Minimizing or Maximizing the Expected Time to Reach Zero

by

D. Heath, Cornell University
S. Orey, University of Minnesota
V. Pestien, University of Miami
and
W. Sudderth*, University of Minnesota
University of Minnesota Tech. Report No. 447
February 1985

* Research supported by National Science Foundation Grants MCS83-01080 (for Orey) and MCS81-0079 (for Sudderth).

Abstract

We treat the following control problems: the process $X_1(t)$ with values in the interval $(-\infty,0]$ (or $[0,\infty)$) is given by the stochastic differential equation

$$dX_1(t) = \mu(t)dt + \sigma(t)dW_t, \; X_1(0) = x_1$$

where the non-anticipative controls μ and σ are to be chosen so that $(\mu(t),\sigma(t))$ remains in a given set $\mathcal{S}$ and the object is to minimize (or maximize) the expected time to reach the origin. The minimization problem had been discussed earlier by Heath, Pestien, and Sudderth under various restrictions on the set $\mathcal{S}$. Here an improved verification lemma is established which is used to solve the minimization and maximization problems for any $\mathcal{S}$. An application to a portfolio problem is discussed.

Key words: Stochastic control, portfolio selection, gambling theory.

American Mathematical Society 1980 subject classification. 60G40, 60J60, 93E20.

1. Introduction.

Consider a real-valued process $\{X_1(t)\}$ given by a stochastic differential equation

$$dX_1(t) = \mu(t)dt + \sigma(t)dW_t, \quad X_1(0) = x_1$$

where $\{W_t\}$ is standard Brownian motion and $\mu(t)$ and $\sigma(t)$ are non-anticipative controls to be chosen so that $(\mu(t),\sigma(t))$ remains in a specified set S. The problems of minimizing or maximizing the expected time to reach the origin are treated in section 3. The minimization problem has been studied in [6] and [2], though with an exponential change of variables putting the problem on (0,1]. For a more detailed discussion, see Remark 2 in section 3.

The solution of these control problems uses a new refinement of the verification lemma of [6], which is proved in section 2. This result should be of independent interest.

Section 4 deals with a portfolio planning problem which turns out to be a special case of the minimization problem. This portfolio problem was originally solved in [2].

2. Continuous-time stochastic control.

The formulation of stochastic control problems given here is adapted from Pestien and Sudderth [6]. Our notation and terminology is the same as theirs, but we consider a more general class of processes and establish a verification lemma more suited to the present applications.

A <u>continuous-time gambling problem</u> is a triple (F,Σ,u) where

(2.1) the <u>state space</u> F is Polish (we shall use a Borel subset of ordinary Euclidean space),

(2.2) the <u>gambling house</u> Σ is a mapping which assigns to each $x \in F$ a non-empty collection $\Sigma(x)$ of processes $X = \{X_t, t \geq 0\}$ with state space F such that $X_0 = x$ and X has right-continuous paths with left-limits,

(2.3) the <u>utility function</u> u is a Borel function from F to the real line.

A process $X \in \Sigma(x)$ is said to be <u>available</u> at x. Each available X is defined on some probability space $(\Omega,\mathcal{F},P)$ and is adapted to an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete sub-sigma fields of $\mathcal{F}$. The probability space and filtration may depend on X.

A player, starting at position $x \in F$, selects a process $X \in \Sigma(x)$ and receives payoff $u(X)$ defined by

(2.4) $$u(X) = E[\limsup_{t\to\infty} u(X_t)].$$

The expectation occurring on the right is assumed to be well-defined for every available process X.

The <u>value function</u> V is defined by

$$V(x) = \sup\{u(X): X \in \Sigma(x)\}$$

for every $x \in F$. A process $X \in \Sigma(x)$ is <u>optimal</u> at x if

$$u(X) = V(x).$$

From now on we shall require that F be a Borel subset of the Euclidean space $\mathbb{R}^d$ having non-empty interior, and each process $X = \{X_t\}$ under consideration will be an <u>Ito process</u> of the form

$$X_t = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW_s \tag{2.5}$$

where $W = \{W_t\}$ is a standard m-dimensional Brownian motion process on $(\Omega, \mathcal{F}, P)$ adapted to increasing, right-continuous σ-fields $\{\mathcal{F}_t\}$, and $\mathcal{F}_t$ is independent of $\{W_{t+s}-W_t,\ s \geq 0\}$. The function $\alpha = \alpha(t,w)$ is to be $\mathbb{R}^d$-valued, progressively measurable, adapted to $\{\mathcal{F}_t\}$ and such that

$$\int_0^t |\alpha(s)|ds < \infty \quad \text{a.s.} \quad \text{for all } t. \tag{2.6}$$

The function $\beta = \beta(t,\omega)$ has as values real $d \times m$ matrices, is progressively measurable, adapted to $\{\mathcal{F}_t\}$, and satisfies

(2.7) $$\int_0^t |\beta(s)|^2 ds < \infty \quad \text{a.s. for all } t.$$

For each pair (a,b), where $a \in \mathbb{R}^d$ is a $d\times 1$ vector and b is a $d\times m$ real-valued matrix, define the differential operator D(a,b) for sufficiently smooth functions $Q:\mathbb{R}^d \to \mathbb{R}$ by

$$D(a,b)Q(y) = Q_x(y)a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} Q_{x_i x_j}(y)(bb')_{ij}$$

where

$$Q_x(y) = \left(\frac{\partial Q}{\partial x_1}, \ldots, \frac{\partial Q}{\partial x_d}\right),$$

$$Q_{x_i x_j} = \frac{\partial^2 Q}{\partial x_i \partial x_j},$$

and b' is the transpose of b.

We now specify $\Sigma(x)$ by specifying the possible values of α and β. To this end, let C(x) be, for each $x \in F$, a non-empty set of pairs (a,b), where $a \in \mathbb{R}^d$ and b is a real $d\times m$ matrix. (The idea is that C(x) is the set from which a player at state x may choose the value of (α,β).) Assume also that every available process X is absorbed at the time T_X of its first exit from F^o, the interior of F. These conditions define a function Σ_C on F where $\Sigma_C(x)$ is the collection of all processes X having paths in F and satisfying (2.5), (2.6), and (2.7) together with

(2.8) $$\big(\alpha(t,\omega),\beta(t,\omega)\big) \in C\big(X_t(\omega)\big) \quad \text{for all } (t,\omega),$$

(2.9) $$\big(\alpha(t,\omega),\beta(t,\omega)\big) = (0,0) \quad \text{for } t \geq T_X(\omega),$$

(2.10) $$C(x) = \{(0,0)\} \quad \text{for } x \in F - F^o.$$

Let Σ be a gambling house such that $\Sigma(x) \subset \Sigma_c(x)$ for every $x \in F$.

The following proposition, which is related to Lemmas 2 and 3 of [6], will be applied in the next two sections.

Proposition. Let G be an open subset of $\mathbb{R}^d$ which contains F. Suppose $Q:G \to \mathbb{R}$ and $Q_n:G \to \mathbb{R}$ for $n = 1,2,\ldots$. Suppose also that each Q_n has continuous second-order derivatives on G and that

(i) $\lim_{n\to\infty} Q_n(x) = Q(x)$ for every $x \in F$.

Assume the following conditions for every $x \in F^o$ and every $X \in \Sigma(x)$:

(ii) $Q(X) \geq u(X)$ where

$Q(X) = E[\limsup_{t\to\infty} Q(X_t)]$ is assumed to be well-defined,

(iii) there exists a sequence $\{k_n\}$ of non-negative constants such that $\lim_{n\to\infty} k_n = 0$ and with probability one, for all n and all $t \geq 0$,

$$D\big(\alpha(t),\beta(t)\big)Q_n(X_t) \leq k_n.$$

(Here α and β are related to X by (2.5).)

(iv) there exist integrable random variables $Z, Y_1, Y_2, \ldots$ such that, for all n and all $t \geq 0$,

$$Z \leq Q_n(X_t) \leq Y_n.$$

Then $Q \geq V$.

The following lemma is the chief tool for the proof of the proposition.

Lemma. Suppose $Q: G \to \mathbb{R}$ has continuous second-order derivatives, $x_o \in F^o$, $X \in \Sigma(x_o)$, and τ is an almost surely finite $\{F_t\}$ - stopping time. Also assume

(i) there is a non-negative constant k such that with probability one, for all $s \geq 0$,

$$D\big(\alpha(s),\beta(s)\big)Q(X_s) \leq k,$$

(ii) there exist integrable random variables Y and Z such that for all $t \geq 0$,

$$Z \leq Q(X_t) \leq Y.$$

Then

$$EQ(X_\tau) \leq Q(x_o) + kE\tau.$$

Proof: Apply Ito's Lemma to write

$$(2.11) \quad Q(X_t) = Q(x_0) - A_t + M_t = Q(x_0) - (kt+A_t) + kt + M_t$$

where

$$A_t = -\int_0^t D\big(\alpha(s),\beta(s)\big)Q(X_s)ds,$$

$$M_t = \int_0^t Q_x(X_s)\beta(s)dW_s.$$

(Here α and β are related to X by (2.5) and satisfy (2.6) and (2.7).)

Assume without loss of generality that $E\tau < \infty$. Hence,

$$(2.12) \quad EQ(X_\tau) = Q(x_0) + kE\tau + E[M_\tau - (k\tau+A_\tau)].$$

It suffices to show that the final expectation in (2.12) is less than or equal to zero. By condition (i), $-(k\tau + A_\tau) \leq 0$. We will show that $EM_\tau \leq 0$. (Notice that EM_τ is well-defined by the first equality in (2.11) and condition (ii).)

Let T_j be a sequence of stopping times such that $\{M_{t\wedge T_j}, F_t\}$ is a uniformly integrable martingale for every j and $T_j \to \infty$ a.s. Let $B_j = [\tau > T_j]$. Then $M_\tau = M_{\tau\wedge T_j}$ on B_j^c, and

$$\int_{B_j^c} M_\tau = \int_{B_j^c} M_{\tau\wedge T_j} = EM_{\tau\wedge T_j} - \int_{B_j} M_{T_j}$$

$$= 0 - \int_{B_j} [Q(X_{T_j}) + A_{T_j} - Q(x_0)]$$

$$\leq \int_{B_j} [-Z + k\tau + Q(x_0)].$$

That is,

$$\int_{B_j^c} M_\tau^+ - \int_{B_j^c} M_\tau^- \leq \int_{B_j} [-Z + k\tau + Q(x_0)]$$

Let $j \to \infty$, and conclude

$$EM_\tau = \int_\Omega M_\tau^+ - \int_\Omega M_\tau^- \leq \int_\emptyset [-Z + k\tau + Q(x_s)] = 0. \quad \square$$

Proof of the proposition: Let $x_0 \in F$ and $X \in \Sigma(x_0)$. By condition (ii) and Lemma 1 of [6], it suffices to show

$$\text{(2.13)} \qquad EQ(X_\tau) \leq Q(x_0)$$

for every almost surely finite stopping time τ.

Assume first that τ is bounded. Then, by the Lemma and Fatou's inequality,

$$EQ(X_\tau) \le \liminf_{n\to\infty} EQ_n(X_\tau)$$

$$\le \lim_{n\to\infty} Q_n(x_0) + (\lim_{n\to\infty} k_n)E\tau$$

$$= Q(x_0).$$

If τ is unbounded, use Fatou again:

$$EQ(X_\tau) \le \liminf_{n\to\infty} EQ(X_{\tau\wedge n})$$

$$\le Q(x_0). \quad \square$$

3. Minimizing or maximizing the expected time to reach zero.

The problems described in the introduction will now be formulated as continuous-time gambling problems in $\mathbb{R}^2$. Consider first the problem of minimizing expected time. The first coordinate, x_1, of the state vector x will correspond to the player's position on $(-\infty,0]$, while the second coordinate, x_2, will represent time.

It is convenient to allow negative as well as positive times and define

$$F = \{x \in \mathbb{R}^2: -\infty < x_1 \le 0\}.$$

Because the object is to minimize expected time, let

$$u(x) = -x_2$$

Recall the notation from section 2. The interior of F is $F^o = (-\infty,0)\times(-\infty,\infty)$

and by our conventions each available process X will be absorbed at

$$T = T_X = \inf\{t: X_1(t) = 0\}.$$

In the present example the set C(x) will not depend on x for $x \in F^o$. Let $\mathcal{S} \in \mathbb{R}\times[0,\infty)$,

(3.0) $$C_0 = \left\{\left(\binom{\mu}{1},\binom{\sigma}{0}\right): (\mu,\sigma) \in \mathcal{S}\right\}$$

and let $C(x) = C_0$ for $x \in F^o$. Every $X \in \Sigma_C(x)$ can be specified by stochastic differential equations

(3.1)
$$\begin{aligned} dX_1(t) &= \mu(t)dt + \sigma(t)dW_t \\ dX_2(t) &= dt \\ X_1(0) &= x_1,\ X_2(0) = x_2 \end{aligned}$$

where μ and σ are progressively measurable and $(\mu(t),\sigma(t)) \in \mathcal{S}$, $t < T$; and $X_t = X_T$ for $t \geq T$. Note that for every $X \in \Sigma_C(x)$ the second coordinate process $\{X_2(t)\}$ increases deterministically at rate 1 up to time T, and by (2.4) and the definition of u

(3.2) $$u(X) = -x_2 - ET$$

Now let

$$(3.3) \qquad \Sigma(x) = \{X \in \Sigma_C(x): u(X) > -\infty\}$$
$$= \{X \in \Sigma_C(x): ET < \infty\}.$$

From (3.2) and (3.3) one sees that

$$(3.4) \qquad V(x_1,x_2) = V(x_1) - x_2$$

where $V(x_1) = V(x_1,0)$. Furthermore, for $x_1 < y_1 < 0$, a strategy starting at x_1 and minimizing the time to 0 must first minimize the time to y_1 and, having gotten there, minimize the time to 0. This argument leads to $V(x_1) = (V(x_1-y_1) + V(y_1)$. Since V is also continuous and vanishes at the origin, one may conclude

$$V(x_1) = \lambda x_1$$

where $\lambda \geq 0$ depends on S. (We omit a formal proof because we will not rely on this formula below.)

If in (3.1) $\mu(t) = \mu(X_1(t))$ and $\sigma(t) = \sigma(X_1(t))$, where μ and σ are measurable real valued functions on $(-\infty,0)$. We say that X is given by a stationary Markovian strategy. For given functions μ and σ then X as defined by (3.1) depends only on the initial conditions, so we may write $u(X) = v(x_1,x_2)$ and from (3.2)

$$v(x_1,x_2) = v(x_1) - x_2 \tag{3.5}$$

where $v(x_1) = v(x_1,0)$. Now u can be obtained explicitly. Assume for simplicity that μ and σ are piecewise continuous functions, and $\sigma(x_1) \geq \sigma_0 > 0$ for all x_1. By definition $v(x_1)$ is simply the negative of the expected time it takes the diffusion to reach the origin if it is started at x_1. If $X \in \Sigma(x)$, then T is finite with probability one and $v(x_1)$ is the limit as $M \to \infty$ of $-v_M(x_1)$, where $v_M(x_1)$ is the expected time to exit the interval $[-M,0]$. Let us set

$$a(x) = \sigma^2(x).$$

Then v_M is determined by

$$\mu v_M' + \frac{1}{2} a v_M'' + 1 = 0; \quad v_M(0) = v_M(-M) = 0.$$

Solving for v_M' and letting $M \to \infty$ gives

$$v'(x_1) = e^{-B(x_1)} \int_{-\infty}^{x_1} e^{B(z)} \frac{2}{a(z)}\, dz \tag{3.6}$$

where

$$B(x_1) = \int_r^{x_1} \frac{2\mu(y)}{a(y)}\, dy$$

and r is an arbitrary point in $(-\infty,0]$. Of course

(3.7) $$v(x_1) = \int_0^{x_1} v'(y)dy.$$

Recall (see e.g. [3]) that the diffusion determined by μ and a has a scale function and speed measure determined respectively by

(3.8) $$dp(x_1) = e^{-B(x_1)}dx_1, \quad dm(x_1) = \frac{2}{a(x_1)} e^{B(x_1)}dx_1.$$

Consider now $\mu(t) \equiv \mu_0$, $\sigma(t) \equiv \sigma_0$, where μ_0 and σ_0 are constants. This will determine a diffusion with $ET < \infty$ if and only if $\mu_0 > 0$, and then

(3.9) $$v(x_1) = \frac{x_1}{\mu_0}$$

which is a special case of (3.6) if $\sigma_0 > 0$ and obvious if $\sigma_0 = 0$.

It is natural, especially in the light of (3.9), to conjecture that an optimal strategy is to choose the drift μ to achieve the supremum

$$M = \sup\{\mu: (\mu,\sigma) \in S \text{ for some } \sigma\}.$$

As is explained in remark 2 below, a similar strategy was proposed by Kelly [4] for certain discrete-time problems. However, these 'Kelly strategies' need not be optimal if the set of possible σ's is unbounded. The exact criterion for our continuous-time problem involves another quantity

$$I = \inf_{\varepsilon>0} \sup\{\mu + \varepsilon\sigma^2 : (\mu,\sigma) \in S\}.$$

Theorem 1. Let $x \in F^o$.

(a) If $0 < M < \infty$ and $I < \infty$ then $V(x) = x_1/M - x_2$.

If in addition $(M,\sigma_0) \in S$, then the process $X \in \Sigma(x)$ with $\mu(t) \equiv M$ and $\sigma(t) \equiv \sigma_0$ is optimal.

(b) If $M \le 0$ and $I < \infty$ then $V(x) = -\infty$.

(c) If $M = \infty$ or $I = \infty$ then $V(x) = -x_2$ (i.e. the origin can be reached in an arbitrarily small expected time.)

Proof. (a) Let $Q(x) = x_1/M - x_2$. It is clear from formulas (3.5) and (3.9) that $Q \le V$. It remains to verify that $Q \ge V$. (Once this is done, the final assertion of (a) will follow from (3.5) and (3.9).) This inequality will be proved by an application of the proposition of section 2.

Let $\{\delta_n\}$ be a sequence of positive numbers decreasing to zero, each of which is small enough so that the quantity

$$I_n = \sup\{\mu + \frac{\delta_n}{2}\sigma^2 : (\mu,\sigma) \in S\}$$

satisfies $I_n < \infty$. (Condition (a) guarantees the existence of the δ_n's.) Notice that $I_n \to M$ as $n \to \infty$.

Define

$$Q_n(x) = \frac{e^{x_1\delta_n} - 1}{\delta_n I_n} - x_2.$$

Now verify the conditions of the proposition. Conditions (i), (ii), and (iii) follow easily. As to condition (iv), observe that in the formula for $Q_n(x)$ the first term on the right is bounded uniformly in x_1 for each fixed n. So $Q_n(X_t)$ is bounded above and below by a constant plus $X_2(t)$, and since $x_2 \le X_2(t) \le x_2 +$ T and $X \in \Sigma(x)$ implies that T is integrable, the proof of (a) is complete.

(b). We reduce the result to (a). Let $\varepsilon > 0$ and consider a new problem based on the set

$$S_\varepsilon = S \cup \{(\varepsilon,0)\}.$$

The quantity corresponding to M for the new problem is $M_\varepsilon = \varepsilon$. Thus part (a) can be applied to obtain the value function

$$V_\varepsilon(x) = \frac{x_1}{\varepsilon} - x_2$$

Clearly $V(x) \le V_\varepsilon(x) \to -\infty$ as $\varepsilon \to 0$.

(c) If $M = \infty$ the desired conclusion, $V(x) = -x_2$ follows easily from (3.9). So assume now that $M < \infty$ and $I = \infty$. Then there exists a sequence (μ_i,σ_i), with

$(\mu_i, \sigma_i) \in S$, $\sigma_i > 0$, and $\sigma_i \uparrow \infty$ and

$$\mu_i \geq -h(a_i)a_i, \; i = 1, 2, \ldots ,$$

where $a_i = \sigma_i^2$ and $h(s)$ is a non-negative function on $[0,\infty)$ which decreases to zero as $s \to \infty$. Let

$$\sigma(x_1) = \sigma_{i(x_1)}, \; \mu(x_1) = \mu_{i(x_1)}$$

where i is a function from $(-\infty,0)$ to the positive integers with $i(x_1)$ increasing rapidly to ∞ as x_1 decreases to $-\infty$. Now use the expression for $v'(x_1)$ given in (3.6). Substituting into the expression for B given after (3.6), with τ taken to be 0,

$$B(x_1) = \int_0^{x_1} \frac{2\mu(y)}{a(y)}\, dy \;=\; -\int_{x_1}^{0} \frac{2\mu_i(y)}{a_i(y)}\, dy$$
$$\leq 2\int_{x_1}^{0} h\left(a_{i(y)}\right) dy \; .$$

So for any $\varepsilon > 0$ we can arrange $B(x_1) < \varepsilon$ for all $x_1 \in (-\infty,0)$ by choosing the function i appropriately. It follows from (3.6) that i can be chosen to make $v'(x_1)$ as small as desired and then (3.7) gives the desired conclusion. (Notice $T < \infty$ with probability one because $p(-\infty) = -\infty$ by (3.8) and σ is bounded below by σ_1.) □

For the maximization problem it seems natural to work on $[0,\infty)$ rather than $(-\infty,0]$ and to think of maximizing the expected time until bankruptcy occurs. Here is the formal definition of the gambling problem:

$$F = \{x \in \mathbb{R}^2 : 0 < x_1 < \infty\},$$
$$u(x) = x_2,$$
$$C(x) = C_0 \text{ for } x \in F^o$$

where C_0 is given by (3.0),

$$\Sigma(x) = \Sigma_C(x).$$

Then, for $x \in F$ and $X \in \Sigma(x)$,

$$u(x) = x_2 + ET$$

where $T = \inf\{t: X_1(t) = 0\}$. As before

$$V(x_1,x_2) = V(x_1) + x_2$$

where

$$V(x_1) = V(x_1,0).$$

It is natural, as it was for the minimization problem to conjecture that on optimal strategy will choose μ to achieve

$$M = \sup\{\mu\colon (\mu,\sigma) \in S \text{ for some } \sigma\}.$$

This time the conjecture is essentially correct.

Theorem 2. Let $x \in F^{o}$.

(a) If $M < 0$, then $V(x) = -x_1/M + x_2$. If in addition $(M,\sigma_0) \in S$, then the process $X \in \Sigma(x)$ with $\mu(t) \equiv M$ and $\sigma(t) \equiv \sigma_0$ is optimal.

(b) If $M \geq 0$, then $V(x) = \infty$.

Proof: Suppose X is given by a stationary Markov strategy $\mu(t) \equiv \mu_0$, $\sigma(t) \equiv \sigma_0$ where μ_0 and σ_0 are constants. Because we have changed from $(-\infty,0]$ to $[0,\infty)$, formulas (3.5) and (3.9) now imply

$$u(X) = -\frac{x_1}{\mu_0} + x_2 \quad \text{if } \mu_0 < 0, \tag{3.10}$$
$$= \infty \quad \text{if } \mu_0 \geq 0.$$

Part (b) of the theorem is immediate. For (a), let $Q(x) = -x_1/M + x_2$. By (3.10), $Q \leq V$. The reverse inequality will be proved by another application of the Proposition of section 2.

Let $\{\beta_n\}$ be a sequence of numbers in the interval (0,1) which increase up to

1. Define

$$Q_n(x) = \frac{e^{\lambda(\beta_n)x_1} - 1}{\log \beta_n} + x_2\beta_n^{x_2}$$

where

$$\lambda(\beta) = \frac{-M - \sqrt{M^2 - 2\sigma_0^2 \log \beta}}{\sigma_0^2}$$

and $\sigma_0 > 0$. (The first term on the right-hand-side in the definition of $Q_n(x)$ is equal to the expectation of $\int_0^T (\beta_n)^s ds$ for a process $\mu(t) \equiv M$, $\sigma(t) \equiv \sigma_0$ and thus corresponds to a discounted payoff.)

Condition (i) of the Proposition is easily verified, and (ii) is obvious because $Q \geq u$. For (iii) let $(a,b) \in C(x)$ where $a = \binom{\mu}{1}$, $b = \binom{\sigma}{0}$ and calculate (with $\beta = \beta_n$)

$$\begin{aligned} D(a,b)Q_n(x) &= \frac{\lambda(\beta)e^{\lambda(\beta)x_1}}{\log \beta}(\mu + \tfrac{1}{2}\lambda(\beta)\sigma^2) + \beta^{x_2}(1 + x_2 \log \beta) \\ &\leq \frac{\lambda(\beta)M}{\log \beta} + \beta^{x_2}(1 + x_2 \log \beta). \end{aligned}$$

The inequality holds because $\mu \leq M$ and $\lambda(\beta) < 0$. Now recall that $X_2(s)$ is an increasing process and (iii) will follow after some calculus. Condition (iv) is an easy consequence of the definition of Q_n together with the facts that $X_1(s) \geq$

0 and $X_2(s) \geq X_2(0)$. □

Remark 1. Since the set S is not assumed to be bounded, and the σ with $(\mu,\sigma) \in S$ are not bounded away from zero, the usual approach via Bellman's equation for the value function V could not be used above. For a continuous time gambling problem as defined in section 2 the Bellman's equation can be written in the form

$$\sup D(a,b)V(x) = 0 \tag{3.11}$$

where the supremum is taken over all $(a,b) \in C(x)$. For the minimization problem of this section (3.4) applies and (3.11) becomes

$$\sup_{(\mu,\sigma)\in S} [\mu V'(x_1) + \tfrac{1}{2}\sigma^2 V''(x_1) - 1] = 0. \tag{3.12}$$

nder condition (c) of Theorem 1 the value function $V(x_1) \equiv 0$ does not satisfy (3.12). Furthermore if $I = \infty$ and $M < \infty$, with $(M,\sigma_0) \in S$ the function x_1/M does solve (3.12), but does not represent the value function. Under condition (a) of the theorem the value function $V(x_1) = x_1/M$ is a solution of (3.12), but this fact does not follow from standard theorems.

Remark 2. Consider the problem of a process on the interval $0 < \tilde{x}_1 \leq 1$ determined by the equation

(3.13) $$\tilde{X}_1(0) = \tilde{x}_1, \; d\tilde{X}_1(t) = \tilde{X}_1(t)[\tilde{\mu}(t)dt + \tilde{\sigma}(t)dW_t]$$

where $\tilde{\mu}(t)$, $\tilde{\sigma}(t)$ are non-anticipating controls required to satisfy $(\tilde{\mu}(t),\tilde{\sigma}(t)) \in \tilde{S}$ and the object is to minimize $\tilde{T} = \inf\{t: \tilde{X}_1(t) = 1\}$. This problem reduces to that of Theorem 1 by the change of variables $X_1(t) = \log \tilde{X}_1(t)$. This follows from Ito's formula, and one finds $\mu(t) = \tilde{\mu}(t) - \sigma^2(t)/2$, $\sigma(t) = \tilde{\sigma}(t)$. So one can formulate the theorem to apply to the $\tilde{X}_1$ process. Note that the role of M is assumed by

$$\tilde{M} = \sup\{\tilde{\mu} - \tilde{\sigma}^2/2: \quad (\tilde{\mu},\tilde{\sigma}) \in \tilde{S}\}$$

and the role of I is taken by

$$\tilde{I} = \inf_{\varepsilon>0} \sup\{\tilde{\mu} - (\tfrac{1}{2} - \varepsilon)\tilde{\sigma}^2: \; (\tilde{\mu},\tilde{\sigma}) \in \tilde{S}\}.$$

The problem for the $\tilde{X}_1$ process was considered in [6] and [2] and solved under some restrictions on $\tilde{S}$. In [6] it was assumed that $\lambda\tilde{S} \subseteq \tilde{S}$ for all $\lambda \geq 0$, while in [2] this assumption was needed only for $0 \leq \lambda \leq 1$.

As discussed in [6] and [2] various models lead to the problem on [0,1]. One of these, the "portfolio problem," will be explained in section 4. In [4] Kelly introduced a plan in discrete time, based on the criterion of maximizing, at each stage, the expected value of the logarithm. This "Kelly criterion" was further studied by Breiman [3] who established certain asymptotic optimality properties. Our theorem may be interpreted to imply that a continuous time

Kelly criterion is in fact optimal under the hypotheses of (a), but not under those of (c).

4. A portfolio problem.

Consider the problem of managing a portfolio of stocks, bonds, and cash so as to minimize the expected time to reach a given total worth. For a simple model, suppose that there is one bond where price B_t at time t satisfies

$$dB_t = r_B B_t dt,$$

and one stock whose price S_t at time t satisfies

$$dS_t = r_S S_t dt + \sigma_S S_t dW_t$$

where r_B, r_S and σ_S are positive constants and $\{W_t\}$ is a standard Brownian motion. A recent paper by Malliaris [5] explains the use of stochastic differential models in finance, and has numerous references to the financial literature. Let $\tilde{X}_1(t)$ be the total fortune of an investor at time t, let $f_S(t)$ be the fraction of that fortune invested in the stock, and let $f_B(t)$ be the fraction invested in the bond. Then $\tilde{X}_1$ satisfies

$$(4.1) \qquad d\tilde{X}_1(t) = X_1(t)[r_S f_S(t) + r_B f_B(t)]dt + \sigma_S f_S(t) dW_t$$

Let

$$\tilde{S} = \{(\tilde{\mu},\tilde{\sigma}) : \tilde{\mu} = r_S f_S + r_B f_B,\ \tilde{\sigma} = \sigma_S f_S,\ f_B \geq 0,\ f_S \geq 0,\ f_B + f_S \leq 1\}.$$

Then (4.1) and $\tilde{X}_1(0) = \tilde{x}_1$ is equivalent to (3.13) and $(\tilde{\mu}(t),\tilde{\sigma}(t)) \in \tilde{S}$. We are in the situation of Remark 2 of section 3. Theorem 1 applies, and one is in case (a). If $r_B > r_S$ obviously $f_B(t) \equiv 1$. If $r_B \leq r_S$ one finds

$$\tilde{M} = \begin{cases} r_B + \dfrac{(r_S - r_B)^2}{2\sigma_S^2} & \text{if } r_S \leq r_B + \sigma_S^2 \\ r_S - \sigma_S^2/2 & \text{otherwise.} \end{cases}$$

The corresponding optimal policies are given by $f_B = 1 - f_S$ and

$$f_S = \begin{cases} \dfrac{r_S - r_B}{\sigma_S^2} & \text{if this is less than 1} \\ 1 & \text{otherwise.} \end{cases}$$

In particular the Kelly strategy is optimal.

REFERENCES

[1] Breiman, Leo (1961). Optimal gambling systems for favorable games. Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley. 1 65-78.

[2] Heath, D. and Sudderth, W. (1984). Continuous time portfolio management: minimizing the expected time to reach a goal. Preprint of Institute for Math. and Appl., University of Minnesota.

[3] Ito, K. and McKean, H. Jr. (1965). Diffusion Processes and Their Sample Paths. Academic Press, New York.

[4] Kelly, J. L. Jr. (1956). A new interpretation of information rate. Bell System Technical Journal 35 917-926.

[5] Malliaris, A. G. (1983). Ito's calculus in financial decision making. SIAM Review 25 481-496.

[6] Pestien, Victor C. and Sudderth, William D. (1983). Continuous-time Red and Black: How to control a diffusion to a goal. Math. of O.R. (to appear).