M, \eta)$ *is a constant that only depends on each* $M_i(\cdot)$ *and* $\eta$*.*

In condition (4.2), $\bar{b}$ can be viewed as the average bid value of a bidder over time. Given that each bidder is at least interested in some fractions of the keywords, this average will go to a certain constant as $n$ becomes large. Also, $\eta$ can be viewed as the ratio between the value of the smallest non-zero bid and the highest bid. In practice, this is often bounded below by a constant by enforcing a reserve price and a maximum price for any single bid. The exact functional form of $F(M, \eta)$ is somewhat complicated, and is given in Proposition 2. Just to give an example, if we choose $M_i(x) = x^p$ $(0 < p < 1)$, then $F(M, \eta) = \frac{2}{\eta^{(2-p)/(1-p)}}$. Therefore, condition (4.2) can be viewed as simply requiring the total number of inputs is large, which is often the case in practice. For example, in the Adwords problem, $n$ is the number of keyword searches in a certain period, and for instance, Google receives more than 5 billion searches per day. Even if we focus on a specific category, the number can still be in the millions. Thus, this condition is reasonable. We note that most learning algorithms in the literature make similar requirements, see Agrawal et al. (2014), Devanur and Hayes (2009), Molinaro and Ravi (2014). Furthermore, as we will show in our numerical tests, our algorithm performs well even for problems with sizes that are significantly smaller than the condition requires, which validates the potential usefulness of our algorithm.

To propose an algorithm that achieves near-optimal performance, the main idea is to utilize the observed data in the allocation process. In particular, since the input data arrives in a random order, using the past input data and projecting it into the future should present a good approximation for the problem. To mathematically capture this idea, we use a primal-dual approach. We obtain the dual optimal solutions to suitably constructed optimization problems and use them to assist with future allocations. We first propose a one-time learning algorithm (OLA, see Section 4.2) that only solves an optimization problem once at time $\epsilon n$. By carefully examining this algorithm, we prove that it achieves near-optimal performance when the inputs satisfy certain conditions. However, the conditions are stronger than those stated in Theorem 15. To improve our algorithm, we further propose a dynamic learning algorithm (DLA, see Section 4.3). The dynamic learning algorithm makes better use of the observed data and updates the dual solution at a geometric pace, that is, at time $\epsilon n$, $2\epsilon n$, $4\epsilon n$ and so on. We show that these resolvings can lift the performance of the algorithm and thus prove Theorem 15. As one will see in the proof of the DLA, the choice of the resolving points perfectly balances the tradeoff between *exploration* and *exploitation*, which are the main tradeoffs in such types of learning algorithms.

It is worth mentioning that a similar kind of dynamic learning algorithm has been proposed in Agrawal et al. (2014) and further studied in Molinaro and Ravi (2014) and Wang (2012). However, those works only focus on linear objectives. In our analysis, the nonlinearity of the objective function presents a non-trivial hurdle since one can no longer simply analyze the revenue generated in each time segment and add them together. In this chapter, we successfully work around this hurdle by a convex duality argument. We believe that our analysis is a non-trivial extension of the previous work.

Moreover, the problem solved has important applications.

The remainder of this chapter is organized as follows. In Section 4.2, we start with a one-time learning algorithm and prove that it achieves near-optimal performance under some mild conditions on the input. The one-time learning algorithm is easy to understand and shows important insights for designing this class of learning algorithms. However, it only achieves a weaker performance than what is stated in Theorem 15. In Section 4.3, we propose a dynamic learning algorithm which makes better use of the data and has a stronger performance. Some numerical test results of our algorithm are presented in Section 4.4, which validate the strength of our algorithm. Section 4.5 concludes this chapter.

## 4.2 One-Time Learning Algorithm

In this section, we first propose a one-time learning algorithm for the online matching problem. We rewrite the offline problem (4.1) as follows:

$$\begin{array}{lll} \text{maximize}_{\boldsymbol{x},\boldsymbol{u}} & \sum_{i=1}^m M_i(u_i) & \\ \text{s.t.} & \sum_{j=1}^n b_{ij}x_{ij} = u_i, & \forall i \\ & \sum_{i=1}^m x_{ij} \le 1, & \forall j \\ & x_{ij} \ge 0, & \forall i,j. \end{array} \tag{4.3}$$

We define the following *dual problem*:

$$\begin{array}{lll} \inf_{\boldsymbol{v},\boldsymbol{y}} & \sum_{j=1}^n y_j + \sum_{i=1}^m (M_i(v_i) - M_i'(v_i)v_i) & \\ \text{s.t.} & y_j \ge b_{ij}M_i'(v_i), & \forall i,j \\ & v_i \ge 0, & \forall i \\ & y_j \ge 0, & \forall j. \end{array} \tag{4.4}$$

Let the optimal value of (4.3) be $P^*$ and the optimal value of (4.4) be $D^*$. In Devanur and Jain (2012), the authors proved the weak duality between (4.3) and (4.4). In the following lemma, we prove that in fact the strong duality holds. The proof of the lemma is relegated to the Appendix.

**Lemma 2.** *$P^* = D^*$. Furthermore, the objective value of any feasible solution to (4.4) is an upper bound of $P^*$.*

Before we describe our algorithm, we define the following partial optimization problem:

$$
\begin{array}{llll}
(\mathbf{P}_\epsilon) & \text{maximize}_{\boldsymbol{x},\boldsymbol{u}} & \sum_{i=1}^{m} M_i(u_i) & \\
 & \text{s.t.} & \sum_{j=1}^{\epsilon n} \frac{b_{ij}}{\epsilon} x_{ij} = u_i, & \forall i \\
 & & \sum_{i=1}^{m} x_{ij} \le 1, & \forall j \\
 & & x_{ij} \ge 0, & \forall i, j.
\end{array}
\tag{4.5}
$$

Now we define the one-time learning algorithm as follows:

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**Algorithm 2. One-Time Learning Algorithm(OLA)**

1. *During the first $\epsilon n$ arrivals, no allocation is made.*
2. *After observing the first $\epsilon n$ arrivals, solve ($\mathbf{P}_\epsilon$) and denote the optimal solutions by $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{u}}$.*
3. *For any $m$ dimensional vector $\boldsymbol{w}, \boldsymbol{q} \ge 0$, define*

$$
x_i(\boldsymbol{w}, \boldsymbol{q}) = \begin{cases} 1 & \text{if } i = \text{argmax}_k \{q_k M_k'(w_k)\} \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}
$$

*Here, ties among $q_k M_k'(w_k)$ are broken arbitrarily. For the $(\epsilon n + 1)$th to the $n$th arrival, the allocation rule $x_{ij} = x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j)$ is used.*

---

Now we provide some intuition for the algorithm. The idea of the algorithm is to use the first $\epsilon n$ inputs to learn an approximate $\hat{\boldsymbol{u}}$ and then use it to make all the future allocations based on the complementarity conditions between the primal and dual problems ((4.3) and (4.4)). Here $\hat{\boldsymbol{u}}$ is solved from ($\mathbf{P}_\epsilon$) which projects the allocation in the first $\epsilon n$ inputs to the entire problem. The decision rule in (4.6) can be explained as choosing the $i$ with the highest product of the nominal bid value $b_{ij}$ and the marginal contribution rate to the total projected reward $M_i'(\hat{\boldsymbol{u}})$. Note that a similar idea has been used to construct algorithms for an online matching problem with linear objective functions (see e.g., Devanur and Hayes (2009), Agrawal et al. (2014), Molinaro and Ravi (2014)). However, the analyses of those algorithms all depend on the linearity of the objective function which we do not possess in this problem. Instead, an analysis with the use of concavity is required in our analysis, making it quite different from those in the prior literature. In the following, we assume without loss of generality that $\max_{i,j} b_{ij} \leq 1$ (we can always scale the inputs to make this hold). We also make a technical assumption as follows:

**Assumption 1.** *The inputs of the problem are in a general position. That is, for any vector* $\boldsymbol{p} = (p_1, ..., p_m) \neq 0$*, there are at most* $m$ *terms among* $\arg\max_i\{b_{ij}p_i\}$*,* $j = 1, ..., n$*, that are not singleton sets.*

The assumption says that we only need to break ties in (4.6) no more than $m$ times. This assumption is not necessarily true for all inputs. However, as pointed out by Devanur and Hayes (2009) and Agrawal et al. (2014), one can always perturb $b_{ij}$ by adding a random variable $\eta_{ij}$ taking uniform distribution on $[0, \eta]$ for some very small $\eta$. By doing so the assumption holds with probability one and the effect to the solution can be made arbitrarily small. Given this assumption and by the complementarity conditions, we have the following lemma, whose proof is in the Appendix.

**Lemma 3.**

$$\epsilon\hat{u}_i - m \leq \sum_{j=1}^{\epsilon n} b_{ij} x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j) \leq \epsilon\hat{u}_i + m.$$

We first prove the following proposition about the performance of the OLA, which relies on a condition of the solution to $(\mathbf{P}_\epsilon)$.

**Proposition 2.** *For any given* $\epsilon \in (0, 1/2)$*, if* $\min_i \hat{u}_i \geq \Omega\left(\frac{m \log(m^2 n/\epsilon)}{\epsilon^3}\right)$*, then the OLA is a* $1-\epsilon$*-competitive algorithm.*

Before we prove Proposition 2, we define some notation.

- We define the optimal offline solution to (4.3) by $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ with optimal value OPT.
- Define $\sum_{j=1}^n b_{ij} x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j) = \bar{u}_i$, note that $\bar{u}_i$ normally does not equal $\hat{u}_i$.

We show the following lemma:

**Lemma 4.** *For any given* $\epsilon \in (0, 1/2)$*, if* $\min_i \hat{u}_i \geq \frac{12m\log(m^2n/\epsilon)}{\epsilon^3}$*, then with probability* $1-\epsilon$*,*

$$(1-\epsilon)\hat{u}_i \leq \bar{u}_i \leq (1+\epsilon)\hat{u}_i, \quad \textit{for all } i. \tag{4.7}$$

**Proof.** The proof will proceed as follows: For any fixed $\hat{\boldsymbol{u}}$, we define that a random sample (the first $\epsilon n$ arrivals) $S$ is *bad* for this $\hat{\boldsymbol{u}}$ if and only if $\hat{\boldsymbol{u}}$ is the optimal solution to (4.5) for this $S$, but $\bar{u}_i < (1-\epsilon)\,\hat{u}_i$, or $\bar{u}_i > (1+\epsilon)\,\hat{u}_i$, for some $i$. First, we show that the probability of a bad sample is small for every fixed $\hat{\boldsymbol{u}}$ (satisfying $\min_i \hat{u}_i \geq \frac{12m\log(m^2n/\epsilon)}{\epsilon^3}$) and $i$. Then, we take a union bound over all distinct $i$ and $\hat{u}_i$s to prove the lemma.

To start with, we fix $\hat{\boldsymbol{u}}$ and $i$. Define $Y_j = b_{ij}x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j)$. By Lemma 3 and the condition on $\hat{u}_i$, we have

$$(1-\epsilon^2)\epsilon\hat{u}_i \leq \epsilon\hat{u}_i - m \leq \sum_{j\in S} Y_j \leq \epsilon\hat{u}_i + m \leq (1+\epsilon^2)\epsilon\hat{u}_i.$$

Therefore, the probability of bad $S$ is bounded by the sum of the following two terms ($N = \{1, 2, ..., n\}$):

$$P\left(\sum_{j\in S} Y_j \leq \epsilon(1+\epsilon^2)\hat{u}_i, \sum_{j\in N} Y_j > (1+\epsilon)\hat{u}_i\right) + P\left(\sum_{j\in S} Y_j \geq \epsilon(1-\epsilon^2)\hat{u}_i, \sum_{j\in N} Y_j < (1-\epsilon)\hat{u}_i\right). \tag{4.8}$$

For the first term, we first define $Z_t = \frac{(1+\epsilon)\hat{u}_i Y_t}{\sum_{j\in N} Y_j}$ and we have

$$P\left(\sum_{j\in S} Y_j \leq \epsilon(1+\epsilon^2)\hat{u}_i, \sum_{j\in N} Y_j > (1+\epsilon)\hat{u}_i\right) \leq P\left(\sum_{j\in S} Z_j \leq \epsilon(1+\epsilon^2)\hat{u}_i, \sum_{j\in N} Z_j = (1+\epsilon)\hat{u}_i\right).$$

Then we have

$$\begin{aligned} P\left(\sum_{j\in S} Z_j \leq \epsilon(1+\epsilon^2)\hat{u}_i, \sum_{j\in N} Z_j = (1+\epsilon)\hat{u}_i\right) \leq & P\left(|\sum_{j\in S} Z_j - \epsilon\sum_{j\in N} Z_j| > \frac{\epsilon^2}{2}\hat{u}_i, \sum_{j\in N} Z_j = (1+\epsilon)\hat{u}_i\right) \\ \leq & P\left(|\sum_{j\in S} Z_j - \epsilon\sum_{j\in N} Z_j| > \frac{\epsilon^2}{2}\hat{u}_i \,\middle|\, \sum_{j\in N} Z_j = (1+\epsilon)\hat{u}_i\right) \\ \leq & 2\exp\left(-\frac{\epsilon^3\hat{u}_i}{4(2+\epsilon)}\right) \leq \frac{\epsilon}{2m(m^2n)^m} \doteq \delta. \end{aligned}$$

Here the second inequality follows from the Hoeffding-Bernstein's inequality for sampling without replacement, see Lemma 9 in the Appendix. Similarly, we can get the same result for the second term in (4.8), which is also bounded by $\delta$. Therefore, the probability of a bad sample is bounded by $2\delta$ for fixed $\hat{\boldsymbol{u}}$ and $i$.

Next, we take a union bound over all distinct $\hat{\boldsymbol{u}}$s. We call $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}'$ *distinct* if and only if they result in different allocations, i.e., $x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j) \neq x_i(\hat{\boldsymbol{u}}', \boldsymbol{b}_j)$ for some $i$, $j$. Denote $M_i'(\hat{u}_i) = v_i$. For each $j$, by the definition in (4.6), the allocation is uniquely defined by the signs of the following terms:

$$b_{ij}v_i - b_{i'j}v_{i'}, \quad \forall 1 \leq i < i' \leq m.$$

There are $m(m-1)/2$ such terms for each $j$. Therefore, the entire allocation profiles for all the $n$ arrivals can be determined by the signs of no more than $m^2n$ differences. Now we find out how many different allocation profiles can arise by choosing different $v$s. By Orlik and Terao (1992), the total number of different profiles for the $m^2n$ differences can not exceed $\left(m^2n\right)^m$. Therefore, the number of distinct $\hat{\boldsymbol{u}}$s is no more than $\left(m^2n\right)^m$. Now we take a union bound over all distinct $\hat{\boldsymbol{u}}$s and $i = 1, \ldots, m$, and Lemma 4 follows. □

Next we show that the OLA achieves a near-optimal solution under the condition in Proposition 2. We first construct a feasible solution to (4.4):

$$\hat{v}_i = \hat{u}_i, \qquad \hat{y}_j = \max_i\{b_{ij}M_i'(\hat{u}_i)\}.$$

By Lemma 2, $\sum_{j=1}^n \hat{y}_j + \sum_{i=1}^m \left(M_i(\hat{u}_i) - M_i'(\hat{u}_i)\hat{u}_i\right)$ is an upper bound of OPT. Thus, we have

$$\begin{aligned}\mathrm{OPT} - \sum_{i=1}^m M_i(\bar{u}_i) &\leq \sum_{i=1}^m \left(M_i(\hat{u}_i) - \hat{u}_iM_i'(\hat{u}_i)\right) - \sum_{i=1}^m M_i(\bar{u}_i) + \sum_{j=1}^n \hat{y}_j \\ &= \sum_{i=1}^m (M_i(\hat{u}_i) - M_i(\bar{u}_i)) + \sum_{i=1}^m \left(\bar{u}_iM_i'(\hat{u}_i) - \hat{u}_iM_i'(\hat{u}_i)\right) - \sum_{i=1}^m \bar{u}_iM_i'(\hat{u}_i) + \sum_{j=1}^n \hat{y}_j \\ &= \sum_{i=1}^m \left(M_i(\hat{u}_i) - M_i(\bar{u}_i) + (\bar{u}_i - \hat{u}_i)M_i'(\hat{u}_i)\right),\end{aligned}$$

where the last equality is because by the allocation rule (4.6):

$$\sum_{j=1}^{n} \hat{y}_j = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j) b_{ij} M_i'(\hat{u}_i) = \sum_{i=1}^{m} \bar{u}_i M_i'(\hat{u}_i).$$

Now, we claim that if condition (4.7) holds,

$$\sum_{i=1}^{m} \big(M_i(\hat{u}_i) - M_i(\bar{u}_i) + (\bar{u}_i - \hat{u}_i)M_i'(\hat{u}_i)\big) \le 2\epsilon \sum_{i=1}^{m} M_i(\bar{u}_i).$$

We consider the following two cases:

- Case 1: $\bar{u}_i \le \hat{u}_i$. In this case,

  $$M_i(\hat{u}_i) - M_i(\bar{u}_i) + (\bar{u}_i - \hat{u}_i)M_i'(\hat{u}_i) \le M_i(\hat{u}_i) - M_i(\bar{u}_i) \le \left|\frac{\hat{u}_i - \bar{u}_i}{\bar{u}_i}\right| M_i(\bar{u}_i) \le 2\epsilon M_i(\bar{u}_i),$$

  where the second inequality holds because of the concavity of $M_i(\cdot)$.

- Case 2: $\bar{u}_i > \hat{u}_i$. In this case,

  $$M_i(\hat{u}_i) - M_i(\bar{u}_i) + (\bar{u}_i - \hat{u}_i)M_i'(\hat{u}_i) \le (\bar{u}_i - \hat{u}_i)M_i'(\hat{u}_i) \le \left|\frac{\bar{u}_i - \hat{u}_i}{\hat{u}_i}\right| M_i(\hat{u}_i) \le \epsilon M_i(\bar{u}_i).$$

  Again, the second inequality is due to the concavity of $M_i(\cdot)$.

Thus, under the condition that $\min_i \hat{u}_i \ge \frac{12m\log(m^2n/\epsilon)}{\epsilon^3}$, with probability $1-\epsilon$,

$$\mathrm{OPT} - \sum_{i=1}^{m} M_i(\bar{u}_i) \;\le\; 2\epsilon \sum_{i=1}^{m} M_i(\bar{u}_i) \;\le\; 2\epsilon\mathrm{OPT},$$

i.e., $\sum_{i=1}^{m} M_i(\bar{u}_i) \ge (1-2\epsilon)\mathrm{OPT}$.

Lastly, we note that the actual allocation in our algorithm for $i$ is $\tilde{u}_i = \sum_{j=\epsilon n+1}^{n} b_{ij} x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j)$ (since we ignore the first $\epsilon n$ arrivals). By Lemma 3, we have

$$\tilde{u}_i = \bar{u}_i - \sum_{j=1}^{\epsilon n} b_{ij} x_i(\hat{\boldsymbol{u}}, \boldsymbol{b}_j) \ge \bar{u}_i - \epsilon(1+\epsilon^2)\hat{u}_i.$$

Thus when condition (4.7) holds, $\tilde{u}_i \ge (1-3\epsilon)\bar{u}_i$. Therefore,

$$\sum_{i=1}^{m} M_i(\tilde{\boldsymbol{u}}) \ge \sum_{i=1}^{m} M_i((1-3\epsilon)\bar{u}_i) \ge (1-3\epsilon)\sum_{i=1}^{m} M_i(\bar{u}_i).$$

The last inequality is due to the concavity of $M_i(\cdot)$s and that $M_i(0) = 0$. Therefore, given $\min_i \hat{u}_i \ge \frac{12m\log(m^2n/\epsilon)}{\epsilon^3}$, with probability $1-\epsilon$,

$$\sum_{i=1}^{m} M_i(\tilde{u}_i) \ge (1-5\epsilon)\text{OPT}.$$

Therefore, Proposition 2 is proved. □

Proposition 2 shows that the OLA is near-optimal under some conditions on $\hat{\boldsymbol{u}}$. However, $\hat{\boldsymbol{u}}$ is essentially an output of the algorithm. Although such types of conditions are not uncommon in the study of online algorithms (e.g., in the result of Devanur and Hayes (2009), Feldman et al. (2010)), it is quite undesirable. In the following, we address this problem by providing a set of sufficient conditions which only depend on the input parameters (i.e., $m$, $n$, $\boldsymbol{b}$s and $M(\cdot)$s). We show that our algorithm achieves near-optimal performance under these conditions. We start with the following lemma.

**Lemma 5.** *For any $C > 0$, suppose the following condition holds:*

$$n \ge \max\left\{\frac{12\log(m/\epsilon)}{\epsilon\bar{b}^2}, \frac{4mCF(M,\eta)}{\epsilon\bar{b}}\right\} \tag{4.9}$$

*where $\bar{b} = \frac{1}{n}\min_i\{\sum_{j=1}^{n} b_{ij}\}$, $\eta = \min_{i,j}\{b_{ij}|b_{ij} > 0\}$ and $F(M,\eta)$ is such that*

$$M_i'(\eta F(M,\eta)C) < \eta M_{i'}'(C), \forall i, i'.$$

*Then with probability $1-\epsilon$, $\hat{u}_i \ge C$, for all $i$.*

The proof of Lemma 5 is relegated to the Appendix (it is proved together with Lemma 8). Now combining Proposition 2 and Lemma 5, we have the following result for the OLA:

**Proposition 3.** *Fix any $\epsilon \in (0, 1/2)$. Suppose*

$$n \ge \max\left\{\frac{12\log(m/\epsilon)}{\epsilon\bar{b}^2}, \frac{4mCF(M,\eta)}{\epsilon\bar{b}}\right\} \tag{4.10}$$

*where $\bar{b} = \frac{1}{n}\min_i\{\sum_{j=1}^{n} b_{ij}\}$, $\eta = \min_{i,j}\{b_{ij}|b_{ij} > 0\}$ and $F(M,\eta)$ is such that*

$$M_i'(\eta F(M,\eta)C) < \eta M_{i'}'(C), \forall i, i' \tag{4.11}$$

*with $C = \frac{12m\log(m^2n/\epsilon)}{\epsilon^3}$. Then the OLA is $1-\epsilon$-competitive under the random permutation model.*

Here we give some comments on the definition of $F(M, \eta)$. The definition of $F(M, \eta)$ basically ensures that we rule out the possibility that one $i$ receives nearly all the allocation while some others receive almost none. Note that such $F(M, \eta)$ always exists and is finite if $\lim_{x\to\infty} M_i'(x) = 0$ for all $i$. In practice, this is usually true as there is usually a upper bound on the possible reward from each bidder $i$. In particular, if $M_i(\cdot) = M(\cdot)$ for all $i$ and $\lim_{x\to\infty} M'(x) = 0$, then one can choose

$$F(M, \eta) = \frac{M'^{-1}(\eta M'(C))}{\eta},$$

where $M'^{-1}(\cdot)$ denotes the inverse function of $M'(\cdot)$. For example, if one chooses $M_i(x) = x^p$ $(0 < p < 1)$, then one can further choose $F(M, \eta) \geq \frac{2}{\eta^{(2-p)/(1-p)}}$. Therefore, in most practical situations, one can view $F(M, \eta)$ as a constant. Finally, we want to remark that the conditions in Lemma 5 (or Proposition 3) are only one set of sufficient conditions which have the nice feature of only depending on the problem inputs. In practice, one can always resort to the condition in Proposition 2 ($\min_i \hat{u}_i \geq \frac{12m \log(m^2 n/\epsilon)}{\epsilon^3}$) if they are more favorable. In addition, as we will show in our numerical tests in Section 4.4, our algorithm performs quite well even if some of the conditions in Lemma 5 are not satisfied. Therefore, the applicability of our algorithm could be well beyond what the conditions require.

## 4.3 Dynamic Learning Algorithm

In the previous section, we introduced the OLA that can achieve near-optimal performance. While the OLA illustrates the ideas of our approach and requires solving a convex optimization problem only once, the conditions it requires to achieve near-optimality are stricter than what we claim in Theorem 15. In this section, we propose an enhanced algorithm that lessens the conditions and thus improves the OLA.

The main idea for the enhancement is the following: In the one-time learning algorithm, we only solve a partial optimization problem once. However, it is possible that there is some error for that solution due to the random order of arrival. If we could modify the solution as we gather more data, we might be able to improve the performance of the algorithm. In the following, we introduce a dynamic learning algorithm based on this idea, which updates the *allocation policy* every time the history doubles; that is, it computes a new $\hat{\boldsymbol{u}}$ at time $t = \epsilon n, 2\epsilon n, 4\epsilon n, \dots$ and uses it to perform the matching for the next time period. We define the following problem:

$$\begin{array}{lll} (\mathbf{P}_\ell) & \text{maximize}_{\boldsymbol{x},\boldsymbol{u}} & \sum_{i=1}^m M_i(u_i) \\ & \text{s.t.} & \sum_{j=1}^{\ell} \frac{n}{\ell} b_{ij} x_{ij} = u_i, \quad \forall i \\ & & \sum_{i=1}^m x_{ij} \le 1, \qquad\quad \forall j \\ & & x_{ij} \ge 0, \qquad\qquad\quad \forall i, j. \end{array}$$

We further define $(\boldsymbol{x}^\ell, \boldsymbol{u}^\ell)$ to be the optimal solution to $(\mathbf{P}_\ell)$.

We define the dynamic learning algorithm as follows:

---

**Algorithm 3. Dynamic Learning Algorithm (DLA)**

1. *During the first $\epsilon n$ arrivals, no allocation is made.*
2. *For $r = 0, 1, \dots$, for $2^r n\epsilon < j \le 2^{r+1} n\epsilon$, set $x_{ij} = x_i(\boldsymbol{u}^\ell, \boldsymbol{b}_j)$ for all $i$, where $\ell = \lceil 2^r n\epsilon \rceil$.*

---

In the following, without loss of generality, we assume that $L = -\log_2 \epsilon$ is an integer (otherwise one can just choose a smaller $\epsilon$ and prove the same result). Define $\ell_k = 2^{k-1}\epsilon n$, $k = 1, \dots, L$, and define $\hat{\boldsymbol{u}}^k = \boldsymbol{u}^{\ell_k}$. We first prove the following proposition:

**Proposition 4.** *If for all $k$, $\min_i \hat{u}_i^k \ge \Omega\left\{\frac{m \log(m^2 n/\epsilon)}{\epsilon^2}\right\}$, then the DLA is $1-\epsilon$-competitive under the random permutation model.*

Before we proceed to the proof, we first define some more notation. We define:

$$\bar{u}_i^k = \sum_{j=\ell_k+1}^{\ell_{k+1}} b_{ij} x_i(\hat{\boldsymbol{u}}^k, \boldsymbol{b}_j), \quad \tilde{u}_i^k = \sum_{j=1}^{n} b_{ij} x_i(\hat{\boldsymbol{u}}^k, \boldsymbol{b}_j), \quad \bar{u}_i = \sum_{k=1}^{L} \bar{u}_i^k.$$

Note that in these definitions, $\bar{u}_i^k$ is the allocated values for $i$ in the period $\ell_k + 1$ to $\ell_{k+1}$ using $\hat{\boldsymbol{u}}^k$, which is the actual allocation in that period. $\tilde{u}_i^k$ is the allocation for $i$ in all periods if $\hat{\boldsymbol{u}}^k$ is used. $\bar{u}_i$ is the actual allocation for $i$ during the entire algorithm. We first prove the following lemma bounding the differences between $\bar{u}_i^k$, $\tilde{u}_i^k$ and $\hat{u}_i^k$.

**Lemma 6.** *If* $\min_i \hat{u}_i^k \geq \frac{16m\log(m^2n/\epsilon)}{\epsilon^2}$*, then with probability* $1-\epsilon$*, for all* $i$*,*

$$\left(1-\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k \leq \frac{n}{\ell_k}\bar{u}_i^k \leq \left(1+\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k \tag{4.12}$$

*and*

$$\left(1-\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k \leq \tilde{u}_i^k \leq \left(1+\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k. \tag{4.13}$$

Lemma 6 shows that with high probability, $\frac{n}{\ell_k}\bar{u}_i^k$, $\tilde{u}_i^k$ and $\hat{u}_i^k$ are close to each other. In particular, when $k$ is small, the factor $(1 \pm \epsilon\sqrt{n/\ell_k})$ is relatively loose while as $k$ increases, the factor becomes tight. The proof of Lemma 6 is similar to that of Lemma 4 and is relegated to the Appendix.

The next lemma gives a bound on the revenue obtained by the DLA.

**Lemma 7.** *If* $\hat{u}_i^k \geq \frac{16m\log(m^2n/\epsilon)}{\epsilon^2}$ *for all* $i$ *and* $k$*, then with probability* $1-\epsilon$*,*

$$\sum_{i=1}^m M_i\left(\frac{n}{\ell_i}\bar{u}_i^k\right) \geq \left(1-6\epsilon\sqrt{\frac{n}{l_k}}\right) OPT.$$

The proof of Lemma 7 can be found in the Appendix.

Finally, we prove Proposition 4. We bound the objective value of the actual allocation. Note that the actual allocation for each $i$ can be written as

$$\sum_{k=1}^L \bar{u}_i^k = \sum_{k=1}^L \alpha_k \frac{n}{\ell_k}\bar{u}_i^k,$$

where $\alpha_k = \frac{\ell_k}{n}$. By the property of concave functions, we have

$$\sum_{i=1}^m M_i\left(\sum_{k=1}^L \bar{u}_i^k\right) = \sum_{i=1}^m M_i\left(\sum_{k=1}^L \alpha_k\frac{n}{\ell_k}\bar{u}_i^k + \left(1-\sum_{k=1}^L\alpha_k\right)\cdot 0\right) \geq \sum_{i=1}^m\sum_{k=1}^L \alpha_k M_i\left(\frac{n}{\ell_k}\bar{u}_i^k\right).$$

By Lemma 7, with probability $1-\epsilon$

$$\begin{aligned}\sum_{i=1}^m\sum_{k=1}^L \alpha_k M_i\left(\frac{n}{\ell_k}\bar{u}_i^k\right) \geq \sum_{k=1}^L \frac{\ell_k}{n}\left(1-6\epsilon\sqrt{\frac{n}{\ell_k}}\right)\text{OPT} =&(1-\epsilon)\text{OPT} - 6\epsilon\sum_{k=1}^L\sqrt{\frac{\ell_k}{n}}\text{OPT}\\ \geq&(1-16\epsilon)\text{OPT},\end{aligned}$$

where the last inequality is because

$$\sum_{k=1}^{L} \sqrt{\frac{\ell_k}{n}} = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} ... \leq 1 + \sqrt{2} \leq 2.5.$$

Therefore, Proposition 4 is proved. □

Similar to Lemma 5, we have the following conditions on the input parameters such that with high probability, the conditions in Proposition 4 hold.

**Lemma 8.** *For any* $C > 0$*, suppose the following condition holds:*

$$n \geq \max\left\{\frac{24\log(m/\epsilon)}{\epsilon \bar{b}^2}, \frac{4mCF(M,\eta)}{\epsilon \bar{b}}\right\} \tag{4.14}$$

*where* $\bar{b} = \frac{1}{n}\min_i\{\sum_{j=1}^{n} b_{ij}\}$*,* $\eta = \min_{i,j}\{b_{ij} | b_{ij} > 0\}$ *and* $F(M,\eta)$ *is such that*

$$M_i'(\eta F(M,\eta)C) < \eta M_{i'}'(C), \forall i, i'.$$

*Then with probability* $1-\epsilon$*,* $\hat{u}_i^k \geq C$*, for all* $i$*.*

The proof of Lemma 8 is given in the Appendix. Finally, we combine Proposition 4 and Lemma 8, and Theorem 15 follows.

The same remark after Lemma 5 applies here. In particular, the conditions in Lemma 8 is only one set of sufficient conditions for our algorithm to achieve the target performance. However, one may also use the conditions in Proposition 4 if they turn out to hold in practice. In the next section, we show that the DLA works well even if the conditions in Lemma 8 are not satisfied.

## 4.4 Numerical Experiments

In this section, we report some numerical test results for our algorithms (both the OLA and the DLA). The objective is to validate the strength of our approaches and investigate the relationship between the performance of our algorithms and the input parameters.

In our numerical tests, we consider the Adwords problem. We assume there are $m$ advertisers (bidders), $n$ keywords arriving sequentially, and $b_{ij}$ is the amount bidder $i$ would like to pay to display his advertisement on keyword search $j$. We introduce a base problem in which we set $m = 50$, $n = 10,000$ and $M_i(x) = x^p$ with $p = 0.9$. The bidding values $b_{ij}$ are generated in the following way:

1. Assume there are $k = 100$ categories of keywords. For each category $k$, there is a base valuation of bidder $i$, denoted by $\bar{b}_{ik}$, which is generated according to the following distribution:

   $$\bar{b}_{ik} = \begin{cases} 0 & \text{with probability } 0.7 \\ U[0.2, 1] & \text{with probability } 0.3, \end{cases}$$

   where $U[a, b]$ denotes a uniformly distributed random variable on $[a, b]$.

2. For each arriving keyword, we first randomly choose a category. The probability for each category $i$, denoted by $\rho_i$, is randomly chosen on the simplex $\{\rho_i | \sum_{i=1}^{k} \rho_i = 1, \rho_i \geq 0\}$. Then if category $k_0$ is chosen, the final bid value for bidder $i$ will be $\bar{b}_{ik_0} \cdot U[0.9, 1.1]$.

Although the way $b_{ij}$ is chosen seems arbitrary, we believe it reflects some major features of the bid values in practice. In the Adwords problem, each bidder is interested in certain categories of keywords. For example, a sport product company is interested in keywords related to sports. The $\bar{b}_{ik}$s represent such interest levels. Then the bidder $i$'s actual bid on such a keyword is the base value $\bar{b}_{ik}$ multiplied by a random number, which reflects some level of idiosyncrasies of each keyword arrival. We also tested other ways to generate $b_{ij}$, and the test results are similar. We will report those test results in the end of this section.

To evaluate the performance of our algorithms, we introduce the notion of Relative Loss (RL) defined as follows:

$$\text{RL} = 1 - \frac{\text{Actual Revenue}}{\text{Offline Optimal Revenue}}.$$

Note that the RL is simply 1 minus the competitive ratio used in our theoretic analysis.

In the numerical experiment, there is one key parameter we need to set in both of our algorithms: $\epsilon$. In Theorem 15 and Proposition 2, we gave sufficient conditions on the inputs such that the algorithms will have expected RL less than $\epsilon$. However, the theoretical results are asymptotic and thus may not represent the best practical choice of $\epsilon$. In Table 4.1 and Figure 1, we first test both our OLA and DLA with different choices of $\epsilon$. We have the following observations from Table 4.1 and Figure 4.1 (each number in Table 4.1 is the average of 100 independent runs, the standard deviations of the results are insignificant compared to the average value):

- For the DLA, choosing a smaller $\epsilon$ improves its performance. There are two reasons for that. First, choosing a smaller $\epsilon$ reduces the loss due to ignoring the first $\epsilon n$ bids. Second, it increases the number of price updates which help the decision maker to refine the decision policy and achieve better performance. Therefore one should choose a smaller $\epsilon$ in the DLA.

- For the OLA, the optimal choice of $\epsilon$ is more subtle. There are two countervailing forces when one chooses a smaller $\epsilon$. On one hand, by choosing a smaller $\epsilon$, the loss due to the failure to allocate any bid during period 1 to $\epsilon n$ becomes smaller, which benefits the algorithm. On the other hand, if $\epsilon$ is too small, the learned price may not be accurate enough which may lead to poor allocation in the remaining periods. In the test example, the optimal choice is $\epsilon = 0.02$.

- The DLA outperforms the OLA for all choices of $\epsilon$.

Next we focus ourselves on the DLA. As shown in Table 4.1, we prefer to choose a smaller $\epsilon$ in the DLA. In the following experiments, we will choose $\epsilon = 0.001$. Next we compare the performance of the DLA to a myopic allocation method which simply allocates each incoming keyword to the bidder with the highest $b_{ij}$ value. We also study the impact of the two parameters, $n$ and $p$, on the performance of our algorithm. We generate 100 instances of the input $b_{ij}$ and compare the average performance. The results of the average RL are shown in Table 4.2 (the standard deviations are shown in the parentheses) as well as in Figures 4.2 and 4.3.

| $\epsilon$ | 0.001 | 0.005 | 0.01 | 0.02 | 0.05 | 0.10 |
|---|---|---|---|---|---|---|
| OLA | 22.02% | 5.17% | 4.07% | 3.71% | 6.10% | 10.26% |
| DLA | 0.47% | 0.84% | 1.27% | 2.15% | 4.89% | 9.41% |

Table 4.1: Performance of the OLA and the DLA for different choices of $\epsilon$

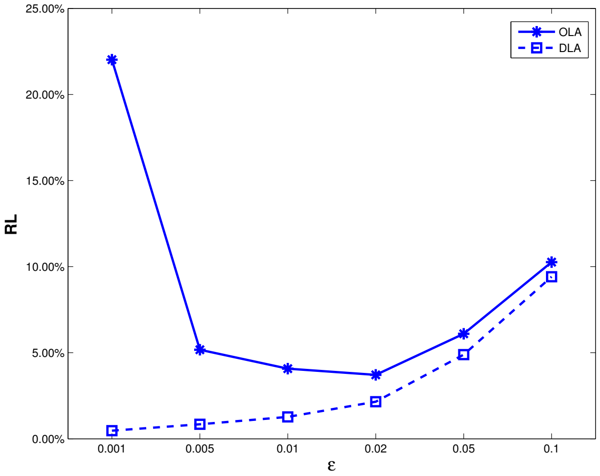

Figure 4.1: Performance of the OLA and the DLA for different choices of $\epsilon$

From Table 4.2, we can see that the DLA consistently performs better than the myopic approach. In particular, the performance of the DLA gradually improves when $n$ increases, while the performance of the myopic approach seems to be insensitive to the size $n$ of the problem. Moreover, even for small values of $n$, the performance of the DLA is still very good. This means that the DLA works well even for problems whose size is much smaller than what Theorem 15 requires. For the parameter $p$, we can see that both the DLA and the myopic algorithm deteriorate when $p$ decreases, but the DLA deteriorates much slower. Finally, we comment that these results are computed when we ignore the first $\epsilon n$ bids. In practice, one does not need to do that and the performance of the DLA would be even better.

Finally, we repeat the above test for different setups of the inputs. We fix the parameters $m = 50, n = 10,000$, and $M_i(x) = x^p$ with $p = 0.9$ in the base problem and generate $b_{ij}$s in the following ways:

| $n$ | 1,000 | 2,000 | 5,000 | 10,000 | 20,000 |
|---|---|---|---|---|---|
| DLA | 1.29% (0.29%) | 0.87% (0.26%) | 0.58% (0.12%) | 0.57% (0.17%) | 0.57% (0.16%) |
| Myopic | 3.07% (0.47%) | 3.03% (0.61%) | 2.90% (0.49%) | 3.11% (0.54%) | 2.96% (0.47%) |

| $p$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|
| DLA | 2.30% (0.82%) | 1.87% (0.67%) | 1.55% (0.56%) | 0.99 % (0.34%) | 0.57% (0.17%) |
| Myopic | 14.38% (2.01%) | 12.46% (1.81%) | 9.46% (1.38%) | 6.51% (1.05%) | 3.11% (0.54%) |

Table 4.2: Performance of the DLA and the myopic policy

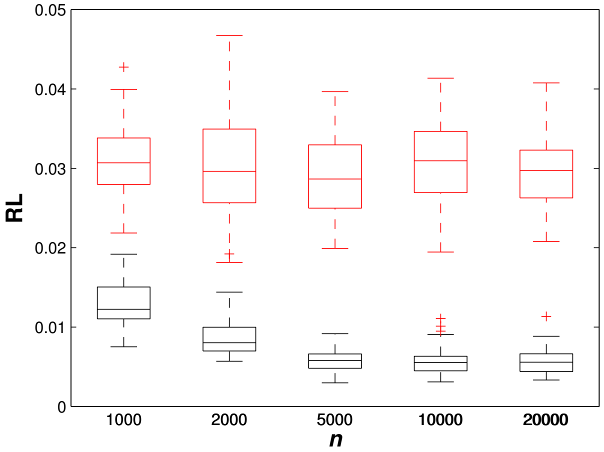

Figure 4.2: Performance of the DLA (the bottom ones) and the myopic policy (the top ones) for different problem sizes

1. $b_{ij}$ follows a normal distribution (truncated at 0 and 1). The parameters of each normal distribution (mean $\mu$ and standard deviation $\sigma$) are randomly generated from a uniform distribution on $[0, 1]$.

2. $b_{ij}$ follows a Beta distribution. The parameters $(\alpha, \beta)$ of the Beta distribution are generated from a uniform distribution on $[0, 1]$.

3. $b_{ij}$ follows a mixed normal and Beta distribution. That is, with probability 0.5, $b_{ij}$ follows a truncated normal distribution with mean 0.5 and standard deviation 0.5, and with probability 0.5, $b_{ij}$ follows a Beta distribution with $\alpha = \beta = 1/2$.

Next, we compare the performance of the DLA (choose $\epsilon = 0.001$) and the myopic

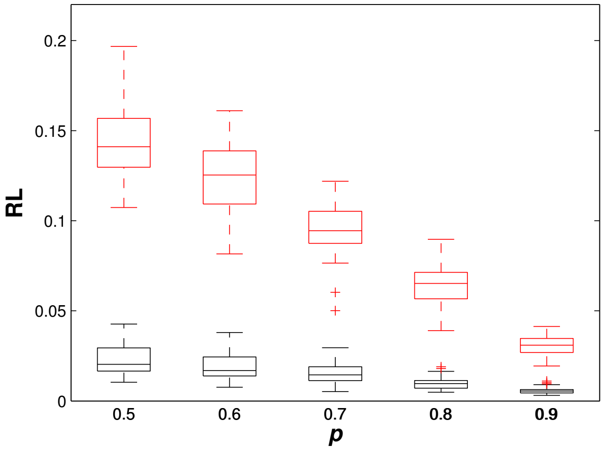

Figure 4.3: Performance of the DLA (the bottom ones) and the myopic policy (the top ones) for different $M(\cdot)$s

algorithm. For each case, we generate 100 instances of the input $b_{ij}$ and compare the average RL. The results are shown in Table 4.3.

| | | | | | | |
|---|---|---|---|---|---|---|
| Case 1 | $n$ | 1,000 | 2,000 | 5,000 | 10,000 | 20,000 |
| | DLA | 1.56% (0.33%) | 0.84% (0.18%) | 0.36% (0.07%) | 0.21% (0.05%) | 0.14% (0.03%) |
| | Myopic | 1.45% (0.45%) | 1.37% (0.47%) | 1.42% (0.45%) | 1.41% (0.48%) | 1.31% (0.47%) |
| | $p$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | DLA | 0.20% (0.04%) | 0.22% (0.06%) | 0.22% (0.05%) | 0.21% (0.03%) | 0.21% (0.05%) |
| | Myopic | 9.10% (2.01%) | 7.98% (2.00%) | 5.71% (1.32%) | 3.38% (0.96%) | 1.44% (0.46%) |
| Case 2 | $n$ | 1,000 | 2,000 | 5,000 | 10,000 | 20,000 |
| | DLA | 1.19% (0.24%) | 0.61% (0.12%) | 0.25% (0.08%) | 0.13% (0.07%) | 0.12% (0.08%) |
| | Myopic | 13.91% (2.07%) | 13.66% (2.48%) | 13.90% (2.07%) | 13.78% (1.97%) | 13.62% (2.12%) |
| | $p$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | DLA | 0.20% (0.16%) | 0.29% (0.20%) | 0.21% (0.12%) | 0.17% (0.11%) | 0.13% (0.07%) |
| | Myopic | 40.57% (6.50%) | 38.30% (5.98%) | 32.45% (4.71%) | 24.67% (3.26%) | 13.46% (2.07%) |
| Case 3 | $n$ | 1,000 | 2,000 | 5,000 | 10,000 | 20,000 |
| | DLA | 1.50% (0.31%) | 0.89% (0.34%) | 0.35% (0.07%) | 0.21% (0.06%) | 0.14% (0.01%) |
| | Myopic | 10.84% (2.50%) | 11.22% (2.64%) | 11.30% (2.44%) | 10.91% (2.54%) | 11.07% (2.85%) |
| | $p$ | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | DLA | 0.21% (0.12%) | 0.22% (0.12%) | 0.15% (0.45%) | 0.15% (0.35%) | 0.21% (0.06%) |
| | Myopic | 37.00% (7.43%) | 34.02% (6.82%) | 30.22% (5.89%) | 20.72% (4.70%) | 10.91% (2.54%) |

Table 4.3: Performance of the DLA and the myopic policy

From Table 4.3, we can see that the DLA outperforms the myopic approach under all the above three setups. The RL of the DLA decreases as the problem size grows, while the RL of the myopic policy is not sensitive to $n$. Also, as $p$ changes, the performance of the DLA is rather stable, while the performance of the myopic algorithm varies a lot. The overall trend of the DLA and the myopic algorithm resembles that in the experiment in the beginning of this section. Finally, we observe that the DLA seems robust toward various problem setups, while the myopic approach does not.

## 4.5 Conclusion

In this chapter, we propose a dynamic learning algorithm for an online matching problem with concave returns. We show that our algorithm achieves near-optimal performance when the data arrives in a random order and satisfies some conditions. Specifically, the order of arrival is uniformly distributed over all the permutations. Our model requires that the number of total arrivals be known a prior and larger than a certain threshold. In practice, the number of arrivals usually satisfies the threshold required here. The analysis is primal-dual based, however, the nonlinear objective function requires us to work around nontrivial hurdles that do not exist in previous work. Numerical experiment results show that our algorithm works well in test problems.

# Chapter 5

# Conclusion and Future Work

In this dissertation, we studied several problems related to pricing and risk-return analysis in an uncoordinated supply chain. In particular, we studied the optimal pricing problem when a supplier sells to a multi-period newsvendor, and obtained the optimal pricing strategy and comparative statics. We also established a model that considers robust decisions in a multi-period newsvendor model under worst-case distribution. We further derived some risk-return analysis in a supplier-retailer relationship and obtained tight bounds of the proportion of profit of a supplier in a supply chain. In addition to the classic newsvendor model with wholesale-price contract, two variations of the newsvendor model are studied, the spot market model and the revenu-sharing model. Both models introduces variability to the supplier's profit function. We examine the optimal pricing and ordering policies in each model, as well as the risk-return trade-offs.

We also studied a version of the online matching problem with concave return functions. We proposed two algorithms, a one-time learning algorithm and a dynamic learning algorithm for such problems. We obtained theoretical performance bounds for these algorithms and demonstrated via numerical experiments that these algorithms are very efficient in practice.

There are several directions for future research:

1. To continue our work on the optimal pricing/ordering policy in the supply chain

model, we could conduct more comparative statics on the supplier and retailer's optimal pricing and ordering policy. For example, we can study the effect of market size on the supplier's profit and the retailer's order quantity, which was suggested in Lariviere and Porteus (2001). In addition, we can also analyze the supplier's optimal pricing strategy and retailer's ordering strategy in a model with multiple suppliers or multiple retailers. Bringing competition to our current model could change some basic assumptions, thus, making our model more interesting.

Another topic we could study is the effect of demand correlation across different periods. Currently, we do not assume correlation in our demand distribution in the distribution-free model. If we know the demand correlation, new constraints need to be introduced to the distribution-free model. Such models would provide some meaningful insights.

2. For the risk-return analysis, we can look at other types of supply chain contracts that share the risk between the supplier and the retailer. One example is the buyback contract. We can adopt a loss-aversion model similar to Zhang et al. (2015). After looking at different supply chain contract setups, we can compare the supply chain performance across different types of contracts, then recommend the supplier or the retailer the best contract type to choose, similar to Pasternack (2002).

   Another possible direction is extending the risk-return analysis to a multi-period newsvendor model. It would be interesting to characterize the relationship between the demand variation and the retailer/supplier's profit variation.

3. For the online learning algorithms, one important direction of future work is the practical performance of such algorithms, especially how such learning type of algorithms compare to the algorithms that focus on the worst-case performance, in practical size of problems. As we mentioned in the body part, such online learning algorithms might be of very large scale. Therefore, having an algorithm that solves optimally and efficiently for large problem size is very important.

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# Appendix A

# Hoeffding-Bernstein's Inequality

In our proofs, we will frequently use the following Hoeffding-Bernstein's Inequality for sampling without replacement:

**Lemma 9** (Theorem 2.14.19 in van der Vaart and Wellner (1996):)**.** *Let* $u_1, u_2, ..., u_r$ *be random samples without replacement from real numbers* $\{c_1, c_2, ..., c_R\}$*. Then for every* $t > 0$*,*

$$P\left(\left|\sum_{i=1}^{r} u_i - r\bar{c}\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2r\sigma_R^2 + t\Delta_R}\right)$$

*where* $\Delta_R = \max_i c_i - \min_i c_i$*,* $\bar{c} = \frac{1}{R}\sum_{i=1}^{R} c_i$*, and* $\sigma_R^2 = \frac{1}{R}\sum_{i=1}^{R}(c_i - \bar{c})^2$*.*

# Appendix B

# Proof of Lemma 2

We first write down the Lagrangian dual of (4.3). By associating $p_i$ to the first set of constraints and $y_j$ to the second set of constraints, the Lagrangian dual of (4.3) is:

$$\begin{array}{lll} \inf_{\boldsymbol{p},\boldsymbol{y}} & \sum_{j=1}^{n} y_j + \sup_{u_i \geq 0} \sum_{i=1}^{m} \left( M_i(u_i) - p_i u_i \right) & \\ \text{s.t.} & y_j \geq b_{ij} p_i, \qquad \forall i & \\ & y_j \geq 0, \qquad \forall j. & \end{array} \tag{B.1}$$

Since the primal problem is convex and only has linear constraints, Slater's condition holds, thus the strong duality theorem holds and (4.3) and (B.1) have the same optimal value. Next we show that (4.4) and (B.1) are equivalent. To show this, assume the range of $M_i'(\cdot)$ on $[0, \infty)$ is $(a_i, b_i]$ or $[a_i, b_i]$ (by the assumption that $M(\cdot)$s are continuously differentiable, it must be either one of these two forms). Now we argue that the optimal $p_i$ must be within $[a_i, b_i]$ in (B.1). First we must have $p_i \geq a_i$, otherwise the term $\sup_{u_i \geq 0} \{M_i(u_i) - p_i u_i\}$ goes to infinity as $u_i$ increases and it cannot be the optimal solution to (B.1). On the other hand, if $p_i > b_i$, the optimal $u_i$ must be 0, and one can always set $p_i = b_i$ and achieves a smaller value of the objective function. Therefore, $p_i \in [a_i, b_i]$ at optimality.

Now if $p_i \in (a_i, b_i]$ at optimality, one can always find one $v_i$ such that $M_i'(v_i) = p_i$, and that $v_i$ must be the optimal solution to $\sup_{u_i} \{M_i(u_i) - p_i u_i\}$ (the optimal solution must be attainable in this case). Therefore, each feasible solution of (B.1) will correspond to a feasible solution of (4.4) and vice versa. The only case left now is when $p_i = a_i$ at optimality. In this case, $\sup_{u_i} \{M_i(u_i) - a_i u_i\} = \lim_{x \to \infty} \{M_i(x) - a_i x\}$. Also, we know that $\lim_{x \to \infty} M_i'(x) = a_i$, therefore, there exists a sequence of feasible solution of (4.4) such that the limit of the objective value equals the objective obtained when $p_i = a_i$ in (B.1). Therefore, the lemma is proved. $\square$

# Appendix C

# Proof of Lemma 3

To prove this lemma, it suffices to show that for each fixed $i$, $x_i(\hat{u}, \boldsymbol{b}_j)$ and $\hat{x}_{ij}$ (recall that $\hat{\boldsymbol{x}} = \{\hat{x}_{ij}\}$ is the optimal solution to (4.5)) differ by no more than $m$ terms. If this is true, then note that $\sum_{j=1}^{\epsilon n} b_{ij}\hat{x}_{ij} = \epsilon\hat{u}_i$ and $0 \leq b_{ij} \leq 1$, the lemma holds.

To show that $x_i(\hat{u}, \boldsymbol{b}_j)$ and $\hat{x}_{ij}$ differ by no more than $m$ terms, we first construct the dual problem of (4.5) (according to (4.4)):

$$\begin{array}{lll} \inf_{\boldsymbol{v},\boldsymbol{y}} & \sum_{j=1}^{\epsilon n} y_j + \sum_{i=1}^{m} (M_i(v_i) - M_i'(v_i)v_i) & \\ \text{s.t.} & y_j \geq \frac{b_{ij}}{\epsilon} M_i'(v_i), & \forall i, j \\ & v_i \geq 0, & \forall i \\ & y_j \geq 0, & \forall j. \end{array}$$

By Lemma 2.1, strong duality holds and thus any optimal solution should satisfy the complementarity conditions. Among them we should have $\hat{x}_{ij}(y_j - \frac{b_{ij}}{\epsilon} M_i'(v_i)) = 0$. Therefore, if there is no tie when we defined $x_i(\hat{u}, \boldsymbol{b}_j)$, we must have $x_i(\hat{u}, \boldsymbol{b}_j) = \hat{x}_{ij}$. By Assumption 1, there are no more than $m$ ties. Thus, Lemma 3 is proved. $\square$

# Appendix D

# Proof of Lemma 6

We first prove (4.12). The idea is similar to the proof of the one-time learning case. For any fixed $\hat{\boldsymbol{u}}^k$, we define that a random sample $S$ (a sequence of arrival) is *bad* if and only if $\hat{\boldsymbol{u}}^k$ is the optimal solution to $(\mathbf{P}_{\ell_{\mathbf{k}}})$ but $\bar{\boldsymbol{u}}^k$ does not satisfy (4.12) for some $i$. First, we show that the probability of a bad sample is small for any fixed $\hat{\boldsymbol{u}}^k$ and fixed $i$. Then we take a union bound over all distinct $\hat{\boldsymbol{u}}^k$s and $i$s to show the result.

Fix $\hat{\boldsymbol{u}}^k$ and $i$. We define $Y_j = b_{ij} x_i(\hat{\boldsymbol{u}}^k, \boldsymbol{b}_j)$. By Lemma 3 and the assumption on $\hat{u}_i^k$, we have

$$\frac{\ell_k}{n}\hat{u}_i^k - \epsilon^2 \hat{u}_i^k \le \sum_{j=1}^{\ell_k} Y_j \le \frac{\ell_k}{n}\hat{u}_i^k + \epsilon^2 \hat{u}_i^k.$$

Therefore, the probability of a bad sample is bounded by the following two terms:

$$\begin{aligned} & P\left(\sum_{j=1}^{\ell^k} Y_j \le \frac{\ell_k}{n}\hat{u}_i^k + \epsilon^2 \hat{u}_i^k, \sum_{j=\ell_k+1}^{\ell_{k+1}} Y_j > \frac{\ell_k}{n}(1+\sqrt{\frac{n}{\ell_k}}\epsilon)\hat{u}_i^k\right) \\ +& P\left(\sum_{j=1}^{\ell^k} Y_j \ge \frac{\ell_k}{n}\hat{u}_i^k - \epsilon^2 \hat{u}_i^k, \sum_{j=\ell_k+1}^{\ell_{k+1}} Y_j < \frac{\ell_k}{n}(1-\sqrt{\frac{n}{\ell_k}}\epsilon)\hat{u}_i^k\right). \end{aligned} \tag{D.1}$$

For the first term, we have

$$\begin{aligned}
&P\left(\sum_{j=1}^{\ell^k} Y_j \le \frac{\ell_k}{n}\hat{u}_i^k + \epsilon^2\hat{u}_i^k, \sum_{j=\ell_k+1}^{\ell_{k+1}} Y_j > \frac{\ell_k}{n}\left(1+\sqrt{\frac{n}{\ell_k}}\epsilon\right)\hat{u}_i^k\right)\\
=&P\left(\sum_{j=1}^{\ell^k} Y_j \le \frac{\ell_k}{n}\hat{u}_i^k + \epsilon^2\hat{u}_i^k, \sum_{j=1}^{\ell_{k+1}} Y_j > \frac{\ell_k}{n}\left(2+\sqrt{\frac{n}{\ell_k}}\epsilon\right)\hat{u}_i^k\right)\\
\le&P\left(|\sum_{j=1}^{\ell_k} Y_j - \frac{1}{2}\sum_{j=1}^{\ell_{k+1}} Y_j| > \frac{\epsilon}{4}\sqrt{\frac{n}{\ell_k}}\frac{\ell_k}{n}\hat{u}_i^k \,\middle|\, \sum_{j=1}^{\ell_{k+1}} Y_j > \frac{\ell_k}{n}\left(2+\sqrt{\frac{n}{\ell_k}}\epsilon\right)\hat{u}_i^k\right)\\
\le&2\exp\left(-\frac{\epsilon^2\hat{u}_i^k}{16}\right) \le \frac{\epsilon}{2m(m^2n)^m} \doteq \delta.
\end{aligned}$$

Here the second inequality follows from Lemma 9, and the third inequality is due to the condition of $\hat{u}_i^k$. Similarly, we can get the bound for the second term in (D.1). Therefore, the probability of a bad sample is bounded by $2\delta$.

Now we take union bound over all distinct $\hat{\boldsymbol{u}}^k$ and $i$. Similar to the proof of Lemma 4, we call $\boldsymbol{u}$s to be *distinct* if they result in different allocations. As argued earlier, there are no more than $(m^2n)^m$ distinct $\boldsymbol{u}$s. Therefore, we know that with probability $1-\epsilon$, (4.12) holds.

Next we prove (4.13). The idea is similar. Fix $\hat{\boldsymbol{u}}^k$ and $i$. We define $Y_j = b_{ij}x_i(\hat{\boldsymbol{u}}^k, \boldsymbol{b}_j)$. Applying Lemma 9, we get

$$\begin{aligned}
&P\left(\sum_{j=1}^{\ell_k} Y_j \le \frac{\ell_k}{n}\hat{u}_i^k + \epsilon^2\hat{u}_i^k, \sum_{j=1}^{n} Y_j > \left(1+\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k\right)\\
\le&P\left(|\sum_{j=1}^{\ell_k} Y_j - \frac{\ell_k}{n}\sum_{j=1}^{n} Y_j| > \frac{\epsilon}{4}\sqrt{\frac{\ell_k}{n}}\hat{u}_i^k \,\middle|\, \sum_{j=1}^{n} Y_j > \left(1+\epsilon\sqrt{\frac{n}{\ell_k}}\right)\hat{u}_i^k\right)\\
\le&\exp\left(-\frac{\epsilon^2\hat{u}_i^k}{16}\right) \doteq \delta.
\end{aligned}$$

Using the same argument as above, Lemma 6 holds. □

# Appendix E

# Proof of Lemma 7

The proof consists of two main steps. First we show that with probability $1 - \epsilon$, the following is true for all $k$:

$$\sum_{i=1}^{m} M_i(\tilde{u}_i^k) \geq \left(1 - 2\epsilon\sqrt{\frac{n}{l_k}}\right) \mathrm{OPT}. \tag{E.1}$$

To show this, we follow a similar step when we prove the optimality of the one-time learning algorithm. Define

$$\hat{v}_i^k = \hat{u}_i^k \text{ and } \hat{y}_i^k = \max_i \{b_{ij} M_i'(\hat{u}_i^k)\}.$$

Since $(v_i^k, y_i^k)$ is a feasible solution to (4.4), we know that

$$\sum_{j=1}^{n} \hat{y}_j^k + \sum_{i=1}^{m} (M_i(\hat{u}_i^k) - M_i'(\hat{u}_i^k)\hat{u}_i^k)$$

is an upper bound of OPT. Therefore, by using the same argument as in (12), we know that

$$\mathrm{OPT} - \sum_{i=1}^{m} M_i(\tilde{u}_i^k) \leq \sum_{i=1}^{m} (M_i(\hat{u}_i^k) - M_i(\tilde{u}_i^k) + (\tilde{u}_i^k - \hat{u}_i^k) M_i'(\hat{u}_i^k)).$$

Now for each term above, we consider two cases. If $\hat{u}_i^k \geq \tilde{u}_i^k$, then

$$M_i(\hat{u}_i^k) - M_i(\tilde{u}_i^k) + (\tilde{u}_i^k - \hat{u}_i^k) M_i'(\hat{u}_i^k) \leq M_i(\hat{u}_i^k) - M_i(\tilde{u}_i^k) \leq \frac{M_i(\tilde{u}_i^k)}{\tilde{u}_i^k} (\hat{u}_i^k - \tilde{u}_i^k)$$

and with probability $1-\epsilon$, this is less than $2\epsilon\sqrt{\frac{n}{\ell_k}}M_i(\tilde{u}_i^k)$; if $\hat{u}_i^k < \tilde{u}_i^k$, then

$$M_i(\hat{u}_i^k) - M_i(\tilde{u}_i^k) + (\tilde{u}_i^k - \hat{u}_i^k)M_i'(\hat{u}_i^k) \le (\tilde{u}_i^k - \hat{u}_i^k)M_i'(\hat{u}_i^k) \le \frac{M_i(\hat{u}_i^k)}{\hat{u}_i^k}(\tilde{u}_i^k - \hat{u}_i^k).$$

Again, with probability $1-\epsilon$, this is less than $2\epsilon\sqrt{\frac{n}{\ell_k}}M_i(\tilde{u}_i^k)$. Therefore, with probability $1-\epsilon$,

$$\sum_{i=1}^{m}(M_i(\hat{u}_i^k) - M_i(\tilde{u}_i^k) + (\tilde{u}_i^k - \hat{u}_i^k)M_i'(\hat{u}_i^k)) \le 2\epsilon\sqrt{\frac{n}{\ell_k}}\mathrm{OPT}.$$

Therefore, (E.1) is proved. Next we show that

$$\sum_{i=1}^{m} M_i(\tilde{u}_i^k) - \sum_{i=1}^{m} M_i(\frac{n}{\ell_k}\bar{u}_i^k) \le 4\epsilon\sqrt{\frac{n}{\ell_k}}\mathrm{OPT}.$$

To see this, by Lemma 6, we know that with probability $1-\epsilon$,

$$\tilde{u}_i^k - \frac{n}{\ell_k}\bar{u}_i^k \le 2\epsilon\sqrt{\frac{n}{\ell_k}}\hat{u}_i^k.$$

Therefore, for each $i$, we have

$$\frac{\tilde{u}_i^k - \frac{n}{\ell_k}\bar{u}_i^k}{\tilde{u}_i^k} \le \frac{2\epsilon\sqrt{\frac{n}{\ell_k}}\hat{u}_i^k}{(1-\epsilon\sqrt{\frac{n}{\ell_k}})\hat{u}_i^k} \le 4\epsilon\sqrt{\frac{n}{l_k}}.$$

Now we analyze $M_i(\tilde{u}_i^k) - M_i(\frac{n}{\ell_k}\bar{u}_i^k)$ for each $i$. We only need to focus on the case when $\tilde{u}_i^k > \frac{n}{\ell_k}\bar{u}_i^k$ (otherwise the difference is less than 0). In this case, by the concavity of $M_i(\cdot)$, we have

$$M_i(\tilde{u}_i^k) - M_i(\frac{n}{\ell_k}\bar{u}_i^k) \le \frac{M_i(\tilde{u}_i^k)}{\tilde{u}_i^k}(\tilde{u}_i^k - \frac{n}{\ell_k}\bar{u}_i^k) \le 4\epsilon\sqrt{\frac{n}{\ell_k}}M_i(\tilde{u}_i^k).$$

Therefore, we have

$$\sum_{i=1}^{m} M_i(\tilde{u}_i^k) - \sum_{i=1}^{m} M_i(\frac{n}{\ell_k}\bar{u}_i^k) \le 4\epsilon\sqrt{\frac{n}{\ell_k}}\sum_{i=1}^{m} M_i(\tilde{u}_i^k) \le 4\epsilon\sqrt{\frac{n}{\ell_k}}\mathrm{OPT}.$$

Together with (E.1), Lemma 7 holds. □

# Appendix F

# Proof of Lemma 5 and Lemma 8

First, we note that Lemma 8 implies Lemma 5 (except for the constant part, which can be strengthened easily by only considering one $\ell_k$ in the following proof). Therefore, it suffices to prove Lemma 8.

We first prove for each $k$, with probability $1 - \epsilon/\log(1/\epsilon)$, $\min_i \hat{u}_i^k > C$. Then we take a union bound to prove Lemma 8. To show that for each $k$, with probability $1 - \epsilon/\log(1/\epsilon)$, $\min_i \hat{u}_i^k > C$, first we show that with probability $1 - \epsilon/\log(1/\epsilon)$, $\sum_{j=1}^{\ell_k} b_{ij} \geq \ell_k \bar{b}/2$ for all $i$. To see this, we use Lemma 9, we have for any $i$,

$$P\left(\left|\sum_{j=1}^{\ell_k} b_{ij} - \frac{\ell_k}{n}\sum_{j=1}^{n} n_{ij}\right| \geq \ell_k \bar{b}/2\right) \leq 2\exp(-\ell_k \bar{b}^2/12) < \frac{\epsilon}{m\log(1/\epsilon)},$$

where the last inequality is due to condition (4.14). Next we show that given $\sum_{j=1}^{\ell_k} b_{ij} \geq \ell\bar{b}/2$, there cannot exist an $i$ such that $\hat{u}_i^k < C$ in the optimal solution to the partial program $(\mathbf{P}_{\ell_\mathbf{k}})$. We prove by contradiction. Let $K = \eta\frac{\epsilon n\bar{b} - 2\epsilon C}{2mC}$. If there exists $i$ such that $\hat{u}_i^k < C$ in the optimal solution, then we argue that there must exist $1 \leq j \leq \ell_k$ such that

1. $j \in S_k = \{j : x_{ij} < 1, b_{ij} > \eta\}$, and
2. There exists $i'$ such that $x_{i'j} > 0$ and $\hat{u}_{i'}^k \geq KC$.

Here these two conditions mean that there must exist a keyword $j$ such that we allocated it (at least partially) to bidder $i'$ whose total allocation had already exceeded $KC$ when we could have allocated it to bidder $i$ whose final allocation is less than $C$.

To see this, we note that we have proved with probability $1-\epsilon/\log(1/\epsilon)$, $\sum_{j=1}^{\ell_k} b_{ij} \geq \ell_k\bar{b}/2$. However, by the definition of $i$, $u_i^k < C$, thus we also have $\sum_{j=1}^{\ell_k} b_{ij}x_{ij} \leq \ell_k C/n$. Therefore, combined with the assumption that $\max_{i,j} b_{ij} \leq 1$, there must exist at least $\ell_k\bar{b}/2 - \ell_k C/n$ $j$s between 1 and $\ell_k$ such that $x_{ij} < 1$ but $b_{ij} \geq \eta$, i.e., $|S_k| \geq \ell_k\bar{b}/2 - \ell_k C/n$.

Next we show that among $j \in S_k$, there exists at least one $j$ such that $x_{i'j} > 0$ while $\hat{u}_{i'}^k \geq KC$ for some $i'$. We define $T_k = \{i : \hat{u}_i^k < KC\}$. We have

$$\sum_{i\in T, j\in S_k} x_{ij} \leq \frac{1}{\eta}\sum_{i\in T, j\in S_k} b_{ij}x_{ij} < \frac{mKC}{\eta}. \tag{F.1}$$

Here the second inequality is because $|T| < m$. However, we also have

$$\sum_{i, j\in S_k} x_{ij} \geq \frac{\ell_k\bar{b}}{2} - \frac{\ell_k C}{n}. \tag{F.2}$$

This is because $M_i(\cdot)$s are increasing, thus each $\sum_i x_{ij}$ must equal 1 at optimality. Then, by taking the difference between (F.1) and (F.2), we have that

$$\sum_{i\notin T, j\in S_k} x_{ij} > \frac{\ell_k\bar{b}}{2} - \frac{\ell_k C}{n} - \frac{mKC}{\eta} \geq 0.$$

The last inequality is by the definition of $K$ and that $\ell_k \geq \epsilon n$ for all $k$. Therefore, there exists $j \in S_k$ such that the bid is allocated to some $i'$ with $\hat{u}_{i'}^k \geq KC$. We denote such $j$ by $j^*$.

Finally, we consider another allocation that increases the allocation of $j^*$ to $i$ while decreasing the allocation to $i'$. The local change of the objective function at this point is:

$$M_i'(\hat{u}_i^k)b_{ij} - M_{i'}'(\hat{u}_{i'}^k)b_{i'j} \geq M_i'(C)\eta - M_{i'}'(KC) > 0$$

where the first inequality is due to the concavity of $M_i(\cdot)$s and the last inequality is due to condition (4.14). However, this contradicts the assumption that the solution is optimal. Thus, Lemma 8 is proved. $\square$

## Appendix G

# Risk-Return Curve of the Retailer under a Three-Point Demand Distribution

Let $\alpha = \sqrt{\frac{3\mu-4t}{\mu-t}}$, $\beta = \frac{t}{\mu-t}$, and $\xi = \sqrt{3+\beta^2+\frac{t(-5+2\sqrt{4-\frac{\mu}{\mu-t}})}{\mu-t}}$. Under a three-point demand distribution with mean $\mu$, the risk-return curve as a function of $t$ is given as follows:

$$\mathrm{RR}(t) = \frac{\mu-t}{2}\left((\mu-t)\left(-\alpha+\beta+\xi\right)+\frac{4\left(t-\frac{1}{2}(\mu-t)\left(-\alpha+\beta+\xi\right)\right)}{2+\alpha-\beta-\xi}+\frac{4(\mu-t)^2}{\mu\left(\alpha\xi-1\right)-t\left(-2+\alpha+\alpha\xi\right)}\right).$$