Singularity Structures of Linear Inviscid Damping and Local Well-posedness of the De Gregorio Model in a Critical Besov Space

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

Wenjie Lu

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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June, 2023

Acknowledgements

My deepest gratitude is extended to the following individuals whose influence, guidance, and support during my time at the University of Minnesota has been immeasurable.

(Advisors) Vladmír Šverák and Hao Jia: Thank you for your benevolent and generous mentorship. Your guidance has immensely deepened my exploration of the academic research in math, and elevated my personal growth as a human being.

(Committee members) Peter Poláčik, Jiaping Wang and Mikhail Safonov. I extend my heartfelt gratitude for your contribution and service on my thesis defense and oral exam committee. Your expertise and guidance have been invaluable during this significant stage of my academic journey.

(Professors and others) Maury Bramson, Jeffrey Calder, Paul Garrett, Kai-Wen Lan, Siran Li, Mitchell Luskin, Arnd Scheel, Craig Westerland, and many, many others.

(Friends) Yichang Liu, Zanbing Dai, Tong Shi, Wei Wang, Mingqian Dai, Xinchen Miao, Lingfei Yi, Kunlun Qi, Shan Chen, Dallas Albritton, Zongyuan Li, Jin Xu, Tianhao Zhang, Zhaolin Li, Shuo Zhang, Jiuzhou Wang, Bo Zhu, Wuzhe Xu and many, many others.

(Family) Mom and Dad: Thank you for your neverending love and support — you are the best.

Abstract

This thesis consists of two parts. The first part of the thesis studies singularity structures of the linear inviscid damping of two-dimensional Euler equations in a finite periodic channel. We introduce a recursive definition of singularity structures which characterize the singularities of the spectrum density function from different sources: the free part and the boundary part of the Green function. As an application, we demonstrate that the stream function exhibits smoothness away from the channel's boundary, yet it presents singularities in close proximity to the boundary. The singularities arise due to the interaction of boundary and interior singularities of the spectrum density function. We also show that the behavior of the initial data and background flow have an impact on the regularity of different components of the stream function.

The second part studies the local-in-time well-posedness of the De Gregorio modification of the Constantin-Lax-Majda model in a Besov space which is critical under the natural scaling of the De Gregorio model. We also develop a Beale-Kato-Majda type blow-up criterion for the De Gregorio model.

Contents

Chapter 1

Introduction

1.1 The Euler equations

The motion of homogeneous incompressible ideal fluid in a domain $\Omega \subset \mathbb{R}^n$ can be described by the following system of Euler equations,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \quad (t,x) \in (0,\infty) \times \Omega, \\ \operatorname{div} u = 0, \quad (t,x) \in (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega. \end{cases} \tag{E}$$

The function $u = (u^1, u^2, \cdots, u^n)$ is the velocity field of the fluid. The scalar function $p(t,x)$ is the scalar pressure. The divergence free condition indicates that the fluid is incompressible. The system (E) is first derived by L. Euler in 1755 [Eul57]. We suppose $n = 2$ or 3 throughout the thesis.

If we add a dissipation term $\mu \Delta u$ to the right-hand side of the first equation in (E), where μ is the viscosity constant, we get the system of Navier-Stokes equations. The regularity or singularity problem of three-dimensional Navier-Stokes equations is one of the seven Clay Millennium problems. We do not study Navier-Stokes equations in this thesis. Interested readers may see e.g. [MT98, CF20, LR02, Maj12] for developments in mathematical studies of Navier-Stokes equations.

The notion of vorticity, $\omega = \nabla \times u$, plays an important role in the study of Euler equations. For the 2D case, the system (E) can be reformulated as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^{\perp}(-\Delta)^{-1}\omega, \\ \omega(0,x) = \omega_0(x), \end{cases} \tag{1.1.1}$$

while the 3D incompressible Euler equations can be written as

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ u = \nabla \times (-\Delta)^{-1}\omega, \\ \omega(0,x) = \omega_0(x). \end{cases} \tag{1.1.2}$$

The second equations of the system (1.1.1) and (1.1.2) are called the Biot-Savart law.

For the 2D incompressible Euler equations (1.1.1), the vorticity ω is transported by the velocity field u. It is well known that (1.1.1) are globally in time well-posed for smooth initial data. See for example [MBO02]. The long time behavior of (1.1.1) such as small scale creation (see for example [KŠ14]), vortex patch solutions (see for example [HMW20]), vortex symmetrization (see for example [IIJ22]), etc., remains an active direction in the mathematical study of fluid dynamics. The first part of this thesis is dedicated to the linear stability of (1.1.1) in a periodic channel with boundaries near the monotone shear flow $(b(y), 0)$. See the section 1.2.2 for an introduction to the background of hydrodynamic stability and linear inviscid damping. We show that the stream function is smooth away from the boundary and develops singularities near boundaries. We characterize the singularities from different sources and show that a loss of regularity can happen when different singularities interact with each other. We also show that the boundary behavior of the initial data and background flow can have an impact on the regularity of different components of the stream function. See the section 1.2.3 for a detailed representation of results of this part.

For the 3D incompressible Euler equations (1.1.2), whether it can develop finite-time singularities from smooth initial data remains one of the major open problems in the study of partial differential equations. The difference is that 3D Euler equations have the vortex stretching term $\omega \cdot \nabla u$ in addition to the advection term $u \cdot \nabla \omega$. Transportation by the divergence free velocity field u cannot cause growth of the vorticity, but the vortex stretching term can lead to amplification of the vorticity. The remarkable work

[Elg21] shows that finite time singularities can develop from $C^{1,\alpha}$ initial data for the system (E) in $\mathbb{R}^3$ with sufficiently small $\alpha > 0$. Numerical studies [LH14b, LH14a] show that 3D axisymmetric Euler equations can potentially develop finite time singularities from smooth initial data. Inspired by the numerical evidence, [CH22] shows that 3D axisymmetric Euler equations in a cylinder can develop a stable nearly self-similar blowup in finite time from smooth initial data. The boundary plays an important role in the blow up mechanism in [CH22].

People develop simplified model equations in order to study the different mechanisms like transportation and vortex stretching in the 3D Euler equation (1.1.2). The one dimensional Constantin-Lax-Majda model with De Gregorio modification is one of these models that receives extensive studies. See the section 1.3.1 for an introduction to the De Gregorio model. The second part of this thesis studies the local in time well-posedness of the De Gregorio model in a Besov space that is critical under the natural scaling of the model. We develop the local in time existence and uniqueness of the De Gregorio model in this critical Besov space. We also develop a Beale-Kato-Majda type blow-up criterion for this model. See the section 1.3.3 for a detailed presentation of results of this part.

1.2 Linear Inviscid Damping

1.2.1 Main equations

Consider the two dimensional Euler equation linearized around a shear flow $(b(y), 0)$, in the periodic channel $(x, y) \in \mathbb{T} \times [0, 1]$:

$$\begin{aligned} &\partial_t \omega + b(y)\partial_x \omega - b''(y)u^y = 0, \\ &\operatorname{div} u = 0 \qquad \text{and} \qquad \omega = -\partial_y u^x + \partial_x u^y, \end{aligned} \tag{1.2.1}$$

with the natural non-penetration boundary condition $u^y|_{y=0,1} = 0$.

For the linearized flow, $\int_{\mathbb{T}\times[0,1]} u^x(x, y, t)\, dxdy$ and $\int_{\mathbb{T}\times[0,1]} \omega(x, y, t)\, dxdy$ are conserved quantities. In this paper, we will assume that

$$\int_{\mathbb{T}\times[0,1]} u_0^x(x, y)\, dxdy = \int_{\mathbb{T}\times[0,1]} \omega_0\, dxdy = 0. \tag{1.2.2}$$

These assumptions can be dropped by adjusting $b(y)$ with a linear shear flow $C_0 y + C_1$. Then one can see from the divergence free condition on u that there exists a stream function $\psi(t,x,y)$ with $\psi(t,x,0) = \psi(t,x,1) \equiv 0$, such that

$$u^x = -\partial_y \psi, \; u^y = \partial_x \psi. \tag{1.2.3}$$

The stream function ψ can be solved through

$$\Delta\psi = \omega, \qquad \psi|_{y=0,1} = 0. \tag{1.2.4}$$

We summarize our equations as follows

$$\begin{cases} \partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi = 0, \\ \Delta\psi(t,x,y) = \omega(t,x,y), \qquad \psi(t,x,0) = \psi(t,x,1) = 0, \\ (u^x, u^y) = (-\partial_y \psi, \partial_x \psi), \\ \omega(0,x,y) = \omega_0(x,y). \end{cases} \tag{1.2.5}$$

for $t \geq 0, (x,y) \in \mathbb{T} \times [0,1]$.

An important property for (1.2.5) is that the evolution for each mode is de-coupled. Indeed, define for $k \in \mathbb{Z}$, $y \in [0,1]$ and $t \geq 0$ the Fourier modes ω_k, ψ_k for the vorticity and stream functions ω and ψ as

$$\begin{aligned} \omega_k(t,y) &:= \int_{\mathbb{T}} e^{-ikx} \omega(t,x,y)\, dx, \quad \omega_0^k(y) := \int_{\mathbb{T}} e^{-ikx} \omega_0(x,y)\, dx, \\ \psi_k(t,y) &:= \int_{\mathbb{T}} e^{-ikx} \psi(t,x,y)\, dx. \end{aligned} \tag{1.2.6}$$

Then the equation (1.2.5) can be written as

$$\begin{cases} \partial_t \omega_k + ikb(y)\omega_k - ikb''(y)\psi_k = 0, \\ (\partial_y^2 - k^2)\psi_k(t,y) = \omega_k(t,y), \qquad \psi_k(t,0) = \psi(t,1) = 0, \\ \omega_k(0,y) = \omega_0^k(y), \end{cases} \tag{1.2.7}$$

for $k \in \mathbb{Z}, y \in [0,1], t \geq 0$. For each $k \in \mathbb{Z} \setminus \{0\}$, we set for any $g \in L^2(0,1)$,

$$L_k g(y) = b(y)g(y) + b''(y) \int_0^1 G_k(y,z) g(z) dz, \tag{1.2.8}$$

where G_k is the Green's function for the operator $k^2 - \frac{d^2}{dy^2}$ on $(0,1)$ with zero Dirichlet boundary condition. Our main assumption is that the linearized operator L_k does not have embedded eigenvalues. Our goal is to obtain a precise asymptotic decomposition of the stream function $\psi_k(t,y), y \in [0,1]$, as $t \to +\infty$.

1.2.2 Background on hydrodynamic stability and inviscid damping

Hydrodynamical stability is a classical topic in mathematical analysis of fluid flows, pioneered by prominent figures such as Rayleigh [Ray95], Kelvin [Kel87], Orr [Orr07], among many others. The main focus was to study stability of important physically relevant flows, such as shear flows and vortices.

In this paper we consider shear flows. There are extensive works on the linear stability property of these flows. In particular, Rayleigh [Ray95] proved that shear flows with no inflection points are spectrally stable. Orr [Orr07] in 1907 observed the t^{-1} decay rate of the velocity when the shear flow is Couette (linear shear), and Case [Cas60] provided a formal proof in the case of a finite channel. See also Lin and Zeng [LZ11] for a sharp version with optimal dependence on the regularity of the initial data.

The observation of Orr can be described roughly as follows. Consider the linearized equation near Couette flow:

$$\partial_t\omega + y\partial_x\omega = 0,\ (x,y)\in\mathbb{T}\times\mathbb{R}.$$

One can solve this equation explicitly and it follows that $\omega(t,x,y)=\omega_0(x-yt,y)$. The equation for the stream function becomes $\Delta\psi(t,x,y)=\omega(t,x,y)=\omega_0(x-yt,y)$ for $(x,y)\in\mathbb{T}\times\mathbb{R}$ and therefore

$$\widetilde{\psi}(t,k,\xi) = -\frac{\widetilde{\omega}(t,k,\xi)}{k^2+|\xi|^2} = -\frac{\widetilde{\omega_0}(k,\xi+kt)}{k^2+|\xi|^2}. \tag{1.2.9}$$

In the above, $\widetilde{h}$ denotes the Fourier transform of h in x, y. Assume that ω_0 is smooth, so $\widetilde{\omega}_0(k,\xi)$ decays fast in k, ξ. Then we can view ξ as

$$\xi = -kt + O(1),$$

and hence $\widetilde{\psi}(t,k,\xi)$ decays like $|k|^{-2}\langle t\rangle^{-2}$ for each $k\neq 0$. Similarly, using the relations $u^x=-\partial_y\psi$ and $u^y=\partial_x\psi$, we conclude that $\widetilde{u^x}$ decays like $|k|^{-1}\langle t\rangle^{-1}$ and $\widetilde{u^y}$ decays like $|k|^{-1}\langle t\rangle^{-2}$ for all $k\neq 0$. Hence, the velocity field decays to another shear flow $(u_\infty(y),0)$.

For general monotone shear flows, the linearized operator becomes more complicated due to the extra term $b''(y)\partial_x\psi$, see (1.2.5), which can not be treated as perturbations. Therefore spectral analysis of the linearized operator is required to understand the dynamical properties of the associated flow. For results on the general spectral property of the linearized operator, we refer to Faddeev [Fad71] and Lin [Lin03]. In the direction of inviscid damping, Stepin [Ste95] proved $t^{-\nu}$ decay of the stream function associated with the continuous spectrum, Rosencrans and Sattinger [RS66] proved t^{-1} decay for analytic monotone shear flows.

Recently, inspired by the remarkable work of Bedrossian and Masmoudi [BM15] on the nonlinear asymptotic stability of shear flows close to the Couette flow in $\mathbb{T} \times \mathbb{R}$ (see also an extension [IJ20a] to $\mathbb{T} \times [0, 1]$), optimal decay estimates for the linear problem received much attention, see e.g. Zillinger [Zil16, Zil17] and references therein for shear flows close to Couette. In an important work, Wei, Zhang and Zhao [WZZ18] obtained the optimal decay estimates for the linearized problem around monotone shear flows, under very general conditions. In [Jia20b] the first author identified the main term in the asymptotics of the stream function.

We also refer the reader to important developments for the linear inviscid damping in the case of non-monotone shear flows [BM10, WZZ19, WZZ20, IIJ22] and circular flows [BCZV19, CZZ19]. See also Grenier et al [GNRS20] for an approach using methods from the study of Schrödinger operators.

Lastly, we mention the result of Zillinger [Zil16] which addresses the boundary effect in the dynamics of (1.2.7), and is perhaps the closest to our results below. In particular, he showed that it is in general not possible to obtain a uniformly smooth "profile" in high Sobolev spaces over $t \in [0, \infty)$. The main assumption in [Zil16] is that the background shear flow is close to the Couette flow. In our case, we need to perform detailed spectral analysis if such an assumption is dropped, using and refining methods introduced in [Jia20b].

1.2.3 Main results

Without loss of generality, we assume the mode $k \geq 1$. In fact, the equations are trivial for $k = 0$, and the case $k > 0$ and $k < 0$ are equivalent up to complex conjugation. Our main goal is to understand the more refined asymptotics of the linearized flow (1.2.7) as $t \to +\infty$, and in particular, to capture precisely the singularity structure of the so-called spectral density functions when the spectral parameter collapses into the continuous spectrum. As a consequence, we obtain an expansion of the stream function

$\psi_k(t,y)$, $y\in[0,1], t\geq 0$ to all orders, as $t\to\infty$, see Theorem 1.2.1- Theorem 1.2.3. As we shall see, the boundary $y\in\{0,1\}$ plays an important role in the long time behavior of $\psi_k(t,y), y\in[0,1], t\geq 0$, and it is the primary difficulty in the study of the refined dynamics of (1.2.7).

In order to capture the optimal dependence on the frequency k in different estimates, we define for integer $n\geq 1$ the weighted Sobolev norm $\|h\|_{H^n_k(\mathbb{R})}$ for $h\in H^n(\mathbb{R})$ as

$$\|h\|_{H^n_k(\mathbb{R})} := \sum_{\alpha=0}^{n} |k|^{n-\alpha}\left\|\partial_v^\alpha h\right\|_{L^2(\mathbb{R})}. \tag{1.2.10}$$

Direct computation shows that

$$\|h\|_{H^n_k(\mathbb{R})} \lesssim |k|^{-m}\|h\|_{H^{n+m}_k(\mathbb{R})} \tag{1.2.11}$$

for all $m\geq 0$.

To state main theorems, we first define the smooth cut-off functions

$$\chi^{in}(y), \chi^{b0}(y), \chi^{b1}(y) \in C_c^\infty(\mathbb{R})$$

such that for $y\in\mathbb{R}$,

$$\chi^{in}(y) = \begin{cases} 1, & \text{if} \quad y\in[1/8, 7/8] \\ 0, & \text{if} \quad y\notin[1/16, 15/16], \end{cases} \tag{1.2.12}$$

$$\chi^{b0}(y) = \begin{cases} 1, & \text{if} \quad y\in[-1/8, 1/8] \\ 0, & \text{if} \quad y\notin[-1/4, 1/4], \end{cases} \tag{1.2.13}$$

and

$$\chi^{b1}(y) = \chi^{b0}(1-y). \tag{1.2.14}$$

Denote $I := [0,1]$. Our first main result is the following characterization of the stream function in the interior of the channel.

Theorem 1.2.1. *For $y\in I$, $k\geq 1$ and $N\in\mathbb{Z}\cap[3,\infty)$, assume that $\omega_0^k\in H^N(I)$. Then there exist functions $\alpha^{in}(t,\cdot,k)$, $\beta^{in}(t,\cdot,k), \gamma^{in}(t,\cdot,k)\in C([1,\infty), H^{N-3}(I))$, such that the unique solution $\psi_k\in C([0,\infty), H^{N+2}(I))$ satisfies for for $y\in I$ and $t\geq 1$,*

$$\psi_k(t,y)\chi^{in}(y) = \frac{e^{-ikb(y)t}}{k^2t^2}\alpha^{in}(t,y,k) + \frac{e^{-ikb(0)t}}{k^2t^2}\beta^{in}(t,y,k) + \frac{e^{-ikb(1)t}}{k^2t^2}\gamma^{in}(t,y,k), \tag{1.2.15}$$

and for every integer $m \in \mathbb{Z} \cap [0, N-3]$ *and* $t \geq 1$,

$$\left\|\alpha^{in}(t,\cdot,k)\right\|_{H_k^m(I)} + \left\|\beta^{in}(t,\cdot,k)\right\|_{H_k^m(I)} + \left\|\gamma^{in}(t,\cdot,k)\right\|_{H_k^m(I)} \lesssim_m \left\|\omega_0^k\right\|_{H_k^{m+3}(I)}. \quad (1.2.16)$$

Remark 1.2.2. *Theorem 1.2.1 implies the existence of not one, but three profiles. The profile* α^{in} *is the "main profile" when no boundary effect is present, see for example [Jia20b]. The profiles* β^{in}, γ^{in} *come from the boundary contribution, and in general do not decay over time with a fast rate, if* ω_0^k *and* b'' *do not vanish (to a correspondingly high order) at the boundary.*

Next we turn to the behavior of $\psi_k(t,y), y \in I, t \geq 1$ near $y = b(0)$, the case of $y = b(1)$ being similar.

Theorem 1.2.3. *For any integers* $N \geq 3$, $k \geq 1$, *and* $y \in [0,1]$, *assume* $\omega_0^k(y) \in H_k^N(I)$. *Then there exist functions* $\alpha^{b0,N}(t,y,k)$, $\beta^{b0,N}(t,y,k), R^{b0,N}(t,y,k) \in C([1,\infty), L^2(I))$ *such that the unique solution* $\psi_k \in C([0,\infty), H^{N+2}(I))$ *satisfies for for* $y \in I$ *and* $t \geq 1$,

$$\psi_k(t,y)\chi^{b0}(y) = \frac{e^{-ikb(y)t}}{k^2t^2}\alpha^{b0,N}(t,y,k) + \frac{e^{-ikb(0)t}}{k^2t^2}\beta^{b0,N}(t,y,k) + \frac{R^{b0,N}(t,y,k)}{(kt)^{N-1}} \quad (1.2.17)$$

with the following properties.

(a) $R^{b0,N}(t,y,k)$ *satisfies the bounds for* $t \geq 1$,

$$\left\|R^{b0,N}(t,\cdot,k)\right\|_{H_k^1(I)} \lesssim_N (1 + \log^{2(N-3)}\langle t\rangle)\left\|\omega_0^k\right\|_{H_k^N(I)}. \quad (1.2.18)$$

(b) There exist functions $\alpha_j(t,y,k) \in C([1,\infty), L^2(I))$ *for* $1 \leq j \leq N-2$, *satisfying the bounds for* $t \geq 1$ *and* $j \in [2, N-2]$,

$$\|\alpha_1(t,\cdot,k)\|_{H_k^1(I)} \lesssim \left\|\omega_0^k\right\|_{H_k^3(I)} \quad (1.2.19)$$

and

$$\|\alpha_j(t,\cdot,k)\|_{L^2(I)} \lesssim (1 + \log^{2(j-1)}\langle t\rangle)\left\|\omega_0^k\right\|_{H_k^{j+2}(I)} \quad (1.2.20)$$

such that for $y \in I$ *and* $t \geq 1$,

$$\alpha^{b0,N}(t,y,k) = \sum_{j=1}^{N-2}\frac{\alpha_j(t,y,k)}{(kt)^{j-1}}. \quad (1.2.21)$$

Furthermore, if $\omega_0^k(0) = 0$, *we have* $\alpha_1(t,y,k) \in H_k^2(I)$ *and* $\alpha_2(t,y,k) \in H_k^1(I)$.

(c) *There exist functions* $\beta_j(t,y,k)\in C([1,\infty),L^2(I))$ *for* $1\le j\le N-2$, *such that for* $y\in I$ *and* $t\ge 1$, *satisfying the bounds for* $t\ge 1$ *and* $j\in[2,N-2]$,

$$\|\beta_1(t,\cdot,k)\|_{H^1_k(I)}\lesssim \left\|\omega_0^k\right\|_{H^3_k(I)} \tag{1.2.22}$$

and

$$\|\beta_j(t,\cdot,k)\|_{L^2(I)}\lesssim (1+\log^{2(j-1)}\langle t\rangle)\left\|\omega_0^k\right\|_{H^{j+2}_k(I)} \tag{1.2.23}$$

such that for $y\in I$ *and* $t\ge 1$,

$$\beta^{b0,N}(t,y,k)=\sum_{j=1}^{N-2}\frac{\beta_j(t,y,k)}{(kt)^{j-1}}. \tag{1.2.24}$$

Furthermore, if $\omega_0^k(0)=0$, *we have* $\beta_1(t,y,k)\equiv 0$ *for* $y\in I$, *and* $\beta_2(t,\cdot,k)\in H^1_k(I)$.

Remark 1.2.4. *We note that the expansion* (1.2.17)-(1.2.23) *allows us to identify the asymptotics of the stream function to all orders. The regularity bounds* (1.2.19) *and* (1.2.23) *on the coefficient functions* α_j,β_j *are sharp in general, unless we assume more vanishing conditions on the initial vorticity* ω_0^k *on the boundary. For example, one of the contributing factor to the term* $\beta_2(t,y,k)$ *is given by the integral for* $y\in I, t\ge 1$ *with* $v=b(y)$,

$$\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ik(w-b(0))t}\varphi_k(v,w)\frac{\log(v+i\epsilon)}{b(0)-w+i\epsilon}\,dw, \tag{1.2.25}$$

where $\varphi\in C_0^\infty(\mathbb{R}^2)$. *It follows that it is not possible to control* (1.2.25) *in* H^1. *Similar considerations apply for* α_2. *On the other hand, if we assume that* ω_0^k *and* b'' *vanish to a high order, then the control we have on* α_j,β_j *can be improved significantly.*

The proof of theorems is based on the study of the singularity structure of the spectral density functions, defined in section 2.1.1. The stream function can be written as an oscillatory integral of the spectral density function, using standard spectral analysis. To obtain the precise singularity structure of the spectral density functions, we make a change of coordinate and extend the problem to the whole space, to simplify the specific form of the singularities both in the interior and near the boundary. The desired characterization of the singularity structures, see section 2.3.2, follows from the limiting absorption principle and a refined analysis of the singularities of Rayleigh equations.

1.3 The De Gregorio Model

1.3.1 Main equations

The Constantin-Lax-Majda(CLM) model introduced in [CLM85] is a one-dimensional equation which models the vortex stretching phenomenon of three-dimensional incompressible fluids. The model is

$$\omega_t = \omega H\omega \tag{1.3.1}$$

where H is the Hilbert transform. It can be considered on the real line $\mathbb{R}$ or on the circle $\mathcal{S}^1$. In this paper we will work on the circle $\mathcal{S}^1$. De Gregorio suggested in [DG96] the following modification of the CLM equation (1.3.1) to add the effects of transport:

$$\begin{cases} \omega_t + u\omega_x = u_x\omega, \quad u_x = H\omega, \quad (t,x) \in (0,\infty)\times\mathcal{S}^1 \\ \omega(0,x) = \omega_0, \quad x \in \mathcal{S}^1. \end{cases} \tag{1.3.2}$$

The equation $u_x = H\omega$ models the Biot-Savart law in the Euler equation.

As pointed out in [JSS17], (1.3.2) can be written in terms of u as follows:

$$\begin{cases} u_t + uu_x = uu_x - \Lambda^{-1}(u\Lambda u_x - u_x\Lambda u) \overset{def}{=} B(u), \quad (t,x) \in (0,\infty)\times\mathcal{S}^1 \\ u(0,x) = u_0, \quad x \in \mathcal{S}^1. \end{cases} \tag{DG}$$

Here the differential operator Λ is defined by $\Lambda \overset{def}{=} H\partial_x$. The symbol of Λ is $-|k|$, that is, $\mathcal{F}(\Lambda u)(k) = -|k|\hat{u}(k)$. Here $\hat{u}(k)$ denotes the kth Fourier coefficient of u. Thus $\Lambda^{-1}u$ is well defined for the periodic function u with vanishing zero mode. One can check that the zero mode of $u\Lambda u_x - u_x\Lambda u$ vanishes, thus the right-hand side of (DG) is well defined. It is easy to see that (1.3.2) and (DG) are equivalent,

1.3.2 Background of wellposedness of the De Gregorio equation.

If we view u and ω as vector fields on $\mathcal{S}^1$, we can rewrite (1.3.2) in terms of usual Lie brackets for vector fields as

$$\omega_t + [u,\omega] = 0,$$

just like the three-dimensional incompressible Euler equation. [CLM85] showed that finite time singularity can occur from smooth initial data for the CLM model (1.3.1).

This is because the CLM equation only models the vortex stretching phenomenon in 3D incompressible ideal fluids. There is no stablizing effect in this model. As a contrast, the De Gregorio model contains a transport term which tends to stablize the solution. Hence there are two competing effects in this model which make the picture complicated. It is conjectured that (1.3.2) enjoys global regularity assuming that the initial data ω_0 is smooth. See discussion in [DG96],[CHK$^+$17], and [EJ17]. We mention here that in a recent paper [LLR18] the authors proved global well-posedness of (1.3.2) with non-negative initial data on $\mathbb{R}$ or $\mathcal{S}^1$. As an interesting contrast, self-similar type singularity has been established in [CHH19] for the De Gregorio model (1.3.2) on $\mathbb{R}$ with compactly supported smooth initial data.

It is clear that the CLM model (1.3.1) is locally-in-time well-posed for initial data ω_0 satisfying the condition $\|\omega_0\|_{L^\infty} + \|H\omega_0\|_{L^\infty} < \infty$. We can see this by considering an equivalent equation for holomorphic functions on the unit disc

$$\dot{f} = -if^2,$$

where f is the holomorphic extension of the function $\omega + iH\omega$ into the unit disk D. However, this also shows that the CLM model is not locally-in-time well-posed for general continuous initial data on the circle because $H\omega$ may not be bounded for continuous ω. For more detailed discussion one can refer to [JSS17]. Similarly, one may expect that the De Gregorio equation (1.3.2) is locally-in-time well-posed in $H^s(\mathcal{S}^1)$ for $s > \frac{1}{2}$ but not in $H^{\frac{1}{2}}(\mathcal{S}^1)$ or $C(\mathcal{S}^1)$. The local well-posedness of (1.3.2) for initial data ω_0 in H^1 is proved in [OSW08], and the authors mentioned in [JSS17] that it is possible to get local well-posedness in $H^{\frac{1}{2}+\epsilon}$ following the idea of [Che98] and [D$^+$01].

1.3.3 Main results

Note that the De Gregorio equation (1.3.2) has the scaling invariance

$$\omega(x,t) \to \omega(\lambda x, t), \qquad u(x,t) \to \frac{1}{\lambda} u(\lambda x, t), \tag{1.3.3}$$

thus a natural critical space for (1.3.2) is $H^{\frac{1}{2}}(\mathcal{S}^1)$. But as mentioned above, this is not a good space to consider the local well-posedness due to failure of the embedding $H^{\frac{1}{2}}(\mathcal{S}^1) \hookrightarrow L^\infty(\mathcal{S}^1)$. In order to get local well-posedness the velocity field u is usually required to be Lipschitz. However, we can still find critical spaces which admit a Lipschitz velocity field in the class of Besov spaces, and $B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1)$ is the space satisfying desired properties: $B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1)$ is continuously embedded into $\mathcal{C}(\mathcal{S}^1)$, the set of continuous functions on $\mathcal{S}^1$, and if $\omega \in B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1)$, we have $H\omega \in B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1)$. Therefore, it is natural to consider local well-posedness of the De Gregorio equation with initial data in $B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1)$. The definition and properties of Besov spaces in the periodic setting will be introduced in section 3.1.

To develop the local in time wellposed-ness of the De Gregorio model in the critical Besov space, we mainly consider the equation (DG) of $u(t,x)$. The corresponding critical Besov space for u is $B^{\frac{3}{2}}_{2,1}(\mathcal{S}^1)$. The advantage of considering (DG) is that we are allowed more regularity when considering the question of uniqueness and continuous dependence on initial data. Our main result is

Theorem 1.3.1. *For any $u_0 \in B^{\frac{3}{2}}_{2,1}(\mathcal{S}^1)$, there exists $T > 0$ which depends only on $\|u_0\|_{B^{\frac{3}{2}}_{2,1}(\mathcal{S}^1)}$ such that the De Gregorio equation (DG) has a unique solution u in the class $C([0,T], B^{\frac{3}{2}}_{2,1}(\mathcal{S}^1)) \cap C^1([0,T], B^{\frac{1}{2}}_{2,1}(\mathcal{S}^1))$. The solution u continuously depends on the initial data u_0. Furthermore, if u blows up at finite time $T^* > 0$, then we have the following Beale-Kato-Majda blow up criterion:*

$$\int_0^{T^*} \|u(t)\|_{B^{\frac{3}{2}}_{2,1}(\mathcal{S}^1)} dt = \infty. \tag{1.3.4}$$

To prove Theorem 1.3.1, we first develop a Gronwall type estimate for a class of linear transport equation with forces in the Besov space. We next obtain estimates of $B(u)$ defined in (DG) with Besov norms. We then apply an iterative scheme to construct the solution to (1.3.2) and show that the solution is unique. Finally, we show the Beale-Kato-Majda type blow up criterion (1.3.4) with another Grownwall type estimate.

Chapter 2

Singularity Structures of the Spectrum Density Function of Linear Inviscid Damping

2.1 Preliminary

2.1.1 Spectral density function

With the linearized operator L_k defined in (1.2.8), the equation (1.2.7) can be formulated as

$$\partial_t \omega_k + ikL_k\omega_k = 0, \quad \omega_k(0,y) = \omega_0^k(y). \tag{2.1.1}$$

By standard theory of spectral projection, we then have

$$\begin{aligned}\omega_k(t,y) &= \frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{i\lambda t}\left[(\lambda + kL_k - i\epsilon)^{-1} - (\lambda + kL_k + i\epsilon)^{-1}\right]\omega_0^k\, d\lambda \\ &= \frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_0^1 e^{-ikb(y_0)t}|b'(y_0)| \times \Big[(-b(y_0) + L_k - i\epsilon)^{-1} \\ &\qquad - (-b(y_0) + L_k + i\epsilon)^{-1}\Big]\omega_0^k\, dy_0.\end{aligned} \tag{2.1.2}$$

We then obtain

$$\begin{aligned}\psi_k(t,y) = &-\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_0^1 e^{-ikb(y_0)t}|b'(y_0)|\int_0^1 G_k(y,z)\\ &\times\Big\{\Big[(-b(y_0)+L_k-i\epsilon)^{-1}-(-b(y_0)+L_k+i\epsilon)^{-1}\Big]\omega_0^k\Big\}(z)\,dzdy_0\\ =&-\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_0^1 e^{-ikb(y_0)t}|b'(y_0)|\Big[\psi^-_{k,\epsilon}(y,y_0)-\psi^+_{k,\epsilon}(y,y_0)\Big]\,dy_0.\end{aligned} \tag{2.1.3}$$

In the above,

$$\begin{aligned}\psi^+_{k,\epsilon}(y,y_0) &:= \int_0^1 G_k(y,z)\Big[(-b(y_0)+L_k+i\epsilon)^{-1}\omega_0^k\Big](z)\,dz,\\ \psi^-_{k,\epsilon}(y,y_0) &:= \int_0^1 G_k(y,z)\Big[(-b(y_0)+L_k-i\epsilon)^{-1}\omega_0^k\Big](z)\,dz.\end{aligned} \tag{2.1.4}$$

We note that $\psi^+_{k,\epsilon}(y,y_0),\psi^-_{k,\epsilon}(y,y_0)$ satisfy for $\iota\in\{+,-\}$

$$\Big(-k^2+\frac{d^2}{dy^2}\Big)\psi^\iota_{k,\epsilon}(y,y_0)-\frac{b''(y)}{b(y)-b(y_0)+i\iota\epsilon}\psi^\iota_{k,\epsilon}(y,y_0)=\frac{-\omega_0^k(y)}{b(y)-b(y_0)+i\iota\epsilon} \tag{2.1.5}$$

which can be reformulated as

$$\begin{aligned}&\psi^\iota_{k,\epsilon}(y,y_0)+\int_0^1 G_k(y,z)\frac{b''(z)\psi^\iota_{k,\epsilon}(z,y_0)}{b(z)-b(y_0)+i\iota\epsilon}dz\\ =&\int_0^1 G_k(y,z)\frac{\omega_0^k(z)}{b(z)-b(y_0)+i\iota\epsilon}dz.\end{aligned} \tag{2.1.6}$$

We summarize the above computation as the following proposition.

Proposition 2.1.1. *For $k\geq 1$ and $(t,y)\in[0,\infty)\times[0,1]$, assume $\omega_k(t,y)$ and $\psi_k(t,y)$ solve the equation (1.2.7) with the initial data $\omega_0^k(y)$. For $y,y_0\in[0,1]$, $\epsilon\in(0,1)$ and $\iota\in\{+,-\}$, let $\psi^\iota_{k,\epsilon}(y,y_0)$ be the solution to*

$$\begin{aligned}&\psi^\iota_{k,\epsilon}(y,y_0)+\int_0^1 G_k(y,z)\frac{b''(z)\psi^\iota_{k,\epsilon}(z,y_0)}{b(z)-b(y_0)+i\iota\epsilon}dz\\ =&\int_0^1 G_k(y,z)\frac{\omega_0^k(z)}{b(z)-b(y_0)+i\iota\epsilon}dz.\end{aligned} \tag{2.1.7}$$

Then we have for $(t,y)\in[0,\infty)\times[0,1]$

$$\psi_k(t,y)=-\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_0^1 e^{-ikb(y_0)t}|b'(y_0)|\Big[\psi^-_{k,\epsilon}(y,y_0)-\psi^+_{k,\epsilon}(y,y_0)\Big]\,dy_0. \tag{2.1.8}$$

2.1.2 Extension of the Green's function

For integers $k \geq 1$, recall that the Green's function $G_k(y,z)$ solves for $y, z \in [0,1]$

$$-\frac{\partial^2}{\partial y^2} G_k(y,z) + k^2 G_k(y,z) = \delta(y-z), \tag{2.1.9}$$

with Dirichlet boundary conditions $G_k(0,z) = G_k(1,z) = 0$. G_k has the explicit formula

$$G_k(y,z) = \frac{1}{k \sinh k} \begin{cases} \sinh(k(1-z)) \sinh(ky) & \text{if } y \leq z, \\ \sinh(kz) \sinh(k(1-y)) & \text{if } y \geq z. \end{cases} \tag{2.1.10}$$

We first make zero extension of $G_k(y,z)$ in the z direction as follows. For $y \in [0,1], z \in \mathbb{R}$, we set

$$G_k(y,z) := \begin{cases} G_k(y,z) & \text{if } z \in [0,1], \\ 0 & \text{if } z \notin [0,1]. \end{cases} \tag{2.1.11}$$

After this extension, $G_k(y,z)$ solves

$$(-\frac{d^2}{dz^2} + k^2) G_k(y,z) = \delta(z-y) - \frac{\sinh(k(1-y))}{\sinh k} \delta(z) - \frac{\sinh(ky)}{\sinh k} \delta(z-1) \tag{2.1.12}$$

for $y \in [0,1], z \in \mathbb{R}$. In order to obtain optimal control of Sobolev norms, we choose a smooth cutoff function $\Psi_k(y)$ satisfying the following assumptions:

(a) For some small $\delta_0 > 0$, $\Psi_k \equiv 1$ on $[0,1], \mathrm{supp}\Psi_k \Subset (-\delta_0/k, 1+\delta_0/k)$ and $0 \leq \Psi_k \leq 1$.

(b) $\Psi_k'(y) > 0$ for $y \in (-\delta_0/k, 0)$, and $\Psi_k'(y) < 0$ for $y \in (1, 1+\delta_0/k)$.

These assumptions are helpful for establishing the limiting absorption principle. We can then extend $G_k(y,z)$ to $(y,z) \in \mathbb{R} \times \mathbb{R}$ by solving

$$(-\frac{d^2}{dz^2} + k^2) G_k(y,z) = \Psi_k(y) \Big[\delta(z-y) - \frac{\sinh(k(1-y))}{\sinh k} \delta(z) - \frac{\sinh(ky)}{\sinh k} \delta(z-1) \Big]. \tag{2.1.13}$$

Lemma 2.1.2. *The extended Green function (also denoted as $G_k(y,z)$) has the explicit expression: for $y, z \in \mathbb{R}$,*

$$G_k(y,z) = \Psi_k(y) \Big[\frac{1}{k} e^{-k|z-y|} - \frac{\sinh(k(1-y))}{k \sinh k} e^{-k|z|} - \frac{\sinh(ky)}{k \sinh k} e^{-k|z-1|} \Big]. \tag{2.1.14}$$

The proof of this lemma is straightforward. One can directly verify that expression (2.1.14) agrees with (2.1.10) for $(y,z)\in[0,1]\times[0,1]$.

We extend the background flow $b(y)$ in a way such that $b(y)$ is smooth, $b'(y)>c$ for $y\in\mathbb{R}$ and some constant $c>0$. In addition, $b''(y)=0$ for $y\notin[-2,2]$. We extend the initial data $\omega_0^k(y)$ to be a smooth function defined for all $y\in\mathbb{R}$ and supported on the interval $[-2,2]$. We can then extend the spectral density function $\psi_{k,\epsilon}^{\iota}(y,y_0)$ to $y\in\mathbb{R}$ by solving for any $y_0\in[0,1]$

$$\psi_{k,\epsilon}^{\iota}(y,y_0)+\int_{\mathbb{R}}G_k(y,z)\frac{b''(z)\psi_{k,\epsilon}^{\iota}(z,y_0)}{b(z)-b(y_0)+i\epsilon}dz=\int_{\mathbb{R}}G_k(y,z)\frac{\omega_0^k(z)}{b(z)-b(y_0)+i\epsilon}dz. \quad (2.1.15)$$

Note that for $y\in[0,1]$, the extended Green function $G_k(y,z)$, see (2.1.14), does not vanish only when $z\in[0,1]$. Therefore, the solution to (2.1.15) agrees with the original spectral density function when $y\in[0,1]$ and $y_0\in[0,1]$.

2.1.3 Change of variables

For $y,y_0,z\in\mathbb{R}$, define the following new variables $v:=b(y), v':=b(z), w:=b(y_0)$. Define the following functions with new variables: for $y,y_0\in\mathbb{R}$,

$$\begin{aligned} &f_0^k(v):=\omega_0^k(y), \quad \mathcal{G}_k(v,v'):=G_k(y,z),\\ &\phi_{k,\epsilon}^{\iota}(v,w):=\psi_{k,\epsilon}^{\iota}(y,y_0), \quad B(v):=b'(y). \end{aligned} \quad (2.1.16)$$

Let $\Theta_{k,\epsilon}^{\iota}(v,w)=\phi_{k,\epsilon}^{\iota}(v+w,w)$. Then for $v,w\in\mathbb{R}$, $\Theta_{k,\epsilon}^{\iota}(v,w)$ solves the following equation: for $v,w\in\mathbb{R}$,

$$\begin{aligned} &\Theta_{k,\epsilon}^{\iota}(v,w)+\int_{\mathbb{R}}\mathcal{G}_k(v+w,v'+w)\frac{\partial_{v'}B(v'+w)\Theta_{k,\epsilon}^{\iota}(v',w)}{v'+i\iota\epsilon}dv'\\ =&\int_{\mathbb{R}}\mathcal{G}_k(v+w,v'+w)\frac{1}{B(v'+w)}\frac{f_0^k(v'+w)}{v'+i\iota\epsilon}dv'. \end{aligned} \quad (2.1.17)$$

2.1.4 Weighted Sobolev norms

For the weighted Sobolev norm $H_k^n(\mathbb{R})$ defined in (1.2.10), we have the following lemma.

Lemma 2.1.3. *For any positive integer k we define the weighted Sobolev space $H_k^n(\mathbb{R})$ as in* (1.2.10)*. Let r be a real number such that $0\le r\le 1$. Let $H^r(\mathbb{R})$ be the usual Sobolev space. Then for any $u\in H_k^1(\mathbb{R})$, we have*

$$\|u\|_{H^r(\mathbb{R})}\lesssim_{r,n}|k|^{r-1}\|u\|_{H_k^1(\mathbb{R})}. \quad (2.1.18)$$

In addition, for any $2 \leq p < \infty$, we have

$$\|u\|_{L^p(\mathbb{R})} \lesssim_p k^{-(\frac{1}{2}+\frac{1}{p})} \|u\|_{H^1_k(\mathbb{R})} \tag{2.1.19}$$

Proof. The proof follows directly from the Gagliardo-Nirenberg interpolation inequality and the definition of the $H^1_k(\mathbb{R})$ in (1.2.10). □

2.2 The limiting absorption principle

Define the following operator for $v, w \in \mathbb{R}, \epsilon \in [-\frac{1}{4}, \frac{1}{4}] \setminus \{0\}$ and $h \in H^1_k(\mathbb{R})$

$$T_{k,\epsilon}h(v,w) := \int_{\mathbb{R}} \mathcal{G}_k(v+w, v'+w)\partial_{v'}B(v'+w)\frac{h(v')}{v'+i\epsilon}dv'. \tag{2.2.1}$$

We have the following lemma.

Lemma 2.2.1. *Assume k is a positive integer and $\epsilon \in [-\frac{1}{4}, \frac{1}{4}] \setminus \{0\}$. For any $h \in H^1_k(\mathbb{R})$ and $w \in \mathbb{R}$, we have*

$$\|T_{k,\epsilon}h(\cdot, w)\|_{H^1_k(\mathbb{R})} \lesssim \frac{k^{-\frac{1}{4}}}{1+|w|}\|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.2}$$

In addition, we have for $w \in \mathbb{R}$,

$$\left\|\partial_v T_{k,\epsilon}h(v,w) + 2\Psi_k(v+w)\frac{\partial_v B(v+w)}{B(v+w)}h(v)\log(v+i\epsilon)\right\|_{W^{1,1}(v\in\mathbb{R})} \lesssim_k \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.3}$$

Proof. For $w \in [b(-10), b(10)]$, using the identity that

$$\partial_{v'} \log(v'+i\epsilon) = \frac{1}{v'+i\epsilon}$$

and taking integration by parts, we get

$$\begin{aligned} T_{k,\epsilon}h(v,w) = &-\int_{\mathbb{R}} \partial_{v'}\mathcal{G}_k(v+w, v'+w)\partial_{v'}B(v'+w)h(v')\log(v'+i\epsilon)dv' \\ &-\int_{\mathbb{R}} \mathcal{G}_k(v+w, v'+w)\partial_{v'}\left[\partial_{v'}B(v'+w)h(v')\right]\log(v'+i\epsilon)dv' \\ :=&T_1+T_2. \end{aligned} \tag{2.2.4}$$

Step 1: estimates of T_1. Using the expression of $G_k(y,z)$ in (2.1.14), we have

$$
\begin{aligned}
T_1 = \Psi_k(v+w)\int_{\mathbb{R}} \Big[& e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\operatorname{sign}(v-v')\frac{\partial_{v'}B(v'+w)}{B(v'+w)}h(v') \\
& \times \log(v'+i\epsilon)\Big]dv' \\
- \Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k} & \int_{\mathbb{R}}\Big[e^{-k|b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(0)) \\
& \times \frac{\partial_{v'}B(v'+w)}{B(v'+w)}h(v')\log(v'+i\epsilon)\Big]dv' \\
- \Psi_k(v+w)\frac{\sinh k(b^{-1}(v+w))}{\sinh k} & \int_{\mathbb{R}}\Big[e^{-k|1-b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(1)) \\
& \times \frac{\partial_{v'}B(v'+w)}{B(v'+w)}h(v')\log(v'+i\epsilon)\Big]dv' \\
:= T_{11}+T_{12}+T_{13}. &
\end{aligned} \tag{2.2.5}
$$

For $w\in[b(-10),b(10)]$, we have the following bounds

$$
\begin{aligned}
\|T_{11}(\cdot,w)\|_{L^\infty(\mathbb{R})} &\lesssim \|h\|_{L^2(\mathbb{R})}\Big(\int_{\mathbb{R}} e^{-2k|b^{-1}(v+w)-b^{-1}(v'+w)|}|\log(v'+i\epsilon)|^2dv'\Big)^{\frac12} \\
&\lesssim \frac{1+\log\langle k\rangle}{k^{\frac12}}\|h\|_{L^2(\mathbb{R})} \lesssim k^{-\frac14}\|h\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.2.6}
$$

Since Ψ_k is compactly supported, (2.2.6) implies that

$$
\|T_{11}(\cdot,w)\|_{L^2(\mathbb{R})} \lesssim k^{-\frac14}\|h\|_{L^2(\mathbb{R})}. \tag{2.2.7}
$$

Note that

$$
\begin{aligned}
\partial_v T_{11}(v,w) = -\frac{k\Psi_k(v+w)}{B(v+w)} & \int_{\mathbb{R}}\Big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{\partial_{v'}B(v'+w)}{B(v'+w)}h(v') \\
& \times \log(v'+i\epsilon)\Big]dv' \\
& + 2\Psi_k(v+w)\frac{\partial_v B(v+w)}{B(v+w)}h(v)\log(v+i\epsilon) \\
& + \partial_v\Psi_k(v+w)\int_{\mathbb{R}}\Big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\operatorname{sign}(v-v') \\
& \times \frac{\partial_{v'}B(v'+w)}{B(v'+w)}h(v')\log(v'+i\epsilon)\Big]dv' \\
& := T_{111}+T_{112}+T_{113}.
\end{aligned} \tag{2.2.8}
$$

By similar computation in (2.2.6), T_{111} and T_{113} are bounded in v for $w \in [b(-10), b(10)]$ with the following estimates

$$\|T_{111}(\cdot,w)\|_{L^2(\mathbb{R})} + \|T_{113}(\cdot,w)\|_{L^2(\mathbb{R})} \lesssim k^{\frac{3}{4}}\|h\|_{L^2(\mathbb{R})} \lesssim k^{-\frac{1}{4}}\|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.9}$$

For T_{112}, using Hölder inequality and Lemma 2.1.3 we have

$$\|T_{112}(\cdot,w)\|_{L^2(\mathbb{R})} \lesssim \|h\|_{L^4(\mathbb{R})} \lesssim k^{-\frac{3}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.10}$$

Hence by definition of the space $H^1_k(\mathbb{R})$, we have

$$\|T_{11}(\cdot,w)\|_{H^1_k(\mathbb{R})} \lesssim |k|^{-\frac{1}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.11}$$

T_{12} and T_{13} can be treated in the same way. Note that by definition of the cutoff function $\Psi_k(v+w)$,

$$\Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k} \quad \text{and} \quad \Psi_k(v+w)\frac{\sinh k(b^{-1}(v+w))}{\sinh k}$$

are bounded uniformly for $k \geq 1$ on the support of $\Psi_k(v+w)$. Hence the same estimate holds for T_{12} and T_{13}. Therefore, we have for $w \in [b(-10), b(10)]$

$$\|T_1(\cdot,w)\|_{H^1_k(\mathbb{R})} \lesssim k^{-\frac{1}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.12}$$

Step2: estimates of T_2. Using the Hölder inequality, we have

$$\begin{aligned} |T_2(v,w)| &\lesssim \|h\|_{H^1(\mathbb{R})} \left\{ \int_{\mathbb{R}} \left(\mathcal{G}_k(v+w, v'+w) \log(v'+i\epsilon) \right)^2 dv' \right\}^{\frac{1}{2}} \\ &\lesssim k^{-\frac{5}{4}} \|h\|_{H^1_k(\mathbb{R})}. \end{aligned} \tag{2.2.13}$$

Hence we have

$$\|T_2(\cdot,w)\|_{L^2(\mathbb{R})} \lesssim k^{-\frac{5}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.14}$$

Taking one derivative in v for T_2, we find that the derivative only acts on the Green function $\mathcal{G}_k$ and the cutoff function Ψ_k which leads to a factor of k. We have

$$\|\partial_v T_2\|_{L^2(\mathbb{R})} \lesssim k^{-\frac{1}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.15}$$

Therefore,

$$\|T_2\|_{H^1_k(\mathbb{R})} \lesssim k^{-\frac{1}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.16}$$

Combining (2.2.16) and (2.2.12), we get for $w \in [b(-10), b(10)]$,

$$\|T_{k,\epsilon}h(\cdot,w)\|_{H^1_k(\mathbb{R})} \lesssim k^{-\frac{1}{4}} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.17}$$

Step 3: the case $w \notin [b(-10), b(10)]$. Assume now $w \notin [b(-10), b(10)]$. Since $\partial_v' B(v'+w)$ is supported in $[b(-2), b(2)]$, we have $|v'| \gtrsim |w|$ on the support of $\partial_{v'} B(v'+w)$. Direct computation shows that

$$\|T_{k,\epsilon}h(\cdot,w)\|_{H^1_k(\mathbb{R})} \lesssim \frac{1}{k|w|} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.18}$$

Combining (2.2.18) and(2.2.17) we get for $w \in \mathbb{R}$,

$$\|T_{k,\epsilon}h(\cdot,w)\|_{H^1_k(\mathbb{R})} \lesssim \frac{k^{-\frac{1}{4}}}{1+|w|} \|h\|_{H^1_k(\mathbb{R})}. \tag{2.2.19}$$

In view of (2.2.8), the most singular component of $\partial_v T_{k,\epsilon}h$ is

$$2\Psi_k(v+w)\frac{\partial_v B(v+w)}{B(v+w)} h(v)\log(v+i\epsilon)$$

. (2.2.3) follows from a similar computation. □

Proposition 2.2.2 (Limiting absorption principle)**.** *There exists $\sigma_0 > 0$ that is sufficiently small such that for $\epsilon \in [-\sigma_0, \sigma_0] \setminus \{0\}$, $k \geq 1$ and $f \in H^1_k(\mathbb{R})$, we have for each $w \in \mathbb{R}$*

$$\left\| f + \int_{\mathbb{R}} \mathcal{G}_k(v+w, v'+w)\partial_{v'} B(v'+w)\frac{f(v')}{v'+i\epsilon}dv' \right\|_{H^1_k(\mathbb{R})} \geq \sigma_0 \|f\|_{H^1_k(\mathbb{R})}. \tag{2.2.20}$$

Proof. Assume (2.2.20) does not hold, we can then find ϵ_j, $w_j \in \mathbb{R}$, k_j and f_j such that $\epsilon_j \to 0+$, $\|f_j\|_{H^1_{k_j}(\mathbb{R})} = 1$ and

$$\lim_{j\to\infty} \left\| f_j + \int_{\mathbb{R}} \mathcal{G}_{k_j}(v+w_j, v'+w_j)\partial_{v'} B(v'+w_j)\frac{f_j(v')}{v'+i\epsilon_j}dv' \right\|_{H^1_{k_j}(\mathbb{R})} = 0. \tag{2.2.21}$$

In view of Lemma 2.2.1, we can assume that k_j and w_j are bounded and thus we can replace k_j simply by some fixed $k \in \mathbb{Z}^+$. Assume $w_j \to w_0$ for some $w_0 \in \mathbb{R}$. By the estimate (2.2.3) and the fact that Ψ_k is compactly supported, we can choose a subsequence of $\{f_j\}$ (also denoted by $\{f_j\}$) such that $f_j \to f$ in $H^1_k(\mathbb{R})$ and $\|f\|_{H^1_k(\mathbb{R})} = 1$. Therefore, we have

$$f(v) + \lim_{j\to\infty} \int_{\mathbb{R}} \mathcal{G}_k(v+w_0, v'+w_0)\partial_{v'} B(v'+w_0)\frac{f(v')}{v'+i\epsilon_j}dv' = 0. \tag{2.2.22}$$

Let $y = b^{-1}(v), y_0 = b^{-1}(w_0)$ and set

$$h(y) = f(v). \tag{2.2.23}$$

We have

$$h(y) + \lim_{j\to\infty} \int_{\mathbb{R}} G_k(y,z) b''(z) \frac{h(z)}{b(z) - b(y_0) + i\epsilon_j} dz = 0. \tag{2.2.24}$$

Recall that $G_k(y,z)$ is given by

$$\begin{aligned} G_k(y,z) &= \Psi_k(y)\left[\frac{1}{k}e^{-k|z-y|} - \frac{\sinh(k(1-y))}{k\sinh k}e^{-k|z|} - \frac{\sinh(ky)}{k\sinh k}e^{-k|z-1|}\right] \\ &:= \Psi_k(y)\widetilde{G}_k(y,z), \end{aligned} \tag{2.2.25}$$

and $\widetilde{G}_k(y,z)$ satisfies for $y, z \in \mathbb{R}$,

$$(k^2 - \frac{d^2}{dy^2})\widetilde{G}_k(y,z) = \delta(y-z). \tag{2.2.26}$$

It follows from (2.2.24) that we can write $h(y)$ as

$$h(y) = \Psi_k(y) g(y) \tag{2.2.27}$$

where $g(y)$ is not identically zero and has the same support as $\Psi_k(y)$. Furthermore, $g(y)$ satisfies for $y \in \mathbb{R}$

$$g(y) + \lim_{j\to\infty} \int_{\mathbb{R}} \widetilde{G}_k(y,z) b''(z) \frac{\Psi_k(z) g(z)}{b(z) - b(y_0) + i\epsilon_j} dz = 0. \tag{2.2.28}$$

We apply $k^2 - \frac{d^2}{dy^2}$ to (2.2.28) and get for $y \in \mathbb{R}$

$$k^2 g(y) - \partial_y^2 g(y) + \lim_{j\to\infty} \frac{b''(y)\Psi_k(y) g(y)}{b(y) - b(y_0) + i\epsilon_j} = 0. \tag{2.2.29}$$

Therefore, for $y \in \mathbb{R}$, in the sense of distributions

$$\begin{aligned} &k^2 g(y) - \partial_y^2 g(y) + \lim_{j\to\infty} \frac{(b(y) - b(y_0)) b''(y)\Psi_k(y) g(y)}{(b(y) - b(y_0))^2 + \epsilon_j^2} \\ &+ iC(y_0)\delta(y - y_0) b''(y_0)\Psi_k(y_0) g(y_0) = 0 \end{aligned} \tag{2.2.30}$$

for some real number $C(y_0) \neq 0$. Multiplying (2.2.30) by $\overline{g(y)}$, integrating over $\mathbb{R}$ and taking the imaginary part, we get

$$b''(y_0)\Psi_k(y_0) g(y_0) = b''(y_0) h(y_0) = 0. \tag{2.2.31}$$

In view of (2.2.31), (2.2.30) is reduced to

$$k^2 g(y) - \partial_y^2 g(y) + \frac{b''(y)\Psi_k(y)g(y)}{b(y)-b(y_0)} = 0 \tag{2.2.32}$$

We next show that $g(y) \equiv 0$ for $y \in \mathbb{R}$. This contradicts with the fact $\|f\|_{H_k^1(\mathbb{R})} = 1$. We consider two cases: $y_0 \in (0,1)$ and $y_0 \notin (0,1)$.

Case I: $y_0 \in (0,1)$. We first show that $g(y) \equiv 0$ for $y \in [0,1]$. Set

$$H(y) = \frac{b''(y)g(y)}{b(y)-b(y_0)}. \tag{2.2.33}$$

By (2.2.31), $H(y) \in L^2(\mathbb{R})$. Assume first $\|H\|_{L^2(0,1)} > 0$. Note that for $y \in [0,1]$, $G_k(y,z) = 0$ for $z \notin [0,1]$ and $\Psi_k(y) = 1$. It follows from (2.2.32) that for $y \in [0,1]$

$$g(y) + \int_0^1 G_k(y,z)H(z)dz = 0, \tag{2.2.34}$$

which leads to

$$(b(y)-b(y_0))H(y) + b''(y)\int_0^1 G_k(y,z)H(z)dz = 0. \tag{2.2.35}$$

This contradicts with the assumption that the linearized operator L_k does not have embedded eigenvalues. Hence $\|H\|_{L^2(0,1)} = 0$. Equation (2.2.32) and the boundary condition $g(0) = g(1) = 0$ give that $g(y) \equiv 0$ for $y \in [0,1]$.

Next, we show that $g(y) \equiv 0$ for $y \geq 0$ and $y \leq 1$. The fact $g(y) \equiv 0$ on $[0,1]$ and (2.2.32) implies that $g \in H^2(\mathbb{R})$ and $g'(0) = g'(1) = 0$. Multiplying (2.2.32) by $g(y)$ and integrating for $y \geq 1$, we get

$$\begin{aligned} 0 = \int_1^\infty \Bigg[& k^2|g(y)|^2 + |\partial_y g(y)|^2 + \frac{(b'(y))^2\Psi_k(y)g^2(y)}{(b(y)-b(y_0))^2} \\ & - \frac{b'(y)\Psi_k'(y)g^2(y)}{b(y)-b(y_0)} - \frac{2b'(y)\Psi_k(y)g_k(y)\partial_y g(y)}{b(y)-b(y_0)}\Bigg] dy. \end{aligned} \tag{2.2.36}$$

By assumption, $0 \leq \Psi_k(y) \leq 1$, we have

$$\begin{aligned} & \int_1^\infty \left[|\partial_y g(y)|^2 + \frac{(b'(y))^2\Psi_k(y)g^2(y)}{(b(y)-b(y_0))^2} - \frac{2b'(y)\Psi_k(y)g(y)\partial_y g(y)}{b(y)-b(y_0)}\right] dy \\ \geq & \int_1^\infty \left[|\partial_y g(y)|^2 + \frac{(b'(y))^2\Psi_k^2(y)g(y)}{(b(y)-b(y_0))^2} - \frac{2b'(y)\Psi_k(y)g(y)\partial_y g(y)}{b(y)-b(y_0)}\right] dy \\ = & \int_1^\infty \left[\partial_y g(y) - \frac{b'(y)\Psi_k(y)g(y)}{b(y)-b(y_0)}\right]^2 dy \geq 0. \end{aligned} \tag{2.2.37}$$

Since $b(y)$ is strictly increasing and we assumed that $\Psi_k'(y) \leq 0$ for $y \geq 1$, we have

$$-\frac{b'(y)\Psi_k'(y)g^2(y)}{b(y)-b(y_0)} \geq 0. \tag{2.2.38}$$

Therefore, it follows from (2.2.36) that $g(y) \equiv 0$ for $y \geq 1$. $g(y) \equiv 0$ for $y \leq 0$ follows from a similar argument.

Case II: $y_0 \notin (0,1)$**.** In this case $g(y) \equiv 0$ for $y \in [0,1]$ still holds since the linearized operator L_k is assumed to have no embedded eigenvalues. Note that $g(y)$ is supported on $(-\delta_0/k, 1+\delta_0/k)$. Multiplying (2.2.32) by $g(y)$ and integraing over the interval $J := (1, 1+\delta_0/k)$, we get

$$k^2 \|g\|_{L^2(J)}^2 + \|\partial_y g\|_{L^2(J)}^2 + \int_1^{1+\delta_0/k} \frac{b''(y)\Psi_k(y)(g(y))^2}{b(y)-b(y_0)} dy = 0. \tag{2.2.39}$$

Assume first $y_0 \in J$ and $g(y_0) \neq 0$. By assumption of $\Psi_k(y)$, $\Psi_k(y_0) \neq 0$. Hence from (2.2.31) we get $b''(y_0) = 0$. Therefore,

$$\begin{aligned}|\int_1^{1+\delta_0/k} \frac{b''(y)\Psi_k(y)(g(y))^2}{b(y)-b(y_0)} dy| &\leq \|g\|_{L^\infty(J)}^2 |\int_1^{1+\delta_0/k} \frac{b''(y)-b''(y_0)}{b(y)-b(y_0)} dy| \\ &\leq \frac{\delta_0}{k} \left\|b''/b'\right\|_{L^\infty(J)} \|g\|_{L^\infty(J)}^2 .\end{aligned} \tag{2.2.40}$$

Since $g(1) = 0$, we have for $y \in J$

$$\begin{aligned}|g(y)| &= |g(y) - g(1)| \\ &= |\int_1^y \partial_t g(t) dt| \\ &\leq (\frac{\delta_0}{k})^{1/2} \|\partial_y g\|_{L^2(J)} .\end{aligned} \tag{2.2.41}$$

It follows from (2.2.39), (2.2.40) and (2.2.41) that

$$k^2 \|g\|_{L^2(J)}^2 + \|\partial_y g\|_{L^2(J)}^2 \leq (\frac{\delta_0}{k})^2 \left\|b''/b'\right\|_{L^\infty(J)} \|\partial_y g\|_{L^2(J)}^2 . \tag{2.2.42}$$

By assumption of the background flow $b(y)$, $b''(y)/b'(y)$ is bounded for all $y \in \mathbb{R}$. Therefore, there exists a $\delta_0 > 0$ sufficiently small such that

$$(\frac{\delta_0}{k})^2 \left\|b''/b'\right\|_{L^\infty(J)} < 1, \tag{2.2.43}$$

which implies $g(y) \equiv 0$ on J.

Next, we assume that $y_0 \in J$ and $g(y_0) = 0$. We have

$$\begin{aligned}
&|\int_1^{1+\delta_0/k} \frac{b''(y)\Psi_k(y)(g(y))^2}{b(y)-b(y_0)} dy| \\
\le& \left\|b''\right\|_{L^\infty(J)} \int_1^{1+\delta_0/k} \frac{\left[g(y)-g(y_0)\right]^2}{|b(y)-b(y_0)|} dy \\
=& \left\|b''\right\|_{L^\infty(J)} \int_1^{1+\delta_0/k} \frac{\left[\int_{y_0}^{y} \partial_s g(s) ds\right]^2}{|b(y)-b(y_0)|} dy \\
\le& \left\|b''\right\|_{L^\infty(J)} \left\|\partial_y g\right\|_{L^2(J)}^2 \int_1^{1+\delta_0/k} \frac{|y-y_0|}{|b(y)-b(y_0)|} dy \\
\le& \frac{\delta_0}{k} \left\|b''\right\|_{L^\infty(J)} \left\|1/b'\right\|_{L^\infty(J)} \left\|\partial_y g\right\|_{L^2(J)}^2 .
\end{aligned} \tag{2.2.44}$$

There exists a δ_0 sufficiently small that

$$\frac{\delta_0}{k} \left\|b''\right\|_{L^\infty(J)} \left\|1/b'\right\|_{L^\infty(J)} < 1, \tag{2.2.45}$$

which also implies that $g(y) \equiv 0$ on J.

Lastly, we assume that $y_0 \notin J$. In this case

$$\begin{aligned}
|\int_1^{1+\delta_0/k} \frac{b''(y)\Psi_k(y)(g(y))^2}{b(y)-b(y_0)} dy| &\le \left\|b''\right\|_{L^\infty(J)} \int_1^{1+\delta_0/k} \frac{\left[g(y)\right]^2}{|b(y)-b(y_0)|} dy \\
&\le \left\|b''\right\|_{L^\infty(J)} \int_1^{1+\delta_0/k} \frac{\left[g(y)\right]^2}{|b(y)-b(1+\delta_0/k)|} dy.
\end{aligned} \tag{2.2.46}$$

Using the fact that $g(1+\delta_0/k) = 0$ and similar arguments in (2.2.44), we can show that $g(y) \equiv 0$ on J if $\delta_0 > 0$ is sufficiently small.

In conclusion, we have showed that $g(y) \equiv 0$ on $J = (1, 1+\delta_0/k)$ if $y_0 \notin [0,1]$ and $\delta_0 > 0$ is sufficiently small. Using the same method we can show that $g(y)$ also vanishes on $(-\delta_0/k, 0)$ for some $\delta_0 > 0$. Therefore, $g(y)$ vanishes for all $y \in \mathbb{R}$ which finishes the proof of the limiting absorption principle. □

For later application, we also need the following result.

Proposition 2.2.3. *Let $\sigma_0 > 0$ be the constant given by Proposition 2.2.2. For $\epsilon \in (-\sigma_0, \sigma_0) \setminus \{0\}$, $k \ge 1$, let $T_{k,\epsilon}$ be the operator defined in* (2.2.1)*. For $w \in \mathbb{R}$, there exist two functions $\Phi^0_{k,\epsilon}(v,w), \Phi^1_{k,\epsilon}(v,w) \in H^1_k(\mathbb{R})$ such that*

$$\Phi^0_{k,\epsilon}(v,w) + T_{k,\epsilon}\Phi^0_{k,\epsilon}(v,w) = \Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k} \tag{2.2.47}$$

and

$$\Phi^1_{k,\epsilon}(v,w) + T_{k,\epsilon}\Phi^1_{k,\epsilon}(v,w) = \Psi_k(v+w)\frac{\sinh(kb^{-1}(v+w))}{\sinh k}. \tag{2.2.48}$$

Proof. The above two equations are similar so we only need to focus on (2.2.47). By Lemma 2.2.1,

$$-T_{k,\epsilon}\left(\Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k}\right) \in H^1_k(\mathbb{R}). \tag{2.2.49}$$

It follows from the limiting absorption principle Proposition 2.2.2 that there exists a unique function $\widetilde{\Psi}_{k,\epsilon}(v,w) \in H^1_k(\mathbb{R})$ such that

$$\widetilde{\Psi}_{k,\epsilon} + T_{k,\epsilon}\left(\widetilde{\Psi}_{k,\epsilon}\right) = -T_{k,\epsilon}\left(\Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k}\right). \tag{2.2.50}$$

Set

$$\Phi^0_{k,\epsilon}(v,w) = \widetilde{\Psi}_{k,\epsilon}(v,w) + \Psi_k(v+w)\frac{\sinh k(1-b^{-1}(v+w))}{\sinh k}. \tag{2.2.51}$$

One can verify that $\Phi^0_{k,\epsilon}(v,w)$ solves (2.2.47). □

2.3 Singularity structures

The analysis of the singularity structure of the spectrum density function $\Theta^\iota_{k,\epsilon}(v,w)$ is the most crucial part to understand the long time behavior of the stream function. Recall that for $v \in \mathbb{R}, w \in [b(0), b(1)]$, $\Theta^\iota_{k,\epsilon}(v,w)$ solves the following equation

$$\begin{aligned}&\Theta^\iota_{k,\epsilon}(v,w) + \int_{\mathbb{R}} \mathcal{G}_k(v+w, v'+w)\frac{\partial_{v'}B(v'+w)\Theta^\iota_{k,\epsilon}(v',w)}{v'+i\iota\epsilon}dv' \\ &= \int_{\mathbb{R}} \mathcal{G}_k(v+w,v'+w)\frac{1}{B(v'+w)}\frac{f^k_0(v'+w)}{v'+i\iota\epsilon}dv'.\end{aligned} \tag{2.3.1}$$

2.3.1 Singularity structure of right-hand side of (2.3.1)

To illustrate the main idea, we start with the analysis of singularities of

$$A_k(v,w) := \int_{\mathbb{R}} \mathcal{G}_k(v+w,v'+w)\frac{h(v'+w)}{v'+i\epsilon}dv', \tag{2.3.2}$$

where $h(\cdot)$ is compactly supported smooth function on the real line. For $v, v' \in \mathbb{R}$, let

$$\mathcal{G}^{fr}_k(v,v') := \Psi_k(b^{-1}(v))\frac{1}{k}e^{-k|b^{-1}(v)-b^{-1}(v')|} \tag{2.3.3}$$

be the *free part* of the Green function, and

$$\mathcal{G}_k^b(v,v') := \mathcal{G}_k(v,v') - \mathcal{G}_k^{fr}(v,v') \tag{2.3.4}$$

be the *boundary part* of the Green function. The boundary part $\mathcal{G}_k^b$ can be further splitted into two parts $\mathcal{G}_k^b(v,v') = \mathcal{G}_k^{b_0}(v,v') + \mathcal{G}_k^{b_1}(v,v')$ where

$$\begin{aligned}\mathcal{G}_k^{b_0}(v,v') &:= -\Psi_k(b^{-1}(v))\frac{\sinh(k(1-b^{-1}(v)))}{k\sinh k}e^{-k|b^{-1}(v')|}\\ &:= \Phi_k^{b0}(v)\frac{e^{-k|b^{-1}(v')|}}{k}\end{aligned} \tag{2.3.5}$$

captures the boundary effect from the side $b(0)$, and

$$\begin{aligned}\mathcal{G}_k^{b_1}(v,v') &:= -\Psi_k(b^{-1}(v))\frac{\sinh(kb^{-1}(v))}{k\sinh k}e^{-k|b^{-1}(v')-1|}\\ &:= \Phi_k^{b1}(v)\frac{e^{-k|b^{-1}(v')-1|}}{k}\end{aligned} \tag{2.3.6}$$

captures the boundary effect from the side $b(1)$. Then we can decompose for $v, w \in \mathbb{R}$

$$\begin{aligned}&A_k(v,w)\\ &= \int_{\mathbb{R}} \mathcal{G}_k^{fr}(v+w,v'+w)\frac{h(v'+w)}{v'+i\epsilon}dv' + \int_{\mathbb{R}} \mathcal{G}_k^{b}(v+w,v'+w)\frac{h(v'+w)}{v'+i\epsilon}dv'\\ &:= A_k^{fr}(v,w) + A_k^b(v,w).\end{aligned} \tag{2.3.7}$$

We study the free term $A_k^{fr}(v,w)$ first, which can be written as

$$A_k^{fr}(v,w) = \Psi_k(v+w)\int_{\mathbb{R}} \frac{1}{k}e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{h(v'+w)}{v'+i\epsilon}dv'. \tag{2.3.8}$$

Taking one derivative in v, we get for $v, w \in \mathbb{R}$

$$\begin{aligned}\partial_v A_k^{fr}(v,w) &= -\Psi_k(v+w)\int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{\operatorname{sign}(v-v')}{B(v+w)}\frac{h(v'+w)}{v'+i\epsilon}dv'\\ &\quad + \partial_v\Psi_k(v+w)\int_{\mathbb{R}} \frac{1}{k}e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{h(v'+w)}{v'+i\epsilon}dv'\\ &:= T_1 + T_2.\end{aligned} \tag{2.3.9}$$

Taking integration by parts, we get

$$
\begin{aligned}
T_1 = &- \Psi_k(v+w) \int_{\mathbb{R}} \Big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \frac{\operatorname{sign}(v-v')}{B(v+w)} h(v'+w) \\
&\qquad \times \partial_{v'} \log(v'+i\epsilon) \Big] dv' \\
= &- \Psi_k(v+w) \int_{\mathbb{R}} \Big[k e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \frac{h(v'+w)}{B(v+w)B(v'+w)} \\
&\qquad \times \log(v'+i\epsilon) \Big] dv' \\
&+ \Psi_k(v+w) \int_{\mathbb{R}} \Big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \operatorname{sign}(v'-v) \frac{\partial_{v'} h(v'+w)}{B(v+w)} \\
&\qquad \times \log(v'+i\epsilon) \Big] dv' \\
&+ \frac{2\Psi_k(v+w)}{B(v+w)} h(v+w) \log(v+i\epsilon)
\end{aligned}
\tag{2.3.10}
$$

Here in the last equation we used the fact that for $v, v' \in \mathbb{R}$

$$
\frac{d}{dv'} \operatorname{sign}(v'-v) = 2\delta(v'-v).
$$

Therefore, we can rewrite for $v, w \in \mathbb{R}$

$$
\partial_v \mathcal{A}_k^{fr}(v,w) = \frac{2\Psi_k(v+w)}{B(v+w)} h(v+w) \log(v+i\epsilon) + \mathcal{R}^1_{k,\epsilon}(v,w), \tag{2.3.11}
$$

where the remainder term $\mathcal{R}^1_{k,\epsilon}(v,w)$ is defined as

$$
\begin{aligned}
\mathcal{R}^1_{k,\epsilon}(v,w) := &- \Psi_k(v+w) \int_{\mathbb{R}} \Big[k e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \frac{h(v'+w)}{B(v+w)B(v'+w)} \\
&\qquad \times \log(v'+i\epsilon) \Big] dv' \\
&+ \Psi_k(v+w) \int_{\mathbb{R}} \Big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \operatorname{sign}(v'-v) \\
&\qquad \times \frac{\partial_{v'} h(v'+w)}{B(v+w)} \log(v'+i\epsilon) \Big] dv' \\
&- \partial_v \Psi_k(v+w) \int_{\mathbb{R}} \Big[\partial_{v'} \big[\frac{1}{k} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} h(v'+w) \big] \\
&\qquad \times \log(v'+i\epsilon) \Big] dv'.
\end{aligned}
\tag{2.3.12}
$$

We observe from the above calculation that when we differentiate $A_k^{fr}(v,w)$ in the variable v, we get a term with $\log(v+i\epsilon)$. This term introduces singularity as $\epsilon \to 0+$. For the remainder term, we also have similar singularity structure. We formulate this rigorously as the following lemma.

Lemma 2.3.1. *Let $N \geq 1$, $k \geq 1$, $\epsilon \in (-1,1) \setminus \{0\}$. For any $h \in H_k^N(\mathbb{R})$, define for $v, w \in \mathbb{R}$*

$$\mathcal{R}_{k,\epsilon}^0(v,w) = \int_{\mathbb{R}} \mathcal{G}^{fr}(v+w, v'+w) \frac{h(v'+w)}{v'+i\epsilon} dv'. \tag{2.3.13}$$

Then for $0 \leq j \leq N-1$ there exist functions $A_k^j(v)$ and $\mathcal{R}_{k,\epsilon}^j(v,w)$ satisfying the following bounds (setting $A_k^0(v,w)=0$)

$$\left\| \partial_v^m A_k^j(v) \right\|_{H_k^1} + \left\| \partial_w^m \mathcal{R}_{k,\epsilon}^j(v,w) \right\|_{L_w^2 H_{k,v}^1} \lesssim_{j,m} \|h\|_{H_k^{j+1+m}} \tag{2.3.14}$$

such that for $0 \leq j \leq N-2$,

$$\partial_v \mathcal{R}_{k,\epsilon}^j(v,w) = A_k^{j+1}(v+w)\log(v+i\epsilon) + \mathcal{R}_{k,\epsilon}^{j+1}(v,w) \tag{2.3.15}$$

Proof. We claim that for $j \geq 1$, there exist smooth functions $\alpha_p^j(v,v',k)$ and $\beta_p^j(v,v',k)$ ($p \in \{0,1\}$) such that for any integers $m, n \geq 0$

$$\left\| \partial_v^m \partial_{v'}^n \alpha_p^j(v,v',k) \right\|_{L^\infty(\mathbb{R}\times\mathbb{R})} + \left\| \partial_v^m \partial_{v'}^n \beta_p^j(v,v',k) \right\|_{L^\infty(\mathbb{R}\times\mathbb{R})} \lesssim k^{j+m-p}, \tag{2.3.16}$$

and we can define $\mathcal{R}_{k,\epsilon}^j(v,w)$ as

$$\begin{aligned} \mathcal{R}_{k,\epsilon}^j(v,w) = & \int_{\mathbb{R}} \Big\{ e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \log(v'+i\epsilon)\,\mathrm{sign}(v-v') \\ & \times \sum_{p=0}^{1} \left[\alpha_p^j(v+w, v'+w, k) \partial_{v'}^p h(v'+w) \right] \Big\} dv' \\ & + \int_{\mathbb{R}} \Big\{ e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \log(v'+i\epsilon) \\ & \times \sum_{p=0}^{1} \left[\beta_p^j(v+w, v'+w, k) \partial_{v'}^p h(v'+w) \right] \Big\} dv'. \end{aligned} \tag{2.3.17}$$

We see from (2.3.9) to (2.3.12) that we can define $\mathcal{R}_{k,\epsilon}^1(v,w)$ by setting

$$\alpha_0^1(v,v',k) = -\frac{\partial_v \Psi_k(v)}{B(v')}, \quad \alpha_1^1(v,v',k) = \frac{\Psi_k(v)}{B(v)} \tag{2.3.18}$$

$$\beta_0^1(v,v',k) = -\frac{k\Psi_k(v)}{B(v)B(v')}, \quad \beta_1^1(v,v',k) = -\frac{\partial_v \Psi_k(v)}{k}, \tag{2.3.19}$$

and (2.3.11) holds with

$$A_k^1(v) = \frac{2\Psi_k(v)h(v)}{B(v)}. \tag{2.3.20}$$

Noting that $\exp -k|b^{-1}(v+w) - b^{-1}(v'+w)|$ is smooth in w, the estimate 2.3.14 follows from (2.3.17), (2.3.20) and the Cauchy-Schwarz inequality.

Assume for some $j_0 \geq 1$ we have constructed $\mathcal{R}_{k,\epsilon}^{j_0}(v,w)$. We have

$$\begin{aligned}
&\partial_v \mathcal{R}_{k,\epsilon}^{j_0}(v,w) \\
&= 2\log(v+i\epsilon)\sum_{p=0}^{1}\left[\alpha_p^j(v+w,v+w,k)\partial_v^p h(v+w)\right] \\
&\quad + \int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\log(v'+i\epsilon)\,\mathrm{sign}(v-v')S(v,v',w)dv' \\
&\quad + \int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\log(v'+i\epsilon)T(v,v',w)dv' \\
&:= A_k^{j_0+1}(v+w)\log(v+i\epsilon) + R_{k,\epsilon}^{j_0+1}(v,w),
\end{aligned} \tag{2.3.21}$$

where

$$\begin{aligned}
S(v,v',w) = &\sum_{p=0}^{1}\left[\partial_v\alpha_p^j(v+w,v'+w,k)\partial_{v'}^p h(v'+w)\right] \\
&- \frac{k}{B(v+w)}\sum_{p=0}^{1}\left[\beta_p^j(v+w,v'+w,k)\partial_{v'}^p h(v'+w)\right]
\end{aligned} \tag{2.3.22}$$

and

$$\begin{aligned}
T(v,v',w) = &\sum_{p=0}^{1}\left[\partial_v\beta_p^j(v+w,v'+w,k)\partial_{v'}^p h(v'+w)\right] \\
&- \frac{k}{B(v+w)}\sum_{p=0}^{1}\left[\alpha_p^j(v+w,v'+w,k)\partial_{v'}^p h(v'+w)\right].
\end{aligned} \tag{2.3.23}$$

Hence we can define $\mathcal{R}_{k,\epsilon}^{j_0+1}(v,w)$ by setting for $p \in \{0,1\}$

$$\alpha_p^{j_0+1}(v,v',k) = \partial_v\alpha_p^{j_0}(v+w,v'+w,k) - \frac{k}{B(v+w)}\beta_p^{j_0}(v+w,v'+w,k), \tag{2.3.24}$$

$$\beta_p^{j_0+1}(v,v',k) = \partial_v\beta_p^{j_0}(v+w,v'+w,k) - \frac{k}{B(v+w)}\alpha_p^{j_0}(v+w,v'+w,k), \tag{2.3.25}$$

We define

$$A_k^{j_0+1}(v) = 2\sum_{p=0}^{1}\left[\alpha_p^{j_0}(v,v,k)\partial_v^p h(v)\right]. \tag{2.3.26}$$

Hence (2.3.15) holds for $j = j_0$. For $0 \le m \le N - j_0 - 2$, it follows that

$$\left\|\partial_v^m A_k^{j_0+1}(v)\right\|_{L^2_w H^1_{k,v}} \lesssim \sum_{q=0}^{m}\sum_{p=0}^{1}\binom{m}{q}\left\|\partial_v^q \alpha_p^{j_0}(v,v,k)\partial_v^p h(v)\right\|_{H^1_k}. \tag{2.3.27}$$

Since (2.3.16) holds for the case $j = j_0$, we have

$$\left\|\partial_v^m A_k^{j_0+1}(v)\right\|_{L^2_w H^1_{k,v}} \lesssim_m \sum_{q=0}^{m}\sum_{p=0}^{1} k^{j_0+q-p}\left\|h\right\|_{H_k^{1+p}} \lesssim_m \left\|h\right\|_{H_k^{j_0+1+m}}. \tag{2.3.28}$$

The estimate of $\mathcal{R}_{k,\epsilon}^{j_0+1}(v,w)$ follows from (2.3.16) ,(2.3.17), (2.3.24), (2.3.25) and the Cauchy-Schwarz inequality. □

The free component $\mathcal{G}_k^{fr}$ of the Green function generates singularities in v when v is close to 0. As a contrast, the boundary components $\mathcal{G}_k^b$ generate singularities in w when w is close to the boundary $b(0)$ or $b(1)$. We have the following lemma establishing the boundary singularity structure near $b(0)$. The proof is similar to that of Lemma 2.3.1.

Lemma 2.3.2. *Let $N \ge 1$, $k \ge 1$, $0 < \epsilon < 1/2$. For any $h \in H_k^N(\mathbb{R})$, define*

$$\mathcal{S}_{k,\epsilon}^0(v,w) = \int_{\mathbb{R}} \mathcal{G}^{b0}(v+w, v'+w)\frac{h(v'+w)}{v'+i\epsilon}dv'. \tag{2.3.29}$$

Then for $1 \le j \le N-1$ there exist functions $B_k^j(v)$ and $\mathcal{S}_{k,\epsilon}^j(v,w)$ satisfying the following bounds (setting $B_k^0(v,w) := 0$)

$$\left\|\partial_v^m B_{k,\epsilon}^j(v)\right\|_{H^1_k} + \left\|\partial_v^m \mathcal{S}_{k,\epsilon}^j(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim_{m,j} \left\|h\right\|_{H_k^{j+1+m}} \tag{2.3.30}$$

such that for $0 \le j \le N-2$,

$$\partial_w \mathcal{S}_{k,\epsilon}^j(v,w) = B_k^{j+1}(v+w)\log(b(0)-w+i\epsilon) + \mathcal{S}_{k,\epsilon}^{j+1}(v,w). \tag{2.3.31}$$

Proof. The proof applies similar arguments to the proof of Lemma 2.3.1. We thus omit the details. □

For the component $\mathcal{G}^{b1}$, it does not generate singularities if the variable w is supported away from the boundary $b(1)$. We have the following lemma.

Lemma 2.3.3. *Let $N \geq 1$, $k \geq 1$, $0 < \epsilon < 1/2$. Assume for some $\delta_0 > 0$, $\varphi_{\delta_0}(w)$ is a smooth function such that $\varphi_{\delta_0}(w) = 0$ for $w \in (b(1-\delta_0), b(1+\delta_0))$. For any $h \in H_k^N(\mathbb{R})$, define*

$$\mathcal{T}_{k,\epsilon}(v,w) = \int_{\mathbb{R}} \mathcal{G}^{b1}(v+w, v'+w) \frac{h(v'+w)\varphi_{\delta_0}(w)}{v'+i\epsilon} dv'. \tag{2.3.32}$$

Then for integers $m, n \geq 0$ and $m+n \leq N-1$, we have

$$\|\partial_v^m \partial_w^n \mathcal{T}_{k,\epsilon}(v,w)\|_{L_w^2 H_{k,v}^1} \lesssim_{m,n} \|h\|_{H_k^{1+m+n}}. \tag{2.3.33}$$

Proof. Define

$$\mathcal{T}_{k,\epsilon}^0(v,w) = \int_{\mathbb{R}} \mathcal{G}^{b1}(v+w, v'+w) \frac{h(v'+w)}{v'+i\epsilon} dv'.$$

By symmetry, a similar conclusion to Lemma 2.3.2 holds for $\mathcal{T}_{k,\epsilon}^0(v,w)$. More specifically, for $1 \leq j \leq N-1$ there exist functions $B_k^j(v)$ and $\mathcal{T}_{k,\epsilon}^j(v,w)$, such that for $0 \leq j \leq N-2$,

$$\partial_w \mathcal{T}_{k,\epsilon}^j(v,w) = B_k^{j+1}(v+w)\log(b(1) - w + i\epsilon) + \mathcal{T}_{k,\epsilon}^{j+1}(v,w). \tag{2.3.34}$$

In addition, we have for $0 \leq j \leq N-1$ (setting $B_k^0(v,w) := 0$) and any $0 \leq m \leq N-j-1$,

$$\left\|\partial_v^m B_{k,\epsilon}^j(v)\right\|_{H_k^1} + \left\|\partial_v^m \mathcal{T}_{k,\epsilon}^j(v,w)\right\|_{L_w^2 H_{k,v}^1} \lesssim_{m,j} \|h\|_{H_k^{j+1+m}}. \tag{2.3.35}$$

However, $\log(b(1) - w + i\epsilon)$ is smooth with bounded derivatives when w is restricted on the support of $\varphi_{\delta_0}(w)$. Therefore, $\mathcal{T}_{k,\epsilon}(v,w) = \mathcal{T}_{k,\epsilon}^0(v,w)\varphi_{\delta_0}(w)$ has the estimate (2.3.33) □

2.3.2 Singularity structures of the spectrum density function

In order to study the singularity of $\Theta_{k,\epsilon}^{\iota}(v,w)$ more clearly, we split the domain of w into three parts. Fix $0 < \delta_0 < 1/10$. There exist three non-negative smooth functions $\Upsilon_1(w)$, $\Upsilon_2(w)$ and $\Upsilon_3(w)$ such that

$$\begin{aligned} &\Upsilon_1(w) = 1 \quad \text{on} \quad [b(-\delta_0), b(\delta_0)], \qquad \Upsilon_2(w) = 1 \quad \text{on} \quad [b(2\delta_0), b(1-2\delta_0)], \\ &\Upsilon_3(w) = 1 \quad \text{on} \quad [b(1-\delta_0), b(1+\delta_0)] \end{aligned} \tag{2.3.36}$$

and

$$\Upsilon_1(w) + \Upsilon_2(w) + \Upsilon_3(w) = 1 \quad \text{on} \quad \mathbb{R}. \tag{2.3.37}$$

For $v, w \in \mathbb{R}$ and $j \in \{1, 2, 3\}$, set

$$\Theta^{j,\iota}_{k,\epsilon}(v, w) := \Theta^{\iota}_{k,\epsilon}(v, w)\Upsilon_j(w). \tag{2.3.38}$$

In order to save notation, we define a function $\tau(m)$ for an integer $m \geq 0$ as

$$\tau(m) = \begin{cases} 1, & \text{if} \quad m = 0 \\ 2, & \text{if} \quad m \geq 1. \end{cases} \tag{2.3.39}$$

Before presenting the results on regularity structures of $\Theta^{\iota}_{k,\epsilon}(v, w)$, we first introduce the following definition of singularity structure when w is close to the boundary $b(0)$.

Definition 2.3.4. *Assume $k \geq 1$, $m \geq 0$, $n \geq 0$, $N \geq 1$ and $1 \leq \alpha \leq N$ are five integers. For any $\epsilon \in (-1, 1) \setminus \{0\}$, $g \in H^N_k(\mathbb{R})$, a function $\Gamma(v, w)$, $v, w \in \mathbb{R}$, is said to have the $\mathcal{F}^{b0}_{m,n,k,\epsilon}(g, \alpha, N)$ type singularity structure if there exists a constant C which depends only on α and N such that*

(a) $\Gamma(v, w)$ has the following estimate

$$\|\Gamma(v, w)\|_{L^2_w H^1_{k,v}(\mathbb{R}^2)} \leq C \|g\|_{H^\alpha_k(\mathbb{R})}. \tag{2.3.40}$$

(b) If $\alpha < N$, then for $1 \leq p \leq \tau(m)$ and $1 \leq q \leq \tau(n)$ such that there exist functions $A_p(v, w)$, $B_q(v, w)$, $R(v, w)$ and $S(v, w)$ such that

$$\begin{aligned} \partial_v \Gamma(v, w) &= \sum_{p=1}^{\tau(m)} A_p(v, w) \log^p(v + i\epsilon) + R(v, w), \\ \partial_w \Gamma(v, w) &= \sum_{q=1}^{\tau(n)} B_q(v, w) \log^q(b(0) - w + i\epsilon) + S(v, w). \end{aligned} \tag{2.3.41}$$

Furthermore, for $1 \leq p \leq \tau(m)$, $A_p(v, w)$ and $R(v, w)$ have the $\mathcal{F}^{b0}_{m+1,n,k,\epsilon}(g, \alpha + 1, N)$ type singularity structure. For $1 \leq q \leq \tau(n)$, $B_q(v, w)$ and $S(v, w)$ have the $\mathcal{F}^{b0}_{m,n+1,k,\epsilon}(g, \alpha + 1, N)$ type singularity structure.

We also define the interior singularity structure when w is away from the boundary points $b(0)$ and $b(1)$.

Definition 2.3.5. *Assume $k \geq 1$, $m \geq 0$, $N \geq 1$ and $1 \leq \alpha \leq N$ are four integers. For any $\epsilon \in (-1,1) \setminus \{0\}$, $g \in H_k^N(\mathbb{R})$, a function $\Gamma(v,w)$, $v, w \in \mathbb{R}$, is said to have the $\mathcal{F}^{in}_{m,k,\epsilon}(g,\alpha,N)$ type singularity structure if there exists a constant C which depends only on α and N such that*

(a) $\Gamma(v,w)$ *has the following estimate*

$$\|\Gamma(v,w)\|_{L^2_w H^1_{k,v}(\mathbb{R}^2)} \leq C \|g\|_{H^\alpha_k(\mathbb{R})}. \tag{2.3.42}$$

(b) If $\alpha < N$, for $1 \leq p \leq \tau(m)$ there exist functions $A_p(v,w)$ and $R(v,w)$ such that

$$\partial_v \Gamma(v,w) = \sum_{p=1}^{\tau(m)} A_p(v,w) \log^p(v + i\epsilon) + R(v,w). \tag{2.3.43}$$

In addition, $A_p(v,w)$ and $R(v,w)$ have the $\mathcal{F}^{in}_{m+1,k,\epsilon}(g,\alpha+1,N)$ type singularity structure.

(c) For any integer $0 \leq q \leq N-\alpha$, $\partial_w^q \Gamma(v,w)$ has the $\mathcal{F}^{in}_{m,k,\epsilon}(g,\alpha+q,N)$ type singularity structure.

Using Definition 2.3.4, we see that the right-hand side of equation (2.3.1) has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,1,N)$ type singularity structure assuming that $f_0^k \in H_k^N(\mathbb{R})$. We have the following proposition.

Proposition 2.3.6. *Assume $g(v) \in H_k^N(\mathbb{R})$ for some integer $N \geq 1$. Then*

$$\Gamma(v,w) := \int_{\mathbb{R}} \mathcal{G}_k(v+w, v'+w) \frac{g(v'+w)\Upsilon_1(w)}{v'+i\epsilon} dv' \tag{2.3.44}$$

has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,1,N)$ type singularity structure.

Proof. The proof follows from Lemma 2.3.1, Lemma 2.3.2 and Lemma 2.3.3. □

We also need the following technical lemmas regarding the singularity structures defined in Definition 2.3.4.

Lemma 2.3.7. *Assume $g \in H_k^{N+j}(\mathbb{R})$ for some $N \geq 1$ and $0 \leq j \leq N-1$. For any $\Gamma(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,N-j,N)$, we have $\Gamma(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,N,N+j)$.*

Proof. The proof follows from the Definition 2.3.4 and the fact that for $m \geq n$

$$H_k^m(\mathbb{R}) \subset H_k^n(\mathbb{R}). \tag{2.3.45}$$

□

Recall that for $k \geq 1$, $v \in \mathbb{R}$, the smooth function Φ_k^{b0} is defined as

$$\Phi_k^{b0}(v) = \Psi_k(v) \frac{\sinh k(1 - b^{-1}(v))}{\sinh k} \tag{2.3.46}$$

Lemma 2.3.8. *Let Φ_k^{b0} be defined as (2.3.46). For any integer $N \geq 1$ and any function $g \in H_k^M(\mathbb{R})$ with $M \geq N$, assume $\alpha(v,w) \in \mathcal{F}_{0,0,k,\epsilon}^{b0}(\Phi_k^{b0}, m, N)$ and $\beta(v,w) \in \mathcal{F}_{0,0,k,\epsilon}^{b0}(g, n, N)$. If there exist a integer $j \geq 1$ that $N - m \geq j$, $N - n \geq j$ and $M \geq m + n + j$, then*

$$\alpha(v,w)\beta(b(0) - w, w) \in \mathcal{F}_{0,0,k,\epsilon}^{b0}(g, m+n, m+n+j). \tag{2.3.47}$$

Proof. We prove the lemma with an induction argument for $j \geq 0$. For $j = 0$, the lemma is reduced to prove that

$$\|\alpha(v,w)\beta(b(0) - w, w)\|_{L_w^2 H_{k,v}^1} \lesssim \|g\|_{H_k^{m+n}}. \tag{2.3.48}$$

As a matter of fact,

$$\begin{aligned}
\|\alpha(v,w)\beta(b(0) - w, w)\|_{L_w^2 H_{k,v}^1} &\lesssim \|\alpha(v,w)\|_{L_w^\infty H_{k,v}^1} \|\beta(b(0) - w, w)\|_{L_w^2} \\
&\lesssim \|\alpha(v,w)\|_{L_w^\infty H_{k,v}^1} \|\beta(v,w)\|_{L_w^2 L_v^\infty} \\
&\lesssim \frac{1}{2} \|\alpha(v,w)\|_{H_{k,w}^1 H_{k,v}^1} \cdot \frac{1}{2} \|\beta(v,w)\|_{L_w^2 H_{k,v}^1} \\
&\lesssim k^m \|g\|_{H_k^n} \\
&\lesssim \|g\|_{H_k^{m+n}}.
\end{aligned} \tag{2.3.49}$$

Assume now the lemma holds for some integer $j_0 \geq 0$. We show that the case $j = j_0 + 1$ also holds. Taking derivative in v, there exist functions $\alpha_1, \alpha_2, \alpha_3$ such that

$$\partial_v \alpha(v,w) = \alpha_1(v,w)\log(v + i\epsilon) + \alpha_2(v,w)\log^2(v + i\epsilon) + \alpha_3(v,w), \tag{2.3.50}$$

and for $1 \leq j \leq 3$, $\alpha_j(v,w) \in \mathcal{F}_{k,\epsilon}^{b0}(\Phi_k^{b0}, m+1, N)$. By assumption $j = j_0 + 1$, we have $N - (m+1) \geq j_0$. Hence the induction assumption implies that $\alpha_j(v,w)\beta(b(0) - w, w) \in \mathcal{F}_{k,\epsilon}^{b0}(g, m+n+1, m+n+1+j_0)$.

We next take one derivative in w. The analysis is the same as above. The only difference is that the w derivative can be taken on $\beta(b(0)-w,w)$, which can generate more terms with $\log(b(0)-w+i\epsilon)$. We omit the details here. In conclusion, we showed that for $j=j_0+1$, $\alpha(v,w)\beta(b(0)-w,w)$ has the $\mathcal{F}^{b0}_{k,\epsilon}(g,m+n,m+n+j_0+1)$ type singularity structure. The lemma is proved. □

We next show that different components of the Green's function $\mathcal{G}_k(v+w,v'+w)$ keep the singularity structures.

Lemma 2.3.9. *Assume $m\geq 0$, $n\geq 0$, $k\geq 1$, $N\geq 0$ and $\lambda\geq 0$ are five integers. For any $g\in H_k^{\lambda+N}(\mathbb{R})$, $\epsilon\in(-1,1)\setminus\{0\}$ and any $\Gamma(v,w)\in\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$. Let $a_k(v,v',w)$ be a smooth function such that for any integers $r,s,t\geq 0$,*

$$|\partial_v^r\partial_{v'}^s\partial_w^t a_k(v,v',w)|\lesssim k^{r+s+t}. \tag{2.3.51}$$

For $1\leq p\leq\tau(n)$, $v,w\in\mathbb{R}$, define

$$\Lambda^p(v,w):=\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}a_k(v,v',w)\,\mathrm{sign}(v'+w-b(0))\log^p(v'+i\epsilon)\Gamma(v',w)dv', \tag{2.3.52}$$

and

$$\Omega^p(v,w):=\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}a_k(v,v',w)\log^p(v'+i\epsilon)\Gamma(v',w)dv'. \tag{2.3.53}$$

Then $\Lambda^p(v,w)$ and $\Omega^p(v,w)$ have the $\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure.

Proof. We prove the lemma with an induction argument for $N\geq 0$. For the case $N=0$, we need to show that

$$\|\Lambda^p(v,w)\|_{L^2_wH^1_k}+\|\Omega^p(v,w)\|_{L^2_wH^1_k}\lesssim\|g\|_{H^\lambda_k}, \tag{2.3.54}$$

which follows from the Cauchy-Schwarz inequality.

Assume for some integer $N_0\geq 0$ the lemma holds for all $N\leq N_0$. We now consider the case $N=N_0+1$. By definition $\Lambda^p(v,w)$ and $\Omega^p(v,w)$ are smooth in v. We only need to consider their derivatives in w. For $\Lambda^p(v,w)$, we have

$$\begin{aligned}
&\partial_w \Lambda^p(v,w)\\
=\;& 2a_k(v,b(0)-w,w)\Gamma(b(0)-w,w)\log^p(b(0)-w+i\epsilon)\\
&-k\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\frac{a_k(v,v',w)}{B(v'+w)}\log^p(v'+i\epsilon)\Gamma(v',w)dv'\\
&+\int_{\mathbb{R}}\Big[e^{-k|b^{-1}(v'+w)|}\partial_w a_k(v,v',w)\,\mathrm{sign}(v'+w-b(0))\\
&\qquad\times\log^p(v'+i\epsilon)\Gamma(v',w)\Big]dv'\\
&+\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}a_k(v,v',w)\,\mathrm{sign}(v'+w-b(0))\\
&\qquad\times\log^p(v'+i\epsilon)\partial_w\Gamma(v',w)dv'\\
:=\;& T_1+T_2+T_3+T_4.
\end{aligned} \tag{2.3.55}$$

For T_1 we have

$$2a_k(v,b(0)-w,w)\Gamma(b(0)-w,w)\in\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N_0+1). \tag{2.3.56}$$

For T_2 and T_3 it follows from the induction assumption that

$$T_2,T_3\in\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda+1,\lambda+N_0+1). \tag{2.3.57}$$

For T_4, since we assumed that $\Gamma(v,w)\in\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N_0+1)$, there exist functions $B(v,w),R(v,w)\in\mathcal{F}^{b0}_{m,n+1,k,\epsilon}(g,\lambda+1,\lambda+N_0+1)$ such that

$$\partial_w\Gamma(v,w)=B(v,w)\log(b(0)-w+i\epsilon)+R(v,w). \tag{2.3.58}$$

Set

$$\begin{aligned}
\widetilde{B}(v,w)=&\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}a_k(v,v',w)\,\mathrm{sign}(v'+w-b(0))\\
&\times\log^p(v'+i\epsilon)B(v',w)dv',
\end{aligned} \tag{2.3.59}$$

and

$$\begin{aligned}
\widetilde{R}(v,w)=&\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}a_k(v,v',w)\,\mathrm{sign}(v'+w-b(0))\\
&\times\log^p(v'+i\epsilon)R(v',w)dv'.
\end{aligned} \tag{2.3.60}$$

Using the induction assumption for $N = N_0$, we have

$$\widetilde{B}(v,w),\ \widetilde{R}(v,w) \in \mathcal{F}^{b0}_{m,n+1,k,\epsilon}(g, \lambda+1, \lambda+N_0+1), \tag{2.3.61}$$

and

$$T_4 = \widetilde{B}(v,w)\log(b(0) - w + i\epsilon) + \widetilde{R}(v,w). \tag{2.3.62}$$

Therefore, we have $\Lambda^p(v,w) \in \mathcal{F}^{b0}_{m,n,k,\epsilon}(g, \lambda, \lambda+N_0+1)$. The analysis of $\Omega^p(v,w)$ is similar. □

Using similar ideas, we can also prove the following two lemmas.

Lemma 2.3.10. *Assume $m \geq 0$, $n \geq 0$, $k \geq 1$, $N \geq 0$ and $\lambda \geq 0$ are five integers. For any $g \in H^{\lambda+N}_k(\mathbb{R})$, $\epsilon \in (-1,1)\setminus\{0\}$ and any $\Gamma(v,w) \in \mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$. Let $a_k(v,v',w)$ be a smooth function such that for any integers $r, s, t \geq 0$,*

$$|\partial^r_v \partial^s_{v'} \partial^t_w a_k(v,v',w)| \lesssim k^{r+s+t}. \tag{2.3.63}$$

For $1 \leq p \leq \tau(n)$, $v, w \in \mathbb{R}$, define

$$\begin{aligned}\Lambda^p(v,w) := \int_{\mathbb{R}} & e^{-k|1-b^{-1}(v'+w)|} a_k(v,v',w)\,\mathrm{sign}(v'+w-b(1)) \\ & \times \log^p(v'+i\epsilon)\Gamma(v',w)\Upsilon_1(w)dv',\end{aligned} \tag{2.3.64}$$

and

$$\Omega^p(v,w) := \int_{\mathbb{R}} e^{-k|1-b^{-1}(v'+w)|} a_k(v,v',w)\log^p(v'+i\epsilon)\Gamma(v',w)\Upsilon_1(w)dv'. \tag{2.3.65}$$

Then $\Lambda^p(v,w)$ and $\Omega^p(v,w)$ have the $\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure.

Proof. Note that on the support of $\Upsilon_1(w)$, w is supported away from the boundary point $b(1)$. Hence the component

$$e^{-k|1-b^{-1}(v'+w)|}\,\mathrm{sign}(v'+w-b(1))$$

actually does not generate any singularity. We can apply an induction argument similar to that in the proof of Lemma 2.3.9 to prove this lemma.

□

Lemma 2.3.11. *Assume $m \geq 0$, $n \geq 0$, $k \geq 1$, $N \geq 0$ and $\lambda \geq 0$ are five integers. For any $g \in H_k^{\lambda+N}(\mathbb{R})$, $\epsilon \in (-1,1) \setminus \{0\}$ and any $\Gamma(v,w) \in \mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$. Let $a_k(v,v',w)$ be a smooth function such that for any integers $r,s,t \geq 0$,*

$$|\partial_v^r \partial_{v'}^s \partial_w^t a_k(v,v',w)| \lesssim k^{r+s+t}. \tag{2.3.66}$$

For $1 \leq p \leq \tau(n)$, $v,w \in \mathbb{R}$, define

$$\begin{aligned} \Lambda^p(v,w) := \int_{\mathbb{R}} & e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} a_k(v,v',w)\,\mathrm{sign}(v-v') \\ & \times \log^p(v'+i\epsilon)\Gamma(v',w)dv', \end{aligned} \tag{2.3.67}$$

and

$$\Omega^p(v,w) := \int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} a_k(v,v',w) \log^p(v'+i\epsilon)\Gamma(v',w)dv'. \tag{2.3.68}$$

Then $\Lambda^p(v,w)$ and $\Omega^p(v,w)$ have the $\mathcal{F}^{b0}_{m,n,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure.

Our main result in this section is the characterization of the singularity structure of the spectrum density function $\Theta^1_{k,\epsilon}(v,w)$ with w close to the boundary point $b(0)$, assuming that the right-hand side of the equation has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure.

Theorem 2.3.12. *For any integer $N \geq 0$, $k \geq 1$, $\epsilon \in (-1,1) \setminus \{0\}$. For any integer $\lambda \geq 1$ and function $g \in H_k^{\lambda+N}(\mathbb{R})$, assume $\Gamma(v,w)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure. Let $\Theta^1_{k,\epsilon}(v,w) := \Theta_{k,\epsilon}(v,w)\Upsilon_1(w)$ solve the following equation*

$$\Theta^1_{k,\epsilon}(v,w) + T_{k,\epsilon}\Theta^1_{k,\epsilon}(v,w) = \Gamma(v,w) \tag{2.3.69}$$

where $T_{k,\epsilon}$ is the operator defined in (2.2.1). Then $\Theta^1_{k,\epsilon}(v,w)$ also has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,\lambda+N)$ type singularity structure.

Proof. If $N=0$, the theorem follows directly from the limiting absorption principle. We apply induction arguments to show that Theorem 2.3.12 holds for any $N \geq 0$. Assume for some $N_0 \geq 0$ the theorem holds for all $0 \leq N \leq N_0$. We now consider the case $N = N_0+1$. We first analyze remainder terms. In order to streamline the notation, we set

$$R^{0,0}(v,w) := \Theta^1_{k,\epsilon}(v,w). \tag{2.3.70}$$

We have the following claim.

Claim: For $0 \le n \le N_0+1$ and $0 \le m \le N_0+1-n$, there exist functions $R^{m,n}(v,w)$ satisfying the following properties.

(a) For $0 \le n \le N_0+1$ and $0 \le m \le N_0+1-n$,

$$\|R^{m,n}(v,w)\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{m+n+\lambda}} \tag{Claim 1}$$

(b) For $0 \le n \le N_0$, there exist a function $A^n(v,w,k) \in \mathcal{F}^{b0}_{1,n,k,\epsilon}(g, \lambda+n+1, \lambda+N_0+1)$ and smooth functions $\alpha^n_j(v,w,k)$, $0 \le j \le n$, such that

$$\begin{aligned}\partial_v R^{0,n}(v,w) = & \sum_{j=0}^{n} \alpha^n_j(v,w,k) R^{0,j}(v,w)\log(v+i\epsilon) \\ & + A^n(v,w,k)\log(v+i\epsilon) + R^{1,n}(v,w),\end{aligned} \tag{Claim 2}$$

and for any integers $p, q \ge 0$,

$$|\partial_v^p \partial_w^q \alpha^n_j(v,w,k)| \lesssim k^{n-j+p+q}. \tag{Claim 3}$$

(c) For $0 \le n \le N_0$, $1 \le m \le N_0-n$, there exist functions $A^{m,n}_j(v,w) \in \mathcal{F}^{b0}_{m,n,k,\epsilon}(g, \lambda+m+n+1, \lambda+N_0+1)$, $j \in \{1,2\}$ and smooth functions $\tau^{m,n}_j(v,w)$, $\eta^{m,n}_j(v,w)$ such that

$$\begin{aligned}&\partial_v R^{m,n}(v,w) \\ =& \sum_{j=0}^{n}\left[\tau^{m,n}_j(v,,w)R^{0,j}(v,w) + \eta^{m,n}_j(v,w)\partial_v R^{0,j}(v,w)\right]\log(v+i\epsilon) \\ & + \sum_{j\in\{1,2\}} A^{m,n}_j(v,w)\log^j(v+i\epsilon) + R^{m+1,n}(v,w).\end{aligned} \tag{Claim 4}$$

Furthermore, for any integer $p, q \ge 0$,

$$|\partial_v^p\partial_w^q g^{m,n}_j(v,w)| \lesssim k^{m+n-j+p+q}, \quad |\partial_v^p\partial_w^q h^{m,n}_j(v,w)| \lesssim k^{m+n-j+p+q-1} \tag{Claim 5}$$

(d) For $0 \le n \le N_0$, there exist a function $B^n(v,w,k) \in \mathcal{F}^{b0}_{k,\epsilon}(g, \lambda+n+1, \lambda+N_0+1)$

and functions $\beta_j^n(v,w,k)$ and $\gamma_j^n(v,w,k)$, $0 \le j \le n$ such that

$$\begin{aligned} &\partial_w R^{0,n}(v,w) \\ = \ & \sum_{j=0}^{n} \beta_j^n(v,w,k) R^{0,j}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\ &+ \sum_{j=0}^{n-1} \gamma_j^n(v,w,k)\partial_v R^{0,j}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\ &+ B^n(v,w,k)\log(b(0)-w+i\epsilon) + R^{0,n+1}(v,w). \end{aligned} \tag{Claim 6}$$

Further more, $\beta_j^n(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n-j+1, N_0+1)$ type singularity structure; $\gamma_j^n(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n-j, N_0+1)$ type singularity structure. The smooth function Φ_k^{b0} is defined in (2.3.46)

(e) For $0 \le n \le N_0$,

$$R^{0,n}(v,w) \in \mathcal{F}^{b0}_{0,n,k,\epsilon}(g, \lambda+n, \lambda+N_0). \tag{Claim 7}$$

(f) For $0 \le n \le N_0 - 1$,

$$R^{1,n}(v,w) \in \mathcal{F}^{b0}_{1,n,k,\epsilon}(g, \lambda+n+1, \lambda+N_0). \tag{Claim 8}$$

Step 1: proof of the case $N = N_0 + 1$ assuming (Claim 1) to (Claim 8) are true. For the case $N = N_0 + 1$, we would like to show that for any $0 \le n \le N_0 + 1$, $R^{0,n}(v,w)$ has the $\mathcal{F}^{b0}_{0,n,k,\epsilon}(g, n+\lambda, \lambda+N_0+1)$ type singularity structure. We apply another induction argument to show this.

For $n = N_0+1$, it follows from (Claim 1) that $R^{0,N_0+1}(v,w)$ has the $\mathcal{F}^{b0}_{0,N_0+1,k,\epsilon}(g, \lambda+N_0+1, \lambda+N_0+1)$ type singularity structure. Assume for some $j_0 \ge 0$, $R^{0,N_0+1-j_0}(v,w)$ has the $\mathcal{F}^{b0}_{0,N_0+1-j_0,k,\epsilon}(g, \lambda+N_0-j_0+1, \lambda+N_0+1)$ type singularity structure. Now let us consider the singularity structure of $R^{0,N_0-j_0}(v,w)$.

It follows from (Claim 6) and (Claim 2) that

$$\begin{aligned}
&\partial_w R^{0,N_0-j_0}(v,w) \\
= &\sum_{j=0}^{N_0-j_0} \beta_j^{N_0-j_0}(v,w,k) R^{0,j}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\
&+ \sum_{j=0}^{N_0-j_0-1} \gamma_j^{N_0-j_0}(v,w,k) R^{0,j}(b(0)-w,w)\log^2(b(0)-w+i\epsilon) \\
&+ \sum_{j=0}^{N_0-j_0-1} \zeta_j^{N_0-j_0}(v,w,k) R^{1,j}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\
&+ B^{N_0-j_0}(v,w,k)\log(b(0)-w+i\epsilon) + R^{0,N_0-j_0+1}(v,w).
\end{aligned} \tag{2.3.71}$$

In addition, $\beta_j^{N_0-j_0}(v,w,k)$ has the $\mathcal{F}_{k,\epsilon}^{b0}(\Phi_k^{b0}, N_0-j_0-j+1, N_0+1)$ type singularity structure; $\gamma_j^{N_0-j_0}(v,w,k)$ and $\zeta_j^{N_0-j_0}(v,w,k)$ have the $\mathcal{F}_{k,\epsilon}^{b0}(\Phi_k^{b0}, N_0-j_0-j, N_0+1)$ type singularity structure. By (Claim 7), $R^{0,j}(v,w) \in \mathcal{F}_{k,\epsilon}^{b0}(g, j+\lambda, N_0+\lambda)$ for $j \le N_0$.

Note that for $j_0 \ge 0$ and $0 \le j \le N_0 - j_0$,

$$(N_0+1) - (N_0-j_0-j+1) \ge j_0 - 1, \quad (N_0+\lambda) - (j+\lambda) \ge j_0. \tag{2.3.72}$$

It follows from Lemma 2.3.8 that for $0 \le j \le N_0 - j_0$,

$$\beta_j^{N_0-j_0}(v,w,k) R^{0,j}(b(0)-w,w) \in \mathcal{F}_{k,\epsilon}^{b0}(g, N_0-j_0+\lambda+1, N_0+1+\lambda).$$

In view of (Claim 8), the same argument holds for $\zeta_j^{N_0-j_0}(v,w,k)R^{1,j}(b(0)-w,w)$. For $\gamma_j^{N_0-j_0}(v,w,k)R^{0,j}(b(0)-w,w)$, we can show that

$$\gamma_j^{N_0-j_0}(v,w,k) R^{0,j}(b(0)-w,w) \in \mathcal{F}_{k,\epsilon}^{b0}(g, N_0-j_0+\lambda, N_0+\lambda)$$

which is the subset of $\mathcal{F}_{k,\epsilon}^{b0}(g, N_0-j_0+\lambda+1, N_0+\lambda+1)$ by Lemma 2.3.7. By the induction assumption, $R^{0,N_0-j_0+1}(v,w)$ has a $\mathcal{F}_{k,\epsilon}^{b0}(g, N_0-j_0+\lambda+1, N_0+\lambda+1)$ type singularity structure. Hence we have finished the analysis of $\partial_w R^{0,N_0-j_0}$.

Next, let us consider $\partial_v R^{0,N_0-j_0}$. From (Claim 2), we have

$$\begin{aligned}
\partial_v R^{0,N_0-j_0}(v,w) = &\sum_{j=0}^{N_0-j_0} \alpha_j^{N_0-j_0}(v,w,k) R^{0,j}(v,w)\log(v+i\epsilon) \\
&+ A^{N_0-j_0}(v,w,k)\log(v+i\epsilon) + R^{1,N_0-j_0}(v,w),
\end{aligned} \tag{2.3.73}$$

and for any integers $p, q \geq 0$,

$$|\partial_v^p \partial_w^q \alpha_j^{N_0-j_0}(v,w,k)| \lesssim k^{N_0-j_0-j+p+q}. \tag{2.3.74}$$

We have for $0 \leq j \leq N_0 - j_0$,

$$\alpha_j^{N_0-j_0}(v,w,k)R^{0,j}(v,w) \in \mathcal{F}^{b0}_{k,\epsilon}(g, N_0-j_0+\lambda, N_0+\lambda) \subset \mathcal{F}^{b0}_{k,\epsilon}(g, N_0-j_0+\lambda+1, N_0+\lambda+1).$$

Hence we only need to analyze $R^{1,N_0-j_0}(v,w)$. Note that the singularity structure of $\partial_w R^{0,N_0-j_0}$ implies the singularity structure of $\partial_w R^{1,N_0-j_0}(v,w)$, since from (2.3.73) we can write

$$\begin{aligned} &\partial_w R^{1,N_0-j_0}(v,w) \\ =\ &\partial_w \Bigg[\partial_v R^{0,N_0-j_0}(v,w) - \sum_{j=0}^{N_0-j_0} \alpha_j^{N_0-j_0}(v,w,k) R^{0,j}(v,w)\log(v+i\epsilon) \\ &- A^{N_0-j_0}(v,w,k)\log(v+i\epsilon)\Bigg]. \end{aligned} \tag{2.3.75}$$

Therefore, we only need to analyze $\partial_v R^{1,N_0-j_0}(v,w)$. It follows from (Claim 4) that

$$\begin{aligned} &\partial_v R^{1,N_0-j_0}(v,w) \\ =\ &\sum_{j=0}^{N_0-j_0} \left[g_j^{1,N_0-j_0}(v,w)R^{0,j}(v,w) + h_j^{1,N_0-j_0}(v,w)\partial_v R^{0,j}(v,w)\right]\log(v+i\epsilon) \\ &+ \sum_{j\in\{1,2\}} A_j^{1,N_0-j_0}(v,w)\log^j(v+i\epsilon) + R^{2,N_0-j_0}(v,w), \end{aligned} \tag{2.3.76}$$

and for any integer $p, q \geq 0$,

$$|\partial_v^p \partial_w^q g_j^{1,N_0-j_0}(v,w)| \lesssim k^{1+N_0-j_0-j+p+q} \tag{2.3.77}$$

and

$$|\partial_v^p \partial_w^q h_j^{1,N_0-j_0}(v,w)| \lesssim k^{N_0-j_0-j+p+q}. \tag{2.3.78}$$

It follows from Lemma 2.3.7 that both $g_j^{1,N_0-j_0}(v,w)R^{0,j}(v,w)$ and $h_j^{1,N_0-j_0}(v,w)R^{1,j}(v,w)$ are in $\mathcal{F}^{b0}_{1,N_0-j_0,k,\epsilon}(g, N_0 - j_0 + \lambda + 2, N_0 + \lambda + 1)$. Hence we only need to analyze $\partial_v R^{2,N_0-j_0}(v,w)$. This procedure only need to be repeated finite times, since we have $R^{j_0+1,N_0-j_0}(v,w) \in \mathcal{F}^{b0}_{j_0+1,N_0-j_0,k,\epsilon}(g, N_0+\lambda+1, N_0+\lambda+1)$ by (Claim 1).

Therefore, we have showed that $R^{0,N_0-j_0}(v,w)$ has the $\mathcal{F}^{b0}_{k,\epsilon}(g,N_0-j_0+\lambda,N_0+\lambda+1)$ type singularity structure, which is the case $j=j_0+1$ in the induction argument. Hence $\Theta^1_{k,\epsilon}(v,w)=R^{0,0}(v,w)$ has the $\mathcal{F}^{b0}_{k,\epsilon}(g,\lambda,N_0+\lambda+1)$ type singularity structure which is the case $N=N_0+1$ for the theorem. We next show that (Claim 1) to (Claim 6) are correct for all $0\le n\le N_0$.

For the rest of the proof, we show that for $0\le n\le N_0+1$ and $0\le m\le N_0+1-m$, there exist functions $R^{m,n}(v,w)$ such that (Claim 1) to (Claim 8) hold.

Step 2: iterative construction of $R^{0,n}(v,w)$ for $0\le n\le N_0$. Set

$$R^{0,0}(v,w)=\Theta^1_{k,\epsilon}(v,w).$$

We claim that for $0\le n\le N_0$ and $0\le p\le n-1$, there exist smooth functions $a^{n,p}_k(v,v',w)$, $b^{n,p}_k(v,v',w)$, $c^{n,p}_k(v,v',w)$ and $d^{n,p}_k(v,v',w)$ such that the solution $R^{0,n}(v,w)$ to the following system satisfying (Claim 6) and (Claim 7).

$$R^{0,n}(v,w)+T_{k,\epsilon}(R^{0,n}(v,w))=\mathcal{R}^{0,n}_1(v,w)+\mathcal{R}^{0,n}_2(v,w)+\mathcal{R}^{0,n}_3(v,w)+\mathcal{R}^{0,n}_4(v,w), \quad \text{(IA 1)}$$

and the right-hand side of (IA 1) has the following expression.

(a) $\mathcal{R}^{0,n}_1(v,w)$ is from the singularity structure of $\Gamma(v,w)$ and

$$\mathcal{R}^{0,n}_1(v,w)\in\mathcal{F}^{b0}_{0,n,k,\epsilon}(g,n+\lambda,N_0+\lambda+1). \quad \text{(IA 2)}$$

(b) $\mathcal{R}^{0,n}_2(v,w)$:

$$\mathcal{R}^{0,n}_2(v,w)=-\sum_{j=1}^{n}\binom{n}{j}\int_{\mathbb{R}}\partial^j_w\big[(\mathcal{G}^{fr}_k+\mathcal{G}^{b1}_k)\partial_{v'}B(v'+w)\big]\frac{R^{0,n-j}(v',w)}{v'+i\epsilon}dv'. \quad \text{(IA 3)}$$

(c) $\mathcal{R}^{0,n}_3(v,w)$:

$$\begin{aligned}\mathcal{R}^{0,n}_3(v,w)=&\int_{\mathbb{R}}e^{-k|b^{-1}(v'+w)|}\,\mathrm{sign}(v'+w-b(0))\log(v'+i\epsilon)\times\\ &\Big[\sum_{p=0}^{n-1}a^{n,p}_k(v,v',w)R^{0,p}(v',w)+\sum_{p=0}^{n-1}b^{n,p}_k(v,v',w)\partial_{v'}R^{0,p}(v',w)\Big]dv'\end{aligned} \quad \text{(IA 4)}$$

where the functions $a^{n,p}_k(v,v',w)$ and $b^{n,p}_k(v,v',w)$ in (IA 4) are smooth functions depending only on the cutoff function Ψ_k and background flow $B(v+w)$ such that for any integers $\alpha,\beta\ge 0$,

$$|\partial^\alpha_v\partial^\beta_w a^{n,p}_k(v,v',w)|\lesssim k^{n-p+\alpha+\beta},\quad |\partial^\alpha_v\partial^\beta_w b^{n,p}_k(v,v',w)|\lesssim k^{n-p-j-1+\alpha+\beta}. \quad \text{(IA 5)}$$

(d) $\mathcal{R}_4^{0,n}(v,w)$:

$$\begin{aligned}\mathcal{R}_4^{0,n}(v,w) = \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\log(v'+i\epsilon)\times\Bigg[\sum_{p=0}^{n-1} c_k^{n,p}(v,v',w)R^{0,p}(v',w)\\ +\sum_{p=0}^{n-1} d_k^{n,p}(v,v',w)\partial_{v'}R^{0,p}(v',w)\Bigg]dv'\end{aligned} \tag{IA 6}$$

where the functions $c_k^p(v,v',w)$ and $d_k^{j,p}(v,v',w)$ in (IA 6) are smooth functions such that for any integers $\alpha,\beta\geq 0$,

$$|\partial_v^\alpha\partial_w^\beta c_k^p(v,v',w)|\lesssim k^{n-p+\alpha+\beta} \quad \text{and} \quad |\partial_v^\alpha\partial_w^\beta d_k^p(v)|\lesssim k^{n-p-1+\alpha+\beta}. \tag{IA 7}$$

For the case $n=0$, since $R^{0,0}(v,w)$ solves

$$R^{0,0}(v,w)+T_{k,\epsilon}R^{0,0}(v,w)=\Gamma(v,w)\in\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,N_0+\lambda+1). \tag{2.3.79}$$

It suffices to take $a_k^{0,p}(v,v',w)$, $b_k^{0,p}(v,v',w)$, $c_k^{0,p}(v,v',w)$ and $d_k^{0,p}(v,v',w)$ to be all zero. We next study $\partial_w R^{0,0}(v,w)$. Taking a derivative in w to the equation (2.3.79), we get

$$\begin{aligned}&(I+T_{k,\epsilon})\partial_w R^{0,0}(v,w)\\ =&\ \partial_w\Gamma(v,w)-\int_{\mathbb{R}}\partial_w\left[\mathcal{G}_k(v+w,v'+w)\partial_{v'}B(v'+w)\right]\frac{R^{0,0}(v',w)}{v'+i\epsilon}dv'\\ :=&\ T_1+T_2.\end{aligned} \tag{2.3.80}$$

For T_2, we have

$$\begin{aligned}T_2=&-\int_{\mathbb{R}}\partial_w\left\{(\mathcal{G}_k^{fr}+\mathcal{G}_k^{b1})\partial_{v'}B(v'+w)\right\}\frac{R^{0,0}(v',w)}{v'+i\epsilon}dv'\\ &-\int_{\mathbb{R}}\partial_w\left\{\mathcal{G}_k^{b0}(v+w,v'+w)\partial_{v'}B(v'+w)\right\}\frac{R^{0,0}(v',w)}{v'+i\epsilon}dv'\\ :=&T_{21}+T_{22}.\end{aligned} \tag{2.3.81}$$

For T_{22}, we deduce using integration by parts that

$$\begin{aligned} T_{22} =&\Phi_k^{b0}(v+w)\partial_v B(b(0))R^{0,0}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\ &+ \Phi_k^{b0}(v+w)\int_{\mathbb{R}}\partial_{v'}\left\{e^{-k|b^{-1}(v'+w)|}\partial_{v'}B(v'+w)R^{0,0}(v',w)\right\} \\ &\qquad\qquad \times \operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)dv' \\ &+ \partial_w\big(\Phi_k^{b0}(v+w)\big)\int_{\mathbb{R}}\partial_{v'}\left\{\frac{e^{-k|b^{-1}(v'+w)|}}{k}R^{0,0}(v',w)\partial_{v'}B(v'+w)\right\} \\ &\qquad\qquad \times \log(v'+i\epsilon)dv' \\ :=&T_{221}+T_{222}+T_{223}. \end{aligned} \tag{2.3.82}$$

By the assumption that $\Gamma(v,w)\in\mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,N_0{+}\lambda{+}1)$, there exist functions $H^1(v,w)$, $S^1(v,w)$ in $\mathcal{F}^{b0}_{0,1,k,\epsilon}(g,\lambda+1,N_0+\lambda+1)$ such that

$$T_1=\partial_w\Gamma(v,w)=H^1(v,w)\log(b(0)-w+i\epsilon)+S^1(v,w). \tag{2.3.83}$$

Set

$$\mathcal{B}^1(v,w)=\Phi_k^{b0}(v+w)\partial_v B(b(0))R^{0,0}(b(0)-w,w)+H^1(v,w), \tag{2.3.84}$$

and

$$\mathcal{S}^1(v,w)=T_{21}+T_{222}+T_{223}+S^1(v,w). \tag{2.3.85}$$

It follows from (2.3.80) that $\partial_w R^{0,0}(v,w)$ solves

$$(I+T_{k,\epsilon})\partial_w R^{0,0}(v,w)=\mathcal{B}^1(v,w)\log(b(0)-w+i\epsilon)+\mathcal{S}^1(v,w). \tag{2.3.86}$$

Take $R^{0,1}(v,w)$ as

$$R^{0,1}(v,w)=(I+T_{k,\epsilon})^{-1}\mathcal{S}^1(v,w). \tag{2.3.87}$$

We have

$$\begin{aligned} &\partial_w R^{0,0}(v,w) \\ =&(I+T_{k,\epsilon})^{-1}\Phi_k^{b0}(v+w)\partial_v B(b(0))R^{0,0}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\ &+(I+T_{k,\epsilon})^{-1}H^1(v,w)\log(b(0)-w+i\epsilon) \\ &+R^{0,1}(v,w). \end{aligned} \tag{2.3.88}$$

Take

$$\beta_0^0(v,w,k)=(I+T_{k,\epsilon})^{-1}\Phi_k^{b0}(v+w)\partial_v B(b(0)) \tag{2.3.89}$$

and

$$B^0(v,w,k) = (I + T_{k,\epsilon})^{-1} H^1(v,w). \tag{2.3.90}$$

Since $\Phi_k^{b0}(v+w)$ is smooth and we have assumed that the theorem holds for $N \le N_0$, we have $\beta_0^0(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,1,N_0+1)$. Since $H^1 \in \mathcal{F}^{b0}_{0,1,k,\epsilon}(g,\lambda+1,N_0+\lambda+1)$, we also have $B^0(v,w,k) \in \mathcal{F}^{b0}_{0,1,k,\epsilon}(g,\lambda+1,N_0+\lambda+1)$. Therefore, (Claim 6) holds for $n=0$.

For $R^{0,1}(v,w)$ defined in (2.3.87), we take

$$a_k^{1,0}(v,v',w) = \Phi_k^{b0}(v+w)\partial^2_{v'}B(v'+w) - \partial_w \Phi_k^{b0}(v+w)\frac{\partial_{v'}B(v'+w)}{B(v'+w)}, \tag{2.3.91}$$

$$b_k^{1,0}(v,v',w) = \Phi_k^{b0}(v+w)\partial_{v'}B(v'+w), \tag{2.3.92}$$

$$c_k^{1,0}(v,v',w) = -k\Phi_k^{b0}(v+w)\frac{\partial_{v'}B(v'+w)}{B(v'+w)} + \frac{\partial_w\Phi_k^{b0}(v+w)}{k}\partial^2_{v'}B(v'+w), \tag{2.3.93}$$

$$d_k^{1,0}(v,v',w) = \frac{\partial_w\Phi_k^{b0}(v+w)}{k}\partial_{v'}B(v'+w). \tag{2.3.94}$$

We see that (IA 1) to (IA 7) holds for $n=1$. From the induction assumption, we have $R^{0,0}(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,\lambda+N_0)$. It follows from Lemma 2.3.9, Lemma 2.3.10 and Lemma 2.3.11 that $\mathcal{S}^1(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda+1,\lambda+N_0)$. Then we have $R^{0,1}(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda+1,\lambda+N_0)$ which is (Claim 7) for the case $n=1$.

Now assume for some integer $n_0 \ge 1$ we have constructed $R^{0,n}(v,w)$ for all $0 \le n \le n_0$ that satisfy (IA 1) to (IA 7), and (Claim 6) holds for all $0 \le n \le n_0 - 1$, (Claim 7) holds for all $0 \le n \le n_0$. We take a derivative in w to the equation (IA 1) for $n = n_0$, and get

$$\begin{aligned} & (I+T_{k,\epsilon})\partial_w R^{0,n_0}(v,w) \\ = \; & -\int_{\mathbb{R}} \partial_w \Big[\mathcal{G}_k(v+w,v'+w)\partial_{v'}B(v'+w)\Big]\frac{R^{0,n_0}(v',w)}{v'+i\epsilon}dv' \\ & + \partial_w\Big[\mathcal{R}_1^{0,n_0}(v,w) + \mathcal{R}_2^{0,n_0}(v,w) + \mathcal{R}_3^{0,n_0}(v,w) + \mathcal{R}_4^{0,n_0}(v,w)\Big] \\ := \; & T_3 + T_4 + T_5 + T_6 + T_7. \end{aligned} \tag{2.3.95}$$

For T_3, we have

$$\begin{aligned} T_3 =& -\int_{\mathbb{R}} \partial_w\Big\{(\mathcal{G}_k^{fr}+\mathcal{G}_k^{b1})\partial_{v'}B(v'+w)\Big\}\frac{R^{0,n_0}(v',w)}{v'+i\epsilon}dv' \\ & -\int_{\mathbb{R}} \partial_w\Big\{\mathcal{G}_k^{b0}(v+w,v'+w)\partial_{v'}B(v'+w)\Big\}\frac{R^{0,n_0}(v',w)}{v'+i\epsilon}dv' \\ :=& T_{31}+T_{32}. \end{aligned} \tag{2.3.96}$$

By induction assumption of (Claim 7), $R^{0,n_0}\in\mathcal{F}^{b_0}_{0,n_0,k,\epsilon}(g,\lambda+n_0,\lambda+N_0)$, we have by Lemma 2.3.10 and 2.3.11

$$T_{31}\in\mathcal{F}^{b_0}_{0,n_0,k,\epsilon}(g,\lambda+n_0+1,\lambda+N_0) \tag{2.3.97}$$

For T_{32}, using integration by parts we get

$$\begin{aligned} T_{32} =& \Phi_k^{b0}(v+w)\partial_v B(b(0))R^{0,n_0}(b(0)-w,w)\log(b(0)-w+i\epsilon) \\ & +\Phi_k^{b0}(v+w)\int_{\mathbb{R}}\partial_{v'}\Big\{e^{-k|b^{-1}(v'+w)|}\partial_{v'}B(v'+w)R^{0,n_0}(v',w)\Big\} \\ & \qquad\qquad \times \operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)dv' \\ & +\frac{\partial_w\big(\Phi_k^{b0}(v+w)\big)}{k}\int_{\mathbb{R}}\partial_{v'}\Big[e^{-k|b^{-1}(v'+w)|}R^{0,n_0}(v',w)\partial_{v'}B(v'+w)\Big] \\ & \qquad\qquad \times\log(v'+i\epsilon)dv'. \end{aligned} \tag{2.3.98}$$

The analysis of $T_4=\partial_w\mathcal{R}_1^{0,n_0}(v,w)$ is straightforward since the induction assumption assumes that $\mathcal{R}_1^{0,n_0}(v,w)$ is from the singularity structure of $\Gamma(v,w)$ and has the $\mathcal{F}^{b0}_{k,\epsilon}(g,\lambda+n_0,\lambda+N_0+1)$ type singularity structure. There exist functions $H^{n_0}(v,w,k)$, $S^{n_0}(v,w,k)$ in $\mathcal{F}^{b0}_{k,\epsilon}(g,\lambda+n_0+1,\lambda+N_0+1)$ such that

$$T_4 = H^{n_0}(v,w,k)\log(b(0)-w+i\epsilon)+S^{n_0}(v,w,k). \tag{2.3.99}$$

For T_5, we have

$$\begin{aligned} & \partial_w\mathcal{R}_2^{0,n_0}(v,w) \\ =& -\sum_{j=1}^{n_0}\binom{n_0}{j}\int_{\mathbb{R}}\partial_w^{j+1}\big[(\mathcal{G}_k^{fr}+\mathcal{G}_k^{b1})\partial_{v'}B(v'+w)\big]\frac{R^{0,n_0-j}(v',w)}{v'+i\epsilon}dv' \\ & -\sum_{j=1}^{n_0}\binom{n_0}{j}\int_{\mathbb{R}}\partial_w^{j+1}\big[(\mathcal{G}_k^{fr}+\mathcal{G}_k^{b1})\partial_{v'}B(v'+w)\big]\frac{\partial_w R^{0,n_0-j}(v',w)}{v'+i\epsilon}dv' \\ :=& T_{51}+T_{52}. \end{aligned} \tag{2.3.100}$$

From (Claim 6) for the case $n \leq n_0 - 1$, we have

$$\begin{aligned} & \partial_w R^{0,n_0-j}(v,w) \\ = & \sum_{p=0}^{n_0-j} \beta_p^{n_0-j}(v,w,k) R^{0,p}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\ & + \sum_{p=0}^{n_0-j-1} \gamma_p^{n_0-j}(v,w,k) \partial_v R^{0,p}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\ & + B^{n_0-j}(v,w) \log(b(0)-w+i\epsilon) + R^{0,n_0-j+1}(v,w). \end{aligned} \tag{2.3.101}$$

In addition, $\beta_p^{n_0-j}(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-j-p+1, N_0+1)$ type singularity structure; $\gamma_p^{n_0-j}(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-j_p, N_0+1)$ type singularity structure. For $0 \leq p \leq n_0 - 1$, set

$$\begin{aligned} \widehat{\beta}_p^{n_0}(v,w,k) = - \sum_{j=1}^{n_0-p} \binom{n_0}{j} \int_{\mathbb{R}} & \partial_w^{j+1} \big[(\mathcal{G}_k^{fr} + \mathcal{G}_k^{b1}) \partial_{v'} B(v'+w) \big] \\ & \times \frac{\beta_p^{n_0-j}(v',w,k)}{v'+i\epsilon} dv', \end{aligned} \tag{2.3.102}$$

$$\begin{aligned} \widehat{\gamma}_p^{n_0}(v,w,k) = - \sum_{j=1}^{n_0-1-p} \binom{n_0}{j} \int_{\mathbb{R}} & \partial_w^{j+1} \big[(\mathcal{G}_k^{fr} + \mathcal{G}_k^{b1}) \partial_{v'} B(v'+w) \big] \\ & \times \frac{\gamma_p^{n_0-j}(v',w,k)}{v'+i\epsilon} dv', \end{aligned} \tag{2.3.103}$$

and

$$\begin{aligned} \widehat{B}^{n_0}(v,w,k) = - \sum_{j=1}^{n_0} \binom{n_0}{j} \int_{\mathbb{R}} & \partial_w^{j+1} \big[(\mathcal{G}_k^{fr} + \mathcal{G}_k^{b1}) \partial_{v'} B(v'+w) \big] \\ & \times \frac{B^{n_0-j}(v',w,k)}{v'+i\epsilon} dv'. \end{aligned} \tag{2.3.104}$$

We have $\widehat{\beta}_p^{n_0}(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-p+1, N_0+1)$ type singularity structure; $\widehat{\gamma}_p^{n_0}(v,w,k)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-p, N_0+1)$ type singularity structure; $\widehat{B}^{n_0}(v,w,k)$

has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(g, \lambda + n_0 + 1, \lambda + N_0 + 1)$ type singularity structure. In addition,

$$\begin{aligned}
T_{52} =& \sum_{p=0}^{n_0-1} \widehat{\beta}_p^{n_0}(v,w,k) R^{0,p}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\
&+ \sum_{p=0}^{n_0-2} \widehat{\gamma}_p^{n_0}(v,w,k) \partial_v R^{0,p}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\
&+ \widehat{B}^{n_0}(v,w,k) \log(b(0)-w+i\epsilon) \\
&- \sum_{j=1}^{n_0} \binom{n_0}{j} \int_{\mathbb{R}} \partial_w^{j+1} \big[(\mathcal{G}_k^{fr} + \mathcal{G}_k^{b1}) \partial_{v'} B(v'+w) \big] \frac{R^{0,n_0-j+1}(v',w)}{v'+i\epsilon} dv' \\
:=& T_{521} + T_{522} + T_{523} + T_{524}.
\end{aligned} \tag{2.3.105}$$

Take

$$\mathcal{R}_2^{0,n_0+1}(v,w) = T_{31} + T_{51} + T_{524}. \tag{2.3.106}$$

For $T_6 = \partial_w \mathcal{R}_3^{0,n_0}(v,w)$, in view of (IA 4), we have

$$\begin{aligned}
T_6 =\ & 2\log(b(0)-w+i\epsilon) \sum_{p=0}^{n_0-1} a_k^{n_0,p}(v, b(0)-w, w) R^{0,p}(b(0)-w,w) \\
&+ 2\log(b(0)-w+i\epsilon) \sum_{p=0}^{n_0-1} b_k^{n_0,p}(v, b(0)-w, w) \partial_v R^{0,p}(b(0)-w,w) \\
&- k \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \frac{\log(v'+i\epsilon)}{B(v'+w)} \times \Bigg[\sum_{p=0}^{n_0-1} a_k^{n_0,p}(v,v',w) R^{0,p}(v',w) \\
&\qquad + \sum_{p=0}^{n_0-1} b_k^{n_0,p}(v,v',w) \partial_{v'} R^{0,p}(v',w) \Bigg] dv' \\
&+ \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \operatorname{sign}(v'+w-b(0)) \log(v'+i\epsilon) \times \partial_w \Bigg[\\
&\sum_{p=0}^{n_0-1} a_k^{n_0,p}(v,v',w) R^{0,p}(v',w) + \sum_{p=0}^{n_0-1} b_k^{n_0,p}(v,v',w) \partial_{v'} R^{0,p}(v',w) \Bigg] dv' \\
:=\ & T_{61} + T_{62} + T_{63} + T_{64}.
\end{aligned} \tag{2.3.107}$$

For T_{64}, we have

$$\begin{aligned}T_{64} =&\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)\times\\&\sum_{p=0}^{n_0-1}\Big[\partial_w a_k^{n_0,p}(v,v',w)R^{0,p}(v',w)+\partial_w b_k^{n_0,p}(v,v',w)\partial_{v'}R^{0,p}(v',w)\Big]dv'\\&+\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)\times\\&\sum_{p=0}^{n_0-1}\Big[a_k^{n_0,p}(v,v',w)\partial_w R^{0,p}(v',w)+b_k^{n_0,p}(v,v',w)\partial_{v'}\partial_w R^{0,p}(v',w)\Big]dv'\\:=&T_{641}+T_{642}.\end{aligned}\tag{2.3.108}$$

Since we assumed that (Claim 6) holds for all $n \le n_0-1$, we have for $p \le n_0-1$

$$\begin{aligned}&\partial_v\partial_w R^{0,p}(v,w)\\=\;&\partial_v\Big(\sum_{j=0}^{p}\beta_j^p(v,w,k)R^{0,j}(b(0)-w,w)\log(b(0)-w+i\epsilon)\\&+\sum_{j=0}^{p-1}\gamma_j^p(v,w,k)\partial_v R^{0,j}(b(0)-w,w)\log(b(0)-w+i\epsilon)\\&+B^p(v,w,k)\log(b(0)-w+i\epsilon)+R^{0,p+1}(v,w)\Big).\end{aligned}\tag{2.3.109}$$

In addition, for $0 \le j \le p$, $\beta_j^p(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, p-j+1, N_0+1)$ and $\gamma_j^p(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, p-j, N_0+1)$. $B^p(v,w,k)$ is from the singularity structure of $\Gamma(v,w)$ and $B^p(v,w,k) \in \mathcal{F}^{b0}_{0,p,k,\epsilon}(g, \lambda+p, \lambda+N_0+1)$. For $0 \le j \le n_0-1$, define

$$\begin{aligned}\widetilde{\beta}_j^{n_0}(v,w,k) =& \sum_{p=j}^{n_0-1}\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)\times\\&\Big[a_k^{n_0,p}(v,v',w)\beta_j^p(v',w,k)+b_k^{n_0,p}(v,v',w)\partial_{v'}\beta_j^p(v',w,k)\Big]dv',\end{aligned}\tag{2.3.110}$$

$$\begin{aligned}\widetilde{\gamma}_j^{n_0}(v,w,k) =& \sum_{p=j+1}^{n_0-1}\int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|}\operatorname{sign}(v'+w-b(0))\log(v'+i\epsilon)\times\\&\Big[a_k^{n_0,p}(v,v',w)\gamma_j^p(v',w,k)+b_k^{n_0,p}(v,v',w)\partial_{v'}\gamma_j^p(v',w,k)\Big]dv',\end{aligned}\tag{2.3.111}$$

and

$$\widetilde{B}^{n_0}(v,w,k) = \sum_{p=0}^{n_0-1} \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \operatorname{sign}(v'+w-b(0)) \log(v'+i\epsilon) \times \Big[a_k^{n_0,p}(v,v',w) B^p(v',w,k) + b_k^{n_0,p}(v,v',w) \partial_{v'} B^p(v',w,k) \Big] dv'. \tag{2.3.112}$$

It then follows from Lemma 2.3.9 that $\widetilde{\beta}_j^{n_0}(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-j+1, N_0+1)$, $\widetilde{\gamma}_j^{n_0}(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(\Phi_k^{b0}, n_0-j, N_0+1)$ and $\widetilde{B}_j^{n_0}(v,w,k) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g, n_0+\lambda+1, N_0+\lambda+1)$, and T_{642} can be written as

$$\begin{aligned} T_{642} = & \sum_{j=0}^{n_0-1} \widetilde{\beta}_j^{n_0}(v,w,k) R^{0,j}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\ & + \sum_{j=0}^{n_0-2} \widetilde{\gamma}_j^{n_0}(v,w,k) \partial_v R^{0,j}(b(0)-w,w) \log(b(0)-w+i\epsilon) \\ & + \widetilde{B}^{n_0}(v,w,k) \log(b(0)-w+i\epsilon) \\ & + \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \operatorname{sign}(v'+w-b(0)) \log(v'+i\epsilon) \\ & \sum_{p=0}^{n_0-1} \Big[a_k^{n_0,p}(v,v',w) R^{0,p+1}(v',w) + b_k^{n_0,p}(v,v',w) \partial_{v'} R^{0,p+1}(v',w) \Big] dv'. \end{aligned} \tag{2.3.113}$$

For T_7, in view of (IA 6), we have

$$\begin{aligned} T_7 = & \ \partial_w \mathcal{R}_4^{0,n}(v,w) \\ = & \ -k \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \operatorname{sign}(v'+w-b(0)) \frac{\log(v'+i\epsilon)}{B(v'+w)} \times \Big[\\ & \sum_{p=0}^{n_0-1} c_k^{n_0,p}(v,v',w) R^{0,p}(v',w) + \sum_{p=0}^{n_0-1} d_k^{n_0,p}(v,v',w) \partial_{v'} R^{0,p}(v',w) \Big] dv' \\ & + \int_{\mathbb{R}} e^{-k|b^{-1}(v'+w)|} \log(v'+i\epsilon) \times \partial_w \Big[\sum_{p=1}^{n_0-1} c_k^{n_0,p}(v,v',w) R^{0,p}(v',w) \\ & + \sum_{p=0}^{n_0-1} d_k^{n_0,p}(v,v',w) \partial_{v'} R^{0,p}(v',w) \Big] dv'. \end{aligned} \tag{2.3.114}$$

The analysis of T_7 is similar to T_{63} and T_{64} above.

Set

$$\begin{aligned}\mathcal{B}^{n_0}(v,w,k) =&\Phi_k^{b0}(v+w)\partial_v B(b(0))R^{0,n_0}(b(0)-w,w)\\&+2\sum_{p=0}^{n_0-1} a_k^{n_0,p}(v,b(0)-w,w)R^{0,p}(b(0)-w,w)\\&+2\sum_{p=0}^{n_0-1} b_k^{n_0,p}(v,b(0)-w,w)\partial_v R^{0,p}(b(0)-w,w)\\&+\sum_{p=0}^{n_0-1}\left[\widehat{\beta}_p^{n_0}(v,w,k)+\widetilde{\beta}_p^{n_0}(v,w,k)\right]R^{0,p}(b(0)-w,w)\\&+\sum_{p=0}^{n_0-2}\left[\widehat{\gamma}_p^{n_0}(v,w,k)+\widetilde{\gamma}_p^{n_0}(v,w,k)\right]\partial_v R^{0,p}(b(0)-w,w)\\&+\widehat{B}^{n_0}(v,w,k)+\widetilde{B}^{n_0}(v,w,k)+H^{n_0}(v,w,k),\end{aligned} \tag{2.3.115}$$

and

$$\mathcal{R}_1^{0,n_0+1}(v,w)=\mathcal{S}^{n_0}(v,w,k) \tag{2.3.116}$$

$$\mathcal{R}_2^{0,n_0+1}(v,w)=T_{31}+T_{51}+T_{524}. \tag{2.3.117}$$

Furthermore, from the analysis of T_{32}, T_6 and T_7, we define $\mathcal{R}_3^{0,n_0+1}(v,w)$ and $\mathcal{R}_4^{0,n_0+1}(v,w)$ by setting for $0\le p\le n_0-1$

$$\begin{aligned}a_k^{n_0+1,p}(v,v',w)&=\partial_w a_k^{n_0,p}(v,v',w)+a_k^{n_0,p-1}(v,v',w)-\frac{kc_k^{n_0,p}(v,v',w)}{B(v'+w)}\\b_k^{n_0+1,p}(v,v',w)&=\partial_w b_k^{n_0,p}(v,v',w)+b_k^{n_0,p-1}(v,v',w)-\frac{kd_k^{n_0,p}(v,v',w)}{B(v'+w)}\\c_k^{n_0+1,p}(v,v',w)&=\partial_w c_k^{n_0,p}(v,v',w)+c_k^{n_0,p-1}(v,v',w)-\frac{ka_k^{n_0,p}(v,v',w)}{B(v'+w)}\\d_k^{n_0+1,p}(v,v',w)&=\partial_w d_k^{n_0,p}(v,v',w)+d_k^{n_0,p-1}(v,v',w)-\frac{kb_k^{n_0,p}(v,v',w)}{B(v'+w)},\end{aligned}$$

and for $p = n_0$

$$
\begin{aligned}
a_k^{n_0+1,n_0}(v,v',w) =& \Phi_k^{b0}(v+w)\partial_{v'}^2 B(v'+w) - \partial_w \Phi_k^{b0}(v+w)\frac{\partial_{v'}B(v'+w)}{B(v'+w)} \\
& + a_k^{n_0,n_0-1}(v,v',w), \\
b_k^{n_0+1,n_0}(v,v',w) =& \Phi_k^{b0}(v+w)\partial_{v'}B(v'+w) + b_k^{n_0,n_0-1}(v,v',w), \\
c_k^{n_0+1,n_0}(v,v',w) =& -k\Phi_k^{b0}(v+w)\frac{\partial_{v'}B(v'+w)}{B(v'+w)} + \frac{\partial_w \Phi_k^{b0}(v+w)}{k}\partial_{v'}^2 B(v'+w) \\
& + c^{n_0,n_0-1}, \\
d_k^{n_0+1,n_0}(v,v',w) =& \frac{\partial_w \Phi_k^{b0}(v+w)}{k}\partial_{v'}B(v'+w) + d_k^{n_0,n_0-1}(v,v',w).
\end{aligned}
$$

It follows from the induction assumption and Lemma 2.3.9, Lemma 2.3.10 and Lemma 2.3.11 that for $1 \le j \le 4$, $\mathcal{R}_j^{0,n_0+1}(v,w) \in \mathcal{F}^{b_0}_{0,n_0+1,k,\epsilon}(g, \lambda+n_0+1, \lambda+N_0)$. We then define

$$B^{n_0}(v,w,k) = (I+T_{k,\epsilon})^{-1}\mathcal{B}^{n_0}(v,w,k) \tag{2.3.118}$$

and

$$R^{0,n_0+1}(v,w) = (I+T_{k,\epsilon})^{-1}\left(\sum_{j=1}^{4}\mathcal{R}_j^{0,n_0+1}(v,w)\right). \tag{2.3.119}$$

We have

$$\partial_w R^{0,n_0}(v,w) = B^{n_0}(v,w,k)\log(b(0)-w+i\epsilon) + R^{0,n_0+1}(v,w). \tag{2.3.120}$$

We apply the assumption that the theorem holds for $N \le N_0$ and get

$$R^{0,n_0+1}(v,w) \in \mathcal{F}^{b_0}_{0,n_0+1,k,\epsilon}(g, \lambda+n_0+1, \lambda+N_0). \tag{2.3.121}$$

Therefore, (Claim 6) holds for the case $n = n_0$ and (Claim 7) holds for the case $n = n_0+1$.

Step 3: construction of $R^{m,n}(v,w)$ for $1 \le m \le N_0 - n$. Fix $0 \le n \le N_0$. For $1 \le m \le N_0 - n$, we construct $R^{m,n}(v,w)$ in the following way.

$$R^{m,n}(v,w) = \Omega_1^{m,n}(v,w) + \Omega_2^{m,n}(v,w) + \Omega_3^{m,n}(v,w) + \Omega_4^{m,n}(v,w) \tag{IB 1}$$

and the terms in (IB 1) have the following expression.

(a) $\Omega_1^{m,n}(v,w)$ is from the singularity structure of $\Gamma(v,w)$, and $\Omega_1^{m,n}(v,w)$ has the $\mathcal{F}_{k,\epsilon}^{b0}(g, m+n+\lambda, N_0+\lambda+1)$ type singularity structure.

(b) Recall that $\mathcal{R}_3^{0,n}(v,w)$ and $\mathcal{R}_4^{0,n}(v,w)$ are defined in (IA 3) and (IA 4). We have

$$\begin{aligned}\Omega_2^{m,n}(v,w) = \; & \partial_v^m \big[\mathcal{R}_3^{0,n}(v,w) + \mathcal{R}_4^{0,n}(v,w)\big] \\ & - \partial_v^m \int_{\mathbb{R}} \mathcal{G}_k^b(v+w, v'+w) \frac{\partial_{v'} B(v'+w) R^{0,n}(v',w)}{v'+i\epsilon} dv' \\ & - \sum_{j=1}^{n} \binom{n}{j} \int_{\mathbb{R}} \partial_v^m \partial_w^j \big[\mathcal{G}_k^{b1}(v+w, v'+w) \partial_{v'} B(v'+w)\big] \\ & \qquad \times \frac{R^{0,n-j}(v',w)}{v'+i\epsilon} dv'.\end{aligned} \tag{IB 2}$$

(c) For $\Omega_3^{m,n}(v,w)$, given smooth functions $g_k^{m,n,j}(v,v',w)$ and $h_k^{m,n,j}(v,v',w)$ $(0 \leq j \leq n)$,

$$\begin{aligned}& \Omega_3^{m,n}(v,w) \\ = \; & \int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \operatorname{sign}(v-v') \log(v'+i\epsilon) \\ & \times \sum_{j=0}^{n} \big[g_k^{m,n,j}(v,v',w) R^{0,j}(v',w) + h_k^{m,n,j}(v,v',w) \partial_{v'} R^{0,j}(v',w)\big] dv'.\end{aligned} \tag{IB 3}$$

In addition, for any integers $p, q \geq 0$ the coefficient functions satisfy

$$\begin{aligned}|\partial_v^p \partial_w^q g_k^{m,n,j}(v,v',w)| &\lesssim k^{m+n-j+p+q}, \\ |\partial_v^p \partial_w^q h_k^{m,n,j}(v,v',w)| &\lesssim k^{m+n-j+p+q-1}.\end{aligned} \tag{IB 4}$$

(d) For $\Omega_4^{m,n}(v,w)$, given smooth functions $\sigma_k^{m,n,j}(v,v',w)$ and $\rho_k^{m,n,j}(v,v',w)$ $(0 \leq j \leq n)$,

$$\begin{aligned}& \Omega_4^{m,n}(v,w) \\ = \; & \int_{\mathbb{R}} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \log(v'+i\epsilon) \\ & \times \sum_{j=0}^{n} \big[\sigma_k^{m,n,j}(v,v',w) R^{0,j}(v',w) + \rho_k^{m,n,j}(v,v',w) \partial_{v'} R^{0,j}(v',w)\big] dv'.\end{aligned} \tag{IB 5}$$

In addition, for any integers $p, q \geq 0$ the coefficient functions satisfy

$$\begin{aligned}|\partial_v^p \partial_w^q \sigma_k^{m,n,j}(v,v',w)| &\lesssim k^{m+n-j+1+p+q}, \\ |\partial_v^p \partial_w^q \rho_k^{m,n,j}(v,v',w)| &\lesssim k^{m+n-j+p+q}.\end{aligned} \tag{IB 6}$$

We claim that for $0 \leq n \leq N_0$ and $1 \leq m \leq N_0 - m$, there exist smooth functions $g_k^{m,n,j}(v,v',w)$, $h_k^{m,n,j}(v,v',w)$, $\sigma_k^{m,n,j}(v,v',w)$ and $\rho_k^{m,n,j}(v,v',w)$ such that the above system generates $R^{m,n}(v,w)$ and (Claim 2) to (Claim 5) holds.

For the case $m=1$, we take a derivative in v to the equation (IA 1) of $R^{0,n}$, and get

$$\partial_v R^{0,n}(v,w) = -\partial_v T_{k,\epsilon}(R^{0,n}) + \partial_v \sum_{j=1}^{4} \mathcal{R}_j^{0,n}(v,w). \tag{2.3.122}$$

For $-\partial_v T_{k,\epsilon}(R^{0,n})$, we have

$$\begin{aligned}
-\partial_v T_{k,\epsilon}(R^{0,n}) =& - \partial_v \int_{\mathbb{R}} \mathcal{G}_k^{fr}(v+w, v'+w) \frac{\partial_{v'} B(v'+w) R^{0,n}(v',w)}{v'+i\epsilon} dv' \\
& - \partial_v \int_{\mathbb{R}} \mathcal{G}_k^{b}(v+w, v'+w) \frac{\partial_{v'} B(v'+w) R^{0,n}(v',w)}{v'+i\epsilon} dv' \\
:=& T_a + T_b.
\end{aligned} \tag{2.3.123}$$

Using integration by parts, we get

$$\begin{aligned}
& T_a \\
=& - \frac{2\Psi_k(v+w)\partial_v B(v+w)}{B(v+w)} R^{0,n}(v,w) \log(v+i\epsilon) \\
& + \frac{\Psi_k(v+w)}{B(v+w)} \int_{\mathbb{R}} \Big\{ \partial_{v'} \big[e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \partial_{v'} B(v'+w) R^{0,n}(v',w) \big] \\
& \qquad \times \mathrm{sign}(v-v') \log(v'+i\epsilon) \Big\} dv' \\
& + \partial_v \Psi_k(v+w) \int_{\mathbb{R}} \Big\{ \partial_{v'} \big[\frac{e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}}{k} \partial_{v'} B(v'+w) R^{0,n}(v',w) \big] \\
& \qquad \times \log(v'+i\epsilon) \Big\} dv' \\
:=& T_{a1} + T_{a2} + T_{a3}.
\end{aligned} \tag{2.3.124}$$

Since $\mathcal{R}_1^{0,n}(v,w) \in \mathcal{F}^{b0}_{0,n,k,\epsilon}(g, \lambda+n, \lambda+N_0+1)$, there exist functions $X^{0,n}(v,w), Y^{0,n}(v,w) \in \mathcal{F}^{b0}_{1,n,k,\epsilon}(g, \lambda+n+1, \lambda+N_0+1)$ such that

$$\partial_v \mathcal{R}_1^{0,n}(v,w) = X^{0,n}(v,w) \log(v+i\epsilon) + Y^{0,n}(v,w). \tag{2.3.125}$$

For $\partial_v \mathcal{R}_2^{0,n}(v,w)$, we have

$$\begin{aligned}
&\partial_v \mathcal{R}_2^{0,n}(v,w) \\
=& -\sum_{j=1}^{n} \binom{n}{j} \int_{\mathbb{R}} \partial_w^j \partial_v \left[\mathcal{G}_k^{fr}(v+w, v'+w)\partial_{v'} B(v'+w)\right] \frac{R^{0,n-j}(v',w)}{v'+i\epsilon} dv' \\
& -\sum_{j=1}^{n} \binom{n}{j} \int_{\mathbb{R}} \partial_w^j \partial_v \left[\mathcal{G}_k^{b1}(v+w, v'+w)\partial_{v'} B(v'+w)\right] \frac{R^{0,n-j}(v',w)}{v'+i\epsilon} dv' \\
:=& T_c + T_d.
\end{aligned} \tag{2.3.126}$$

Using integration by parts, we have

$$\begin{aligned}
&T_{c1} \\
=& -\sum_{j=1}^{n} \binom{n}{j} \partial_w^j \left[\frac{2\Psi_k(v+w)\partial_v B(v+w)}{B(v+w)}\right] R^{0,n-j}(v,w) \log(v+i\epsilon) \\
&+\sum_{j=1}^{n} \binom{n}{j} \int_{\mathbb{R}} \partial_{v'} \left[\partial_w^j \left(\frac{\Psi_k(v+w)}{B(v+w)} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|} \partial_{v'} B(v'+w)\right)\right. \\
&\qquad \left. \times R^{0,n-j}(v',w)\right] \mathrm{sign}(v-v') \log(v'+i\epsilon) dv' \\
&+\sum_{j=1}^{n} \binom{n}{j} \int_{\mathbb{R}} \partial_{v'} \left[\partial_w^j \left(\partial_v \Psi_k(v+w) \frac{e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}}{k} \partial_{v'} B(v'+w)\right)\right. \\
&\qquad \left. \times R^{0,n-j}(v',w)\right] \log(v'+i\epsilon) dv' \\
:=& T_{c1} + T_{c2} + T_{c3}.
\end{aligned} \tag{2.3.127}$$

Take

$$\Omega_1^{1,n}(v,w) = Y^{0,n}(v,w) \tag{2.3.128}$$

$$\Omega_2^{1,n}(v,w) = \partial_v(\mathcal{R}_3^{0,n}(v,w) + \mathcal{R}_3^{0,n}(v,w)) + T_b + T_d. \tag{2.3.129}$$

From T_{a2}, T_{a3}, T_{c2} and T_{c3}, we define $\Omega_3^{1,n}(v,w)$ and $\Omega_4^{1,n}(v,w)$ by setting for $0 \le j \le n$

$$\begin{aligned}
g_k^{1,n,j}(v,w) =& \Bigg\{ \partial_w^{n-j}\left(\frac{\Psi_k(v+w)}{B(v+w)} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\partial_{v'}^2 B(v'+w)\right) \\
&- \partial_w^{n-j}\left(\partial_v \Psi_k(v+w) e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{\partial_{v'}B(v'+w)}{B(v'+w)}\right)\Bigg\} \\
&\times \binom{n}{j} e^{k|b^{-1}(v+w)-b^{-1}(v'+w)|}, \\
h_k^{1,n,j}(v,w) =& \partial_w^{n-j}\left(\frac{\Psi_k(v+w)}{B(v+w)} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\partial_{v'} B(v'+w)\right) \\
&\times \binom{n}{j} e^{k|b^{-1}(v+w)-b^{-1}(v'+w)|}, \\
\sigma_k^{1,n,j}(v,w) =& \Bigg\{ - k\partial_w^{n-j}\left(\frac{\Psi_k(v+w)}{B(v+w)} e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}\frac{\partial_{v'}B(v'+w)}{B(v'+w)}\right) \\
&+ \partial_w^{n-j}\left(\partial_v \Psi_k(v+w) \frac{e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}}{k}\partial_{v'}^2 B(v'+w)\right)\Bigg\} \\
&\times \binom{n}{j} e^{k|b^{-1}(v+w)-b^{-1}(v'+w)|}, \\
\rho_k^{1,n,j}(v,w) =& \partial_w^{n-j}\left(\partial_v \Psi_k(v+w) \frac{e^{-k|b^{-1}(v+w)-b^{-1}(v'+w)|}}{k}\partial_{v'} B(v'+w)\right) \\
&\times \binom{n}{j} e^{k|b^{-1}(v+w)-b^{-1}(v'+w)|}.
\end{aligned}$$

Set $R^{1,n}(v,w)$ as

$$R^{1,n}(v,w) = \sum_{j=1}^{4} \Omega_j^{1,n}(v,w). \tag{2.3.130}$$

We see that (Claim 2) holds for all $0 \le n \le N_0$ with

$$\alpha_j^n(v,w,k) = \binom{n}{j} \partial_w^{n-j}\left[\frac{2\Psi_k(v+w)\partial_v B(v+w)}{B(v+w)}\right]. \tag{2.3.131}$$

for $0 \le j \le n$.

Assume for some integer $m_0 \ge 1$, we have constructed functions $R^{m,n}(v,w)$ for $m \le m_0$, and (Claim 2) to (Claim 5) holds for $m = m_0 - 1$. Taking a derivative in v to $R^{m_0,n}$, we get

$$\partial_v R^{m_0,n} = \partial_v \left[\Omega_1^{m_0,n}(v,w) + \Omega_2^{m_0,n}(v,w) + \Omega_3^{m_0,n}(v,w) + \Omega_4^{m_0,n}(v,w)\right] \tag{2.3.132}$$

We only need to analyze $\partial_v \Omega_3^{m_0,n}(v,w)$ and $\partial_v \Omega_3^{m_0,n}(v,w)$. It turns out the we can define $R^{m_0+1,n}(v,w)$ by setting

$$g_k^{m_0+1,n,j}(v,v',w) = \partial_v g_k^{m_0,n,j}(v,v',w) - k\frac{\sigma_k^{m_0,n,j}(v,v',w)}{B(v+w)}, \tag{2.3.133}$$

$$h_k^{m_0+1,n,j}(v,v',w) = \partial_v h_k^{m_0,n,j}(v,v',w), \tag{2.3.134}$$

$$\sigma_k^{m_0+1,n,j}(v,v',w) = -k\frac{g_k^{m_0,n,j}(v,v',w)}{B(v+w)} + \partial_v \sigma_k^{m_0,n,j}(v,v',w), \tag{2.3.135}$$

$$\rho_k^{m_0+1,n,j}(v,v',w) = \partial_v \rho_k^{m_0,n,j}(v,v',w). \tag{2.3.136}$$

Then (Claim 4) and (Claim 5) holds with

$$\tau_j^{m_0,n}(v,w) = 2g_k^{m_0,n,j}(v,w), \quad \eta_j^{m_0,n}(v,w) = 2h_k^{m_0,n,j}(v,w). \tag{2.3.137}$$

Step 4: estimate of $R^{m,n}$ for $0 \le n \le N_0+1$, $m = N_0+1-m$. In order to complete the proof, we only need to prove (Claim 1) for $0 \le n \le N_0+1$, $m = N_0+1-m$ under the assumption that $\Gamma(v,w) \in \mathcal{F}^{b0}_{0,0,k,\epsilon}(g,\lambda,\lambda+N_0+1)$. We first show that

$$\left\|R^{1,N_0}(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{\lambda+N_0+1}}. \tag{2.3.138}$$

By (Claim 7), for $0 \le n \le N_0$ we have

$$\left\|R^{0,n}(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{\lambda+n}}. \tag{2.3.139}$$

It follows from (Claim 2) that $R^{1,N_0} \in L^2_v L^2_w$ is defined and is given by (IB 1) for $m=1$ and $N = N_0$. We have

$$\left\|R^{1,N_0}(v,w)\right\|_{L^2_w L^2_v} \lesssim \sum_{j=1}^{4} \left\|\Omega_j^{1,N_0}(v,w)\right\|_{L^2_w L^2_v} \lesssim \frac{1}{k}\|g\|_{H^{\lambda+N_0+1}}. \tag{2.3.140}$$

Differentiate $R^{1,N_0}(v,w)$ in v and we get

$$\partial_v R^{1,N_0}(v,w) = 2h_k^{1,N_0,N_0}(v,v,w)\partial_v R^{0,N_0}\log(v+i\epsilon) + R(v,w), \tag{2.3.141}$$

where

$$\|R(v,w)\|_{L^2_v L^2_w} \lesssim \|g\|_{H_k^{\lambda+N_0+1}}. \tag{2.3.142}$$

By (Claim 2), there exists a function $M(v,w) \in L^2_w H^1_{k,v}$ such that

$$\|M(v,w)\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{\lambda+N_0+1}}, \tag{2.3.143}$$

and

$$\partial_v R^{0,N_0}(v,w) = M(v,w)\log(v+i\epsilon) + R^{1,N_0}(v,w). \tag{2.3.144}$$

From (2.3.141), we deduce that R^{1,N_0} sovles the following equation

$$\begin{aligned}\partial_v R^{1,N_0}(v,w) =& \big[2h_k^{1,N_0,N_0}(v,v,w)\log(v+i\epsilon)\big]R^{1,N_0}(v,w)\\ &+ 2h_k^{1,N_0,N_0}(v,v,w)M(v,w)\log^2(v+i\epsilon) + R(v,w).\end{aligned} \tag{2.3.145}$$

Since $|h_k^{1,N_0,N_0}| \lesssim 1$, we have

$$\begin{aligned}&\left\|\partial_v R^{1,N_0}(v,w)\right\|_{L^2_w L^2_v}\\ \lesssim& \left\|2h_k^{1,N_0,N_0}(v,v,w)M(v,w)\log^2(v+i\epsilon) + R(v,w)\right\|_{L^2_w L^2_v}\\ \lesssim& \|g\|_{H_k^{\lambda+N_0+1}}.\end{aligned} \tag{2.3.146}$$

Therefore,

$$\left\|R^{1,N_0}(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{\lambda+N_0+1}}. \tag{2.3.147}$$

Then by (IA 1) to (IA 7), $R^{0,N_0+1}(v,w)$ is also defined and

$$\left\|R^{0,N_0+1}(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim \|g\|_{H_k^{\lambda+N_0+1}}. \tag{2.3.148}$$

The estimate of $R^{m,n}(v,w)$ for $m = N_0+1-n$ and $0 \le n \le N_0-1$ follows directly from (IB 1) to (IB 6) and estimates of $R^{0,p}(v,w)$ and $R^{1,p}(v,w)$, $0 \le p \le n$. □

Theorem 2.3.13. *For any $k \ge 1$, $N \ge 1$, $0 < \epsilon < 1$, $\iota \in \{+,-\}$, assume $f_0^k \in H_k^N(\mathbb{R})$ and $\Theta^\iota_{k,\epsilon}(v,w)$ solves*

$$\begin{aligned}&\Theta^\iota_{k,\epsilon}(v,w) + \int_{\mathbb{R}} \mathcal{G}_k(v+w,v'+w)\frac{\partial_{v'}B(v'+w)\Theta^\iota_{k,\epsilon}(v',w)}{v'+i\iota\epsilon}dv'\\ =& \int_{\mathbb{R}} \mathcal{G}_k(v+w,v'+w)\frac{1}{B(v'+w)}\frac{f_0^k(v'+w)}{v'+i\iota\epsilon}dv'.\end{aligned} \tag{2.3.149}$$

Then $\Theta^{2,\iota}_{k,\epsilon}(v,w) = \Theta^\iota_{k,\epsilon}(v,w)\Upsilon_2(w)$ has the $\mathcal{F}^{in}_{0,k,\epsilon}(f_0^k,1,N)$ type singularity structure.

Proof. It follows from Proposition 2.3.6 and Theorem 2.3.12 that $\Theta^{2,\iota}_{k,\epsilon}(v,w)$ has the $\mathcal{F}^{b0}_{0,0,k,\epsilon}(f_0^k,1,N)$ type regularity structure. Noting that on the support of $\Upsilon_2(w)$, $w - b(0) > \delta$ for some $\delta > 0$. Hence $\Upsilon_2(w)\log(b(0)-w+i\epsilon)$ is a Schwarz function. The definition of $\mathcal{F}^{b0}_{0,0,k,\epsilon}(f_0^k,1,N)$ type singularity structure implies that $\Theta^{2,\iota}_{k,\epsilon}(v,w)$ has the $\mathcal{F}^{in}_{0,k,\epsilon}(f_0^k,1,N)$ type singularity structure. □

Remark 2.3.14. *The major terms in the singularity structure of* $\Theta^{\iota}_{k,\epsilon}$ *are explicitly computable. As a matter of fact, one can verify with integration by parts that*

$$\partial_v \Theta^{\iota}_{k,\epsilon} = A^{\iota}_{k,\epsilon}(v,w)\log(v+i\epsilon) + R^{\iota}_{k,\epsilon}(v,w) \tag{2.3.150}$$

where $A^{\iota}_{k,\epsilon}$ *is given explicitly by*

$$A^{\iota}_{k,\epsilon}(v,w) = \frac{2\partial_v B(v+w)}{B(v+w)}\Theta^{\iota}_{k,\epsilon}(v,w) - \frac{2}{B^2(v+w)}f^k_0(v+w). \tag{2.3.151}$$

Taking a derivative in w*, we have*

$$\begin{aligned}&\partial_w \Theta^{\iota}_{k,\epsilon}\\ =&D^{0,\iota}_{k,\epsilon}(v,w)\log(b(0)-w+i\epsilon) + D^{1,\iota}_{k,\epsilon}(v,w)\log(b(1)-w+i\epsilon) + S^{\iota}_{k,\epsilon}(v,w),\end{aligned} \tag{2.3.152}$$

where for j *in*$\{0,1\}$*,*

$$D^{j,\iota}_{k,\epsilon}(v,w) = \Phi^j_{k,\iota\epsilon}(v,w)\frac{2f^k_0(b(j))}{(b'(j))^2} - \Phi^j_{k,\iota\epsilon}(v,w)\frac{2b''(j)\Theta^{\iota}_{k,\epsilon}(b(j)-w,w)}{b'(j)}, \tag{2.3.153}$$

and $\Phi^j_{k,\epsilon}(v,w)$ *is defined in Proposition 2.2.3.*

Proposition 2.3.15. *Under the same condition of Theorem 2.3.12, if we further assume that* $N \geq 2$ *and* $f^k_0(b(0)) = 0$*, then* $\partial_w \Theta^{1,\iota}_{k,\epsilon}(v,w)$ *has the* $\mathcal{F}^{b0}_{k,\epsilon}(f^k_0, 2, N)$ *type singularity structure.*

Proof. Note that under the change of variable $v = b(y)$, $w = b(y_0)$,

$$\Theta^{\iota}_{k,\epsilon}(b(0)-w,w) = \psi^{\iota}_{k,\epsilon}(0,y_0) = 0.$$

Hence when $f^k_0(b(0)) = 0$, we see from (2.3.153) that the major term $D^{0,\iota}_{k,\epsilon}(v,w) \equiv 0$. Therefore, it follows from Theorem 2.3.12 that $\partial_w \Theta^{1,\iota}_{k,\epsilon}(v,w)$ has the $\mathcal{F}^{b0}_{k,\epsilon}(f^k_0, 2, N)$ type singularity structure. □

2.4 Proof of the main theorem

2.4.1 Technical lemmas

In order to capture the regularity of different components of the stream function, we need the following lemmas.

Lemma 2.4.1. *Suppose $h(v,w)$ is supported on $(-L,L)\times(-L,L)$ for some $L>0$ and $h(v,w)$ is $H^{\frac{1}{2}+\delta_0}$ in v and H^s in w for some $s\geq 0$ and $\delta_0>0$. For any non-negative integer p', $\epsilon>0$ and $k\in\mathbb{Z}$, set*

$$g^{p'}_{k,\epsilon}(t,v)=\int_{\mathbb{R}} e^{-ikwt}h(v-w,w)\frac{\log^{p'}(v-w+i\epsilon)}{v-w+i\epsilon}dw. \tag{2.4.1}$$

Then there exists a function $\alpha^{p'}_k(t,v)$ such that $g^{p'}_{k,\epsilon}(t,v)$ converges to

$$g^{p'}_k(t,v):=e^{-ikvt}\alpha^{p'}_k(t,v). \tag{2.4.2}$$

in distribution as $\epsilon\to 0+$. In addition, following estimates hold.

(a) If $p'=0$, then

$$\left\|\alpha^0_k(t,\cdot)\right\|_{H^s}\lesssim_{\delta,s,\delta_0}\|h(v,w)\|_{H_v^{1/2+\delta_0}H_w^s}. \tag{2.4.3}$$

(b) If $p'>0$, then for any $0<\delta<\delta_0$ we have

$$\left\|\alpha^{p'}_k(t,\cdot)\right\|_{H^{s-\delta}}\lesssim_{\delta,s,p',\delta_0}\left|1+\log^{p'}\langle kt\rangle\right|\|h(v,w)\|_{H_v^{1/2+\delta_0}H_w^{s-\delta/2}}. \tag{2.4.4}$$

Proof. Set

$$\mu(v,\xi)=\int_{\mathbb{R}} h(v-w,w)e^{-iw\xi}dw. \tag{2.4.5}$$

Take

$$z^{p'}_\epsilon(w)=\varphi(w)\frac{\log^{p'}(w+i\epsilon)}{w+i\epsilon} \tag{2.4.6}$$

where $\varphi(w)$ is a smooth cutoff function with $\varphi(w)=1$ on $(-L,L)$. We have

$$z^{p'}_\epsilon(w)=\frac{d}{dw}\left[\varphi(w)\log^{p'+1}(w+i\epsilon)\right]-\frac{d\varphi(w)}{dw}\log^{p'+1}(w+i\epsilon) \tag{2.4.7}$$

It follows from Lemma A.1.1 in the Appendix that

$$|\mathcal{F}(z^{p'}_\epsilon)(\xi)|\lesssim 1+\log^{p'}\langle\xi\rangle. \tag{2.4.8}$$

We have

$$g^{p'}_{k,\epsilon}(t,v)=e^{-ikvt}\int_{\mathbb{R}}\mu(v,\xi)\widehat{z^{p'}_\epsilon}(kt-\xi)e^{iv\xi}d\xi. \tag{2.4.9}$$

Define

$$\alpha^{p'}_{k,\epsilon}(t,v)=\int_{\mathbb{R}}\mu(v,\xi)\widehat{z^{p'}_\epsilon}(kt-\xi)e^{iv\xi}d\xi. \tag{2.4.10}$$

We have

$$\widehat{\alpha^{p'}_{k,\epsilon}}(t,\eta) = \int_{\mathbb{R}} \widetilde{h}(\eta-\xi,\eta)\widehat{z^{p'}_{\epsilon}}(kt-\xi)d\xi. \tag{2.4.11}$$

Here $\widetilde{h}(\xi,\eta)$ denotes the Fourier transform of $h(v,w)$ in both variables. If $p' > 0$, for any $\delta > 0$ we apply Cauchy-Schwarz inequality and get

$$\begin{aligned}
&\int_{\mathbb{R}} \langle\eta\rangle^{2s-2\delta}\left|\widehat{\alpha^{p'}_{k,\epsilon}}(t,\eta)\right|^2 d\eta \\
\lesssim_{s,\delta,\delta_0,p'} &\int_{\mathbb{R}} \left[\langle\eta\rangle^{2s-2\delta}\int_{\mathbb{R}} \frac{1}{\langle\eta-\xi\rangle^{1+2\delta_0}}\left|1+\log^{2p'}\langle kt-\xi\rangle\right| d\xi \right. \\
&\qquad \left. \times \int_{\mathbb{R}} \langle\eta-\xi\rangle^{1+2\delta_0}|\widetilde{h}(\eta-\xi,\eta)|^2 d\xi\right] d\eta \\
\lesssim_{s,\delta,\delta_0,p'} &\int_{\mathbb{R}}\int_{R} \langle\eta\rangle^{2s-\delta}\langle\xi\rangle^{1+2\delta_0}\left|1+\log^{2p'}\langle kt\rangle\right||\widetilde{h}(\xi,\eta)|^2 d\xi d\eta \\
\lesssim_{s,\delta,\delta_0,p'} &\left|1+\log^{2p'}\langle kt\rangle\right| \|h(v,w)\|^2_{H_v^{1/2+\delta_0}H_w^{s-\delta/2}}.
\end{aligned} \tag{2.4.12}$$

If $p' = 0$, we have for $\xi \in \mathbb{R}$

$$|\widehat{z^{p'}_{\epsilon}}(\xi)| \lesssim 1. \tag{2.4.13}$$

Then (2.4.3) follows directly from (2.4.11) and the Cauchy-Schwarz inequality. □

Similarly, we also have the following Lemma. The idea of proof is similar to Lemma 2.4.1

Lemma 2.4.2. *Suppose $h(v,w)$ is compactly supported on $(-L,L)\times(-L,L)$ for some $L > 0$ and $h(v,w)$ is $H^{\frac{1}{2}+\delta_0}$ in w and H^s in v for some $s \geq 0$ and $\delta_0 > 0$. For any non-negative integer p', $\epsilon > 0$ and $k \geq 1$, set*

$$g^{p'}_{k,\epsilon}(t,v) = \int_{\mathbb{R}} e^{-ikwt}h(v-w,w)\frac{\log^{p'}(b(0)-w+i\epsilon)}{b(0)-w+i\epsilon}dw. \tag{2.4.14}$$

Then there exists a function $\beta^{p'}_k(t,v)$ such that $g^{p'}_{k,\epsilon}(t,v)$ converges to

$$g^{p'}_k(t,v) := e^{-ikb(0)t}\beta^{p'}_k(t,v). \tag{2.4.15}$$

in distribution as $\epsilon \to 0+$. In addition, following estimates hold.

(a) If $p' = 0$, then

$$\left\|\beta^0_k(t,\cdot)\right\|_{H^s} \lesssim_{\delta,s,\delta_0} \|h(v,w)\|_{H_w^{1/2+\delta_0}H_v^s}. \tag{2.4.16}$$

(b) If $p' > 0$, then for any $0 < \delta < \delta_0$ we have

$$\left\|\beta^{p'}_k(t,\cdot)\right\|_{H^{s-\delta}} \lesssim_{\delta,s,p',\delta_0} \left|1+\log^{p'}\langle kt\rangle\right| \|h(v,w)\|_{H_w^{1/2+\delta_0}H_v^{s-\delta/2}}. \tag{2.4.17}$$

2.4.2 Proof of main theorems

By (2.1.3) and the change of variables (2.1.16) $v = b(y)$, we have for $t \geq 0$, $v \in \mathbb{R}$

$$\begin{aligned}\phi_k(t,v) &:= \psi_k(t, b^{-1}(v)) \\ &= -\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_{b(0)}^{b(1)} e^{-ikwt}\Big[\Theta^-_{k,\epsilon}(v-w,w) - \Theta^+_{k,\epsilon}(v-w,w)\Big]dw.\end{aligned} \tag{2.4.18}$$

Since $\Theta^\iota_{k,\epsilon}(v,w)$ is smooth for w outside of $[b(0), b(1)]$, we can safely rewrite (2.4.18) as

$$\phi_k(t,v) = -\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} e^{-ikwt}\Big[\Theta^-_{k,\epsilon}(v-w,w) - \Theta^+_{k,\epsilon}(v-w,w)\Big]dw. \tag{2.4.19}$$

By (2.3.38) we can seperate the range of w into three parts to capture different type of singularities and rewrite (2.4.19) as

$$\begin{aligned}\phi_k(t,v) &= -\frac{1}{2\pi i}\sum_{j=1}^{3}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} e^{-ikwt}\Big[\Theta^{j,-}_{k,\epsilon}(v-w,w) - \Theta^{j,+}_{k,\epsilon}(v-w,w)\Big]dw \\ &:= \sum_{j=1}^{3}\phi^j_k(t,v).\end{aligned} \tag{2.4.20}$$

Using integration by parts, we get for $j \in \{1,2,3\}$

$$\begin{aligned}&\phi^j_k(t,v) \\ =& -\frac{1}{2\pi i}\frac{1}{(-ikt)^2}\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}\frac{d^2}{dw^2}\Big[\Theta^{j,-}_{k,\epsilon}(v-w,w) - \Theta^{j,+}_{k,\epsilon}(v-w,w)\Big]dw \\ =& -\frac{1}{2\pi i}\frac{1}{(-ikt)^2}\Bigg\{\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}\Big[\partial_v^2\Theta^{j,-}_{k,\epsilon}(v-w,w) - \partial_v^2\Theta^{j,+}_{k,\epsilon}(v-w,w)\Big]dw \\ &+ \lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}\Big[2\partial_v\partial_w\Theta^{j,-}_{k,\epsilon}(v-w,w) - 2\partial_v\partial_w\Theta^{j,+}_{k,\epsilon}(v-w,w)\Big] \\ &- \lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}\Big[\partial_w^2\Theta^{j,-}_{k,\epsilon}(v-w,w) - \partial_w^2\Theta^{j,+}_{k,\epsilon}(v-w,w)\Big]dw\Bigg\} \\ :=& T^j_1 + T^j_2 + T^j_3.\end{aligned} \tag{2.4.21}$$

Proof of Theorem 1.2.1

For the new variable $v = b(y)$, we define the cutoff function $\mu^{in}(v) = \chi^{in}(v)$. As in (2.4.20), we write

$$\phi_k(t,v)\mu^{in}(v) = \phi^1_k(t,v)\mu^{in}(v) + \phi^2_k(t,v)\mu^{in}(v) + \phi^3_k(t,v)\mu^{in}(v). \tag{2.4.22}$$

Step 1: analysis of T_1^2. We first analyze $\phi_k^2(t,v)\mu^{in}(v)$. For T_1^2 defined in (2.4.21), it follows from Theorem 2.3.13 that there exist functions $A_{k,\epsilon}^{j,\iota}(v,w)$, $1 \le j \le 4$, and a remaining term $R_{k,\epsilon}^{\iota}(v,w)$ such that

$$\partial_v^2 \Theta_{k,\epsilon}^{2,\iota}(v,w) = \frac{A_{k,\epsilon}^{1,\iota}(v,w)}{v+i\iota\epsilon} + \sum_{j=2}^{4} A_{k,\epsilon}^{j,\iota}(v,w)\log^{j-1}(v+i\iota\epsilon) + R_{k,\epsilon}^{\iota}(v,w), \tag{2.4.23}$$

and for any integer $m \ge 0$

$$\left\|\partial_w^m A_{k,\epsilon}^{1,\iota}(v,w)\right\|_{L_w^2 H_{k,v}^1} \lesssim \left\|f_0^k\right\|_{H_k^{2+m}}, \tag{2.4.24}$$

$$\sum_{j=2}^{4} \left\|\partial_w^m A_{k,\epsilon}^{j,\iota}(v,w)\right\|_{L_w^2 H_{k,v}^1} + \left\|\partial_w^m R_{k,\epsilon}^{\iota}(v,w)\right\|_{L_w^2 H_{k,v}^1} \lesssim \left\|f_0^k\right\|_{H_k^{3+m}}. \tag{2.4.25}$$

By Lemma 2.4.1, there exists a function $\varpi_{2,1}^{\iota}(t,v,k)$ such that

$$\lim_{\epsilon\to 0+} \int_{\mathbb{R}} e^{-ikwt} A_{k,\epsilon}^{1,\iota}(v-w,w)\frac{1}{v-w+i\iota\epsilon}dw = e^{-ikvt}\varpi_{2,1}^{\iota}(t,v,k), \tag{2.4.26}$$

and for any $\delta > 0$

$$\|\partial_v^m \varpi_2^{\iota}(t,\cdot,k)\|_{L^2} \lesssim_{m,\delta} \left\|A_{k,\epsilon}^{\iota}(v,w)\right\|_{H_v^{1/2+\delta}H_w^m} \lesssim_m \left\|f_0^k\right\|_{H_k^{m+2}}. \tag{2.4.27}$$

Set

$$\widetilde{R}_{k,\epsilon}^{\iota}(v,w) = \sum_{j=2}^{4} A_{k,\epsilon}^{j,\iota}(v,w)\log^{j-1}(v+i\iota\epsilon) + R_{k,\epsilon}^{\iota}(v,w). \tag{2.4.28}$$

It follows from (2.4.24) and (2.4.25) that

$$\left\|\partial_w^m \widetilde{R}_{k,\epsilon}^{\iota}(v,w)\right\|_{L_w^2 L_v^2} \lesssim_m \left\|f_0^k\right\|_{H_k^{m+3}}. \tag{2.4.29}$$

Therefore, by change of variable $\tilde{w} = v - w$, we have

$$\begin{aligned}\lim_{\epsilon\to 0+} \int_{\mathbb{R}} e^{-ikwt}\widetilde{R}_{k,\epsilon}^{\iota}(v-w,w)dw = e^{-ikvt} \lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{ik\tilde{w}t}\widetilde{R}_{k,\epsilon}^{\iota}(\tilde{w},v-\tilde{w})d\tilde{w} \\ := e^{-ikvt}\varpi_{2,2}^{\iota}(t,v,k),\end{aligned} \tag{2.4.30}$$

and for any $m \ge 0$,

$$\left\|\partial_v^m \varpi_{2,2}^{\iota}(t,\cdot,k)\right\|_{L^2(\mathbb{R})} \lesssim_m \left\|f_0^k\right\|_{H_k^{3+m}(\mathbb{R})}. \tag{2.4.31}$$

Step 2: analysis of T_3^2. For T_3^2, $\Theta_{k,\epsilon}^{2,\iota}(v,w)$ is smooth in w since w is supported away from the boundary. We have by change of variable $\tilde{w} = v - w$

$$
\begin{aligned}
&T_3^2 \mu^{in}(v) \\
=&\frac{1}{2\pi i(kt)^2} \lim_{\epsilon\to 0+} \int_{\mathbb{R}} e^{-ikwt}\Big[\partial_w^2 \Theta_{k,\epsilon}^{2,-}(v-w,w) - \partial_w^2 \Theta_{k,\epsilon}^{2,+}(v-w,w)\Big]\mu^{in}(v)dw \\
=&\frac{e^{-ikvt}}{2\pi i(kt)^2} \lim_{\epsilon\to 0+} \int_{\mathbb{R}} e^{ik\tilde{w}t}\Big[\partial_w^2 \Theta_{k,\epsilon}^{2,-}(\tilde{w},v-\tilde{w}) - \partial_w^2 \Theta_{k,\epsilon}^{2,+}(\tilde{w},v-\tilde{w})\Big]\mu^{in}(v)d\tilde{w}.
\end{aligned}
\tag{2.4.32}
$$

Now $v - \bar{w}$ is bounded away from $b(0)$ and $b(1)$, thus $\partial_w^2 \Theta_{k,\epsilon}^{2,\iota}(\bar{w}, v-\bar{w})$ is smooth in v. It follows from Theorem 2.3.13 that there exists a function $\varpi_{2,3}^{\iota}(t,v,k)$ such that

$$T_3^2 \mu^{in}(v) = \frac{1}{2\pi i}\frac{e^{-ikvt}}{k^2t^2}\varpi_{2,3}^{\iota}(t,v,k) \tag{2.4.33}$$

and for any integer $m \geq 0$

$$\left\|\partial_v^m \varpi_{2,3}^{\iota}(t,\cdot,k)\right\|_{L^2} \lesssim_m \left\|f_0^k\right\|_{H_k^{m+3}}. \tag{2.4.34}$$

Step 3: analysis of T_2^2. For T_2^2, the singularity for $\partial_v\partial_w \Theta_{k,\epsilon}^{2,\iota}(v,w)$ is of $\log(v+i\iota\epsilon)$ type. Hence we can repeat the analysis of T_3^2 and deduce that there exist a function $\varpi_{2,4}^{\iota}(t,v,k)$ such that

$$T_2^2 \mu^{in}(v) = \frac{1}{2\pi i}\frac{e^{-ikvt}}{k^2t^2}\varpi_{2,4}^{\iota}(t,v,k) \tag{2.4.35}$$

and

$$\left\|\partial_v^m \varpi_{2,4}^{\iota}(t,\cdot,k)\right\|_{L^2} \lesssim_m \left\|f_0^k\right\|_{H_k^{m+3}}. \tag{2.4.36}$$

Set

$$\varpi_2(t,v,k) := \frac{1}{2\pi i}\sum_{j=1}^{4}\left[\varpi_{2,j}^{-}(t,v,k) - \varpi_{2,j}^{+}(t,v,k)\right], \tag{2.4.37}$$

and

$$\alpha^{in}(t,y,k) := \varpi_2(t,b(y),k). \tag{2.4.38}$$

We have

$$\psi_k^2(t,y)\chi^{in}(y) = \frac{e^{-ikb(y)t}}{k^2t^2}\alpha^{in}(t,y,k), \tag{2.4.39}$$

and for any integer $m \geq 0$,

$$\left\|\partial_y^m \alpha_k^{in}(t,\cdot,k)\right\|_{L^2} \lesssim_m \left\|f_0^k\right\|_{H_k^{m+3}(\mathbb{R})}. \tag{2.4.40}$$

Step 4: analysis of $\psi_k^1(t,y)\chi^{in}(y)$ and $\psi_k^3(t,y)\chi^{in}(y)$. By symmetry we only need to analyze $\psi_k^1(t,y)\chi^{in}(y)$. For $\Theta_{k,\epsilon}^{1,\iota}(v-w,w)$, w is supported close to $b(0)$ while v is supported away from the boundary. Hence $|v-w|>\delta_0$ for some $\delta_0>0$. It follows from Theorem 2.3.12 that $\Theta_{k,\epsilon}^{1,\iota}(v-w,w)$ is smooth in v, and

$$\psi_k^1(t,y)\chi^{in}(y)=\frac{e^{-ikb(0)t}}{k^2t^2}\beta^{in}(t,y,k), \tag{2.4.41}$$

with

$$\left\|\partial_y^m\beta^{in}(t,y,k)\right\|_{L^2}\lesssim_m\left\|f_0^k\right\|_{H_k^{m+3}} \tag{2.4.42}$$

for any integer $m\geq 0$. This finishes the proof of Theorem 1.2.1.

Proof of Theorem 1.2.3

For the variable $v=b(y)$, $y\in\mathbb{R}$, we define the cutoff function

$$\mu^{b0}(v):=\chi^{b0}(y).$$

Similar to the proof of Theorem 1.2.1, we write for $t\geq 0, v\in\mathbb{R}$

$$\phi_k(t,v)\mu^{b0}(v)=\big(\phi_k^1(t,v)+\phi_k^2(t,v)+\phi_k^3(t,v)\big)\mu^{b0}(v). \tag{2.4.43}$$

For $\phi_k^1(t,v)$, we apply integration by parts and get

$$\begin{aligned}&\phi_k^1(t,v)\\&=-\frac{1}{2\pi i}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}e^{-ikwt}\Big[\Theta_{k,\epsilon}^{1,-}(v-w,w)-\Theta_{k,\epsilon}^{1,+}(v-w,w)\Big]dw\\&=\frac{1}{2\pi(kt)^2 i}\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}e^{-ikwt}\frac{d^2}{dw^2}\Big[\Theta_{k,\epsilon}^{1,-}(v-w,w)-\Theta_{k,\epsilon}^{1,+}(v-w,w)\Big]dw\end{aligned} \tag{2.4.44}$$

It follows from Theorem 2.3.12 that

$$\begin{aligned}&\frac{d^2}{dw^2}\big[\Theta_{k,\epsilon}^{1,\iota}(v-w,w)\big]\\=\;&A_{k,\epsilon}^{\iota}(v-w,w)\frac{\Upsilon_1(w)}{v-w+i\iota\epsilon}+B_{k,\epsilon}^{\iota}(v-w,w)\frac{\Upsilon_1(w)}{b(0)-w+i\iota\epsilon}\\&+\mathcal{R}_{2,k,\epsilon}^{\iota}(v-w,w).\end{aligned} \tag{2.4.45}$$

Here $A^{\iota}_{k,\epsilon}(v,w)$ is given by (2.3.151) and $B^{\iota}_{k,\epsilon}(v,w)$ is given by (2.3.153). The cutoff function $\Upsilon_1(w)$ is defined in (2.3.36). $\mathcal{R}^{\iota}_{2,k,\epsilon}(v,w)$ can be written as

$$\begin{aligned}\mathcal{R}^{\iota}_{2,k,\epsilon}(v,w) = & \sum_{1\le p'+q'\le 3} M^{p',q',\iota}_{2,k,\epsilon}(v,w)\log^{p'}(v+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)\\ & + \widetilde{R}^{\iota}_{2,k,\epsilon}(v,w),\end{aligned}\tag{2.4.46}$$

where

$$\sum_{1\le p'+q'\le 3}\left\|M^{p',q',\iota}_{2,k,\epsilon}(v,w)\right\|_{L^2_w H^1_{k,v}} + \left\|\widetilde{R}^{\iota}_{2,k,\epsilon}(v,w)\right\|_{L^2_w H^1_{k,v}} \lesssim \left\|f^k_0\right\|_{H^3_k}.\tag{2.4.47}$$

By Theorem 2.3.12, $A^{\iota}_{k,\epsilon}(v,w)$ is in H^1_k for both v and w. It follows from Lemma 2.4.1 that there exist a function $\alpha^{\iota}_1(t,v,k)$ such that

$$\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}A^{\iota}_{k,\epsilon}(v-w,w)\frac{\Upsilon_1(w)\mu^{b0}(v)}{v-w+i\iota\epsilon}dw = e^{-ikvt}\alpha^{\iota}_1(t,v,k),\tag{2.4.48}$$

and

$$\|\alpha^{\iota}_1(t,\cdot,k)\|_{H^1_k} \lesssim \left\|f^k_0\right\|_{H^3_k}.\tag{2.4.49}$$

Similarly, we deduce from Lemma 2.4.2 that there exist a function $\beta^{\iota}_1(t,v,k)$ such that

$$\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}B^{\iota}_{k,\epsilon}(v-w,w)\frac{\Upsilon_1(w)\xi^{b0}(v)}{b(0)-w+i\iota\epsilon}dw = e^{-ikb(0)t}\beta^{\iota}_1(t,v,k),\tag{2.4.50}$$

and,

$$\|\beta^{\iota}_1(t,\cdot,k)\|_{H^1_k} \lesssim \left\|f^k_0\right\|_{H^3_k}.\tag{2.4.51}$$

Set

$$\alpha_1(t,y,k) = \frac{1}{2\pi i}(\alpha^-_1(t,v,k)-\alpha^+_1(t,v,k))\tag{2.4.52}$$

and

$$\beta_1(t,y,k) = \frac{1}{2\pi i}(\beta^-_1(t,v,k)-\beta^+_1(t,v,k)).\tag{2.4.53}$$

We have, with one more integration by part,

$$\begin{aligned}&\phi^1_k(t,v)\mu^{b0}(v)\\ =& \frac{1}{(kt)^2}\left[e^{-ikvt}\alpha_1(t,v,k)+e^{-ikb(0)t}\beta_1(t,v,k)\right]\\ &-\frac{1}{2\pi(kt)^3}\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt}\frac{d}{dw}\left[\mathcal{R}^-_{2,k,\epsilon}(v-w,w)-\mathcal{R}^+_{2,k,\epsilon}(v-w,w)\right]dw.\end{aligned}\tag{2.4.54}$$

For the remaining term $\mathcal{R}^{\iota}_{2,k,\epsilon}(v,w)$, it follows from (2.4.46) and Theorem 2.3.12 that

$$\begin{aligned}
&\frac{d}{dw}\mathcal{R}^{\iota}_{2,k,\epsilon}(v-w,w)\\
=&\sum_{0\le p'+q'\le 2} A^{\iota,2,p',q'}_{k,\epsilon}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{v-w+i\iota\epsilon}\\
&+\sum_{0\le p'+q'\le 2} B^{\iota,2,p',q'}_{k,\epsilon}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{b(0)-w+i\iota\epsilon}\\
&+\mathcal{R}^{\iota}_{3,k,\epsilon}(v-w,w).
\end{aligned} \tag{2.4.55}$$

In addition, we have

$$\sum_{0\le p'+q'\le 2}\left[\left\|A^{\iota,2,p',q'}_{k,\epsilon}(v,w)\right\|_{L^2_wH^1_{k,v}}+\left\|B^{\iota,2,p',q'}_{k,\epsilon}(v,w)\right\|_{L^2_wH^1_{k,v}}\right]\lesssim\left\|f^k_0\right\|_{H^3_k}. \tag{2.4.56}$$

The remaining term $\mathcal{R}^{\iota}_{3,k,\epsilon}(v-w,w)$ has singularities in the form of $\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)$ $(1\le p'+q'\le 5)$.

In general, we can repeat the process in (2.4.54) and take one more derivative for the remaining term for each step. We have for the n-th remaining term,

$$\begin{aligned}
&\frac{d}{dw}\mathcal{R}^{\iota}_{n,k,\epsilon}(v-w,w)\\
=&\sum_{0\le p'+q'\le 2(n-1)} A^{\iota,n,p',q'}_{k,\epsilon}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{v-w+i\iota\epsilon}\\
&+\sum_{0\le p'+q'\le 2(n-1)} B^{\iota,n,p',q'}_{k,\epsilon}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{b(0)-w+i\iota\epsilon}\\
&+\mathcal{R}^{\iota}_{n+1,k,\epsilon}(v-w,w),
\end{aligned} \tag{2.4.57}$$

and for $0\le p'+q'\le 2(n-1)$ the following estimate holds

$$\left\|A^{\iota,n,p',q'}_{k,\epsilon}(v,w)\right\|_{L^2_wH^1_{k,v}}+\left\|B^{\iota,n,p',q'}_{k,\epsilon}(v,w)\right\|_{L^2_wH^1_{k,v}}\lesssim\left\|f^k_0\right\|_{H^{n+1}_k}. \tag{2.4.58}$$

The appearance of log terms in the numerator makes the regularity different from (2.4.45). When $q'>0$, the term

$$\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{v-w+i\iota\epsilon}$$

indicates the interaction of interior singularities and boundary singularities. Using Lemma 2.4.1, we deduce that there exists a function $\alpha_n^{p',q',\iota}(t,v,k)$ such that

$$\begin{aligned}&\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt} A_{k,\epsilon}^{\iota,n,p',q'}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{v-w+i\iota\epsilon}\\ &\qquad\times\xi^{b0}(v)\Upsilon_1(w)dw\\ =\;& e^{-ikvt}\alpha_n^{p',q',\iota}(t,v,k).\end{aligned} \tag{2.4.59}$$

To study the regularity of $\alpha_n^{p',q',\iota}(t,v,k)$, we consider the following two cases.

(a) There is no interaction between interior singularities and boundary singularities. In this case, $q'=0$. Then it follows from Lemma 2.4.1 and Theorem 2.3.12 that for any $\delta>0$

$$\begin{aligned}\left\|\alpha_n^{p',0,\iota}(t,\cdot,k)\right\|_{H^{1-\delta}} &\lesssim_{\delta,p',n} (1+\log^{p'}\langle t\rangle)\left\|A_{k,\epsilon}^{\iota,n,p',0}(v,w)\right\|_{H^1_{k,v}H^1_{k,w}}\\ &\lesssim_{\delta,p',n} (1+\log^{p'}\langle t\rangle)\left\|f_0^k\right\|_{H_k^{n+2}}.\end{aligned} \tag{2.4.60}$$

(b) There exists interaction between interior singularities and boundary singularities. In this case, $q'>0$ and it follows from Lemma A.1.1 that $A_{k,\epsilon}^{\iota,n,p',q'}(v,w)\log^{q'}(b(0)-w+i\iota\epsilon)$ is only $H^{1/2-}$ in the variable w. Therefore, Lemma 2.4.1 and Theorem 2.3.12 implies that

$$\begin{aligned}\left\|\alpha_n^{p',q',\iota}(t,\cdot,k)\right\|_{L^2} &\lesssim_{p',n} (1+\log^{p'}\langle t\rangle)\left\|A_{k,\epsilon}^{\iota,n,p',q'}(v,w)\right\|_{H^1_{k,v}H^1_{k,w}}\\ &\lesssim_{p',q',n} (1+\log^{p'}\langle t\rangle)\left\|f_0^k\right\|_{H_k^{n+2}}.\end{aligned} \tag{2.4.61}$$

Similarly, we have

$$\begin{aligned}&\lim_{\epsilon\to 0+}\int_{\mathbb{R}} e^{-ikwt} B_{k,\epsilon}^{\iota,n,p',q'}(v-w,w)\frac{\log^{p'}(v-w+i\iota\epsilon)\log^{q'}(b(0)-w+i\iota\epsilon)}{b(0)-w+i\iota\epsilon}dw\\ =&e^{-ikb(0)t}\beta_n^{p',q',\iota}(t,v,k).\end{aligned} \tag{2.4.62}$$

When $p'=0$, we have

$$\left\|\beta_n^{0,q',\iota}(t,\cdot,k)\right\|_{H^{1-\delta}} \lesssim_{\delta,q',n} (1+\log^{q'}\langle t\rangle)\left\|f_0^k\right\|_{H_k^{n+2}}. \tag{2.4.63}$$

When $p' > 0$, we have

$$\left\| \beta_n^{p',q',\iota}(t,\cdot,k) \right\|_{L^2} \lesssim_{p',q',n} (1+\log^{q'}\langle t\rangle) \left\| f_0^k \right\|_{H_k^{n+2}}. \tag{2.4.64}$$

For $2 \le n \le N-2$, set

$$\alpha_n(t,y,k) = \frac{1}{2\pi i^{n+2}} \sum_{0\le p'+q'\le n-1} \left[\alpha_n^{p',q',-}(t,b(y),k) - \alpha_n^{p',q',+}(t,b(y),k) \right] \tag{2.4.65}$$

and

$$\beta_n(t,y,k) = \frac{1}{2\pi i^{n+2}} \sum_{0\le p'+q'\le n-1} \left[\beta_n^{p',q',-}(t,b(y),k) - \beta_n^{p',q',+}(t,b(y),k) \right]. \tag{2.4.66}$$

We have

$$\|\alpha_n(t,\cdot,k)\|_{L^2} + \|\beta_n(t,\cdot,k)\|_{L^2} \lesssim (1+\log^{2(n-1)}\langle t\rangle) \left\| f_0^k \right\|_{H_k^{n+2}} \tag{2.4.67}$$

and

$$\begin{aligned} &\psi_k^1(t,y)\chi^{b0}(y) \\ = \;& e^{-ikb(y)t} \sum_{n=1}^{N-2} \frac{\alpha_n(t,y,k)}{(kt)^{n+1}} + e^{-ikb(0)t} \sum_{n=1}^{N-2} \frac{\beta_n(t,y,k)}{(kt)^{n+1}} + \frac{R_{N-2}(t,y,k)}{(kt)^{N-1}}. \end{aligned} \tag{2.4.68}$$

Here

$$R_{N-2}(t,y,k) = \lim_{\epsilon\to 0+} \frac{1}{2\pi i^N} \int_{\mathbb{R}} e^{-ikwt} \left[\mathcal{R}^-_{N-1,k,\epsilon}(b(y)-w,w) - \mathcal{R}^+_{N-1,k,\epsilon}(b(y)-w,w) \right] dw. \tag{2.4.69}$$

Noting that $\mathcal{R}^\iota_{N-1,k,\epsilon}(v-w,w)$ only has log-type singularity, we have

$$\|R_{N-2}(t,\cdot,k)\|_{H_k^1} \lesssim_N (1+\log^{2(N-3)}\langle t\rangle) \left\| f_0^k \right\|_{H_k^N}. \tag{2.4.70}$$

For $\psi_k^2(t,y)\chi^{b0}(y)$ and $\psi_k^3(t,y)\chi^{b0}(y)$, decomposition (2.4.68) still holds since we can keep taking integration by parts and applying Lemma 2.4.1 and Lemma 2.4.2. But in this case, v and w are supported on non-overlapping intervals. From the proof of Theorem 1.2.1, we know that $\psi_k^2(t,y)\chi^{b0}(y)$ and $\psi_k^3(t,y)\chi^{b0}(y)$ are smooth in y. Therefore, when y is near the boundary 0, only $\psi_k^1(t,y)$ contributes singularity in y.

If we further assume that $\omega_0(0) = 0$, that is, $f_0^k(b(0)) = 0$ after change of variables, then from Proposition 2.3.15 we know that $\partial_w \Theta_{k,\epsilon}^{1,\iota}(v,w)$ is in H_k^1 for both variables v and w. In addition, the major term $B_{k,\epsilon}^{\iota} = 0$ in (2.4.45). In this situation, we have

$$\begin{aligned}
&\frac{d}{dw}\left[\Theta_{k,\epsilon}^{1,\iota}(v-w,w)\right] \\
&= 2\Psi_k(v)\frac{\partial_v B(v)}{B(v)}\Theta_{k,\epsilon}^{1,\iota}(v-w,w)\log(v-w+i\epsilon) + R_{k,\epsilon}(v-w,w),
\end{aligned} \tag{2.4.71}$$

and $R_{k,\epsilon}(v,w)$ has the $\mathcal{F}_{1,1,k,\epsilon}^{b_0}(f_0^k, 2, N)$ type singularity structure. Now $\alpha_1^{0,0,\iota}(t,v,k)$ is given by

$$\alpha_1^{0,0,\iota}(t,v,k) = \lim_{\epsilon\to 0+} 2\Psi(v)\frac{\partial_v B(v)}{B(v)}\int_{\mathbb{R}} e^{-ikwt}\frac{\Theta_{k,\epsilon}^{1,\iota}(v-w,w)}{v-w+i\iota\epsilon}dw. \tag{2.4.72}$$

Now $\Theta_{k,\epsilon}^{1,\iota}(v,w)$ is H_k^1 in v and H_k^2 in w. It follows from Lemma 2.4.1 that

$$\left\|\alpha_1^{0,0,\iota}(t,\cdot,k)\right\|_{H_k^2} \lesssim \left\|f_0^k\right\|_{H_k^4}. \tag{2.4.73}$$

$\beta_1^{0,0,\iota}(t,v,k) = 0$ since the major term $B_{k,\epsilon}^{\iota} = 0$.

If we take two more derivatives in w to the equation (2.4.71), we observe that there will be no interaction of log singularities. Therefore, α_2 and β_2 are in H_k^1.

Chapter 3

Local Well-posedness of the De Gregorio

3.1 A few facts on Besov spaces and some notations

We first state some basic facts on Besov spaces $B^s_{p,r}(\mathcal{S}^1)$ in the periodic setting. Periodic Besov spaces are defined in a similar way to the usual Besov spaces on $\mathbb{R}^1$, and they have similar properties. By standard Littlewood-Paley theory, there exist two non-negative smooth even functions φ and χ, supported on the annulus $\{\xi \in \mathbb{R} |\ 1/2 < |\xi| < 4/3\}$ and the ball $\{\xi \in \mathbb{R} |\ |\xi| < 3/4\}$ respectively, such that

$$\forall \xi \in \mathbb{R},\ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \tag{3.1.1}$$

$$\forall \xi \in \mathbb{R} \setminus \{0\},\ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1. \tag{3.1.2}$$

The *dyadic blocks* Δ_j are defined for integers $j \geq -1$ by

$$\Delta_{-1} u \stackrel{def}{=} \chi(D) u = \sum_{k \in \mathbb{Z}} \chi(k) \hat{u}(k) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) u(x-y) dy \tag{3.1.3}$$

and for $j \geq 0$

$$\Delta_j u \stackrel{def}{=} \varphi(2^{-j} D) u = \sum_{k \in \mathbb{Z}} \varphi(2^{-j} k) \hat{u}(k) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_j(y) u(x-y) dy, \tag{3.1.4}$$

where

$$g(x)=\sum_{k\in\mathbb{Z}}\chi(k)e^{ikx} \quad \text{and} \quad h_j(x)=\sum_{k\in\mathbb{Z}}\varphi(2^{-j}k)e^{ikx}. \tag{3.1.5}$$

From definition (3.1.3) to (3.1.5) and the fact that φ and χ are compactly supported, it follows that $\Delta_j\Delta_{j'}=0$ if $|j-j'|\geq 2$. The *low frequency cut-off operators* S_j is defined as

$$S_ju=\chi(2^{-j}D)u\overset{def}{=}\sum_{k\in\mathbb{Z}}\chi(2^{-j}k)\hat{u}(k)e^{ikx}=\frac{1}{2\pi}\int_{-\pi}^{\pi}g_j(y)u(x-y)dy, \tag{3.1.6}$$

where

$$g_j(x)=\sum_{k\in\mathbb{Z}}\chi(2^{-j}k)e^{ikx}. \tag{3.1.7}$$

Note that by (3.1.1) we have $\chi(0)=1$, thus we deduce from (3.1.7) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}g_j(x)dx=\chi(0)=1. \tag{3.1.8}$$

With operators defined above, we get the following *Littlewoold-Paley decomposition*:

$$u=\sum_{j\geq -1}\Delta_ju. \tag{3.1.9}$$

Definition 3.1.1. *For $s\in\mathbb{R},(p,r)\in[1,\infty]^2$ the Besov space $B^s_{p,r}(\mathcal{S}^1)$ consists of all distributions on $\mathcal{S}^1$ such that the following quantity is finite:*

$$\|u\|_{B^s_{p,r}(\mathcal{S}^1)}\overset{def}{=}\begin{cases}\left(\sum_{j\geq -1}(2^{js}\|\Delta_ju\|_{L^p})^r\right)^{\frac{1}{r}}, & r<\infty\\ \sup_{j\geq -1}2^{js}\|\Delta_ju\|_{L^p}, & r=\infty\end{cases}.$$

In this paper we will abbreviate $B^s_{p,r}(\mathcal{S}^1)$ as $B^s_{p,r}$. We have the following properties of periodic Besov spaces. The proof is quite similar to the non-periodic case and is given in the appendix.

Proposition 3.1.2. *The following properties hold:*

i) Generalized derivatives: Let $f\in C^\infty(\mathbb{R})$ be a homogeneous function of degree $m\in\mathbb{Z}\cap[0,\infty)$. There exists a constant C depending only on f such that $\|f(D)u\|_{B^{s-m}_{p,r}}\leq C\|u\|_{B^s_{p,r}}$.

ii) Fatou property. If the sequence $\{f_k\}_{k\in\mathcal{N}}$ is uniformly bounded in $B^s_{p,r}$ and converges *weakly in* $\mathcal{D}'$ *to* f, *then* $f \in B^s_{p,r}$ *and* $\|f\|_{B^s_{p,r}} \leq \liminf_{k\to\infty} \|f_k\|_{B^s_{p,r}}$. *Here* $\mathcal{D}'$ *is the space of distributions on* $\mathcal{S}^1$.

iii) Embedding property. Suppose $1 \leq p_1 \leq p_2 \leq \infty$ *and* $1 \leq r_1 \leq r_2 \leq \infty$. *Then for any real number* s, *the space* $B^s_{p_1,r_1}$ *is continuously embedded in* $B^{s-\frac{1}{p_1}+\frac{1}{p_2}}_{p_2,r_2}$. *If* p *is finite, then* $B^{\frac{1}{p}}_{p,1}$ *is continuously embedded into* L^∞. *In addition, for all* $q \in [1,\infty]$, *the space* L^q *is continuously embedded in the space* $B^0_{q,\infty}$.

We will also use the paradifferential calculus in the periodic setting. Operators are defined in a similar way to the non-periodic case.

Definition 3.1.3. *The paraproduct of periodic functions* u *and* v *is defined by*

$$T_u v \overset{def}{=} \sum_j S_{j-1} u \Delta_j v.$$

The remainder of u *and* v *is defined by*

$$R(u,v) \overset{def}{=} \sum_{|k-j|\leq 1} \Delta_k u \Delta_j v.$$

With operators defined above, the following *Bony decomposition* holds true:

$$uv = T_u v + T_v u + R(u,v). \tag{3.1.10}$$

We have the following results concerning the continuity of the paraproduct and remainder operators. The proof is very similar to the non-periodic case, and we give a brief proof in the appendix for completeness. One can refer to [BCD11] for a detailed representation on non-periodic paradifferential calculus.

Proposition 3.1.4. *i) For any real number* s *and any* $(p,r) \in [1,\infty]^2$ *there exists a constant* C *such that for any* (u,v) *in* $L^\infty \times B^s_{p,r}$,

$$\|T_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}.$$

Moreover, for any (s,t) *in* $\mathbb{R} \times (-\infty, 0)$ *and any* (p, r_1, r_2) *in* $[1,\infty]^3$, *we have, for any* $(u,v) \in B^t_{\infty,r_1} \times B^s_{p,r_2}$,

$$\|T_u v\|_{B^{s+t}_{p,r}} \leq C\|u\|_{B^t_{\infty,r_1}}\|v\|_{B^s_{p,r_2}} \quad \text{with} \quad \frac{1}{r} \overset{def}{=} \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}.$$

ii) Let (s_1, s_2) *be in* $\mathbb{R}^2$ *and* (p_1, p_2, r_1, r_2) *be in* $[1, \infty]^4$. *Assume that*

$$\frac{1}{p} \stackrel{def}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{ and } \quad \frac{1}{r} \stackrel{def}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$

If $s_1 + s_2$ *is positive, then there exists a constant* C *such that for any* (u, v) *in* $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(u, v)\|_{B^{s_1+s_2}_{p,r}} \leq C\|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$

If $r = 1$ *and* $s_1 + s_2 \geq 0$, *then for* (u, v) *in* $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$,

$$\|R(u, v)\|_{B^{s_1+s_2}_{p,\infty}} \leq C\|u\|_{B^{s_1}_{p_1,r_1}} \|v\|_{B^{s_2}_{p_2,r_2}}.$$

With Bony decomposition and Proposition 3.1.4, one can prove the following estimate for product of two functions in Besov spaces.

Corollary 3.1.5. *If* $(s, p, r) \in (0, \infty) \times [1, \infty]^2$ *satisfies* $s > 1/p$ *or* $s = 1/p$ *and* $r = 1$, *then* $L^\infty \cap B^s_{p,r}$ *is an algebra. Moreover, there exists a constant* C *such that*

$$\|uv\|_{B^s_{p,r}} \leq C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\|v\|_{L^\infty}\big).$$

We will need the following lemma when tackling the uniqueness of De Gregorio equation in $B^{\frac{3}{2}}_{2,1}$.

Lemma 3.1.6. *i) There exists a constant* C *such that for any* $u \in B^{\frac{1}{2}}_{2,1}$, $v \in B^{-\frac{1}{2}}_{2,\infty}$, *we have*

$$\|uv\|_{B^{-\frac{1}{2}}_{2,\infty}} \leq C \|u\|_{B^{\frac{1}{2}}_{2,1}} \|v\|_{B^{-\frac{1}{2}}_{2,\infty}}.$$

ii) There exists a constant C *such that for any* $u \in B^{\frac{1}{2}}_{2,1}$, $v \in B^{\frac{1}{2}}_{2,\infty} \cap L^\infty$, *we have*

$$\|uv\|_{B^{\frac{1}{2}}_{2,\infty} \cap L^\infty} \leq C\|u\|_{B^{\frac{1}{2}}_{2,1}} \|v\|_{B^{\frac{1}{2}}_{2,\infty} \cap L^\infty}.$$

Proof. i) Note that $B^{\frac{1}{2}}_{2,1}$ is contained in the dual space of $B^{-\frac{1}{2}}_{2,\infty}$. Hence in order to prove the statement i) we only need to show

$$\|uw\|_{B^{\frac{1}{2}}_{2,1}} \leq C\|u\|_{B^{\frac{1}{2}}_{2,1}} \|w\|_{B^{\frac{1}{2}}_{2,1}}$$

for any $u, w \in B^{\frac{1}{2}}_{2,1}$, which follows directly from Corollary 3.1.5.
ii) Since $u \in B^{\frac{1}{2}}_{2,1}$ which is embedded into L^∞, we get $uv \in L^\infty$. Applying Bony decomposition, we write

$$uv = T_u v + T_v u + R(u, v).$$

It follows from Proposition 3.1.4 i) that

$$\|T_u v\|_{B^{\frac{1}{2}}_{2,\infty}} \leq C\|u\|_{L^\infty}\|v\|_{B^{\frac{1}{2}}_{2,\infty}} \leq C\|u\|_{B^{\frac{1}{2}}_{2,1}}\|v\|_{B^{\frac{1}{2}}_{2,\infty}},$$

and

$$\|T_v u\|_{B^{\frac{1}{2}}_{2,\infty}} \leq \|T_v u\|_{B^{\frac{1}{2}}_{2,\infty}} \leq C\|v\|_{L^\infty}\|u\|_{B^{\frac{1}{2}}_{2,1}}.$$

For the remainder term $R(u, v)$ we derive from Proposition 3.1.4 ii) that

$$\|R(u, v)\|_{B^1_{1,1}} \leq C\|u\|_{B^{\frac{1}{2}}_{2,1}}\|v\|_{B^{\frac{1}{2}}_{2,\infty}}.$$

Since $B^1_{1,1}$ is continuously embedded into $B^{\frac{1}{2}}_{2,\infty}$, we thus have completed the proof of ii). □

For convenience of later statements, it is helpful to define a bounded operator L on $B^s_{2,r}$ as

$$Lu \stackrel{def}{=} \mathcal{F}^{-1}(|\mathcal{F}(u)|),$$

that is, $\mathcal{F}(Lu)(k) = |\mathcal{F}(u)(k)|$ for every integer k. One can check by Definition 3.1.1 and Parseval's identity that for any $u \in B^s_{2,r}$, $\|Lu\|_{B^s_{2,r}} \leq C\|u\|_{B^s_{2,r}}$ for some independent constant C.

Notation. Throughout this paper, we use C to denote a constant which may change from line to line. We use L^p to denote $L^p(\mathcal{S}^1)$, the usual Lebesgue space on the circle. For a Banach space X, we denote by $L^r(0, T; X)$ the set of measurable functions on $(0, T)$ valued in X such that the map $t \mapsto \|u(t)\|_X$ belongs to $L^r(0, T)$. Similarly, we will use $C([0, T); X)$ as notation for the set of continuous functions on $[0, T)$ valued in X, and $C^1([0, T); X)$ for the functions u in $C([0, T); X)$ differentiable with respect to t such that $\partial_t u$ also belongs to $C([0, T); X)$.

3.2 Local well-posedness

In this section, we are going to prove the local well-posedness of (DG) with initial data u_0 in $B^{\frac{3}{2}}_{2,1}$. The idea is similar to [D+01], where the authors proved the local well-posedness of Camassa-Holm equation in non-periodic Besov spaces. We will first derive an a priori estimate in $B^s_{2,1}$ for linear transport equations. Next, we apply this estimate to construct a solution in $B^{\frac{3}{2}}_{2,1}$ by a standard iteration process. We then get uniqueness and continuous dependence on initial data by considering the estimate in a lower regularity space.

3.2.1 Linear transport equations

Consider the following linear transport equations:

$$\begin{cases} \partial_t f + v\partial_x f = F, \\ f|_{t=0} = f_0. \end{cases} \tag{LT}$$

The estimate of solutions to (LT) in periodic Besov spaces plays an important role in our proof of local well-posedness. Here the periodic function f is transported by the velocity field v with bounded derivative. We have the following Grownwall type estimate. The idea of proof originates from [Che98], where the author applied Bony decomposition to get the Hölder estimate of the linear transport system (LT).

Proposition 3.2.1. *Suppose that $s > 0, 1 \le r \le \infty$ and $r = 1$ if $s = \frac{3}{2}$. Let v be a function on $\mathcal{S}^1$ such that $\partial_x v$ belongs to $L^1(0,T; B^{\frac{1}{2}}_{2,1})$ if $s < 3/2$ or to $L^1(0,T; B^{s-1}_{2,1})$ if $s \ge 3/2$. Suppose also that $f_0 \in B^s_{2,r}, F \in L^1(0,T; B^s_{2,r})$ and that $f \in L^\infty(0,T; B^s_{2,r}) \cap C(0,T;\mathcal{D}')$ solves (LT). Then there exists a constant C depending only on s such that the following inequality holds:*

$$\|f(t)\|_{B^s_{2,r}} \le e^{CV(t)}\left(\|f_0\|_{B^s_{2,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{2,r}} d\tau\right), \tag{3.2.1}$$

where $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{\frac{1}{2}}_{2,1}} d\tau$ if $s < \frac{3}{2}$ and $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{s-1}_{2,1}} d\tau$ else. Furthermore, f belongs to $C([0,T]; B^s_{2,r})$.

Proof. Split (LT) in dyadic blocks according to Littlewood-Paley decomposition:

$$(\partial_t + v\partial_x)\Delta_j f = \Delta_j F + R_j, \quad \Delta_j f_{|t=0} = \Delta_j f_0, \tag{3.2.2}$$

where R_j is the remainder term $[v, \Delta_j]\partial_x f$. Using standard energy arguments and integration by parts, we get

$$\begin{aligned}\|\Delta_j f(t)\|_{L^2} \leq \int_0^t \Big(\|\Delta_j F(\tau)\|_{L^2} + \|R_j\|_{L^2} + \frac{1}{2}\|\partial_x v(\tau)\|_{L^\infty}\|\Delta_j f(\tau)\|_{L^2}\Big) d\tau \\ + \|\Delta_j f_0\|_{L^2}.\end{aligned} \tag{3.2.3}$$

Multiply both sides of (3.2.3) by 2^{js} and take the l^r norm. It follows that

$$\begin{aligned}\|f(t)\|_{B^s_{2,r}} \leq & \|f_0\|_{B^s_{2,r}} + \int_0^t \|F(\tau)\|_{B^s_{2,r}} d\tau \\ & + \int_0^t \Big(\sum_{j\in\mathbb{Z}} 2^{js}\|R_j(\tau)\|_{L^2} + \frac{1}{2}\|\partial_x v(\tau)\|_{L^\infty}\|f(\tau)\|_{B^s_{2,r}}\Big) d\tau.\end{aligned}$$

We claim that

$$\|R_j\|_{L^2} \leq C c_j 2^{-js}\|\partial_x v\|_{B^{\frac{1}{2}}_{2,1}}\|f\|_{B^s_{2,r}} \qquad \text{if} \qquad s < \frac{3}{2}, \tag{3.2.4}$$

and

$$\|R_j\|_{L^2} \leq C c_j 2^{-js}\|\partial_x v\|_{B^{s-1}_{2,1}}\|f\|_{B^s_{2,r}} \qquad \text{if} \qquad s > \frac{3}{2} \quad \text{or} \quad s = \frac{3}{2}, r = 1. \tag{3.2.5}$$

Here c_j are non-negative terms such that $\sum_j |c_j|^r = 1$. Assume first that the claim holds true, then we can get estimate (3.2.1) conveniently by Grownwall's lemma. Thus it suffices to prove the claim (3.2.4) and (3.2.5).

Similar to [D+01] and [Che98], we apply Bony decomposition to decompose R_j into five terms $R_j = \sum_{k=1}^5 R_j^k$ with $R_j^1 = T_v\Delta_j\partial_x f - \Delta_j T_v \partial_x f$, $R_j^2 = T_{\partial_x \Delta_j f} v$, $R_j^3 = -\Delta_j T_{\partial_x f} v$, $R_j^4 = \partial_x R(v, \Delta_j f) - \Delta_j \partial_x R(v, f)$, and $R_j^5 = \Delta_j R(\partial_x v, f) - R(\partial_x v, \Delta_j f)$. Here we used the fact that $R(v, \partial_x f) = \partial_x R(v, f) - R(\partial_x v, f)$. Let's prove the estimate (3.2.4) and (3.2.5) term by term.

By definition of the paraproduct and the fact that φ and χ are compactly supported, we have

$$R_j^1 = [T_v, \Delta_j]\partial_x f = \sum_{|j-j'|\leq 4} [\Delta_j, S_{j'-1}(v)]\Delta_{j'}\partial_x f.$$

For a periodic function u and $j \geq 0$, we derive from (3.1.4) that

$$[\Delta_j, S_{j'-1}(v)]u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_j(y)\Big(S_{j'-1}(v)(x-y) - S_{j'-1}(v)(x)\Big) u(x-y)dy. \tag{3.2.6}$$

Hence Lemma A.2.2 in the Appendix and Young's inequality yield that

$$\begin{aligned}\left\|[\Delta_j, S_{j'-1}(v)]u\right\|_{L^2} &\le C\left\|\partial_x v\right\|_{L^\infty}\left\|\int_{-\pi}^{\pi}|h_j(y)yu(x-y)|dy\right\|_{L^2}\\ &\le C\left\|\partial_x v\right\|_{L^\infty}\left\|u\right\|_{L^2}\int_{-\pi}^{\pi}|h_j(y)y|dy\\ &\le C2^{-j}\left\|\partial_x v\right\|_{L^\infty}\left\|u\right\|_{L^2}.\end{aligned}$$

Replacing u with $\Delta_{j'}\partial_x f(t)$ in (3.2.6), we get

$$\begin{aligned}\|[\Delta_j, S_{j'-1}(v)]\Delta_{j'}\partial_x f\|_{L^2} &\le C2^{-j}\|\Delta_{j'}\partial_x f(t)\|_{L^2}\|\partial_x v\|_{L^\infty}\\ &\le C\|\Delta_j f\|_{L^2}\|\partial_x v\|_{L^\infty}.\end{aligned}$$

For the last inequality we used the fact $|j-j'|\le 4$. For $j=-1$, the same estimate holds true since there are only finite low frequencies involved. Thus we complete the estimate of R_j^1.

Let's turn to the estimate of the second term. Since $S_{j'-1}\Delta_j = 0$ when $j'\le j$, we have

$$\begin{aligned}R_j^2 &= \sum_{j'\ge j+1} S_{j'-1}\Delta_j\partial_x f\Delta_{j'}v\\ &= \Delta_{j-1}\Delta_j\partial_x f\sum_{j'\ge j+1}\Delta_{j'}v + \Delta_j\Delta_j\partial_x f\sum_{j'\ge j+2}\Delta_{j'}v + \Delta_{j+1}\Delta_j\partial_x f\sum_{j'\ge j+3}\Delta_{j'}v.\end{aligned}$$

Hence

$$\left\|R_j^2\right\|_{L^2} \le \sum_{|k-j|\le 1}\left\|\Delta_k\Delta_j\partial_x f\right\|_{L^2}\left\|v - S_{k+2}v\right\|_{L^\infty}.$$

To bound $\|v-S_{k+2}v\|_{L^\infty}$, we first note that $\int_{-\pi}^{\pi} h_j(x)dx = 0$ since $\varphi(0)=0$, thus we deduce that

$$\begin{aligned}\|\Delta_k v\|_{L^\infty} &= \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} h_k(y)v(x-y)dy\right\|_{L^\infty}\\ &= \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} h_k(y)(v(x-y)-v(x))dy\right\|_{L^\infty}\\ &\le C\left\|\partial_x v\right\|_{L^\infty}\int_{-\pi}^{\pi}|h_k(y)y|dy\\ &\le C2^{-k}\left\|\partial_x v\right\|_{L^\infty}.\end{aligned}$$

Here we applied Lemma A.2.2 with $r = 1, m = 1$. Thus we have

$$\sum_{|k-j|\le 1} \|v - S_{k+2}v\|_{L^\infty} \le C2^{-j} \|\partial_x v\|_{L^\infty}.$$

Note that for $|k - j| \le 1$,

$$\|\Delta_k \Delta_j \partial_x f\|_{L^2} \le C \|\Delta_j \partial_x f\|_{L^2} \le C2^j \|\Delta_j f\|_{L^2}.$$

Therefore,

$$\|R_j^2\|_{L^2} \le C\|\Delta_j f\|_{L^2}\|\partial_x v\|_{L^\infty}$$

which completes the estimate of R_j^2.

Next let's tackle the estimate of R_j^3. Assume first $s > \frac{3}{2}$ or $s = \frac{3}{2}, r = 1$. By Proposition 3.1.4 i) we have

$$\begin{aligned}\|R_j^3\|_{L^2} &\le c_j 2^{-js}\|T_{\partial_x f} v\|_{B^s_{2,r}} \\ &\le Cc_j 2^{-js}\|\partial_x f\|_{L^\infty}\|v\|_{B^s_{2,r}} \\ &\le Cc_j 2^{-js}\|f\|_{B^s_{2,r}}\|v\|_{B^s_{2,1}}.\end{aligned}$$

When $0 < s < \frac{3}{2}$, we apply Proposition 3.1.4 i) and Proposition 3.1.2 iii) and deduce that

$$\begin{aligned}\|R_j^3\|_{L^2} &\le c_j 2^{-js}\|T_{\partial_x f} v\|_{B^s_{2,r}} \\ &\le Cc_j 2^{-js}\|\partial_x f\|_{B^{s-\frac{3}{2}}_{\infty,r}}\|v\|_{B^{\frac{3}{2}}_{2,\infty}} \\ &\le Cc_j 2^{-js}\|f\|_{B^s_{2,r}}\|v\|_{B^{\frac{3}{2}}_{2,1}}.\end{aligned}$$

R_j^4 and R_j^5 are relatively easy to estimate. For the term $\partial_x R(v, \Delta_j f)$, we deduce from Proposition 3.1.4 ii) that

$$\begin{aligned}\|\partial_x R(v, \Delta_j f)\|_{L^2} &\le C\|v\|_{B^1_{\infty,\infty}}\|\Delta_j f\|_{L^2} \\ &\le Cc_j 2^{-js}\|\partial_x v\|_{L^\infty}\|f\|_{B^s_{2,r}}.\end{aligned}$$

We used the fact that $L^\infty \hookrightarrow B^0_{\infty,\infty}$ in the last inequality. For the same reason,

$$\begin{aligned}\|\Delta_j \partial_x R(v, f)\|_{L^2} &\le Cc_j 2^{-js}\|R(v, \Delta_j f)\|_{B^{s+1}_{2,r}} \\ &\le Cc_j 2^{-js}\|v\|_{B^1_{\infty,\infty}}\|f\|_{B^s_{2,r}} \\ &\le Cc_j 2^{-js}\|\partial_x v\|_{L^\infty}\|f\|_{B^s_{2,r}}.\end{aligned}$$

Therefore, we get the estimate of the fourth term: $\|R_j^4\|_{L^2} \le Cc_j 2^{-js}\|\partial_x v\|_{L^\infty}\|f\|_{B^s_{2,r}}$.

For R_j^5, we apply again Proposition 3.1.4 ii) to get

$$\begin{aligned}\|\Delta_j R(\partial_x v, f)\|_{L^2} &\le Cc_j 2^{-js}\|R(\partial_x v, f)\|_{B^s_{2,r}}\\ &\le Cc_j 2^{-js}\|\partial_x v\|_{L^\infty}\|f\|_{B^s_{2,r}},\end{aligned}$$

and

$$\|R(\partial_x v, \Delta_j f)\|_{L^2} \le C\|\partial_x v\|_{L^\infty}\|\Delta_j f\|_{L^2},$$

Hence

$$\|R_j^5\|_{L^2} \le Cc_j 2^{-js}\|\partial_x v\|_{L^\infty}\|f\|_{B^s_{2,r}}.$$

Combining estimates of all five terms $R_j^k, 1 \le k \le 5$, we have proved claim (3.2.4) and (3.2.5) and thus the estimate (3.2.1). One may use exactly the same method as Proposition A.1 in [D+01] to get continuity in time of f. Thus we complete the proof of Proposition 3.2.1. □

3.2.2 Local existence and a blow-up criterion

In order to apply Proposition 3.2.1 to get local existence of $B^{\frac{3}{2}}_{2,1}$ solutions to (DG), we need to gain estimate of $\|B(u)\|_{B^{\frac{3}{2}}_{2,1}}$. Specifically, we have the following lemma.

Lemma 3.2.2. *Recall that the operator Λ is defined by $\Lambda = H\partial_x$ where H is the Hilbert transform. For any $u \in B^{\frac{3}{2}}_{2,1}$, $B(u) = uu_x - \Lambda^{-1}(u\Lambda u_x - u_x\Lambda u)$ also belongs to $B^{\frac{3}{2}}_{2,1}$, and there exists a constant C which is independent of u such that*

$$\|B(u)\|_{B^{\frac{3}{2}}_{2,1}} \le C\|u\|^2_{B^{\frac{3}{2}}_{2,1}}. \tag{3.2.7}$$

Proof. First note that by Proposition 3.1.2 i) and iii), we have $\Lambda^{-1}(u_x\Lambda u) \in B^{\frac{3}{2}}_{2,1}$. Thus it remains to prove that $uu_x - \Lambda^{-1}(u\Lambda u_x) \in B^{\frac{3}{2}}_{2,1}$, which is further reduced to prove that $E(u) \overset{def}{=} \Lambda(uu_x) - u\Lambda u_x \in B^{\frac{1}{2}}_{2,1}$. Taking Fourier transform of $E(u)$, we have

$$\begin{aligned}\mathcal{F}\big(E(u)\big)(k) &= -|k|\sum_{l\in\mathbb{Z}}\hat{u}(k-l)il\hat{u}(l) + \sum_{l\in\mathbb{Z}}\hat{u}(k-l)|l|il\hat{u}(l)\\ &= -i\sum_{l\in\mathbb{Z}}(|k|-|l|)\hat{u}(k-l)l\hat{u}(l).\end{aligned}$$

Thus

$$|\mathcal{F}\big(E(u)\big)(k)| \le \sum_{l\in\mathbb{Z}} |k-l||\hat{u}(k-l)|\cdot|l||\hat{w}(l)|. \tag{3.2.8}$$

Note that $|l||\hat{u}(l)| = \mathcal{F}(Lu_x)(l)$ by definition of operator L, thus from (3.2.8) we get

$$|\mathcal{F}\big(E(u)\big)(k)| \le \mathcal{F}\big((Lu_x)^2\big)(k). \tag{3.2.9}$$

Then for each dyadic block, Plancherel theorem yields that

$$\begin{aligned}\|\Delta_j E(u)\|_{L^2} &= \|\varphi(2^{-j}\cdot)\mathcal{F}(E(u))(\cdot)\|_{l^2}\\ &\le \|\varphi(2^{-j}\cdot)\mathcal{F}\big((Lu_x)^2\big)(\cdot)\|_{l^2}\\ &= \|\Delta_j\big((Lu_x)^2\big)\|_{L^2}.\end{aligned}$$

Since L is a bounded operator on $B^{\frac{1}{2}}_{2,1}$, we deduce that

$$\begin{aligned}\|E(u)\|_{B^{\frac{1}{2}}_{2,1}} &= \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\Delta_j E(u)\|_{L^2}\\ &\le \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\|\Delta_j\big((Lu_x)^2\big)\|_{L^2}\\ &= \|(Lu_x)^2\|_{B^{\frac{1}{2}}_{2,1}}\\ &\le C\|u\|^2_{B^{\frac{3}{2}}_{2,1}},\end{aligned}$$

which completes the proof of this lemma.

□

With Proposition 3.2.1 and Lemma 3.2.2, we can now prove the local existence of solutions to (DG) in $B^{\frac{3}{2}}_{2,1}$ with a standard iterative process. Let S_k be the lower frequency cutoff operators defined by (3.1.6). Let's construct a sequence of functions $\{u^k\}_{k\ge 0}$ in an iterative way. Start with $u^0 = 0$, let u^k be the solution to the following linear transport equation:

$$\begin{cases}\partial_t u^k + u^{k-1}\partial_x u^k = B(u^{k-1}),\\ u^k|_{t=0} = S_k(u_0).\end{cases} \tag{3.2.10}$$

By Proposition 3.2.1 and Lemma 3.2.2, there exists a constant C such that

$$\|u^k(t)\|_{B^{\frac{3}{2}}_{2,1}} \le e^{CV^{k-1}(t)}\bigg(\|S_k(u_0)\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t e^{-CV^{k-1}(\tau)}\|u^{k-1}(\tau)\|^2_{B^{\frac{3}{2}}_{2,1}}\,d\tau\bigg), \tag{3.2.11}$$

where $V^{k-1}(t) = \int_0^t \|u^{k-1}(s)\|_{B^{\frac{3}{2}}_{2,1}} ds$. Fix $T > 0$ such that $2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} T < 1$, we claim that for every $k \geq 0, t \in [0,T]$,

$$\|u^k(t)\|_{B^{\frac{3}{2}}_{2,1}} \leq \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{1 - 2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} t}. \tag{3.2.12}$$

Actually, assume the claim holds true for some $k \geq 0$, then

$$CV^k(t) \leq -\frac{1}{2}\ln(1 - 2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} t).$$

Using the fact that $\|S_k(u_0)\|_{B^{\frac{3}{2}}_{2,1}} \leq \|u_0\|_{B^{\frac{3}{2}}_{2,1}}$, we get

$$\begin{aligned}
\|u^{k+1}(t)\|_{B^{\frac{3}{2}}_{2,1}} &\leq e^{CV^{k-1}(t)}\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t e^{CV^{k-1}(t)-CV^{k-1}(\tau)}\|u(\tau)\|^2_{B^{\frac{3}{2}}_{2,1}} d\tau \\
&\leq \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{\sqrt{1 - 2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} t}} + \frac{1}{\sqrt{1 - 2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} t}} \int_0^t \frac{C\|u_0\|^2_{B^{\frac{3}{2}}_{2,1}}}{(1 - 2C\|u\|_{B^{\frac{3}{2}}_{2,1}} \tau)^{\frac{3}{2}}} d\tau \\
&= \frac{\|u_0\|_{B^{\frac{3}{2}}_{2,1}}}{1 - 2C\|u_0\|_{B^{\frac{3}{2}}_{2,1}} t}.
\end{aligned} \tag{3.2.13}$$

Note that (3.2.12) is true for $k = 0$, thus it holds for every $k \geq 0$ by induction. Therefore, the sequence $\{u^k\}$ is uniformly bounded in $C([0,T], B^{\frac{3}{2}}_{2,1})$ by Proposition 3.2.1. From equation (3.2.10) we know that $\partial_t u_k$ belongs to $B^{\frac{1}{2}}_{2,1}$, hence $\{u^k\}$ is uniformly bounded in $C([0,T], B^{\frac{3}{2}}_{2,1}) \cap C^1([0,T], B^{\frac{1}{2}}_{2,1})$. By Proposition 3.1.2 iii), $\{u^k\}$ and $\{\partial_t u^k\}$ are uniformly bounded in $C([0,T], L^2)$. Since embedding $B^{\frac{3}{2}}_{2,1} \cap L^2 \hookrightarrow L^2$ and $B^{\frac{1}{2}}_{2,1} \cap L^2 \hookrightarrow L^2$ are compact due to the compactness of $\mathcal{S}^1$, after passing to a subsequence u_k converges strongly to a limit u in $L^\infty([0,T], L^2)$. Then by Proposition 3.1.2 iii), $u \in L^\infty([0,T], B^{\frac{3}{2}}_{2,1})$. In fact, by interpolation u_k converges strongly to $L^\infty([0,T], B^s_{2,1})$ for every $0 < s < \frac{3}{2}$. Furthermore, $B(u_k)$ converges strongly to $B(u)$ in $L^\infty([0,T], B^{\frac{1}{2}}_{2,1})$, which also implies that $B(u) \in L^\infty([0,T], B^{\frac{3}{2}}_{2,1})$. This can be shown by inequality (3.2.18) in the next subsection. Therefore, u solves the De Gregorio equation (DG) with initial data u_0, and $u \in C([0,T], B^{\frac{3}{2}}_{2,1}) \cap C^1([0,T], B^{\frac{1}{2}}_{2,1})$ according to Proposition 3.2.1. We thus have completed proof of the local existence part of Theorem 1.3.1.

For the solution u we construct using the above method, the following estimate holds true

$$\|u(t)\|_{B^{\frac{3}{2}}_{2,1}} \le e^{CV(t)}\left(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + C\int_0^t e^{-CV(\tau)}\|u(\tau)\|^2_{B^{\frac{3}{2}}_{2,1}}\,d\tau\right). \tag{3.2.14}$$

Here $V(t) = \int_0^t \|u(\tau)\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau$. Thus Grownwall's lemma yields that

$$\|u(t)\|_{B^{\frac{3}{2}}_{2,1}} \le \|u_0\|_{B^{\frac{3}{2}}_{2,1}} e^{CV(t)}. \tag{3.2.15}$$

From this we can readily get the Beale-Kato-Majda blow-up criterion (1.3.4).

3.2.3 Uniqueness and continuous dependence on initial data

Assume $u, v \in C([0,T], B^{\frac{3}{2}}_{2,1}) \cap C^1([0,T], B^{\frac{1}{2}}_{2,1})$ are two solutions of (DG) with initial data u_0 and v_0 respectively. Let $w = u - v$, then w solves the following equation

$$\begin{cases} w_t + uw_x = \Big[\Lambda^{-1}\big(\Lambda(uw_x) - u\Lambda w_x + \Lambda(wv_x) - w\Lambda v_x + \Lambda wv_x + \Lambda uw_x\big) \\ \qquad\qquad\qquad -wv_x\Big], \\ w|_{t=0} = u_0 - v_0. \end{cases} \tag{3.2.16}$$

For simplicity of notation, let's denote the right-hand side of (3.2.16) by $M(u,v,w)$. We can not expect $M(u,v,w)$ belongs to $B^{\frac{3}{2}}_{2,1}$ since the nonlinearity causes a loss of derivative in this situation. One may want to work on $B^{\frac{1}{2}}_{2,1}$, but this space is the endpoint case for many estimates involving Besov norms. Instead, we work on a weaker space $B^{\frac{1}{2}}_{2,\infty} \cap L^\infty$. Note that in our setting w naturally belongs to $B^{\frac{1}{2}}_{2,\infty} \cap L^\infty$ since $w = u - v$ actually lies in $B^{\frac{3}{2}}_{2,1}$. We have the following lemma. The idea of proof is similar to that of Lemma 3.2.2.

Lemma 3.2.3. *There exists a constant C such that for every $u \in B^{\frac{3}{2}}_{2,1}, w \in B^{\frac{1}{2}}_{2,\infty}$ the following two estimates hold true*

$$\|\Lambda(uw_x) - u\Lambda w_x\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\,\|u\|_{B^{\frac{3}{2}}_{2,1}}\,\|w\|_{B^{\frac{1}{2}}_{2,\infty}},$$

$$\|\Lambda(wu_x) - w\Lambda u_x\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\,\|u\|_{B^{\frac{3}{2}}_{2,1}}\,\|w\|_{B^{\frac{1}{2}}_{2,\infty}}.$$

Proof. Taking Fourier transform of $\Lambda(uw_x) - u\Lambda w_x$, we have for every $k \in \mathbb{Z}$

$$\mathcal{F}(\Lambda(uw_x) - u\Lambda w_x)(k) = \sum_{l\in\mathbb{Z}} \Big(-|k|\hat{u}(l)i(k-l)\hat{w}(k-l) + \hat{u}(l)|k-l|i(k-l)\hat{w}(k-l)\Big).$$

Similar to the proof of Lemma 3.2.2, we get

$$|\mathcal{F}(\Lambda(uw_x) - u\Lambda w_x)(k)| \le \sum_{l\in\mathbb{Z}} |l\hat{u}(l)(k-l)\hat{w}(k-l)|,$$

which implies that

$$\|\mathcal{F}(\Lambda(uw_x) - u\Lambda w_x)(k)\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\,\|u_x w_x\|_{B^{-\frac{1}{2}}_{2,\infty}}$$

by similar arguments used in the proof of Lemma 3.2.2. Note that by assumption $u_x \in B^{\frac{1}{2}}_{2,1}, w_x \in B^{-\frac{1}{2}}_{2,\infty}$, then we deduce by Lemma 3.1.6 that

$$\|\mathcal{F}(\Lambda(uw_x) - u\Lambda w_x)(k)\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\,\|u\|_{B^{\frac{3}{2}}_{2,1}}\,\|w\|_{B^{\frac{1}{2}}_{2,\infty}}.$$

The second estimate follows from exactly the same argument.

□

Note that by Lemma 3.1.6,

$$\|\Lambda w v_x + \Lambda u w_x\|_{B^{-\frac{1}{2}}_{2,\infty}} \le C\,\|w\|_{B^{\frac{1}{2}}_{2,\infty}}\,(\|u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{3}{2}}_{2,1}}),$$

and

$$\|wv_x\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \le C\|w\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\|v\|_{B^{\frac{3}{2}}_{2,1}}.$$

Combining this with Lemma 3.2.3 we get estimate

$$\|M(u,v,w)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \le C\,\|w\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\,(\|u\|_{B^{\frac{3}{2}}_{2,1}} + \|v\|_{B^{\frac{3}{2}}_{2,1}}). \tag{3.2.17}$$

Estimate (3.2.17) allows us to apply Proposition 3.2.1 to equation (3.2.16) with $s = 1/2, r = \infty$ and get

$$\begin{aligned}\|w(t,\cdot)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \le & e^{CU(t)}\Big[C\int_0^t e^{-CU(\tau)}\|w(\tau,\cdot)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\big(\|u(\tau,\cdot)\|_{B^{\frac{3}{2}}_{2,1}} + \|v(\tau,\cdot)\|_{B^{\frac{3}{2}}_{2,1}}\big)\\ & + \|w_0\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty}\Big],\end{aligned}$$

where $U(t) = \int_0^t \|u(\tau)\|_{B^{\frac{3}{2}}_{2,1}} d\tau$. Grownwall's lemma yields that

$$\|w(t,\cdot)\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} \le \|w_0\|_{B^{\frac{1}{2}}_{2,\infty}\cap L^\infty} e^{CW(t)}, \tag{3.2.18}$$

where $W(t) = \int_0^t (\|u(\tau,\cdot)\|_{B^{\frac{3}{2}}_{2,1}} + \|v(\tau,\cdot)\|_{B^{\frac{3}{2}}_{2,1}}) d\tau$. From estimate (3.2.18) we get the desired result of uniqueness and continuous dependence on initial data. We remark that continuity with respect to initial data actually holds true in $C([0,T], B^s_{2,1}) \cap C^1([0,T], B^{s-1}_{2,1})$ for every $\frac{1}{2} \le s < \frac{3}{2}$ by an interpolation argument.

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Appendix A

Technical Lemmas and Proof

A.1 Singularity Structures

Lemma A.1.1. *Assume $f(x) \in H^1(\mathbb{R})$ and f is supported on $(-L, L)$ for some $L > 0$. For integer $m \geq 1$ and $\epsilon \in (-1, 1) \setminus \{0\}$, let*

$$h_{m,\epsilon}(x) = f(x) \log^m(x + i\epsilon). \tag{A.1.1}$$

We have the following decay estimate of the Fourier transform of $h_{m,\epsilon}$: for $\xi \in \mathbb{R}$,

$$|\widehat{h}_{m,\epsilon}(\xi)| \lesssim_L \|f\|_{H^1} \frac{1 + \log^{m-1}\langle\xi\rangle}{\langle\xi\rangle}. \tag{A.1.2}$$

Proof. Without loss of generality, assume $\epsilon > 0$. To make notations simple, set

$$u_\epsilon(x) := \log(x + i\epsilon). \tag{A.1.3}$$

Applying the residue theorem, we get

$$i\xi\mathcal{F}(u_\epsilon)(\xi) = \int_{\mathbb{R}} \frac{1}{x + i\epsilon} e^{-ix\xi} dx = -2\pi i e^{-\epsilon\xi} \mathbf{1}_{\xi>0}. \tag{A.1.4}$$

Hence for $|\xi| \geq 1$,

$$|\mathcal{F}(u_\epsilon)(\xi)| \lesssim \frac{1}{|\xi|}. \tag{A.1.5}$$

Let $\varphi(x)$ be a smooth function supported on $(-2L, 2L)$ such that $\varphi(x) = 1$ on $(-L, L)$. We first show that

$$|\mathcal{F}(\varphi u_\epsilon)(\xi)| \lesssim_L \frac{1}{\langle\xi\rangle}. \tag{A.1.6}$$

Let $\chi(x)$ be a smooth cutoff function supported on $(-2,2)$ such that $\chi(x)=1$ on $(-1,1)$. We have

$$\begin{aligned}\mathcal{F}(\varphi u_\epsilon) &= \int_{\mathbb{R}} \widehat{\varphi}(\xi-\eta)\chi(\eta)\widehat{u_\epsilon}(\eta)d\eta + \int_{\mathbb{R}} \widehat{\varphi}(\xi-\eta)(1-\chi(\eta))\widehat{u_\epsilon}(\eta)d\eta \\ &:= T_1 + T_2.\end{aligned} \tag{A.1.7}$$

Assume first $|\xi| \geq 10$. For T_1, since η is supported on $(-2,2)$, φ is a Schwartz function and $\widehat{u_\epsilon}$ is a tempered distribution, we have for any integer $N \geq 1$,

$$|T_1(\xi)| \lesssim_{N,L} |\xi|^{-N}. \tag{A.1.8}$$

For T_2, we have

$$\begin{aligned}|T_2(\xi)| &\lesssim \int_{|\eta|\geq 1} |\widehat{\varphi}(\xi-\eta)||\widehat{u_\epsilon}(\eta)|d\eta \\ &\lesssim \int_{|\eta|\geq 1} |\widehat{\varphi}(\xi-\eta)|\frac{1}{|\eta|}d\eta \\ &= \left(\int_{1\leq|\eta|\leq|\xi|/2} + \int_{|\eta|\geq|\xi|/2}\right)\frac{|\widehat{\varphi}(\xi-\eta)|}{|\eta|}d\eta \\ &:= T_{21} + T_{22}.\end{aligned} \tag{A.1.9}$$

For T_{21} we have

$$\begin{aligned}|T_{21}(\xi)| &\lesssim |\widehat{\varphi}(\xi/2)| \int_{1\leq|\eta|\leq|\xi|/2} \frac{1}{|\eta|}d\eta \\ &\lesssim \frac{1}{\langle\xi\rangle}.\end{aligned} \tag{A.1.10}$$

For T_{22} we have

$$\begin{aligned}|T_{22}(\xi)| &\lesssim \frac{1}{|\xi|}\int_{\mathbb{R}} |\widehat{\varphi}(\eta)|d\eta \\ &\lesssim \frac{1}{|\xi|}.\end{aligned} \tag{A.1.11}$$

For $|\xi| \leq 10$, we note that

$$|\mathcal{F}(\varphi u_\epsilon)(\xi)| \lesssim \|\varphi u_\epsilon\|_{L^1} \lesssim 1. \tag{A.1.12}$$

Hence for $\xi \in \mathbb{R}$, we have

$$|\mathcal{F}(\varphi u_\epsilon)(\xi)| \lesssim \frac{1}{\langle\xi\rangle}. \tag{A.1.13}$$

We next show that if $f \in H^1(\mathbb{R})$ and is supported on $(-L, L)$, we have

$$|\mathcal{F}(fu_\epsilon)(\xi)| \lesssim \frac{\|f\|_{H^1}}{\langle \xi \rangle}. \tag{A.1.14}$$

In fact,

$$\begin{aligned} |\mathcal{F}(fu_\epsilon))(\xi)| &= |\mathcal{F}(f\varphi u_\epsilon)(\xi)| \\ &= |\int_{\mathbb{R}} \widehat{f}(\xi - \eta)\widehat{\varphi u_\epsilon}(\eta)d\eta| \\ &\lesssim \|f\|_{H^1} \left(\int_{\mathbb{R}} \frac{1}{\langle \xi - \eta \rangle^2} \frac{1}{\langle \eta \rangle^2} d\eta \right)^{\frac{1}{2}} \\ &\lesssim \frac{\|f\|_{H^1}}{\langle \xi \rangle}. \end{aligned} \tag{A.1.15}$$

Assume now for some $m_0 \geq 1$, the estimate (A.1.2) holds for $m = m_0$. For the case $m = m_0 + 1$, we have

$$\begin{aligned} &|\mathcal{F}(f(x)\log^{m_0+1}(x + i\epsilon))(\xi)| \\ =& |\mathcal{F}(f(x)\log^{m_0}(x + i\epsilon)u_\epsilon(x))(\xi)| \\ =& |\int_{\mathbb{R}} \mathcal{F}\big(f(x)\log^{m_0}(x + i\epsilon)\big)(\xi - \eta) \cdot \widehat{u_\epsilon}(\eta)d\eta| \\ \lesssim_L& \|f\|_{H^1} \int_{\mathbb{R}} \frac{1 + \log^{m_0-1}\langle \xi - \eta \rangle}{\langle \xi - \eta \rangle} \frac{1}{\langle \eta \rangle} d\eta \\ =& \left(\int_{|\eta|<|\xi|/2} + \int_{|\xi|/2 \leq |\eta| \leq 2|\xi|} + \int_{|\eta|>2|\xi|} \right) \frac{1 + \log^{m_0-1}\langle \xi - \eta \rangle}{\langle \xi - \eta \rangle} \frac{1}{\langle \eta \rangle} d\eta \\ :=& T_3 + T_4 + T_5. \end{aligned} \tag{A.1.16}$$

For T_3, we have

$$\begin{aligned} |T_3| &\lesssim_L \|f\|_{H^1} \frac{1 + \log^{m_0-1}\langle \xi \rangle}{\langle \xi \rangle} \int_{|\eta|<|\xi|/2} \frac{1}{\langle \eta \rangle} d\eta \\ &\lesssim_L \|f\|_{H^1} \frac{1 + \log^{m_0}\langle \xi \rangle}{\langle \xi \rangle}. \end{aligned} \tag{A.1.17}$$

For T_4, we have

$$\begin{aligned} |T_4| &\lesssim_L \frac{\|f\|_{H^1}}{\langle \xi \rangle} \int_{\frac{|\xi|}{2} < |\eta| \leq 2|\xi|} \frac{1 + \log^{m_0-1}\langle \xi - \eta \rangle}{\langle \xi - \eta \rangle} d\eta \\ &\lesssim_L \frac{\|f\|_{H^1}}{\langle \xi \rangle} \int_{|\eta| \leq 3|\xi|} \frac{1 + \log^{m_0-1}\langle \eta \rangle}{\langle \eta \rangle} d\eta \\ &\lesssim_L \|f\|_{H^1} \frac{1 + \log^{m_0}\langle \xi \rangle}{\langle \xi \rangle}. \end{aligned} \tag{A.1.18}$$

For T_5, we have

$$\begin{aligned}|T_5| &\lesssim \|f\|_{H^1} \int_{|\eta|\geq 2|\xi|} \frac{1+\log^{m_0-1}\langle\eta\rangle}{\langle\eta\rangle^2} d\eta \\ &\lesssim \|f\|_{H^1} \frac{1+\log^{m_0-1}\langle\xi\rangle}{\langle\xi\rangle}.\end{aligned} \tag{A.1.19}$$

Therefore, the estimate (A.1.2) holds for the case $m = m_0 + 1$. This finishes the proof of the lemma. □

A.2 Besov Spaces

We mainly consider one-dimensional periodic Besov spaces, but statements in this section can be extended naturally to higher dimensions. Although Fourier transform on the circle does not enjoy the natural scaling property, we still have Bernstein type estimate from which we are able to get desired Sobolev embedding property. We will need the following lemma.

Lemma A.2.1. *Let $\beta(x)$ be a smooth function with supp$\beta \in]a, b[$, where $0 < a < b < \infty$. For every $j \in \mathcal{N}^+$, define a smooth function $d_j(x)$ on $[0, \pi]$ as*

$$d_j(x) = \sum_{k\in\mathcal{Z}} \beta(2^{-j}k)\cos(kx).$$

Then for any non-negative integer n, there exists a constant $C(n, \beta)$ which is independent of j and x, such that $\forall x \in (0, \pi]$

$$|d_j(x)| \leq C(n, \beta) 2^{-(n-1)j} x^{-n}. \tag{A.2.1}$$

Proof. If $n = 0$, (A.2.1) holds with constant $C = (b - a)\|\beta\|_{L^\infty}$. To study the case $n > 0$, first recall that for positive integers $l < m$ and $x \in (0, \pi]$

$$\sum_{k=l}^{m} \cos(kx) = \frac{\sin(m+\frac{1}{2})x - \sin(l-\frac{1}{2})x}{2\sin(\frac{x}{2})}, \tag{A.2.2}$$

and

$$\sum_{k=l}^{m} \sin(kx) = \frac{\cos(l-\frac{1}{2})x - \cos(m+\frac{1}{2})x}{2\sin(\frac{x}{2})}. \tag{A.2.3}$$

By definition,

$$d_j(x) = \sum_{k=[2^j a]+1}^{[2^j b]} \beta(2^{-j}k)\cos(kx),$$

where $[x]$ denotes the largest integer smaller than x.

Let

$$A_m^1 = \sum_{k=[2^j a]+1}^{m} \cos(kx) = \frac{\sin(m+\frac{1}{2})x - \sin([2^j a]+\frac{1}{2})x}{2\sin(\frac{x}{2})},$$

then by Abel's partial summation formula,

$$d_j(x) = A^1_{[2^j b]}\beta(2^{-j}[2^j b]) - \sum_{k=[2^j a]+1}^{[2^j b]-1} A_k^1 \left(\beta(2^{-j}(k+1)) - \beta(2^{-j}k)\right). \tag{A.2.4}$$

Since β is compactly supported on (a,b), there exists some $J > 0$ such that for every $j > J$, $\beta(2^{-j}[2^j b]) = \beta(2^{-j}([2^j a]+1)) = 0$. Hence we deduce from (A.2.4) that

$$d_j(x) = -\sum_{k=[2^j a]+1}^{[2^j b]-1} \frac{\sin(k+\frac{1}{2})x}{2\sin(\frac{x}{2})}\{\beta(2^{-j}(k+1)) - \beta(2^{-j}k)\}. \tag{A.2.5}$$

Note that

$$\sin(x) \geq \frac{1}{2}x, \qquad \forall x \in [0, \frac{\pi}{2}]$$

and

$$|\beta(2^{-j}(k+1)) - \beta(2^{-j}k)| \leq 2^{-j}\left\|\beta'\right\|_{L^\infty},$$

hence

$$\begin{aligned} |d_j(x)| &\leq (b-a)2^j \cdot \frac{4}{x} \cdot 2^{-j}\|\beta'\|_{L^\infty} \\ &= \frac{C}{x}, \end{aligned}$$

which proves (A.2.1) for $n = 1$.

The case $n = 2$ will be treated similarly. Let

$$A_m^2 = \sum_{k=[2^j a]+1}^{m} \frac{\sin(k+\frac{1}{2})x}{2\sin(\frac{x}{2})} = \frac{\cos([2^j a]+1)x - \cos(m+1)x}{4\sin^2(\frac{x}{2})}.$$

By (A.2.5) and Abel's summation formula we get

$$d_j(x) = \sum_{k=[2^j a]+1}^{[2^j b]-2} A_k^2 \{\beta(2^{-j}(k+2)) - 2\beta(2^{-j}(k+1)) + \beta(2^{-j}k)\}$$
$$= -\sum_{k=[2^j a]+1}^{[2^j b]-2} \frac{\cos(k+1)x}{4\sin^2(\frac{x}{2})} \{\beta(2^{-j}(k+2)) - 2\beta(2^{-j}(k+1)) + \beta(2^{-j}k)\}. \quad \text{(A.2.6)}$$

Here again we assume that j is large enough. Since the second order difference

$$|\beta(2^{-j}(k+2)) - 2\beta(2^{-j}(k+1)) + \beta(2^{-j}k)| \le 2^{-2j} \left\|\beta''\right\|_{L^\infty},$$

we get $|d_j(x)| \le C2^{-j}x^{-2}$ as desired. From the above proof we see that for arbitrary positive integer n, the boundary terms will vanish as long as $j > J(n)$ for some positive integer $J(n)$, and we will get a nth order difference of $\beta(2^{-j}\cdot)$ which gives us the desired decay in j. Thus the lemma is proved. □

Lemma A.2.2. *Suppose $d_j(x)$ satisfies estimate (A.2.1) on $]0,\pi[$. Then for every $1 \le r \le \infty$, $0 \le m < \infty$, there exists a constant $C(m,r)$ such that the following estimate holds:*

$$\|x^m d_j\|_{L^r(0,\pi)} \le C(m,r)2^{-j(m-1+\frac{1}{r})}. \quad \text{(A.2.7)}$$

Proof. In the case $r = 1$, we take $n = m+2$ in Lemma A.2.1 and get

$$|d_j(x)| \le C2^{-(m+1)j}x^{-(m+2)}.$$

Then

$$\|x^m d_j\|_{L^1} = \int_0^{2^{-j}\pi} |x^m d_j(x)| + \sum_{k=0}^{j-1} \int_{2^{-j+k}\pi}^{2^{-j+k+1}\pi} |x^m d_j(x)|$$
$$\le C2^j \int_0^{2^{-j}\pi} x^m + C2^{-(m+1)j} \sum_{k=0}^{j-1} \int_{2^{-j+k}\pi}^{2^{-j+k+1}\pi} \frac{1}{x^2}$$
$$\le C2^{-mj} + C2^{-(m+1)j} \sum_{k=0}^{j-1} 2^{j-k}$$
$$\le C2^{-mj}$$

as desired. Here in the second step we also use the fact $|d_j(x)| \le C2^j$.
For $r = \infty$, take $n = m$ in Lemma A.2.1 and we get

$$\|x^m d_j\|_{L^\infty} \le C2^{-(m-1)j}.$$

For $1 < r < \infty$, (A.2.7) follows from an interpolation argument. □

Remark A.2.3. *If smooth function $\beta(x)$ has compact support on $[0, b[$ and $\beta(0) = 1$, and $d_j(x)$ is defined exactly the same as Lemma A.2.1, then we have estimate*

$$d_j(x) \le C \min\{2^j, 1 + 2^{-j}x^{-2}\}.$$

The proof is exactly the same as Lemma A.2.1 but now the boundary term at 0 does not vanish. Hence we can also get

$$\|d_j\|_{L^1(0,\pi)} \le C$$

for some constant C independent of j.

Proof of Proposition 3.1.2. i) Note that $f(D)u(x) \overset{def}{=} \sum_k f(k)\hat{u}e^{ikx}$, thus assertion i) follows directly from Definition 3.1.1.
ii) For every $\eta \in C^\infty(\mathcal{S}^1)$, Parseval's identity yields that

$$\langle \Delta_j u_n,\ \eta\rangle = \langle u_n,\ \Delta_j \eta\rangle \longrightarrow \langle f,\ \Delta_j \eta\rangle = \langle \Delta_j f,\ \eta\rangle.$$

Thus $\Delta_j u_n \rightharpoonup \Delta_j f$, and

$$\|\Delta_j f\|_{L^p} \le \liminf_{n\to\infty} \|\Delta_j u_n\|_{L^p}.$$

Therefore, f also belongs to $B^s_{p,r}$ and

$$\|f\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

iii) Let $\beta(x)$ be a smooth even function supported on the annulus $\{\xi | 1/4 < |\xi| < 2\}$, and $\beta = 1$ on $\{\xi | 1/2 \le |\xi| \le 4/3\}$. By definition of the dyadic block, we have

$$\begin{aligned}\Delta_j u &= \sum_{k\in\mathcal{Z}} \beta(2^{-j}k)\varphi(2^{-j}k)\hat{u}(k)e^{ikx} \\ &= d_j * \Delta_j u.\end{aligned}$$

Here $d_j(x)=\sum_{k\in\mathcal{Z}}\beta(2^{-j}k)e^{ikx}=2\sum_{k\geq 0}\beta(2^{-j}k)\cos(kx)$. Hence for $p_2\geq p_1$, Young's inequality and Lemma A.2.2 yield that

$$\begin{aligned}\|\Delta_j u\|_{L^{p_2}} &\leq \|\Delta_j u\|_{L^{p_1}}\|d_j\|_{L^{\frac{p_1p_2}{p_1p_2+p_1-p_2}}}\\ &\leq C2^{j(\frac{1}{p_1}-\frac{1}{p_2})}\|\Delta_j u\|_{L^{p_1}}.\end{aligned}$$

Note that $l^{r_1}(\mathcal{Z})$ is continuously embedded into $l^{r_2}(\mathcal{Z})$ since $r_1\leq r_2$, thus the first assertion is proved. The second assertion follows from the first one and the fact that $u=\sum_j\Delta_j u$ for u satisfying condition (1.2.2). To prove the last assertion, we can write $\Delta_j u=h_j*u$ where $h_j(x)$ is defined by (3.1.5). Then the assertion follows from Young's inequality and Lemma A.2.2. □

Proof of Proposition 3.1.4. i) By definition, $\mathcal{F}(S_{j-1}u\Delta_j v)$ is supported on $2^j\mathcal{C}$, where annulus $\mathcal{C}=\{\xi\in\mathbb{R}|1/8<|\xi|<41/24\}$. Thus $\Delta_{j'}(S_{j-1}u\Delta_j u)$ vanishes when $|j-j'|>4$, and we only need to estimate $\|S_{j-1}u\Delta_j v\|_{L^p}$ for every $j\geq 0$. By definition,

$$S_{j-1}u=\chi(2^{-(j-1)}D)u=g_{j-1}*u,$$

where $g_j(x)=\sum_{k\in\mathcal{Z}}\chi(2^{-j}k)e^{ikx}$. Then by the remark of Lemma A.2.2, $\|g_j\|_{L^1}$ is bounded by some constant C independent of j. Hence Young's inequality yields that

$$\|S_{j-1}u\|_{L^\infty}\leq C\|u\|_{L^\infty},$$

from which we can easily conclude that $\|T_u v\|_{B^s_{p,r}}\leq C\|u\|_{L^\infty}\|v\|_{B^s_{p,r}}$.
Furthermore, if $u\in B^t_{\infty,r_1}$ with $t<0$, we have

$$\Big(\sum_{j\geq 1}2^{jtr_1}\|S_{j-1}u\|_{L^\infty}\Big)^{1/r_1}\leq C(1+\frac{1}{|t|})\|u\|_{B^t_{\infty,r_1}}. \tag{A.2.8}$$

Actually, we can write

$$\begin{aligned}2^{jt}\|S_{j-1}u\|_{L^\infty} &\leq 2^{jt}\sum_{j'\leq j-1}\|\Delta_{j'}u\|_{L^\infty}\\ &\leq \sum_{j'\leq j-1}2^{(j-j')t}2^{j't}\|\Delta_{j'}u\|_{L^\infty}.\end{aligned}$$

Since t is negative, (A.2.8) follows from Young's inequality for series convolution.
ii) The proof of these two estimates is exactly the same as the non-periodic case, thus we omit it. One can refer to Theorem 2.52 of [BCD11] for detailed proof. □