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CONVERGENCE RESULTS FOR THE HOMOGENIZATION OF FLOW IN FRACTURED POROUS MEDIA

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Abstract. In a previous paper (Vernescu [20]) by means of matched asymptotic expansions, the homogenized behaviour of the flow of a viscous, incompressible fluid in a fractured porous medium was studied. The present paper is devoted to the convergence proof of the homogenization process. The microscopic model consists of the viscous flow around porous blocks. The macroscopic behaviour is obtained by means of the epi-convergence of the functionals that describe the microscopic behaviour. The homogenized equations can be of Stokes, Brinkman or Darcy type, depending on the dimensions of the blocks relatively to the dimensions of the fractures. The macroscopic behaviour exhibits the interaction in between the flow in the porous blocks and the flow in the fractures.

Key words. homogenization, fractured porous media, epi-convergence.

AMS(MOS) subject classifications. 76S05, 35B40

1. Introduction. A fractured porous medium can be viewed as being formed by porous blocks that are contained in a net of interconnected fractures. Thus there is an interaction in between the flow in the porous blocks and the flow in the fractures. The model studied in this paper is the one introduced in Ene and Vernescu [11] (see also Ene [10]). Different models have been considered by Arbogast [2] and Lewy [13].

The microscopic problem is described by the fluid flow around porous obstacles. The porous blocks are supposed to be periodically distributed in a bounded domain $\Omega \subset \mathbb{R}^N (N = 2,3)$ with a 2ε period, their characteristic length being $2a \quad (0 < a < \varepsilon)$, we have thus a problem depending on two parameters: a and ε, and we study its behaviour as ε tends to zero. For a fixed ε the problem represents the Stokes flow problem around porous obstacles, for which the existence and uniqueness was studied in Ene and Vernescu [11].

On the boundary of the porous blocks only the pressure and the normal component of the velocity are continuous. The jump of the tangential component of the velocity gives to this problem its special feature. The technical difficulty consists in dealing with a sequence of solutions that have jumps on a set that depends on the parameter ε.

The problem is formulated as a minimum problem on a product space that does not depend on ε:

$$\begin{aligned} &F_\varepsilon(\underline{\mathbf{v}}^\varepsilon) = \min_H F_\varepsilon(\underline{\mathbf{w}}) \\ (1.1) \qquad &F_\varepsilon(\underline{\mathbf{w}}) = \frac{1}{2}\left(\int_\Omega (\text{curl } \mathbf{w}^2)^2 + a^{-2}\int_\Omega k(\mathbf{w}^1)^2\right) + I_H(\underline{\mathbf{w}}) \end{aligned}$$

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where $F_\varepsilon : H \to \mathbb{R}$, $\underline{\mathbf{w}} = (\mathbf{w}^1, \mathbf{w}^2) \subset H = (H_0(\text{div}\ , \Omega) \cap \{\text{div}\ \mathbf{w} = 0\}) \times H_0(\text{curl}\ , \Omega)$ and I_H is the indicator functional of the set:

(1.2)
$$\{\underline{\mathbf{v}} \in H / \mathbf{v}^1 = \mathbf{v}^2 \text{ in } \Omega_{1\varepsilon}, \quad \mathbf{v}^2 = 0 \text{ in } \Omega_{2\varepsilon}\}$$

where $\Omega_{1\varepsilon}, \Omega_{2\varepsilon}$ are the subsets of Ω represented by fractures, and respectively by porous blocks.

With the aid of a corresponding penalized problem and the corresponding local problems, the epi-convergence (Γ-convergence) of F_ε is studied in the case $a = O(\varepsilon^3)$ and $a = o(\varepsilon^3)$.

In general, a family $(F_\varepsilon)_\varepsilon$ of functionals on a first countable topological space X is said to be epi-convergent to F if:

i) for every $x \in X$ there exists $x^\varepsilon \to x$ such that:

(1.3)
$$F(x) = \lim_\varepsilon F^\varepsilon(x^\varepsilon)$$

ii) for every $x \in X$ and every $x^\varepsilon \to x$ we have:

(1.4)
$$F(x) \leq \liminf_\varepsilon F^\varepsilon(x^\varepsilon)$$

and we write $F = \lim_\varepsilon {}_e F^\varepsilon$. If (F^ε) is epi-convergent to F then (see Attouch [3]) for any minimizing sequence $(x^\varepsilon)_\varepsilon$:

$$F^\varepsilon(x^\varepsilon) \leq \inf F^\varepsilon(x) + \varepsilon$$

and for any convergent subsequence $(x^{\varepsilon_n}) \subset (x^\varepsilon)$, $x^{\varepsilon_n} \to x$ we have:

(1.5)
$$F(x) = \min_{y \in X} F(y) \quad \text{and} \quad F^{\varepsilon_n}(x^{\varepsilon_n}) \to F(x).$$

We obtain the epi-limit of (F^ε) to be $F : H \to \mathbb{R}$ given by:

(1.6)
$$F(\underline{\mathbf{w}}) = \frac{1}{2}\left(\int_\Omega (\text{curl}\ \mathbf{w}^2)^2 + m(\mathbf{w}^1 K \mathbf{w}^1)\right) + I(\underline{\mathbf{w}})$$

with $I : H \to \mathbb{R}$ the indicator functional of the set:

$$\{\underline{\mathbf{v}} \in H / \mathbf{v}^1 = \mathbf{v}^2 \text{ in } \Omega\}$$

and $m = \lim_{\varepsilon \to 0} \dfrac{a}{\varepsilon^3}$. Thus the homogenized problem is of Stokes or Brinkman type:

(1.7)
$$\begin{cases} -\Delta \mathbf{v} + mK\mathbf{v} = \mathbf{f} - \text{grad}\ p & \text{in } \Omega \\ \text{div}\ \mathbf{v} = 0 & \text{in } \Omega \end{cases}$$

where K is a symmetric, positive matrix, dependent on k, the inverse of the permeability of the blocks. In the case of the spherical blocks K is equal to a constant times the identity matrix, where the constant is given by the generalized Stokes formula (Gheorghiţă [9]) for the drag for the flow around spherical porous bodies.

The case of a having the same order as ε was studied in Ene and Vernescu [11], where the homogenized behaviour is expressed by a Darcy law for a modified permeability tensor, depending on the permeability of the blocks.

If $k \to \infty$ one can obtain the classical case of impermeable inclusions (see Sanchez-Palencia [16]) for which the convergence was proved by Tartar [18] in the case $a = O(\varepsilon)$ and by Brillard [4] in the cases $a = O(\varepsilon^3), a = o(\varepsilon^3)$(see also Allaire [1]). The homogenized behaviour was given by a Darcy, Brinkman or Stokes law. The analysis of the transition between the asymptotic structures corresponding to the different situations can be found in Sanchez-Palencia [17] and Geymonat and Sanchez-Palencia [8].

2. Statement of the problem. Let $\Omega \subset \mathbb{R}^N (N = 2,3)$ be a bounded open set and $\Gamma = \partial\Omega$ of class C^2. If $Y = (-1,1)^N$ and $Y_1, Y_2 \subset Y$, $\overline{Y}_2 \subset Y$, $Y_1 = Y - \overline{Y}_2$ with $S = \partial Y_2$ smooth, we define:

$$\Omega_{2\varepsilon} = \left\{ x \in \Omega / x = 2n\varepsilon + ay, n \in \mathbb{Z}^N, y \in Y_2 \right\} \tag{2.1}$$

$$\Omega_{1\varepsilon} = \Omega - \overline{\Omega}_{2\varepsilon} \tag{2.2}$$

subsets of Ω obtained from Y_1, Y_2 by homotety, where $\varepsilon \geq a = a(\varepsilon) > 0$, and let $S_\varepsilon = \partial\Omega_{1\varepsilon} \cap \partial\Omega_{2\varepsilon}$.

The domain $\Omega_{1\varepsilon}$ represents the subset of Ω formed by the fractures and $\Omega_{2\varepsilon}$ the subset formed by the porous blocks. The non-dimensional equations that describe the flow are:

$$\begin{aligned} -\Delta \mathbf{v}^\varepsilon &= \mathbf{f} - \operatorname{grad} p^\varepsilon && \text{in } \Omega_{1\varepsilon} \\ a^{-2} k \mathbf{v}^\varepsilon &= \mathbf{f} - \operatorname{grad} p^\varepsilon && \text{in } \Omega_{2\varepsilon} \\ \operatorname{div} \mathbf{v}^\varepsilon &= 0 && \text{in } \Omega \end{aligned} \tag{2.3}$$

with the boundary and transmission conditions (see Gheorghiţă [9]):

$$\mathbf{v}^\varepsilon\big|_{\Omega_{1\varepsilon}} \times \mathbf{n} = 0, \quad [\mathbf{v}^\varepsilon \cdot \mathbf{n}] = 0, \quad [p^\varepsilon] = 0 \text{ on } S_\varepsilon \tag{2.4}$$

$$\mathbf{v}^\varepsilon = 0 \text{ on } \Omega \tag{2.5}$$

where $\mathbf{v}, p, \mathbf{f}, k$ represent respectively the velocity, the pressure, the body forces, the inverse of the permeability and [ ] represents the jump on S_ε.

The problem (2.3) – (2.5) can be viewed in the more general setting of Stokes flow around porous obstacles, for which an existence and uniqueness result has been proved in

Ene and Vernescu [11], by proving a coercivity result for the first term in the left side of the variational formulation:

$$
\text{(2.6)} \qquad \begin{aligned} &\text{find } \mathbf{v}^\varepsilon \in H_\varepsilon \text{ , so that:} \\ &\int_{\Omega_{1\varepsilon}} \text{curl } \mathbf{v}^\varepsilon \cdot \text{curl } \mathbf{w} + a^{-2} \int_{\Omega_{2\varepsilon}} k\mathbf{v}^\varepsilon \cdot \mathbf{w} = \int_\Omega \mathbf{f} \cdot \mathbf{w}, \quad \text{for all } \mathbf{w} \in H_\varepsilon \end{aligned}
$$

where $H_\varepsilon = \{\mathbf{v} \in L^2(\Omega)/\mathbf{v} \in H^1(\Omega_{1\varepsilon}), \text{div } \mathbf{v} = 0 \text{ in } \Omega, \mathbf{v} = 0 \text{ on } \Gamma, \mathbf{v} \times \mathbf{n} = 0 \text{ on } S_\varepsilon\}$. The proof relies on the characterization of the spaces curl $H^1(\Omega)$ and (curl $H^1(\Omega))^\perp$ given in Foias and Temam [12].

In the sequel we shall consider problem (2.6) in the form:

$$
\text{(2.7)} \qquad \begin{aligned} &\mathcal{F}_\varepsilon(\mathbf{v}^\varepsilon) = \min_{H_\varepsilon} \mathcal{F}_\varepsilon(\mathbf{w}) \\ &\mathcal{F}_\varepsilon(\mathbf{w}) = \frac{1}{2}\left(\int_{\Omega_{1\varepsilon}} (\text{curl } \mathbf{w})^2 + a^{-2} \int_{\Omega_{2\varepsilon}} k(\mathbf{w})^2 \right) - \int_\Omega \mathbf{f} \cdot \mathbf{w} \end{aligned}
$$

and also the corresponding penalized problem:

$$
\text{(2.8)} \qquad \begin{aligned} &\mathcal{F}_{\varepsilon\lambda}(\mathbf{v}^\varepsilon) = \min_{V_\varepsilon} \mathcal{F}_{\varepsilon\lambda}(\mathbf{w}) \\ &\mathcal{F}_{\varepsilon\lambda}(\mathbf{w}) = \mathcal{F}_\varepsilon(\mathbf{w}) + \frac{1}{2\lambda^2} \int_\Omega (\text{div } \mathbf{w})^2 \end{aligned}
$$

where $H_\varepsilon \subset V_\varepsilon = \{\mathbf{v} \in L^2(\Omega)/\mathbf{v} \in H^1(\Omega_{1\varepsilon}), \mathbf{v} = 0 \text{ on } \Gamma, \mathbf{v} \times \mathbf{n} = 0 \text{ on } S_\varepsilon\}$. The existence of a unique solution for (2.8) can be similarly proved.

The sequences of solutions of (2.7) and (2.8) are bounded and this yields from the next lemma. We shall denote by $|\ |_D$ the $L^2(D)$ norm.

LEMMA 2.1. *There exists a constant $C > 0$, independent of ε, so that for each $\varepsilon > 0$ and for each $\mathbf{v} \in V_\varepsilon$:*

$$
\text{(2.9)} \qquad |\mathbf{v}|_{\Omega_{1\varepsilon}} \leq C \left(|\text{curl } \mathbf{v}|_{\Omega_{1\varepsilon}} + |\text{div } \mathbf{v}|_\Omega + a^{-1}|\mathbf{v}|_{\Omega_{2\varepsilon}} \right)
$$

Proof. Let $\mathbf{v} \in V_\varepsilon$ and $\mathbf{u}$ defined by:

$$
\text{(2.10)} \qquad \Delta\mathbf{u} = 0 \text{ in } \Omega_{2\varepsilon}, \quad \mathbf{u} \times \mathbf{n} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} \text{ on } S_\varepsilon.
$$

Then if $\mathbf{w} = \mathbf{v}$ in $\Omega_{1\varepsilon}$ and $\mathbf{w} = \mathbf{u}$ in $\Omega_{2\varepsilon}$ we have $\mathbf{w} \in H_0^1(\Omega)$ and:

$$
\text{(2.11)} \qquad |\mathbf{w}|_\Omega \leq C(|\text{div } \mathbf{w}|_\Omega + |\text{curl } \mathbf{w}|_\Omega)
$$

From (2.10), by passing to the variable $y = x/a$ we have:

$$|\text{div}_y \mathbf{u}|_{Y_2} + |\text{curl}_y \mathbf{u}|_{Y_2} \le C\|\mathbf{v}\cdot\mathbf{n}\|_{H^{-1/2}(S)} \le$$
$$\le C(|\mathbf{v}|_{Y_2} + |\text{div }\mathbf{v}|_{Y_2})$$

By changing back the variable we obtain:

$$|\text{div }\mathbf{u}|_{aY_2} + |\text{curl }\mathbf{u}|_{aY_2} \le C(a^{-1}|\mathbf{v}|_{aY_2} + |\text{div }\mathbf{v}|_{aY_2}) \tag{2.12}$$

and hence from (2.11) and (2.12) we get the conclusion. □

In order to study the asymptotic behaviour of the problems (2.7) and (2.8) a formulation on a space not depending on ε is more suitable. Let us introduce the following spaces:

$$H = (H_0(\text{div }, \Omega) \cap \{\mathbf{w}|\text{div }\mathbf{w} = 0\}) \times H_0(\text{curl }, \Omega) \tag{2.13}$$
$$V = H_0(\text{div }, \Omega) \times H_0(\text{curl }, \Omega) \tag{2.14}$$

where:

$$H_0(\text{div }, \Omega) = \{\mathbf{w} \in L^2(\Omega)|\text{div }\mathbf{w} \in L^2(\Omega), \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$$
$$H_0(\text{curl }, \Omega) = \{\mathbf{w} \in L^2(\Omega)|\text{curl }\mathbf{w} \in L^2(\Omega), \mathbf{w}\times\mathbf{n} = 0 \text{ on } \partial\Omega\}$$

If I_H (resp. I_V) denotes the indicator functional of the space:

$$\{\underline{\mathbf{w}} \in H(\text{resp. } \underline{\mathbf{w}} \in V)|\underline{\mathbf{w}} = (\mathbf{w}^1, \mathbf{w}^2), \quad \mathbf{w}^1 = \mathbf{w}^2 \text{ in } \Omega_{1\varepsilon}, \quad \mathbf{w}^2 = 0 \text{ in } \Omega_{2\varepsilon}\}$$

then the problems (2.7) and (2.8) are respectively equivalent to:

$$(\text{P}_\varepsilon) \quad \begin{cases} F_\varepsilon(\underline{\mathbf{v}}^\varepsilon) & = \min\limits_{\text{H}} F_\varepsilon(\underline{\mathbf{w}}) \\ F_\varepsilon(\underline{\mathbf{w}}) & = \frac{1}{2}\left(\int\limits_\Omega (\text{curl }\mathbf{w}^2)^2 + a^{-2}\int\limits_{\Omega_{2\varepsilon}} k(\mathbf{w}^1)^2\right) - \int\limits_\Omega f(\mathbf{w}^1)^2 + I_H(\underline{\mathbf{w}}) \end{cases}$$

$$(\text{P}_{\varepsilon\lambda}) \quad \begin{cases} F_{\varepsilon\lambda}(\underline{\mathbf{v}}^{\varepsilon\lambda}) & = \min\limits_{\text{V}} F_{\varepsilon\lambda}(\underline{\mathbf{w}}) \\ F_{\varepsilon\lambda}(\underline{\mathbf{w}}) & = \frac{1}{2}\left(\int\limits_\Omega (\text{curl }\mathbf{w}^2)^2 + \frac{1}{\lambda^2}(\text{div }\mathbf{w}^1)^2 + a^{-2}\int\limits_{\Omega_{2\varepsilon}} k(\mathbf{w}^1)^2\right) - \\ & \quad - \int\limits_\Omega f(\mathbf{w}^1)^2 + I_V(\underline{\mathbf{w}}) \end{cases}$$

The equivalence is based on the remark:

$$\underline{\mathbf{v}} \in H_\varepsilon \iff \underline{\mathbf{v}} = (\mathbf{v}, \mathbf{v}\cdot\chi(\Omega_{1\varepsilon})) \in H$$

which is obvious because $H_\varepsilon \subset H_0(\text{div }, \Omega)$ and because by changing a function belonging to H_ε with zero in $\Omega_{2\varepsilon}$ one obtaines a function from $H_0(\text{curl }, \Omega)$.

The following lemma is a direct consequence of Lemma 2.1.:

LEMMA 2.2. *The sequences* $(\underline{\mathbf{v}}^\varepsilon), (\underline{\mathbf{v}}^{\varepsilon\lambda})$ *of the solutions of* (P_ε) *and* $(P_{\varepsilon\lambda})$ *are bounded:*

$$\|\underline{\mathbf{v}}^\varepsilon\|_H \le C, \qquad |\mathbf{v}^{1\varepsilon}|_{\Omega_{2\varepsilon}} \le aC \tag{2.15}$$

$$\|\underline{\mathbf{v}}^{\varepsilon\lambda}\|_V \le C, \qquad |\mathbf{v}^{1\varepsilon\lambda}|_{\Omega_{2\varepsilon}} \le aC \tag{2.16}$$

3. The local problems. Let us consider the local variational problems corresponding to (P_ε) and $(\mathrm{P}_{\varepsilon\lambda})$:

$(\mathrm{LP}_\varepsilon)$

$$\begin{cases} \text{find } \mathbf{t}^i_\varepsilon \in H(\text{div}\,, B(\varepsilon)) \cap H(\text{curl}\,, B(\varepsilon) - aY_2) \text{ solution for :} \\ \min\left\{ \int\limits_{B(\varepsilon)-aY_2} (\text{curl}\ \mathbf{t}^i)^2 + a^{-2} \int\limits_{aY_2} k(\mathbf{t}^i)^2 \middle| \mathbf{t}^i = \mathbf{e}_i \text{ on } \partial B(\varepsilon), \mathbf{t}^i \times \mathbf{n} = 0 \right. \\ \qquad \left. \text{on } \partial aY_2, \text{div } \mathbf{t}^i = 0 \text{ in } B(\varepsilon) \right\} \end{cases}$$

$(\mathrm{LP}_{\varepsilon\lambda})$

$$\begin{cases} \text{find } \mathbf{w}^i_{\varepsilon\lambda} \in H(\text{div}\,, B(\varepsilon)) \cap H(\text{curl}\,, B(\varepsilon) - aY_2) \text{ solution for :} \\ \min\left\{ \int\limits_{B(\varepsilon)-aY_2} (\text{curl}\ \mathbf{w}^i)^2 + \frac{1}{\lambda^2} \int\limits_{B(\varepsilon)} (\text{div}\ \mathbf{w}^i)^2 + a^{-2} \int\limits_{aY_2} k(\mathbf{w}^i)^2 \middle| \mathbf{w}^i = \mathbf{e}_i \text{ on } \partial B(\varepsilon), \right. \\ \qquad \left. \mathbf{w}^i \times \mathbf{n} = 0 \text{ on } \partial aY_2 \right\} \end{cases}$$

where $B(\varepsilon)$ is the ball having radius ε contained in εY, and $\mathbf{e}_i$ is the i-th unitary vector of the cartezian base. The functions $\mathbf{t}^i_\varepsilon$ and $\mathbf{w}^i_{\varepsilon\lambda}$ can be extended in $\varepsilon Y - B(\varepsilon)$ by $\mathbf{e}_i$ and by periodicity to $\mathbb{R}^N$.

By passing from the x variable to $y = x/a$, in $(\mathrm{LP}_{\varepsilon\lambda})$ we get in each period:

$$\begin{cases} \text{curl (curl } \mathbf{w}^i) - \frac{1}{\lambda^2}\text{grad (div } \mathbf{w}^i) = 0 & \text{in } B - Y_2 \\ -\frac{1}{\lambda^2}\text{grad (div } \mathbf{w}^i) + k\mathbf{w}^i = 0 & \text{in } Y_2 \\ \mathbf{w}^i|_{B-Y_2} \times \mathbf{n} = 0, \quad [\mathbf{w}^i \cdot \mathbf{n}] = 0, \quad [\text{div } \mathbf{w}^i] = 0 & \text{on } \partial Y_2 \\ \mathbf{w}^i = \mathbf{e}_i & \text{on } \partial B \end{cases} \tag{3.1}$$

where $B = B\left(\frac{\varepsilon}{a}\right)$ and the derivatives are with respect to y.

The following lemma will show the behaviour of $\mathbf{w}^i$ for $N = 3$ in the case $Y_2 = B(1)$. For $N = 2$ the computations are similar.

LEMMA 3.1. *If* $N = 3$ *and* $Y_2 = B(1)$ *and if* (r, θ, φ) *are the spherical coordinates, with* θ *the angle between* y *and* $\mathbf{e}_i$ *then:*

$$\mathbf{w}^i = f^i(r)\cos\theta\mathbf{e}_r + g^i(r)\sin\theta\mathbf{e}_\theta \tag{3.2}$$

where:

(3.3)

$$f^i(r) = \begin{cases} \frac{2A}{r^3} + \frac{2B}{r} + C + (3\lambda^2 - 1)Dr^2 & \text{in } B - Y_2 \\ \frac{E}{(\nu r)^3}(\nu^2 r^2 \sinh \nu r + 2\nu r + 2\nu r \cosh \nu r - 2 \sinh \nu r) & \text{in } Y_2 \end{cases}$$

(3.4)

$$g^i(r) = \begin{cases} \frac{A}{r^3} - (1 + \lambda^2)\frac{B}{r} - C + (2 + \lambda^2)Dr^2 & \text{in } B - Y_2 \\ \frac{E}{(\nu r)^3}(\nu r \cosh \nu r - \sinh \nu r) & \text{in } Y_2 \end{cases}$$

with $\nu^2 = \lambda^2 k$. A, B, C, D, E *depend on* $\eta = \frac{\varepsilon}{a}$ *and* λ *so that:*

(3.5)
$$\lim_{\eta \to \infty} B = b_\lambda, \quad \lim_{\eta \to \infty} D = 0, \quad \lim_{\lambda \to 0} b_\lambda = -\frac{B}{2}\frac{k}{1 + 2k} = -K$$

and where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ *denote the orthogonal base in spherical coordinates.*

Proof. From (3.2) (omitting the superscript i) if we denote by:

(3.6)
$$u = g' + \frac{1}{r}(f + g) \qquad v = f' + \frac{2}{r}(f + g)$$

we obtain:

(3.7)
$$\text{curl } \mathbf{w} = u \sin \theta \mathbf{e}_\varphi \qquad \text{div } \mathbf{w} = v \cos \theta$$

and thus, from (3.1), u satisfies the homogeneous Euler equation:

$$r^2 u'' + 2ru' - 2u = 0$$

and $v = \lambda^2(ru' + u)$.

Thus we obtain:

(3.8)
$$u = \frac{2B}{r^2} + 5Dr \qquad v = \lambda^2 \left(-\frac{2B}{r^2} + 10Dr \right)$$

From (3.6) and (3.8) we get $(3.3)_1$ and $(3.4)_1$, by integrating a linear second order differential equation.

In order to compute the form of $\mathbf{w}$ in Y_2 from (3.2) and $(3.1)_2$ we obtain:

(3.9)
$$f = -(g + rg') \qquad rg'' + 4g' - \nu^2 rg = 0$$

where from, using the substitutions $\rho = \nu r$, $h = g\rho^{3/2}$ we get:

$$\rho^2 h'' + \rho h' - \left(\frac{g}{4} + \rho^2\right) h = 0$$

Thus if I_n denotes the modified Bessel functions of first kind:

$$h = EI_{3/2}(\rho) + HI_{-3/2}(\rho)$$

where:

$$I_{3/2}(\rho) = \sqrt{\frac{2}{\pi}}\rho^{-3/2}(\rho\cosh\rho - \sinh\rho)$$
$$I_{-3/2}(\rho) = \sqrt{\frac{2}{\pi}}\rho^{-3/2}(\rho\sinh\rho - \cosh\rho)$$

and hence

$$g = \frac{1}{(\nu r)^3}\left(\nu r(E\cosh\nu r + H\sinh\nu r) - (E\sinh\nu r + H\cosh\nu r)\right) \tag{3.10}$$

By imposing that $\lim g(r) < \infty$ we get $H = 0$.

The constants are obtained by imposing the transmission and boundary conditions $(3.1)_4$ and $(3.1)_5$ and (3.5) follows. □

LEMMA 3.2. *If $N = 3$ and $Y_2 = B(1)$, the solutions of the local problem $(LP_{\varepsilon\lambda})$ have the following properties:*

1. $\mathbf{w}^i_{\varepsilon\lambda}\chi_1 \to \mathbf{e}_i$ *strongly in* $L^2(\Omega)$, *for* $\varepsilon \to 0$
   $\mathbf{w}^i_{\varepsilon\lambda} \to \mathbf{e}_i$ *strongly in* $L^2(\Omega)$, *for* $\varepsilon \to 0$
2. *if* $a = o(\varepsilon^3)$ *then for* $\varepsilon \to 0$:
   $\operatorname{div}\mathbf{w}^i_{\varepsilon\lambda} \to 0$ *strongly in* $L^2(\Omega)$, $\quad \operatorname{curl}(\mathbf{w}^i_{\varepsilon\lambda}\chi_1) \to 0$ *strongly in* $L^2(\Omega)$
3. *if* $a = O(\varepsilon^3)$ *then for* $\varepsilon \to 0$:
   $\operatorname{div}\mathbf{w}^i_{\varepsilon\lambda} \to 0$ *weakly in* $L^2(\Omega)$, $\quad \operatorname{curl}(\mathbf{w}^i_{\varepsilon\lambda}\chi_1) \to 0$ *weakly in* $L^2(\Omega)$
   $\sum(\operatorname{curl}\mathbf{w}^i_{\varepsilon\lambda} \times \mathbf{n} + \frac{1}{\lambda^2}\operatorname{div}\mathbf{w}^i_{\varepsilon\lambda}\mathbf{n})|_{\partial B(\varepsilon)} \to -\pi b_\lambda \mathbf{e}_i$ *strongly in* $H^{-1}(\Omega)$

where the sum is taken over all the periods $aY \subset \Omega$.

We have denoted by $\chi_1 = \chi(\Omega_{1\varepsilon}), \chi_2 = \chi(\Omega_{2\varepsilon})$.

Proof. 1. By defining $\mathbf{h}^i_{\varepsilon\lambda} = \mathbf{w}^i_{\varepsilon\lambda}$ in $\Omega_{1\varepsilon}$, $\mathbf{h}^i_{\varepsilon\lambda} = f(r)\cos\theta\mathbf{e}_r$ in $\Omega_{2\varepsilon}$ we have $\mathbf{h}^i_{\varepsilon\lambda} \in H^1_0(\Omega)$ and $(\mathbf{h}^i_{\varepsilon\lambda})$ bounded in $H^1_0(\Omega)$, thus the sequence is weakly convergent. From:

$$(\mathbf{h}^i_{\varepsilon\lambda} - \mathbf{e}_i)\chi(\Omega - \cup B(\varepsilon)) = 0$$

because the characteristic function is weakly $*$ convergent in $L^\infty(\Omega)$ and $\mathbf{h}^i_{\varepsilon\lambda}$ is strongly convergent in $L^2(\Omega)$ we get $\mathbf{h}^i_{\varepsilon\lambda} \to \mathbf{e}_i$ in $H^1_0(\Omega)$. We also have:

$$\int_\Omega (\mathbf{w}^i_{\varepsilon\lambda}\chi_1 - \mathbf{e}_i)^2 = \int_\Omega (\mathbf{h}^i_{\varepsilon\lambda}\chi_1 - \mathbf{e}_i)^2$$

but the second member converges to zero because $\chi_1 \to 1$ weakly $*$ in $L^\infty(\Omega)$, thus the first convergence in 1. is proved.

The second convergence in 1. is obvious because $\chi_2 \to 0$ weakly $*$ in $L^\infty(\Omega)$.

2. Integrating by parts and using (3.1) we get.

$$
\begin{aligned}
&|\text{curl } \mathbf{w}^i_{\varepsilon\lambda}\chi_1|^2_\Omega + \frac{1}{\lambda^2}|\text{div } \mathbf{w}^i_{\varepsilon\lambda}|^2_\Omega + a^{-2}|\mathbf{w}^i_{\varepsilon\lambda}\chi_2|^2_\Omega = \\
&\qquad = \sum \int_{\partial B(\varepsilon)} \mathbf{e}_i \left(\text{curl } \mathbf{w}^i_{\varepsilon\lambda} \times \mathbf{n} + \frac{1}{\lambda^2}\text{div } \mathbf{w}^i_{\varepsilon\lambda}\ \mathbf{n}\right)
\end{aligned} \tag{3.11}
$$

where the sum is taken over all balls $B(\varepsilon) \subset \Omega$. The right hand term in (3.12) can be estimated to be of order

$$
\text{meas } \Omega \frac{a}{\varepsilon^3}\left(-\frac{b_\lambda}{4}\right) \tag{3.12}
$$

and thus the convergence is proved.

3. The first two assertions result from (3.11) and the convergence of $\mathbf{w}^i_\varepsilon$ to $\mathbf{e}_i$ in $L^2(\Omega)$. For the third convergence, from (3.2) we have:

$$
\begin{aligned}
\text{curl } \mathbf{w}^i &= \left(\frac{2B}{r^2} + 5Dr\right)\sin\theta \mathbf{e}_\varphi \quad \text{in } B - Y_2 \\
\text{div } \mathbf{w}^i &= \begin{cases} \lambda^2\left(-\frac{2B}{r^2} + 10Dr\right)\cos\theta & \text{in } B - Y_2 \\ \frac{E}{\nu r^2}(\sinh \nu r - \nu r \cosh \nu r)\cos\theta & \text{in } Y_2 \end{cases}
\end{aligned} \tag{3.13}
$$

and thus the sum is of the form:

$$
(-2B\mathbf{e}_i + 4Da\mathbf{s})\sum \varepsilon\delta^\varepsilon \tag{3.14}
$$

where $\mathbf{s}$ is a constant vector and δ^ε is the Dirac measure on $\partial B(\varepsilon)$. From a result of Cioranescu and Murat [5]:

$$
\sum \varepsilon\delta^\varepsilon \to \frac{\pi}{2} \quad \text{strongly in } H^{-1}(\Omega) \tag{3.15}
$$

From (3.5), (3.14), (3.15) the convergence is proved. □

The next two results are the analogous of the previous two, but for the solutions of (LP_ε), that satisfy in each Y period:

$$
\begin{cases}
\text{curl }(\text{curl } \mathbf{t}^i) = -\text{grad } s^i & \text{in } B - Y_2 \\
\text{div } \mathbf{t}^i = 0 & \text{in } B \\
k\mathbf{t}^i = -\text{grad } s^i & \text{in } Y_2 \\
\mathbf{t}^i|_{B-Y_2} \times \mathbf{n} = 0, [\mathbf{t}^i \cdot \mathbf{n}] = 0, [s^i] = 0 & \text{on } \partial Y_2 \\
\mathbf{t}^i = \mathbf{e}_i & \text{on } \partial B
\end{cases} \tag{3.16}
$$

LEMMA 3.3. *If $N = 3$ and $Y_2 = B(1)$ then the solutions of the local problem (LP_ε) have the following properties:*

$$\mathbf{t}^i = h^i(r)\cos\theta \mathbf{e}_r + i^i(r)\sin\theta \mathbf{e}_\theta, \quad s^i = j^i(r)\cos\theta \tag{3.17}$$

with:

$$h^i(r) = \begin{cases} \frac{2G}{r^3} + \frac{2I}{r} + J - Lr^2 & \text{in } B - Y_2 \\ -M & \text{in } Y_2 \end{cases} \tag{3.18}$$

$$i^i(r) = \begin{cases} \frac{G}{r^3} - \frac{I}{r} - J + 2Lr^2 & \text{in } B - Y_2 \\ M & \text{in } Y_2 \end{cases} \tag{3.19}$$

$$j^i(r) = \begin{cases} \frac{2I}{r^2} - 10Lr & \text{in } B - Y_2 \\ kMr & \text{in } Y_2 \end{cases} \tag{3.20}$$

where G, I, J, L, M depend on $\eta = \varepsilon/a$ and can be obtained from A, B, C, D, E respectively when $\lambda \to 0$. We have:

$$\lim_{\eta\to\infty} I = -\frac{3}{2}\frac{k}{1+2k} = -K \quad \lim_{\eta\to\infty} J = 0 \tag{3.21}$$

$$\text{curl } \mathbf{t}^i = \left(\frac{2I}{r^2} + 5Lr\right)\sin\theta \mathbf{e}_\varphi \tag{3.22}$$

Proof. It can be observed that the functionals in ($\mathrm{LP}_{\varepsilon\lambda}$) form an increasing sequence of functions with respect to λ, thus (Attouch [3]) $(\mathbf{w}^i_{\varepsilon\lambda}, \mathbf{w}^i_{\varepsilon\lambda}\chi_1) \to (\mathbf{t}^i_\varepsilon, \mathbf{t}^i_\varepsilon\chi_1)$weakly in V for $\lambda \to 0$. □

LEMMA 3.4. *The solutions of the local problem (LP_ε) have the following properties:*

1. $\mathbf{t}^i_\varepsilon\chi_1 \to \mathbf{e}_i$ *strongly in* $L^2(\Omega)$, $\mathbf{t}^i_\varepsilon \to \mathbf{e}_i$ *strongly in* $L^2(\Omega)$
2. *if* $a = o(\varepsilon^3)$ *then* :

   curl $\mathbf{t}^i_\varepsilon\chi_1 \to 0$ *strongly in* $L^2(\Omega)$
3. *if* $a = O(\varepsilon^3)$ *then:*

   curl $\mathbf{t}^i_\varepsilon\chi_1 \to 0$ *weakly in* $L^2(\Omega)$

$$\frac{1}{(2\varepsilon)^3}\left(\int_{B(1)-B(a)} \text{curl } \mathbf{t}^i_\varepsilon \cdot \text{curl } \mathbf{t}^j_\varepsilon + a^{-2}\int_{B(a)} k\mathbf{t}^i_\varepsilon \cdot \mathbf{t}^j_\varepsilon\right) \to \frac{3\pi}{2}\frac{k}{1+2k}\mathbf{e}_i \cdot \mathbf{e}_j$$

Remark. For a non-spherical Y_2 we can use the $\mathbf{t}^i_\varepsilon$ solutions for the spherical case as comparison functions. The results in lemma 3.4. are still valid. In this case K is a positive, symmetric matrix having as i-th column the drag force exerted on the porous obstacle Y_2 when the velocity of the fluid is equal to $\mathbf{e}_i$ at infinity.

4. The homogenized problem. Before studying the epiconvergence of the functionals F_ε we will prove the following:

LEMMA 4.1. *If* $\underline{\mathbf{u}}^\varepsilon = (\mathbf{u}^{1\varepsilon}, \mathbf{u}^{2\varepsilon}) \in V$ *so that* $I_\varepsilon(\underline{\mathbf{u}}^\varepsilon) = 0$ *and* $\underline{\mathbf{u}}^\varepsilon \to \underline{\mathbf{u}} = (\mathbf{u}^1, \mathbf{u}^2)$ *weakly in* V*, then:*

i. $\mathbf{u}^1 = \mathbf{u}^2 = \mathbf{u}$ *in* Ω

ii. $\mathbf{u}^{2\varepsilon}\phi \to \mathbf{u}\phi$ *strongly in* $L^2(\Omega)$*, for all* $\phi \in \mathcal{D}(\Omega)$.

Proof. i. We have:

$$\mathbf{u}^{1\varepsilon} \to \mathbf{u}^1 \text{ weakly in } L^2(\Omega), \mathbf{u}^{2\varepsilon} \to \mathbf{u}^2 \text{ weakly in } L^2(\Omega), \mathbf{u}^{2\varepsilon} = \mathbf{u}^{1\varepsilon}\chi_1 \tag{4.1}$$

and thus:

$$(\mathbf{u}^{1\varepsilon} - \mathbf{u}^{2\varepsilon}, \phi)_\Omega = (\mathbf{u}^{1\varepsilon}\chi_2, \phi)_\Omega \leq C|\mathbf{u}^{1\varepsilon}|_\Omega(\text{ meas } \Omega_{2\varepsilon})^{1/2} \to 0$$

for all $\phi \in \mathcal{D}(\Omega)$ and thus $\mathbf{u}^1 = \mathbf{u}^2$.

ii. By compensated compactness (Murat [14]), because $(\mathbf{u}^{2\varepsilon})$ is bounded in $H(\text{curl }, \Omega)$ and $(\mathbf{u}^{1\varepsilon})$ is bounded in $H(\text{div }, \Omega)$ we get:

$$\int_\Omega (\mathbf{u}^{2\varepsilon}\phi)^2 = \int_\Omega \mathbf{u}^{2\varepsilon}\mathbf{u}^{1\varepsilon}\phi^2 \to \int_\Omega (\mathbf{u}\phi)^2$$

and thus $\mathbf{u}^{2\varepsilon}\phi$ converges weakly and in norm to $\mathbf{u}\phi$. □

We study the epiconvergence of $F_{\varepsilon\lambda}$ and F_ε without considering the continuous perturbation $(\mathbf{f},\mathbf{w}^1)$, because if F_α is epi-convergent to F and G is continuous then $(F_\alpha + G)$ is epi-convergent to $(F + G)$(see Attouch [3])

Let I_0 be the indicator functional of the set:

$$\{\underline{\mathbf{v}} \in V/\mathbf{v}^1 = \mathbf{v}^2\}$$

and $F_\lambda : V \to R$ given by:

$$F_\lambda(\underline{\mathbf{v}}) = \int_\Omega \left((\text{curl } \mathbf{v}^2)^2 + \frac{1}{\lambda^2}(\text{div } \mathbf{v}^1)^2 - m\pi b_\lambda(\mathbf{v}^1)^2\right) + I_0(\underline{\mathbf{v}})$$

where $m \in \mathbf{R}$.

THEOREM 4.2. $(F_{\varepsilon\lambda})_\varepsilon$ *is epi-convergent to* F_λ*, where* $m = 0$ *if* $a = o(\varepsilon^3)$ *and* $m = 1$ *if* $a = O(\varepsilon^3)$.

Proof. For proving the first assertion (1.3) from the definition of epi-convergence, let $\underline{\mathbf{v}} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with $I_0(\underline{\mathbf{v}}) = 0$ (otherwise the inequality is obvious). We define:

$$\underline{\mathbf{v}}^\varepsilon = (\mathbf{v}^{1\varepsilon}, \mathbf{v}^{2\varepsilon}), \quad \mathbf{v}^{1\varepsilon} = v_i\mathbf{w}^i_\varepsilon, \quad \mathbf{v}^{2\varepsilon} = \mathbf{v}^{1\varepsilon}\chi_1$$

so that $I_\varepsilon(\underline{\mathbf{v}}^\varepsilon) = 0$. We split $F_{\varepsilon\lambda}$ into three terms:

$$F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon) = I_1 + I_2 + I_3$$

where

$$I_1 = \int_\Omega (v_i \text{curl } \mathbf{w}_\varepsilon^i \chi_1)^2 + \frac{1}{\lambda^2}(v_i \text{div } \mathbf{w}_\varepsilon^i)^2 + a^{-2}k(v_i \mathbf{w}_\varepsilon^i)^2$$

$$I_2 = 2\int_\Omega -v_i \text{curl } (\mathbf{w}_\varepsilon^i \chi_1)\cdot(\mathbf{w}_\varepsilon^i \chi_1 \times \text{grad } v_j) + \frac{1}{\lambda^2} v_i \text{div } \mathbf{w}_\varepsilon^i \mathbf{w}_\varepsilon^j \cdot \text{grad } v_j.$$

$$I_3 = \int_\Omega (\mathbf{w}_\varepsilon^i \chi_1 \times \text{grad } v_i)^2 + \frac{1}{\lambda^2}(\mathbf{w}_\varepsilon^i \cdot \text{grad } v_i)^2$$

From Lemma 3.2 we get:

$$I_2 \to 0, \quad I_3 \to \int_\Omega (\text{curl } \mathbf{v})^2 + \frac{1}{\lambda^2}(\text{div } \mathbf{v})^2.$$

If $a = o(\varepsilon^3)$ we get $I_1 \to 0$. If $a = O(\varepsilon^3)$ by integrating by parts and using (3.1) we get:

$$I_1 \to -\pi b_\lambda \int_\Omega (\mathbf{v})^2$$

and thus we have proved that

$$(4.2) \qquad F_\lambda(\underline{\mathbf{v}}) = \lim_\varepsilon F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon), \qquad \text{for all } \underline{\mathbf{v}} \in \mathcal{D}(\Omega)\times\mathcal{D}(\Omega).$$

But $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega) \subset V$ dense (see Duvaut and Lions [7]), thus for $\underline{\mathbf{v}} \in V$ we get $(\underline{\mathbf{v}}_k) \subset \mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ so that $\underline{\mathbf{v}}_k \to \underline{\mathbf{v}}$ strongly. As before, for each k, we construct $\underline{\mathbf{v}}_k^\varepsilon \to \underline{\mathbf{v}}_k$ weakly in V_1. We have:

$$F_{\varepsilon\lambda}(\underline{\mathbf{v}}_k^\varepsilon) \to F_{\varepsilon\lambda}(\underline{\mathbf{v}}_k) \to F_\lambda(\underline{\mathbf{v}})$$

and by using a diagonalization lemma (Attouch [3, pp. 37]), there exists an increasing function $\varepsilon \to k(\varepsilon)$ so that:

$$(\underline{\mathbf{v}}_{k(\varepsilon)}^\varepsilon, F_{\varepsilon\lambda}(\underline{\mathbf{v}}_{k(\varepsilon)}^\varepsilon)) \to (\underline{\mathbf{v}}, F_\lambda(\underline{\mathbf{v}})) \text{in } L^2(\Omega)\times\mathbb{R}$$

By choosing $\underline{\mathbf{v}}^\varepsilon = \underline{\mathbf{v}}_{k(\varepsilon)}^\varepsilon$ we get:

$$\|\underline{\mathbf{v}}^\varepsilon\| \leq F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon) < C(\lambda)$$

so $\underline{\mathbf{v}}^\varepsilon \to \underline{\mathbf{v}}$ in V weakly and $F_\lambda(\underline{\mathbf{v}}) = \lim F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon)$.

For proving the second part (1.4) let $\underline{\mathbf{v}} \in V$ with $I_0(\underline{\mathbf{v}}) = \infty$ and $\underline{\mathbf{v}}^\varepsilon \to \underline{\mathbf{v}}$weakly in V. Then by lemma (4.1) $I_\varepsilon(\underline{\mathbf{v}}^\varepsilon) = \infty$ excepting eventually a finite number of terms, and thus $F_\lambda(\underline{\mathbf{v}}) = F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon) = \infty$.

Let $\underline{\mathbf{v}} \in V$ with $I_0(\underline{\mathbf{v}}) = 0$ and let $\underline{\mathbf{v}}^\varepsilon \to \underline{\mathbf{v}}$ weakly in V with $I_\varepsilon(\underline{\mathbf{v}}^\varepsilon) = 0$ (for at least a subsequence). We have $\underline{\mathbf{v}}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon\chi_1)$ and $\underline{\mathbf{v}} = (\mathbf{v}, \mathbf{v})$. Let $\mathbf{u} \in \mathcal{D}(\Omega)$, $\underline{\mathbf{u}} = (\mathbf{u}, \mathbf{u})$, $\underline{\mathbf{u}}^\varepsilon = (u_i \mathbf{w}_\varepsilon^i, u_i \mathbf{w}_\varepsilon^i \chi_1)$. We next use the inequality:

$$F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon) \geq -F_{\varepsilon\lambda}(\underline{\mathbf{u}}^\varepsilon) + 2I_4 \tag{4.3}$$

with $I_4 = \int_\Omega \left(\text{curl } \mathbf{u}^{2\varepsilon} \cdot \text{curl } \mathbf{v}^{2\varepsilon} + \frac{1}{\lambda^2} \text{div } \mathbf{u}^{1\varepsilon} \text{div } \mathbf{v}^{1\varepsilon} + a^{-2} k \mathbf{u}^{1\varepsilon} \cdot \mathbf{v}^{1\varepsilon} \chi_2\right)$ and from the first part of the proof we get:

$$\lim_\varepsilon F_{\varepsilon\lambda}(\underline{\mathbf{u}}^\varepsilon) = F_\lambda(\underline{\mathbf{u}}). \tag{4.4}$$

In the case $a = O(\varepsilon^3)$, we split I_4 into three terms: $I_4 = I_5 + I_6 + I_7$ where:

$$I_5 = \int_\Omega -\text{curl } \mathbf{v}^{2\varepsilon}(\mathbf{w}_\varepsilon^i \chi_1 \times \text{grad } u_i) + \frac{1}{\lambda^2} \text{div } \mathbf{v}^{1\varepsilon} \mathbf{w}_\varepsilon^i \cdot \text{grad } u_i$$

$$I_6 = -\int_\Omega (\mathbf{v}^{2\varepsilon} \times \text{grad } u_i) \text{curl } \mathbf{w}_\varepsilon^i \chi_1 + \frac{1}{\lambda^2} \text{grad } u_i \cdot \mathbf{v}^{1\varepsilon} \text{div } \mathbf{w}_\varepsilon^i$$

$$I_7 = \langle \sum \left(\text{curl } \mathbf{w}^i \times \mathbf{n} + \frac{1}{\lambda^2} \text{div } \mathbf{w}^i \mathbf{n}\right), u_i \mathbf{v}^{1\varepsilon} \rangle$$

By using again Lemma 3.2, we get:

$$I_5 \to \int_\Omega \text{curl } \mathbf{v} \cdot \text{curl } \mathbf{u} + \frac{1}{\lambda^2} \text{div } \mathbf{v} \text{div } \mathbf{u}$$

$$I_7 \to -\pi b_\lambda \int_\Omega \mathbf{v} \cdot \mathbf{u}$$

and $I_6 \to 0$, by passing to the limit in the first term and by splitting the second in two, on Ω_1 and on Ω_2 from lemma 4.1, lemma 3.2 and $(3.13)_2$.

In the case $a = o(\varepsilon^3)$ we get directly:

$$I_4 \to 2 \int_\Omega \left(\text{curl } \mathbf{u} \cdot \text{curl } \mathbf{v} + \frac{1}{\lambda^2} \text{div } \mathbf{u} \ \text{ div } \mathbf{v}\right)$$

By making $\mathbf{u} \to \mathbf{v}$ we get:

$$\lim F_{\varepsilon\lambda}(\underline{\mathbf{v}}^\varepsilon) \geq F_\lambda(\underline{\mathbf{v}}) \tag{4.5}$$

□

THEOREM 4.3. *$(F_\lambda)_\lambda$ is epi-convergent to F where $F : H \to \mathbb{R}$,*

$$F(\underline{\mathbf{v}}) = \int_\Omega (\text{curl } \mathbf{v}^2)^2 + mK(\mathbf{v}^1)^2 + I(\underline{\mathbf{v}}) \tag{4.6}$$

and $m = 0$ if $a = o(\varepsilon^3)$, $m = 1$ if $a = O(\varepsilon^3)$

Proof. The sequence $(F_\lambda)_\lambda$ is increasing because $1/\lambda^2$ and $-b_\lambda$ is increasing (for small λ). We have thus (Attouch [3]):

$$\lim_{\lambda} {}_e F_\lambda = \sup_\lambda (clF_\lambda)$$

But F_λ is convex and inferior semicontinuous and thus:

$$\lim_{\lambda} {}_e F_\lambda = \sup_\lambda F_\lambda = F.$$

□

We have thus proved the following convergences:

$$\begin{array}{ccc} F_{\varepsilon\lambda} & \longrightarrow & F_\lambda \\ \downarrow & & \downarrow \\ F_\varepsilon & & F \end{array}$$

The convergence $F_{\varepsilon\lambda} \to F_\varepsilon$ is similar to theorem 4.3. We are interested to prove that the scheme can be closed and we will use the result of theorem 4.2. for proving one of the inequalities:

LEMMA 4.4. $\lim_e F_\varepsilon \geq F$

Proof. The proof is obvious by passing to the limit first with ε and then with λ in the inequality $F_\varepsilon \geq F_{\varepsilon\lambda}$. □

We are now able to prove the main result of the paper:

THEOREM 4.5. *$(F_\varepsilon)_\varepsilon$ is epi-convergent to F.*

For proving this theorem we have to prove the converse inequality than in lemma 4.4. Thus we have to prove that for any $\underline{\mathbf{v}} \in H$ there exists a sequence $(\underline{\mathbf{v}}^\varepsilon)_\varepsilon, \underline{\mathbf{v}}^\varepsilon \to \underline{\mathbf{v}}$ weakly in H so that $\limsup_\varepsilon F_\varepsilon(\underline{\mathbf{v}}^\varepsilon) \leq F(\underline{\mathbf{v}})$. For this we need the local problem (LP_ε) for the divergence free space and the following two results (see (Marchenko and Hrouslov [15] and Brillard [4])

LEMMA 4.6. *Let $a > 0$ and $\mathbf{u} \in L^2(B(a))$, $\text{div } \mathbf{u} = 0$ in $B(a)$. Then there exists $\tilde{u} \in H^1(B(a))$ so that:*

$$\text{div } \tilde{\mathbf{u}} = 0, \quad \text{curl } \tilde{\mathbf{u}} = \mathbf{u} \text{ in } B(a), \quad \tilde{\mathbf{u}} \cdot \mathbf{n} = 0 \text{ on } \partial B(a)$$
$$|\tilde{\mathbf{u}}|_{B(a)} \leq Ca|\mathbf{u}|_{B(a)}, \quad |\text{grad } \tilde{\mathbf{u}}|_{B(a)} \leq C|\mathbf{u}|_{B(a)}.$$

Let $\phi \in \mathcal{D}(B(\varepsilon))$ with supp $\phi \subset B(a^{2/3}), \phi = 1$ in $B(a)$, and let us denote by ϕ_ε the periodic function obtained by extending ϕ. Let $\mathbf{v} \in \mathcal{D}(\Omega)$ and let $\mathbf{v}(x_{\varepsilon I})$ be the value of $\mathbf{v}$ in the centre $x_{\varepsilon I}$ of each period $aY_I \subset \Omega$. Then:

LEMMA 4.7. *If $\tilde{\mathbf{v}}_{\varepsilon I}$ is the function corresponding to $\mathbf{v} - \mathbf{v}(x_{\varepsilon I})$ by the previous lemma then:*

$$\sum_I \text{curl } (\tilde{\mathbf{v}}_{\varepsilon I}\phi_\varepsilon) \to 0 \text{ strongly in } H^1(\Omega)$$

Proof of Theorem 4.5. Let $\underline{\mathbf{v}} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ with $I(\underline{\mathbf{v}}) = 0$. We construct $\underline{\mathbf{v}}^\varepsilon = (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon\chi_1)$ where $\mathbf{v}^\varepsilon$ is defined in each period Y_I by:

$$\mathbf{v}^\varepsilon = \begin{cases} \mathbf{v} & \text{in } Y_I - B_I(\varepsilon) \\ \mathbf{v} + (\mathbf{t}_\varepsilon^i - \mathbf{e}_i)v_i(x_{\varepsilon I}) - \text{curl } (\tilde{\mathbf{v}}_{\varepsilon I}\phi_\varepsilon) & \text{in } B_I(\varepsilon) - B_I(a) \\ \mathbf{t}_\varepsilon^i v_i(x_{\varepsilon I}) & \text{in } B_I(a) \end{cases}$$

One can observe that in each of the three subdomains of $Y_I, \mathbf{v}^\varepsilon$ is divergence free and $[\mathbf{v}^\varepsilon \cdot \mathbf{n}] = 0$ on $B_I(\varepsilon)$ and $B_I(a)$, so that $\mathbf{v}^\varepsilon \in H(\text{div }, \Omega)$. Similarly $\mathbf{v}^\varepsilon\chi_1 \in H(\text{curl }, \Omega)$ and thus $\mathbf{v}^\varepsilon$ is well defined.

We have $\underline{\mathbf{v}}^\varepsilon$ bounded in H, so that from:

$$(\underline{\mathbf{v}}^\varepsilon - \underline{\mathbf{v}})\chi_{\cup_I(Y_I - B_I(\varepsilon))} = 0$$

and lemma 4.1. we get $\underline{\mathbf{v}}_\varepsilon \to \underline{\mathbf{v}}$ weakly in H.

We next want to pass to the limit in:

$$F_\varepsilon(\underline{\mathbf{v}}^\varepsilon) = \int_\Omega (\text{curl } \mathbf{v})^2\chi_1 + \sum_I v_i(x_{\varepsilon I}) \left(a^{-2} \int_{B(a)} k\mathbf{t}_\varepsilon^i \cdot \mathbf{t}_\varepsilon^j + \right.$$
$$\left. + \int_{B(\varepsilon)-B(a)} \text{curl } \mathbf{t}_\varepsilon^i \cdot \text{curl } \mathbf{t}_\varepsilon^j \right) + I_1$$

From lemma 4.7. and 3.4. we get $I_1 \to 0$. Because $\mathbf{v}$ is smooth

$$(2\varepsilon)^3 \sum_I v_i(x_{\varepsilon I})v_j(x_{\varepsilon I}) \to \int_\Omega v_i v_j$$

and thus from lemma 3.4. we get:

$$F_\varepsilon(\underline{\mathbf{v}}^\varepsilon) \to \int_\Omega (\text{curl } \mathbf{v})^2 + mK(\mathbf{v})^2 = F(\underline{\mathbf{v}})$$

The proof is completed by using a diagonalization argument similar to the one in theorem 4.2. □

We have thus proved that the homogenized behaviour of the problem (2.3) is given by the problem (1.7). Therefore if $a = O(\varepsilon^3)$ we get by homogenization a Brinkman type law, the supplementary term exhibiting the interaction of the two flows: the flow in the fractures and the flow in the porous blocks. If $a = o(\varepsilon^3)$ the porous blocks are two small to perturbe the flow, thus the homogenized behaviour is given by a Stokes type law. In the case $a = O(\varepsilon)$ the homogenized equations are of Darcy type, the permeability tensor depending on both the geometry and permeability of the porous blocks (see Ene and Vernescu [11]).

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