Understandings of Proportionality as a Mathematical Structure and Psychological Aspects of Proportional Reasoning in Community College Mathematics Students

A Dissertation
SUBMITTED TO THE FACULTY OF
UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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May 2015

Acknowledgements

I extend my deepest gratitude to my advisors, Dr. Thomas Post and Dr. Kathy Cramer. You challenged me, encouraged me, and supported me in all efforts surrounding my doctoral degree and my research. I am fortunate to have had such wonderful advisors.

To my committee members, thank you. Dr. Terry Wyberg, you are a model of excellence that I strive for in my own teaching, research, and course development. Dr. Sashank Varma, I am inspired by your academic excellence, and I learned many things about being an academic through your example. Dr. Julie Brown, thank you for welcoming me to the world of design based research, and helping me strengthen my efforts and bring rigor to this study.

Dr. Tamara Moore, you encouraged me to pursue my doctoral degree, and knew just how to support me through all seasons of my graduate studies. Thank you for always being there for me.

To my students – past, present, and future – you inspire me to better understand how people come to know mathematics, so that I can be a stronger teacher. It is a great privilege to be a student of my students.

To the teachers and students who engaged in this study with me, thank you.

Thank you to my colleagues. Your encouragement, kind words, and occasional hugs helped me accomplish this dissertation. Thank you to all my teachers who walked beside me in my educational journey from my most early years through graduate school. Thank you to the many graduate students whose friendships made this pursuit fun. Thank you to my friends and family who listened to me talk graduate school, and proportionality over and over again.

Thank you to my parents, Charles and Diane, my first teachers. You inspired me to love learning, love mathematics, and love teaching. You supported me in many ways as I pursued this degree. Thank you to my parents-in-law, Terry and Colleen. I am lucky to call you family. The love, support, and encouragement of both sets of parents helped me through many challenges along the way and made this work possible.

To my daughters, Catherine and Elizabeth, thank you for bringing me so much happiness and playfulness as I pursued this dream.

Stavig, thank you for literally walking this dissertation out with me. Step-by-step, we got it done.

Mike, your love and support helped me return to my grounding many times during this process. You were beside me throughout this entire journey. You experienced the ups and the downs with me. Your belief in me has meant more than I could ever put in words. I love you. Thank you.

Thank you to God, my Lord and Savior. You have called and equipped me to serve others as a mathematics teacher. You continue to open doors of opportunity for me to work with a diversity of students, and I am grateful for each and every experience.

Dedication

This thesis is dedicated to my family. To Michael, my husband and my best friend. To our daughters, Catherine and Elizabeth. You fill my heart with joy. I love you.

Abstract

Proportionality and proportional reasoning play pivotal roles in the foundation of algebra and higher-level mathematics study. Proportionality is a mathematical structure that models the relationship within contextual situations in which two quantities, *x* and *y*, change together in ways that the rate between the quantities stays the same, such as speed or density. Proportional reasoning involves the psychological underpinnings that facilitate the interpretation, sense making, and operational flexibility necessary for working with proportion related situations. The development of these understandings and reasoning processes is both mathematically and psychologically complex. Although there has been much research surrounding the ways children come to understand proportionality and reason proportionally (e.g. Lamon, 2007; Lesh, et al., 1987; Lobato et al. 2010; Post et al., 1988), there is a need for research into the ways that these concepts and reasoning processes emerge in older students and adults (e.g. Lamon, 2007; Mesa, Wladis, & Watkins, 2014; Sitomer et al., 2012).

This study explored the relationships between understandings of proportionality and proportional reasoning processes in community college mathematics students, and the teaching and learning activities that support their construction in post-secondary developmental mathematics students. The study employed design experiment methodology that included two two-week teaching experiments (Cobb et al., 2003; Cobb & Steffe, 1983/2011; Gravemeijer & van Eerde, 2009).

The findings showed that the understanding and interpretation of rate relationships are central to a connected understanding of proportionality and flexible proportional reasoning processes. This key understanding was characteristic of college-level mathematics students, and successfully constructed by developmental mathematics students through the teaching experiment. The interpretation of a $y = mx$ functional relationship in proportional contexts served to stabilize the understandings and reasoning processes of developmental students and facilitated reasoning processes similar to those of college algebra students. These results provide evidence that a non-traditional approach to the treatment of proportionality in developmental mathematics contexts can effectively build connected and meaningful understandings that will support student success in college-level mathematics courses. The two teaching experiments allowed for observation based modifications to a Hypothetical Learning Trajectory (Simon, 1995) consistent with the tenants of design study.

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Chapter 1: Introduction

Rationale

A majority of students entering two-year community colleges are underprepared for college-level math and place into developmental mathematics courses (Bailey, Jeong, & Cho, 2010). Developmental mathematics courses serve an important role in postsecondary education because they open access to higher-level math and academic programs that require mathematical literacy. However, many students who initially enroll in developmental mathematics courses struggle and subsequently fail to complete their developmental mathematical course trajectories and enroll in a college-level math course (Adelman, 2006; Bailey et al., 2010). This failure produces significant academic, professional and personal consequences with lasting effects on student economic mobility, personal goals of achievement, and perceptions of ability.

Interest in the reform of developmental mathematics within community colleges has increased as schools explore the roles and outcomes of supplemental instruction, the effects of study skills courses, and the implementation of broader student support services. Entire developmental mathematics programs have been redesigned into modular formats, curricula accelerated or slowed down, and course trajectories changed in efforts to improve student success and retention. Most of these reform efforts fail to look deeply into the actual relationships among curriculum, pedagogy, and student learning within developmental mathematics (Mesa, Wladis, & Watkins, 2014; Stigler, Givvin, & Thompson, 2010). Therefore, many of these efforts have been marginally successful, at best, and actual improvements to the teaching and learning of mathematics within developmental mathematics courses and programs have remained elusive (Sitomer et al., 2012).

The American Mathematical Association of Two-Year College's (AMATYC) Research Committee identified the relationships among curriculum, instruction, and student understandings as key focus areas in their 2012 proposed research agenda (Sitomer et al., 2012). Examples of such research includes inquiry into the mathematics learning trajectories of developmental mathematics students (referring to the actual ways students learn mathematics, not course taking trajectories), and focus on the ways mathematical misconceptions change as students learn in developmental mathematics courses (Mesa et al, 2014, p. 182). Such lines of research done in developmental mathematics contexts are not yet widely represented in the cannon of literature in mathematics education, postsecondary education, nor adult mathematics learning (Mesa et al., 2014). This research can "push the boundaries" of existing K-12 mathematics and postsecondary education research and models (Mesa et al., 2014, p. 182) and holds the promise of improving student learning experiences in ways that will better prepare students for higher-level mathematics and mathematics related coursework.

Proportional reasoning has been identified as a significant area for research and improvement in developmental mathematics teaching and learning (National Center on Education and the Economy [NCEE], 2013; Sitomer et al., 2012). Proportionality is a multiplicative structure in which two quantities change together in ways that the ratio between the two quantities stays the same. Proportion related understandings are the ways students understand and interpret the multiplicative relationships within a proportional situation. Proportional reasoning involves the psychological underpinnings that facilitate the interpretation, sense making, and operational flexibility necessary when working with proportion related situations. Together, proportion related understandings and proportional reasoning inform the approaches students take when operating with proportion related problems.

Lamon (2007) identified the following questions as rich areas of inquiry into proportion related understandings and proportional reasoning processes:

- What are the connections between proportional reasoning…and proportionality?
- What constitutes understanding of proportionality?
- What are some of the benchmarks of understanding between proportional reasoning and proportionality?
- Do algorithms preclude reasoning, or can older students develop useful knowledge about the central multiplicative structures? (p. 662).

The rationale for this study is the need for effective learning experiences in postsecondary mathematics courses that support the development of underprepared students for further mathematics, and the gap in existing research that can inform such experiences. Proportion related understanding and reasoning processes provide some of the fundamental underpinnings of algebraic thinking needed for higher-level mathematics and mathematics related coursework. Therefore, the construction of these understandings and reasoning processes in the context of developmental mathematics students are important areas of inquiry.

This study explored the relationships between proportion related understandings and proportional reasoning processes in developmental mathematics students and the teaching and learning activities that support their construction. Following an overview of relevant literature on proportionality and proportional reasoning, the research questions guiding the study are presented with a description of the method of inquiry and data collection used in the study.

Relevant Literature

Proportion related understandings and reasoning processes play important roles in the mathematical development of students. These understandings and processes develop in prealgebra mathematics and form the foundation for algebraic thinking and understanding (Lamon, 2007; Lesh, Post, & Behr, 1988; Lobato, Ellis, Charles, & Zbiek, 2010). Lesh et al. (1988) describe proportional reasoning as the capstone of arithmetic and cornerstone of higher-level mathematics (p. 94). The significance of proportion related understandings and proportional reasoning processes is identified and emphasized in the mathematics standards documents published by the American Mathematical Association of Two-Year Colleges [AMATYC] (1995), the National Council of Teachers of Mathematics [NCTM], 1989, 2000), and in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010).

Proportionality and proportional reasoning.

Vergnaud (1983) presented an important model for proportional situations in his framework for the study of multiplicative structures that was used to guide the inquiry in this study. Vergnaud's model is based on a system of contextual magnitudes, or *measure spaces*, such as people, objects, costs or distances. In this framework, a proportional relationship is defined as a multiplicative relationship between quantities in two measure spaces. A missing value problem is presented using Vergnaud's (1983) notation in Figure 1.1.

Joshua purchases 3 chocolates for 75 cents. The chocolates cost the same amount. How much do 12 chocolates cost?

Number of Chocolates	Cents
3	75
12	x

Figure 1.1. Measure space representation of a missing value problem.

Two multiplicative relationships always exist between and within measure spaces defining a proportional relationship: a function relationship between measure spaces, and a scalar relationship within measure spaces.

Invariant constant of proportionality and unit rate.

Within any proportion related situation, the quantities between contextual measures (e.g. x = *number of chocolates* and y = *cents*) are related through the function, $y = mx$. That is, the magnitude of one quantity is a constant multiple of the other. The constant of proportionality, m, is an invariant unit rate that defines the multiplicative relationship between the values of y and x. This relationship can be used to determine the cost of 12 chocolates as follows: 12 chocolates $\times \frac{\text{25 cents}}{\text{1 chocolate}} = 300$ cents. This is an example of the *unit rate approach* to operating within a proportional situation.

The reciprocal of a unit rate can also be used to define a function relationship within a proportional situation. In this example, the alternative unit rate of $\frac{1}{25}$ chocolate per 1 cent is quantitatively, contextually, and relationally complex. It is also more complicated to use to solve the problem presented because it would require a division as opposed to a multiplication operation to determine the cost of 12 chocolates. This unit rate, though mathematically accurate, is contextually difficult to interpret and normally of lesser interest.

Covariance and factor of change multiplicative relationships.

A scalar multiplicative relationship exists within each contextual measure in a proportional situation. This multiplicative relationship can be observed in the problem presented as $3 \text{ chocolates } \times 4 = 12 \text{ chocolates}$. This scalar relationship can be extended to determine the cost of 12 chocolates through the multiplication $75 \text{ cents} \times 4 = 300 \text{ cents}$. This approach to operating in a proportional situation is called the *factor of change approach*. This approach can be used to solve for the price of any other quantity of chocolates. For example, starting with 3 chocolates for 75 cents, the number of chocolates and the number of cents can be multiplied by 2 to determine the rate pair 6 chocolates for 75 cents. The usefulness of this approach is often limited to proportional situations in which familiar, integer scalar factors of change can be applied (Cramer & Post, 1993; Karplus et al., 1983).

It is important to note that the unit rate that defines a functional relationship between measure spaces is invariant, but the scalar multiplicative relationship within measure spaces changes according to rate pairs selected.

Role of proportionality in mathematical understanding and reasoning.

Prealgebra concepts that relate to and are refined through the development of proportional reasoning include ratios and rates, fraction equivalence, long division, place value, percent, and measurement conversion (Lesh et al., 1988). These concepts involve some of the most fundamental and important understandings in early mathematics. The mathematical structures and connections among these arithmetic concepts must be deeply understood and operational to students both qualitatively and quantitatively in order to facilitate the development of proportion related understandings.

Many aspects of proportion related understandings and proportional reasoning processes are important to the learning of higher-level mathematics. These include an understanding of the algebraic representation of proportionality as a linear function, $y = mx$, and flexible approaches to working with multiple modes of representation (e.g. tables, graphs, symbols, pictures) and contexts (Post et al., 1988). Additionally, proportion related understandings and proportional reasoning processes formalize, abstract and generalize ideas of multiplicative structures, equivalence, and function.

Curricular Approaches to Proportionality.

Traditionally, proportional situations have been represented in American mathematics curricula through the use of proportions in which two rate pairs are related through equivalence, A/B = C/D (Lamon, 2007; Lobato et al., 2010; Post et al., 1988). Research has demonstrated that students who can solve problems involving proportions do not necessarily reason proportionally (e.g. Lesh et al., 1988; Lamon, 2007; Lobato et al., 2010). Traditional curricular treatment of proportionality through the definitions of proportions as equivalent rate pairs, and the procedure of cross multiply and divide to solve missing values problems, as shown in Figure 1.2, is poorly understood and disconnected from the informal understandings and natural operations of students in proportion related situations (Cramer & Post, 1993; Karplus et al., 1983; Post et al., 1988; Vergnaud, 1983).

Joshua purchases 3 chocolates for 75 cents. The chocolates cost the same amount. How much do 12 chocolates cost?

$$\frac{75 \text{ cents}}{3 \text{ chocolates}} = \frac{x \text{ cents}}{12 \text{ chocolates}}$$

$$(75 \text{ cents}) \times (12 \text{ chocolates}) = (x \text{ cents}) \times (3 \text{ chocolates})$$

$$\frac{(75 \text{ cents}) \times (12 \text{ chocolates})}{(3 \text{ chocolates})} = x \text{ cents}$$

$$300 = x$$

Figure 1.2. Missing value problem solved with the standard algorithm.

Different approaches to the curricular treatment of proportionality and proportional reasoning should be built and implemented to better support the development of these understandings and processes (Cramer & Post, 1993; Karplus et al., 1983; Lobato et al., 2009; Post et al., 1988; Vergnaud, 1983). These approaches must include a variety of proportion related situations and tasks, and provide ample time and experiences for students to construct proportion related understandings and reasoning processes through intuitive strategies before more procedural approaches are introduced (Cramer & Post, 1993; Karplus et al., 1983; Vergnaud, 1983). This study explored new approaches to the construction of proportion related understanding and proportional reasoning processes in the context of post-secondary developmental mathematics classrooms.

Research Questions

The purpose of this study was to analyze the evolving understandings of proportionality and the psychological aspects of proportional reasoning that inform developmental mathematics student approach to proportion related problem solving situations, and the teaching and learning that supports their construction. The following questions and subquestions guided the research study:

(1) What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?

   a. What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?

   b. What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?

   c. What understandings of proportionality as a multiplicative structure facilitate flexible and successful approaches to problem solving situations that are proportional in nature?

   d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understandings and reasoning processes?

(2) How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?

(3) What differences, if any, exist between developmental mathematics student and college level mathematics student proportion related understandings and reasoning processes?

This study employed design experiment methodology to explore the relationships among student understandings of proportionality as a mathematical structure, and the psychological aspects of proportionality reasoning. A design experiment focuses on co-development of theory surrounding domain-specific learning processes, and aspects of teaching and learning that support the targeted processes (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 10). A non-experimental design (Sadish et al., 2002) guided the quantitative elements of the study. A non-experimental design measures a presumed cause and effect, but does not include design elements such as random assignment, control groups or pretests.

The setting of the study was a public, open-door community college in the Midwest. The study was conducted in developmental mathematics courses, replicated across two semesters, with two different groups of students. The study was conducted in a Mathematical Reasoning course in the first iteration. The Mathematical Reasoning course prepared students to enroll in a college level Liberal Arts Mathematics course. The study was conducted in an Introductory Algebra course in the second iteration. The Introductory Algebra course prepared students to either enroll in a college level Liberal Arts Mathematics Course, or a developmental Intermediate Algebra course in preparation for future enrollment in a college level College Algebra course. Two groups of college level mathematics students, Liberal Arts Mathematics and College Algebra, were selected for inquiry into the third research question.

Summary

Proportion related understanding and reasoning processes are important mathematical constructs that play pivotal roles in the mathematical development of students. The evolution of proportion related understandings and reasoning processes is both mathematically and psychologically complex. Yet, proportionality remains traditionally taught in most postsecondary developmental mathematics curricula through the definition of proportions as equivalent rate pairs, $A/B = C/D$, and the procedure of cross-multiply and divide. Both mathematically and procedurally, the traditional approach in isolation is disconnected from the more robust multiplicative structure of proportionality, and intuitive and meaningful approaches that students use to reason proportionally that make use of a more connected definition of proportionality (Cramer & Post, 1993; Karplus et al., 1983; Post et al., 1988; Vergnaud, 1983). This study served to increase what is known about the connections between proportion related understandings and reasoning processes by describing the ways they evolve in postsecondary developmental mathematics students.

Overview of the Following Chapters

Chapter 2 discusses existing research in the areas of proportionality and proportional reasoning, and presents a theoretical model of a connected understanding of the mathematical characteristics of proportionality. The psychological aspects involved in proportional reasoning are presented in a theoretical model that connects to the multiplicative structures that define proportionality. Following the review of literature, Chapter 3 discusses the research methodology, and describes the research site, participants, data instruments, and the processes used to analyze the data. Chapter 4 provides the analysis of the quantitative and qualitative data in the study. The results of the study and implications of the results, limitations to the study, and possibilities for further research are then discussed in Chapter 5.

Chapter 2 Review of the Literature

The purpose of this study was to analyze the evolving understandings of proportionality and the psychological aspects of proportional reasoning that inform developmental mathematics student approach to proportion related problem solving situations. The developmental mathematics context for this research is not yet widely represented in the cannons of literature in mathematics education, postsecondary education, nor adult mathematics learning (Mesa et al., 2014). Therefore this research both builds upon and “pushes the boundaries” of existing K-12 mathematics and postsecondary education research and models (Mesa et al., 2014, p. 182).

The literature review that follows discusses research surrounding proportionality and proportional reasoning, and how students come to understand and operate with the multiplicative relationships in proportion related problem-solving tasks. Much of the existing research in this area was conducted in contexts with children and pre- and in-service teachers. These studies serve as the theoretical basis on which the models of developmental student understandings and reasoning processes are built and refined.

The review of literature is divided into four parts.

- First, Vergnaud’s (1983) measure space conceptualization of proportional situations is presented, and the multiplicative relationships within proportional situations are discussed. This conceptualization serves as the basis of this research.
- Second, proportion related problems and proportional reasoning processes are discussed.

- Third, theoretical models of understandings of proportionality as a multiplicative structure, and connections among and between these understandings and the psychological aspects of proportional reasoning are presented. These models serve as the theoretical framework of the study.
- Fourth, curricular approaches to proportionality and proportional reasoning are discussed.

Measure Space Model of Proportionality

Vergnaud (1983) presented an important model of proportionality in his framework for the study of multiplicative structures. The model is based on a system of contextual magnitudes, or *measure spaces*, such as people, objects, costs or distances. In this framework, a proportional relationship is defined as a multiplicative relationship between quantities in two measure spaces. Two multiplicative relationships always exist between and within measure spaces defining a proportional relationship: a function relationship between measure spaces and a scalar relationship within measure spaces. Vergnaud's (1983) notation is presented in Figure 2. In this model, M_1 and M_2 represent measure spaces, and A, B, C, and D are the quantities that create rate pairs in a proportion, $A/B = C/D$.

The measure space model of proportionality serves as the mathematical basis of the study. Throughout the study, the terms rate and rate pair will refer to a ratio of quantities between measure spaces.

M_1	M_2
A	B
C	D

M_1 = [Contextual Magnitude 1], M_2 = [Contextual Magnitude 2]

Figure 2.1. Measure space representation of a proportional situation.

Vergnaud (1983) presents a general case of missing value problems called a *rule-of-three* isomorphism of measure problem. An example of a rule-of-three problem is presented in Figure 2.2. The function relationship between measure spaces and a scalar relationship within measure spaces can be utilized to solve the problem and will be illustrated below.

Joshua purchases 3 chocolates for 75 cents. The chocolates cost the same amount. How much do 12 chocolates cost?

Number of Chocolates	Cents
3	75
12	x

Figure 2.2. Rule-of-three problem.

Between measure space functional relationships.

A function relationship that relates the quantities between measure spaces in the cost of chocolates problem presented in Figure 2.2 can be defined by multiplying the number of chocolates by the *unit rate* of 25 cents per 1 chocolate to obtain the number of cents for the total cost of chocolates. The unit rate is derived from the division of 75 cents by 3 chocolates. Once derived, the unit rate can be used to define a linear function relationship that characterizes the proportional situation, $y = 25x$. This relationship can be used to determine the cost of 12 chocolates as follows:

$12 \text{ chocolates } \times \dfrac{25 \text{ cents}}{1 \text{ chocolate}} = 300 \text{ cents}$. This type of reasoning is an example of the *unit rate approach* to proportionality and will be further discussed in the review of literature on proportional reasoning.

The reciprocal of a unit rate can also be used to define a functional relationship within a proportional situation. In this example, the alternative unit rate of $\frac{1}{25}$ chocolate per 1 cent, although mathematical correct, is quantitatively, contextually, and relationally more complex for interpretation than the unit rate of 25 cents per 1 chocolate. The unit rate of $\frac{1}{25}$ chocolate per 1 cent is also more complicated to use to solve the problem presented in Figure 2.2 because it would require a division as opposed to a multiplication operation to determine the cost of 12 chocolates. The choice of which unit rate to use, and how to interpret and manipulate a rate when solving proportion related problems are central to proportional reasoning processes, but are challenging tasks for students (Lamon, 2007; Post et al., 1988).

Within measure space scalar multiplicative relationships.

In the cost of chocolates example presented in Figure 2.2, a scalar numerical multiplicative relationship can be observed as $3 \text{ chocolates } \times 4 = 12 \text{ chocolates}$. This scalar relationship can be extended to the "cents" measure space to determine the cost of 12 chocolates through the multiplication $75 \text{ cents} \times 4 = 300 \text{ cents}$. This type of reasoning can be used to solve for the price of any other quantity of chocolates. For example, starting with 3 chocolates for 75 cents, the number of chocolates and the number of cents can be multiplied by 2 to determine the rate pair of 6 chocolates for 150 cents. This approach to reasoning in a proportional situation is called the *factor of change approach*, and will also be discussed in the literature review on proportional reasoning.

As opposed to the invariant unit rate multiplicative relationship between measure spaces, the scalar multiplicative relationship within measure spaces is not defined by a constant factor. Instead, the factor relating any two quantities within a measure space changes according to the pair of quantities being related. Thus, the within measure space scalar multiplicative relationship provides an alternative quantitative representation of how two related quantities covary, or change together, in ways that preserve equivalence among the rate pairs in a proportional situation.

Proportional Reasoning

Proportional reasoning involves the psychological underpinnings that facilitate the interpretation, sense making, and operational flexibility necessary for working with proportion related situations. The key to proportional reasoning is the ability to discern and operate with a multiplicative relationship between a pair of quantities and extend the same relationship to other pairs of quantities (Lamon, 2007). It involves the interpretation of rates and their reciprocals, the perception and understanding of covariant relationships, and the ability to simultaneously mentally store and operate with several pieces of information (Lesh et al., 1988). Proportional reasoners are able to distinguish between proportional and non-proportional situations, reason qualitatively about proportional situations, and overcome the effects of unfamiliar settings and numerically complex relationships (Cramer & Post, 1993). Additionally, strong proportional reasoners have access to multiple strategies and are flexible in their thought and approach to reasoning within proportional situations (Cramer & Post, 1993).

Proportion related problems.

Three types of proportion related problems have been used to assess student thinking in proportional situations: missing value problems, comparison problems and qualitative reasoning problems. Missing value problems, as presented in the cost of chocolates problem in Figure 2.2, are problems in which three of four quantities are given in a proportional situation and the task is to determine the missing fourth value. Comparison problems are proportional situations that require the identification of the order relationship between two rate pairs, *A/B (<, =, or >) C/D*. Qualitative reasoning problems are problems that require inference into the direction and intensity of change (i.e. will a rate decrease, increase or stay the same when its quantitative components change?).

Unit rate approach to solving proportion related problems.

All standard missing value problems can be solved using a unit rate approach: a unit rate is determined by a simple division across measure spaces, and then the unit rate is used as a multiplier to determine the missing solution value. For example, in the cost of chocolates problem presented in Figure 2.2, it was given that 3 chocolates cost 75 cents. A unit rate was determined by dividing: 75 cents ÷ 3 chocolates = 25 cents per 1 chocolate. Then, the unit rate was used as a multiplier to determine the cost of 12 chocolates by multiplying 12 chocolates by 25 cents per 1 chocolate. This procedure can be applied to any number of chocolates.

All comparison problems in which two rate pairs are compared can also be solved using unit rates (Post et al., 1988). The unit rate facilitates the comparison of *intensive quantities*, that is, quantities whose units are not typically directly counted nor measured (Schwartz, 1983), by establishing the "per unit" amounts for each rate, *A/B* and *C/D,* in answering the question of "how many for one?" For example, consider a comparison problem in which the fuel consumption of two vehicles is compared. Vehicle A travels 180 miles and consumes 12 gallons of gasoline. Vehicle B travels 144 miles and consumes 8 gallons of gasoline. Computing the unit rate of number of miles per 1 gallon for each vehicle allows a comparison of the fuel efficiency of the vehicles. Vehicle A travels 15 miles per 1 gallon of gasoline, and vehicle B travels 18 miles per 1 gallon of gasoline. Therefore, vehicle B is the more efficient vehicle. In this comparison, the miles per gallon is an intensive quantity that is not directly counted or measured.

The unit rate approach is the most intuitive approach used in proportional reasoning situations (Cramer & Post, 1993; Karplus et al., 1983; Heller, Post, & Behr, 1985; Post et al., 1988). The functionality and multiplicative structure embedded in the unit rate approach to proportional reasoning, as well as it psychological advantages, makes it an important area of focus in the development of proportional reasoning (Cramer, Post, & Currier, 1993; Heller et al., 1985; Karplus et al, 1983; Post et al., 1988). This approach is strongly connected to understandings of the meaning and role of the constant of proportionality, which are foundational to the understanding of the general multiplicative structure of proportionality (Cramer et al., 1993; Lamon, 2007).

Factor of change approach to solving proportion relates problems.

The factor of change approach, or a "times as many" approach for solving missing value problems is another a useful and intuitive approach used in proportional reasoning. This method utilizes the scalar multiplicative relationships within measure spaces. The method applies the reasoning that if one quantity is "*a*" times another within a given measure space, the missing quantity must also be "*a*" times as many as its corresponding quantity in the other measure space. For example, in the cost of chocolates problem presented in Figure 2.2, a factor of change was determined within the chocolates measure space as 3 chocolates times 4 is 12 chocolates. The "times 4" factor of change was then applied to the cents measure space to determine the cost of 12 chocolates by computing 75 cents times 4 is 300 cents.

Factor of change reasoning can also be conceptualized across rate pairs represented in a proportion. To maintain equality, if one component is "*a*" times its corresponding component in another rate pair, the missing component must also be "*a*" times as many as its corresponding component in the rate pair. For example, representing the cost of chocolates problem with a proportion, the factor of change reasoning across rate pairs can be identified as follows: $\frac{75\text{ cents}}{3\text{ chocolates}} \underset{\times 4}{\overset{\times ?}{=}} \frac{x\text{ cents}}{12\text{ chocolates}}$. Therefore,

$$\frac{75\text{ cents}}{3\text{ chocolates}} \underset{\times 4}{\overset{\times 4}{=}} \frac{300\text{ cents}}{12\text{ chocolates}}.$$

Reasoning with a factor of change scalar multiplicative relationship is connected to understandings of equivalence. This approach requires the identification and manipulation of covariant relationships in which two measures change together by the same scale factor, while the across measure space rate remains invariant. When this type of reasoning is employed without reference to measure space labels, the approach is a "fraction approach" (Cramer & Post, 1993) meaning the rates are treated as fractions and multiplicative rules of equivalent fractions are applied. The fraction approach is thought to be more complex and algorithmic in comparison to the factor of change approach because it lacks a real world grounding for the student.

Building up strategies and rate tables approaches.

Factor of change reasoning often emerges through pattern recognition in an "building up" strategy that utilizes partitioning and iteration, and often involves additive thinking. Using this approach, a student might reason that 3 chocolates are 75 cents, so 6 chocolates are 150 cents, 9 chocolates are 225 cents and 12 chocolates are 300 cents. This type of reasoning is shown in the rate table presented in Figure 5.

Joshua purchases 3 chocolates for 75 cents. The chocolates cost the same amount. How much do 12 chocolates cost?

Number of Chocolates	Cents
3	75
6	150
9	225
12	300

Figure 2.3. Rate table illustrating the building up strategy approach.

The building up approach to solving a proportion related problem is not a strong indicator of proportional reasoning (Lesh et al., 1988). The approach is not necessarily multiplicative, and the invariance of across measure space rate (i.e. equivalence of rates) is not necessarily considered. Some students employ additive reasoning in iterative appraoches in which they think of adding together numbers of chocolates (e.g. 3 + 3 + 3 + 3 = 12) and adding together number of cents (e.g. 75 + 75 + 75 + 75 = 300) to build to a given quantitative pair (e.g. 12 chocolates are 300 cents). Other students use multiplicative reasoning in which they think of "doubling," "tripling," or "quadrupaling" their quantities. The building up strategy, in both its additive and multiplicative approaches, becomes complicated when integer scalar multiples are not readily available. The usefulness of the building up approach is usually limited to situations in which "convenient numbers," which are familiar integral multiples of one another, are presented in the problem.

Rate tables, similar to the table presented in Figure 2.3, provide important representations of proportion related situations (and examples of non-proportional situations). Students intuitively approach proportion related problems by reasoning with rates within rate tables when the quantitative elements of a problem are convenient (e.g. integer scalar multiples are involved). Rate tables can open access to the development of understandings of the multiplicative structure of proportionality.

Building up approaches are naturally used by students and may be necessary to individual development of proportion related understandings, reasoning processes, and problem solving approaches (Lobato et al., 2010). When carefully approached in guided learning experiences, these strategies can be connected to multiplicative relationships within measure spaces that provide an anchor for proportional reasoning when the quantitative aspects of a situation are numerically complex.

Qualitative reasoning.

Qualitative reasoning tasks elicit proportional reasoning using qualitative rather than quantitative information. Lamon (2007) presents the following example of a qualitative reasoning task. “Yesterday you shared some cookies with some friends. Today, you share fewer cookies with more friends. Will everyone get more, less or the same amount as they received yesterday?” (p. 631). In this example, qualitative reasoning would determine that each person would receive fewer cookies than her or she received the previous day. Qualitative reasoning is an important ability that can be accessed by successful proportional reasoners to assess the reasonableness of solutions in proportionally structured problems.

Qualitative reasoning is the product of a reasoner’s level of experience and understanding with real world contexts and rational number concepts (Lamon, 2007; Heller, Post & Behr, 1985). It depends on a deep understanding of the relationship between the numerator and denominator in a fraction. Table 1 presents the possible directional changes in the value of a rate when the numerator and denominators of the rate change. There are two indeterminate cases when the directional effects of componential changes to a rate depend on the particular quantitative changes that are made. These cases arise when the numerator and denominator increase or decrease together.

Table 2.1.

Change in the Value of a Rate Based on Changes in its Numerator and Denominator

	Numerator		
Denominator	Stays Same	Increase	Decrease
Stays Same	No Change	Increase	Decrease
Increase	Decrease	Indeterminate	Decrease
Decrease	Increase	Increase	Indeterminate

The development of proportional reasoning.

The development of proportional reasoning often occurs over an extended period of time (Lamon, 2007; Lobato et al., 2010). Lobato et al. (2010) identified transitions that students regularly make as they develop in their approaches to proportion related problem solving tasks. The first shift is the organization of a proportional situation in ways that account for two measures simultaneously, instead of focusing on each measure individually. The second transition is a shift in which students begin to move away from additive comparisons to multiplicative comparisons across measures. The third transition is a shift from composed – unit strategies that involve partitioning and iteration in building up processes, to multiplicative approaches involving comparisons between and within measures. The fourth transition is the shift towards extending the multiplicative comparisons to many elements of the equivalence class that characterizes a proportional situation.

The Abstract Meaning of Proportionality

Proportionality is a mathematical structure that models the relationship within contextual situations in which two quantities, x and y, change together in ways that the rate between the quantities stays the same, such as speed or density. In proportional situations, the ratio of one quantity to the other, y/x, remains invariant as the components, y and x, change by the same factor (Lobato et al., 2010). All corresponding rate pairs, (x,y), in a proportional situation create an equivalence class. The linear function relationship $y = mx$ governs the covariation between the two quantities in a proportional situation. That is, the magnitude of one quantity is a constant multiple of the other. Graphically, the (x,y) rate pairs lie on the line $y = mx$, which passes through the origin.

The constant of proportionality, m, is an invariant rate that defines the multiplicative relationship between the values of y and x. Contextually it is an intensive quantity, which is a quantity not typically counted nor measured directly (Schwartz, 1988). Its composed unit is a quotient that relates the units of the measures y and x. Graphically, it is the slope of the line for the graph of $y = mx$. Quantitatively, it defines an equivalence class corresponding to all rate pairs in a particular proportional situation. That is, $m = y/x$ for all (x,y) rate pairs in the $y = mx$ proportional relationship. The value of m is a unit rate that is the result of a simple division, $m = y \div x$. Here, the unit rate is interpreted as the number of y's per one x.

The reciprocal of the constant of proportionality, $1/m$, is also a constant of proportionality, x/y, that can be used to express a proportional relationship between x and y. Here, the value of $x \div y$ is interpreted as the number of x's per one y. Thus, two invariant rates (y/x or x/y) define any particular proportional situation, each rate with its separate contextual interpretation.

A Model of Understanding Proportionality as a Multiplicative Structure

Mathematical understanding is constructed when students make connections among and within ideas, concepts and procedures. Hiebert and Carpenter (1992) define mathematical understanding as connections within a structured internal network. The connections and relationships that form networks of understanding are formed by comparing similarities and differences within and across representations of mathematical ideas and the subsumption and inclusion of ideas into organized schema. As representations are rearranged, old connections are modified while new connections are made. These notions of understanding are anchored in the cognitive constructivist theories of Piaget (1970) that conceptualize learning as a process of organization and reorganization of schema in response to experience.

Proportionality is a robust mathematical structure and the development of understandings of proportionality involves the connection of five central multiplicative constructs that characterize the structure. These constructs and the connected understandings that are investigated in this study are shown in Figure 2.3. This model of proportionality served as the mathematical context that guided student experiences and the measurement of student understanding of the multiplicative structure of proportionality in this study. A connected understanding of proportionality is an understanding of the individual multiplicative constructs of proportionality and the mathematical relationships among and within the constructs. This level of understanding has the potential to support flexible and meaningful proportional reasoning processes.

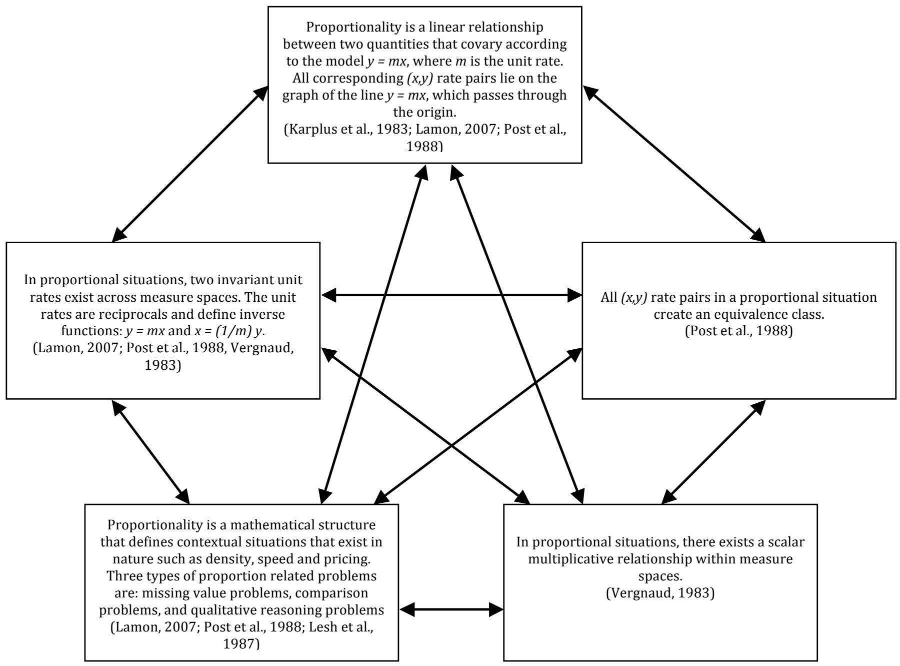


Figure 2.3 A theoretical model of connected understandings of proportionality as a multiplicative structure.

Connecting Proportionality and Proportional Reasoning

This study examines some of the connections between the mathematical constructs that define an understanding of the multiplicative structure of proportionality as presented in Figure 2.3, and the psychological processes underlying proportional reasoning. The intervention aspect of this design experiment was anchored in existing research and theory that supports the idea that proportional reasoning requires connected understandings of the multiplicative relationships that define proportionality as a mathematical structure. These understandings include the $y = mx$ function relationship, covariance and invariance, equivalence, and contextual situations that are proportional in nature. Conversely, a connected understanding of the mathematical structure of proportionality requires flexible proportional reasoning abilities including the differentiation between proportional and non-proportional situations, the manipulation of covariant relationships, multiplicative thinking, the interpretation of rates and their reciprocals, and the ability to make multiple comparisons and simultaneously store and process multiple pieces of information (Lesh et al., 1988; Post et al., 1988).

Table 2.2 presents both the multiplicative structures that define the theoretical model of a connected understanding of proportionality as presented in Figure 2.4, and core psychological constructs involved in proportional reasoning. The grouping of multiplicative constructs of proportionality and psychological constructs of proportional reasoning indicated in Table 2.2 illustrate key relationships between proportion related understandings and proportional reasoning processes. It is suggested here that the psychological aspects of proportional reasoning and understandings of the five multiplicative constructs of proportionality are all interrelated. That is, the characteristics that define proportional reasoning are connected to one another, as are the elements of the multiplicative structure of proportionality. Further, connections exist between each of the constructs that define the multiplicative structure of proportionality and each of the characteristics that define proportional reasoning.

Table 2.2.

The Multiplicative Structure of Proportionality and the Psychological Aspects of Proportional Reasoning

Multiplicative Constructs that Define the Mathematical Structure of Proportionality	Psychological Aspects of Proportional Reasoning
Proportionality is a linear relationship between two quantities that covary according to the model $y = mx$, where m is the unit rate. All corresponding (x,y) rate pairs lie on the graph of the line $y = mx$, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).
	Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).
In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: $y = mx$ and $x = (1/m)\ y$ (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).
All (x,y) rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).
In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)
Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).
	Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).
	Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).

Three aspects of proportional reasoning are targeted in this research:

- Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).

- The invariant relationship between two variables, *x* and *y*, can be extended to other equal multiples of *x* and *y* using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).
- Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).

The relationships between these aspects of proportional reasoning and specific understandings of proportionality as a multiplicative structure were explored in this study. The model of connected understandings presented in Figure 2.4 and the hypothesized connections between these understandings and the psychological aspects of proportional reasoning as presented in Table 2.2 served as the theoretical framework that guided the intervention and the inquiry in this design experiment.

Curricular Treatment of Proportionality

Traditional curricular approach.

Traditionally, proportionality has been represented in American curriculum through the use of proportions in which two rate pairs are related through equivalence, A/B = C/D. Students are then taught to solve missing value problems through the standard algorithm of cross multiply and divide (Lamon, 2007; Post et al., 1988). This approach towards the development of understandings of proportionality and proportional reasoning is poorly understood and disconnected from student informal understandings and natural operations in proportional situations (Cramer & Post, 1993; Karplus et al., 1983; Post et al., 1988). Moreover, research demonstrates that students who can solve problems involving proportions using the standard algorithm do not necessarily reason proportionally (Lesh et al., 1988; Lamon, 2007). The traditional treatment of proportionality does not connect several core ideas that are central to the mathematical structure of proportionality including the function $y = mx$ and its multiple representations, connected ideas of covariance and invariance, and the interpretation and relationship between two unit rates that define a proportional situation. Traditional treatment of proportionality is therefore thought to be wholly inadequate.

New curricular approaches.

Different approaches to the curricular treatment of proportionality and proportional reasoning should be built and enacted to better support the development of these important understandings and processes (Cramer & Post, 1993; Karplus, 1983; Post et al., 1988). New curricular treatment should encourage multiple approaches to operating with proportional reasoning tasks including the unit rate and factor of change approaches. Utilizing multiple approaches to proportional reasoning will help students anchor their reasoning in natural understandings of proportional relationships and connect their processes to the mathematical structure of proportionality. Additionally, a variety of proportional situations and tasks, including missing value problems, numerical comparison tasks, and qualitative reasoning tasks should be included in a proportionality curriculum.

In early experiences with proportional situations, students often ignore parts of the data or apply incorrect reasoning and operations to proportion related problems. For example, research has shown that when proportional reasoning is quantitatively difficult (i.e. existence of non-integer multiplicative relationships), additive reasoning often replaces proportional reasoning (e.g. Karplus et al. 1983; Lesh et al., 1987; Cramer et al., 1993). Consider the following quantitatively difficult missing value problem presented as a proportion without units: $2/5 = x/9$. Incorrect additive reasoning would determine that $x = 6$ because $5 + 4 = 9$, so $2 + 4 = 6$, or $2 + 3 = 5$, so $6 + 3 = 9$. An effective curriculum would incorporate opportunities for students to differentiate between proportional and non-proportional situations in order to build an understanding of when multiplicative reasoning should be used as opposed to in the enactment of an effective curriculum to nudge students toward multiplicative reasoning that can be generalized from additive thinking (e.g. building up strategies) in proportion related problem solving.

Research shows that numerical complexity and problem type effect student performance on proportional reasoning tasks (e.g. Harel, Behr, Post & Lesh, 1991; Heller et al., 1985; Karplus et al., 1983; Lamon, 2007; Noelting, 1980a, 1980b; Tourniaire & Pulos, 1985). The development of multiplicative thinking that can be generalized into connected understandings of the structure of proportionality is best developed in situations presented with familiar numbers (i.e. integer factors). Therefore, particular attention should be brought to the sequencing of tasks to begin with more familiar contexts and numbers and to progress to less familiar context and challenging numbers through a carefully sequenced variety of problem tasks in a proportionality curriculum (Cramer & Post, 1993).

After a connected understanding of proportionality is developed, the procedural approach of the standard algorithm can be introduced and connected to the more meaningful proportional reasoning approaches (Cramer & Post, 1993). Connection to the unit rate and a scalar factor of change can be made through the restructurings shown in Figure 2.4. These connections can be challenging for students due to their rigorous structural nature. Therefore, care and attention must be brought to instruction centering on these connections as well as to when the standard algorithm should be introduced.

$$\frac{75 \text{ cents}}{3 \text{ chocolates}} = \frac{x \text{ cents}}{12 \text{ chocolates}}$$

$$(75 \text{ cents}) \times (12 \text{ chocolates}) = (x \text{ cents}) \times (3 \text{ chocolates})$$

Unit Rate

$$\frac{(75 \text{ cents})}{(3 \text{ chocolates})} \times (12 \text{ chocolates}) = x \text{ cents}$$

Unit Rate ↓

$$\frac{(25 \text{ cents})}{(1 \text{ chocolate})} \times (12 \text{ chocolates}) = x \text{ cents}$$

Scalar Factor of Change

$$(75 \text{ cents}) \times \frac{(12 \text{ chocolates})}{(3 \text{ chocolates})} = x \text{ cents}$$

Factor of Change ↓

$$(75 \text{ cents}) \times 4 = x \text{ cents}$$

$$300 = x$$

Figure 2.4. Connecting the standard algorithm to unit rate and scalar factor of change.

Psychological considerations of representations and translations.

In order to become effective and efficient proportional reasoners, students must eventually form a generalized conception of proportionality that is inclusive of a variety of numerical, contextual and representational modes of proportional structures. Multiplicative reasoning in proportional situations gradually emerges through pattern recognition of proportionality in multiple contexts and modes of representation (e.g. tables, graphs, real world contexts). When proportional reasoning successfully emerges, the multiplicative structure that governs the relationship between measure spaces in a proportional situation is recognized and quantitative and qualitative operational fluency develops (Lesh et al., 1988).

Real world instances of proportional situations often involve translation between and within modes of representations. In the process of solving proportional reasoning tasks, multiple modes of representation may be utilized to interpret, reason about and manipulate a proportional situation. For example, students may paraphrase, draw diagrams, or write equations to assist in their reasoning (Lesh et al., 1988).

The *Lesh Translation Model* was developed to model and interpret external representations and connections between and within modes of representation depicteg forms of mathematical concepts (Lesh, Post & Behr, 1987). The model was based on the theories of multiple representations and embodiments presented by Bruner (1966) and Dienes (1960). In the Lesh Translation Model student understanding of a concept rests in the ability to represent a concept through multiple modes of representation and the ability to translate between and within the modes of representation (Lesh & Doerr, 2003; Lesh et al., 1987). This study utilized the Lesh Translation Model presented in Figure 2.6 to guide the learning experiences of students and the measurement of student understanding of proportionality and proportional reasoning processes. Particular focus was brought to real world situations (contexts), spoken words, written symbols (numbers, tables, and equations), and pictures (graphs).

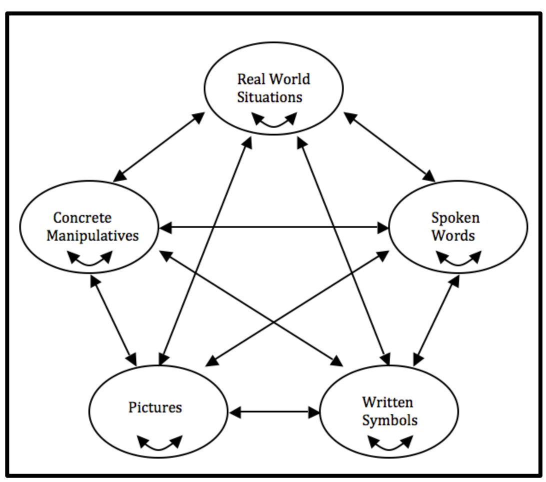


Figure 2.6. Lesh Translation Model

Standards-based learning environments.

The National Council of Teachers of Mathematics (NCTM, 2000) present recommendations for standards-based learning environments in which students meaningfully learn mathematics by building new knowledge anchored in experience and prior knowledge. These effective learning environments honor the importance of multiple strategies and ideas, communication, problem solving, representation, connections, reasoning and sense making. These ideals are also found in the standards for teaching and learning presented in the *Common Core Standards* (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010) and by the American Mathematical Association of Two-Year Colleges (AMATYC, 1995, 2006). Standards-based learning environments open access to mathematics for all students, and provide opportunities for students to learn in collaboration with others.

Standards-based learning environments are built on the tenets of social constructivism. In standards-based learning environments, the development of mathematical understandings is considered a dynamic process existing in social contexts in which students learn through engaging in mathematical thinking and discourse with others. These ideas are anchored in classical theory including cognitive constructivist theories of Piaget (1970) and the theories of Vygotsky (1978) on the socially situated aspects to the construction of knowledge.

Standards-based learning experiences center on both the processes and the products of learning mathematics (Bruner, 1971) and emphasize conceptual understanding, multiple representations, and connections. Learning mathematics with understanding is generative in that it provides a strong basis for future mathematical experiences. It promotes remembering through the focus on connections among concepts and enhances transfer among mathematical tasks, concepts and contexts (Hiebert & Carpenter, 1992).

The intervention aspect of this study was enacted using standards-based practices in which students often engaged in carefully sequenced problem-solving tasks in small groups, and students shared their ideas and strategies in ways that built on the work of other students. Class discussions were structured using the five practices of productive mathematics discussions (Stein & Smith, 2011): anticipating student strategies and understandings, close monitoring of student work in class, purposeful selection and sequencing of student generated strategies and ideas for whole class discussion, and careful attention to connecting student strategies and ideas in ways that helped students understand the mathematical structure of proportionality.

Summary

Proportion related understandings and proportional reasoning are central components in the mathematical foundation for the study of algebra and higher-level mathematics (Lesh et al., 1988; Lamon, 2007; Lobato et al., 2010; NCEE, 2013; NCTM, 1989, 2000). The National Council of Teachers of Mathematics (1989) emphasized the importance of proportional reasoning in their *Curriculum and Evaluation Standards for School Mathematics* by describing it to be "of such great importance that it merits whatever time and effort must be expended to assure its careful development" (p. 82). The analysis presented in this literature review shows that the development of a connected understanding of proportionality and proportional reasoning processes are both mathematically and psychologically complex.

In most traditional American school mathematics curricula, the treatment of proportionality and processes of proportional reasoning are often reduced to rote teaching and learning of the standard algorithm of cross multiply and divide for solving missing value problems presented in a manner similar to $A/B = X/D$, solve for X (Post et al., 1988). Limitations to this curricular and pedagogical approach include the treatment of proportional reasoning as a routine procedure disconnected from many of the mathematical constructs of proportionality, its lack of meaning in real world contexts, and its limited basis for transferable understanding to important algebraic concepts such as linear functions, equivalence class, and graphical interpretations (Lamon, 2007; Post et al., 1988). Therefore, it is understandable why many secondary students and adults fail to reason proportionally (Lamon, 2007; Post et al., 1988). This failure should be considered a significant contributor to the number of students entering two-year community colleges that are not prepared for college level mathematics and must enroll in pre-college level developmental mathematics courses.

Unfortunately, the curricula and teaching methods often found within developmental mathematics programs at the postsecondary level are dominated by the teaching of disconnected procedures to be rotely learned through drill (Stigler et al., 2010). Developmental mathematics students who struggle with proportional reasoning are often presented the exact same material, in the same way, as they were in their previous experiences, and are expected to build meaning. Given this nature of developmental mathematics courses, the high failure and low retention rates that characterize developmental mathematics programs (e.g. Adelman, 2006; Bailey et al., 2010) are not surprising. In order to improve the learning experiences and outcomes of students in developmental mathematics programs, curricular and pedagogical transformations must be made.

The rationale for this study is the need for effective learning environments and experiences in post-secondary mathematics courses that support the development of underprepared students for college-level math. Given the central role that proportion related understandings and reasoning play in the foundation of higher-level mathematics, it is imperative that efforts be made to improve the way students who struggle with understanding proportionality engage in proportional reasoning experiences so that they have access to algebra and opportunities to be successful in mathematics and higher education.

Changes to the curricular treatment of proportionality and proportional reasoning must be made in order to meet this challenge. These changes include an emphasis on the development of intuitive proportional reasoning strategies prior to procedural strategies, and the presentation of a variety of proportional reasoning experiences to students through carefully sequenced contexts and representations. These changes will support the development of student understandings of the mathematical structure of proportionality and flexible approaches to proportional reasoning.

This study increases what is known about how proportion related understandings and proportional reasoning evolve in developmental mathematics students. Further, the study analyzes and refines a Hypothetical Learning Trajectory (Simon, 1995) built around the mathematical structure of proportionality and research based understandings of the development of proportional reasoning processes. The study utilizes the theoretical framework of understanding connections within and between the multiplicative constructs that define the structure of proportionality as presented in Figure 2.3, and the definition of psychological aspects of proportional reasoning as presented in Table 2.2 to guide the design of the intervention component of the study and the theoretical inquiry.

In the next chapter, the method of inquiry for this study will be discussed, including the Hypothetical Learning Trajectory, the details of the intervention, quantitative and qualitative inquiry methods of analysis, and the theoretical underpinnings of each of these components.

Chapter 3 Research Methodology

This study employed design experiment methodology to explore the relationships among student understandings of proportionality as a mathematical structure, and the psychological aspects of proportionality reasoning in developmental mathematics students. A design experiment focuses on the co-development of theory surrounding domain-specific learning processes, and aspects of teaching and learning that support the targeted processes (Cobb, et al., 2003, p. 10). The following questions and subquestions guided the research study:

(1) What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?

  a. What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?

  b. What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?

  c. What understandings of proportionality as a multiplicative structure facilitate flexible and successful approaches to problem solving situations that are proportional in nature?

  d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understandings and reasoning processes?

(2) How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?

(3) What differences, if any, exist between developmental mathematics student and college level mathematics student proportion related understandings and reasoning processes?

Table 3.1 describes five core multiplicative structures that define proportionality and eight psychological aspects of proportional reasoning. The codes listed in the table were used in the development of the study and in the analysis of the data.

Table 3.1

The Mathematical Structure of Proportionality and the Psychological Aspects of Proportional Reasoning, and Codes Used to Represent the Mathematical Constructs and Psychological Aspects

Code	Multiplicative Constructs that Define the Mathematical Structure of Proportionality	Psychological Aspects of Proportional Reasoning	Code
1	Proportionality is a linear relationship between two quantities that covary according to the model $y = mx$, where m is the unit rate. All corresponding (x,y) rate pairs lie on the graph of the line $y = mx$, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	A
		Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	B
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: $y = mx$ and $x = (1/m)\ y$ (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	C
3	All (x,y) rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	D
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	E
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	F
		Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	G
		Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	H

This chapter describes the research methods employed in the study. First, the theoretical underpinnings of design experiment methodology are discussed and applied to the present study. Second, issues of methodological rigor are addressed. Third, the intervention component of the study is described, including the Hypothetical Learning

Trajectory (HLT) (Simon, 1995) that guided the intervention. Fourth, the research setting and participants are described. Fifth, a description of the data collection instruments, procedures, and analysis is made.

Design Experiment Methodology

Design experiments are conducted to develop theories about domain-specific learning processes and the teaching and learning activities and practices that support them. The methodology is interventionist, taking place in carefully constructed and well-defined contexts. New forms of learning are engineered and redesigned through iterative cycles of development, testing, and revision in design experiments, and the theories developed are interpreted in the context of the designs themselves. That is, the context, instruments, and activities involved in the teaching and learning activities are central to the theories developed in design experiments (Cobb et al., 2003).

The purpose of this research was to increase what is known about developmental mathematics student understandings of proportionality as a mathematical structure, and the ways specific understandings support the psychological aspects of proportional reasoning. A Hypothetical Learning Trajectory (HLT) describing teaching and learning activities that may support the emergence of connected understandings of proportionality and proportional reasoning processes was created and refined through the study.

This design experiment took the form and structure of a teaching experiment (Cobb et al., 2003; Cobb & Steffe, 1983/2011; Gravemeijer & van Eerde, 2009) in which models of student understandings of proportionality were developed, tested, and revised in actual classroom settings in a series of teaching episodes. The teaching experiment structure enabled the exploration of student understandings and reasoning processes within authentic classroom contexts, and, importantly, a close examination of the actual models of proportionality used by students as they grew in their understandings.

The research was conducted in developmental mathematics classroom settings with the researcher serving as the teacher during a two-week instructional intervention targeting the development of proportion related understandings. The classroom teachers served as observers of the teaching episodes and collaborators in cycles of analysis of the HLT. Student interviews were conducted with students to gain further insight into the development of student understandings. Written assessment data was analyzed to further explore differences in student understandings.

The HLT that guided the teaching experiment presented a non-traditional curricular and pedagogical approach to the construction of proportion related understandings and reasoning processes. The HLT was built around the model of proportionality and the psychological aspects of proportional reasoning presented in the literature review and Table 3.1. The ways student understandings and reasoning processes evolved were examined through student interviews conducted at selected points in the intervention, and cycles of analysis of the HLT following each class meeting and retrospective analysis following each intervention (Gravemeijer, 1999). The analysis of the HLT provided greater understanding of both the ways students come to understand proportionality and reason proportionally, as well as the aspects of the learning ecology (Cobb & Steffe, 1983/2011; Steffe & Thompson, 2000) that support their development.

A non-experimental design (Shadish, Cook, & Campbell, 2002) guided the quantitative elements of the research study. A non-experimental design measures a presumed cause and effect, but does not include design elements such as random assignment, control groups, or pretests. The purpose of the study was the development of models of student understanding, and hypotheses generation about the ways students come to understand proportionality and reasoning proportionally. Therefore, the non-experimental design was appropriate for the purpose and position of the study.

Establishing Methodological Rigor

Cobb and Gravemeijer (2008) organize specific issues of rigor around to three phases of design research specific to mathematical design experiments with groups of students in a classroom: preparing for the experiment, experimenting to support learning, and conducting retrospective analyses of the data (p. 68). The issues of methodological rigor present in each phase of the experiment are addressed in the following subsections, organized by phase of the design experiment.

Preparing for the design experiment.

When preparing for a design experiment, the researcher must establish instructional goals in existing research and theory and detail how the goals are situated in the context or in contrast to traditional curriculum, instruction, and approaches to learning in the specific content domain (Cobb & Gravemeijer, 2008). Prior to the development of the specific research questions and goals that guided this experiment, the researcher conducted an in-depth literature review, and pilot interviews with developmental mathematics students. These activities established the theoretical framework of a connected understanding of proportionality and the psychological aspects of proportional reasoning that guided the study. A review of traditional curricular treatment of proportionality in both existing research and examples of developmental mathematics curricula was conducted to position the research questions and intervention as a new approach to the teaching and learning of proportionality and proportional reasoning.

Next, the instructional starting points for an intervention must be identified both in consideration of student prior learning and existing research (Cobb & Gravemeijer, 2008). The instructional starting point for the first iteration of the intervention was a lesson targeted at the development of understandings and interpretation of the reciprocal unit rates that can define a function relationship between two measure spaces in a proportional situation. The unit rate was selected as the starting point of the intervention due to its importance to the structure of proportionality (Vergnaud, 1983), and its role in the development of student understandings and reasoning processes (Karplus et al, 1983; Cramer et al, 1993; Lamon, 2007; Post et al., 1988). It was assumed that students would have prior understandings of the ratio and rate subconstruct of rational numbers (Kieren, 1976) that would serve as the knowledge anchor for the lesson. However, as will be discussed in the analysis of the HLT in Chapter 4, this assumption was incorrect, and the HLT was subsequently adjusted to address these understandings in the first intervention, and a new starting point was established for the HLT that guided the second intervention.

The learning processes, including the norms of the academic environment in which the intervention is conducted, must be clearly articulated and connected through the HLT that guides the intervention (Cobb & Gravemeijer, 2008). The development of the HLT was guided by the theoretical framework of connected understandings of proportionality (as shown in Figure 2.3), and existing research and theory in the development of meaningful approaches to proportional reasoning processes (e.g. Karplus et al, 1983; Cramer et al, 1993; Lamon, 2007; Post et al., 1988). The development took place over a period of several months, in which the researcher engaged in regular and disciplined collaborative conversations with other mathematics educators with particular expertise in the areas of proportionality and proportional reasoning. Table 3.2 outlines the theoretical grounding of the HLT that guided the first iteration of the study.

The intervention aspect of this study was enacted using standards-based practices, as defined in Chapter 2. Students engaged in carefully sequenced problem-solving tasks in small groups, and shared their ideas and strategies in ways that built on the work of other students. The learning environment was carefully created so as to support the development of flexible approaches to proportional reasoning with the intent of connecting the mathematical structure of proportionality through proportional reasoning experiences. Therefore, multiple reasoning approaches were encouraged, and students connected their approaches to the mathematical structure of proportionality through discourse with other students.

Conducting the design experiment.

To attain the dual goals of developing theory and practice, the researcher must connect all processes of data collection and methods of analysis to the theoretical framework used to interpret these understandings and processes. In particular, the iterative cycles of design and analysis that characterize of design experiment methodology must be described (Cobb & Gravemeijer, 2008).

This study consisted of two iterations of a design experiment in classroom-based contexts. The first iteration was conducted in fall semester 2014, in two developmental Mathematical Reasoning classrooms. The second iteration was conducted in spring semester 2015 in one developmental Introductory Algebra classroom. During both implementations, the researcher and the classroom teacher engaged in pre- and post-lesson debriefing sessions in which fieldnotes, researcher-teacher journaling, and student work were reviewed. These sessions were collaborative, and instructional decisions were made that adapted and adjusted the learning activities to fit the immediate learning needs and contexts of the real classroom and students. The role of the classroom-teacher was of particular importance in these mini-cycles of analysis in order to obtain a perspective other than that of the researcher-teacher of what learning goals were achieved or not achieved in a particular enacted lesson. The mini-cycles of analysis that occurred in these meetings and key outcomes of the analysis are presented in Chapter 4.

Between the first and second iteration of the design experiment, a rigorous analysis of written assessment and interview data that measured student understandings and reasoning processes was conducted and implications of this analysis further refined the HLT that guided the second iteration of the design experiment. The integration of this analysis is described in Chapter 4.

Retrospective analysis.

Following the cycles of analysis that occur during a design experiment, retrospective analysis is used to position the theoretical and practical outputs of the experiment as a particular case in a broader context framed by existing theory (Cobb & Gravemeijer, 2008). Several methodological issues arise surrounding this phase of analysis: issues of argumentative grammar, trustworthiness, repeatability, and generalizability (Cobb & Gravemeijer, 2008, p. 83). Each of these issues is addressed below.

Argumentative grammar is the logic and reasoning that characterizes a particular research method (Kelly, 2004). It is the responsibility of a design researcher to clearly articulate the characteristics that determine the targeted learning processes and ways they develop within a design experiment and why these characteristics are central to the development of the domain-specific learning processes studied. It must be made clear how particular characteristics are purposefully supported in the intervention. These responsibilities present two challenges to the designer and researcher in a classroom based design experiment: the documentation of major shifts that occurred in student reasoning in the intervention, and the demonstration of how particular aspects and norms of the classroom environment influenced the evolution of student learning (Cobb & Gravemeijer, 2008).

The mathematical constructs that define a connected understanding of proportionality and the psychological aspects that characterize proportional reasoning were defined in the theoretical framework developed in Chapter 2. This framework grounded the HLT that guided the intervention of the design experiment, and the construction of the data measurement tools, and the analysis of the data. In particular, student interview data was used to demonstrate the evolution of student understanding and reasoning during the intervention. The analysis of this data, presented in Chapter 4, demonstrates when particular connected understanding emerged within students, and the shifts in proportional reasoning that occurred alongside the understandings. This analysis was then incorporated into the analysis of the HLT and consistencies and inconsistencies within student work samples and student interview data were identified and interpreted through the learning tasks of individual lessons.

Trustworthiness is a measure of qualitative validity of a study that describes the extent to which the outcomes of the study are reasonable, justifiable, and accurate (Creswell, 2003; Lincoln & Guba, 1985; Cobb & Gravemeijer, 2008). In this design study, trustworthiness was addressed in two ways. First, multiple types and methods of data were collected and analyzed including written assessment data, interview data, student work samples, a researcher-teacher journal, and field notes. Triangulation among data sources allowed for patterns and inferences to emerge and connected through multiple measures. Second, data was systematically analyzed in cycles throughout both iterations of the study: starting with analysis meetings between the researcher and the classroom teachers during the interventions, and followed by cycles of coding and analysis of interview data and quantitative analysis of written assessment data. The analysis was documented throughout all cycles, and critiqued by mathematics educators with expertise in the area of proportionality and proportional reasoning.

Cobb and Gravemeijer (2008) state that it is important for design researchers to aim to develop instructional designs that can support learning in other settings. Therefore, it is the responsibility of the researcher to carefully delineate the aspects of the learning processes that have the potential to be extended to other settings (p. 88 – 89). In this study, the third HLT that is presented in Chapter 4 is an aspect of the study that can be used in other settings. The third HLT can be used as a starting point for other researchers and educators to use as they approach they develop new methods and tools that support the teaching and learning of proportionality and proportional reasoning.

Generalizability refers to the ways the outputs of a design experiment can inform teaching and research efforts in other settings. To achieve generalizability, design research must be tightly connected to domain-specific, instructional theories and replicated in various settings (Cobb & Gravemeijer, 2008). All aspects of this design study were developed and analyzed through the theoretical framework of connected understandings of proportionality and the psychological aspects of proportional reasoning. Replication across two semesters, in two different developmental mathematics classes supported the generalizability of the study.

Intervention

Research has demonstrated that traditional curricular treatment of proportionality is poorly understood and disconnected from student's informal understandings and natural operations in proportional situations (Cramer & Post, 1993; Karplus et al., 1983; Post et al., 1988). The teaching and learning experiences in this design experiment provided students with opportunities to operate with a variety of proportion related situations using intuitive strategies before more procedural approaches were introduced (Cramer & Post, 1993). The researcher served as the teacher during the interventions. The classroom teachers served as observers and interacted with students as they completed problem-solving tasks.

Hypothetical Learning Trajectory.

There are three components to a Hypothetical Learning Trajectory (HLT): learning goals, learning activities, and the predicted learning processes (Simon, 1995, p. 136). Hypothetical Learning Trajectories are theoretically based in the mathematical structures of the content to be learned, and in the pedagogical approaches that support student learning. The first HLT is presented in Figure 3.1. The theoretical grounding of the first HLT is outlined in Table 3.2.

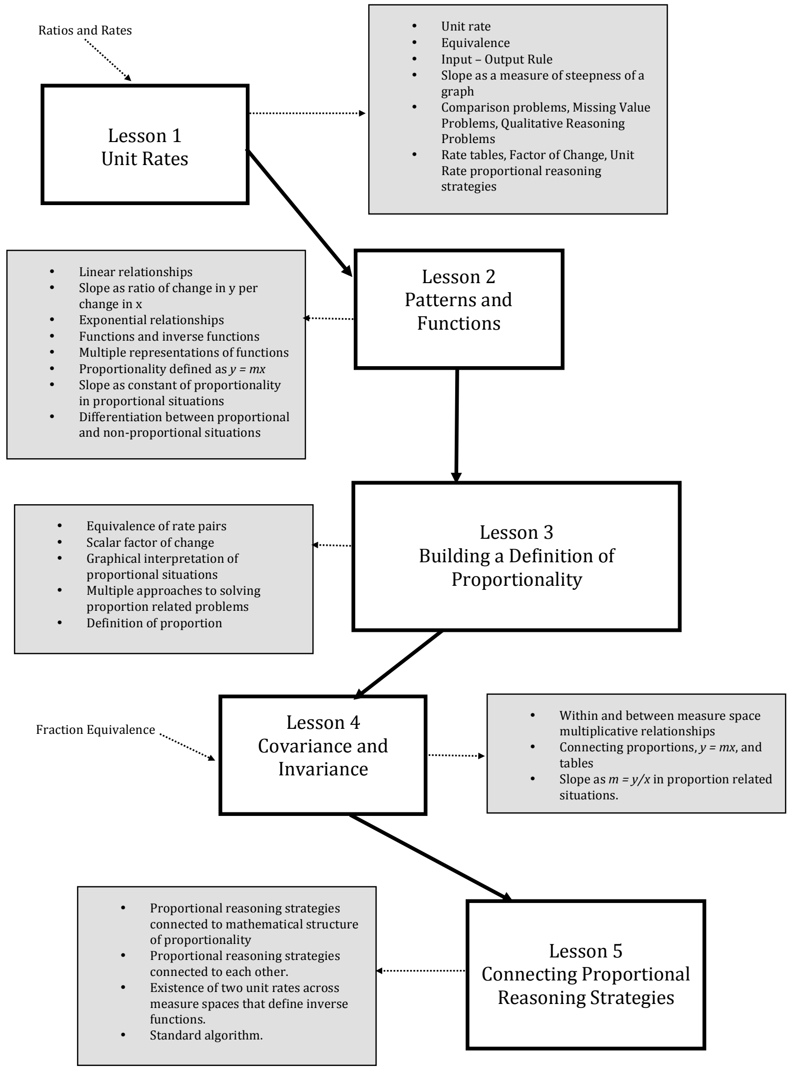


Figure 3.1 First Hypothetical Learning Trajectory.

Table 3.2

Theoretical Grounding of the First Hypothetical Learning Trajectory around Multiplicative Constructs Outlined in Table 3.1

Lesson	Multiplicative Constructs Targeted	Problem Solving Approaches Addressed	Predicted Learning Outcomes	Theoretical Grounding*
Unit Rate	2	- Unit Rate - Rate tables - Factor of Change	- Unit rate - Equivalence - Measure space interpretation of proportional situations - Input – Output Rule - Slope as a measure of steepness of a graph - Comparison problems - Missing Value Problems - Qualitative Reasoning Problems	- Unit rate is central to the interpretation of rate and proportional reasoning (Cramer et al, 1993; Lamon, 2007; Post et al., 1988; Vergnaud, 1983) [C]. - The interpretation of unit rate is challenging (Lamon, 2007). - Students should experience working with a variety of proportion related problem types when building understandings of proportionality (Cramer & Post, 1993) [F & G].
Patterns and Functions	1	- Unit Rate - Rate tables	- Linear relationships - Multiple representations of functions - Proportionality defined as $y = mx$ - Slope as constant of proportionality in proportional situations - Differentiation between proportional and nonproportional situations - Slope as ratio of change in y per change in x - Exponential relationships - Functions and inverse functions	- Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983) [B]. - Proportional reasoning requires the differentiation between proportional and non-proportional situations (Cramer et al., 1993; Post et al., 1988) [A].

Table 3.2 continued.

Lesson	Multiplicative Structures Targeted	Problem Solving Approaches Addressed	Predicted Learning Outcomes	Theoretical Grounding*
Building a Definition of Proportionality	1, 5	- Unit Rate - Rate tables - Factor of Change	- Equivalence of rate pairs - Scalar factor of change - Graphical interpretation of proportional situations - Multiple approaches to solving proportion related problems - Definition of proportion	- The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009) [D].
Covariance and Invariance	3, 4	- Unit Rate - Rate tables - Factor of Change	- Within and between measure space multiplicative relationships - Connecting proportions, $y = mx$, and tables - Slope as $m = y/x$ in proportion related situations.	- The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988) [E & H].
Connecting Proportional Reasoning Strategies	1, 2, 3, 4, 5	- Unit Rate - Rate tables - Factor of Change - Standard Algorithm	- Proportional reasoning strategies connected to mathematical structure of proportionality - Proportional reasoning strategies connected to each other. - Existence of two unit rates across measure spaces that define inverse functions.	- Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988) [G]. - Research has demonstrated that focus on intuitive strategies applied to multiple representations and contexts of proportional situations can facilitate the development of proportional reasoning (Cramer et al., 1993; Karplus et al., 1983; Lesh et al. 1988) [F & G]. - The standard algorithm can be introduced and connected to more meaningful proportional reasoning approaches (Cramer & Post, 1993) [D & E & F & G].

* Psychological aspects of proportional, as outlined in Table 3.1 reasoning are identified by code.

The HLT was revised through cycles of analysis on the conjectured learning activities, and the actual learning experiences and outcomes that occurred in the intervention (Gravemeijer, 1999). Prior to each class meeting, the researcher and classroom teacher met to discuss the activities, anticipated learning processes, and key ideas that were to be connected in the lesson. Following each class meeting, the researcher again met with the classroom teacher to discuss classroom observations, student work, and student reasoning processes demonstrated in class. During these meetings, targeted understandings and processes that were connected, or not connected were identified. The HLT and subsequent lessons were revised accordingly. These revisions are articulated in Chapter 4.

Curriculum.

The curriculum used in the first iteration of the design experiment was developed around the HLT presented in Figure 3.1 and Table 3.2. The curriculum consisted of five lessons, each designed for two fifty-minute class periods. The lessons provided students experience with different types of proportional reasoning tasks including missing value problems, comparison problems and qualitative reasoning problems.

Proportion related problem solving strategies emphasized in the curriculum began with intuitive approaches including the use of rate tables, the unit rate approach, and the factor of change approach. Students were encouraged to solve problems multiple ways, and were instructed to not use the standard algorithm of cross multiply and divide until the final lesson. Each lesson targeted specific multiplicative constructs that compose the mathematical structure of proportionality. Rate tables, equations, graphs, and a variety of contextual situations were used. A sample lesson is presented in Appendix A.

Many of the tasks that were included in the curriculum came from existing curricula and literature, and were used with permission of their authors and publishing companies when applicable. The five resources from which the intervention drew problem contexts and learning tasks were *Grade 7 Connected Mathematics Project 3 (CMP3)* (Lappan, G, Phillips, B., D., Fey, J. T., Friel, S. N., 2014), *Functions and Proportionality* (Cramer, 2014), NCTM's *Classroom Activities for Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook,* (Bright G. W. & Litwiller B. (Eds.), 2002), the personal collection of Kathleen Cramer (2014), and the personal collection of Tom Post (Post, 2014). Additionally, to maintain consistency with the class structure outside of the intervention, an online homework assignment was given for each lesson consisting of problems from the adopted text for the Mathematical Reasoning course, *Elementary and Intermediate Algebra, 5th Edition* (Tussy, A.S., & Gustafson, R.D., 2012).

Research Setting

The setting for the study was a public, suburban community college in the Midwest. The college is an open-door institution in which the entrance requirements include a high school diploma, GED, or equivalent. Student headcount at the college on the 10th day of the fall 2014 semester was 7,665 students, with a full-time equivalent count of 4,963. The three-year completion rate, as measured by graduation or transfer by the end of the third spring after entry into the college was 45.7% for students first enrolled in fall of 2010. In the fiscal year 2013, 18.9% of students as a percent of total credit headcount were students of color.

Course descriptions.

The intervention was replicated across two semesters, in two different developmental mathematics courses, and each with two different groups of students. Between the fall 2014 and spring 2015 semesters, the developmental course trajectories and curriculum changed at the college. The two developmental courses involved in the study and their position in the mathematics course trajectories are shown in Figure 3.2.

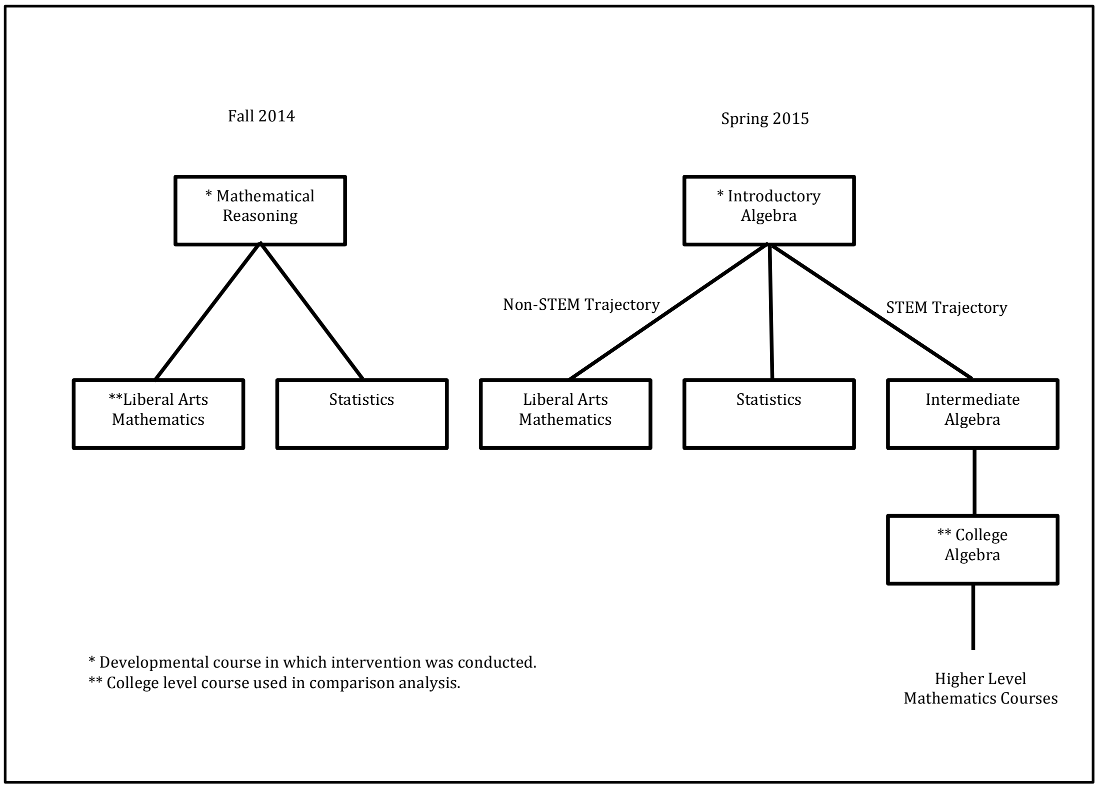


Figure 3.2. Mathematics Course Trajectories

The fall intervention was conducted in a developmental Mathematical Reasoning course that prepared students to enroll in a college level Liberal Arts Mathematics course. Students enrolled in the Mathematical Reasoning course were neither intending to enroll in a College Algebra course, nor pursue a STEM intensive field. The spring intervention was conducted in a developmental Introductory Algebra course that prepared students for two different course trajectories: a college level Liberal Arts Mathematics course for non-STEM fields of study, and a trajectory through Intermediate Algebra (a second developmental course) followed by College Algebra for STEM intensive fields of study. Both developmental courses prepared students to enroll in a college level Statistics course. The learning objectives for the Mathematical Reasoning, Introductory Algebra, College Algebra, and the Liberal Arts Mathematics courses are provided in Appendix B. Table 3.3 describes the core content in the Mathematical Reasoning course and the Introductory Algebra course, and highlights content differences between the two courses. Table 3.4 describes the core content in the Liberal Arts Mathematics Course.

Table 3.3

Developmental Mathematics Course Content. Content Specifically Addressed in Teaching Experiments Represented in Bold.

<table>
<tr><th></th><th>Mathematical Reasoning Course</th><th>Introductory Algebra Course</th></tr>
<tr><td>Common Content</td><td colspan="2">Proportions
First-degree equations
Linear relationships
Graphical approaches to problem solving
Exponents
Operations with polynomials
Pythagorean Theorem
Mathematical modeling
Sequences</td></tr>
<tr><td>Different Content</td><td>Functions and inverse functions
Exponential functions
Logarithmic functions</td><td>Similarity
Linear inequalities
Factoring
Math study skills</td></tr>
</table>

Table 3.4

College Level Math Course Content. Content Specifically Addressed in Teaching Experiments Represented in Bold.

	Liberal Arts Mathematics	College Algebra
Common Content	**Variation** **Mathematical modeling** Linear programming Sequences Counting Exponential growth and decay	
Different Content	Logic Sets Patterns and symmetry Probability Statistics Personal finance Voting and apportionment methods Graph theory	**Functions and Function Inverses** **Multiple representations of Functions** Polynomial and Rational Functions Algebra of Functions Systems of linear equations

Mathematics placement.

The college utilized the ACCUPLACER college placement exam (http://accuplacer.collegeboard.org/) to determine student mathematics placement. The elementary algebra component of the ACCUPLACER test measures student ability to perform algebraic operations and solve problems. Targeted content of the elementary algebra component includes: operations with rational numbers, operations with algebraic expressions, and solutions of equations, inequalities and word problems. An elementary algebra subscore of 0-75 determined student placement into developmental mathematics. The arithmetic component of the ACCUPLACER test further delineated developmental student placement into the targeted developmental courses. In the three months prior to the start of the fall 2014 semester, a total of 1824 students at the college took the ACCUPLACER exam. Of the students who tested during this time period, 19.6% placed at the level of the Mathematical Reasoning and Introductory Algebra courses.

An Analysis of Co-Variance (ANCOVA) was considered for the statistical analysis of quantitative data with student elementary algebra subscores of the ACCUPLACER exam as the selected covariate. The intent of the inclusion of the covariate measure was to statistically adjust student test scores to order to control for some of the variability in arithmetic and algebraic operational proficiency between students.

After the covariate was obtained, it was dismissed for two reasons. First, many students did not have current ACCUPLACER scores on file with the college because their placement scores were either more than two years old, or they did not take the assessment. Missing ACCUPLACER scores eliminated over 25% of the students from the samples. Second, ACCUPLACER scores were determined to not be a good measure of prior student arithmetic and algebraic operational proficiency. Students may have entered their mathematics course trajectory at different courses than their current course, thus developing mathematical proficiencies that were not available at the time of their placement assessment.

Participants.

The participants in the study were students enrolled in Mathematical Reasoning, Introductory Algebra, College Algebra, and Liberal Arts Mathematics courses at the college. All students in the selected classrooms were invited to participate in the study.

Students who chose not to participate in the study took the same assessments, completed the same course work, but their work was not included in the analysis.

In order to use a student's data for the study, the student had to give the researcher written informed consent. The study used data from 81 students (30 Mathematical Reasoning students, 17 Introductory Algebra students, 18 Liberal Arts Mathematics students, 26 College Algebra Students).

Purposeful sampling was used to select groups of students from each course for task interviews to provide "information-rich cases" (Patton, 2002, p. 230) for inquiry into their proportional reasoning processes. Six students were sampled from the two sections of Mathematical Reasoning in fall 2014 and six students were sampled from one section of Introductory Algebra in spring 2015 based on their performance on the 15 item written pretest. The sample included students with low (6 or lower), medium (between 7 and 10), and high (11 or higher) scores on the written assessment. Four students were sampled from the Liberal Arts Mathematics and College Algebra groups, again based on their assessment scores (low and high).

Data Sources and Methods

Quantitative data included student scores from a written assessment covering proportionality. This data was used to generate descriptive and inferential statistics that illustrate developmental student understandings of proportionality before and after the intervention. This data was also used to assess whether or not there exist evidence of statistically significant differences between developmental and college level mathematics student performance on the written proportionality assessment.

Qualitative methods were used to further explore student understandings and reasoning processes, and support the analysis and revision of the HLT that guided the intervention. Student interviews were used to provide insight into the development of student understandings of proportionality, and aspects of proportional reasoning. Student work samples, observational field notes, and a researcher journal guided the analysis of the HLT.

Table 3.5 connects each research question or subquestion with the data sources that were used for its analysis.

Table 3.5

Data Sources Used for Each Research Question

Research Question	Data Sources
(1) What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?	
a. What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?	• Proportionality assessment • Task interview data • Student work samples • Field notes • Researcher journal
b. What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?	• Proportionality assessment • Task interview data • Student work samples • Field notes • Researcher journal
c. What understandings of proportionality facilitate a flexible approach to problem solving situations that are proportional in nature?	• Proportionality assessment • Task interview data • Student work samples • Field notes • Researcher journal
d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understanding and reasoning processes?	• Proportionality assessment • Task interview data • Student work samples • Field notes • Researcher journal
(2) How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?	• Proportionality assessment • Task interview data • Student work samples • Field notes • Researcher journal
(3) What differences, if any, exist between developmental mathematics student and college level student proportion related understandings and reasoning processes?	• Proportionality assessment • Task interview data

Quantitative Methods

Quantitative methods were used to address two focuses of inquiry of this research. Student performance data from a written assessment were used to determine if there was evidence to suggest (1) statistical differences in developmental student understandings before and after the intervention, and (2) statistical differences in student understanding between developmental and college level students.

Quantitative data collection instrument and procedures.

Written assessment.

The written proportionality assessment consisted of 15 multiple-choice items. Table 3.5 shows the targeted multiplicative constructs of proportionality by item. Table 3.6 shows targeted psychological aspects of proportional reasoning by item. The codes in each table are the codes outlined in Table 3.3. A detailed item analysis of the assessment items is provided in Appendix C.

Table 3.6

Multiplicative Constructs of Proportionality Targeted by Written Assessment Item

	Proportionality Structures Targeted				
Item	1	2	3	4	5
1	X	X			
2	X				
3	X				
4					X
5			X	X	
6			X	X	
7			X	X	
8					X
9				X	
10	X	X			
11			X		X
12		X			X
13	X	X			
14	X	X			
15	X	X	X		
Total	7	6	5	4	4

Table 3.7

Psychological Aspects of Proportional Reasoning Targeted by Written Assessment Item

	Psychological Aspect of Proportional Reasoning							
Item	A	B	C	D	E	F	G	H
1	X	X	X	X		X	X	X
2	X	X	X			X	X	X
3	X							
4			X		X	X	X	X
5	X			X	X	X	X	X
6	X			X	X	X	X	X
7				X	X	X	X	X
8			X		X	X	X	X
9	X				X	X	X	X
10	X	X	X			X	X	X
11			X			X	X	X
12			X			X	X	X
13		X	X	X	X	X	X	X
14		X	X	X	X		X	X
15	X		X	X	X	X	X	X
Total	8	5	10	7	9	13	14	14

Expert validation was used to assess the content validity of the assessment. Content validity is a non-statistical type of validity in which an expert confirms that an item on an assessment matches the specific content the item is intended to measure. Holistically, content validity confirms whether an entire assessment's content covers a strong sample of the domain to be measured (Anastasi & Urbina, 1997, p. 114 - 115). Each item on the written assessment was reviewed by several mathematics educators. The distribution of multiplicative constructs targeted by items was also reviewed. Items were field tested prior to the study and several items were revised.

To assess the internal consistency reliability of the items, Chronbach's alpha was calculated (Cronbach, 1951). Cronbach's alpha is a measure of inter-item correlation of items on a test designed to measure the same construct. Cronbach's alpha varies from 0 to

1 (although it can be negative). The intercorrelations among test items are maximized when all items measure the same construct (e.g. proportionality), therefore high values of Cronbach's alpha are often indicative of an internally reliable assessment.

The written assessment was designed to measure understanding of the multiplicative characteristics that compose the mathematical structure of proportionality. Cronbach's alpha was computed using the Fall 2014 student assessment data including the pretest scores of the Mathematical Reasoning and the Liberal Arts Mathematics students. A total of 48 student tests were included in the sample. Cronbach's alpha was 0.69, demonstrating acceptable internal consistency in psychological testing (Cohen, Manion, & Morrison, 2007).

The readability of the assessment was an important consideration because many developmental mathematics students are emerging readers. Readability was assessed using the Flesch-Kincaid Grade Level Test (Kincaid, Aagard, O'Hara, & Cottrell, 1981). The Flesch-Kincaid Grade Level Test is a computerized test that utilizes a metric that computes the grade level readability of text based on average sentence length (ASL) and average number of syllables per word (ASW) according to the formula $\text{Grade Level} = 0.39(ASL) + 11.8(ASW) - 15.59$. The Flesch-Kinkaid Grade Level for the assessment was 6.1. The reading level was reviewed by a developmental reading faculty member at the college, and was determined appropriate for the targeted developmental population of the study.

Written assessment administration.

The researcher administered the assessment to all students in the targeted classes in their regular classroom settings. The 50-minute assessment was administered to developmental students as a pre and posttest surrounding the intervention, and once to Liberal Arts Mathematics and College Algebra students. Students who required testing accommodations, such as extended time, took the assessment in the college's testing center. Students were allowed to use a calculator of their choosing on the assessment.

The pretest was administered in the developmental mathematics courses after students completed a unit on arithmetic of rational numbers, but prior to extensive problem solving applications and work with proportional relationships. Students were encouraged to do their best on the pretest, and were informed that demonstrating effort and detailing reasoning and approach on the exam would allow them to be included in the pool of interview participants. This provided extra motivation for student efforts because interview participants received a monetary honorarium. As a posttest, the written assessment was scored as a class test for student grades.

The assessment was given to Liberal Arts Mathematics and College Algebra students prior to their study of linear functions and variation. Students received extra credit for completing the assessment. Students were encouraged to do their best on the assessment, showing their work or describing their reasoning. Motivation for strong effort by the students was encouraged by the opportunity to be included in the pool of interview participants, and receive an honorarium.

Quantitative data analysis.

Developmental and college level mathematics student understandings.

Student performance on the written assessment was used to measure student understandings of proportionality. A non-experimental design (Shadish, Cook, & Campbell, 2002) guided this inquiry. Group mean scores from the developmental mathematics pretest, posttest and Liberal Arts Mathematics test were compared using an Analysis of Variance (ANOVA) test. Group means were contrasted in post-hoc analysis using Benjamini – Hochberg adjusted p-values (Benjamini – Hochberg, 1995; What Works Clearinghouse, 2015).

Illustrating changes in developmental mathematics student understandings before and after the intervention.

The quantitative data collection and analysis for illustrating changes in developmental mathematics student understandings of proportionality before and after the intervention follows a pre-experimental one-group pretest - posttest design (Campbell & Stanley, 1966), illustrated in Figure 3.3.

$$N \qquad O_1 \qquad X \qquad O_2$$

Figure 3.3. Pre-Experimental One-Group Pretest-Posttest Design

Many threats to validity and generalizability are present in this design (Campbell & Stanley, 1966) due to lack of randomization and control group. The intent of the present study was to provide evidence in support of hypothesis formation, rather than to test hypotheses with claims of broad generalizability. Therefore, these threats to validity are not relevant to purpose of the study.

The analysis of written assessment data was used to describe developmental student understandings of proportionality before and after each iteration of the intervention. Group mean scores for the pre and posttest were compared using a dependent t-test for paired samples to identify differences in student understandings of proportionality. Additionally, gain scores (Posttest Score – Pretest Score) for each iteration of the study were compared using an independent t-test to provide additional information about the effects of each intervention. Developmental posttest scores were compared to college level mathematics pretest scores using ANOVA analysis.

Qualitative Methods

Qualitative methods were used to strengthen the trustworthiness of the study and produce a description of the evolution of student understandings (Creswell, 2003; Lincoln & Guba,1985). Student interviews were used to detail the specific ways students understand proportionality and the psychological aspects to their reasoning processes. Student work samples, classroom observation field notes, and a journal kept by the researcher were used in the analysis of the HLT that guided the intervention. In the following sections, each data collection instrument is discussed, followed by an overview of the data analysis.

Interviews.

Task-based interviews (Goldin, 2000) were conducted with a purposeful sample (Patton, 2002) of students from the developmental courses and the Liberal Arts Mathematics courses. The interviews were problem-based interviews and examined the mathematical behavior of each student. The items on the interviews targeted the mathematical and psychological elements defined in Table 3.1. A detailed analysis of interview items, including anticipated student approach, is provided in Appendix D. Interview protocols were designed to elicit student thinking that could provide insight into student understandings and psychological reasoning processes. Interview protocols are presented in Appendix E.

Interview items were reviewed by several other mathematics educators to assess content validity. Several items were piloted prior to the study. Items were revised or replaced as needed.

Interview administration.

The researcher conducted all interviews with students on the college campus. Interviews were approximately twenty-five minutes long, and conducted with subjects individually. Each item was read aloud by the researcher, and presented in written form on a student work page. The researcher provided clarification if a student had questions about an item, and also assisted students in their reasoning processes if a student was stuck. Students were allowed to use a calculator of their choosing during the interviews.

Three interviews were conducted with each developmental mathematics student in the sample. The first interview was a baseline interview prior to classroom instruction. The second interview was conducted midway through the intervention for all developmental mathematics students. The third interview was conducted the week following the completion of the intervention.

One interview was conducted each college level mathematics student in the sample. Interviews took place the week following the written assessment, prior to student study of linear relationships and variation. The interview followed the same structure as the baseline interview used with developmental mathematics students, with the addition of one item from the second developmental mathematics interview, and one item from the third developmental mathematics interview. These additional items were included to sequence a common context of unit pricing, with minor adaptations of the items increasing numerical complexity.

Student written work constructed in interviews was used to provide further evidence of student understanding and reasoning processes when solving proportion related tasks. Written work served as supporting evidence of student thinking that were coded in the interview transcripts.

Interview samples.

The sample of subjects for the interviews was selected based on student performance on the 15-item proportionality written assessment. Six students were sampled form the developmental courses for each iteration of the study: two students with low (6 or lower) scores, two students with medium (7 to 10) scores, and two students with high (11 or higher) scores.

Table 3.8 provides a description of each of the developmental mathematics students who were interviewed. Four students were sampled from each of the Liberal Arts Mathematics classes, again based on their assessment scores. The Liberal Arts Mathematics samples consisted of two students with low (6 or lower) and two students high (11 or higher) scores. Table 3.9 provides a description of each of the Liberal Arts Mathematics students who were interviewed. All names of students have been changed.

Within each performance level, students were selected based on the detail they showed in their work, with the goal to select students who would be willing to discuss their approaches to problem solving in an interview setting. Additionally, if something unique stood out in a student's work, such as a reliance on one particular approach to proportional reasoning, the student was considered for the interview sample. For example, Kim (a medium student) was selected for interviews because she demonstrated both the unit rate and factor of change approaches to solving missing value problems, but used additive approaches to reasoning in numerically complex problems.

Table 3.8

Developmental Mathematics Interview Sample

	Name	Pretest Score	Posttest Score	Thinking on Pretest
Fall 2014				
Low	Sharia	4	11	Did not coordinate rates when approaching missing value problems, additive thinking demonstrated
	Sarah	6	7	Considered rates, was sometimes able to reason with a unit rate approach, demonstrated additive thinking, reasoning complicated by numerical complexity
Medium	Aaron	9	10	Flexibly utilized a factor of change approach, demonstrated notions of unit rate, successfully reasoned with unit rate in two problems
	Kim	10	14	Used both unit rate and factor of change approaches in reasoning, demonstrated additive thinking on similarity items that were numerically complex
High	Ashley	12	12	Reasoned accurately with unit rate and factor of change approaches, demonstrated understanding of equivalence in comparison problems
	Jeff	13	14	Reasoned accurately with unit rate and factor of change approaches, identified equal rates in proportion situations
Spring 2015				
Low	Patience	6	8	Did not consistently coordinate rates when approaching missing value problems, additive thinking demonstrated.
	Shannon	6	12	Reasoned with factor of change approaches, considered rates, demonstrated additive thinking.
Medium	Duncan	10	11	Used both unit rate and factor of change approaches in reasoning, demonstrated additive thinking on similarity items that were numerically complex
	Lizzy	10	11	Flexibly utilized a factor of change approach, demonstrated notions of unit rate, set up proportions accurately, additive thinking demonstrated.
High	Catie	13	13	Utilized standard algorithm, detailed work setting up proportions.
	Timothy	11	13	Reasoned accurately with unit rate and factor of change approaches, identified equal rates in proportion situations

Table 3.9

College Level Mathematics Interview Sample

	Name	Assessment Score	Thinking on Pretest
Fall 2014 – Liberal Arts Mathematics			
Low	Beth	5	Did not consistently coordinate rates, demonstrated additive thinking, utilized building up approaches, some instances of using a factor of change approach and a unit rate approach.
	Richard	9*	Did not consistently coordinate rates, demonstrated additive thinking, used unit rate and factor of change approaches to solve problems
High	Kristopher	14	Reasoned accurately with unit rate and factor of change approaches, identified equal rates in proportion situations.
	Tamara	15	Reasoned accurately with unit rate approaches, identified equal rates in proportion situations, utilized standard algorithm.
Spring 2015 – College Algebra			
Low	Cassi	4	Did not consistently coordinate rates, demonstrated additive thinking, utilized building up approaches, additive thinking demonstrated.
	Crystal	6	Reasoned accurately with unit rate and factor of change approaches, identified equal rates in proportion situations. Building up approach utilized.
High	Jennifer	15	Reasoned accurately with unit rate approaches, identified equal rates in proportion situations, utilized standard algorithm.
	Chad	12	Reasoned accurately with unit rate and factor of change approaches, identified equal rates in proportion situations. Interpreted a qualitative reasoning problem graphically.

* Although Richard's score was above the low bound of 6, he was considered an alternate, and subsequently interviewed, based on the detail of work shown on his assessment and the instances of additive thinking and periodic failure to coordinate rates as he approached problems.

***Interview analysis*.**

The categories detailed in Table 3.1 guided the analysis of interview data. All interviews were recorded and transcribed, and a detailed analysis of student understanding and reasoning was completed. Individual interview transcripts were first analyzed before comparisons across transcripts were made. Transcript analysis was conducted by using two coding strategies, starting with predetermined codes as presented in Table 3.1, followed by an open coding process adding codes as they become apparent from the data (Corbin & Strauss, 2008; Miles & Huberman, 1994).

Connected understandings were identified when a student used two or more multiplicative constructs of proportionality in support of each other when describing his or her thinking. Connections between psychological aspects of proportional reasoning, and understandings of the mathematical structure of proportionality were identified when there was evidence that a student was utilizing a specific mathematical construct in support of a particular aspect of reasoning. Psychological aspects [A], [D], and [G] were selected for the connection analysis because they specifically relate to research subquestions (1a), (1b), and (1c) respectively. The data were organized into tables according to connections demonstrated. This organization presented general patterns in the data and an estimate of relative importance and subtle differences (Miles & Huberman, 1994) among and between the observed understandings and reasoning processes.

The example below illustrates an incident that was assigned to the connections of pre-determined codes shown in Table 3.1 of understanding multiplicative constructs [2] and [5], and psychological aspects of proportional reasoning [A], [C], [F], [G], [H]:

Researcher: Steph and Matt are racecar drivers. They tested their cars' fuel efficiency driving at race speeds on an oval racetrack used for a long distance car race. Steph's car used 16.3 gallons of gas on a 61.8 mile drive. Matt's car used 13.2 gallons of gas on a 54.12 mile drive. Whose car had the better fuel efficiency, were they the same, or is it impossible to tell?

Aaron: This should be easy to tell because you just divide the miles by the gallons he used…so this one is 3.79….[this one is] 4.1 miles per gallon. So, Matt's car [had] better fuel efficiency.

Researcher: Great. [Are these rates] proportional?

Aaron: No, 'cause Matt's car burned less gas per mile than

[Steph's] car.

In this excerpt, Aaron overcame the numerical complexity of the problem and successfully interpreted the unit rate of miles per gallon. When asked if the situation was proportional, Aaron flexibly reasoned with the reciprocal rates by interpreting the gallons per mile rate for Matt's car was less than the gallons per mile rate for Steph's car, also connecting to his understanding of proportional as equal rate.

Limitations to the interview data.

Two limitations were present to the analysis of the interview data. The first limitation relates to the reliability of the interview analysis. The second limitation was the product of the complexity of researching in authentic contexts.

The researcher conducted all the coding and analysis of the interview data. To address this limitation, the researcher engaged in multiple rounds of coding. Additionally, the researcher's coding and analyses were regularly reviewed by several experts in mathematics education.

During the fall iteration of the study, Sharia's first interview was conducted after the first lesson of the intervention that focused on unit rate. Therefore, the interview was not a true measure of her learning prior to the intervention. Sharia's written assessment and first interview showed that she had an unstable notion of rate, and did not consistently coordinate nor reason with rates when solving proportion related problems. This made the concept of unit rate mostly uninterpretable in the first lesson. Therefore, the limitation of Sharia's first interview data present, but minimal.

Student work samples.

Student work samples, including in-class activity sheets, short formative assessments, and homework assignments, were collected during the intervention. Student work was reviewed by both the researcher and the classroom teachers and included in the analysis that refined and revised the HLT.

Field notes and researcher journal.

The classroom teachers observed each lesson during the 2-week interventions and completed field notes. The researcher kept a journal in which each lesson in the intervention was reviewed. Field notes and journal entries were reviewed during the meetings between the researcher and the classroom teachers following each day of instruction, and were used in the analysis of the HLT.

Timeline

The curriculum implementation and data collection phases of the design experiment occurred in the first half of the semester for both iterations. The schedules of each iteration of the intervention and data collection are presented in Table 3.10 and Table 3.11.

Table 3.10

Fall 2014 Intervention Implementation and Data Collection Schedule

August	25	26	27	28	29 Introduce Study to Dev. Students, Informed Consent
September	1	2 Written Pretest Given to Dev. Students	3 *Interview 1 with Dev. Students*	4 *Interview 1 with Dev. Students*	5 *Interview 1 with Dev. Students*
	8 Intervention Starts Lesson 1	9 Lesson 1	10 Lesson 2	11 Lesson 2	12 Lesson 3 *Interview 2 with Dev. Students*
	15 Lesson 3 *Interview 2 with Dev. Students*	16 Lesson 4 *Interview 2 with Dev. Students*	17 Lesson 4	18 Lesson 5	19 Intervention Ends Lesson 5
	22 Written Posttest Given to Dev. Students	23 *Interview 3 with Dev. Students*	24 *Interview 3 with Dev. Students*	25 *Interview 3 with Dev. Students*	26
October	29	30	1	2 Introduce Study to Liberal Arts Math, Informed Consent	3
	6	7 Written assessment given to Liberal Arts Students	8	9 *Interviews with Liberal Arts Math students*	10
	13 *Interviews with Liberal Arts Math students*	14 *Interviews with Liberal Arts Math students*	15	16	17

Table 3.11

Spring 2015 Intervention Implementation and Data Collection Schedule

January	12	13	14 Introduce Study to Dev. Students, Informed Consent	15	16 Introduce Study to College Algebra Students, Informed Consent
	19	20	21 Written Pretest Given to Dev. Students	22 *Interview 1 with Dev. Students*	23 *Interview 1 with Dev. Students* Written Pretest Given to College Algebra Students
	26 Intervention Starts Lessons 1 & 2 *Interview 1 with Dev. Students*	27 *Interviews with College Algebra students*	28 Lesson 3 *Interviews with College Algebra students*	29 *Interview 2 with Dev. Students* *Interviews with College Algebra students*	30 *Interview 2 with Dev. Students* *Interviews with College Algebra students*
February	2 Lesson 4 *Interview 2 with Dev. Students*	3	4 Intervention Ends Lesson 5	5 *Interview 3 with Dev. Students*	6 *Interview 3 with Dev. Students*
	9 Written Posttest Given to Dev. Students *Interview 3 with Dev. Students*	10	11	12	13

The following chapters discuss the data analysis and results of the study. Chapter 4 describes the analysis of the quantitative and qualitative data, the revision and refinement of the HLT, and the theoretical implications of the data analysis. Chapter 5 will discuss the results of the research study and the implications of the results for the teaching and learning of proportionality in developmental mathematics contexts.

Chapter 4 Data Analysis and Results

This study employed design experiment methodology to explore the relationships among understandings of proportionality as a mathematical structure, and the psychological aspects of proportionality reasoning in developmental mathematics students. The following questions and subquestions guided the research study:

(1) What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?

   a. What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?

   b. What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?

   c. What understandings of proportionality as a multiplicative structure facilitate flexible and successful approaches to problem solving situations that are proportional in nature?

   d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understandings and reasoning processes?

(2) How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?

(3) What differences, if any, exist between developmental mathematics student and college level mathematics student proportion related understandings and reasoning processes?

Multiple forms of data were collected and analyzed to address the research questions. Quantitative data included student scores from a 15 item written assessment covering proportionality. This data was used to generate descriptive and inferential statistics that illustrate developmental student understandings of proportionality before and after the intervention aspect of the study. This data was also used to assess whether or not there exist evidence of statistically significant differences between developmental and college level mathematics student performance on the written proportionality assessment. Qualitative methods were used to further explore student understandings and reasoning processes, and support the analysis and revision of the Hypothetical Learning Trajectory (HLT) that guided the intervention. Student interviews were used to provide insight into the development of student understandings of proportionality, and aspects of proportional reasoning. Student work samples, observational field notes, and a researcher journal guided the analysis of the HLT.

This chapter reports the data analysis and results of the study. The data analysis is organized into three main sections. Section one provides the quantitative analysis of written assessment data. Section two reports the qualitative analysis of student interviews organized by developmental mathematics student interviews and college level mathematics student interviews. Section three reports the analysis and refinement of the HLT that guided the classroom intervention. Throughout the analysis sections, results that relate to specific research questions are identified and noted. The chapter concludes with a summary of the results, organized by research question.

Section One: Written Assessment Data Analysis

A written assessment on proportionality was given to developmental and college level mathematics students. The test was administered as a pre and posttest to the developmental mathematics students surrounding the teaching experiment intervention. The test was administered to college level mathematics students as pretest, prior to their work with proportionality, direct variation, and linear functions. Table 4.1 presents the mean scores for the student groups on the written assessment.

Table 4.1

Mean Scores and Standard Deviations on the 15 Item Proportionality Written Assessment for Developmental (Mathematics Reasoning, Introductory Algebra) and College Level (Liberal Arts, College Algebra) Mathematics Students

	Pretest			Posttest		
Group	*M*	*SD*	*n*	*M*	*SD*	*n*
Mathematical Reasoning	8.20	2.55	30	10.87	2.03	30
Intermediate Algebra	8.35	2.03	17	10.35	1.90	17
Liberal Arts Mathematics	10.33	3.05	18	--	--	--
College Algebra	10.77	2.61	26	--	--	--

The pretest scores of the college level groups (Liberal Arts Mathematics, College Algebra) were higher than the pretest scores of the developmental mathematics groups (Mathematical Reasoning, Introductory Algebra). College Algebra students scored slightly higher than Liberal Arts Mathematics students, and Introductory Algebra students performed slightly higher than Mathematical Reasoning students on the pretest. An Analysis of Variance (ANOVA) test was performed on the pretest data to determine if mean differences exist among the groups. The results of the ANOVA test are summarized in Table 4.2. The data suggest there is a student group effect on the pretest scores. The effect size was calculated to provide an estimate of the relative size of the differences attributable to the student groups. There was found to be a medium effect size, $\eta^2 = .18$, with 95% CI [0.04, 0.29] among the four groups (Cohen, 1988).

Table. 4.2

One-way Analysis of Variance (ANOVA) Summary Table for the Effects of Student Group, Developmental (Mathematical Reasoning, Introductory Algebra) and College Level (Liberal Arts Mathematics, College Algebra) on Pretest Performance

Source	*df*	*SS*	*MS*	*F*	*p*	η^2
Between-group	3	126.24	42.08	6.28	< .001	.18
Within-group	87	583.30	6.71			
Total	90	709.54				

Post hoc analysis was conducted using Benjamini-Hochberg pairwise contrasts (Benjamini & Hochberg, 1995; What Works Clearinghouse, 2015). Based on the results from the post hoc analysis, it is likely that developmental mathematics students and college level mathematics students differ in their pretest performance. These results relate to research question 3. The standardized mean differences, confidence intervals, and *p* values for the four pairwise contrasts of developmental and college level mathematics student performance are illustrated in Table 4.3. There are likely no differences, on average, between developmental student group performances ($p = .85$). Similarly, there are likely no differences, on average between college level group performances ($p = .70$).

Table 4.3

Standardized Mean Difference, Confidence Intervals, and Benjamini-Hochberg Adjusted p-values for Pairwise Contrasts of Group Pretest Performance

		95% CI		
Contrast	Mean Difference	LL	UL	*p*
College Algebra – Mathematical Reasoning	2.57	0.83	4.30	.001
Liberal Arts Math – Mathematical Reasoning	2.13	0.20	4.06	.009
College Algebra – Introductory Algebra	2.42	0.40	4.44	.007
Liberal Arts Math – Introductory Algebra	1.98	- 0.21	4.17	.026

Dependent *t* tests for paired samples were used to determine if differences in pre and posttest performances existed for the developmental groups for each iteration of the teaching experiment. The data suggested there was likely a significant difference between the pre and posttest performance of Mathematical Reasoning students, $t(29)= 7.28$, $p < 0.001$. With 95% confidence, the mean posttest score was between 1.92 and 3.42 points higher than the mean pretest score for Mathematical Reasoning students. Similarly, the data suggest there was likely a significant difference between the pre and posttest performance of spring Introductory Algebra students, $t(16)= 4.97$, $p < 0.001$. With 95% confidence, the mean posttest score was between 1.15 and 2.85 points higher than the mean pretest score for Introductory Algebra students. The results of the t tests are summarized in Table 4.4.

Table 4.4

Differences Between Pre and Posttest Performance on Written Assessment for Developmental Mathematics Students

	Pretest		Posttest					Cohen's
Group	*M*	*SD*	*M*	*SD*	*df*	*t*	*p*	*d*
Mathematics Reasoning	8.20	2.55	10.87	2.03	29	7.28	< .001	1.16
Introductory Algebra	8.35	2.03	10.35	1.90	16	4.97	< .001	1.01

An independent *t* test was conducted on the gain scores (posttest score – pretest score) for the Mathematical Reasoning (*M* = 2.67, *SD* = 2.01) and the Introductory Algebra (*M* = 2.0, *SD* = 1.66) groups to test for significant differences. The results of the *t* test were not found to be statistically significant, $t(38.8) = 1.23$, $p = .23$. This result suggests that, even though the Mathematical Reasoning student group had a higher gain score than the Introductory Algebra student group, there was not a significant difference between the two groups. Thus the first and second learning trajectories may have had similar effect on the gains in student performance on the written assessment.

When considering the developmental posttest data and the college level pretest data, the results of an ANOVA test were not found to be statistically significant, $F(3, 87) = 0.29$, $p = 0.83$. This suggest that, following the teaching experiment interventions built around the first and second learning trajectories, there was not a significant difference in the performance among the developmental student groups nor between the developmental student groups and the college level students groups on the written assessment. The results of the pre and post-data t-tests and the ANOVA that analyzed mean differences among the developmental mathematics posttest data and the college level mathematics pretest data were used in the analysis supporting research question 2.

Section Two: Interview Data Analysis

Interview data analysis was conducted by combining two coding strategies, starting with predetermined codes and adding others as they become apparent from the data (Corbin & Strauss, 2008; Miles & Huberman, 1994). In this research, "open coding" refers to the process in which specific problem solving approaches and mathematical behaviors were identified in interview transcripts. The codes that emerged during this process provided information related to student thinking, but were not defined in predetermined codes. Predetermined codes are presented in Table 4.5. Codes that emerged during open coding are described in Table 4.6. For all coding, individual interviews were first analyzed before results across interviews were compiled.

Table 4.5

Predetermined Codes Representing the Mathematical Structure of Proportionality and the Psychological Aspects of Proportional Reasoning

Code	Multiplicative Constructs that Define the Mathematical Structure of Proportionality	Psychological Aspects of Proportional Reasoning	Code
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	A
		Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	B
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	C
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	D
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	E
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	F
		Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	G
		Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	H

Table 4.6

Codes that Emerged from the Data

	Code	Description	Example
Not Indicative of Multiplicative Thinking	Missed Rate	Failure to coordinate or interpret a rate	Steph's car used 16.3 [gallons]…61.8 miles. Then [Matt's] would be 13.2 gallons and 54.12. Okay. Let's see. Obviously [Steph] has more gallons than [Matt], and she drove more miles than he…The difference with [the miles] is in 7.68, and the difference between the gas is 3.1, technically, gallons. Okay…I'm not sure [which car had better efficiency].
	Same As	Interpretation of the relationship between two measure spaces as equality	So, 3 days vacation = 15 weeks. So, 6 days = 30 weeks.
	Additive Thinking	Use of additive thinking instead of multiplicative thinking	This one added three and this one added three…Then, yeah, I would say it's proportional, because if they got three, and they got three…then it's equal because you're using the same number.
	Building Up	Use of partitioning and iteration without considering the rate relationship between measure spaces	If I work 3 days for every 15 weeks, and it's 35 [weeks], so ... 15 and 15 is 30… oh, that's too many, that's 45…I don't understand that extra ... I'm at 30 [weeks]. That 5 [weeks].
	Standard Algorithm	Use of standard algorithm of cross multiply and divide	For the whole 15 weeks of work, he gets 3 days paid. So to figure out how many days he would have to work, it would be proportionate again, so I just cross multiply and divide.
Indicative of Multiplicative Thinking	Reciprocal Rates	Transitioning between reciprocal rates within a problem context	So this one is 3.79… [this one is] 4.1 miles per gallon. So, Matt's car has better fuel efficiency…Matt's car burned less gas per mile than [Steph's] car.
	Corrective Thinking	Use of proportional reasoning to correct thinking or change approach	Recipe A is stronger, 'cause there's two cups here that are watered down by 1. Oh, they are equal. They're equal…'cause there's two concentrated cups here and one [water]. So, that cuts it down by a third. But its half of it. And this one has four concentrates, and there are two [waters], which is half of it.

The number of tasks in which specific mathematical understandings of proportionality and psychological aspects of proportional reasoning (outlined in Table 4.5) appeared was counted. Individual student counts were combined into performance level counts based on the written pretest (low, medium, high) and were organized in frequency tables to identify general patterns of reasoning and understanding (Miles & Huberman, 1994). Emergent codes were similarly counted and organized.

Connected understandings were identified when a student used two or more multiplicative constructs of proportionality in support of each other when describing his or her thinking. Jeff, a high performing (HP) developmental student, demonstrated a connected understanding of a graphical representation of a proportional relationship (Construct 1), the existence of invariant rate across measure spaces (Construct 2), and equivalence (Construct 3) in the following excerpt.

Researcher: Here is a graph that shows the number of weeks that are worked and the number of vacation days that are earned. The (x,y) pair (4.5 days, and 22.5 weeks) is shown. Write the ratio, as a fraction, y/x for this pair.

Jeff: The rate would be [5 over 1], I would guess.

Researcher: How did you get the rate?

Jeff: 'Cause if you follow the line, *it crosses at certain intersections, 1 and 5, 2 and 10, 3 and 15 4 and 20 and that kind of stuff. So it is the same rate all the way down, 1 and 5 is a fairly simple rate to use instead of 4 and a half over 22 and a half.*

The connections between Constructs 1, 2, and 3 were identified when Jeff listed several elements of the equivalence class as points on the line: (1,5), (2,10), (3,15), (4,20), and identified the unit rate 5/1 that characterized the equivalence.

Sometimes students demonstrated multiple connections within a response to a single interview task. After a detailed coding of connections, codes were collapsed around the connections listed in Table 4.7. The connections were selected as central because of the role they appeared to play in the development of student understandings and reasoning. Examples of each central connected understandings are provided in Appendix F.

Connections between psychological aspects of proportional reasoning and understandings of the mathematical structure of proportionality were identified and counted. Psychological aspects A, D, and G were selected for this round of coding because they specifically relate to the research subquestions. This is not to imply that the psychological aspects are mutually exclusive of each other. In fact, data suggest that aspects often emerge in conjunction with each other. For example, psychological aspects F, G and H often appeared together.

The multiplicative construct(s) that supported the targeted aspects of student reasoning were identified for each instance in which a psychological aspect was demonstrated in an individual interview. An example of a connection between the differentiation of proportional and non-proportional situations (Psychological Aspect A), and an understanding of invariant unit rate across measure spaces (Construct 2) was identified in Ashley's (HP) first interview.

> Researcher: There is a sale on chocolate candies. 1 piece costs 15 cents. 2 pieces cost 27 cents. 3 pieces cost 39 cents, and 4 pieces cost 51 cents. Is this situation proportional? Why or why not?
>
> Ashley: … I want to say *it is not proportional because the more pieces you buy, the less you pay per piece*.

Ashley determined that the situation was not proportional by determining that the unit rate, interpreted as the cost per piece of candy, changed based on the number of chocolates purchased.

Developmental mathematics interview analysis.

The frequency tables used to organize observed understandings, aspects of reasoning and student approach are presented in Tables 4.7 – 4.11. Following the presentation of the tables, a discussion of the series of interviews is made including illustrative examples of student understanding, reasoning, and approach. In the discussion the results across iterations of the study are combined in a single narrative. Differences observed across iterations are noted when applicable.

Table 4.7

Understandings of the Multiplicative Constructs that Define Proportionality (outlined in Table 4.5) Demonstrated by Developmental Mathematics Students Grouped by Student Performance (Low, Medium, High) on Written Assessment

	Multiplicative	Interview 1			Interview 2			Interview 3		
	Constructs*	Low	Medium	High	Low	Medium	High	Low	Medium	High
Fall	1	0	4	6	7	7	8	5	7	8
	2	4	12	14	13	17	12	13	21	18
	3	8	8	12	5	12	9	6	11	10
	4	4	7	9	3	7	4	1	1	1
	5	8	19	18	6	18	18	17	20	19
Spring	1	0	1	4	2	12	12	7	13	15
	2	1	9	13	13	10	15	13	18	18
	3	10	6	8	7	12	12	8	8	12
	4	1	5	7	1	2	5	1	0	2
	5	12	15	21	10	11	18	11	19	21

* Cell entries are frequency counts.

Table 4.8

Psychological Aspects of Proportional Reasoning (outlined in Table 4.5) Demonstrated by Developmental Mathematics Students Grouped by Student Performance (Low, Medium, High) on Written Assessment

	Psychological	Interview 1			Interview 2			Interview 3		
	Aspect*	Low	Medium	High	Low	Medium	High	Low	Medium	High
Fall	A	7	12	16	13	11	16	11	21	18
	B	1	5	7	7	8	9	6	7	7
	C	4	10	14	8	15	13	13	21	18
	D	7	11	12	7	12	13	11	12	13
	E	3	11	12	6	14	14	9	10	10
	F	6	21	22	8	18	17	17	22	20
	G	4	16	19	7	18	16	17	22	20
	H	4	20	21	5	18	18	17	22	20
Spring	A	6	14	19	13	13	15	10	17	21
	B	0	2	1	1	11	12	8	12	15
	C	1	8	5	11	12	12	14	17	17
	D	7	7	11	10	9	8	7	12	14
	E	0	6	9	10	9	12	8	14	16
	F	7	9	15	13	13	19	10	19	21
	G	7	9	16	13	13	19	10	19	21
	H	8	8	15	13	13	19	10	19	21

* Cell entries are frequency counts.

Table 4.9

Characteristics of Student Thinking that Emerged from the Data (Outlined in Table 4.6) Demonstrated by Developmental Mathematics Students, Grouped by Student Performance (Low, Medium, High) on Written Assessment

		Interview 1			Interview 2			Interview 3		
	Code	Low	Medium	High	Low	Medium	High	Low	Medium	High
Fall	Missed Rate	13				1		2		
	Same As	9	1							
	Additive Thinking	12	3	1	2			2		
	Building Up	6	2	3				1		
	Standard Algorithm									
	Reciprocal Rate		3	2	1	2	1	3	4	5
	Corrected Thinking	1	1	1	4	1	2	1		2
Spring	Missed Rate	9	6		5	3		6		
	Same As	5		1						
	Additive Thinking	3	3			1				
	Building Up	5		2	1					
	Standard Algorithm		1	4						
	Reciprocal Rate					1	1	2	4	5
	Corrected Thinking				1	1	1		2	2

* Cell entries are frequency counts.

Table 4.10

Observed Connected Understandings of the Multiplicative Constructs that Define Proportionality (outlined in Table 4.5) Demonstrated by Developmental Students, Grouped by Student Performance (Low, Medium, High) on Written Assessment

	Connected Understandings*	Interview 1			Interview 2			Interview 3		
		Low	Medium	High	Low	Medium	High	Low	Medium	High
Fall	1 & 2		2	1		4	2	1	6	5
	2 & 3		1	7	3	2	13	1	3	4
	2 & 5	1	7	1	1	3	1	4	5	3
	3 & 4	1	3	4		3			1	
	1 & 2 & 3		1	6	2	1	2	2	2	5
	2 & 3 & 4			3	1	3	4		1	
	Other		1			2	8	3		1
	Total	2	15	22	7	18	30	11	18	18
Spring	1 & 2					4	3		2	3
	2 & 3			3	1	4	15	1	1	2
	2 & 5		2		2	1	1	1	1	
	3 & 4	1	3	3		3	2			
	1 & 2 & 3		1				6	5	7	10
	2 & 3 & 4			2	1	1	1	1		1
	Other			4						
	Total	1	6	12	4	13	28	8	11	16

* Cell entries are frequency counts.

Table 4.11

Observed Connections among Psychological Aspects of Proportional Reasoning and Understandings of Proportionality as Outlined in Table 4.5, Grouped by Student Performance (Low, Medium, High) on Written Assessment

		Psychological Aspect A									Psychological Aspect D									Psychological Aspect G								
		Low			Medium			High			Low			Medium			High			Low			Medium			High		
		Interview			Interview			Interview			Interview			Interview			Interview			Interview			Interview			Interview		
	Understanding*	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
Fall	1		1				1		1												1							
	2	3	12	10	11	7	10	10	15	8	1	5	8	6	7	6	4	3	5		4	11	9	8	12	11	7	7
	3	4			1			1			6		1	1			1			3			2	1				
	4													2	1	1		1			1		1	1			1	
	5			1				4		1										1		2			1	2		2
	1 & 2			1			2			5																		
	2 & 3			1	1	4	1			1			1	1	2	3	4	6	3			1	2	4	3	4	5	5
	2 & 5			1			3			2												1	1	2	3		1	3
	3 & 4								1					1	2		1	1					1	2		1	1	
	1 & 2 & 3						2			1						2	1	1	1		1				2	1	1	1
	2 & 3 & 4							1							2				1						1			
	Other			2									1				1		1			2						1
Spring	1					2																		2				
	2		10		2	3	5	13	5	5		10		1	2	3	2	2	1		10		2	2	4	4	3	5
	3	5	1		5			4			6	1		2			1			3	1		2			1		
	4																											
	5	1			4			1	1	1										2			1	1	2	3		1
	1 & 2		1			2	4		2	3		1			2	2		2	1		1	1		2	4		2	3
	2 & 3			1			1			1			1			1	3		1		1	1			1	3		1
	2 & 5			1	1				1					1						1		1	2	1				
	3 & 4								1		1			2			3	1		1			1			2	1	
	1 & 2 & 3			5	1	3	7		4	10			5	1	3	7		6	10			5	1	3	7		6	10
	2 & 3 & 4			1					1	1			1		1		2	1	1			1		1		2	2	1
	Other																											

* Cell entries are frequency counts.

Developmental mathematics interview 1.

The first interview was conducted prior to the start of the two-week intervention and served as a measure of student understanding and reasoning before instruction.

A noted difference observed between iterations of the study was the construction of *A/B = C/D* proportions and the use of the standard algorithm by students prior to the intervention. These representations and approaches were primarily observed in medium and high performing (MP and HP respectively) students in the second iteration, both on their written pretest and in their first interview. Of the students interviewed in which this was observed, 3 of the 4 completed their most recent math class the previous semester. In these prior courses, proportionality was included in the curriculum, and likely addressed through a traditional approach based on review of the textbook used. Notably, the interview transcripts revealed similar understandings, misunderstandings and reasoning by students across iterations despite the differences in written representation used.

Coordinating quantities and interpreting rates.

Low performing (LP) students struggled to coordinate relationships across measure spaces. When relationships were coordinated, they were not consistently organized into rates. Instead, low performing students often demonstrated a *same as* interpretation of the relationship in which the quantities from different contextual measures were interpreted as being the "same as," or "equal," to each other. This reasoning appeared to be the result of and understanding of equivalence class (Construct 3) that was disconnected from the existence of an invariant rate relationship between contextual measures (Construct 2).

The following excerpts illustrate differences in student thinking between low and high performing student reasoning due to a *same as* interpretation of a proportional relationship. When working with the following proportion related context, Sharia (LP) did not identify a rate relationship, even after prompting and guiding by the researcher. She became stuck in her additive approaches to reasoning as demonstrated in her attempt to *build up* using *additive thinking*.

Researcher: An employee earns 3 days of paid vacation for every 15 weeks of work…How many vacation days does the employee receive for working 35 weeks?

Sharia: ...3 days for every 15 weeks, and it's 35, so…15 and 15 is 30, that's 2 weeks. Then I still have…oh wait, where have I seen 15 somewhere? Oh, that's too many, that's 45. So I know that's 2 days. I don't understand that extra…I'm at 30. That 5.

High performing students tended to readily compute a unit rate and use the unit rate when solving a problem whether using a building up, scalar factor of change, or unit rate approach, illustrating a connection between understanding of Construct 2 and Psychological Aspects D and G. For example, in comparison Sharia's limited additive approach, Jeff (HP) readily transitioned to a unit rate interpretation of the relationship between weeks of work and vacation days to solve the problem. After an immediate correct response to the question of 7 days, Jeff explained his thinking in the excerpt below.

Jeff: Well, I figured, how many times 15 came out of 35 weeks, 'cause that's the same concept with those. And seeing as 15 doesn't come out of there an even amount of times, the remainder was 5. I took 15 divided by the amount of days they get off during the fifteen weeks. So I took two 15 weeks and I got six days [of vacation] for the 2 fifteen weeks that went in there. Then, *one more day for the five [weeks] of work*.

Jeff started with an *additive* approach (35 weeks – 15 weeks – 15 weeks = 5 weeks) towards *building up*, then transitioned to compute the unit rate of 5 weeks of work per 1 day of vacation. Jeff used the unit rate to complete his building up approach and solve the problem.

In the follow item, Jeff wrote “$X \times 5 = W$” as a rule that could be used to solve for the number of weeks that must be worked to receive any number of paid vacation days. When asked what the 5 meant, he replied, “5 is the rate ‘cause for every 5 [weeks] you work, you get one day off.” Thus, even though Jeff solved the missing value problem using a *building up* approach, he formed and interpreted a unit rate relationship and was able to use the functional multiplicative relationship to generalize the relationship (Constructs 1 & 2), supporting Psychological Aspect D. When asked to write rule that could be used to solve for the number of weeks that must be worked to receive any number of paid vacation days, Sharia wrote “15 weeks = 3 days” articulating her *same as* interpretation that limited her ability to generalize the relationship.

Connecting unit rate and equivalence.

The identification and interpretation of an invariant unit rate relationship between contextual measures (Construct 2), and its connection to equivalent rate pairs (Construct 3) had a central role supporting flexible and successful student reasoning and approach when working with proportion related contexts (Psychological Aspect G). All students demonstrated instances in which they identified a relationship across contextual measures showing an understanding of equivalence class (Construct 3) in proportional situations. However, only students who connected an invariant rate relationship to the equivalence class (Construct 2 & 3) were able to consistently differentiate between proportional and non-proportional situations (Psychological Aspect A), and extend an invariant rate relationship to other rate pairs (Psychological Aspect D). For example, Tim (HP) organized a rate relationship between the quantities and was able to list several rate pairs that would be proportional to each other that would have made the pricing proportional.

> Researcher: There is a sale on chocolate candies. 1 piece costs 15 cents. 2 piece costs 27 cents. 3 piece cost 39 cents, and 4 pieces cost 51 cents. Is this situation proportional? Why or why not?
>
> Tim: *The rate is totally different* so it really isn't proportional. If you bought *2 pieces it should be 30*, or *if you bought another one it should be 45*.

Differences existed in the connections between understandings of unit rate, equivalent rate pairs, and scalar multiplicative relationships (Constructs 2 & 3 & 4) between medium and high performing students. High performing students demonstrated stabilized connected understandings among these three constructs that facilitated flexible approaches to problem solving in contextually or numerically complex problem settings Psychological Aspects D & G). Medium performing students did not.

Consider the following excerpt in which Kim (MP) became stuck in her approach to solving a missing value problem using a factor of change approach.

> Kim: So … if I know I get 3 PTO days for every 15 weeks of work… I want to know how many weeks I need to work to get 10 days [of PTO], *I was going to take 10 days divided by 3*, I'm not sure why I did that, um… *What I should do is what I did on the last problem. 15 weeks of work divided by 3 days* of work will give me the value of….this is a hard one…[laughs].
>
> [Kim stops work.]
>
> Researcher: It is. Yeah, it is. Why don't you do that multiplication that you have down right now and just…get that down and then take another approach.

Kim: So that is 49.95.

Researcher: Does that seem somewhat reasonable, just in your gut, from what you have?

Kim: Yes, because 15 weeks, 3, 3, times 3 is nine which is about 10, and 3 time 15 is 45, so I think that is about right. So, I think that is correct. Is it?

Prior to becoming stuck, Kim considered going to an invariant unit rate, as she had done in the previous problem. Kim went on to solve the problem using a unit rate approach with help from the researcher, but it was never clear that she trusted her scalar factor of change approach actually had solved the problem. She was able to utilize both functional and scalar multiplicative relationships, but may have interpreted them as distinct from each other, and did not trust that both approaches could be used to solve missing value problems in the same context. Therefore, Kim had an understanding of and invariant unit rate relationship across measure spaces (Construct 2), but it was not strongly connected to her understanding of covariance with a scalar factor of change within measure spaces (Construct 4) and equivalent rate pairs (Construct 3). Thus, she was unable to use flexible approaches and reasoning processes (Psychological Aspect G) to extend the invariant rate relationship two other equal rate pairs (Psychological Aspect D).

Interpreting the $y = mx$ functional relationship.

Medium and high performing students demonstrated an understanding of the $y = mx$ functional relationship between measure spaces (Construct 1) and were able to identify and invariant rate relationship connecting Constructs 1 and 2. However, high performing students demonstrated these connections with greater frequency, and more naturally used multiple representations of rate to interpret and solve problems in proportion related situations (i.e. without prompting by the researcher). This may have been a product of prior learning and understanding. Differences were noted between the first and second iteration in terms of this interpretation. This may have been the influence of recent work with the traditional approach towards proportionality of setting up proportions and the use of the standard algorithm.

Catie, an important case.

Catie (HP) was a student subject in the second iteration. She regularly used the standard algorithm in her first interview as she had similarly done in her pretest. She was meticulous when setting up equal rates, and always took care to include units on her representations. However, it was noted that she did not consistently interpret the rate relationships with which she worked. Consider the following excerpt, in which she was asked to compute a unit rate.

Researcher: An employee receives 3 days of paid vacation for every 15 weeks of work….how many weeks must the employee work to earn one day of paid vacation?

Catie: How many weeks for one day. I would set up another problem [proportion] like this.

Researcher: Okay.

Catie: We want one day. We need to find out weeks. What I would do is take my beginning set here, so 3 days is to 15 weeks. We would do 1 times 15 is 15, divided by 3 would be 5…Five weeks.

[Catie's work is shown in Figure 4.1.]

Researcher: Now, if I were to take your paper and pencil away…and I said, "An employee earned 3 days paid vacation for every 15 weeks of work, how many weeks must the employee work to earn one day of paid vacation?" What would you do?

Catie:	I would probably pray. [Laughs] If you took away…okay. I would have to have you repeat what you're telling me.
Researcher:	Okay.
Catie:	I would basically be trying to do this [cross multiply and divide] in my head.

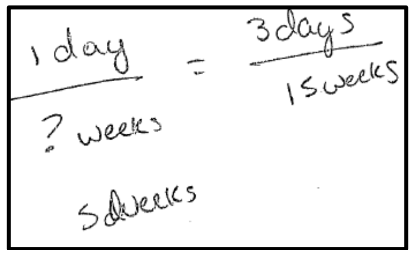


Figure 4.1. Catie's approach to determining "How much for one?"

Catie was able to solve the problem, but did not understand that she was computing a unit rate. In particular, she did not identify that employing a one-step division of 15 weeks divided by 3 days would yield the unit rate of 5 weeks per 1 day. She had an algorithm that worked for her when solving proportion related problems, but she was not able to flexibly approach problems in ways that allowed her to apply meaningful reasoning with the contexts involved.

Developmental mathematics interview 2.

The second interview was conducted midway through the two-week intervention, after students had worked with rates and proportionality was defined as a mathematical relationship in which one variable is a constant multiple of the other variable, $y = mx$. All students demonstrated more connections in their understandings of the multiplicative constructs that define proportionality, and could use their understandings to more clearly articulate their thinking. Understanding of unit rate relationships across measure spaces (Construct 2), continued to serve as a central understanding in many observed connected understandings.

Coordinating quantities and interpreting rates.

All levels of students more consistently coordinated quantities and interpreted rates in this interview than they had in the first interview. Open coding revealed that students did not apply *additive thinking* to proportion related problem solving tasks as frequently as they had in the first interview. This change occurred alongside a shift away from *same as* reasoning and *building up* strategies and a shift towards coordinating rates across measure spaces (Construct 2). Low students differentiated between proportional and non-proportional situations (Psychological Aspect A) by considering a unit rate (Construct 2) with greater frequency than they had in the first interview.

In the first iteration of the study, low students were not always able to form nor interpret a rate relationship as compared to low students in the second iteration. The connections between the scalar multiplicative relationships within measure spaces and the invariant rate relationships across measure spaces (Constructs 2 & 4) were often unstable in their thinking. The excerpt below illustrates this point of development in Sharia (LP).

Researcher: John and Mary make lemonade concentrate by mixing spoonfuls of sugar and spoonfuls of lemon juice. John makes his concentrate by using 3 spoonfuls of sugar and 9 spoonfuls of lemon juice. Mary makes her concentrate by using 6 spoonfuls of sugar and 15 spoonfuls of lemon juice. Whose lemonade concentrate is sweeter, John's or Mary's? Do they taste the same, or is it impossible to tell?

Sharia: Which is sweeter? That would be Mary's?

Researcher: Tell me why.

Sharia: Because [John's] has less sugar, but it has 9 spoonfuls of lemon juice. [Mary's] has more sugar and more lemon juice stuff. And just thinking about when I am making stuff, this one [Mary's] would be sweeter, and this one [John's] would be more bitter.

Researcher: So my next question for you, how much lemon juice would Mary need with her 6 spoonfuls of sugar to make her concentrate taste just like John's?

Sharia: So she wouldn't have to decrease them? So how could you make hers if she has more spoonfuls of everything? He has less.

After direction by the researcher towards a scalar factor of change, Sharia was able to solve the problem. In the first interview, Sharia did not coordinate quantities when solving comparison problems presented in the context of mixing orange juice (Noelting, 1980), and had employed *additive thinking*. Her reasoning in the excerpt above shows a significant shift towards consideration of a rate relationship between measure spaces (Construct 2) and multiplicative thinking; however, she still did not have understandings that supported Psychological Aspects D and G.

Interpreting the $y = mx$ functional relationship.

Understandings of the functional relationship between measure spaces (Construct 1) began to emerge across student performance levels. Students were able articulate reciprocal relationships between the two unit rates in a proportion related situation, and explain how each rate could be used in either a one-step multiplication or division problem to solve for a missing value (Psychological Aspect G). In the excerpt below, Jeff (HP) demonstrates a strong connection between the understanding of reciprocal unit rates, equivalent rate pairs, and the functional relationship that defines a proportional relationship (Constructs 1 & 2 & 3). This connection supported his ability to flexibly extend proportional relationships to equivalent rate pairs (Psychological Aspects D and G) when solving the missing value problems presented in Figure 4.2.

Researcher: Can you tell me a little about how you were using the rate differently?

Jeff: Well, for this one I knew how many hours he worked and I was trying to find how much he made from that, so I was talking the hours times 9.5 to see how much he made from that. Whereas, this one it told me the amount of money he made but not how long he worked, so I took the amount of money he made divided by the rate of how much money he makes per hour, and that's how I got the hours.

Researcher: Ok, how did you know to divide in the second case? What pushed you towards division. You were right to do so.

Jeff: I don't know, it made more sense in my head to do it that way, *because you had to divide down in order to multiply by the reciprocal, but that seems like more work than just dividing with what I have already*.

Researcher: Alright it's the same context, but what I'd like you to do is write an input-output rule that can determine the amount of money James earns for working any number of hours.

Jeff: So I have M, which stands for the total amount of money, equals 9.5 times H because you can just take the number of hours times the 9.5 and get how much money he made the entire shift.

Researcher: Okay, so looking at your rule, and you can use your rule to sort of help with your next explanation, tell me again why when you were solving for the number of hours, you divided by 9.5.

Jeff: *Because I had to get the H alone on that side so I had to divide the 9.5 over there*.

Jeff's understanding of the reciprocal relationship between the two unit rates in the situation was that one rate would cause a quantitative change of increase, and the other would cause a quantitative decrease. This understanding supported his emerging understanding of the inverse relationship between the two functions that can generalize the proportional situation marking a deeper understanding of Construct 2 than he demonstrated in the first interview.

In the second iteration of the intervention, it was identified that medium and low students were able to flexibly transition between using a unit rate in a one-step multiplication or division to solve for different quantities (e.g. hours worked or dollars earned), but struggled to connect a one-step division to a one-step multiplication by a reciprocal rate. There were cases in which operation with a functional relationship became mechanical through the following process: first a rate was identified and interpreted, next a one-step multiplication was computed, if the result did not make quantitative sense, a one-step division by the same rate was computed. Lizzy (LP) demonstrated this approach when solving the missing value problem in the excerpt below.

Researcher:	A trail mix company mixes 2 pounds of dried fruit for every 5 pounds of nuts for their signature mix. The company is going to make a large batch of their signature mix that contains 70 pounds of nuts. How many pounds of dried fruit will the company use in the batch?
Lizzy:	I need to divide 5 and 2…it equals 2.5.
Researcher:	What does that 2.5 mean in terms of nuts and fruit?
Lizzy:	It means 2.5 pound of nuts per 1 pound of fruit.
Researcher:	All right, now continue on. How many pounds of dried fruit does the company use in the batch when they use 70 pounds of nuts?
	[Lizzy computes 70 x 2.5 = 175 using calculator]
Researcher:	Does that number make sense?
Lizzy:	*No. It's the other way around probably. That makes more sense.*
	[Lizzy computes 70 / 2.5 = 28 using calculator]
Lizzy:	28…which is how many pounds of dried fruit it will be.

With prompting from the researcher, Lizzy was able to compute the reciprocal rate of 0.4 pound of dried fruit per 1 pound of nuts, but struggled to contextually interpret the rate and set up a one-step multiplication rule that could solve the problem. Lizzy was able to identify that her first computation was incorrect by considering the quantitative elements of the situation, but her understandings were not strongly connected to ideas of reciprocal rates within the context.

Connecting unit rate, equivalence, and covariant relationships.

Students demonstrated more flexible thought and approach in their problem solving (Psychological Aspect G), often transitioning between factor of change and unit rate approaches when solving proportion related problems. In several instances, students were able to explain that a scalar factor of change could change the quantities within measure spaces in ways that the ratio between rate pairs stayed the same. The excerpt below demonstrates how Jeff (HP) supported a factor of change approach with an understanding of invariant unit rate, articulating a strong connected understanding of multiplicative Constructs 2, 3, and 4.

Researcher: A trail mix company mixes 2 pounds of dried fruit for every 5 pounds of nuts for their signature mix. The company is going to make a large batch of their signature mix that contains 70 pounds of nuts. How many pounds of dried fruit will the company use in the batch?

Jeff: I know how [many] pounds of nuts they want to use in it, so I took that divided by the 5 there for how much their recipe is, which is 14. So I just took 2 times 14 to get the 28 pounds, so that way the recipe is the same.

Researcher: …Now I want you to think about this in terms of the rate of 5 pounds of nuts to 2 pounds of fruit. How did you use that rate to solve this problem?

Jeff: Well, you have the 5 over 2, and 70 over, you don't know yet, that's what you are trying to find out. You are finding the scalar of it, which is the 14. In order to get 5 to 70 you have to multiply that by 14. *So if you multiply that part, you have to multiply the other part by the same number to get the same ratio*.

Catie, an important case.

Catie did not use the standard algorithm in the second interview. Instead, Catie flexibly utilized a unit rate approach when solving missing value problems. She demonstrated a particularly strong interpretation of rate when working with qualitative and comparison problems and was able to easily switch between reciprocal rates when discussion her reasoning. Her interpretation of rate allowed her to differentiation between proportional and non-proportional situations and relationships (Psychological Aspect A), by determining if a rate relationship was staying the same or changing (Construct 2). It was possible that Catie came to the intervention with strong notions of rate that were stabilized when introduced to ideas of unit rate and the $y = mx$ function relationship (Constructs 1 & 2), enabling more flexible and meaningful approach to problem solving (Psychological Aspect G) as is demonstrated in the excerpt below.

Researcher: A trail mix company mixes 2 pounds of dried fruit per every 15 pounds of nuts in their signature mix. The company is going to make a large batch of their signature trail mix that contains 70 pounds of nuts. How many pounds of dried fruit will the company use in the batch?

Catie: Yep, we've got 2 pounds dried fruit over 5 pounds nuts and they want to get to 70 pounds of nuts, so we need to figure out how much dried fruit. Okay, so *I'm going to do a unit rate*, I think. If I take 2 divided by 5 ... What if I did this: *If I did 1 pound of dried fruit on the bottom, that would give me 2.5 pounds of nuts*….That would be unit rates, so now we need to figure out 70 pounds of nuts, *so we will take 70 divided by 2.5, which is 28*, so they need 28 pounds of dried fruit.

Researcher: Great. Is this situation proportional? Why or why not?

Catie: It is. The *70 pounds of nuts is just a scaled version of the 2.5 pounds or 5 pounds*.

Researcher: Yep. All right, so right now, I see two different rates. You have pounds of dried fruit to the pounds of nuts and you have pounds of nuts to pounds of dried fruit. Can you tell me about the relationship between those two rates?

Catie: Well, I think that they would be reciprocal….But I think that they're just opposite.

Researcher: Okay…How could you use the rate that you have …the 2.5 pounds of nuts to 1 pound of dried fruit to set up a rule that could solve [a problem] like if I gave you any number of pounds of dried fruit, how many pounds of nuts would you get?

Catie: Okay. Let's see ... You want to find out dried fruits?...Well, *you would just multiply the [unit] rate by number of pounds of dried fruit*, I think.

Researcher: Great. Now if I switched the input and output, how would you use that rate to determine the number of pounds of dried fruit, if I gave you as an input this many nuts?

Catie: *Instead of multiplying, you would divide*.

In this excerpt, Catie immediately computes both unit rates, then flexibly uses the unit rate that was more quantitatively comfortable for her (2.5 pounds of nuts per 1 pound of dried fruit) (Construct 1), to solve the problem. She notes ideas of covariance and scalar multiplication by stating that the she could "scale up" to get to 70 pounds of nuts from the 2.5 pounds of nuts or the 5 pounds of nuts (Construct 3). She also was able to generate two generalized rules that could be used to solve for an unknown number of pounds of fruit or nuts using the unit rate of 2.5 pounds of nuts per 1 pound of dried fruit (Construct 1).

Developmental mathematics interview 3.

The third interview was conducted after the close of the instructional intervention. All students were able to solve and discuss a diversity of problems including missing value problems, comparison problems, and qualitative reasoning problems (Construct 5). The understanding and interpretation of invariant rate relationships helped students differentiate between proportional and non-proportional situations and (Psychological Aspect A), and to extend an invariant rate relationship to equal rate pairs when working with proportion related problems (Psychological Aspect D). The depth of understanding of the rate relationships were illustrated in the ability of students to identify, interpret, and operate with *reciprocal rates* in proportion related problems.

Identifying proportional reasoning.

Students were asked to differentiate between proportional and non-proportional reasoning in a series of items based on the "Populations of Towns" problem context from the 1996 National Assessment of Educational Progress (NAEP) Mathematics Assessment (U.S. Department of Education, 2014), shown in Figure 4.2.

In 2000, the populations of Town A and Town B were 5,000 and 6,000 respectively. The 2010 populations of Town A and Town B were 8,000 and 9,000 respectively. Brian claims that from 2000 to 2010 the populations of the two towns grew by the same amount, 3000 people. Was Brian reasoning proportionally about the population growth? How can you tell?

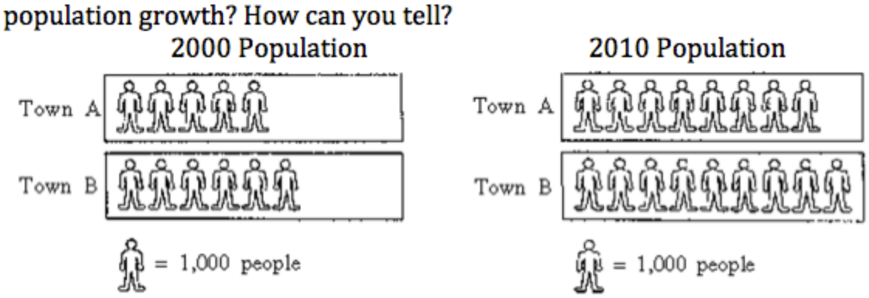


Figure 4.2. Population of Towns context and question.

In response to the question posed in Figure 4.2, most medium and high performing students were able to determine that Brian was not reasoning proportionally. Jeff (HP) used his understanding of the rate relationship between the change in population and beginning population of each town to apply qualitative reasoning that determined that Brian was reasoning additively, not proportionally.

Jeff: It was not proportional. Because, sure, they each went up by the 3000 people, but *it's not the same percentage for how much each town grew over the same time period*…This town went from 5000 to 8000, so they didn't start off with as much and they didn't end up with as much either. *The proportion for how much their town grew was different because they had less to start with*. The fact that they had [grown by] the same amount of people, meant the town grew more [proportionally].

Kim (MP) used her connected understanding of the $y = mx$ functional relationship in a proportion related situation and the unit rate relationship across measure spaces (Constructs 1 & 2) to determine that Brian was not reasoning proportionally.

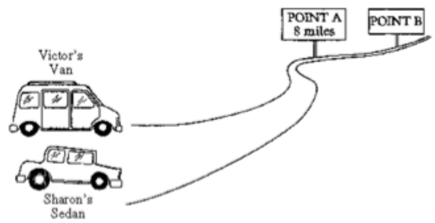
Kim: No, because he's adding 3,000 to each instead of using a one-step multiplication.

Students who stated that Brian was reasoning proportionally failed to notice he was thinking additively instead of multiplicatively, and defined proportional as meaning same additive change or simply identified Brian's reasoning as "correct thinking."

Interpreting and reasoning with rates.

In the proportion related problem presented in Figure 4.3, students were asked to compare the time it took two vehicles to travel 8 miles (U.S. Department of Education, 2014). The problem elicited thinking and reasoning connected to comparison and missing value problems.

Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes. Both cars start at the same time and from the same location. Will Sharon's sedan reach point A, 8 miles away, before, at the same time as, or after Victor's van?



Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Figure 4.3. Victor's Van and Sharon's Sedan problem.

Most students correctly solved the problem by simply comparing the speeds of Victor's Van and Sharon's Sedan and identifying they were traveling at the same rate, demonstrating a Psychological Aspect G supported by multiplicative Construct 2. This understanding supported deep reasoning in which the invariant rate was identified as characteristic of all distance and time pairs in the equivalence class, connecting understandings of Constructs 2 and 3. Shannon (LP) demonstrates this connection. After computing the unit rates for each vehicle (as 1.25 minutes per mile), she explains through a follow up question that although the quantities presented for Victor's Van and Sharon's Sedan are different, they rate at which the vehicles travel is the same.

Shannon: They will both get there at the same time.

Researcher: Are the rates at which the vehicles travel proportional to each other? Why or why not?

Shannon: Yes, because they have different numbers, like the 20 miles and [8] miles, but they come up with the same [unit] rate.

Interpreting the y = mx functional relationship.

The ability to interpret and reason with a functional relationship between measure spaces (Construct 1) was observed in each student's interview. Multiple representations of the functional relationships were accessed by all students, and often appeared in conjunction with reasoning about the rate relationships across measure spaces, demonstrating connection between understandings of Constructs 1 and 2.

In a subsequent question that utilized the context of Victor's van presented in Figure 4.3, students were asked to construct a graph representing the relationship between time and miles travelled for Victor's van. After a graph was constructed, students were asked to identify how the rate at which Victor's van travelled was represented in the graph. In Sarah's (LP) interview, she demonstrated a deep understanding of unit rate (Construct 2) that was connected to the functional relationship between measure spaces (Construct 1).

Researcher: How is the rate at which Victor's van travels represented in the graph?

Sarah: Steady.

Researcher: Tell me more about that.

Sarah: It's not varying, it's not going up or down, *it's a straight line from starting at zero to my second point* [referencing the point (8,10)].

Researcher: Cool. You said that it starts at zero, I would call that a y-intercept because it is crossing your vertical axis.

Sarah: Yep.

Researcher: Where it's touching this part, [referencing the origin] tell me what that means, that it's going through zero in terms of miles and minutes.

Sarah: *If he's at zero, he hasn't gone any miles and he hasn't gone any minutes.*

Researcher: Great, good. When the input is 1 mile what would the output be? When you're horizontally at one mile what would your vertical measure be?

Sarah: 0.8. [Note: Stated without being computed.]

Researcher: You got it. How is that related to the rate?

Sarah: *That is your unit rate*.

Sarah articulated that the function is linear and the graph passes through the origin, which means the function models a proportional situation. Further, she identifies two ways the unit rate was represented in the graph: the slope of the line, in the rate pair (1, 0.8).

This excerpt show exceptional growth in her understanding of proportionality specific to unit rate. In her first interview, Sarah struggled to form rate relationships when coordinating quantities across measure spaces and employed *same as* reasoning instead of forming rates. This excerpt shows that Sarah now had connected understandings of equivalence and rate (Constructs 2 & 3) and demonstrates how her interpretation of a graphical representation of a proportion related situation (Constructs 1 & 5) stabilized her understanding.

Central connections.

Many of the connections that students demonstrated in the interview involved the identification and interpretation of an invariant unit rate relationships across measure spaces (Construct 2). Connections between Constructs 2 and 3, and Constructs 2 and 5 occurred with high frequencies and supported Psychological Aspects A, D and G. The function interpretation of proportion related contexts (Construct 1) served to facilitate these connections and enable students to operate with novel problems and contexts that involved varying levels of numerical complexity and was enhanced when connected to an understanding of rate (Construct 2).

Developmental mathematics interview discussion.

The identification and interpretation of rate (Construct 2) within a proportional situation is central to the foundation of a connected understanding of proportionality. It enables the differentiation between proportional and non-proportional situations (Psychological Aspect A), and supports flexible thought and approach to problem solving situations (Psychological Aspect G). The emergence of understanding of rate marked an important conceptual transition from additive to multiplicative thinking for low performing students. It also marked a transition in student approach towards more efficient and flexible multiplicative approaches when working with proportion related problems. These results apply to research questions 1a, 1c, and 1d.

The connections of understandings of invariant rate and equivalent rate pairs (Constructs 2 & 3) further stabilized student understandings of proportionality, approach to solving proportion related problems, and psychological aspects of proportional reasoning. These connections supported the extension of an invariant rate relationship to other equal multiples of the quantities using proportional reasoning (Psychological Aspect D). When this connection was made, students were able to overcome numerical complexity (e.g. non-integer relationships between and within measure spaces). Multiple representations of the functional relationships within proportion related contexts served to further stabilize student understandings of rate and equivalence in proportion related contexts (Constructs 1 & 2 & 3). These results apply to research question 1b and 1c.

As students demonstrated stronger connections among the multiplicative constructs that define proportionality to their understanding of rate (Construct 2), they began to more regularly use unit rate approaches when operating with proportion related problems. Ideas of covariance (Construct 4) were observed, but were strongest when considered in conjunction with the invariance of the rate relationship. These results apply to research questions 1b, 1c, and 1d.

All of the results that emerged from the developmental mathematics student interview data were used in the retrospective analysis of the HLT, and directly applied to research question 2.

College level mathematics interview analysis.

Tables 4.12 – 4.16 show the organized frequency counts that resulted from the coding, organized by semester and course. Discussion of the interviews is organized by Liberal Arts Mathematics (LAM) and College Algebra (CA) students and conclude with a discussion comparing and contrasting the observed understandings and reasoning processes of each group.

Table 4.12

Understandings of the Multiplicative Constructs that Define Proportionality (outlined in Table 4.5) Demonstrated by College Level Mathematics Students, Grouped by Student Performance (Low, High) on the Written Assessment

	Multiplicative Construct*	Low	High
Liberal Arts Mathematics	1	5	6
	2	20	22
	3	14	18
	4	3	7
	5	18	23
College Algebra	1	6	12
	2	16	26
	3	14	14
	4	8	8
	5	20	26

*Cell entries are frequency counts.

Table 4.13

Psychological Aspects of Proportional Reasoning (outlined in Table 4.5) Demonstrated by College Level Mathematics Students, Grouped by Student Performance (Low, High) on the Written Assessment

	Psychological Aspect*	Low	High
Liberal Arts Mathematics	A	18	19
	B	5	6
	C	19	23
	D	12	19
	E	11	13
	F	19	26
	G	19	26
	H	19	26
College Algebra	A	22	26
	B	11	12
	C	13	21
	D	16	18
	E	12	18
	F	21	26
	G	21	26
	H	21	26

*Cell entries are frequency counts.

Table 4.14

Characteristics of Student Thinking that Emerged from the Data (Outlined in Table 4.6) Demonstrated by College Level Mathematics Students, Grouped by Student Performance (Low, High) on Written Assessment

	Code	Low	High
Liberal Arts Mathematics	Missed Rate	5	
	Same As	3	2
	Additive Thinking	3	
	Building Up	5	3
	Standard Algorithm	5	3
	Reciprocal Rate		3
	Corrected Thinking	3	
College Algebra	Missed Rate	3	
	Same As		
	Additive Thinking	3	
	Building Up	1	2
	Standard Algorithm		
	Reciprocal Rate	1	3
	Corrected Thinking	2	

Table 4.15

Observed Connected Understandings of the Multiplicative Constructs that Define Proportionality in College Level Mathematics Students, Grouped by Student Performance (Low, High) on Written Assessment

	Connected Understandings	Low	High
Liberal Arts Mathematics	1 & 2		
	2 & 3	9	9
	2 & 5	2	8
	3 & 4	2	2
	1 & 2 & 3	3	3
	2 & 3 & 4	1	4
	Other		
	Total	17	26
College Algebra	1 & 2	1	3
	2 & 3	7	6
	2 & 5	2	3
	3 & 4	1	
	1 & 2 & 3	1	5
	2 & 3 & 4	2	4
	Other		1
	Total	14	22

Table 4.16

Observed Connections among Psychological Aspects of Proportional Reasoning and Understandings of Proportionality (Outlined in Tables 4.5) in College Level Mathematics Students, Grouped by Student Performance (Low, High) on the Written Assessment

		Psychological Aspect A		Psychological Aspect D		Psychological Aspect G	
	Understanding	Low	High	Low	High	Low	High
Liberal Arts Mathematics	1						
	2	7	6	1	2	4	1
	3		1	2	2	2	2
	4						
	5	1					
	1 & 2						
	2 & 3	9	10	10	9	10	7
	2 & 5	2				2	7
	3 & 4		2		1		2
	1 & 2 & 3				1		3
	2 & 3 & 4	1		1	4	1	4
	Other						
College Algebra	1	1		1		1	
	2	6	7	1	1	3	1
	3						
	4	4	1	3	1	3	1
	5					1	
	1 & 2		3		5		5
	2 & 3	6	2	6	3	6	3
	2 & 5	2	4	2	1	2	1
	3 & 4						
	1 & 2 & 3	1	5	1	5	1	5
	2 & 3 & 4	2	4	2	4	2	4
	Other						

Liberal Arts Mathematics interview discussion.

Coordinating quantities and interpreting rates.

Liberal Arts Mathematics (LAM) students demonstrated strong understandings of invariant rate relationships (Construct 2) that exists across measure spaces in proportion related situations. This understanding was coupled with strong rational number understandings and operations that supported their organization, interpretation, and operation when working with proportion related problems.

LAM students tended to rely on a proportion representation and their rational number skills, and scaling (Understanding 4) when organizing and solving problems. Differences in understanding were noted between low and high performing LAM students in both the consistency in which rates were coordinated.

Beth (LP) tended to focus in on individual measures before considering a rate relationship across measure spaces. This was noted in her tendency toward *additive thinking* and factor of change approaches when solving missing value problems, and failure to coordinate rates (*missed rate*) in qualitative reasoning and comparison problems. Consider her approach to the comparison problem below.

Researcher: Steph and Matt are racecar drivers. They tested their cars' fuel efficiency driving at race speeds on an oval racetrack used for a long distance car race. Steph's car used 16.3 gallons of gas on a 61.8 mile drive. Matt's car used 13.2 gallons of gas on a 54.12 mile drive. Whose car had the better fuel efficiency, were they the same, or is it impossible to tell?

Beth: I think Steph's car would have the better efficiency because she went a few more miles than Matt did, and Matt was very close to where Steph was at fewer miles.

Researcher: Okay, can you tie that down numerically by doing some computations?

Beth: Computations, what do you mean?

Researcher: So, if you were to find their exact fuel efficiency, what would you do?

Beth: Uhhh, I guess I would start by writing what their cars took.

Researcher: Sure. So, Steph had 16.3 gallons, 61.8 miles.

Beth: Yes, and Matt has 13.2 gallons and he drove for 54.12 miles.

[Beth constructs a representation of the situation shown in Figure 4.4.]

Researcher: So tell me again how you did your comparison. Let me see the numbers.

Beth: See the numbers? Okay, Steph, she had 16.3 gallons and she went 61.8 miles. Matt's had less, 13.2 gallons and he drove 54.12.

Researcher: So Matt used less gallons and drove fewer miles. Okay, and then Steph used more gas and drove more miles.

Beth: Actually, I want to change my answer looking at it this way, having it all written down. Because, yes, he had less miles, but it seems like he went a little farther with it because he was only at…what was it? 54 miles, and he only used 13.2 [gallons]. So looking at it, it looks like he would have gone farther.

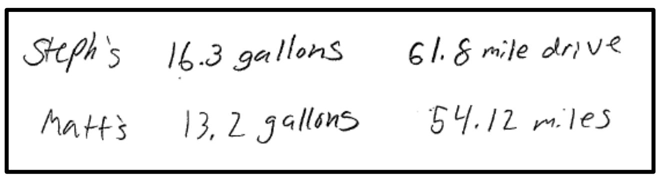


Figure 4.4. Beth's representation of a comparison problem.

When she first approached the problem, Beth did not coordinate rates, but instead attempted to reason with the differences of the quantities in each measure space (gallons and miles), illustrating both *additive thinking* and a failure to coordinate a rate (*missed rate*). With further questioning and prompting by the researcher, Beth was able to reconsider the problem (*corrected thinking*) and correct her thinking by forming a rate relationship and estimating the rates using multiplicative thinking to estimate fuel efficiency.

Indication of strong rational number understandings.

Differences in understanding were noted between low and high performing LAM students in both the consistency in which rates were coordinated, and in the strength of connections between understandings of equivalence (Construct 3) and scalar factors of change within measure spaces (Construct 4) to an invariant unit rate. High performing

LAM students were able to articulate deep and flexible understandings of rational numbers, including their interpretation as ratios and rates, and an understandings of their role as multiplicative operators (Kieren, 1976).

Consider the following excerpt from Kristopher's (HP) interview in which he transitions between reciprocal scalar ratios in order to set up a one-step multiplication that can solve the missing value problem.

> Researcher: An employee receives 3 days paid vacation for every 15 weeks of work. How many vacation days does the employee receive for working 35 weeks?
>
> Kristopher: Well, he is receiving 3 days paid vacation per 15 weeks worked. So if he is receiving 35 weeks, then he would do, how many times 15 goes into 35, and then times that by 3 to get how many days vacation he would get out of 35 weeks.
>
> Researcher: Great, so show me how you would do that.
>
> [Note: Kristopher computes 15 / 35 = 0.42]
>
> Kristopher: 35, oh, yeah, that would be 35 divided by 15 yeah. 2.33 repeated, times 3…you get about 7 days, 6.99, so about 7 days.
>
> Researcher: All right, so first question, sort of how you were solving it. Initially you said, 15 weeks divided by 35 weeks and you got .42 for that.
>
> Kristopher: mmhmmm.
>
> Researcher: uhm, and you said, oh it should be the other one.
>
> Kristopher: yeah, it should be the other one.
>
> Researcher: How did you determine that wasn't the one you had.
>
> Kristopher: Well, *if you times 0.4 by 3, its obviously going to be less days*, um, and so if you have 35 weeks out of 15, obviously going to be more. 'Cause *you either gotta divide, or its going to be the other one*.

Kristopher used his understanding of rational number as an operator to help guide his approach in scaling. He knew that he needed a scalar multiplier that would increase the number of days of vacation because more weeks had been worked, demonstrating Psychological aspects G and H supported by a strong understanding of multiplicative Construct 4. His reference to the relationship between division and multiplication by a reciprocal and the quantitative changes they produce demonstrated a particularly rich understanding of the ratio interpretation of rational number.

Interpreting the $y = mx$ functional relationship.

LAM students were able to interpret and work with several representations of the $y = mx$ functional relationship (Construct 1). However, they were not able to articulate clear connections of the functional relationship to other multiplicative constructs that define proportionality without prompting by the researcher. This may be the result of developing their proportion related understandings and reasoning processes in traditional teaching and learning experiences that did not target these connections and limited experience with multiple representations of functions.

The standard algorithm.

Tamara (HP) utilized the *standard algorithm* several times in her interview. Each time she used the standard algorithm, she grounded her strategy with language or written work that demonstrated an exceptionally strong understanding of an invariant unit rate (Construct 2). For example, Figure 4.5 shows her written work in which she first computes a unit rate (without prompting) before using the standard algorithm.

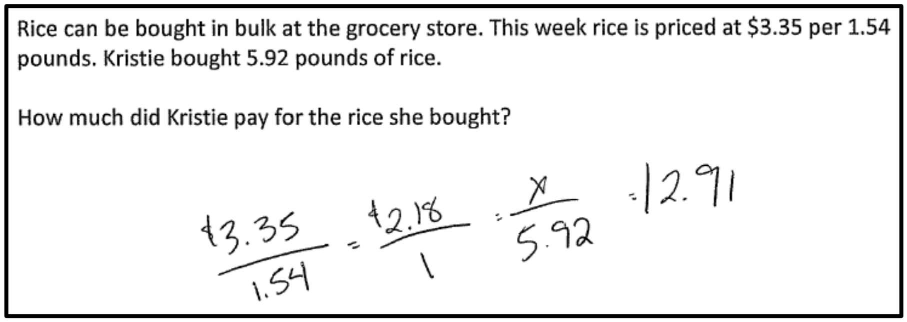


Figure 4.5. Standard algorithm supported by a unit rate.

Tamara's definition of proportionality was a situation in which an invariant rate relationship existed (Construct 2). The symbolic representation of her work often included proportions, an example of which is shown in Figure 4.8.

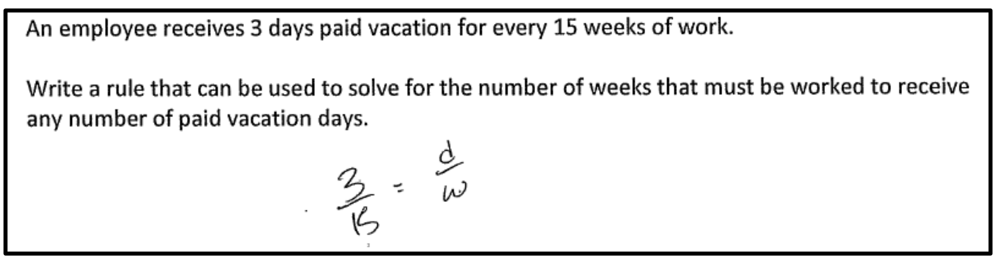


Figure 4.6. A proportion as a generalized rule.

Although Tamara successfully utilized the standard algorithm several times in her interview, she also flexibly transitioned between unit rate and factor of change problem solving approaches (Psychological Aspect G). She regularly applied a scalar factor of change approach when numerically complex scalar multiples were involved (e.g. multiplication by a scalar factor of 7/3). Further, she was able to articulate how scalar multiples maintained an invariant rate in a proportional context. In the following excerpt, Tamara verifies that two rates are not equal by computing a factor of change for each measure.

Researcher:	There is a sale on chocolate candies. 1 piece costs 15 cents. 2 pieces cost 27 cents. 3 pieces cost 39 cents, and 4 pieces cost 51 cents. Is this situation proportional? Why or why not?
Tamara:	Let me double check. No.
Researcher:	What you did there is … 27 divided by 15. And right away, how did that tell you that it is not proportional?
Tamara:	Well, I already had it in my head, that if one piece cost 15 cents, to be proportional, two pieces would have to cost 30 cents. But I just had to double check to make sure.
Researcher:	So you were just checking by saying, 27 cents divided by 15 cents, and if it was proportional…
Tamara:	*It would have been two, but it was 1.8*.

In this excerpt, Tamara begins a list of equal rate pairs (Construct 3). She checked her initial estimation that the rates were not equal by computing the scalar multiple *27 / 15 = 1.8*. Her check verified that the situation was not proportional because the number of chocolates had doubled, but the costs had not (Construct 4).

Tamara demonstrated strong connected understandings between an invariant unit rate, equivalence class, and scalar factors of change (Constructs 2 & 3 & 4). It is likely that this type of understanding supported her interpretation of proportionality through a symbolic interpretation of proportions, and her accurate standard algorithm approach to problem solving.

College Algebra interview discussion.

Coordinating quantities and interpreting rates.

College algebra (CA) students consistently coordinated quantities and formed rate relationships in their interviews. Unit rates were computed across performance levels, often in the early approaches and operation with a problem or context demonstrating a strong understanding of Construct 2. CA students demonstrated deep rational number understandings and operation that supported their work in problems, allowing them to discuss connections between the covariant and invariant rate relationships within proportional situations (Psychological Aspect E). CA students also demonstrated strong rational number understandings and operation that supported their approach.

Consider the approach of Cassi (LP) when solving a missing value problem.

Researcher: An employee receives three days paid vacation for every fifteen weeks of work. How many vacation days does the employee receive for working thirty-five weeks?

Cassi: What I usually do is sometimes make a table, so an employee receives three days for every fifteen weeks. How many vacation days does an employee receive for working thirty-five weeks. Let's see…

[Cassi constructs table shown in Figure 4.7]

Technically, I usually take fifteen divided by three equals five. So if the difference is then five, then I can take five times thirty-five. Wait, no. Does that make sense? Thirty-five divided by five, so seven days.

Researcher: …Is the relationship between the days of vacation and the weeks of work proportional? Why or why not?

Cassi: …I think yes, because it's the *same ratio*.

Researcher: Great. Now, you used the rate of fifteen weeks per three days and then you divided to get five. Tell me about what that rate means after you do the division.

Cassi: …Technically, it's just a different…I don't know if you can call it a *unit*, but it's just the difference between the two.

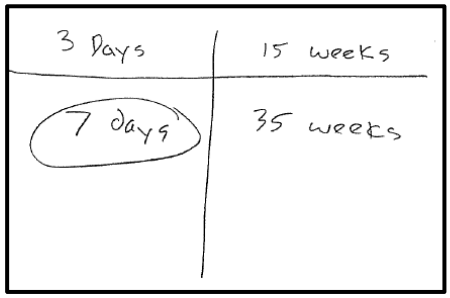


Figure 4.7. Table approach to solving a missing value problem.

Cassi began her approach with a table, working with the relationship between weeks of work and days of vacation using a numeric representation. She immediately computed a unit rate so that she could interpret how many weeks of work earn one day of vacation, calling that the “difference between the two.” She then used her rate to set up a one-step division rule to determine the number of days of vacation used, tapping into the $y = mx$ function rule. In the following problem, Cassi used her unit rate to solve for the number of weeks worked to earn 10 vacation works and set up a one-step multiplication. When asked about how she used her rate differently, she accessed ideas of *reciprocal rates* and the connection between the one-step multiplication and one-step division (Construct 2).

Researcher: In the last two problems you used the rate of five weeks per one day of vacation, but you used it differently. In this one you said ten days times five weeks per one day is fifty. Now on the previous problem, when you figured out how many days that the employee got for thirty-five weeks of work, you divided. Tell me about how you used that rate differently or why you knew to do that.

Cassi: Technically, it helps for me to look on both sides of the question because, for example, *each question asks you to do or find something different*. Here, I had to find weeks instead of days. Because of that, I used the ratio, I guess, *in the opposite direction* or…*reciprocal*.

Although she used the $y = mx$ function relationship flexibly to solve for different quantities, Cassi was unable to write a concise input-output rule when asked to write a rule that generalized the situation. Instead, she first created a table of values, then constructed a rule using words saying "In order to receive three days of paid vacation, [the employee] has to work fifteen weeks…For example, if a person wants to receive six days of vacation, then it means that person needs to work thirty weeks because that number needs to double. *The ratio stays the same*, except it's just doubling." Cassi's rule indicated a connection between invariant unit rate, scalar factor of change, and equivalence (Constructs 2 & 3 & 4), but was not clearly conceptualized in the same structure (i.e. a one step multiplication or division function rule) that she actually used to solve the previous two problems.

Accessing and operating with a functional relationship.

The $y = mx$ functional relationship (Construct 1) was prominent in CA student understanding, reasoning and operation in proportional situations. CA students regularly interpreted proportion related problems through a functional relationship and deftly operated with multiple modes of its representation. This was often done without prompting by the researcher. High performing students, in particular, articulated and connected the multiplicative constructs that define proportionality through and to the $y = mx$ functional relationship, and employed these understandings to solve missing value and qualitative reasoning problems.

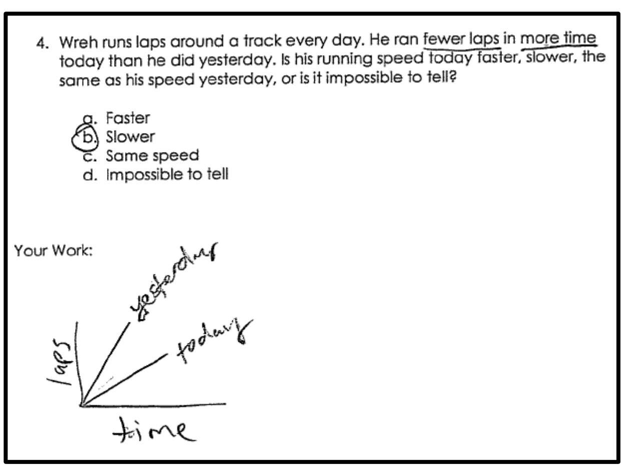
When working with the vacation days context, Jennifer (HP) actually articulates the phrase "directly related" meaning directly proportional or direct variation when discussing her scaling approach.

Researcher: You said that since you multiplied the 3 days vacation times 3, you were going to multiply the 15 weeks of work times 3. Can you tell me about why you would do that same multiplication to both your days and your weeks?

Jennifer: Because they're related. There's a term for it in math, I can't remember it right now, when two umber are like, is it inverse? No, inverse is the opposite. *Directly related*. When one goes up by a multiple, the other one goes up by that same multiple.

When asked to write a rule that would solve for the number of weeks that must be worked to receive any number of vacation days, Jennifer wrote $y = 5x$. When presented with a graph of the $y = 5x$ model of the context, Jennifer connected the slope of the line to the invariant rate relationship of all (x,y) rate pairs. She flexibly operated with the quantities in the problems because she had a connected understanding of the invariant unit rate relationship, equivalence of rate pairs, and scalar multiplicative relationships within the context of a proportional situation. Her understanding, was always interpreted through or connected to the $y = mx$ functional relationship.

Chad (HP) approached a qualitative reasoning using a graphical representation of proportional relationships on his written assessment illustrated in Figure 4.8. When asked about his approach to the problem, Chad gave the following explanation.


4. Wreh runs laps around a track every day. He ran fewer laps in more time today than he did yesterday. Is his running speed today faster, slower, the same as his speed yesterday, or is it impossible to tell?

a. Faster
b. Slower
c. Same speed
d. Impossible to tell

Your Work:

Figure 4.8. Graphical approach to qualitative reasoning.

Chad:	If you figured out the slope of this one (referencing yesterday's line), [it] would be a greater slope than today…
Researcher:	What about the situation makes [yesterday's] slope greater than [today's]?
Chad:	The slope would represent his unit rates or how fast he's going. The slope would be less so it'd be less [laps] per [time] than yesterday…if you were to divide the time by the laps it would be less than yesterday's.

The graphical representation of the function relationship $d = rt$ served as an anchor in which Chad could reason about the Wreh's running rates in the context of the qualitative reasoning problem. Throughout his interview, Chad regularly defined and operated with a $y = mx$ functional relationship symbolically and numerically. Thus, the functional relationship was a central and meaningful understanding of proportionality (Understanding 1) that facilitated his operation with the covariant and invariant relationships and flexible approach when solving proportion related problems (Psychological Aspects D and G).

Summary of college level math interviews.

A strong understanding of invariant unit rate (Understanding 2) was central to the connected understandings observed in College Level Math students. Both LAM and CA students consistently coordinated quantities into rates and were able to effectively interpret rates in meaningful ways that guided their approach when operating with proportion related problems. Similarly, both LAM and CA students demonstrated strong rational number understandings and operations that facilitated their approaches.

The key difference between the two groups was the understanding and role of the functional relationship, $y = mx$, in their reasoning and approach. CA students naturally and regularly used the functional relationship to both interpret proportional situations and approach missing value problems, where as LAM students tended to rely on their understanding of rate and interpretation of ratio. Contributing to this difference may be prior learning, understanding, and experience with function relationships. However, this result serves to support the role of the unit rate approach to solving missing value problems as an important focus of developmental mathematics curriculum, teaching, and learning for courses leading to College Algebra because it provides an opportunity for students to operate with function relationships while building proportional reasoning capacities and refining rational number understandings.

All of the results from this portion of the analysis directly apply to research question 3. The results were also used in the retrospective analysis of the HLT following each iteration.

Section Three: Analysis of the Hypothetical Learning Trajectory (HLT)

The HLT was analyzed in meetings between the researcher and the classroom teachers following each teaching episode in each iteration of the teaching experiment. Field notes, the researcher journal, and student work were reviewed to gain insight into which learning goals were accomplished, missed, or needed further development. Lessons were adjusted accordingly. Interview data was incorporated into the retrospective analyses of the HLT.

The analysis and results that appear in this section of the paper directly apply to research question 2. All analyses on the HLT discussed in the following sections are specific to the participant sample consisting of students who gave informed consent for their data to be used in the study. The term “students” refers to the participant sample, not to the entire classes of students in which the intervention was conducted.

First iteration analysis.

Lesson 1: Unit Rates.

The learning goal and focus of Lesson 1 was the development of understanding and interpretation of unit rate (Construct 2). By the close of the lesson, students were able to compute a unit rate once a rate was formed. However, students struggled to interpret unit rates, and often failed to identify which unit rate could be used to solve a missing value problem using a one-step multiplication. Many students struggled to organize two different (reciprocal) rates within a given proportional situation, and could not articulate a rate’s unique contextual interpretation.

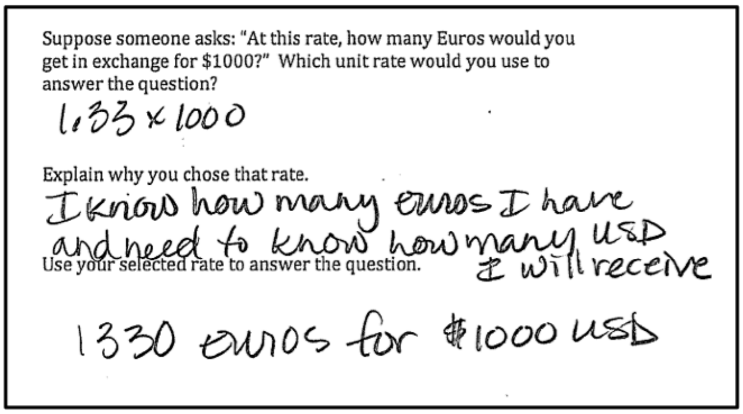
Examples of student difficulties are illustrated in the following discussion. All problems selected as examples of student work come from the lesson’s written homework assignment. The assignment was adapted from proportional reasoning tasks developed by Cramer (2014a) that related to the context presented in Figure 4.9.

While traveling through Europe last summer, Erin could exchange 20 U.S. dollars for 15 Euros.

Figure 4.9. Currency exchange context.

Students were asked to identify two different rates embedded in the context: 20 U. S. Dollars / 15 Euros and the reciprocal rate of 15 Euros per 20 U.S. Dollars. Only half the students were able to identify reciprocal rates. Instead, 21% of students identified a rate and wrote its corresponding unit rate as a second rate, not recognizing that the unit rate was equivalent to the rate they had constructed.

Students struggled to correctly use rates to solve missing value problems on the written homework. The choice of which rate to use was difficult, as was the choice of operation (multiply or divide) to solve the problem. An example of a student using an incorrect unit rate to set up a one-step multiplication problem to solve a missing value problem is given Figure 4.10.


Suppose someone asks: "At this rate, how many Euros would you get in exchange for $1000?" Which unit rate would you use to answer the question?

1.33 × 1000

Explain why you chose that rate.

I know how many euros I have and need to know how many USD I will receive

Use your selected rate to answer the question.

1330 euros for $1000 USD

Figure 4.10. Incorrect use of a unit rate.

In this example, the student uses the rate of 1.33 U.S. Dollar per 1 Euro in a one step multiplication in an attempt to change $1000 to Euros, not recognizing that this rate would be used to change Euros back to dollars in this fashion. Instead, the student could have divided by the rate used, or equivalently, multiplied by the reciprocal rate of 0.75 Euros per 1 U.S. Dollar. The student did not appear to reason with the context nor with the quantitative elements of the problem, as both would have indicated that the number of Euros received for $1000 would be less than the number of dollars. Similar work by other students further demonstrated the importance of an understanding and interpretation of rate in operation with proportion related problems.

Students were instructed to not use the standard algorithm in this lesson. However, the algorithm did appear on the written assignment. One of the most common errors noted in the standard algorithm was the equating of non-equivalent, reciprocal rates in a proportion, as demonstrated in Figure 4.11.

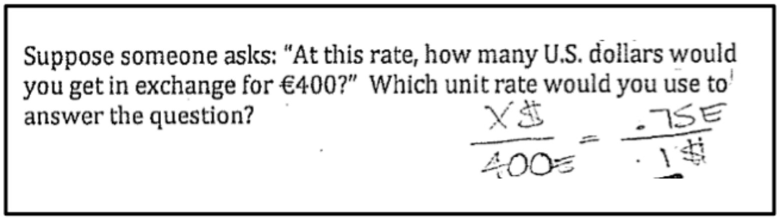


Figure 4.11. Incorrect attempt at setting up a proportion using reciprocal rates.

A new lesson was developed and implemented to build stronger understandings of rate before moving forward in the intervention. The context presented in Figure 4.12 guided all activities in the new lesson.

Sou runs around a track every day after school.

She can run about 12 laps in 15 minutes.

Figure 4.12. Running laps context.

The reciprocal rates of 1.25 minutes per 1 lap, and 0.8 laps per 1 minute served as the focus of the lesson. Students interpreted the rates numerically, contextually and graphically. Students engaged in comparison and qualitative reasoning tasks, solved missing value problems, and wrote one-step multiplication rules that generalized the situation. The adjusted lesson was successful, as was demonstrated by the way students were able to engage in the problem solving tasks. Following the lesson, over 90% of students were able to revise and complete the currency exchange written homework. The adjusted lesson was determined to be a more appropriate starting point for the learning trajectory.

Lesson 2: Patterns and Functions.

The learning goal guiding Lesson 2 was the development of the concept of function in support of the interpretation of proportionality as a linear function (Construct 1). Proportionality was introduced as a specific case of a linear function in which one variable is multiplied by the same constant or scalar to get the value of the other variable, $y = mx$.

The original lesson plan for Lesson 2 included exponential modeling. However, based on the analysis of the first lesson, it was determined that focusing on linear function relationships would better serve the students. Therefore, exponential functions were not included in the lesson.

As was seen in the first lesson, students struggled with the choice of which rate (y/x or x/y) to use as when solving missing value problems using a unit rate approach. However, the pedagogical emphasis on discourse and multiple representations allowed students to compare reciprocal rate choices, and the operations (multiple or divide) both numerically and contextually. Student conversations served to refine and stabilize student understandings of unit rate. Early ideas of inverse functions were developed in student discourse surrounding the reciprocal rates.

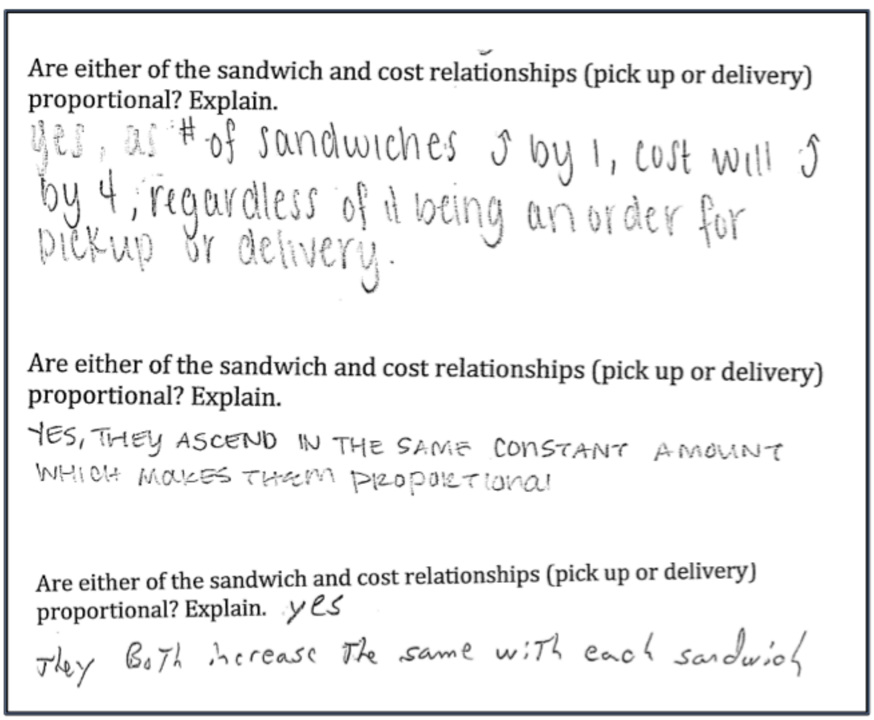
Analysis of student work showed an overgeneralization of constant rate of change (a characteristic of all linear functions of the form $y = mx + b$) to mean proportional. Student written homework centered on the context presented in Figure 4.13.

A sandwich shop charges $4 per sandwich,

plus a flat fee of $5 if the shop delivered the sandwiches.

Figure 4.13. Sandwich shop context.

Students interpreted the cost for pick up, and the cost for delivery contextually, numerically (using tables), symbolically (writing function rules), and graphically. Following a series of tasks and questions about the cost of sandwiches, only 15% of students were able to identify which linear relationship was proportional: $y = 4x$ or $y = 4x + 5$. Figure 4.14 shows evidence of student overgeneralization of the slope as a rate of change in ways that mischaracterize non-proportional situations as proportional.


Are either of the sandwich and cost relationships (pick up or delivery) proportional? Explain.
yes, as # of sandwiches ↑ by 1, cost will ↑ by 4, regardless of it being an order for pickup or delivery.

Are either of the sandwich and cost relationships (pick up or delivery) proportional? Explain.
YES, THEY ASCEND IN THE SAME CONSTANT AMOUNT WHICH MAKES THEM PROPORTIONAL

Are either of the sandwich and cost relationships (pick up or delivery) proportional? Explain. yes
They Both increase The same with each sandwich

Figure 4.14. Overgeneralization of slope as constant of proportionality in a non-proportional context.

The observed overgeneralization of slope as a constant of proportionality motivated a change to the subsequent lesson (Lesson 3) to include analysis of the unit rate in both proportional and non-proportional situations. When the context of the sandwich shop returned on an adjusted homework assignment following Lesson 3, 25% of students (compared to 15% before the adjusted lesson) were able to differentiate between the proportional and non-proportional relationships.

Although many students failed to differentiate between proportional and non-proportional situations following Lesson 2, those that did differentiate between to the two cases demonstrated different understandings of proportionality. Figures 4.15, 4.16, and 4.17 show three samples of student work demonstrating different understandings of proportionality. These samples demonstrated opportunities for classroom discourse in future lessons, and subsequent iterations of the intervention. Asking student to differentiate between proportional and non-proportional situations can foster structured discussions that connect the multiplicative characteristics that define proportionality.

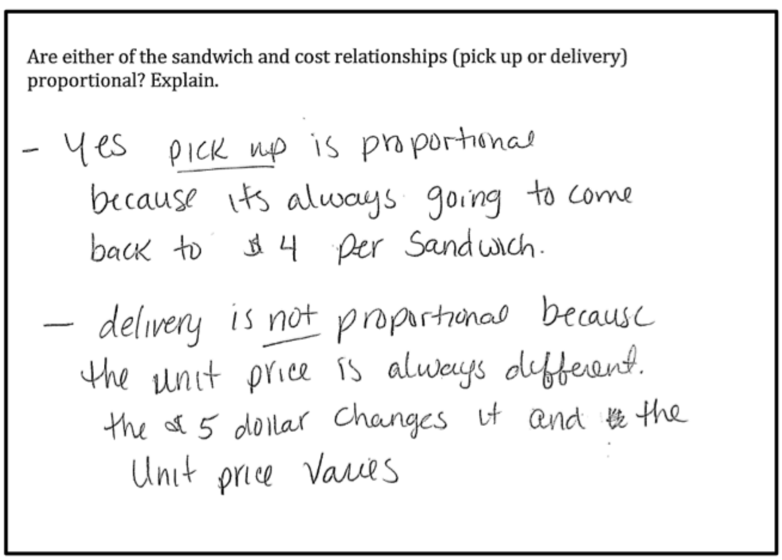


Figure 4.15. Connection of proportionality as a mathematical structure that defines contextual situations and the existence of an invariant unit rate in proportional situations (Construct 5).

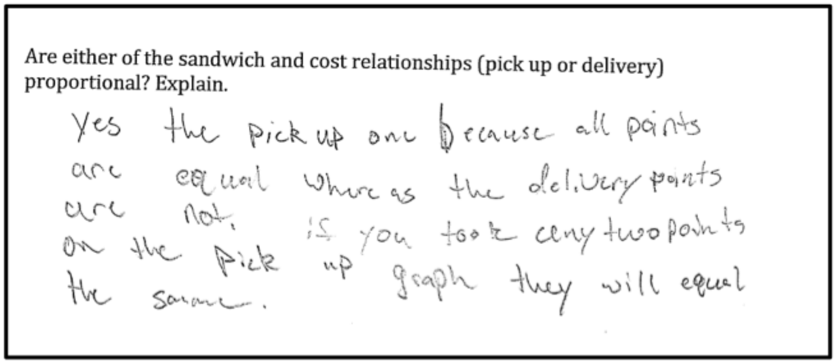


Figure 4.16. Demonstration of connected understanding that all (x,y) rate pairs lie on the line $y = mx$, and all (x,y) rate pairs create an equivalence class (Constructs 1 & 3).

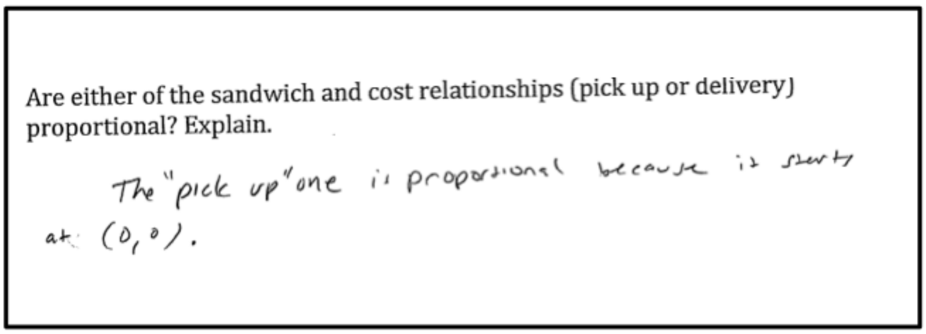


Figure 4.17. Understanding that the graphical representation of a proportional linear relationship passes through the origin (Construct 1).

Lesson 3: Building a Definition of Proportionality.

The learning goal that guided Lesson 3 was the development of connected understandings among the multiplicative constructs that define the mathematical structure of proportionality (Constructs 1, 2, 3, 4, & 5). The unit rate construct played a central role in the lesson. All constructs were related through the ideas of invariant rate and the linear function $y = mx$. Students solved missing value problems of varying contexts and numerical complexity. Students were asked to use multiple approaches, solving each problem at least two ways, but were not to cross multiply and divide.

The cost of gum missing value problem (Cramer, 2014a) presented in Figure 4.18, served as an anchor for the generalization of the mathematical structure of proportionality.

Alex and Peter got equally good buys for bubble gum.
Alex bought 8 pieces for 24¢.

How many pieces did Peter buy for 6¢?

Figure 4.18. Cost of gum missing value problem.

Connecting the contextual interpretation of a unit rate, to the correct operation (multiplication or division) to be used to solve a missing value problem using the rate was challenging for students. Contributing to the challenge was an unstable understanding of the reciprocal relationship between the two unit rates in a proportional situation. It was decided that the second half of Lesson 3 would target these ideas through a structured discussion and direct instruction anchored in student unit rate approaches to solving the missing value problem presented in Figure 4.18.

Following the discussion and direct instruction, students were asked to reason about the mathematical relationship $y = 3x$. Ideas of covariance and invariance were emphasized when working with the context free model. Equivalence of (x,y) rate pairs were emphasized through table, graphical, and proportion equation representations.

At the close of the lesson, students were presented with the context-free proportional relationship $y = (5/2)\ x$ and asked to write three things they knew about the relationship using graphs, words, numbers or equations. Student responses on the task illustrated understandings to three targeted multiplicative constructs. Table 4.18 provides work samples illustrating each construct.

Table 4.17

Sample Student Responses to the Task of Writing Three Things about the relationship y = 5/2 x Organized by Multiplicative Construct

Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).
- The slope is $\frac{5}{2}$ - the unit rate is also $\frac{5}{2}$
In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).
· if $x=1$, $y=2.5$ the reciprocal of $\frac{y}{x}$ or $\frac{2.5}{1}$ is $\frac{x}{y}$ or $\frac{1}{2.5}$ any number you use for x times 2.5 = y. $y \div 2.5 = x$
All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).
$\frac{y}{x} = \frac{5}{2}$ or $\frac{y}{x} = 2.5$ · all x,y rates will be equivalent

Lesson 4: Covariance and Invariance.

Students explored the covariant and invariant relationships within proportional situations in this lesson (Constructs 2, 3, & 4). Students first considered a context that related distance and time through an invariant rate. Attention was brought to how the rate across measure spaces stayed the same when the quantities within measure spaces change by the same scalar factor. Students explored the covariant and invariant multiplicative relationships using numeric, graphical interpretations of the situation. Similarity was used as an application and context for proportionality.

At the close of the lesson, students wrote proportions four ways for a set of two equal rate pairs related to the distance, rate and time context as shown in Figure 4.19.

What are four ways a proportion can be written to represent the rate pairs (6 second, 9 meters) and (18 seconds, 27 meters)?

$$\frac{9\ meters}{6\ seconds} = \frac{27\ meters}{18\ seconds} \qquad \frac{6\ seconds}{9\ meters} = \frac{18\ seconds}{27\ meters}$$

$$\frac{9\ meters}{27\ meters} = \frac{6\ seconds}{18\ seconds} \qquad \frac{27\ meters}{9\ meters} = \frac{18\ seconds}{6\ seconds}$$

Figure 4.19. Four proportions representing equal rate pairs.

The task helped students stabilize their contextual interpretation of ratios and rates because it brought a focus to symbolic and numeric representations of the invariant rate functional relationships between measure spaces and the scalar multiplicative relationships within measure spaces. The task was challenging for students, but the discourse within small groups of students was rich as they struggled with the ideas of rate (i.e. function) and scalar relationships.

Lesson 5: Connecting Proportional Reasoning Strategies.

This lesson reviewed the mathematical characteristics of proportionality, connecting them to approaches to solving proportion related problems. Students were asked to solve the distance, rate and time missing value problem (Cramer, 2014a) presented in Figure 4.20 three different ways and then discuss their approaches in small groups.

Steve and Mark were driving equally fast along a country road.
It took Steve 20 minutes to drive 4 miles.

How long did it take Mark to go 12 miles?

Figure 4.20. Missing value problem to be solved three different ways.

Prior to solving the missing value problem, 93% of students were able to identify two different rates that exist in the proportional situation: 20 minutes / 4 miles, and 4 miles / 20 minutes. Focus was brought to the unit rate throughout the lesson, including the existence of two unit rates in a proportional context that define inverse function relationships between measure spaces: *(# of minutes) = 5 ×(# of miles)* and *(# of miles) = 0.2 × (# of minutes)*. Students discussed how each equation could be used to solve the problem by either a one-step multiplication or a one-step division.

Students were able to successfully solve the problem using multiple approaches, and small group conversations demonstrated that students were connecting their approaches to the multiplicative constructs of invariant unit rate, equivalent rates, the $y = mx$ function relationship across measures, and scalar factors of change within measures. The standard algorithm was introduced and interpreted through connections to the unit rate approach and factor of change approach to solving missing value problems. Figure 4.21 shows samples of student work that demonstrated multiple approaches to solving the missing value problem in Figure 4.20.

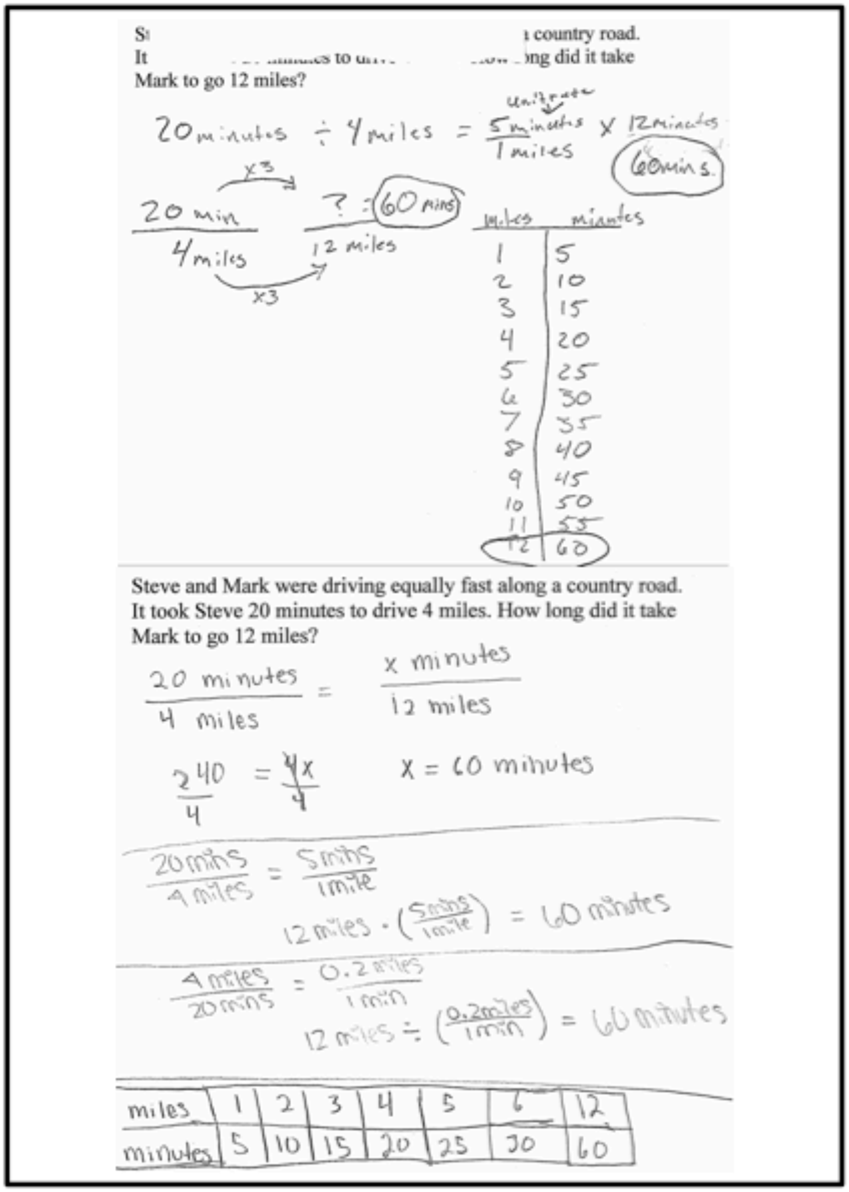


Figure 4.21. Multiple approaches to solving a missing value problem.

At the close of Lesson 5, students were asked to complete a five minute free write describing what they knew about the mathematical structure of proportionality. Most students discussed equivalent rates, and ideas of covariance by a factor of change. Many students were able to discuss these ideas by constructing examples using different contexts including speed, similarity and unit pricing.

Retrospective analysis of the first HLT.

The coordination and interpretation of rate relationships across measure spaces emerged as a central understanding to the development of proportion related understandings and reasoning processes. This understanding of rate appears in conjunction with a shift from additive to multiplicative thinking in proportion related situations. Evidence emerged in the cycles of analysis on the HLT, and in student interviews that this important understanding cannot be assumed present in developmental mathematics students. Therefore, study of proportionality should begin with the development of multiplicative structure and interpretation of rate with developmental mathematics students. It was hypothesized that qualitative reasoning problems, comparison problems may open access to reasoning with rates that will support further development of the unit rate construct.

Analysis of student work and interviews showed graphical representations of proportional relationships to be particularly supportive in the development of rich understandings of invariant rate, and connected understandings among rate, equivalence, and function (Constructs 1 & 2 & 3). Therefore, graphical interpretations of proportion related situations were selected to be enhanced in learning tasks and structured discourse in the second learning trajectory and intervention. Additionally, patterns and functions will no longer be concentrated in one lesson, but instead spread throughout all lessons. Non-proportional situations will be represented using multiple representations of functions, including graphs, and students will be more directly instructed to consider unit rates when differentiating between proportional and non-proportional relationships.

The results presented that guided revisions and refinements to the HLT are also applicable to research question 1d.

Second Hypothetical Learning Trajectory (HLT).

A revised HLT based on the first HLT presented in Figure 3.1 was created based on the analysis presented above. The second HLT, presented in Figure 4.22 and Table 4.19, guided the development and implementation of the second intervention.

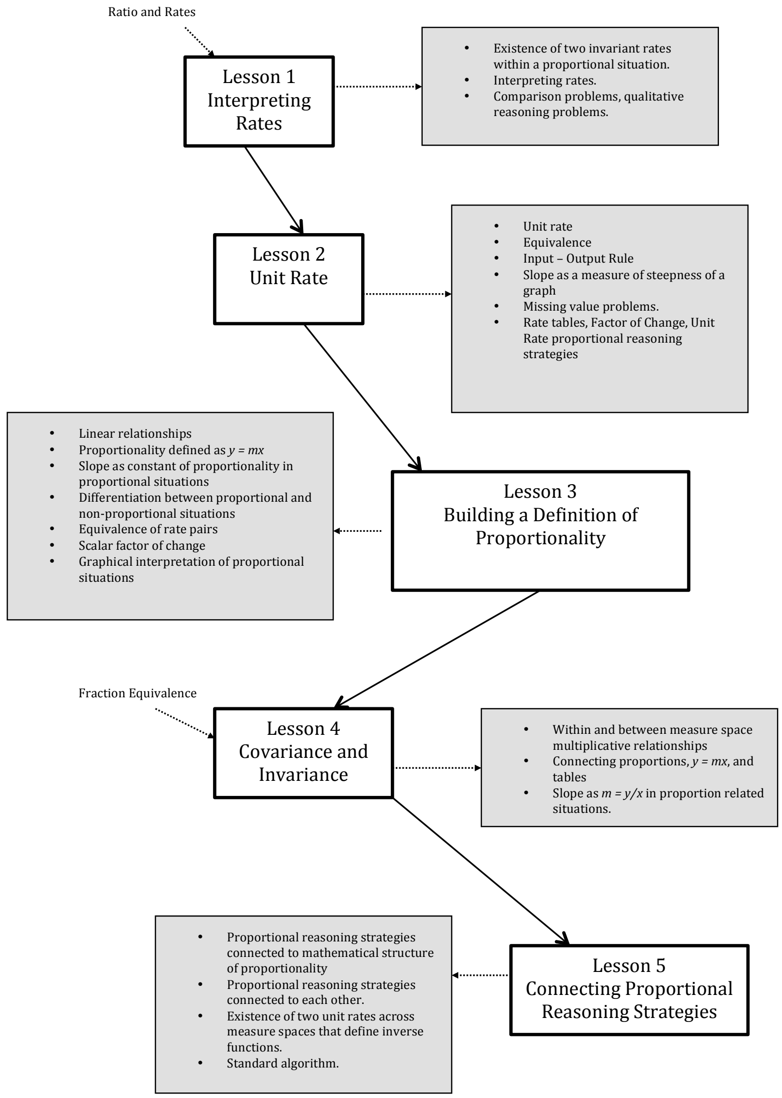


Figure 4.22. Second Hypothetical Learning Trajectory.

Table 4.18

Theoretical Grounding of Second Hypothetical Learning Trajectory

Lesson	Multiplicative Constructs Targeted*	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding
Interpreting Rates	2	- Unit Rate - Rate tables	- Existence of two invariant rates within a proportional situation - Interpretation of rates - Measure space interpretation of proportional situations - Comparison problems - Missing Value Problems - Qualitative Reasoning Problems	- The coordination of rate relationships across measure spaces is foundational to proportional reasoning processes (Post et al., 1988; Karplus, et al., 1984; Lamon, 2007; Lobato, et al., 2009; Vergnaud, et al., 2009) [C]. - The interpretation of unit rate is challenging (Lamon, 2007) [C]. - Students should experience working with a variety of proportion related problem types when building understandings of proportionality (Cramer & Post, 1993) [F&G].
Unit Rate	1, 2	- Unit Rate	- Unit rate - Equivalence - Input – Output Rule - Linear function - Multiple representations of functions - Proportionality defined as $y = mx$ - Slope as constant of proportionality and steepness of graph in proportional situations	- Unit rate is central to the interpretation of rate and proportional reasoning (Cramer et al, 1993; Lamon, 2007; Post et al., 1988; Vergnaud, 1983) [C]. - The interpretation of unit rate is challenging (Lamon, 2007) [C]. - Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983) [B].

Table 4.18 continued.

Lesson	Multiplicative Constructs Targeted	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding*
Building a Definition of Proportionality	1, 5	- Unit Rate - Rate tables - Factor of Change	- Equivalence of rate pairs - Scalar factor of change - Graphical interpretation of proportional situations - Multiple approaches to solving proportion related problems - Definition of proportion - Differentiation between proportional and non-proportional situations	- The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009) [D & H]. - Proportional reasoning requires the differentiation between proportional and non-proportional situations in order to build an understanding of when proportional reasoning should be used (Cramer et al., 1993; Post et al., 1988) [A].
Covariance and Invariance	3, 4	- Unit Rate - Rate tables - Factor of Change	- Within and between measure space multiplicative relationships - Connecting proportions, $y = mx$, and tables - Slope as $m = y/x$ in proportional situations.	- The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988) [E & H].

Table 4.18 continued.

Lesson	Multiplicative Structures Targeted	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding
Connecting Proportional Reasoning Strategies	1, 2, 3, 4, 5	- Unit Rate - Rate tables - Factor of Change - Standard Algorithm	- Proportional reasoning strategies connected to mathematical structure of proportionality - Proportional reasoning strategies connected to each other. - Existence of two unit rates across measure spaces that define inverse functions.	- Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988) [G]. - Research has demonstrated that focus on intuitive strategies applied to multiple representations and contexts of proportional situations can facilitate the development of proportional reasoning (Cramer et al., 1993; Karplus et al., 1983; Lesh et al. 1988) [F&G]. - The standard algorithm can be introduced and connected to more meaningful proportional reasoning approaches (Cramer & Post, 1993) [D & E & F & G].

* Mathematical constructs of proportionality and psychological aspects of proportional reasoning are identified by Code defined in Table 4.5.

Second Iteration Analysis.

Lesson 1: Interpreting Rates.

The first lesson focused on the construction and interpretation of rates. The lesson began with price comparison problem in which students determined the better buy for several dozen tomatoes. The problem successfully generated conversations about rate as a relationship between two quantities, and the interpretation of rate contextually and quantitatively.

A context involving running laps was then introduced (see Figure 4.23). Students worked with the reciprocal rates of 1.25 minutes per 1 lap, and 0.8 laps per 1 minute. Students solved missing value problems using a unit rate approach and discussed the selection of each rate in one-step multiplication rules. These conversations centered on the changing of quantities using contextual and quantitative interpretations of the situation and multiplication rules. Rational number ideas were discussed including their role as a multiplicative operator (Keiren, 1976) as multiplying an input of laps by the rate of 1.25 minutes per 1 lap produced a "bigger" output where multiplying an input of minutes by the rate of 0.8 laps per 1 minute produced a "smaller" output.

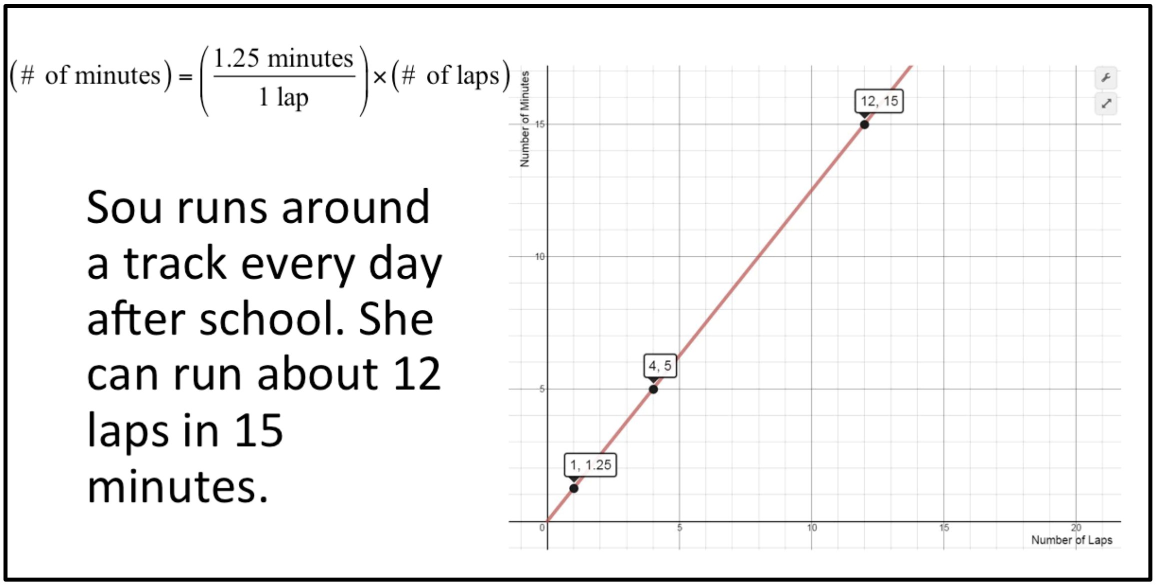


Figure 4.23. Representations of a function defining an input-output relationship between the number of laps and the number of minutes run.

Prior to instruction, the researcher had decided to focus on contextual interpretation of rate in the lesson and structured the conversations around the rates as 1.25 minutes per 1 lap, and 0.8 laps per 1 minute with less focus on the reduced fraction representations of 5/4 minutes per 1 lap and 4/5 laps per 1 minute. Observation of student conversations indicated that this was a missed opportunity because a more balanced approach would have provided opportunity for students to deepen their conversations about the reciprocal relationship between the two rates embedded in the context.

The graphical interpretation of the function rules developed in this part of the lesson, an example of which is presented in Figure 4.23, was particularly meaningful to students as they interpreted Sou's speed as the steepness of the line (slope), and identified the equivalence of all *(x,y)* rate pairs through division $y \div x$. The unit rate was identified through interpretation of the point (1, 1.25) as the number of minutes, 1.25, it took Sou to run 1 lap.

Students then worked with qualitative reasoning tasks involving two different contexts, currency exchange and mixing lemonade. These tasks were developed by Cramer (2014b), and examples of tasks are presented in Figure 4.24. Prior to engaging in these problems, the reciprocal exchange rates of pounds per dollar and dollars per pound were interpreted in a whole class discussion. Students were given less direction surrounding qualitative reasoning problems that involved the strength of a lemonade taste. Most students formed part-part rates: cups of lemonade concentrate per cups of water or the reciprocal rate of cups of water per cups of lemonade concentrate. Following individual work on the qualitative reasoning problems, students sorted each problem into the cases of increase (or stronger), decrease (or weaker), stay the same, or impossible to tell.

The London Bank gave fewer British pounds in exchange for fewer U. S. dollars this week than it did last week. Did the exchange rate, £/$, increase, decrease, stay the same, or can't you tell?
Alice has a recipe for making lemonade. She mixes some lemonade concentrate with some water. She decides to change the recipe, keeping the amount of lemonade concentrate the same and increasing the amount of water. Does the lemonade taste stronger, weaker, the same, or can't you tell?

Figure 4.24. Currency exchange and lemonade qualitative reasoning problems.

Group discussion was rich and meaningful as students began to identify and wrestle with some previously made incorrect generalized rules about rates. One incorrect notion that was observed by both the classroom teacher and the researcher was the idea that the first quantity presented in a rate context must be in the numerator in a rate relationship. For example, in the lemonade context presented in Figure 4.24, some students thought they had to form the rate as cups of lemonade per cups of water because lemonade was presented first in the context. Several student groups wrested with this idea as some students believed it to be a "rule" that "made things work."

The lemonade qualitative reasoning tasks were particularly useful in these discussions as students were able to meaningfully interpret the reciprocal rates (concentrate per water and water per concentrate) and discuss change in the strength of lemonade as the quantities changed using both rates. Instructor prompting opened conversation and reasoning about the reciprocal exchange rate of dollars per pound, but the context was not as easily interpreted. Thus, the two contexts were strengthened by being presented together, as students were able to wrestle with challenging ideas (e.g. as one rate increases, the reciprocal rate decreases) in a familiar context, then immediately refine and formalize ideas using a more challenging context.

Lesson 2: Unit Rate.

The learning goal of Lesson 2 was the further development of understanding and interpretation of unit rate (Construct 2). By the close of the lesson, students were able to interpret unit rates, and select a unit rate to be used to solve a missing value problem using a one-step multiplication. The previous lesson on interpretation of rate served the students well as they were able to efficiently organize two different (reciprocal) rates within a given proportional situation, and they were able to flexibly compare rates by using a variety of strategies. For example, the context presented in Figure 4.25, students flexibly compared the strength of the two lemonade mixtures using the following approaches: computing unit rates, comparing the rates when either the tablespoons of mix were equal (e.g. when 4 tablespoons of mix are used, how many cups of water are used in each recipe?), or the cups of water were equal (e.g. when 35 cups of water are used, how many tablespoons of mix are used for each recipe?).

Lemonade Mix

Below are two recipes for making lemonade. Recipe 1 calls for 2 tablespoons (tbsp.) of mix for each 5 cups of water. Recipe 2 calls for 4 tbsp. of mix for every 7 cups of water. Complete the table to determine the amount of mix and water for the given numbers of batches.

Figure 4.25. Lemonade mix context (Roy, 2002).

Multiple representations of the context, including a table of values, graphs, written ratios and rates, and equations allowed students to discuss the lemonade strength of each recipe using rich mathematical language and discuss detailed aspects of proportional relationships including how quantities (lemonade mix and water) change together through scalar multiplication in ways that keep the invariant rate between the quantities the same.

The currency exchange homework assignment that was given in the first iteration of the intervention (context presented in Figure 4.9) was assigned as the first written homework assignment. Students were asked to identify two different rates embedded in the context: 20 U. S. Dollars per 15 Euros and the reciprocal rate of 15 Euros per 20 U.S. Dollars. Over 90% of students were able to identify reciprocal rates this iteration, compared to half of the students in the first iteration. The majority of students were able to select a unit rate and correctly set up a one step multiplication problem to solve a missing value problem. Therefore, the sequence of Lessons 1 and 2 in which the organization and interpretation of rate was developed prior to the focus on unit rate appeared to better support the learning of the students in the second iteration of the study than the original sequencing in the first learning trajectory.

Lesson 3: Building a Definition of Proportionality.

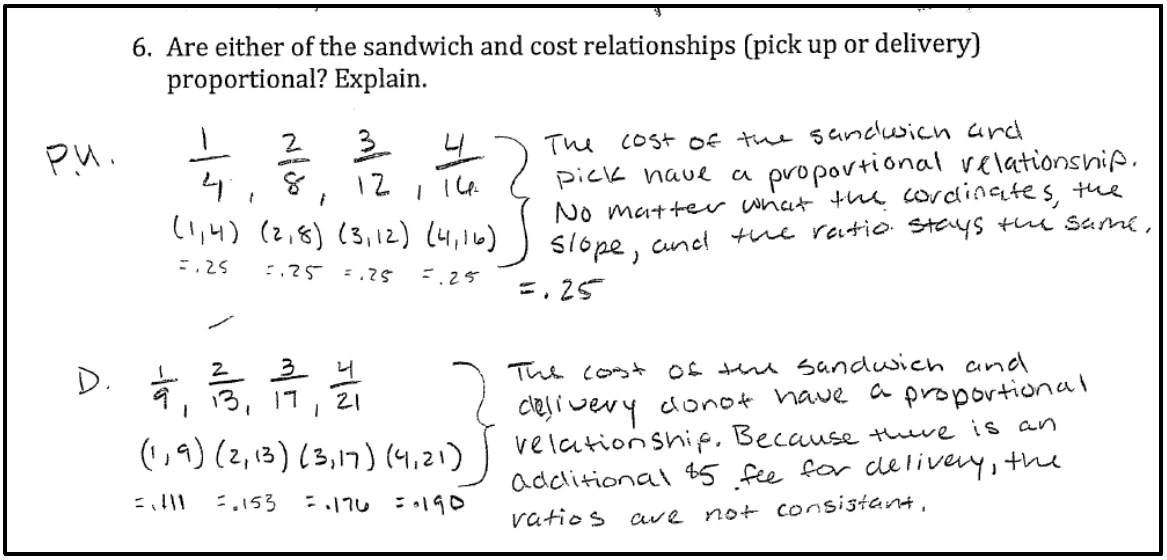
The focus of Lesson 3 was the further development of an understanding invariant rate and the multiplicative relationships within the linear function $y = mx$. Proportionality was defined as a relationship between two variables in which one variable is a constant multiple of the other. Students solved missing value problems of varying contexts and numerical complexity using multiple approaches (e.g. building up, factor of change, unit rate), solving each problem at least two ways, but were instructed to not to cross multiply and divide.

Students readily used unit rate strategies when solving missing value problems. However, it was observed that students were not connecting the inverse function rules to reciprocal rates. Even though students successfully identified two unit rates within a proportional situation, they struggled to construct two different multiplication rules that could be used to solve missing value problems, $y = mx$ and $x = (1/m)y$. Students instead used a single rate, often the first one interpreted, to set up both a multiplication and division rule without consideration of the reciprocal rate that existed within the context.

Although their approach was correct, their difficulty in reasoning with reciprocal rates, and the disconnect between the function rules were noted as places of unstable understanding. This may be the result of the emphasis in the first two lessons on computing a unit rate and utilizing a decimal representation of rate in most examples and discourse. This was noted as an area for improvement in the learning trajectory, and future lessons were adjusted to incorporate more fraction representations of rate.

Following their work with missing value problems, students were asked to reason about the context free mathematical relationship $y = 3x$. Vergnaud's measure space notation (1983) was introduced and used to identify scalar and rate multiplicative relationships in a proportional situation. Equivalence of (x,y) rate pairs were emphasized through table, graphical, and proportion equation representations. Students struggled with writing proportions, and proportions were selected for greater emphasis in the remaining two lessons.

The lesson's homework included Jen's Sandwich shop context similar to the homework given after the functions lesson in the first iteration of the teaching experiment. The final question on the homework asked students to identify if there was a proportional relationship between the number of sandwiches purchased and the total cost of the purchase as shown in Figure 4.13. About half of the students were able to identify the proportional cost structure citing either an invariant unit rate relationship, the relationship $y = mx$, or by making observations about the graphical representation of the linear relationships. An example of student work, identifying an invariant rate relationship while referencing points on a line (Constructs 1 & 2 & 3) is shown in Figure 4.26.


6. Are either of the sandwich and cost relationships (pick up or delivery) proportional? Explain.

P.U. $\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{4}{16}$

(1,4) (2,8) (3,12) (4,16)

=.25 =.25 =.25 =.25

The cost of the sandwich and pick have a proportional relationship. No matter what the cordinates, the slope, and the ratio stays the same, =.25

D. $\frac{1}{9}, \frac{2}{13}, \frac{3}{17}, \frac{4}{21}$

(1,9) (2,13) (3,17) (4,21)

=.111 =.153 =.176 =.190

The cost of the sandwich and delivery donot have a proportional relationship. Because there is an additional $5 fee for delivery, the ratios are not consistant.

Figure 4.26. Identification of an invariant unit rate relationship, referencing points on a line, in the identification of a proportional relationship.

Lesson 4: Covariance and Invariance.

Students explored covariant and invariant relationships within proportional situations (Constructs 2, 3, & 4) in Lesson 4. A context that related distance and time was first considered with a focus on interpretation of invariant rate using tables and graphs. Students solved missing value problems, and discussed ideas of what changes and what stays the same. Using Vergnaud's (1983) measure space notation, the researcher led a direct instruction lesson in which the invariant rates across measures and the scalar rates within measures were identified.

At the close of the lesson, students wrote proportions four ways for a set of two equal rate pairs related to the distance, rate and time context as was shown in Figure 4.19. The task was challenging for students, and most were unable to form proportions that compared equal ratios (e.g. meters to meters and seconds to seconds). The formation of four different proportions was selected as a starting activity and discussion in Lesson 5 based on student performance on the task.

It was observed in the lesson that students were able to readily form two unit rate relationships, but struggled to work with reciprocal rates that were not unit rates. It was decided that time would be spent on the reciprocal relationship between two rates with an emphasis on the quantitative aspects of the rates in Lesson 5.

Lesson 5: Connecting Proportional Reasoning Strategies.

The focus of Lesson was the connections between the mathematical structure of proportionality and approaches used when solving proportion related problems. Students worked with the distance, rate and time context developed by Cramer (2014a), presented in Figure 4.27. Students engaged in multiple tasks including identifying reciprocal rates, writing proportions, and solving a missing value problem.

Driving Speeds

Steve and Mark were driving equally fast along a country road. It took Steve 20 minutes to drive 4 miles.

How long did it take Mark to go 12 miles?

Figure 4.27. Driving speed context.

All students were able to identify two different rates that exist in the proportional situation: 20 minutes / 4 miles, and 4 miles / 20 minutes and represent each using a unit rate. Slightly more than half of the students were able to identify two proportions when asked to write four proportions that describe the relationship within the context. Only two students were able to identify four proportions. Students engaged in small group discourse about the proportions. It was observed that students more easily reasoned about proportions that related rates, than proportions that related scalar ratios. When solving missing value problems throughout the lesson, students rarely formed or reasoned with proportions. Instead, students more often utilized a one-step multiplication or division to solve problems.

Focus was brought to unit rate throughout the lesson, including the existence of two unit rates in a proportional context that define inverse function relationships between measure spaces: *(# of minutes) =5×(# of miles)* and *(# of miles) =0.2×(# of minutes)*. Students were asked to consider the rate of 0.2 miles per minute as the fraction of 1/5 mile per minute to draw focus to the reciprocal relationship between the two rates and the inverse relationship between the function relationships. Students struggled with this idea in small group work, and it was noted as an opportunity for development in the third hypothetical learning trajectory.

Students were able to successfully use multiple approaches to solve the missing value problem presented in Figure 4.27 similar to what was seen in the first iteration of the teaching experiment. Small group discourse was rich as students connected ideas of division to multiplication by a reciprocal rate when using unit rate approaches. The standard algorithm was again introduced and interpreted through connections to the unit rate approach and factor of change approach to solving missing value problems.

At the close of Lesson 5, students were asked to complete a five minute free write describing what they knew about the mathematical structure of proportionality. Most students discussed equivalent rates, and ideas of covariance by a factor of change. Students constructed both tables and graphs and detailed ways that invariant rates can be identified within each.

Retrospective analysis of the second HLT.

The coordination and interpretation of rate relationships across measure spaces again emerged as a central understanding to the development of proportion related understandings and reasoning processes. An early focus on the formation of rate relationships and the contextual interpretations of reciprocal rate relationships appeared to be an appropriate and meaningful place to begin learning trajectory. Qualitative reasoning problems, combined with early graphical representations of proportional situations allowed for rich and meaningful discourse about rate that supported the development of further proportion related understandings.

A missed opportunity in the Second HLT was an early development of the quantitative relationships of reciprocal rates, and their connection to the $y = mx$ and $x = (1/m)\ y$ functional relationships. The decision to focus on contextual interpretation using decimal representations of the rates came at the expense of the refinement of important rational number concepts and operations (e.g. the connection between division and multiplication by a reciprocal), the stability of function ideas, and flexibility of approach to problem solving. The third learning trajectory will detail a more balanced approach to these representations (decimal and fraction) throughout the entire trajectory.

A difference noted in understanding and approach to proportion related problem solving between Liberal Arts Mathematics and College Algebra students was the interpretation of and reasoning with a $y = mx$ functional relationship. Coupled with the challenge students encountered in connecting reciprocal rates to the inverse functional relationships within a proportional situation, the idea of function and inverse function was hypothesized as an area of development for the third learning trajectory.

It was decided that the third learning trajectory should include a sixth lesson in which ideas of function are further developed, with proportionality serving as an exemplar of function. The intent of the lesson is to further stabilize the connected understandings of the $y = mx$ functional relationship to the other mathematical constructs that define proportionality. This was identified as an important area of development for the third learning trajectory so as to better connect proportionality to the broader notions of functions and linear functions in particular. It was hypothesized that additional non-proportional situations, including non-linear situations (e.g. exponential, quadratic) may serve the development of ideas of mathematical relation and function and the $y = mx$ functional relationship. The lesson should be represented using multiple representations of functions, including graphs.

The results presented that guided revisions and refinements to the second HLT are also applicable to research questions 1b and 1d.

Third Hypothetical Learning Trajectory (HLT).

A revised HLT based on the second HLT presented in Figure 4.22 was created based on the analysis presented above. The third HLT is presented in Figure 4.28 and Table 4.20.

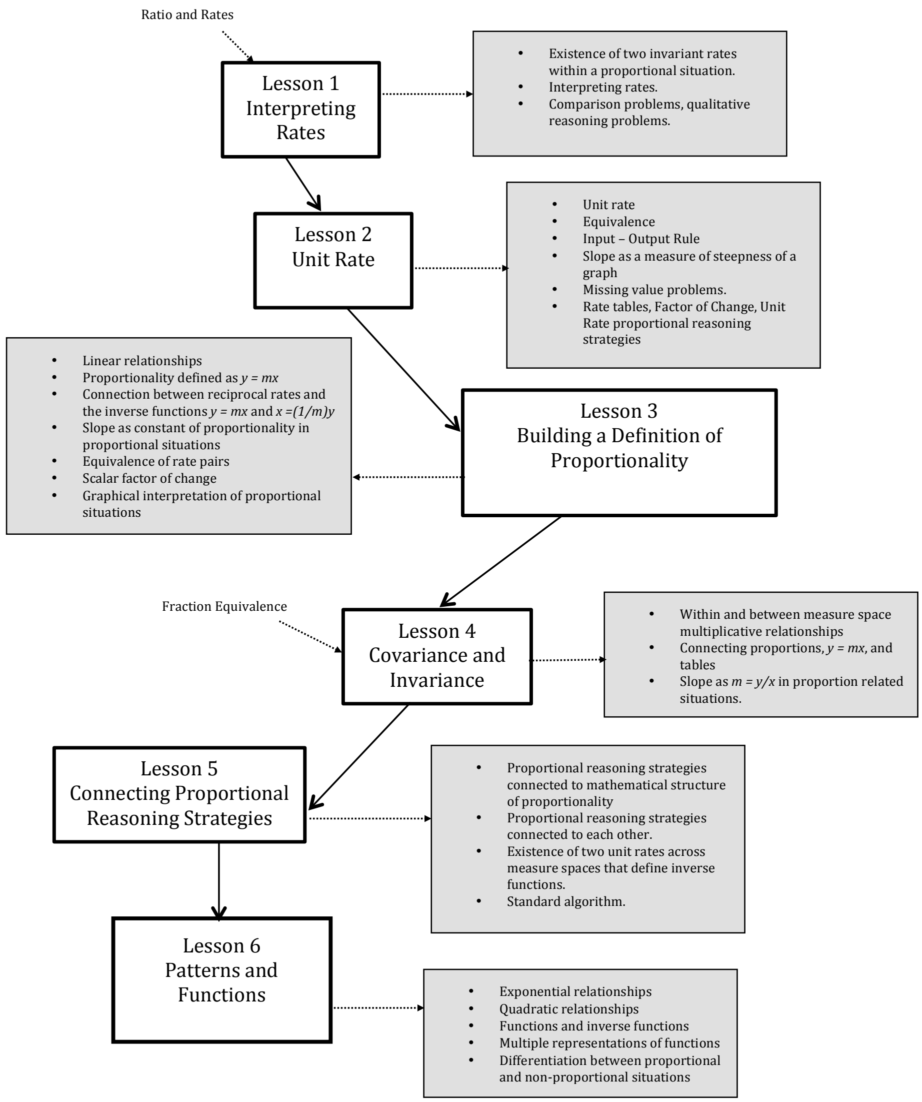


Figure 4.28. Third Hypothetical Learning Trajectory.

Table 4.19

Theoretical Grounding of Third Hypothetical Learning Trajectory

Lesson	Multiplicative Constructs Targeted*	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding
Interpreting Rates	2	- Unit Rate - Rate tables	- Existence of two invariant rates within a proportional situation - Interpretation of rates - Quantitative and qualitative interpretation and reasoning with decimal and fraction representations of rates - Measure space interpretation of proportional situations - Comparison problems - Missing Value Problems - Qualitative Reasoning Problems	- The coordination of rate relationships across measure spaces is foundational to proportional reasoning processes (Post et al., 1988; Karplus, et al., 1984; Lamon, 2007; Lobato, et al., 2009; Vergnaud, et al., 2009) [C]. - The interpretation of unit rate is challenging (Lamon, 2007) [C]. - Students should experience working with a variety of proportion related problem types when building understandings of proportionality (Cramer & Post, 1993) [F&G].
Unit Rate	1, 2	- Unit Rate	- Unit rates interpreted using both decimal and fraction representations - Equivalence - Input – Output Rule - Notions of doing and undoing through multiplication and division used in input-output rules - Linear function - Multiple representations of functions - Proportionality defined as $y = mx$ - Slope as constant of proportionality and steepness of graph in proportional situations	- Unit rate is central to the interpretation of rate and proportional reasoning (Cramer et al, 1993; Lamon, 2007; Post et al., 1988; Vergnaud, 1983) [C]. - The interpretation of unit rate is challenging (Lamon, 2007) [C]. - Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983) [B].

Table 4.19 continued.

Lesson	Multiplicative Constructs Targeted	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding*
Building a Definition of Proportionality	1, 5	- Unit Rate - Rate tables - Factor of Change	- Equivalence of rate pairs - Scalar factor of change - Graphical interpretation of proportional situations - Multiple approaches to solving proportion related problems - Definition of proportion - Connection between reciprocal rates and the inverse functions $y = mx$ and $x = (1/m)y$	- The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009) [D & H]. - Proportional reasoning requires the differentiation between proportional and non-proportional situations in order to build an understanding of when proportional reasoning should be used (Cramer et al., 1993; Post et al., 1988) [A].
Covariance and Invariance	3, 4	- Unit Rate - Rate tables - Factor of Change	- Within and between measure space multiplicative relationships - Connecting proportions, $y = mx$, and tables - Slope as $m = y/x$ in proportional situations.	- The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988) [E & H].

Table 4.19 continued.

Lesson	Multiplicative Structures Targeted	Problem Solving Approaches Addressed	Understandings and Tasks Introduced	Theoretical Grounding
Connecting Proportional Reasoning Strategies	1, 2, 3, 4, 5	- Unit Rate - Rate tables - Factor of Change - Standard Algorithm	- Proportional reasoning strategies connected to mathematical structure of proportionality - Proportional reasoning strategies connected to each other. - Existence of two unit rates across measure spaces that define inverse functions.	- Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988) [G]. - Research has demonstrated that focus on intuitive strategies applied to multiple representations and contexts of proportional situations can facilitate the development of proportional reasoning (Cramer et al., 1993; Karplus et al., 1983; Lesh et al. 1988) [F&G]. - The standard algorithm can be introduced and connected to more meaningful proportional reasoning approaches (Cramer & Post, 1993) [D & E & F & G].
Patterns and Functions	1, 2, 5	- Unit Rate	- Differentiation between proportional and non-proportional situations - Functions and inverse functions - Multiple representations of functions - Exponential relationships - Quadratic relationships	- Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983) [B]. - Proportional reasoning requires the differentiation between proportional and non-proportional situations (Cramer et al., 1993; Post et al., 1988) [A].

* Mathematical constructs of proportionality and psychological aspects of proportional reasoning are identified by Code defined in Table 4.5.

Section 4: Results Summarized by Research Question

Research Question 1. What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?

a. What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?

The data from this study showed three key understandings that support the differentiation between proportional and non-proportional situations for all students. These understandings are invariant unit rate (Construct 2), equivalence of rate pairs (Construct 3), and the interpretation of the $y = mx$ function (Construct 1). The connection between invariant unit rate and equivalence of rate pairs marked an important shift in thinking that enabled students to interpret proportional situations and reason proportionally using multiplicative approaches. The connection was present in both high and low performing college level mathematics students, and was developed across all levels of developmental mathematics students in the intervention.

The interpretation of a proportion related context through the $y = mx$ functional relationship (Construct 1) supported the differentiation between proportional and non-proportional situations by College Algebra students. This interpretation also emerged as central to this ability in developmental mathematics students through the intervention. Liberal Arts Mathematics students did not naturally reason using the $y = mx$ functional relationship and instead often relied on their strong rational number reasoning ability to recognize equal rates or identify scalar factors of change as characteristics of proportionality. Therefore, the $y = mx$ functional relationship may be an important focal area of study in developmental mathematics contexts so as to better support students for future algebraic study and STEM pathways that require a college algebra and beyond.

b. What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?

A strong understanding of unit rate (Construct 2) was found to be central to proportional reasoning with the invariant relationship between measure spaces for all students. Interpretation of a unit rate provides both a quantitative and contextual anchor for the rate's extension to other equal multiples of *x* and *y* using multiplicative approaches that are not limited by partitioning and iteration.

Samples of student work from and student interview data showed that students may identify and understand early notions of equivalence relationships (Construct 3) prior to their identification and interpretation of rate relationships. This was identified in the data from this study when low performing students first interpreted proportional relationships through the use of *same as* reasoning in which two different measures were related through equality (e.g. $4.50 *is* 3 pounds) without the consideration of rate relationships (e.g. $4.50 *per* 3 pounds). *Same as* reasoning was identified to be recognition of equivalence independent from consideration of rate relationships. A *same as* interpretation facilitated composed-unit comparisons, building up approaches through decomposition and additive iteration, and the use of scalar multiplicative approaches when easily recognized integer factors of change were present in a problem. However, until students coordinated a multiplicative rate relationship between two measures (Construct 2), they did not shift towards multiplicative interpretations and approaches when working with proportional situations. Moreover, the *same as* interpretation supported incorrect use of additive approaches. The key understanding that must emerge to support the extension of an invariant rate relationship between two variables x and y, to other equal multiples of x and y is the understanding of rate through interpretation of unit rate (Understanding 2). When stabilized through a connected to an understanding of equivalence (Construct 3), the understanding supports multiple approaches to the extension of the rate relationship to other equal multiples of x and y.

This result is supported by what is known about the ways that children transition to become proportional reasoners. Lobato et al. (2010) identified several important shifts in understanding and approach that children typically make in this development. First, students shift from reasoning with one quantity to coordinating two quantities. Second, students shift from additive to multiplicative comparisons. Next, students regularly move through composed unit strategies that may be additive or multiplicative to multiplicative operations that can create infinite equivalent ratios using any real number as a ratio multiplier.

c. ***What understandings of proportionality as a multiplicative structure facilitate flexible and successful approaches to problem solving situations that are proportional in nature?***

A strong understanding and interpretation of unit rate relationships supported flexible and successful problem solving approaches across measure spaces and equivalence of rate pairs (Constructs 2 and 3) in all students. Strong proportional reasoning was demonstrated in the approaches of students who were able to form and interpret reciprocal rates and reason about their relationships both quantitatively and

contextually. A connected understanding of the functional relationships across measure spaces and the scalar multiplicative relationships within measure spaces (Understandings 2 and 4) strengthened both unit rate and factor of change approaches to problem solving. However, as this connected understanding emerged, students tended to more regularly use unit rate approaches. Reasoning with the functional relationship
$y = mx$ (Construct 1) supported the development of the understandings of the invariant and covariant relationships within proportional situations and opened rich approaches towards problem solving that involved flexible translation among different modes of representation of the function.

d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understandings and reasoning processes?

The understanding of rate, and the ability to form and interpret rate relationships (Construct 2) were identified as foundational to proportion related understandings and reasoning processes in all students. This understanding marked an important transition between additive and multiplicative thinking and also demonstrated prerequisite rational number knowledge that is central to the understanding of proportionality. The understanding of invariant rate, and interpretation of unit rate were foundational to the most powerful connected understandings made.

When the understanding of rate was connected to equivalence (Construct 3), students are able to differentiate between proportional and non-proportional tasks and reason flexibly. Therefore, the connected understanding of rate and equivalence

(Constructs 2 and 3) is central to proportion related understandings and reasoning processes and serves as an important transitional understanding in their development.

Student interview data and student work samples from the intervention show that recognition and understanding of equivalence may appear prior to the regular formation of rates in developing proportional reasoners. Developmental mathematics students often used building up approaches involving additive reasoning, and early multiplicative approaches involving scaling with integer scalar factors of change (Construct 4) when solving missing value problems. Once the notion of invariant rate was strongly connected to equivalence, developmental students shifted their approach toward generalizable unit rate approaches, and were able to flexibly transition and blend approaches to solve problems. The use of scalar factors of change appeared to be most powerful when connected to an understanding of invariant unit rate (i.e. something is staying the same).

College Algebra students demonstrated connected understandings between the function relationship, $y = mx$, and the proportion related contexts (Constructs 1 and 5). These connections were also connected to their interpretation of unit rate (Construct 2). These understandings and connections were successfully constructed by developmental students in the teaching experiment. By the conclusion of the experiment, developmental students could articulate their thinking both contextually and quantitatively using multiple representations of the function relationship (graphs, tables, equations), and tapped into their understandings of the function relationship to help them solve problems and flexibly transition through multiple approaches. Therefore, this research supports prior research finding that students naturally use unit rate approaches and can build understandings of function through work with proportion related contexts (Cramer & Post, 2003; Karplus et al., 1983; Post et al., 1988).

Research Question 2. How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?

The Third Hypothetical Learning Trajectory (HLT) shown in Figure 4.28 and described in Table 4.19 was developed through this study as a meaningful way for developmental mathematics students to construct proportion related understandings and reasoning processes. Some of the key elements of HLT are an initial focus on the interpretation of rate and unit rate (Construct 2), the development of unit rate problem solving strategies that involve multiple representations of the $y = mx$ functional relationship (Construct 1), and the inclusion of multiple contexts and space for the development of multiple approaches in student led problem solving and discourse (Construct 5). The interpretation of proportional situations through the $y = mx$ functional relationship served to stabilize notions of invariant rate, equivalence, and flexible operation with the scalar within measure space multiplicative relationships and the functional across measure space multiplicative relationships.

It was identified in the first iteration of the design experiment that developmental mathematics students did not have stable understandings of ratio and rate prior to their work with proportionality. Early work with qualitative reasoning problems helped students develop these understandings and served as a stronger starting point in the second iteration of the study. The interpretation of proportional situations through the *y* = *mx* functional relationship served to stabilize notions of invariant rate, equivalence, and flexible operation with the scalar within measure space multiplicative relationships and the functional across measure space multiplicative relationships in both iterations of the study. Understandings of proportionality and function that were developed in the second iteration that was guided by the second HLT supported the inclusion of a third lesson that presents non-linear mathematical relationships and functions that may further stabilize student understandings and position the HLT as a point of access for more focal areas of study commonly found in developmental algebra courses.

Research Question 3. What differences, if any, exist between developmental mathematics student and college level mathematics student proportion related understandings and reasoning processes?

The data in the study suggest differences exist between developmental and college level mathematics student proportion related understandings and reasoning processes. Quantitative analysis of the written assessment data showed that there are likely statistically significant differences among group performance on a written assessment targeting proportionality with college level math students preforming higher than developmental math students $F(3,87) = 6.28$, $p < .001$. The data suggest there is a student group effect on the pretest scores. The effect size was calculated to provide an estimate of the relative size of the differences attributable to the student groups. There was found to be a medium effect size, $\eta^2 = .18$, with 95% CI [0.04, 0.29] among the groups (Cohen, 1988).

Interview data showed that college level mathematics students had stronger understandings of unit rate (Construct 2), and stronger connections between the invariant unit rate relationship and equivalence (Constructs 2 and 3) than developmental mathematics students prior to the instruction. These results may be indicative of stronger rational number thinking and knowledge, and understandings of the ratio sub-construct (Kieren, 1976) in particular. College Algebra students had a strong understanding of the $y = mx$ functional relationships (Construct 1) between measure spaces, and regularly used this structure to interpret and operate with proportional situations. This may be the result of stronger notions of function, and linear functions in particular.

Developmental mathematics students demonstrated more connections in their understandings of the multiplicative constructs that define proportionality at the close of the intervention than college level mathematics students. Of particular note was the way that developmental mathematics students interpreted and organized proportion related contexts and problems through different modes of representation of the $y = mx$ function relationship.

Notably, the standard algorithm of cross-multiply and divide did not appear in a conceptually meaningful way in either the developmental or college level mathematics student interviews. When the algorithm did appear, it was treated algorithmically and disconnected from the context of the problem at hand. When it appeared in college level mathematics student work, it did so with the explanation that "it always works" and was anchored in a strong understanding of equal rates. When it appeared in a developmental mathematics interview, it served as an algorithm that circumvented the reasoning about rates and the interpretation of unit rate.

These results support prior research that has demonstrated the importance of multiple representations of function to the development of student understanding (Lesh et al., 1987), and the position of proportionality as a "watershed topic" that supports future algebraic thinking and learning (Post et al., 1988).

Chapter 5: Discussion, Implications, and Future Research Directions

Proportionality and proportional reasoning play pivotal roles in the foundation of algebra and higher-level mathematics and mathematics related study. The development of these understandings and reasoning processes is both mathematically and psychologically complex. Although there has been much work surrounding the ways children come to understand proportionality and reason proportionally (e.g. Lamon, 2007; Lesh, et al., 1987; Lobato et al. 2010; Post et al., 1988), there is a need for research into the ways that the ways these concepts and reasoning processes emerge in older students and adults (e.g. Lamon, 2007; Mesa, Wladis, & Watkins, 2014; NCEE, 2013; Sitomer et al., 2012).

The purpose of this study was to analyze the evolving understandings and reasoning processes that community college students bring to proportion related problem solving situations, and the teaching and learning activities that support their construction. The study increased what is known about the connections between proportion related understandings and reasoning processes and concludes with a Hypothetical Learning Trajectory (HLT) describing teaching and learning activities that support this growth in developmental mathematics students.

Proportionality is a mathematical structure that models the relationship within contextual situations in which two quantities, x and y, change together in ways that the rate between the quantities stays the same, such as speed or density. Proportional reasoning involves the psychological underpinnings that facilitate the interpretation, sense making, and operational flexibility necessary for working with proportion related situations. The key to proportional reasoning is the ability to discern and operate with a multiplicative relationship between a pair of quantities and extend the same relationship to other pairs of quantities (Lamon, 2007).

Table 5.1 describes five core multiplicative structures that define proportionality and eight psychological aspects of proportional reasoning. The model of connected understandings developed in the literature review and presented in Figure 2.4, and the hypothesized connections between these understandings and the psychological aspects of proportional reasoning as presented in Table 5.1 served as the theoretical framework that guided the intervention and this inquiry. The codes listed in the table were used in the development of the study and in the analysis of the data. Psychological Aspects B, D, and G were targeted in this study.

Table 5.1

The Mathematical Structure of Proportionality and the Psychological Aspects of Proportional Reasoning, and Codes Used to Represent the Mathematical Constructs and Psychological Aspects

Code	Multiplicative Constructs that Define the Mathematical Structure of Proportionality	Psychological Aspects of Proportional Reasoning	Code
1	Proportionality is a linear relationship between two quantities that covary according to the model $y = mx$, where m is the unit rate. All corresponding (x,y) rate pairs lie on the graph of the line $y = mx$, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	A
		Proportional reasoning involves the recognition and use of a function relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	B
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: $y = mx$ and $x = (1/m)\ y$ (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	C
3	All (x,y) rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	D
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	E
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	F
		Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	G
		Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	H

This study employed design experiment methodology that included a two-week teaching experiment (Cobb et al., 2003; Cobb & Steffe, 1983/2011; Gravemeijer & van Eerde, 2009). A design experiment focuses on the co-development of theory surrounding domain-specific learning processes, and aspects of teaching and learning that support the targeted processes (Cobb et al., 2003, p. 10). The design study was structured as a teaching experiment and iterated across two semesters with two different groups of developmental mathematics students: a Mathematical Reasoning class, and an Introductory Algebra class. A Liberal Arts Mathematics class and a College Algebra class were included in the study to gather insight into the differences in understandings and reasoning between developmental and college level mathematics students.

The teaching experiment structure enabled the exploration of student understandings and reasoning processes within authentic classroom contexts, and, importantly, a close examination of the actual models of proportionality used by students as they grew in their understandings. The teaching experiment was guided by a HLT that was refined and revised through cycles of analysis during and retrospective analysis following each iteration of the study. The teaching and learning experiences in the classroom intervention portion of the study provided students with opportunities to operate with a variety of proportion related situations using intuitive strategies before more procedural approaches were introduced (Cramer & Post, 1993).

This chapter begins with a summary of the major findings of the study, organized by research question. A discussion of some of the implications of the study on teaching and learning is then presented. Next, the limitations of the study are described. The chapter concludes with a discussion of the position of the study, followed by a presentation of potential areas and questions for future research.

Major Findings Organized by Research Question

In the following discussion, mathematical constructs of proportionality by code presented in Table 5.1.

Research Question 1. What understandings of proportionality as a multiplicative structure are central to the development of flexible and robust proportional reasoning processes?

a. ***What understandings of proportionality as a multiplicative structure enable the differentiation between proportional and non-proportional situations?***

The data from this study showed three key understandings that support the differentiation between proportional and non-proportional situations for all students. These understandings are invariant unit rate (Construct 2), equivalence of rate pairs (Construct 3), and the interpretation of the $y = mx$ function (Construct 1). The connection between invariant unit rate and equivalence of rate pairs marked an important shift in thinking that enabled students to interpret proportional situations and reason proportionally using multiplicative approaches. The connection was present in both high and low performing college level mathematics students, and was developed across all levels of developmental mathematics students in the intervention.

The interpretation of a proportion related context through the $y = mx$ functional relationship (Construct 1) supported the differentiation between proportional and non-proportional situations by College Algebra students. This interpretation also emerged as central to this ability in developmental mathematics students through the intervention. Liberal Arts Mathematics students did not naturally reason using the $y = mx$ functional relationship and instead often relied on their strong rational number reasoning ability to recognize equal rates or identify scalar factors of change as characteristics of proportionality. Therefore, the $y = mx$ functional relationship may be an important focal area of study in developmental mathematics contexts so as to better support students for future algebraic study and STEM pathways that require a college algebra and beyond.

b. *What understandings of proportionality as a multiplicative structure support proportional reasoning with an invariant relationship between two variables, x and y, and its extension to other equal multiples of x and y?*

A strong understanding of unit rate (Construct 2) was found to be central to proportional reasoning with the invariant relationship between measure spaces for all students. Interpretation of a unit rate provides both a quantitative and contextual anchor for the rate's extension to other equal multiples of x and y using multiplicative approaches that are not limited by partitioning and iteration.

Samples of student work from and student interview data showed that students may identify and understand early notions of equivalence relationships (Construct 3) prior to their identification and interpretation of rate relationships. This was identified in the data from this study when low performing students first interpreted proportional relationships through the use of *same as* reasoning in which two different measures were related through equality (e.g. $4.50 *is* 3 pounds) without the consideration of rate relationships (e.g. $4.50 *per* 3 pounds). *Same as* reasoning was identified to be recognition of equivalence independent from consideration of rate relationships. A *same as* interpretation facilitated composed-unit comparisons, building up approaches through decomposition and additive iteration, and the use of scalar multiplicative approaches when easily recognized integer factors of change were present in a problem. However, until students coordinated a multiplicative rate relationship between two measures (Construct 2), they did not shift towards multiplicative interpretations and approaches when working with proportional situations. Moreover, the *same as* interpretation supported incorrect use of additive approaches. The key understanding that must emerge to support the extension of an invariant rate relationship between two variables *x* and *y*, to other equal multiples of *x* and *y* is the understanding of rate through interpretation of unit rate (Understanding 2). When stabilized through a connected to an understanding of equivalence (Construct 3), the understanding supports multiple approaches to the extension of the rate relationship to other equal multiples of *x* and *y*.

This result is supported by what is known about the ways that children transition to become proportional reasoners. Lobato et al. (2010) identified several important shifts in understanding and approach that children typically make in this development. First, students shift from reasoning with one quantity to coordinating two quantities. Second, students shift from additive to multiplicative comparisons. Next, students regularly move through composed unit strategies that may be additive or multiplicative to multiplicative operations that can create infinite equivalent ratios using any real number as a ratio multiplier.

c. What understandings of proportionality as a multiplicative structure facilitate flexible and successful approaches to problem solving situations that are proportional in nature?

A strong understanding and interpretation of unit rate relationships supported flexible and successful problem solving approaches across measure spaces and equivalence of rate pairs (Constructs 2 and 3) in all students. Strong proportional reasoning was demonstrated in the approaches of students who were able to form and interpret reciprocal rates and reason about their relationships both quantitatively and contextually. A connected understanding of the functional relationships across measure spaces and the scalar multiplicative relationships within measure spaces (Understandings 2 and 4) strengthened both unit rate and factor of change approaches to problem solving. However, as this connected understanding emerged, students tended to more regularly use unit rate approaches. Reasoning with the functional relationship $y = mx$ (Construct 1) supported the development of the understandings of the invariant and covariant relationships within proportional situations and opened rich approaches towards problem solving that involved flexible translation among different modes of representation of the function.

d. Are there specific connections within and between the multiplicative constructs that characterize the mathematical structure of proportionality that serve as important transitions in the development of proportion related understandings and reasoning processes?

The understanding of rate, and the ability to form and interpret rate relationships (Construct 2) were identified as foundational to proportion related understandings and reasoning processes in all students. This understanding marked an important transition between additive and multiplicative thinking and also demonstrated prerequisite rational number knowledge that is central to the understanding of proportionality. The understanding of invariant rate, and interpretation of unit rate were foundational to the most powerful connected understandings made.

When the understanding of rate was connected to equivalence (Construct 3), students are able to differentiate between proportional and non-proportional tasks and reason flexibly. Therefore, the connected understanding of rate and equivalence

(Constructs 2 and 3) is central to proportion related understandings and reasoning processes and serves as an important transitional understanding in their development.

Student interview data and student work samples from the intervention show that recognition and understanding of equivalence may appear prior to the regular formation of rates in developing proportional reasoners. Developmental mathematics students often used building up approaches involving additive reasoning, and early multiplicative approaches involving scaling with integer scalar factors of change (Construct 4) when solving missing value problems. Once the notion of invariant rate was strongly connected to equivalence, developmental students shifted their approach toward generalizable unit rate approaches, and were able to flexibly transition and blend approaches to solve problems. The use of scalar factors of change appeared to be most powerful when connected to an understanding of invariant unit rate (i.e. something is staying the same).

College Algebra students demonstrated connected understandings between the function relationship, $y = mx$, and the proportion related contexts (Constructs 1 and 5). These connections were also connected to their interpretation of unit rate (Construct 2). These understandings and connections were successfully constructed by developmental students in the teaching experiment. By the conclusion of the experiment, developmental students could articulate their thinking both contextually and quantitatively using multiple representations of the function relationship (graphs, tables, equations), and tapped into their understandings of the function relationship to help them solve problems and flexibly transition through multiple approaches. Therefore, this research supports prior research finding that students naturally use unit rate approaches and can build understandings of function through work with proportion related contexts (Cramer & Post, 2003; Karplus et al., 1983; Post et al., 1988).

Research Question 2. How can teaching and learning activities be structured in ways that support the emergence of connected understandings of proportionality and proportional reasoning processes in developmental mathematics students?

The Hypothetical Learning Trajectory (HLT) shown in Figure 5.1 was developed and refined through this study as a meaningful way for developmental mathematics students to construct proportion related understandings and reasoning processes. The third HLT represents the revisions and refinement across both iterations of the experiment, and is presented in greater detail in Figure 5.2. Some of the key elements of the third HLT are the initial focus on the interpretation of rate and unit rate (Construct 2), the development of unit rate problem solving strategies that involve multiple representations of the $y = mx$ functional relationship (Construct 1), and the inclusion of multiple contexts and space for the development of multiple approaches in student led problem solving and discourse (Construct 5). The interpretation of proportional situations through the $y = mx$ functional relationship served to stabilize notions of invariant rate, equivalence, and flexible operation with the scalar within measure space multiplicative relationships and the functional across measure space multiplicative relationships.

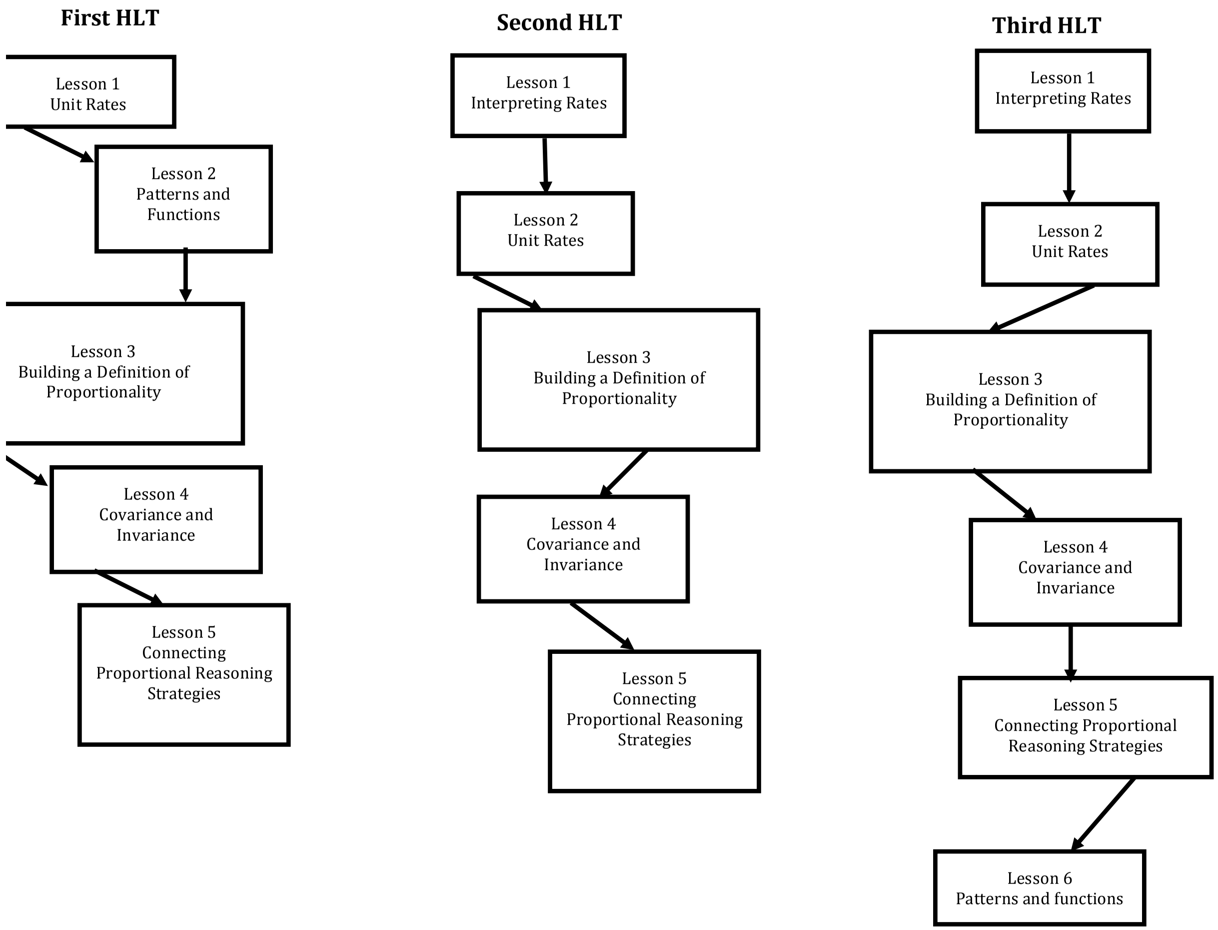


Figure 5.1. First, second, and third versions of Hypothetical Learning Trajectory (HLT)

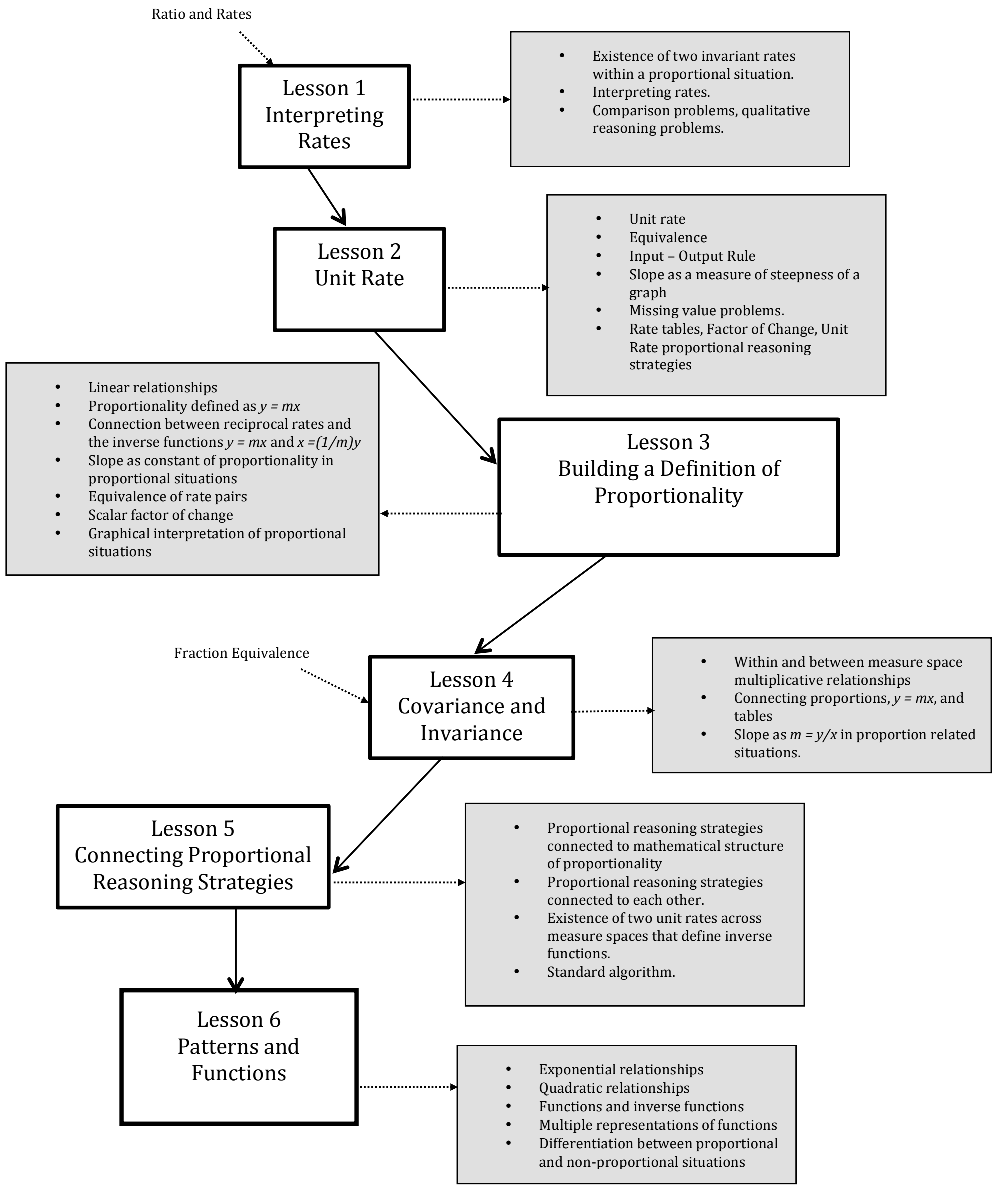


Figure 5.2. Third Hypothetical Learning Trajectory.

It was identified in the first iteration of the design experiment that developmental mathematics students did not have stable understandings of ratio and rate prior to their work with proportionality. Early work with qualitative reasoning problems helped students develop these understandings and served as a stronger starting point in the second iteration of the study. The interpretation of proportional situations through the $y = mx$ functional relationship served to stabilize notions of invariant rate, equivalence, and flexible operation with the scalar within measure space multiplicative relationships and the functional across measure space multiplicative relationships in both iterations of the study. Understandings of proportionality and function that were developed in the second iteration that was guided by the second HLT supported the inclusion of a third lesson that presents non-linear mathematical relationships and functions that may further stabilize student understandings and position the HLT as a point of access for more focal areas of study commonly found in developmental algebra courses.

Research Question 3. What differences, if any, exist between developmental mathematics student and college level mathematics student proportion related understandings and reasoning processes?

The data in the study suggest that differences exist between developmental and college level mathematics student proportion related understandings and reasoning processes. Quantitative analysis of the written assessment data showed that there are statistically significant differences among group performance on a written assessment targeting proportionality with college level math students preforming higher than developmental math students $F(3,87) = 6.28$, $p < .001$. The data suggest there is a student group effect on the pretest scores. The effect size was calculated to provide an estimate of the relative size of the differences attributable to the student groups. There was found to be a medium effect size, $\eta^2 = .18$, with 95% CI [0.04, 0.29] among the groups (Cohen, 1988).

Interview data showed that college level mathematics students had stronger understandings of unit rate (Construct 2), and stronger connections between the invariant unit rate relationship and equivalence (Constructs 2 and 3) than developmental mathematics students prior to the instruction. These results may be indicative of stronger rational number thinking and knowledge, and understandings of the ratio sub-construct (Kieren, 1976) in particular. College Algebra students had a strong understanding of the $y = mx$ functional relationships (Construct 1) between measure spaces, and regularly used this structure to interpret and operate with proportional situations. This may be the result of stronger notions of function, and linear functions in particular.

Developmental mathematics students demonstrated more connections in their understandings of the multiplicative constructs that define proportionality at the close of the intervention than college level mathematics students. Of particular note was the way that developmental mathematics students interpreted and organized proportion related contexts and problems through different modes of representation of the $y = mx$ function relationship. These results support prior research that has demonstrated the importance of multiple representations of function to the development of student understanding (Lesh et al., 1987; Dienes, 1960), and the position of proportionality as a “watershed topic” that supports future algebraic thinking and learning (Post et al., 1988).

Notably, the standard algorithm of cross-multiply and divide did not appear in a conceptually meaningful way in either the developmental or college level mathematics student interviews. When the algorithm did appear, it was treated algorithmically and disconnected from the context of the problem at hand. When it appeared in college level mathematics student work, it did so with the explanation that "it always works" and was anchored in a strong understanding of equal rates. When it appeared in a developmental mathematics interview, it served as an algorithm that circumvented the reasoning about rates and the interpretation of unit rate.

Implications of Study

Proportion related understandings and reasoning processes provide some of the fundamental underpinnings of algebraic thinking needed for higher-level mathematics and mathematics related coursework. This research identified opportunities for the development of meaningful and effective approaches to the teaching and learning of these important understandings and processes in post-secondary developmental mathematics contexts that can prepare students for college level mathematics.

The traditional treatment of proportionality through the standard algorithm of cross-multiply and divide, often developed without connection to context, falls far short of developing connected understandings of the robust multiplicative structure of proportionality. Such treatment of proportionality failed many developmental mathematics students in their prior mathematics studies and different results from the same treatment cannot be expected. The prerequisite rational number thinking that supports the construction of proportions and multiplicative thinking necessary to approach proportion related problem tasks cannot be assumed in developmental contexts similar to the settings of this study. It should not be assumed that developmental mathematics students have a stable understanding of ratio and rate. Approaching proportionality through the introduction of proportions as equal ratios falls short in two ways: its conceptual basis (ratio) may not be stable in students, and attempts to meaningfully interpret approaches to problem solving in proportion related situations may be thwarted by student reliance on additive thinking.

This study demonstrated that proportionality presents an opportunity for students to build meaningful understandings of ratio and rate and connect these understandings to a larger mathematical structure. Approaching proportion related problems in ways that encourage multiple approaches, and facilitating learning environments in which students communicate about the mathematics of their approach allows students to identify and connect the multiplicative relationships that exist within proportion related situations, both scalar and functional. Facilitating classroom discourse that highlights the connections between what changes and what stays the same in proportional situations holds the potential to support the development of all students.

When students begin to coordinate quantities in proportion related situations, the identification of an equivalence relationship may appear independent of the formation of a rate relationship between the quantities. This presents several challenges for both teaching and learning. First, as has been identified in previous literature (e.g. Lobato et al. 2010), it is easy to overestimate the understanding of students by their facility with proportion related situations when the quantitative elements present opportunities for comfortable partitioning and iteration (e.g. doubling, halving). Second, interpretation of proportion related situations through equality, independent of the formation of a rate relationship, may reinforce additive approaches (e.g. building up using additive thinking) and prevent generalization to the $y = mx$ functional relationship.

This study presented evidence in support of new approaches to the teaching and learning of proportionality and proportional reasoning that are better suited for the developmental mathematics contexts. Focusing on ratio and rate in the early work with proportionality supports the development of rational number interpretation and understanding and multiplicative thinking necessary to the grounding for proportion related problem solving and proportional reasoning. This study points to the importance of increasing numerical complexity in student work, and the ways it supports shifts towards multiplicative thinking in postsecondary developmental mathematics students, as has been demonstrated in previous study of the ways children develop proportion related understandings and reasoning processes (e.g. Lobato et al. 2010; Post et al, 1988).

Approaching proportionality through an understanding of unit rate opens access to students of all levels of understanding and better serves to strengthen and stabilize understandings of ratio and rate while developing multiplicative thinking and enabling students to connect and reason with multiple embodiments of the $y = mx$ functional relationship. Keeping the $y = mx$ functional relationship central to the teaching and learning activities introduces students to the concept of function while further stabilizing their understanding of rate and the invariant and covariant multiplicative relationships characteristic of proportionality. Such approach builds a strong foundation for future study of linear functions in particular, and functions in general, which can support students as they progress in their college level mathematical studies.

Limitations of the Study

The research goal of the study was the development of models of developmental mathematics student proportion related understandings and reasoning processes. The results of the study leveraged what is known about the evolution of these constructs in children (e.g. Lamon, 2007; Lobato et al., 2010; Post et al., 1988), but should be uniquely interpreted in the context of postsecondary developmental education.

The scale of the study was small, and the researcher served as the instructor in all classroom based activities. Additionally, the researcher did all the coding of the student data so there was no inter-rater reliability. Control groups were not used in the quantitative analysis of data. Therefore, the generalizability of the results of this study is limited. Instead, the models of student thinking and hypothesized differences between developmental and college level mathematics student understandings and reasoning processes that emerged in the study provide fruitful ground for further model development, and hypothesis testing.

Positioning the Research and Future Research Directions

The research presented in this study was exploratory, model, and hypothesis building work. Therefore, the results and models presented are intended to inform future work that builds towards generalizable models, and should be interpreted as specific to the context in which the research was conducted.

In the context of curriculum research and reform, this work serves as formative research. In his framework guiding the development of research-based curriculum, Clements (2007) identified the need for classroom-based teaching experiments that track and evaluate student learning in the formative research stage. In this stage of curriculum development, the research focus is on student learning in context of the curriculum developer's vision. That is, generalization is not the goal at this stage of development.

The presented models of student understanding and Hypothetical Learning Trajectory in this study provide the grounding for future studies and curriculum development. The cyclical relationship between curriculum development and empirical feedback through research allow insight into what reformed curriculum can offer in support of effective student learning experiences (Clements, 2007). These studies may including large-scale classroom based teaching experiments.

Future studies surrounding the development of proportion related understandings and proportional reasoning may look at some of the particular ways multiple embodiments of the $y = mx$ function relationship support the cognitive shifts towards multiplicative thinking and the development of more connected models of understanding the multiplicative relationships in proportion related situations. For example, in what ways does the interpretation of rate pairs as (x,y) points on a graph support the development of an understanding of equivalence in a proportion related situation? How does the interpretation of slope both graphically and contextually support the connection between the invariant and covariant multiplicative relationships in proportion related situations? How does operation with the $y = mx$ and $x = (1/m)y$ functional relationships stabilize notions of ratio and rate in developmental mathematics students?

The prealgebra constructs that support understandings of proportionality and proportional reasoning, and the algebraic structures that connect to proportionality (e.g. linear function, inverse function, slope, rate of change) may be considered in more contextually inclusive studies. For example, in what ways does a connected understanding of the multiplicative constructs that define proportionality support work with linear functions? How can an understanding of the invariant $m = y/x$ relationship in a proportional situation support the development of the $m = (y1 - y2) / (x1 - x2)$ relationship for all linear functions?

Conclusion

The major findings from this research identified key understandings and connections between understandings of the multiplicative structure of proportionality that support the psychological aspects of proportional reasoning. The results provide additional support to prior research studies that demonstrate the important role of the understanding of rate and interpretation of rate play in proportional reasoning. The data in the study demonstrated ways that students may evolve through the process of organizing quantities using an equality or *same as* interpretation to a more accurate and generalizable understanding of a rate relationship. The study showed the ways incorporating work with the $y = mx$ function model of proportionality support the development of student understandings of multiple constructs that define proportionality and proportional reasoning processes. Moreover, the development of facility with the function model can serve as an exemplar of function, providing a conceptual basis for student understanding of functions in general, and linear functions in particular.

The study concluded with a meaningful learning trajectory of teaching and learning that supports the construction of connected understandings of proportionality as a mathematical structure and robust and flexible proportional reasoning processes in developmental mathematics students. Together, the results of the study provide a basis for future hypothesis testing, theoretical model generating, and the development of new curricular approaches to the teaching and learning of proportionality and the development of proportional reasoning in developmental mathematics contexts.

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Appendix A: Sample Intervention Lesson Plan

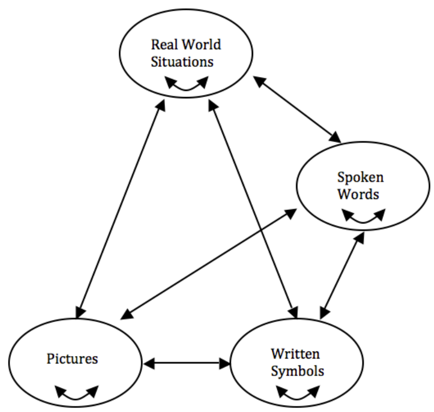
Lesson 5: Connecting Proportional Reasoning Strategies

Lesson Overview	**Proportionality Constructs Targeted**
This lesson reviews the mathematical characteristics of proportionality by looking at different strategies used when solving proportional tasks. The standard algorithm is introduced and interpreted through the unit rate approach and factor of change approach to solving missing value problems.	Proportionality is a linear relationship between two quantities that covary according to the model $y = mx$, where m is the unit rate. All corresponding (x,y) rate pairs lie on the graph of the line $y = mx$, which passes through the origin In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: $y = mx$ and $x = (1/m)\ y$. All (x,y) rate pairs in a proportional situations create an equivalence class. In proportional situations, there exists a scalar multiplicative relationship within measure spaces. Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems.

Hypothetical Learning Trajectory
A proportional situation that relates distance travelled and time strings throughout this lesson. The lesson begins by focusing in on the two rates d/t and t/d that are embedded in the proportional situation. Next a missing value problem is presented that involves a function relationship between time and distance through, *time = (4 minutes per mile) x distance,* where the rate is interpreted in minutes per mile. Students are asked to solve the problem two different ways and then discuss their approaches in small groups. Several approaches are targeted including a ratio table approach, the unit rate approach, the factor of change approach, a fraction approach

and the standard algorithm. Table approaches and additive strategies should be presented, but will be reinterpreted multiplicatively and connected to the multiplicative structure of proportionality. Prior to introduction of the standard algorithm, and its interpretation through unit rates and scalar factor of change, students will generate four different proportions that describe the multiplicative structure of the missing value problem. Connection to the multiplicative structure of proportionality is then brought to the standard algorithm through direct instruction. Focus will be brought to the unit rate throughout the lesson, including the existence of two unit rates in the proportional context that define inverse function relationships between measure spaces.

Representations and Translations Targeted
Real World Situations, Written Symbols (Equations, Numbers, Tables), Pictures (Graphs), Spoken Words



Instructional Materials	Student Materials	Other Materials
Lesson power point presentation.	Problem of the Day 5B Student Activity Sheet 5A Student Practice Page 5A	

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Teaching Actions	Notes
Problem of the Day Steve and Mark were driving equally fast along a country road. It took Steve 20 minutes to drive 4 miles. Write two different rates that describe Steve's speed.	Provide students with the **Problem of the Day 5B** sheet. This proportional situation was written by Cramer (2014). The two rates that describe Steve's speed are 20 minutes per 5 miles and 5 miles per 20 minutes. After students determine the two different rates, ask students to write each rate as a unit rate and interpret each rate. The unit rates are 4 minutes per 1 mile, and 0.25 miles per 1 minute.
Looking at Multiple Strategies **Problem Presentation** Steve and Mark were driving equally fast along a country road. It took Steve 20 minutes to drive 4 miles. How long did it take Mark to go 12 miles?	Provide students **Student Activity Sheet 5A.**
Is this a proportional situation? Why or why not?	The problem is proportional because it involves an invariant unit rate, Steve and Mark are driving equally fast.
Ask students to solve the problem 2 different ways. After about 5 minutes, have students discuss their approaches in small groups. Select groups to present each of the following approaches:	
Rate Table Using the rate of 20 minutes for 4 miles, students can build up to 60 miles for 12 miles using addition. Miles: 4 8 12 Minutes: 20 40 60 Discuss this approach first, and connect other	It is useful to have a rate table available that shows the following rate pairs: Miles: 4 12 Minutes: 20 60

approaches to the rate table, showing unit rate and scalar multiplication rules and defining the function t= (5 minutes per mile) x m.	The rate table can be used to show within and between measure space multiplications and will be useful when students construct four proportions that describe the situation prior to the description of the standard algorithm.
Unit Rate If the rate is 20 minutes for 4 miles, then the unit rate is 5 minutes per 1 mile. If Mark is also driving this speed for 12 miles, it would take him 12 miles x 5 minutes / miles = 60 minutes. Ask how students determined which unit rate to use.	This strategy involves two steps – division to determine a unit rate and then multiplication to determine the missing value. Each proportional problem has two unit rates. When using the unit rate approach, the first step is determining which unit rate is appropriate for the question.
Factor of Change If it takes 20 miles to drive 4 miles, then to go 3 times as far (12 miles), Mark needs to drive for 3 times the amount of time. 20 minutes x 3 = 60 minutes.	A scalar factor does not have units attached. Doubling, tripling, etc. are scalar operations. This strategy relies on a multiplicative relationship within "measure space." The scalar factor is not constant. For example, we could reason that It takes Mark 40 minutes to drive 8 miles by doubling rather than tripling Steve's time and distance.
Function Approach Using the information in the problem, an explicit function is written that relates time and distance through the relationship t = (5 minutes per mile) x m. This relationship is then used to compute the time for Mark to drive 12 miles.	This is a constant function rule between measure spaces (miles to minutes), and is the general case for
Fraction Strategy	

Set up equivalent fractions (without explicitly stating units) and apply strategies of deriving equivalent fractions of multiplying the numerator and denominator by the same factor (3). 20/ 4 = ?/12 **Standard Algorithm** Set up a proportion, determine the cross products, and solve the resulting equation for the missing value. $\frac{20 \text{ minutes}}{4 \text{ miles}} = \frac{? \text{ minutes}}{12 \text{ miles}}$ $(20 \text{ minutes}) \times (12 \text{ miles}) = (? \text{ minutes}) \times (4 \text{ miles})$ $\frac{(20 \text{ minutes}) \times (12 \text{ miles})}{(4 \text{ miles})} = ? \text{ minutes}$ $60 \text{ minutes} = ? \text{ minutes}$ Tell students that we will return to the standard algorithm after taking a brief tour through the multiplicative relationships within the proportional situation. **Four Proportions** Ask students to write four different proportions that show the relationship between Steve and Mark's drive. $\frac{20 \text{ minutes}}{4 \text{ miles}} = \frac{60 \text{ minutes}}{12 \text{ miles}}$ $\frac{4 \text{ miles}}{20 \text{ minutes}} = \frac{12 \text{ miles}}{60 \text{ minutes}}$ $\frac{20 \text{ minutes}}{60 \text{ minutes}} = \frac{4 \text{ miles}}{12 \text{ miles}}$ $\frac{60 \text{ minutes}}{20 \text{ minutes}} = \frac{12 \text{ miles}}{4 \text{ miles}}$ We will now use these multiplicative relationships to bring meaning to the standard algorithm in terms of how things change and what things stay the same.	the problem that asks how many minutes it takes to drive any number of miles. The unit rate has an important role in the rule as it is the constant of proportionality, the slope the line, and coefficient in the rule. In this approach, the measure space units are dropped and rational number reasoning is used. Note that this approach loses the content and students may struggle to connect the context back to the problem. The units are not dropped when the proportion is set up, but are ignored when computing the cross product. Note that the equivalence of rate pairs allows us to set up a proportion because all rate pairs are equivalent. The approach is to be unpacked and connected to the unit rate and factor of change approaches. Connect each proportion with the multiplicative

Connecting the standard algorithm to the Unit Rate, and Factor of Change approach.

Unit Rate Connection

$$\frac{20\text{ minutes}}{4\text{ miles}} = \frac{?\text{ minutes}}{12\text{ miles}}$$

$$(20\text{ minutes}) \times (12\text{ miles}) = (?\text{ minutes}) \times (4\text{ miles})$$

$$\frac{(20\text{ minutes}) \times (12\text{ miles})}{(4\text{ miles})} = ?\text{ minutes}$$

$$(12\text{ miles}) \times \overset{\text{Steve's rate of minutes to miles}}{\frac{(20\text{ minutes})}{(4\text{ miles})}} = ?\text{ minutes}$$

$$(20\text{ minutes}) \times \overset{\text{Unit Rate}}{\frac{(5\text{ minutes})}{(1\text{ mile})}} = ?\text{ minutes}$$

$$60\text{ minutes} = ?\text{ minutes}$$

Factor of Change Connection

$$\frac{20\text{ minutes}}{4\text{ miles}} = \frac{?\text{ minutes}}{12\text{ miles}}$$

$$(20\text{ minutes}) \times (12\text{ miles}) = (?\text{ minutes}) \times (4\text{ miles})$$

$$\frac{(20\text{ minutes}) \times (12\text{ miles})}{(4\text{ miles})} = ?\text{ minutes}$$

$$(20\text{ minutes}) \times \overset{\text{Ratio of miles to miles}}{\frac{(12\text{ miles})}{(4\text{ miles})}} = ?\text{ minutes}$$

$$(20\text{ minutes}) \times \overset{\text{Scalar Factor of Change}}{3} = ?\text{ minutes}$$

$$60\text{ minutes} = ?\text{ minutes}$$

Discuss under what conditions one method could be preferred over another. (personal preference, presence of integer multiples, context, decimals, etc.)

If the question asked was "How many miles did Mark travel in 60 minutes?" How would the unit rate approach change? Write a rule that can be used to solve for the number of miles that can be traveled given any number of minutes.

relationships within and between the measure spaces:

Miles:	4	12
Minutes:	20	60

In the rate proportions, we are looking at the invariant multiplicative relationships between measure spaces or across time and distance rate pairs. In the scalar proportions we are looking at the multiplicative relationship within measure spaces or between rate pairs.

Although the cross product of miles x minutes is uninterpretable, we can bring interpretation back to the problem through the identification of the unit rate and the scalar factor of change within the standard algorithm approach.

The other unit rate would be used, in particular – the more useful unit rate for this problem is 1/5 mile per 1 minute.

$d = 1/5\ t$
Show how this is the inverse of the function $t = 5t$ on a rate table.
\

Show What You Know Ali bought 3 ounces of ground coffee beans for 60 cents. What two unit rates are embedded in this statement? What does each one mean? At this cost, how many ounces of ground coffee beans can Ali buy for 75 cents? If Ali buys 8 ounces of ground coffee beans, how much will it cost? Write one or two sentences explaining your thinking in each problem.	Provide student **Show What You Know 5A** page. 3 ounces / 60 cents = 0.05 pound / 1 cent 60 cents / 3 ounces = 20 cents per ounce (0.05 pounds per 1 cent) x (75 cents) = 3.75 ounces (8 ounces) x (20 cents per 1 ounce) = 160 cents.

Appendix B: Learning Objectives for Mathematics Courses

Mathematical Reasoning Learning Objectives

1. Simplify numerical and algebraic expressions involving the operations of addition, subtraction, multiplication, and division of real numbers and the order of operations.
2. Simplify an/or evaluate an algebraic expression utilizing order of operations, including exponentiation, along with the commutative, associative and distributive properties.
3. Express numbers using scientific or standard notation.
4. Translate applied problems in one or two variables and provide a solution through algebraic manipulation.
5. Solve first-degree equations.
6. Utilize first-degree equations to solve application problems.
7. Sketch the graph of a linear equation in two variables on a rectangular coordinate plane using the x- and y-intercepts and/or other ordered pairs.
8. Sketch the graph of a line satisfying given conditions involving ordered pairs and/or slope.
9. Solve a system of linear equations in two variables by substitution, elimination and graphing.
10. Use linear equations to build mathematical models.
11. Interpret graphs and use graphs to solve math modeling scenarios
12. Solve formulas for a specific variable
13. Solve word problems, including percent problems
14. Perform basic operations on polynomials expressing the answer in simplified form.
15. Simplify radical expressions involving square root and use the simplest radical form or decimal form to express answers.
16. Use the quadratic formula to find real solutions to quadratic equations
17. Solve power equations
18. Solve proportions
19. Solve literal equations
20. Identify a function and its domain and range
21. Evaluate expressions involving function notation
22. Given a simple one-to-one function, find its corresponding inverse function
23. Factor out GCF from a polynomial
24. Utilize the Pythagorean Theorem in problem solving
25. Solve simple exponential equations using logarithms
26. Solve simple logarithmic equations using the definition of a logarithm
27. Use a calculator to perform basic operations and find powers, roots, and logarithmic values.
28. Graph simple exponential functions of the form f(x) = a*b^x
29. Write the terms of a sequence given its general term
30. Find the general term of a sequence
31. Solve applications that involve sequences

32. Identify arithmetic sequences and their common difference
33. Identify geometric sequences and their common ratios
34. Fundamental counting principle

Intermediate Algebra Learning Objectives

1. Utilize math specific study skills pertaining to but not limited to: time management, homework, test preparation, test taking, note taking and math anxiety
2. Identify the difference between STEM and non-STEM pathways in mathematics
3. Simplify numerical expressions involving the operations of addition, subtraction, multiplication and division of real numbers and the order of operations
4. Simplify and/or evaluate an algebraic expression utilizing order of operations, including exponentiation, and square roots, along with the commutative, associative, and distributive properties
5. Solve linear equations in one variable algebraically
6. Translate applied problems in one variable and provide a solution through algebraic manipulation
7. Solve applied problems by evaluating known formulas including but not limited to geometry, statistics, finance, exponential growth and science
8. Solve applied problems by interpreting graphs
9. Simplify numerical and algebraic expressions utilizing the properties of integer exponents
10. Simplify radical expressions involving square root and use the simplest radical form or decimal form to express answers
11. Express numbers using scientific notation or standard notation
12. Perform basic operations on polynomials expressing the answer in simplified form
13. Solve applied problems using proportions
14. Identify similar triangles and use related properties in problem solving
15. Utilize the Pythagorean Theorem in problem solving
16. Sketch the graph of a line satisfying given conditions involving ordered pairs and/or slope
17. Write the equation of a line in slope-intercept form satisfying given conditions involving ordered pairs and/or slope
18. Solve application problem using linear modeling
19. Solve application problems interpreting slope as rate of change
20. Solve linear inequalities in one variable algebraically displaying the solution graphically or using interval notation
21. Solve compound inequalities, intersection, and union of subsets and draw Venn diagrams
22. Graph a linear inequalities in two variables
23. Solve a system of linear equations or inequalities in two variables by graphing
24. Completely factor polynomial expressions, not including sum and difference of cubes
25. Sole quadratic equations using factoring or the square root method

26. Use quadratic formula to solve quadratic equations with real solutions
27. Write the terms in a sequence given its general term
28. Identify the recursion formula for a sequence
29. Solve applications that involve sequences
30. Utilize summation notation I defining or generating a discrete series

College Algebra Learning Objectives

1. Identify, transform, and/or produce the graph for a given function (including constant, linear, polynomial, parabolic, cubic, square root, absolute value, logarithmic, exponential and the rational function $y=1/x$).
2. Identify, transform, and/or produce the graph of a circle.
3. Find an equation of a line given sufficient information.
4. Translate an applied problem into an equation or inequality and provide a solution through algebraic manipulation.
5. Interpret an expression, equation, or inequality by utilizing a graph, table, or diagram.
6. Define a function along with its domain and range.
7. Combine functions through the operations of addition, subtraction, multiplication, division, and composition.
8. Determine the inverse for a given function.
9. Solve any equation of first or second degree.
10. Solve an exponential equation.
11. Solve a logarithmic equation.
12. Solve a system of linear equations in two or three variables.
13. Solve a system of inequalities.
14. Solve a linear programming problem.
15. State the definition of an infinite sequence.
16. Find a particular term or sequence of terms for a particular infinite sequence.
17. State the definition of an arithmetic sequence and give examples thereof.
18. State the definition of a geometric sequence and give examples thereof.
19. Work back and forth readily between expanded and closed forms of summation notation.
20. Expand a binomial raised to natural number power les than six.
21. Apply the definition(s) of the Fundamental Counting Principle, a permutation and a combination to counting problems as appropriate.
22. Apply the concepts of experiment, outcome, and sample space to a given model.
23. State the definition of probability of an event for a given sample space and apply such to simple problems.
24. Determine if a mathematical argument is valid using definitions, field properties, and theorems.
25. Create, analyze, and discuss the validity of a mathematical model for a set of data.

26. Use a graphing utility and interpret the results where applicable in the above outcomes.
27. Solve problems involving direct, inverse and joint variation.

Liberal Arts Mathematics Learning Objectives

1. Discuss, analyze, and solve problems from a variety of mathematical topics related to the liberal arts (e.g. voting theory, apportionment, personal finance, growth models, data collection, statistics, probability, graph theory, linear programming, set theory, sequences, patterns, symmetry)
2. Develop quantitative tools necessary for analyzing and understanding contemporary issues
3. Develop higher order problem-solving skills an d apply them to real-world situation
4. Collect, organize, and present quantitative information
5. Develop critical thinking skills needed to analyze and evaluate quantitative information and assess its reliability
6. Understand and use appropriate mathematical terminology and notation
7. Appreciate the ways in which mathematic helps us better model and understand the world around us
8. Understand and explain historical perspectives of various areas of mathematics

Appendix C: Written Assessment Item Analysis

In the following item analysis, items are coded according to the multiplicative structures of proportionality and aspects of proportional reasoning targeted in each item. Code labels are provided in the table below.

Code	Proportionality as a Multiplicative Structure	Psychological Aspects of Proportional Reasoning	Code
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	A
		Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	B
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	C
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	D
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	E
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	F
		Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	G
		Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	H

Item 1

In 8 ounces of a popular sports drink, there are 48 calories. In 16 ounces there are 96 calories. How many calories are in 20 ounces of the sports drink?

	Foil	Rationale
A.	60	(# of ounces) + 40 calories Incorrect additive strategy based on the first pair of ounces and calories. (8 + 40 = 48)
* B.	120	Unit rate is 48 / 8 = 6 calories per 1 ounce (20 ounces) x (6 calories per 1 ounce) = 120 calories Unit rate approach.
C.	144	Additive reasoning in which the calories increase by 48 as the ounces increase. 96 + 48 = 144. Possible incorrect extension of factor of change approach.
D.	192	The amount of calories and the amount of ounces double as the ounces increase. 96 x 2 = 192 Possible incorrect extension of factor of change approach.
E.	None of the above. Amount of calories:_______________	The amount of calories is not represented in the foils.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2	A, B, C, D, F, G, H

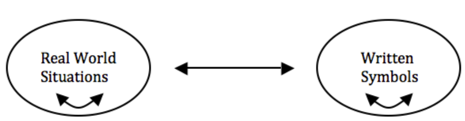
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Familiar context of nutrition labeling. Unit rate not given. May be misled by the integer factor of change present between the rate pairs given.

Multiplicative Structure Analysis

Ounces	Calories
8	48
16	96
20	?

Integer between measure space unit rate of 6 calories / 1 ounce available.
Unit rate must be derived. (Vergnaud Schema 3).
No integer within measure space scalar factor for necessary computation to 20 ounces.

Translations Anticipated in Solution Process



Item 2

In 8 ounces of a popular sports drink, there are 48 calories. In 16 ounces there are 96 calories. Which rule can be used to determine the number of calories in any number of ounces of the sports drink?

	Foil	Rationale
A.	(# of Ounces) + 48 = (# of Calories)	Generalization of an incorrect additive approach to the problem.
B.	(# of Ounces) + 24 = (# of Calories)	Incorrect generalization of an additive approach to the problem. 24 calories per 4 ounces. 20 = 16 + 4 ounces. 96 + 24 calories = 120 calories.
*C.	(# of Ounces) x 6 = (# of Calories)	Utilizes the invariant unit rate of 6 calories per 1 ounce to structure the y = mx relationship.
D.	(# of Ounces) x 2 = (# of Calories)	The amount of calories and the amount of ounces double together. Generalization of incorrect factor of change approach.
E.	None of the above. Rule: ______________	The rule is not represented in the foils.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	A, B, C, F, G, H

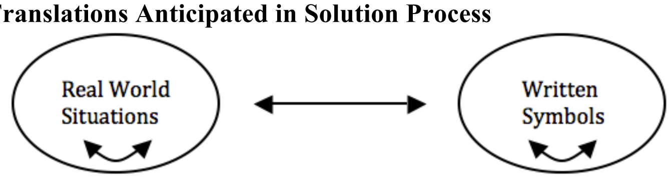
Problem Type	Notes on Context and Problem Type
Generalization to the y = mx rule that can be used to solve for any rate pair in the given proportion related situation.	Familiar context of unit pricing. Non-routine task. Unfamiliarity with notation possible.

Multiplicative Structure Analysis

Ounces	Calories
8	48
16	96
x	6x

Integer between measure space unit rate of 6 calories / 1 ounce available.
Unit rate must be derived.
Misleading integer factor of change given in the table.

Translations Anticipated in Solution Process



Item 3

Sue and Julie were running equally fast around a track. Sue started running first. When Sue had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?

	Foil	Rationale
A.	45	$15 \times \left(\frac{9}{3}\right) = 45$ Application of proportional reasoning to a situation that is not proportional.
B.	27	$9 \times \left(\frac{9}{3}\right) = 27$ Incorrect application of proportional reasoning to a situation that is not proportional.
C.	24	15 + 9 = 24 Sue has run 9 more laps than Julie.
* D.	21	15 + 6 = 21 Sue has run 6 more laps than Julie.
E.	None of the above. Number of Laps: ______________	The number of laps that Sue had run is not represented in the foils above.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	A

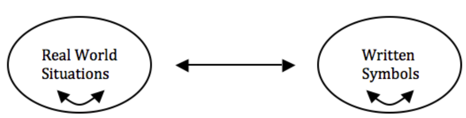
Problem Type	Notes on Context and Problem Type
Non-proportional task.	Familiar context of running laps.

Multiplicative Structure Analysis

Sue's Laps		Julie's Laps
3	+6	9
15	+6	?

If the situation were proportional, there would be both an integer unit rate and integer factor of change available. This is a quantitative distracter that could bring about a misapplication of proportional reasoning to a non-proportional task.

Translations Anticipated in Solution Process



Source: Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research Implications. In D. Owens (Ed.) *Research Ideas for the Classroom* (pp. 159 – 178). New York: MacMillan Publishing Company..

Item 4

Wreh runs laps around a track every day. He ran fewer laps in more time today than he did yesterday. Is his running speed today faster, slower, the same as his speed yesterday, or can it not be determined?

	Foil	**Rationale**
A.	Faster	Guess. Possible reasoning could be running for longer time translated to faster running.
* B.	Slower	Decreasing the amount of distance (laps) covered and increasing the amount of time running means Wreh was running slower.
C.	Same speed	Guess. Possible reasoning could be that decreasing the distance and increasing the time balance each other so the speed is the same.
D.	Cannot be determined	Quantities must be given to answer this problem. Or this is an indeterminate case.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, E, F, G, H

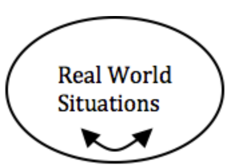
Qualitative reasoning	**Notes on Context and Problem Type**
Qualitative reasoning problem.	Familiar context of speed. Non-routine task.

Multiplicative Structure Analysis

	Numerator (Laps)		
Denominator (Time)	Stays Same	Increase	**Decrease**
Stays Same	No Change	Increase	Decrease
Increase	Decrease	Indeterminate	**Decrease**
Decrease	Increase	Increase	Indeterminate

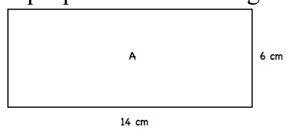
The numerator decreased and the denominator increased, therefore the rate decreased.

Translations Anticipated in Solution Process

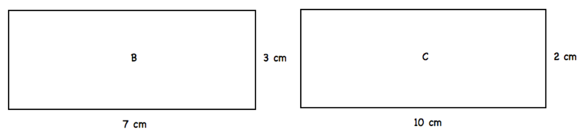


Item 5

Similar figures have the same shape. That is, they have the same angles and their corresponding side lengths are proportional. Rectangle A is shown below.



Which, if any, of the following rectangles are similar to Rectangle A?



	Foil	Rationale
A.	Both Rectangle B and Rectangle C	Both rectangles have dimensions that change in the same (within rectangle) way from rectangle A.
* B.	Rectangle B only	Identification of a scalar factor of change of ½.
C.	Rectangle C only	Identifies that both length and width decreased by 4 cm. Incorrect application of additive structures.
D.	Neither Rectangle B nor Rectangle C	Possible reasoning is that both rectangles have lesser dimensions than Rectangle A, therefore neither is similar.

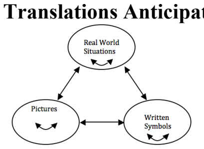
Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
3, 4	A, D, E, F, G, H

Problem Type	Notes on Context and Problem Type
Comparison Problem (are ratios equivalent?)	Background knowledge in geometry helpful. It is anticipated that most students will approach this task by creating ratios of corresponding sides between rectangles. Some may compare between measure space unit rates. Scaling down is challenging, some student may identify the integer factor of change of x2 and use that in their reasoning.

Multiplicative Structure Analysis

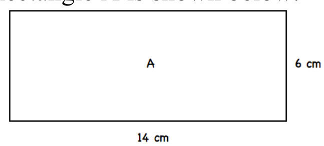
	Length	Width	
Rectangle A	6	14	
	x ½	x ½	Scalar multiple within measure spaces is the same in rectangles A and B, i.e. 1/2
Rectangle B	3	7	
Rectangle C	2	10	Scalar multiple of change within measure spaces is not the same in rectangles A and C. i.e. 1/3 and 5/7.

Translations Anticipated in Solution Process

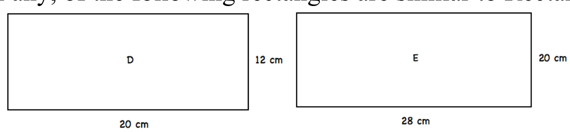


Item 6

Similar figures have the same shape. That is, they have the same angles and their corresponding side lengths are proportional. Rectangle A is shown below.



Which, if any, of the following rectangles are similar to Rectangle A?



	Foil	Rationale
A.	Both Rectangle D and Rectangle E	Both rectangles have dimensions that change in the same (within rectangle) way from rectangle A.
B.	Rectangle D only	Identifies that both length and width increase by 6 cm. Incorrect application of additive structures.
C.	Rectangle E only	Identifies that both length and width increase by 14 cm. Incorrect application of additive structures.
* D.	Neither Rectangle D nor Rectangle E	No scalar factor of change can be found among the rectangles.

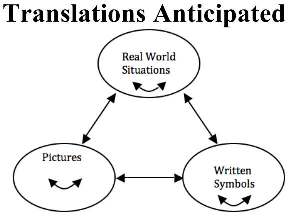
Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
3, 4	A, D, E, F, G, H

Problem Type	Notes on Context and Problem Type
Comparison Problem (are ratios equivalent?)	Background knowledge in geometry helpful. It is anticipated that most students will approach this task by creating ratios of corresponding sides between rectangles. Some may compare between measure space unit rates.

Multiplicative Structure Analysis

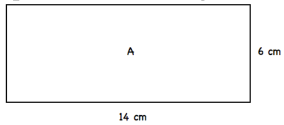
	Length	Width	
Rectangle A	6	14	
Rectangle D	12	20	Scalar multiple within measure spaces is not the same in rectangles A and D, i.e. 12/6 and 20/14.
Rectangle E	20	28	Scalar multiple within measure spaces is not the same in rectangles A and E, i.e. 20/6 and 28/14.

Translations Anticipated in Solution Process

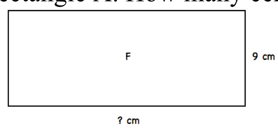


Item 7

Similar figures have the same shape. That is, they have the same angles and their corresponding side lengths are proportional. Rectangle A is shown below.



Rectangle F is similar to Rectangle A. How many centimeters wide is Rectangle F?



	Foil	Rationale
A.	17	Extending incorrect additive reasoning. 6 + 3 = 9, so 14 + 3 = 17.
* B.	21	Identified a scalar factor of change of 1 ½ or 3/2. 14 x (3/2) = 21.
C.	28	Doubled the width. Incorrect factor of change. Tempting guess.
D.	42	Tripled the width. Incorrect factor of change. Tempting guess due to the role that 3 played in the additive and multiplicative reasoning that could derive a length of 9. (6 + 3 = 9, 6 x (3/2) = 9)
E.	None of the above. Centimeters: _______________	The width of Rectangle F is not represented in the foils above.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
3, 4	D, E, F, G, H

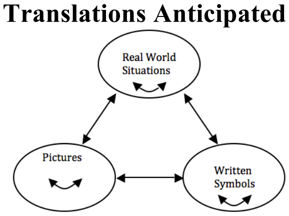
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Background knowledge in geometry helpful. It is anticipated that most students will approach this task using a factor of change strategy.

Multiplicative Structure Analysis

	Length	Width	
Rectangle A	6	14	
	X(3/2)	x (3/2)	Scalar multiple within measure space
Rectangle F	9	?	

Non-integer between measure space unit rates. Unit rate not directly given. (Vergnaud Schema 3). Non-integer within measure space scalar factors (3/2 and 2/3).

Translations Anticipated in Solution Process



Item 8

The Bank of Europe gave more Euros (€) in exchange for more U.S. Dollars ($) this week than it did last week. Did the exchange rate, €/$, increase, decrease, stay the same, or can it not be determined?

	Foil	**Rationale**
A.	Increase	Possible reasoning could be more Euros increases the exchange rate for longer time translated to faster running.
B.	Decrease	Possible reasoning could be more Dollars decreases the exchange rate for longer time translated to faster running.
C.	Stayed the same	Guess. Possible reasoning could be that both increased, so the rate stayed the same.
* D.	Cannot be determined	When both the numerator and denominator increase, the amount by which each increases determines the change in the exchange rate.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, E, F, G, H

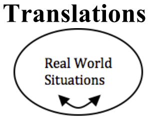
Qualitative reasoning	**Notes on Context and Problem Type**
Qualitative Reasoning Problem.	Challenging context of currency exchange. Non-routine task.

Multiplicative Structure Analysis

	Numerator (Euros)		
Denominator (Dollars)	Stays Same	**Increase**	Decrease
Stays Same	No Change	Increase	Decrease
Increase	Decrease	**Indeterminate**	Decrease
Decrease	Increase	Increase	Indeterminate

The numerator decreased and the both increased, case is indeterminate.

Translations Anticipated in Solution Process



Source: Cramer, K. (2014). *Rational Number Concepts and Proportionality*. Unpublished curriculum, Department of Curriculum and Instruction, University of Minnesota, Minneapolis, MN.

Item 9

Emily is ordering pizza for a neighborhood block party. She estimates that 3 pizzas will be enough for 10 people.

How many pizzas should Michelle order for 60 people?

	Foil	Rationale
A.	180	180 = 3 pizzas x (60 people). Incorrect interpretation of the number of pizzas in the given rate pair as a unit rate of pizzas per person.
B.	20	20 = 60 / 3 incorrect scalar factor of change as people were divided by pizzas. It could be an incorrect rate in which quantities from two different rate pairs were used.
*C.	18	3 pizzas * 6 = 18 pizzas. Use of within the "people" measure space that can be extended to the "pizza" measure space.
D.	6	This is the integer within measure space scalar factor of change. It would be identified, but then not utilized to solve the problem.
E.	None of the above. Number of pizzas: ______________	The number of pizzas is not represented in the foils.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
4	A, E, F, G, H

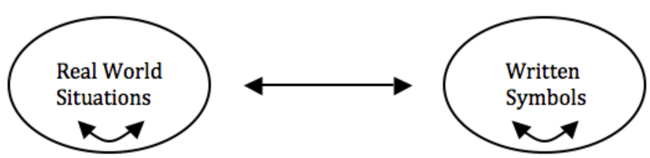
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Familiar problem setting. Numbers facilitate factor of change approach.

Multiplicative Structure Analysis

People	Pizzas	
10	3	
		Scalar factor of change of x 6 within the "people" measure space that can be extended to the "pizza" measure space.
60	?	

Non-integer between measure space unit rates of 10/3 and 3/10. Unit rate must be derived (Vergnaud Schema 3).
Integer within measure space scalar factor.

Translations Anticipated in Solution Process



Item 10

Emily is ordering pizza for a neighborhood block party. She estimates that 3 pizzas will be enough for 10 people.

Which rule can be used to determine the number of pizzas Michelle should purchase for any given number of people?

	Foil	**Rationale**
A.	(# of Pizzas) = 3 x (# of People)	Uses the 3 pizzas part of the between measure space rate that is given.
B.	(# of Pizzas) = 6 x (# of People)	Scalar factor of change from 6a utilized instead of invariant unit rate.
*C.	(# of Pizzas) = (3/10) x (# of People)	(# of Pizzas) = (3 pizzas per 10 people) x (# of People)
D.	(# of Pizzas) = (10/3) x (# of People)	Utilizes the reciprocal of the unit rate that should be used in this rule.
E.	None of the above. Rule: __________	The rule is not represented in the answer choices above.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	A, B, C, F, G, H

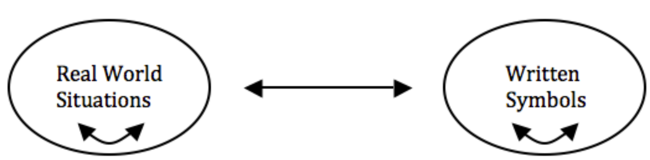
Problem Type	**Notes on Context and Problem Type**
Generalization to the y = mx rule that can be used to solve for any rate pair in the given proportion related situation.	Non-routine task. Unfamiliarity with notation possible. Previous problem primes students to think about using the scalar factor of change in the rule. Students must be able to overcome this complexity.

Multiplicative Structure Analysis

People	Pizzas	
10	x (3/10) = 3	
		Scalar factor of change of x 6
# of people	x (3/10) = # of Pizzas	

Non-integer between measure space unit rates of 10/3 and 3/10. Unit rate must be derived (Vergnaud Schema 3).
Integer within measure space scalar factor.

Translations Anticipated in Solution Process



Item 11

Orange juice is made by mixing cups of orange juice concentrate and cups of water. Recipes are described below according to the number of cups of orange juice concentrate and the number of cups of water that are mixed to make the juice. Each cup contains the same amount of liquid.

Cup of Orange Juice Concentrate

Cup of Water

Which recipe has the strongest orange taste, do the recipes taste the same, or is it impossible to tell?

Recipe A

Recipe B

	Foil	Rationale
*A.	Recipe A	2/3 > ½, alternatively 2/5 > 1/3. Or more cups in total make stronger orange taste.
B.	Recipe B	2/3 < 1/2. Or fewer cups in total make stronger orange taste.
C.	Taste the same	Both mixtures use part orange juice concentrate and part water.
D.	Impossible to tell	Different total amount of cups are used, so the rates cannot be compared.

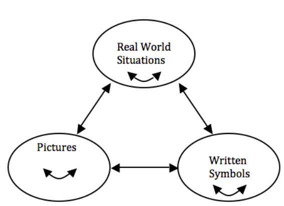
Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
3, 5	C, F, G, H

Problem Type	Notes on Context and Problem Type
Comparison Problem.	Familiar mixture context. Non-routine task based on presentation. Ordered pairs with two corresponding terms multiples of one another. (Noelting, 1980)

Multiplicative Structure Analysis

Cups of OJ Concentrate	Cups of Water	Unit Rate
2	3	2/3 OJ per 1 Water
1	2	1/2 OJ per 1 Water

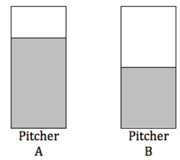
Translations Anticipated in Solution Process



Source: Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Educational Studies in Mathematics, 11, 217 – 235.

Item 12

Pitcher A and Pitcher B contain orange juice that tastes the same. The pitchers are the same size. If one cup of water is added to each pitcher, which pitcher will contain the orange juice with the stronger orange taste, do they taste the same, or is it impossible to tell?



	Foil	Rationale
* A.	Pitcher A	The water added to Pitcher A has less of a diluting effect because there is more juice to begin with, that is the orange to water ratio would be greater in Pitcher A than in Pitcher B.
B.	Pitcher B	The water added to Pitcher B would have less of a diluting effect because there is less juice to begin with, so the orange to water ratio would be greater in Pitcher B than in Pitcher B.
C.	Taste the same	The same amount of water was added to orange juice that tastes the same, so the final juice would also stay the same.
D.	Impossible to tell	Different total amount of juice are involved in each pitcher, so the tastes cannot be compared.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
2, 3, 4, 5	C, F, G, H

Problem Type	**Notes on Context and Problem Type**
Qualitative Comparison Problem.	Familiar mixture context. Non-routine task based on presentation.

Multiplicative Structure Analysis

	Cups of OJ Concentrate	Cups of Water	Unit Rate
Pitcher A	aX (a > 1)	aY (a > 1)	X/Y OJ per 1 Water
Pitcher B	X	Y	X/Y OJ per 1 Water

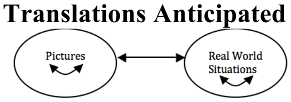
Following 1 cup of water added:

	Cups of OJ Concentrate	Cups of Water	Unit Rate
Pitcher A	aX (a > 1)	aY + 1 (a > 1)	aX/(aY+1) OJ per 1 Water
Pitcher B	X	Y + 1	X/(Y+1) OJ per 1 Water

$\frac{aX}{aY+1} > \frac{aX}{aY+a}$ and $\frac{aX}{aY+a} = \frac{aX}{a(Y+1)} = \frac{X}{Y+1}$.

Therefore, $\frac{aX}{aY+1} > \frac{X}{Y+1}$.

Translations Anticipated in Solution Process



Source: Billings, E. M. H. (2002). Cocoa. In G. W. Bright, & B. Litwiller (Eds.), *Classroom Activities for Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (pp. 38 – 40). Reston, VA: National Council of Teachers of Mathematics.

Item 13

There is a proportional relationship between the time an air conditioner runs and the amount of energy it uses. When air conditioner runs for 3.51 hours, it uses 10.88 kWh of energy. How much energy does the air conditioner use when it runs for 4.62 hours?

	Foil	**Rationale**
A.	8.27 kWh	(kWh of energy) = (10.88 kWh/4.62 h) * (3.51 hours)
B.	11.99 kWh	Incorrect additive strategy. Difference in hours, 4.62 – 3.51 = 1.11 hours, was added to the kWh of energy used when the air conditioner ran for 3.51 hours. That is, 10.88 + 1.11 = 11.99.
*C.	14.32 kWh	(kWh of energy) = (10.88 kWh/3.51 h) * (4.62 hours)
D.	The answer cannot be determined.	The quantitative elements of this problem make it an unsolvable problem.
E.	Other Amount of energy: __________	The answer is not represented in the given foils.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	B, C, D, E, F, G, H

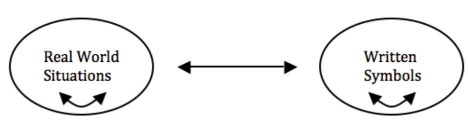
Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Unit rate not given. No integer unit rate or factor of change.

Multiplicative Structure Analysis

Hours		Kilowatt-hours
3.51	x(10.88/3.51)	10.88
4.62	x(10.88/3.51)	14.32

Unit rate must be derived.
No integer within measure space scalar factor or between measure space unit rate.

Translations Anticipated in Solution Process



Item 14

There is a proportional relationship between the values in Column X and the values in Column Y. Which expression represents the missing value in Column B for any given values of a, b, and c?

Column X	Column Y
A	B
C	?

	Foil	Rationale
A.	b + (c – a)	Additive reasoning in which the same value is added to each row.
B.	c + b	Additive reasoning in which the rows increase by a value of c.
C.	$\left(\frac{a}{b}\right) \times c$	Incorrect use of a unit rate.
*D.	$\left(\frac{b}{a}\right) \times c$	Using the unit rate of b/a, the missing value is found by multiplying the value in column x by (b/a).
E.	Cannot be determined.	There are no numbers given in the situation, so the question is unanswerable.

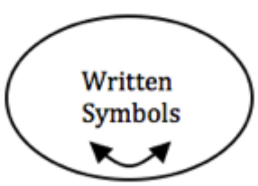
Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2	B, C, D, E, F, G, H

Problem Type	Notes on Context and Problem Type
Missing Value Problem	Unit rate not given.

Multiplicative Structure Analysis

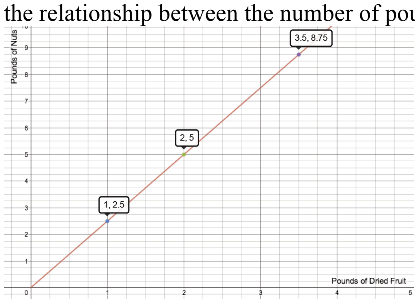
Column X		Column Y
A	x(B/A)	B
C	x(B/A)	$\left(\frac{b}{a}\right) \times c$

Translations Anticipated in Solution Process



Item 15

A trail mix company mixes 5 pounds of nuts per every 2 pounds of dried fruit in their signature mix. The graph shows the relationship between the number of pounds of nuts and the number of pounds of dried fruit in their signature mix.



Which, if any, of the following rates represent the relationship between the number of pounds of nuts and the number of pounds of dried fruit in the signature mix?
Rate A: (2.5 pounds of nuts) / (1 pound of dried fruit)
Rate B: (5 pounds of nuts) / (2 pounds of dried fruit)
Rate C: (8.75 pounds of nuts) / (3.5 pounds of dried fruit)

	Foil	Rationale
A.	Rate A only	This is the unit rate of number of pounds of nuts per one pound of dried fruit.
B.	Rate B only	The rate is 5 pounds of nuts per 2 pounds of dried fruit
C.	Rates A and B only	The unit rate and the rate as was stated in the stem 5 pounds of nuts per 2 pounds of dried fruit.
D. *	Rates A, B, and C	Each of the rates represent a rate pair for the signature mix. The rates are equivalent.
E.	None of the rate	None of the rates represent the relationship

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2, 3	A, C, D, E, F, G, H

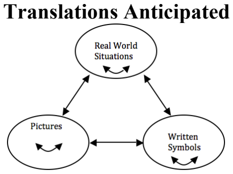
Problem Type	Notes on Context and Problem Type
Graphical proportional reasoning problem.	This is a non-routine proportional reasoning problem. Targets the concept of equivalence and the linear relationship constructs within proportional situations.

Multiplicative Structure Analysis

Dried fruit pounds		Nut Pounds
1	x 5/2	2.5
2	x 5/2	5
3.5	x 5/2	8.75

Non-integer between measure space unit rates of 5/2 and 2/5. Unit rate must be derived (Vergnaud Schema 3).
Non-integer within measure space scalar factor.

Translations Anticipated in Solution Process



Item Summary

Proportionality Structures Targeted

Code	Proportionality as a Multiplicative Structure	Items
1	Proportionality is a linear relationship between two quantities that covary according to the model $y = mx$, where m is the unit rate. All corresponding (x,y) rate pairs lie on the graph of the line $y = mx$, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	1, 2, 3, 10, 13, 14, 15
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: $y = mx$ and $x = (1/m)\, y$ (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	1, 10, 12, 13, 14, 15
3	All (x,y) rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	5, 6, 7, 11, 15
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	5, 6, 7, 9
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	4, 8, 11, 12

Proportionality Structures Targeted By Item

	Proportionality Structures Targeted				
Item	1	2	3	4	5
1	X	X			
2	X				
3	X				
4					X
5			X	X	
6			X	X	
7			X	X	
8					X
9				X	
10	X	X			
11			X		X
12		X			X
13	X	X			
14	X	X			
15	X	X	X		
Total	7	6	5	4	4

Psychological Aspects of Proportional Reasoning Targeted

Code	Psychological Aspects of Proportional Reasoning	Items
A	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	1, 2, 3, 5, 6, 9, 10, 15
B	Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	1, 2, 10, 13, 14
C	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	1, 2, 4, 8, 10, 11, 12, 13, 14, 15
D	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	1, 5, 6, 7, 13, 14, 15
E	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	4, 5, 6, 7, 8,9, 13, 14, 15
F	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	1, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15
G	Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
H	Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15

Psychological Aspects of Proportional Reasoning Targeted by Item

	Psychological Aspect of Proportional Reasoning							
Item	A	B	C	D	E	F	G	H
1	X	X	X	X		X	X	X
2	X	X	X			X	X	X
3	X							
4			X		X	X	X	X
5	X			X	X	X	X	X
6	X			X	X	X	X	X
7				X	X	X	X	X
8			X		X	X	X	X
9	X				X	X	X	X
10	X	X	X			X	X	X
11			X			X	X	X
12			X			X	X	X
13		X	X	X	X	X	X	X
14		X	X	X	X		X	X
15	X		X	X	X	X	X	X
Total	8	5	10	7	9	13	14	14

Solution Key

	Key				
Item	A	B	C	D	E
1		X			
2			X		
3				X	
4		X			
5		X			
6				X	
7		X			
8				X	
9			X		
10			X		
11	X				
12	X				
13			X		
14				X	
15				X	
Total	2	4	4	5	0

Appendix D: Interview Item Analysis

In the following item analyses, interview items are coded according to the multiplicative structures of proportionality and aspects of proportional reasoning targeted in each item. Code labels are provided below.

Code	Proportionality as a Multiplicative Structure	Psychological Aspects of Proportional Reasoning	Code
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	A
		Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	B
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	C
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	D
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	E
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	F
		Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	G
		Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	H

Developmental Mathematics Interview 1 Item Analyses

Interview 1 Item 1.1

An employee receives 3 days paid vacation for every 15 weeks of work. How many paid vacation days does the employee receive for working 35 weeks?

Solution: (35 weeks of work) x (3 vacation days / 15 weeks of work) = 7 vacation days

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	A, B, C, D, F, G, H

Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Familiar context of work and vacation time. Unit rate must be derived. Non-integer unit rate of 1/5 to be used in multiplication.

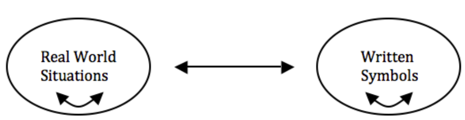
Multiplicative Structure Analysis

Weeks of Work		Vacation Days
15	x 1/5	3
35		?

Anticipated Reasoning Processes and Approaches

Unit rate multiplicative.
Additive reasoning.
Estimation.
Standard algorithm.

Translations Anticipated in Solution Process



Interview 1 Item 1.2

An employee receives 3 days paid vacation for every 15 weeks of work. How many weeks must an employee work to receive 10 days paid vacation?

Solution: (10 vacation days) x (5 weeks of work/ 1 vacation day) = 50 weeks of work

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	A, B, C, D, F, G, H

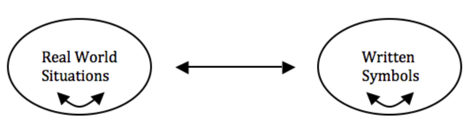
Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Familiar context of work and vacation time. Unit rate must be derived. Rate presented as vacation days per weeks of work. The unit rate that can be used in a one-step multiplication problem is the reciprocal of this rate. Integer unit rate of 5 weeks of work per 1 day of vacation.

Multiplicative Structure Analysis

Vacation Days		Weeks of Work
3	X 5	15
10	X 5	50

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Additive reasoning.
Estimation.
Standard algorithm.

Translations Anticipated in Solution Process



Interview 1 Item 1.3

An employee receives 3 days paid vacation for every 15 weeks of work. Write a rule that can be used to solve for the number of weeks that must be worked to receive any number of paid vacation days.

Solution: (# of vacation days) x (5 weeks of work/ 1 vacation day) = # of weeks of work

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	A, B, C, F, G, H

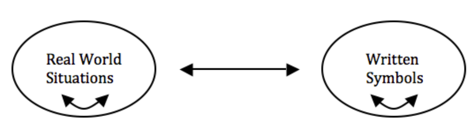
Problem Type	**Notes on Context and Problem Type**
Generalization to the y = mx rule that can be used to solve for any rate pair in the given proportion related situation.	Familiar context of work and vacation time. Unit rate must be derived. Rate presented as vacation days per weeks of work. The unit rate that can be used in a one-step multiplication problem is the reciprocal of this rate. Integer unit rate of 5 weeks of work per 1 day of vacation.

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Additive reasoning.
Standard algorithm.

Multiplicative Structure Analysis

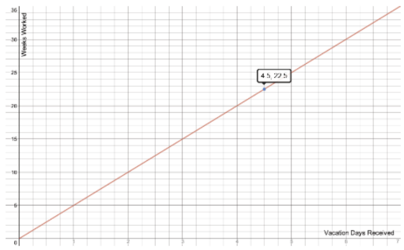
Vacation Days		Weeks of Work
3	X 5	15
x		5x

Translations Anticipated in Solution Process



Interview 1 Item 1.4

An employee receives 3 days paid vacation for every 15 weeks of work. This is a graph that shows the relationship between the number of vacation days that are received and the number of weeks that are worked. The (x,y) point (4.5, 22.5) is shown on the graph. Write the ratio of y/x for this (x,y) pair. What does this rate mean?



Solution: The ratio is 22.5 weeks of work / 4.5 days of vacation. This ratio relates the weeks of work per vacation days. Division would give the rate of 5 weeks of work per 1 day of vacation.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2, 3	C, D

Problem Type	Notes on Context and Problem Type
Graphical interpretation of a proportion related situation.	Familiar context of work and vacation time. Unit rate must be derived. Rate presented as vacation days per weeks of work. The unit rate that can be used in a one-step multiplication problem is the reciprocal of this rate. Integer unit rate of 5 weeks of work per 1 day of vacation.

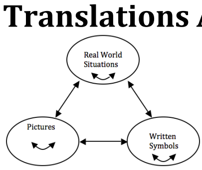
Anticipated Reasoning Processes and Approaches

Unit rate multiplicative.

Multiplicative Structure Analysis

Weeks of Work		Vacation Days
22.5	X 1/5	4.5

Translations Anticipated in Solution Process



Interview 1 Item 2.1

There was a sale on chocolate candies. One piece cost 15¢, two pieces cost 27¢, three pieces cost 39¢, and four pieces cost 51¢. Is this situation proportional? Why or why not?

Solution:
The solution is not proportional. The rate of cents per piece of candy is not invariant.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1	A

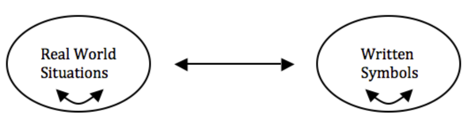
Problem Type	**Notes on Context and Problem Type**
Non-proportional situation	Familiar context of sale pricing. Unit rates are not equivalent for (x, y) pairs of chocolates and costs.

Mathematical Structure Analysis

The relationship between the number of chocolates and the number of cents is linear, but not proportional. It follows the relationship: (# of cents) = 3 cents + (12 cents per chocolate) x (# of chocolates).

Anticipated Reasoning Processes and Approaches
Unit rate misapplication.
Additive reasoning.
Estimation.

Translations Anticipated in Solution Process



Interview 1 Item 3.1

Cece and Sou ran laps around a track. Cece ran for more time that Sou, and they both ran the same amount of laps. Who was the faster runner, were the running at the same speed, or is it impossible to tell?

Solution: Sou was the faster runner.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, E, F, G, H

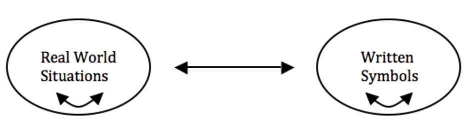
Problem Type	**Notes on Context and Problem Type**
Qualitative Reasoning Problem	No numbers available. For r = d/t comparison, Cece and Sou have the same d, but Cece has a greater t.

Multiplicative Structure Analysis

	Numerator (Laps)		
Denominator (Change in time comparing Cece to Sou's time)	**Stays Same**	Increase	Decrease
Stays Same	No Change	Increase	Decrease
Increase	**Decrease**	Indeterminate	Decrease
Decrease	Increase	Increase	Indeterminate

Anticipated Reasoning Processes and Approaches
Qualitative reasoning.
Numeric assignment.
Contextual reference in reasoning.

Translations Anticipated in Solution Process



Interview 1 Item 3.2

Cece ran fewer laps today in less time than she did yesterday. How does her running speed today compare to her speed yesterday?

Solution: The answer cannot be determined by the information given.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, E, F, G, H

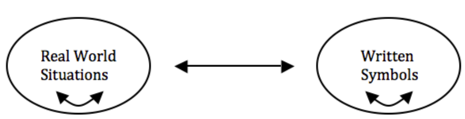
Problem Type	**Notes on Context and Problem Type**
Qualitative Reasoning Problem	For r = d/t comparison, today Cece had the greater d, and greater t. This is an indeterminate case for qualitative reasoning.

Multiplicative Structure Analysis

	Numerator (Laps)		
Denominator (Time)	Stays Same	Increase	**Decrease**
Stays Same	No Change	Increase	Decrease
Increase	Decrease	Indeterminate	Decrease
Decrease	Increase	Increase	**Indeterminate**

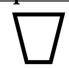
Anticipated Reasoning Processes and Approaches
Qualitative reasoning.
Numeric assignment.
Contextual reference in reasoning.

Translations Anticipated in Solution Process



Interview 1 Item 4.1

Orange juice is made by mixing cups of orange juice concentrate and cups of water. Recipes are described below according to the number of cups of orange juice concentrate and the number of cups of water that are mixed to make the juice.

Cup of Orange Juice Concentrate	Cup of Water
	

Which recipe has the strongest orange taste, do the recipes taste the same, or is it impossible to tell?

Recipe A	Recipe B
	

Solution: The recipes taste the same. 4/2 = 2/1 are equivalent rate pairs.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
2, 3, 4	C, D, E, F, G, H

Problem Type	**Notes on Context and Problem Type**
Comparison Problem.	Familiar mixture context. Non-routine task based on presentation.

Multiplicative Structure Analysis

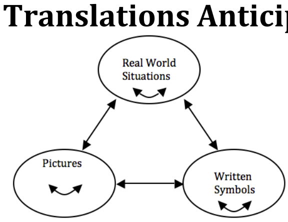
Cups of OJ Concentrate	Cups of Water	Unit Rate
4	2	2 OJ per 1 Water
2	1	2 OJ per 1 Water

Any equivalence class (Noelting, 1980).

Anticipated Reasoning Processes and Approaches

Unit rate comparison.
Scalar factor of change.
Cross Multiplication.

Translations Anticipated in Solution Process



Source: Noelting, 1980

Interview 1 Item 4.2

Orange juice is made by mixing cups of orange juice concentrate and cups of water. Recipes are described below according to the number of cups of orange juice concentrate and the number of cups of water that are mixed to make the juice.

Cup of Orange Juice Concentrate

Cup of Water

Which recipe has the strongest orange taste, do the recipes taste the same, or is it impossible to tell?

Recipe A

Recipe B

Solution: Recipe A. 2/3 > 3/5 or 2/1 > 3/2.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
2, 3	C, F, G, H

Problem Type	**Notes on Context and Problem Type**
Comparison Problem.	Familiar mixture context. Non-routine task based on presentation.

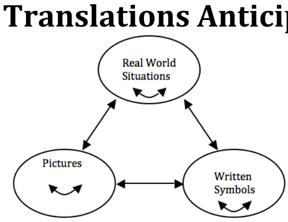
Multiplicative Structure Analysis

Cups of OJ Concentrate	Cups of Water	Unit Rate
2	1	2 OJ per 1 Water
3	2	3/2 OJ per 1 Water

Ordered pairs with two corresponding terms multiples of one another (Noelting, 1980).

Anticipated Reasoning Processes and Approaches
Unit rate comparison.

Translations Anticipated in Solution Process



Source: Noelting, 1980

Interview 1 Item 5.1

Steph and Matt are racecar drivers. They tested their cars' fuel efficiency driving at race speeds on an oval racetrack used for a long distance car race. Steph's car used 16.3 gallons of gas on a 61.8 mile drive. Matt's car used 13.2 gallons of gas on a 54.12 mile drive. Whose car had the better fuel efficiency, were they the same, or is it impossible to tell?

Solution: Matt's car had better fuel efficiency.
61.8 miles / 16.3 gallons = 3.79 mpg for Steph's car. 54.12 miles / 13.2 gallons = 4.1 mpg for Matt's car.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
2, 3	A, C, G

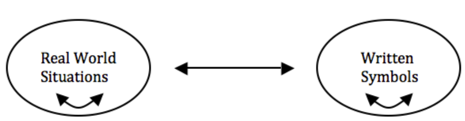
Problem Type	**Notes on Context and Problem Type**
Comparison Problem.	Consumption rate comparison. Familiar context. No integer unit rate or factor of change available. Unequal rate comparison.

Anticipated Reasoning Processes and Approaches
Unit rate comparison.
Estimation.
Cross multiplication.

Multiplicative Structure Analysis

Miles	Gallons	Unit Rate
61.8	16.3	3.79
54.12	13.2	4.1

Translations Anticipated in Solution Process



Interview 1 Item 6.1

Rice can be bought in bulk at the grocery store. This week rice is priced at $3.00 per pound. Kristie bought 4 pounds of rice. How much did Kristie pay for the rice she bought?

Solution: (4 pounds of rice) x (3 dollars per 1 pound of rice) = $12.00

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 4	A, B, C, D, E, F, G, H

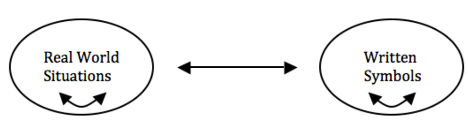
Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Familiar context of unit pricing in grocery shopping. Unit rate is provided. Integer factor of change available.

Multiplicative Structure Analysis

Pounds of rice		Price of rice
1	X 3	3
4	X3	12

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Factor of change.
Additive reasoning.
Estimation.
Standard algorithm.

Translations Anticipated in Solution Process



Interview 1 Item Summary

Proportionality Structures Targeted

Code	Proportionality as a Multiplicative Structure	Items
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	1.1, 1.2, 1.3, 1.4, 2.1, 6.1
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	1.1, 1.2, 1.3, 1.4, 4.1, 4.2, 5.1, 6.1
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	1.4, 4.1, 4.2, 5.1
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	4.1, 6.1
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	3.1, 3.2

Proportionality Structures Targeted By Item

	Proportionality Structures Targeted				
Item	1	2	3	4	5
1.1	X	X			
1.2	X	X			
1.3	X	X			
1.4	X	X	X		
2.1	X				
3.1					X
3.2					X
4.1		X	X	X	
4.2		X	X		
5.1		X	X		
6.1	X	X		X	
Total	6	8	4	2	2

Psychological Aspects of Proportional Reasoning Targeted

Code	Psychological Aspects of Proportional Reasoning	Items
A	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	1.1, 1.2, 1.3, 2.1, 5.1, 6.1
B	Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	1.1, 1.2, 1.3, 6.1
C	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	1.1, 1.2, 1.3, 1.4, 3.1, 3.2, 4.1, 4.2, 5.1, 6.1
D	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	1.1, 1.2, 1.4, 4.1, 6.1
E	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	3.1, 3.2, 4.1, 6.1
F	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	1.1, 1.2, 1.3, 3.1, 3.2, 4.1, 4.2, 6.1
G	Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	1.1, 1.2, 1.3, 3.1, 3.2, 4.1, 4.2, 5.1, 6.1
H	Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	1.1, 1.2, 1.3, 3.1, 3.2, 4.1, 4.2, 6.1

Psychological Aspects of Proportional Reasoning Targeted by Item

	Psychological Aspect of Proportional Reasoning							
Item	A	B	C	D	E	F	G	H
1.1	X	X	X	X		X	X	X
1.2	X	X	X	X		X	X	X
1.3	X	X	X			X	X	X
1.4			X	X				
2.1	X							
3.1			X		X	X	X	X
3.2			X		X	X	X	X
4.1			X	X	X	X	X	X
4.2			X			X	X	X
5.1	X		X				X	
6.1	X	X	X	X	X	X	X	X
Total	6	4	10	5	4	8	9	8

Developmental Mathematics Interview 2 Item Analyses

Interview 2 Item 1.1

James has a part-time job at a restaurant and is paid $9.50 for each hour he works. The chart below reflects his earnings from his most recent work shifts. Complete the chart.

Hours Worked	Money Earned in Dollars
2	$19.00
6	?
?	$71.25

Solution: (6, 57), (7.5, 71.25)

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 3, 4	A, B, C, D, E, F, G, H

Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Familiar context of work and pay. Non-integer unit rate given.

Multiplicative Structure Analysis

Hours Worked		Dollars	
2	$\xrightarrow{\times 9.5}$	19	
$\downarrow \times 3$		$\downarrow \times 3$	Factor of change
6		57	
7.5	$\xleftarrow{\times \frac{1}{9.5}}$	71.25	

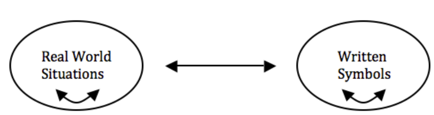
Anticipated Reasoning Processes and Approaches

Unit rate multiplicative.

Additive reasoning.

Factor of change.

Translations Anticipated in Solution Process



Interview 2 Item 1.2

James has a part-time job at a restaurant and is paid $9.50 for each hour he works. The chart below reflects his earnings from his most recent work shifts. Write an input-output rule that can determine the amount of money James earns for working any number of hours.

Hours Worked	Money Earned in Dollars
2	$19.00
6	?
?	$71.25

Solution: (# of dollars) = (9.5 dollars per 1 hour worked) x (# of hours worked)

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1	A, B, C, F, G, H

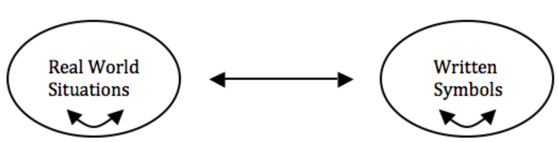
Problem Type	**Notes on Context and Problem Type**
Generalization to the y = mx rule that can be used to solve for any rate pair in the given proportion related situation.	Familiar context of work and pay. Non-integer unit rate given.

Multiplicative Structure Analysis

Hours Worked		Dollars
2	→ ×9.5	19
6		57
7.5		71.25

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.

Translations Anticipated in Solution Process



Interview 2 Item 2.1

The population of a bacteria culture increases by 50% each hour. The table shows the population at different time intervals during an observation. Is this situation proportional? Why or why not?

Time in hours	Number of Bacteria
0	192
1	288
2	432
3	648
4	972

Solution: The situation is not proportional. It is exponential.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	A

Problem Type	Notes on Context and Problem Type
Non-proportional situation	Exponential growth problem.

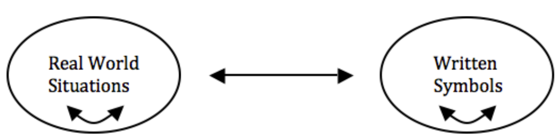
Multiplicative Structure Analysis

$(\#\ \text{of bacteria}) = 192 \times 1.5^{(\#\ \text{of hours})}$

Anticipated Reasoning Processes and Approaches

Unit rate approach.
Additive reasoning.
Factor of change approach.
Multiplicative reasoning using an exponential model.

Translations Anticipated in Solution Process



Interview 2 Item 3.1

The Bank of Europe gave fewer Euros (€) in exchange for more U.S. Dollars ($) this week than it did last week. Did the exchange rate, €/$, increase, decrease, stay the same, or can it not be determined?

Solution: The exchange rate, €/$, decreased.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, E, F, G, H

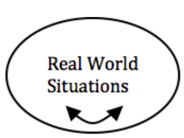
Problem Type	**Notes on Context and Problem Type**
Qualitative Reasoning Problem.	Currency exchange context may be difficult. Non-routine task.

Multiplicative Structure Analysis

	Numerator (Euros)		
Denominator (Dollars)	Stays Same	Increase	**Decrease**
Stays Same	No Change	Increase	Decrease
Increase	Decrease	Indeterminate	**Decrease**
Decrease	Increase	Increase	Indeterminate

Anticipated Reasoning Processes and Approaches
Unit rate comparison.

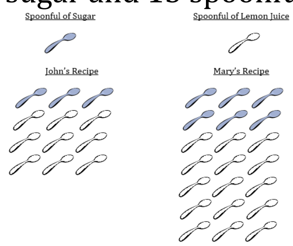
Translations Anticipated in Solution Process



Based on: Cramer, 2014.

Interview 2 Item 4.1

John and Mary make lemonade concentrate by mixing spoonfuls of sugar and spoonfuls of lemon juice. John makes his concentrate by using 3 spoonfuls of sugar and 9 spoonfuls of lemon juice. Mary makes her concentrate by using 6 spoonfuls of sugar and 15 spoonfuls of lemon juice.



Whose lemonade concentrate is sweeter, John's or Mary's, do they taste the same, or is it impossible to tell?

Solution: Mary's concentrate is sweeter 6/15 > 3/9.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
2, 3	C, F, G, H

Problem Type	**Notes on Context and Problem Type**
Comparison Problem.	Familiar mixture context. Non-routine task based on presentation.

Multiplicative Structure Analysis

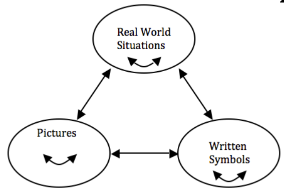
Spoonfuls of Sugar	Spoonfuls of Lemon Juice	Unit Rate
3	9	3/9 spoonfuls of sugar per 1 spoonful of lemon juice
6	15	6/15 spoonfuls of sugar per 1 spoonful of lemon juice

Same after reducing one pair: or extraction of (1,1) unit (Noelting, 1980).

Anticipated Reasoning Processes and Approaches

Unit rate comparison.

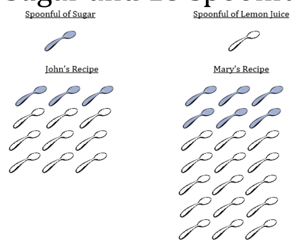
Translations Anticipated in Solution Process



Source: Karplus, Pulos & Stage, 1983.

Interview 2 Item 4.2

John and Mary make lemonade concentrate by mixing spoonfuls of sugar and spoonfuls of lemon juice. John makes his concentrate by using 3 spoonfuls of sugar and 9 spoonfuls of lemon juice. Mary makes her concentrate by using 6 spoonfuls of sugar and 15 spoonfuls of lemon juice.



How much lemon juice would Mary need with her 6 spoonfuls of sugar to make her concentrate taste just like John's?

Solution: Mary would need 18 spoonfuls of sugar for her concentrate to taste the same as John's.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 3, 4	A, C, D, E, F

Problem Type	**Notes on Context and Problem Type**
Missing Value Problem.	Familiar mixture context. Non-routine task based on presentation.

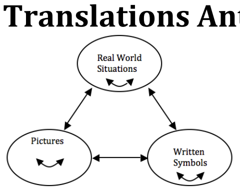
Multiplicative Structure Analysis

Spoonfuls of Sugar		Spoonfuls of Lemon Juice
3	X3	9
↓ x2		↓ x2
6	X3	18

Anticipated Reasoning Processes and Approaches

Unit rate approach.
Factor of Change approach.
Building up approach.
Standard algorithm.

Translations Anticipated in Solution Process



Source: Karplus, Pulos & Stage, 1983.

Interview 2 Item 5.1

A trail mix company mixes 2 pounds of dried fruit per every 5 pounds of nuts in their signature mix. The company is going to make a large batch of their signature trail mix that contains 70 pounds of nuts. How many pounds of dried fruit will the company use in the batch?

Solution: 28 pounds of dried fruit.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2, 3, 4	A, B, C, D, E, F, G, H

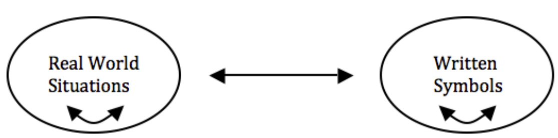
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Mixture context. Non-integer unit rates.

Multiplicative Structure Analysis

Pounds of Nuts		Pounds of Dried Fruit	
5	$\xrightarrow{\times\frac{2}{5}}$	2	
$\downarrow\times 14$		$\downarrow\times 14$	Factor of change
70	$\xrightarrow{\times\frac{2}{5}}$	?	

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Additive reasoning.
Factor of change.

Translations Anticipated in Solution Process



Interview 2 Item 5.2

A trail mix company mixes 2 pounds of dried fruit per every 5 pounds of nuts in their signature mix. Sketch a graph of the relationship between the number of pounds of dried fruit and the number of pounds of nuts that the company uses in their signature mix.

Select an (x,y) point on the graph. What is the meaning of the y/x relationship in terms of pounds of dried fruit and pounds of nuts?

Solution: It is the relationship between the number of pounds of nuts per number of pounds of dried fruit. Unit rates: 2/5 pounds of dried fruit per 1 pound of nuts or 5/2 pounds of nuts per 1 pound of dried fruit.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2	B, C

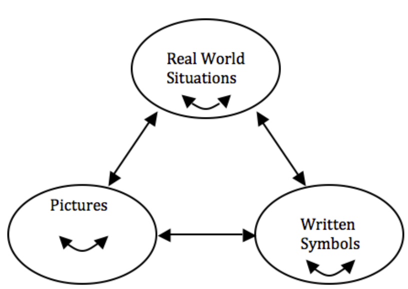
Problem Type	Notes on Context and Problem Type
Graphical interpretation of a proportion related situation.	Mixture context. Non-integer unit rates.

Multiplicative Structure Analysis

Pounds of Nuts		Pounds of Dried Fruit
5	$\xrightarrow{\times\frac{2}{5}}$ $\xleftarrow{\times\frac{5}{2}}$	2

Anticipated Reasoning Processes and Approaches
Unit rate.

Translations Anticipated in Solution Process



Interview 2 Item 5.3

A trail mix company mixes 2 pounds of dried fruit per every 5 pounds of nuts in their signature mix. Sketch a graph of the relationship between the number of pounds of dried fruit and the number of pounds of nuts that the company uses in their signature mix.

Select another (x,y) point on the graph. What is the meaning of the y/x relationship in terms of pounds of dried fruit and pounds of nuts? How are these two y/x rates realted?

Solution: It is the relationship between the number of pounds of nuts per number of pounds of dried fruit. Unit rates: 2/5 pounds of dried fruit per 1 pound of nuts or 5/2 pounds of nuts per 1 pound of dried fruit. The rate pairs are equivalent.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 3	B, C, D, H

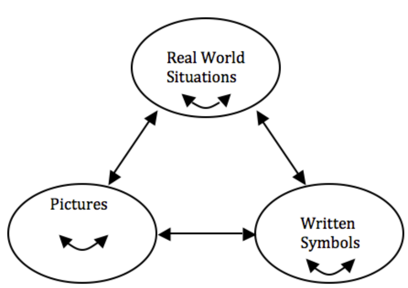
Problem Type	**Notes on Context and Problem Type**
Graphical interpretation of a proportion related situation.	Mixture context. Non-integer unit rates.

Multiplicative Structure Analysis

Pounds of Nuts		Pounds of Dried Fruit
5	→ $\times\frac{2}{5}$ ← $\times\frac{5}{2}$	2

Anticipated Reasoning Processes and Approaches
Unit rate.

Translations Anticipated in Solution Process



Interview 2 Item 6.1

Rice can be bought in bulk at the grocery store. This week rice is priced at $6.00 per 4 pounds. Kristie bought 3 pounds of rice. How much did Kristie pay for the rice she bought?

Solution: (3 pounds of rice) x (6 dollars per 4 pound of rice) = $4.50

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2, 4	A, B, C, D, E, F, G, H

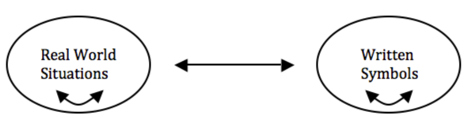
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Familiar context of unit pricing in grocery shopping. Non-integer unit rate must be derived. Non-integer factor of change.

Multiplicative Structure Analysis

Pounds of rice		Price of rice
4	X 6/4	6
3	X 6/4	4.5

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Estimation.
Standard algorithm.

Translations Anticipated in Solution Process



Interview 2 Item Summary

Proportionality Structures Targeted

Code	Proportionality as a Multiplicative Structure	Items
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	1.1, 1.2, 2.1, 4.2, 5.1, 5.2, 5.3, 6.1
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	1.1, 4.1, 4.2, 5.1, 5.2, 5.3, 6.1
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	1.1, 4.1, 4.2, 5.1, 5.3
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	4.2, 5.1, 6.1
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	3.1

Proportionality Structures Targeted By Item

	Proportionality Structures Targeted				
Item	1	2	3	4	5
1.1	X	X	X	X	
1.2	X				
2.1	X				
3.1					X
4.1		X	X		
4.2	X	X	X	X	
5.1	X	X	X	X	
5.2	X	X			
5.3	X	X	X		
6.1	X	X		X	
Total	8	7	5	4	1

Psychological Aspects of Proportional Reasoning Targeted

Code	Psychological Aspects of Proportional Reasoning	Items
A	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	1.1, 1.2, 2.1, 4.2, 5.1, 6.1
B	Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	1.1, 1.2, 5.1, 5.2, 5.3, 6.1
C	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	1.1, 1.2, 3.1, 4.1, 4.2, 5.1, 5.2, 5.3, 6.1
D	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	1.1, 4.2, 5.1, 5.3, 6.1
E	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	1.1, 3.1, 4.2, 5.1, 6.1
F	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	1.1, 1.2, 3.1, 4.1, 4.2, 5.1, 6.1
G	Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	1.1, 1.2, 3.1, 4.1, 5.1, 6.1
H	Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	1.1, 1.2, 4.1, 5.1, 5.3, 6.1

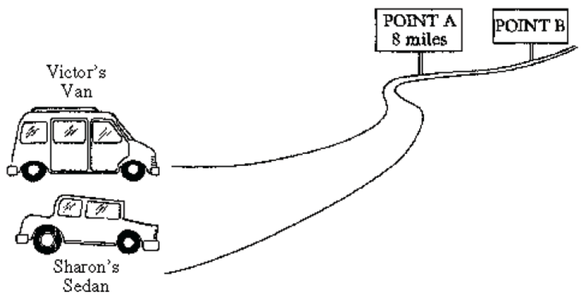
Psychological Aspects of Proportional Reasoning Targeted by Item

	Psychological Aspect of Proportional Reasoning							
Item	A	B	C	D	E	F	G	H
1.1	X	X	X	X	X	X	X	X
1.2	X	X	X			X	X	X
2.1	X							
3.1			X		X	X	X	
4.1			X			X	X	X
4.2	X		X	X	X	X		
5.1	X	X	X	X	X	X	X	X
5.2		X	X					
5.3		X	X	X				X
6.1	X	X	X	X	X	X	X	X
Total	6	6	9	5	5	7	6	6

Developmental Mathematics Interview 3 Item Analyses

Interview 3 Item 1.1

Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes. Both cars start at the same time and from the same location. Will Sharon's sedan reach point A, 8 miles away, before, at the same time as, or after Victor's van?



Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Solution: The sedan will reach point A at the same time as the van.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 3, 4	A, B, C, D, E, F, G, H

Problem Type	**Notes on Context and Problem Type**
Comparison Problem	Familiar context of speed. Non-integer unit rate. Equivalent rates.

Multiplicative Structure Analysis

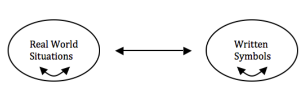
Distance		Time	
8	$\xrightarrow{\times \frac{10 \text{ miles}}{8 \text{ minutes}}}$ $\xrightarrow{\times \frac{25 \text{ miles}}{20 \text{ minutes}}}$	10	
$\downarrow \times \frac{5}{2}$		$\downarrow \times \frac{5}{2}$	Factor of Change
20		25	

Anticipated Reasoning Processes and Approaches

Unit rate multiplicative.

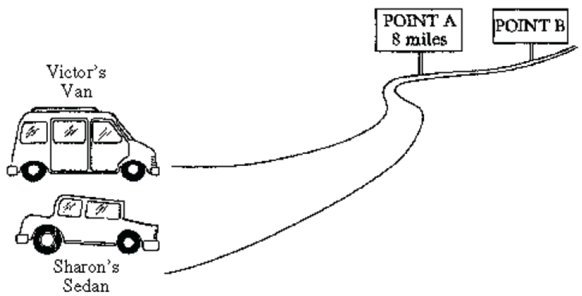
Factor of change.

Translations Anticipated in Solution Process



Interview 3 Item 1.2

Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes. What are two rates that can describe the relationship between miles and minutes for the travel of Victor's van?



Source: NAEP, 2014.

Solution: 8 miles / 10 minutes, 10 minutes / 8 miles.

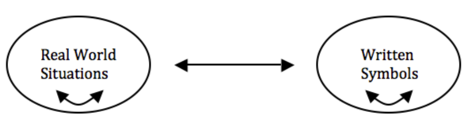
Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
2	B, C

Problem Type	Notes on Context and Problem Type
Rate interpretation problem	Familiar context of speed. Non-integer unit rate.

Multiplicative Structure Analysis

Distance		Time
8	→ $\times \frac{10 \text{ minutes}}{8 \text{ miles}}$	10
	→ $\times \frac{25 \text{ minutes}}{20 \text{ miles}}$	
20	← $\times \frac{8 \text{ miles}}{10 \text{ minutes}}$	25
	← $\times \frac{20 \text{ miles}}{25 \text{ minutes}}$	

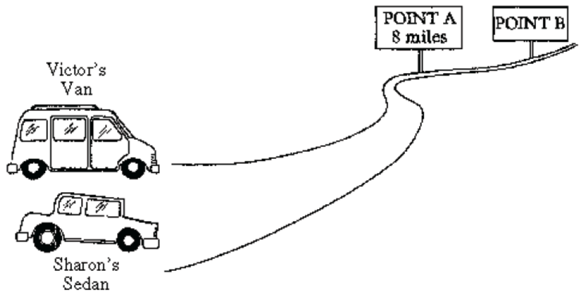
Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Translations Anticipated in Solution Process



Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Interview 3 Item 1.3

Victor’s van travels at a rate of 8 miles every 10 minutes. Sharon’s sedan travels at a rate of 20 miles every 25 minutes. Write an input-output rule that gives the number of miles Victor’s van travels for any given number of minutes.



Solution: (# of miles) = (8 miles / 10 minutes) x (# of minutes)

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	B, C, D, F

Problem Type	Notes on Context and Problem Type
Generalization of proportionality structure problem	Familiar context of speed. Non-integer unit rate.

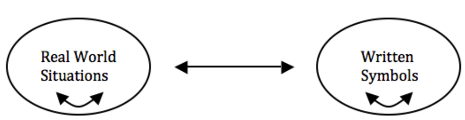
Multiplicative Structure Analysis

Distance		Time
8	→ $\times \frac{10 \text{ minutes}}{8 \text{ miles}}$	10
	→ $\times \frac{25 \text{ minutes}}{20 \text{ miles}}$	
20	← $\times \frac{8 \text{ miles}}{10 \text{ minutes}}$	25
	← $\times \frac{20 \text{ miles}}{25 \text{ minutes}}$	

Anticipated Reasoning Processes and Approaches

Unit rate multiplicative.

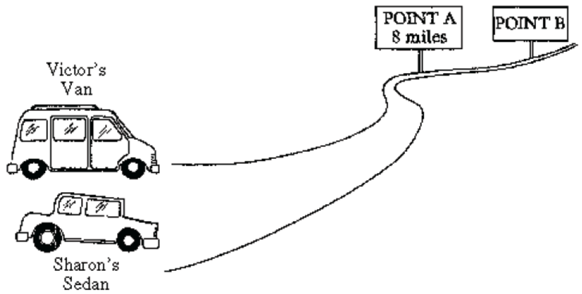
Translations Anticipated in Solution Process



Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Interview 3 Item 1.4

Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes. Draw a graph that represents the relationship between miles traveled and time for Victor's van.



Solution: Graph of (# of miles) = (8 miles / 10 minutes) x (# of minutes) or of (# of minutes) = (10 minutes / 8 miles) x (# of miles).

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	B, C

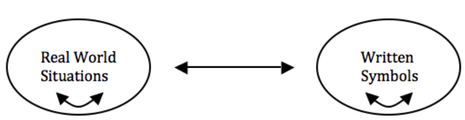
Problem Type	Notes on Context and Problem Type
Graphical interpretation of a proportion related situation.	Familiar context of speed. Non-integer unit rates.

Multiplicative Structure Analysis

Distance		Time
8	→ $\times \frac{10 \text{ minutes}}{8 \text{ miles}}$	10
	→ $\times \frac{25 \text{ minutes}}{20 \text{ miles}}$	
20	← $\times \frac{8 \text{ miles}}{10 \text{ minutes}}$	25
	← $\times \frac{20 \text{ miles}}{25 \text{ minutes}}$	

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.

Translations Anticipated in Solution Process

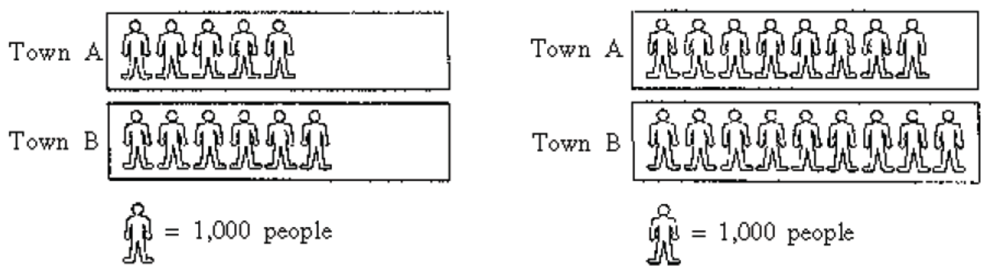


Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Interview 3 Item 2.1

In 2000, the populations of Town A and Town B were 5,000 and 6,000 respectively. The 2010 populations of Town A and Town B were 8,000 and 9,000 respectively. Brian claims that from 2000 to 2010 the populations of the two towns grew by the same amount, 3000 people. Was Brian reasoning proportionally about the population growth? How can you tell?

2000 Population 2010 Population



Solution: Brian was not reasoning proportionally, he was reasoning additively. The population of each town grew by 3000 people. (# of people in the year 2000) + 3000 people = (# of people in the year 2010)

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1	A

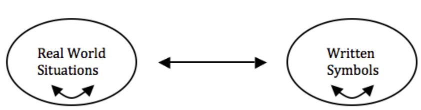
Problem Type	Notes on Context and Problem Type
Identifying a non-proportional linear relationship.	Familiar context.

Mathematical Structure Analysis

2000 population		2010 population	
5,000	→ +3000	8,000	Town A
6,000	→ +3000	9,000	Town B

Anticipated Reasoning Processes and Approaches
Additive reasoning.

Translations Anticipated in Solution Process

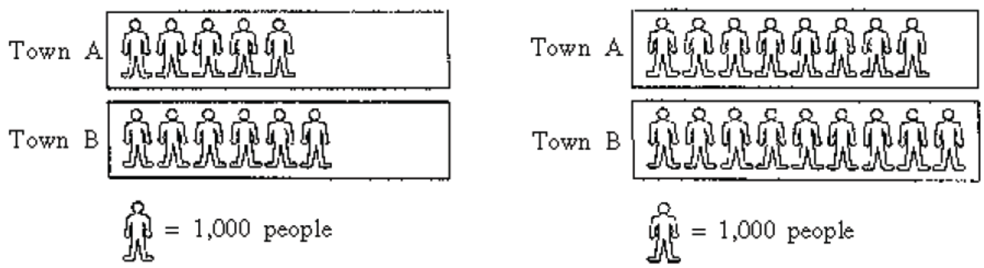


Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Interview 3 Item 2.2

In 2000, the populations of Town A and Town B were 5,000 and 6,000 respectively. The 2010 populations of Town A and Town B were 8,000 and 9,000 respectively. Darlene claims that from 2000 to 2010 the population of the Darlene claims that from 2000 to 2010 the population of the Town A had grown more. Darlene computed that the 2010 population of Town A was 160% of its 2000 population and the 2010 population of Town B was 150% of its 2000 population. Was Darlene reasoning proportionally about the population growth? How can you tell?

2000 Population 2010 Population



Solution: 8000/5000 > 9000/6000. Percent change may be computed. 60% change for Town A, 50% change for Town B. Both cases are comparing rates.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 5	A, B, C, F, G, H

Problem Type	**Notes on Context and Problem Type**
Comparison Problem	Rate of population growth must be derived and then compared.

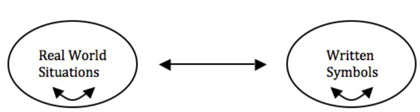
Multiplicative Structure Analysis

2000 population		2010 population	
5,000	$\xrightarrow{\times \frac{8000}{5000}}$	8,000	Town A
6,000	$\xrightarrow{\times \frac{9000}{6000}}$	9,000	Town B

Anticipated Reasoning Processes and Approaches

Unit rate comparison.
Additive reasoning.
Computation of a percent change.

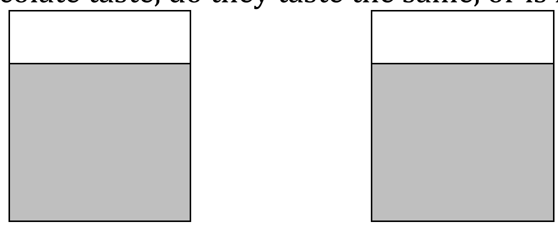
Translations Anticipated in Solution Process



Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Interview 3 Item 3.1

Thermos A contains cocoa with a weaker chocolate taste. When one scoop of cocoa mix is added to both Thermos A and Thermos B, which thermos contains the cocoa with the stronger chocolate taste, do they taste the same, or is it impossible to tell?



Thermos A   Thermos B

Source: Billings, 2002.

Solution: Thermos B has the stronger taste.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
5	C, F, G, H

Problem Type	**Notes on Context and Problem Type**
Qualitative Reasoning Problem.	Mixture context is familiar. Challenging problem type.

Multiplicative Structure Analysis

	Scoops of Mix	Cups of Water	Unit Rate
Pitcher A	X	Y	X/Y Scoops cocoa mix per 1 Water
Pitcher B	aX (a > 1)	Y	aX/Y Scoops cocoa mix per 1 Water

Following 1 scoop of cocoa mix added:

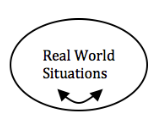
	Cups of OJ Concentrate	Cups of Water	Unit Rate
Pitcher A	X+1	Y	(X+1)/Y Scoops cocoa mix per 1 Water
Pitcher B	aX + 1 (a > 1)	Y	(aX+1)/Y OJ per 1 Water

aX < X, thus, (aX+1) > (X+1). Therefore, (aX+1)/Y > (X+1)/Y.

Anticipated Reasoning Processes and Approaches

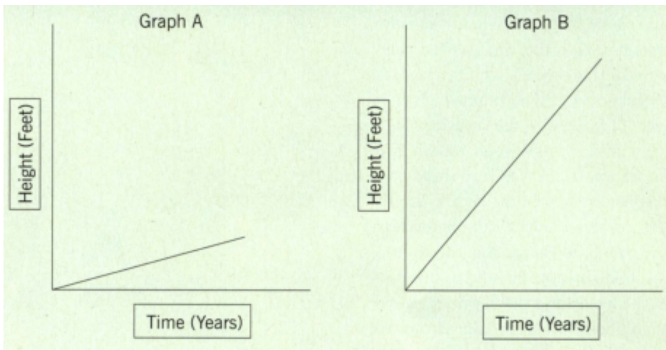
Unit rate comparison.

Translations Anticipated in Solution Process



Interview 3 Item 4.1

The growth of two different plants are shown in the graphs. The axes on the graphs are scaled the same, but the scaling is not shown. Which plant grew faster, did the plants grow at the same rate, or is it impossible to tell?



Source: Joram & Oleson, 2008, p. 261.

Solution: Plant B grew faster because the slope of the line is steeper, indicating a greater rate of change.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2	A, B, C

Problem Type	**Notes on Context and Problem Type**
Qualitative Comparison Problem.	Familiar context of growth. Non-routine task based on presentation.

Multiplicative Structure Analysis

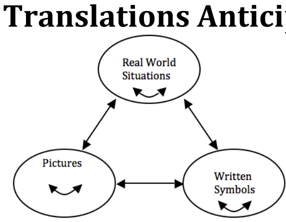
	Height	Years	Unit Rate
Plant A	A	X	A/X Feet per 1 year.
Plant B	B	X	B/X Feet per 1 year.

B > A, therefore, B/X > A/X.

Anticipated Reasoning Processes and Approaches

Unit rate comparison.

Translations Anticipated in Solution Process



Interview 3 Item 5.1

The Kikehs are planning a road trip during their semester break. The scale on the map they are using to plan their trip is 2.5 cm = 75 miles. The distance between their destination city and their current city is 18.75 cm on the map. What are two rates that can be used to describe the relationship between the number of centimeters on the map and the number of miles on land?

Solution: 2.5cm / 75 miles and 75 miles / 2.5 cm.

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
2	B, C

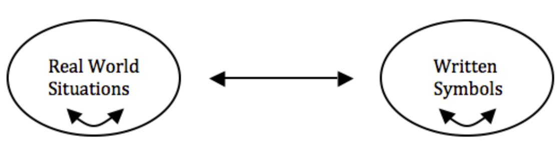
Problem Type	Notes on Context and Problem Type
Rate interpretation problem	Familiar context of map scaling. Integer unit rate of 30 available. Non-integer factor of change.

Multiplicative Structure Analysis

Distance on Map in cm		Distance on Land in Miles	
2.5	$\xrightarrow{\times \frac{75\text{ miles}}{2.5\text{ cm}}}$ $\xleftarrow{\times \frac{2.5\text{ cm}}{75\text{ miles}}}$	75	
↓ ×7.5		↓ ×7.5	Factor of change
18.75	$\xrightarrow{\times \frac{75\text{ miles}}{2.5\text{ cm}}}$	562.5	

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.

Translations Anticipated in Solution Process



Interview 3 Item 5.2

The Kikehs are planning a road trip during their semester break. The scale on the map they are using to plan their trip is 2.5 cm = 75 miles. The distance between their destination city and their current city is 18.75 cm on the map. How many miles will the Kikehs drive to reach their destination if they follow the route on their map?

Solution: 562.5 miles.

Proportionality Structures Targeted	**Aspects of Proportional Reasoning Targeted**
1, 2, 3, 4	A, B, C, D, E

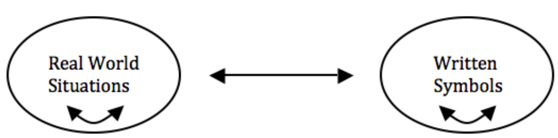
Problem Type	**Notes on Context and Problem Type**
Missing Value Problem	Familiar context of map scaling. Integer unit rate of 30 available. Non-integer factor of change.

Multiplicative Structure Analysis

Distance on Map in cm		Distance on Land in Miles	
2.5	$\xrightarrow{\times \frac{75 \text{ miles}}{2.5 \text{ cm}}}$	75	
$\downarrow \times 7.5$		$\downarrow \times 7.5$	Factor of change
18.75	$\xrightarrow{\times \frac{75 \text{ miles}}{2.5 \text{ cm}}}$	562.5	

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.

Translations Anticipated in Solution Process



Interview 3 Item 6.1

Rice can be bought in bulk at the grocery store. This week rice is priced at $3.35 per 1.54 pounds. Kristie bought 5.92 pounds of rice. How much did Kristie pay for the rice she bought?

Solution: (5.92 pounds of rice) x (3.35 dollars per 1.54 pound of rice) = $12.88

Proportionality Structures Targeted	Aspects of Proportional Reasoning Targeted
1, 2	A, B, C, D, E, F, G, H

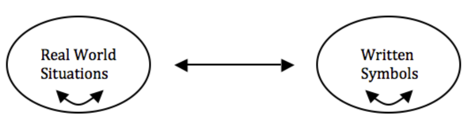
Problem Type	Notes on Context and Problem Type
Missing Value Problem	Familiar context of unit pricing in grocery shopping. Non-integer unit rate must be derived. Non-integer factor of change.

Multiplicative Structure Analysis

Pounds of rice		Price of rice
1.54	X 3.35/1.54	3.35
5.92	X 3.35/1.54	12.88

Anticipated Reasoning Processes and Approaches
Unit rate multiplicative.
Estimation.
Standard algorithm.

Translations Anticipated in Solution Process



Interview 3 Item Summary

Proportionality Structures Targeted

Code	Proportionality as a Multiplicative Structure	Items
1	Proportionality is a linear relationship between two quantities that covary according to the model *y = mx*, where *m* is the unit rate. All corresponding *(x,y)* rate pairs lie on the graph of the line *y = mx*, which passes through the origin (Karplus et al., 1983; Lamon, 2007; Post et al., 1988).	1.1, 1.3, 1.4, 2.1, 2.2, 4.1, 5.2, 6.1
2	In proportional situations, two invariant unit rates exist across measure spaces. The unit rates are reciprocals and define inverse functions: *y = mx* and *x = (1/m) y* (Lamon, 2007; Post et al., 1988, Vergnaud, 1983).	1.1, 1.2, 2.2, 4.1, 5.1, 5.2, 6.1
3	All *(x,y)* rate pairs in a proportional situation create an equivalence class (Post et al., 1988).	1.1, 5.2
4	In proportional situations, there exists a scalar multiplicative relationship within measure spaces (Vergnaud, 1983).	1.1
5	Proportionality is a mathematical structure that defines contextual situations that exist in nature such as density, speed and pricing. Three types of proportion related problems are: missing value problems, comparison problems, and qualitative reasoning problems (Lamon, 2007; Post et al., 1988; Lesh et al., 1987).	2.2, 3.1

Proportionality Structures Targeted By Item

	Proportionality Structures Targeted				
Item	1	2	3	4	5
1.1	X	X	X	X	
1.2		X			
1.3	X				
1.4	X				
2.1	X				
2.2	X	X			X
3.1					X
4.1	X	X			
5.1		X			
5.2	X	X	X		
6.1	X	X			
Total	8	7	2	1	2

Psychological Aspects of Proportional Reasoning Targeted

Code	Psychological Aspects of Proportional Reasoning	Items
A	Proportional reasoning requires the differentiation between proportional and non-proportional situations (Post et al., 1988).	1.1, 2.1, 2.2, 4.1, 5.2, 6.1
B	Proportional reasoning involves the recognition and use of a functional relationship between measure spaces (Karplus et al., 1983; Lamon, 2007; Vergnaud, 1983).	1.1, 1.2, 1.3, 1.4, 2.2, 4.1, 5.1, 5.2, 6.1
C	The interpretation of rates (as demonstrated through interpretation of unit rate) and their reciprocals can be made both quantitatively and qualitatively when reasoning proportionally (Post et al., 1988).	1.1, 1.2, 1.3, 1.4, 2.2, 3.1, 4.1, 5.1, 5.2, 6.1
D	The invariant relationship between two variables, x and y, can be extended to other equal multiples of x and y using proportional reasoning (Karplus et al., 1983; Lamon, 2007; Lobato et al., 2009).	1.1, 1.3, 5.2, 6.1
E	The identification and utilization of covariant and invariant relationships and multiplicative thinking are central to proportional reasoning processes. (Lamon, 2007; Post et al., 1988)	1.1, 5.2, 6.1
F	Proportional reasoning enables the use of proportionality as a mathematical model to organize appropriate real world contexts and the use of qualitative reasoning to guide approach and determine reasonableness of solutions (Post et al., 1988).	1.1, 1.3, 2.2, 3.1, 6.1
G	Proportional reasoning involves flexible thought and approach in problem solving situations and can overcome quantitative and qualitative complexities (Post et al., 1988).	1.1, 2.2, 3.1, 6.1
H	Proportional reasoning involves the ability to make multiple comparisons and simultaneously store and process several pieces of information. (Post et al., 1988).	1.1, 2.2, 3.1, 6.1

Psychological Aspects of Proportional Reasoning Targeted by Item

	Psychological Aspect of Proportional Reasoning							
Item	A	B	C	D	E	F	G	H
1.1	X	X	X	X	X	X	X	X
1.2		X	X					
1.3		X	X	X		X		
1.4		X	X					
2.1	X							
2.2	X	X	X			X	X	X
3.1			X			X	X	X
4.1	X	X	X					
5.1		X	X					
5.2	X	X	X	X	X			
6.1	X	X	X	X	X	X	X	X
Total	6	4	10	5	4	8	9	8

Appendix E: Interview Protocols

Developmental Mathematics Interview 1 Protocol

For each item, provide student with paper with written stem. The interviews are semi-structured task interviews. The stems and initial questions will be asked, but the follow-up questions are meant to be adapted based on individual student responses.

1. An employee receives 3 days paid vacation for every 15 weeks of work.

   Question 1: How many vacations days does the employee receive for working 35 weeks?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   What does the rate of 3 vacation days per 15 weeks of work tell you?
   What are other ways you can represent the rate of 3 days paid vacation for every 15 weeks of work?

   Question 2: How many weeks must an employee work to receive 10 days paid vacation?

   Follow-up questions:
   Why did you select the rate you used in this problem?
   How did the rate you used help you solve this problem?

   Question 3: Write a rule that can be used to solve for the number of weeks that must be worked to receive any number of paid vacation days.

   Follow-up questions:
   How did you select the rate you used in this problem?
   Write a problem that can be solved using the rule.

   Question 4: This is a graph that shows the relationship between the number of vacation days that are received and the number of weeks that are worked. The (x,y) point (4.5, 22.5) is shown on the graph. Write the ratio of y/x for this (x,y) pair. What does this rate mean?

Follow-up question:
Choose another point on the line. What is the ratio of y/x? What is the same in this ratio and the ration 22.5/4.5? What is different?

2. There was a sale on chocolate candies. One piece cost 15¢, two pieces cost 27¢, three pieces cost 39¢, and four pieces cost 51¢.

   Question 1: Is this situation proportional? Why or why not?

3. Cece and Sou ran laps around a track.

   Question 1: Cece ran for more time that Sou, and they both ran the same amount of laps. Who was the faster runner, were the running at the same speed, or is it impossible to tell?

   Follow-up questions:
   Are Cece and Sou's running rates proportional to each other? Why or why not?
   How did you determine your answer?

   Question 2: Cece ran fewer laps today in less time than she did yesterday. How does her running speed today compare to her speed yesterday?

   Follow-up questions:
   Is Cece's running rates proportional to eachother? Why or why not?
   How did you determine your answer?

4. Orange juice is made by mixing cups of orange juice concentrate and cups of water. Recipes are described below according to the number of cups of

orange juice concentrate and the number of cups of water that are mixed to make the juice.

Cup of Orange Juice Concentrate — Cup of Water

Question 1: Which recipe has the strongest orange juice taste, do the recipes taste the same, or is it impossible to tell?

Recipe A — Recipe B

Source: Noelting, 1980.

Follow-up questions:
Are the rates proportional to each other? Why or why not?
How did you determine your answer?
Can you make another recipe that would taste the same?

Question 2: Which recipe has the strongest orange taste, do the recipes taste the same, or is it impossible to tell?

Recipe A — Recipe B

Follow-up questions:
Are the rates proportional to each other? Why or why not?
How did you determine your answer?
How could you change Recipe A so that it tastes the same as Recipe B?

5. Steph and Matt are racecar drivers. They tested their cars' fuel efficiency driving at race speeds on an oval racetrack used for a long distance car race. Steph's car used 16.3 gallons of gas on a 61.8 mile drive. Matt's car used 13.2 gallons of gas on a 54.12 mile drive.

   Question: Whose car had better fuel efficiency, were they the same or is it impossible to tell?

   Follow-up questions:
   Is these rates proportional to each other? Why or why not?

What other rate can you make in this situation? Write both rates as unit rates.
What does the unit rate of miles per gallon tell you?
What does the unit rate of gallons per mile tell you?

6. Rice can be bought in bulk at the grocery store. This week rice is priced at $3.00 per pound. Kristie bought 4 pounds of rice.

   Question: How much did Kristie pay for the rice she bought?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   Did you use a rate to solve this problem? / How could you use a rate to solve this problem?
   What is another way you could solve this problem?

Developmental Mathematics Interview 2 Protocol

For each item, provide student with paper with written stem. The interviews are semi-structured task interviews. The stems and initial questions will be asked, but the follow-up questions are meant to be adapted based on individual student responses.

1. James has a part-time job at a restaurant and is paid $9.50 for each hour he works. The chart below reflects his earnings from his most recent work shifts. Complete the chart.

Hours Worked	Money Earned in Dollars
2	$19.00
6	?
?	$71.25

Question 1: Is this situation proportional? Why or why not?

Question 2: Complete the chart.

Follow-up questions:
How did you use rates differently / how did you select different rates to use to fill in the chart?

Question 2: Write an input-output rule that can determine the amount of money James earns for working any number of hours.

Follow-up questions:
How did you select the rate you used in this problem?
Write a problem that can be solved using the rule.

2. The population of a bacteria culture increases by 50% each hour. The table shows the population at different time intervals during an observation.

Time in hours	Number of Bacteria
0	192
1	288
2	432
3	648
4	972

Question 1: Is the situation proportional? Why or why not?

3. The Bank of Europe gave fewer Euros (€) in exchange for more U.S. Dollars ($) this week than it did last week.

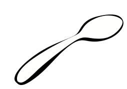
   Question 1: Did the exchange rate, €/$, increase, decrease, stay the same, or can it not be determined?

   Follow-up questions:
   Are the exchange rates proportional to each other? Why or why not?
   How did you determine your answer?

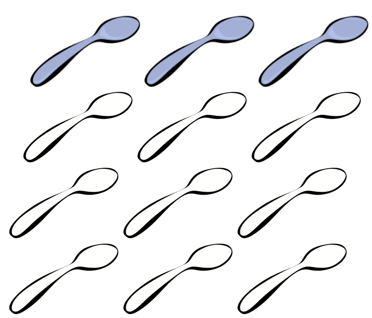
4. John and Mary make lemonade concentrate by mixing spoonfuls of sugar and spoonfuls of lemon juice. John makes his concentrate by using 3 spoonfuls of sugar and 9 spoonfuls of lemon juice. Mary makes her concentrate by using 6 spoonfuls of sugar and 15 spoonfuls of lemon juice.

   Spoonful of Sugar

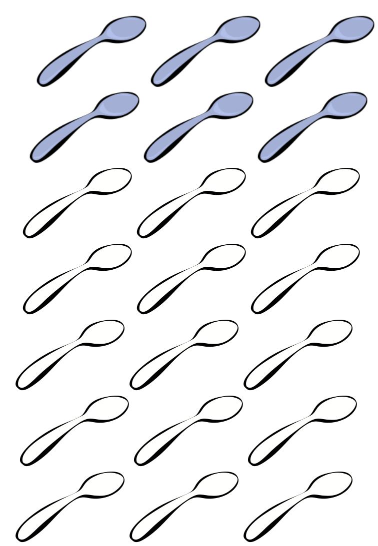
   Spoonful of Lemon Juice



   John's Recipe



   Mary's Recipe



   Question 1: Whose lemonade concentrate is sweeter, John's or Mary's, do they taste the same, or is it impossible to tell?

   Question 2: How did you come up with this answer?

Question 3: (If it was determined the lemonades have unequal tastes) How much lemon juice would Mary need with her 6 spoonfuls of sugar to make her concentrate taste just like John's?

Question 4: Please explain your answer.

Source: Karplus, Pulos & Stage, 1983.

5. A trail mix company mixes 2 pounds of dried fruit per every 5 pounds of nuts in their signature mix. The company is going to make a large batch of their signature trail mix that contains 70 pounds of nuts.

   Question 1: Is the relationship between the number of pounds of dried fruit and the number of nuts proportional? Why or why not?

   Question 2: How many pounds of dried fruit will the company use in the batch?

   Follow-up questions:
   What other rate can you make in this situation? Write both rates as unit rates.
   What does each unit rate tell you?
   Write a rule whose output is the number of pounds of dried fruit that the company would use for any given number of pounds of nuts.

   Question 3: Sketch a graph of the relationship between number of pounds of dried fruit and the number of pounds of nuts that the company uses in their signature mix.

   Follow-up questions:
   Select an (x,y) point on the graph. What is the meaning of the y/x relationship in terms of pounds of dried fruit and pounds of nuts?
   Select another (x,y) point on the graph. What is the meaning of the y/x relationship terms of pounds of dried fruit and pounds of nuts?
   How are these two y/x rates related?

6. Rice can be bought in bulk at the grocery store. This week rice is priced at $6.00 per 4 pounds. Kristie bough 3 pounds of rice.

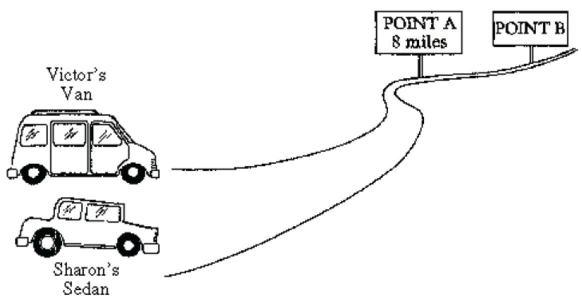
   Question 1: How much did Kristie pay for the rice she bought?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   How did you use rates in this problem?

Developmental Mathematics Interview 3 Protocol

For each item, provide student with paper with written stem. The interviews are semi-structured task interviews. The stems and initial questions will be asked, but the follow-up questions are meant to be adapted based on individual student responses.

1. Victor’s van travels at a rate of 8 miles every 10 minutes. Sharon’s sedan travels at a rate of 20 miles every 25 minutes.



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Question 1: If both cars start at the same time and from the same location, will Sharon’s sedan reach point A, 8 miles away, before, at the same time as, or after Victor’s van?

Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

Follow-up questions:
Are the rates at which the vehicles travel proportional to each other? Why or why not?

Question 2: What are two rates that can describe the relationship between miles and minutes for the travel of Victor’s van?

Follow-up questions:
Write each rate as a unit rate.
What does each rate tell you?
Describe the mathematical relationship between the two rates.

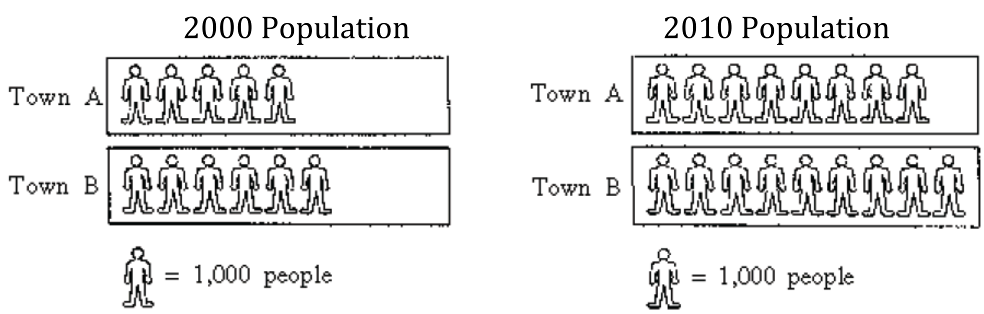
Question 3: Write an input-output rule that gives the number of miles Victor’s van travels for any given number of minutes.

Follow-up Questions:
How did you select the rate you selected for your rule?
Write a problem that can be solved using your rule.

Question 4: Draw a graph that represents the relationship between miles traveled and time for Victor's van.

Follow-up Questions:
How is the rate at which Victor's van travels represented in the graph?
What is the y-intercept of the graph? What is the meaning of the y-intercept?
When the input is 1, what is the output? How is this related to the rate in miles per hour at which Victor's van travels?

2. In 2000, the populations of Town A and Town B were 5,000 and 6,000 respectively. The 2010 populations of Town A and Town B were 8,000 and 9,000 respectively.



Question 1: Brian claims that from 2000 to 2010 the populations of the two towns grew by the same amount, 3000 people. Was Brian reasoning proportionally about the population growth? How can you tell?

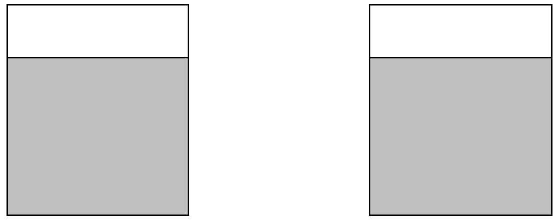
Follow-up questions:
How can you describe Brian's reasoning about the population growth of the two towns?

Question 2: Darlene claims that from 2000 to 2010 the population of the Town A had grown more. Darelene computed that the 2010 population of Town A was 160% of its 2000 population and the 2010 population of Town B was 150% of its 2000 population. Was Darlene reasoning proportionally about the population growth? How can you tell?

Follow-up questions:
How can you describe Darlene's reasoning about the population growth of the two towns?

Source: U.S. Department of Education, Institute of Education Sciences, National Center for Education Statistics, National Assessment of Educational Progress (NAEP), 1996 Mathematics Assessment.

3. Thermos A contains cocoa with a weaker chocolate taste.



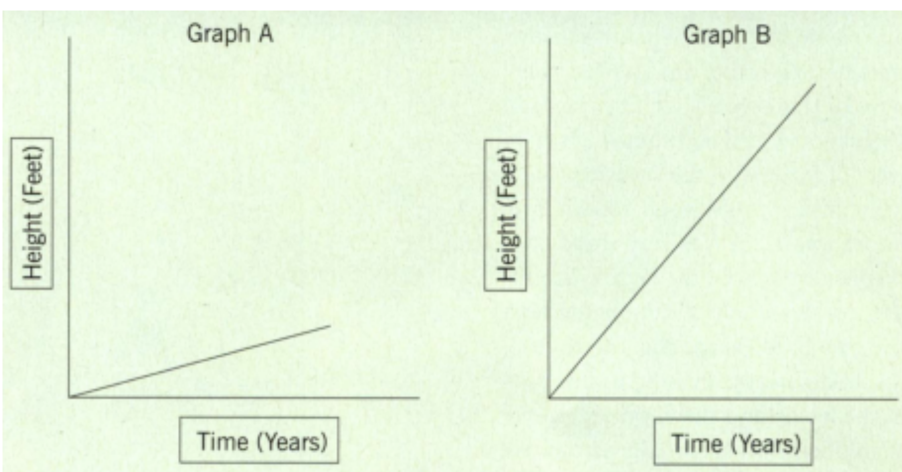
Thermos A | Thermos B

Source: Billings, 2002.

Question 1: When one cup of hot water is added to both Thermos A and Thermos B, which thermos contains the cocoa with the stronger chocolate taste, do they taste the same, or is it impossible to tell?

Follow-up questions:
How did you determine your answer?

4. The growth of two different plants are shown in the graphs. The axes on the graphs are scaled the same, but the scaling is not shown.



Source: Joram & Oleson, 2008, p. 261.

Question 1: Which plant grew faster, did the plants grow at the same rate, or is it impossible to tell?

Follow-up questions:
How did you determine your answer?

Is there a proportional relationship between the height of each plant and the time the plant grew? Why or why not?

5. The Kikehs are planning a road trip during their semester break. The scale on the map they are using to plan their trip is 2.5 cm = 75 miles.

   Question 1: What are two rates that can be used to describe the relationship between the number of centimeters on the map and the number of miles on land?

   Follow-up questions:
   Write each rate as a unit rate.
   What does each rate tell you?
   Is the relationship between the number of centimeters on the map and the number of miles on land proportional? Why or why not?

   Question 2: The distance between their destination city and their current city is 18.75 cm on the map. How many miles will the Kikehs drive to reach their destination if they follow the route on their map?

   Follow-up questions:
   Which rate did you use to solve this problem?
   Write a rule whose output is the number of centimeters on the map that correspond with any given number of miles on land.
   How did you decide which rate to use in your rule?

6. Rice can be bought in bulk at the grocery store. This week rice is priced at $3.35 per 1.54 pounds. Kristie bought 5.92 pounds of rice.

   Question: How much did Kristie pay for the rice she bought?

   Follow-up Questions:
   Is this situation proportional? Why or why not?
   How can rates be used to solve the problem?

College Level Mathematics Interview Protocol

For each item, provide student with paper with written stem. The interviews are semi-structured task interviews. The stems and initial questions will be asked, but the follow-up questions are meant to be adapted based on individual student responses.

1. An employee receives 3 days paid vacation for every 15 weeks of work.

   Question 1: How many vacations days does the employee receive for working 35 weeks?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   What does the rate of 3 vacation days per 15 weeks of work tell you?
   What are other ways you can represent the rate of 3 days paid vacation for every 15 weeks of work?

   Question 2: How many weeks must an employee work to receive 10 days paid vacation?

   Follow-up questions:
   Why did you select the rate you used in this problem?
   How did the rate you used help you solve this problem?

   Question 3: Write a rule that can be used to solve for the number of weeks that must be worked to receive any number of paid vacation days.

   Follow-up questions:
   How did you select the rate you used in this problem?
   Write a problem that can be solved using the rule.

   Question 4: This is a graph that shows the relationship between the number of vacation days that are received and the number of weeks that are worked. The (x,y) point (4.5, 22.5) is shown on the graph. Write the ratio of y/x for this (x,y) pair. What does this rate mean?

Follow-up question:
Choose another point on the line. What is the ratio of y/x? What is the same in this ratio and the ration 22.5/4.5? What is different?

2. There was a sale on chocolate candies. One piece cost 15¢, two pieces cost 27¢, three pieces cost 39¢, and four pieces cost 51¢.

   Question 1: Is this situation proportional? Why or why not?

3. Cece and Sou ran laps around a track.

   Question 1: Cece ran for more time that Sou, and they both ran the same amount of laps. Who was the faster runner, were the running at the same speed, or is it impossible to tell?

   Follow-up questions:
   Are Cece and Sou's running rates proportional to each other? Why or why not?
   How did you determine your answer?

   Question 2: Cece ran fewer laps today in less time than she did yesterday. How does her running speed today compare to her speed yesterday?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   How did you determine your answer?

4. Orange juice is made by mixing cups of orange juice concentrate and cups of water. Recipes are described below according to the number of cups of

orange juice concentrate and the number of cups of water that are mixed to make the juice.

Cup of Orange Juice Concentrate | Cup of Water

Question 1: Which recipe has the strongest orange juice taste, do the recipes taste the same, or is it impossible to tell?

Recipe A | Recipe B

Follow-up questions:
Are the rates proportional to each other? Why or why not?
How did you determine your answer?
Can you make another recipe that would taste the same?

Question 2: Which recipe has the strongest orange taste, do the recipes taste the same, or is it impossible to tell?

Recipe A | Recipe B

Follow-up questions:
Are the rates proportional to each other? Why or why not?
How did you determine your answer?
How could you change Recipe A so that it tastes the same as Recipe B?

Source: Noelting, 1980.

5. Steph and Matt are racecar drivers. They tested their cars' fuel efficiency driving at race speeds on an oval racetrack used for a long distance car race. Steph's car used 16.3 gallons of gas on a 61.8 mile drive. Matt's car used 13.2 gallons of gas on a 54.12 mile drive.

   Question: Whose car had better fuel efficiency, were they the same or is it impossible to tell?

   Follow-up questions:
   Are the rates proportional to each other? Why or why not?

What other rate can you make in this situation? Write both rates as unit rates.
What does the unit rate of miles per gallon tell you?
What does the unit rate of gallons per mile tell you?

6. Rice can be bought in bulk at the grocery store.

   Question 1: This week rice is priced at $3.00 per pound. Kristie bought 4 pounds of rice. How much did Kristie pay for the rice she bought?

   Follow-up questions:
   Is this situation proportional? Why or why not?
   Did you use a rate to solve this problem? / How could you use a rate to solve this problem?
   What is another way you could solve this problem?

   Question 2: This week rice is priced at $6.00 per 4 pounds. Kristie bought 3 pounds of rice. How much did Kristie pay for the rice she bought?

   Question 3: This week rice is priced at $3.35 per 1.54 pounds. Kristie bought 5.92 pounds of rice. How much did Kristie pay for the rice she bought?

Appendix F: Examples of Connected Understandings

Connected Understanding of Constructs 1 & 2

In the following excerpt, Sarah explains how the rate at which a van travels is represented in a graphical interpretation of the $d = rt$ functional relationship by identifying it as the slope of the line and interpreting the point (1, 0.8) as a unit rate.

Researcher: How is the rate at which Victor's van travels represented in the graph?

Sarah: Steady.

Researcher: Tell me more about that.

Sarah: It's not varying, it's not going up or down, *it's a straight line from starting at zero to my second point* [referencing the point (8,10)].

Researcher: Cool. You said that it starts at zero, I would call that a y-intercept because it is crossing your vertical axis.

Sarah: Yep.

Researcher: Where it's touching this part, [referencing the origin] tell me what that means, that it's going through zero in terms of miles and minutes.

Sarah: *If he's at zero, he hasn't gone any miles and he hasn't gone any minutes.*

Researcher: Great, good. When the input is 1 mile what would the output be? When you're on horizontally at one mile what would your vertical measure be?

Sarah: 0.8. [Note: Stated without being computed.]

Researcher: You got it. How is that related to the rate?

Student: *That is your unit rate*.

Connected Understanding of Constructs 2 & 3

In the following excerpt, Shannon determines that two vehicles are traveling at the same rate. She began her work by computing the the unit rates for each vehicle (as 1.25 minutes per mile), she explains through a follow up question that although the quantities presented for the vehicles are different, the rate at which the vehicles travel is the same.

Shannon: They will both get there at the same time.

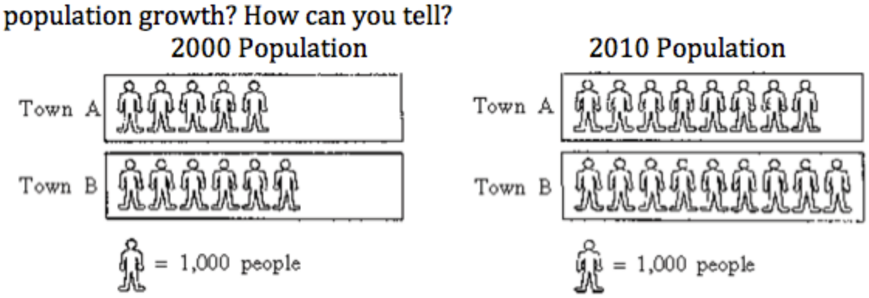
Researcher: Are the rates at which the vehicles travel proportional to each other? Why or why not?

Shannon: Yes, because they have different numbers, like the 20 miles and [8] miles, but they come up with the same [unit] rate.

Connected Understanding of Constructs 2 & 5

Students were asked to differentiate between proportional and non-proportional reasoning in a series of items based on the "Populations of Towns" problem context from the 1996 National Assessment of Educational Progress (NAEP) Mathematics Assessment (U.S. Department of Education, 2014), shown below.

In 2000, the populations of Town A and Town B were 5,000 and 6,000 respectively. The 2010 populations of Town A and Town B were 8,000 and 9,000 respectively. Brian claims that from 2000 to 2010 the populations of the two towns grew by the same amount, 3000 people. Was Brian reasoning proportionally about the population growth? How can you tell?



Jeff: It was not proportional. Because, sure, they each went up by the 3000 people, but *it's not the same percentage for how much each town grew over the same time period*…This town went from 5000 to 8000, so they didn't start off with as much and they didn't end up with as much either. *The proportion for how much their town grew was different because they had less to start with*. The fact that they had [grown by] the same amount of people, meant the town grew more.

Jeff (HP) used his understanding of the rate relationship between the change in population and beginning population of each town to apply qualitative reasoning about a rate (Construct 2) in a way that was embedded in the context of the problem (Construct 5) that determined that Brian was reasoning additively, not proportionally.

Connected Understanding of Constructs 3 & 4

Researcher: There is a sale on chocolate candies. 1 piece costs 15 cents. 2 pieces cost 27 cents. 3 pieces cost 39 cents, and 4 pieces cost 51 cents. Is this situation proportional? Why or why not?

Tamara: Let me double check. No.

Researcher: What you did there is … 27 divided by 15. And right away, how did that tell you that it is not proportional?

Tamara: Well, I already had it in my head, that if one piece cost 15 cents, to be proportional, two pieces would have to cost 30 cents. But I just had to double check to make sure.

Researcher: So you were just checking by saying, 27 cents divided by 15 cents, and if it was proportional…

Tamara: *It would have been two, but it was 1.8.*

In this excerpt, Tamara begins a list of equal rate pairs (Construct 3). She checked her initial estimation that the rates were not equal by computing the scalar multiple *27 / 15 = 1.8*. Her check verified that the situation was not proportional because the number of chocolates had doubled, but the costs had not (Construct 4).

Connected Understanding of Constructs 1 & 2 & 3

Researcher: Here is a graph that shows the number of weeks that are worked and the number of vacation days that are earned. The (x,y) pair (4.5 days, and 22.5 weeks) is shown. Write the ratio, as a fraction, y/x for this pair.

Jeff: The rate would be [5 over 1], I would guess.

Researcher: How did you get the rate?

Jeff: 'Cause if you follow the line, *it crosses at certain intersections, 1 and 5, 2 and 10, 3 and 15 4 and 20 and that kind of stuff. So it is the same rate all the way down, 1 and 5 is a fairly simple rate to use instead of 4 and a half over 22 and a half.*

The connections between Constructs 1, 2, and 3 were identified when Jeff listed several elements of the equivalence class as points on the line: (1,5), (2,10), (3,15), (4,20), and identified the unit rate 5/1 that characterized the equivalence.

Connected Understandings of Constructs 2 & 3 & 4

Researcher: A trail mix company mixes 2 pounds of dried fruit for every 5 pounds of nuts for their signature mix. The company is going to make a large batch of their signature mix that contains 70 pounds of nuts. How many pounds of dried fruit will the company use in the batch?

Jeff: I know how [many] pounds of nuts they want to use in it, so I took that divided by the 5 there for how much their recipe is, which is 14. So I just took 2 times 14 to get the 28 pounds, so that way the recipe is the same.

Researcher: …Now I want you to think about this in terms of the rate of 5 pounds of nuts to 2 pounds of fruit. How did you use that rate to solve this problem?

Jeff: Well, you have the 5 over 2, and 70 over, you don't know yet, that's what you are trying to find out. You are finding the scalar of it, which is the 14. In order to get 5 to 70 you have to multiply that by 14. *So if you multiply that part, you have to multiply the other part by the same number to get the same ratio*.

This excerpt demonstrates how Jeff (HP) supported a factor of change approach with an understanding of invariant unit rate by explaining the need to multiply the quantities from each measure by the same ratio to maintain a equal rate when determining equal rate pairs.