Asymptotics for the biharmonic equation near the tip of a crack

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1 Introduction

A mathematical model of a crack in a 2-dimensional uniform elastic medium occupying a bounded domain Ω consists of the following system (see [9]):

$$\varphi \in H^2(\Omega), \tag{1.1}$$

$$\Delta^2\varphi = 0 \quad \text{in } \Omega \setminus \Gamma(t), \tag{1.2}$$

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \quad \text{from both sides of } \Gamma(t), \tag{1.3}$$

$$\varphi = g, \quad \frac{\partial \varphi}{\partial n} = h \quad \text{on } \partial\Omega, , \tag{1.4}$$

where $\varphi = \varphi(x) = \varphi(x_1, x_2)$ is the stress function. Here Γ is the crack which, for simplicity, we shall take to be a curve of the form

$$x_2 = f(x_1), \quad -1 \leqslant x_1 \leqslant 0 \tag{1.5}$$

contained in Ω except for its initial point $(-1, f(-1))$, which lies on $\partial\Omega$; we shall also assume, for simplicity, that

$$f(0) = 0, \quad f'(0) = 0. \tag{1.6}$$

We are interested in the behavior of φ near the origin.

From basic work by Kondratév and Oleinik [14] [15] it follows that if f is in $C^1[-\delta_0, 0]$ for some $\delta_0 > 0$, then

$$\varphi \in C^{3/2} \quad \text{near the origin}, \tag{1.7}$$

and

$$|\varphi(x)| \leqslant C|x|^{3/2}, \tag{1.8}$$

$$|\nabla\varphi(x)| \leqslant C|x|^{1/2}. \tag{1.9}$$

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On the other hand if $f \in C^\infty[-\delta_0, 0]$ then, by [5] (and some of the papers referenced therein), φ has an asymptotic expansion near the origin. In the special case where

$$f(x_1) \equiv 0 \quad \text{for } -\delta_0 \leqslant x_1 \leqslant 0$$

the expansion is given explicitly [22] (see also [6; §16] [8; Part II, Chap. 7]) by

$$\begin{aligned}\varphi(r,\theta) \;&=\; \sum_{k=1}^{\infty} r^{k/2+1}\Big[a_k \cos\Big(\frac{k}{2}+1\Big)\theta + b_k \cos\Big(\frac{k}{2}-1\Big)\theta \\ &\quad + c_k \sin\Big(\frac{k}{2}-1\Big)\theta + d_k \sin\Big(\frac{k}{2}+1\Big)\theta\Big],\end{aligned} \tag{1.10}$$

where

$$\delta_0 b_{k+2} + a_k = \frac{2}{\pi\delta_0^{k/2+1}} \int_{-\pi}^{\pi} \psi(\delta_0,\theta)\cos\Big(\frac{k}{2}+1\Big)\theta d\theta \quad \text{for } k \geqslant 2,$$

$$\delta_0\Big(\frac{k}{2}+2\Big)b_{k+2} + \Big(\frac{k}{2}+1\Big)a_k = \frac{2}{\pi\delta_0^{k/2}} \int_{-\pi}^{\pi} \psi_r(\delta_0,\theta)\cos\Big(\frac{k}{2}+1\Big)\theta d\theta \quad \text{for } k \geqslant 2$$

(the formulae for b_1, b_2, b_3, a_1 are little different) and similar relations hold for c_k, d_k. Hence we get

$$|a_k| + |b_k| \leqslant \frac{C}{\delta_0^k} \int_{-\pi}^{\pi} \{|\psi(\delta_0,\theta)| + |\psi_r(\delta_0,\theta)|\} d\theta,$$

and the same inequality holds for $|c_k| + |d_k|$. It follows that the series (1.10) is uniformly convergent for $0 \leqslant r \leqslant \theta\delta_0$, for any $\theta < 1$.

From (1.10) we get

$$\begin{aligned}\varphi(r,\theta) \;&=\; A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + A_3 r^2 B_3(\theta) \\ &\quad + A_4 r^{5/2} B_4(\theta) + A_5 r^{5/2} B_5(\theta) + O(r^3),\end{aligned} \tag{1.11}$$

where

$$\begin{aligned}&B_1(\theta) = \cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta, \quad B_2(\theta) = \sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta, \quad B_3(\theta) = \sin^2\theta, \\ &B_4(\theta) = \cos\frac{5}{2}\theta - 5\cos\frac{1}{2}\theta, \quad B_5(\theta) = \sin\frac{5}{2}\theta - \sin\frac{1}{2}\theta;\end{aligned} \tag{1.12}$$

note that $r^2 B_3(\theta) = x_2^2$.

The main purpose of this paper is to establish asymptotic expansions of up to order $r^{3-\eta}$, under very weak assumptions on the regularity of $f(x_1)$. Our results are:

(i) If $f \in C^{1+\alpha}[-\delta_0, 0]$, then

$$\varphi(x) = A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + O(r^{2-\eta}) \tag{1.13}$$

for any $\eta > 0$ such that $\alpha + \eta > 1/2$.

(ii) If $f \in C^{2+\alpha}[-\delta_0, 0]$, then

$$\begin{aligned}\varphi(r,\theta) &= A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + A_3 r^2 B_3(\theta) + A_4 r^{5/2} B_4(\theta) \\ &\quad + A_5 r^{5/2} B_5(\theta) - 2A_2 r^{5/2} f''(0) \cos \frac{1}{2}\theta + O(r^{3-\eta})\end{aligned} \tag{1.14}$$

for any $\eta > 0$ such that $\alpha + \eta > 1/2$.

Note that if $f''(0) = 0$, then the expansion (1.14) agrees with that of (1.11).

The first two terms in the above expansion are called, in fractural mechanics, the *stress intensity factors*; or mode I and mode II of fracture [16; p. 24].

The proof of the estimates (i), (ii) require maximum principles for biharmonic solutions in a domain whose boundary has a singular point. Such estimates are established in §2; for related results see Remark 2.3. In §3 we give a proof of (i) and, in §5, a proof of (ii). In §4 we derive an additional regularity result for φ, near the tip $O = (0,0)$, for f in $C^{1+\alpha}$, namely:

$$\frac{\varphi(x)}{|x|^{3/2}} \leqslant C\Big(|x|^{2\alpha} + \frac{d^2(x)}{|x|^2}\Big) \quad \text{near the origin,} \tag{1.15}$$

where $d(x) = \operatorname{dist}(x, \Gamma)$. This improves the inequality that can be obtained by sub-Schauder $(2-\eta)$-estimates (stated in §9, Example 2), if $|x|^{\alpha+\eta'} < d(x)/|x| \ll 1$ for some $\eta' > 0$.

The remaining part of the paper is concerned with an application of some of the above results to the crack propagation problem, where the tip of the crack is moving in time according to (see [9])

$$\dot{X}(t) = h(|J(X(t))|)|J(X(t))|; \tag{1.16}$$

where $X(t)$ is the tip of the crack at time t, $h(s)$ is a given function, and $J(X(t))$ is the limit of J-integrals taken along circles that shrink to $X(t)$. The model is described in §6. In §7 we prove that the crack propagation problem with $C^{1+\alpha}$ crack is equivalent to the following geometric problem:

Find an extension $x_2 = f(x_1), -1 \leqslant x_1 \leqslant \tau$ $(\tau > 0)$ of Γ which is $C^{1+\alpha}$ such that, at each intermediate value $x_1 = s$, the coefficient $A_2 = A_2(s)$ in the asymptotic expansion about the tip $X(s) = (s, f(s))$ satisfies:

$$A_2(s) \equiv 0. \tag{1.17}$$

In §8 we make a few comments on this problem, which we hope to pursue in a future work. The paper concludes with an appendix in which we have assembled several sub-Schauder estimates used in this paper.

2 Maximum principles

In this section we establish estimates for solutions of $\Delta^2\varphi = f$ in a bounded domain Ω in terms of the supremum of the boundary values of φ and $\partial\varphi/\partial\nu$ on $\partial\Omega$. When $\partial\Omega$ is C^4 such an estimate is well known:

$$\|\varphi\|_{W^{1,\infty}(\Omega)} \leqslant C\Big(\|\varphi\|_{L^\infty(\partial\Omega)} + \|\nabla\varphi\|_{L^\infty(\partial\Omega)} + \|f\|_{L^1(\Omega)}\Big). \tag{2.1}$$

This estimate was derived by Miranda [18] when Ω is 2-dimensional, and by Agmon [2] for Ω which is n-dimensional; Agmon has actually extended the results to elliptic operators of any order with variable coefficients.

For our purposes we need to deal with domains whose boundary has a singularity as, for example, in Figure 1 below. However we begin with a local version of the type (2.1).

Let Ω be a bounded n-dimensional domain, S an open subset of $\partial\Omega$, S in C^4, and D a subdomain of Ω such that

$$\overline{D} \subset \Omega \cup S_0 \quad \text{where } S_0 \subset \partial\Omega,\ \overline{S}_0 \subset \text{int} S.$$

Lemma 2.1 *Under the above assumptions there is a constant C such that for any function φ in $L^p(\Omega) \cap C^1(\Omega \cup S)$, $p > 1$, if*

$$\Delta^2 \varphi = f \qquad \textit{in } \Omega, \tag{2.2}$$

then

$$\|\varphi\|_{W^{1,\infty}(D)} \leqslant C\Big(\|\varphi\|_{L^\infty(S)} + \|\nabla\varphi\|_{L^\infty(S)} + \|\varphi\|_{L^p(\Omega)} + \|f\|_{L^1(\Omega)}\Big). \tag{2.3}$$

Proof. Set

$$\Lambda = \|\varphi\|_{L^\infty(S)} + \|\nabla\varphi\|_{L^\infty(S)} + \|\varphi\|_{L^p(\Omega)} + \|f\|_{L^1(\Omega)}.$$

Without loss of generality we may assume that $f = 0$; otherwise we subtract from φ the special solution of (2.2)

$$\frac{1}{8\pi} \int_\Omega |x-y|^2 \log \frac{1}{|x-y|} f(y) dy. \tag{2.4}$$

Introduce domains $D_1 \subset D_2 \subset \Omega$ with

$$\begin{aligned} &\overline{D} \subset D_1, \quad \overline{D}_1 \subset D_2, \quad \overline{D}_2 \subset \Omega \cup S_0, \\ &\partial D \cap S \subset \text{int}(\partial D_1 \cap S), \quad \partial D_1 \cap S \subset \text{int}(\partial D_2 \cap S). \end{aligned}$$

Take a C^∞ function ζ such that

$$\begin{aligned} \zeta &= 1 \quad \text{in } \overline{D}_1, \\ \zeta &= 0 \quad \text{in } \overline{\Omega} \setminus D_2 \end{aligned}$$

and introduce a domain $\widetilde{D}$ with C^4 boundary such that

$$D_1 \subset \widetilde{D} \subset D_2, \quad \text{and } \zeta \equiv 0 \quad \text{in } \Omega \setminus \widetilde{D}.$$

Consider the function $v = \zeta\varphi$. It satisfies

$$\Delta^2 v = \sum_{|\alpha| \leqslant 3} \zeta_\alpha D^\alpha \varphi$$

and

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\widetilde{D} \setminus S.$$

Write $v = v_1 + v_2$ where

$$\Delta^2 v_1 = 0 \quad \text{in } \widetilde{D},$$
$$v_1 = v, \quad \frac{\partial v_1}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\widetilde{D}.$$

By the Miranda-Agmon maximum principle

$$\|v_1\|_{W^{1,\infty}(\widetilde{D})} \leqslant C\Big(\|\zeta\varphi\|_{L^\infty(\partial\widetilde{D})} + \|\nabla(\zeta\varphi)\|_{L^\infty(\partial\widetilde{D})}\Big) \leqslant C\Big(\|\varphi\|_{L^\infty(S)} + \|\nabla\varphi\|_{L^\infty(S)}\Big). \tag{2.5}$$

To estimate v_2 we apply Theorem 8.1 of [1] which says: If for any $w \in C^4(\widetilde{D})$ with $w = \frac{\partial w}{\partial \nu} = 0$ on $\partial\widetilde{D}$ there holds

$$\Big|\int_{\widetilde{D}} v_2 \Delta^2 w dx\Big| \leqslant \lambda |w|_{W^{4-k,p'}(\widetilde{D})} \quad (\frac{1}{p} + \frac{1}{p'} = 1) \tag{2.6}$$

for some integer k, $0 \leqslant k \leqslant 4$, then

$$|v_2|_{W^{k,p}(\widetilde{D})} \leqslant C\lambda; \tag{2.7}$$

here v_2 is an arbitrary function (say in $L^p(\widetilde{D})$), λ is a constant depending on v_2, and C is a constant independent of v_2, λ.

We shall apply this result to the function v_2 defined above as $v - v_1$. By integration by parts,

$$\begin{aligned}\int_{\widetilde{D}} v_2 \Delta^2 w dx &= \int_{\widetilde{D}} \Delta^2 v_2 \cdot w dx = \int_{\widetilde{D}} \sum_{|\alpha|\leqslant 3} \zeta_\alpha D^\alpha \varphi \cdot w dx \\ &= \int_{\widetilde{D}} \sum_{|\alpha|\leqslant 3} \varphi(-1)^\alpha D^\alpha(\zeta_\alpha w) dx\end{aligned}$$

so that (2.6) holds with $k = 1$, $\lambda = C\|\varphi\|_{L^p}$. Consequently, by (2.7),

$$\|v_2\|_{W^{1,p}(\widetilde{D})} \leqslant C\|\varphi\|_{L^p(\Omega)}.$$

Combining this with (2.5) we get

$$\|\varphi\|_{W^{1,p}(\widetilde{D})} \leqslant \|v\|_{W^{1,p}(\widetilde{D})} \leqslant C\Lambda.$$

We can now repeat the above argument with the smaller domains D_1, $\widetilde{D}$, D_2 (still containing D) and $k = 2$ in (2.6). We get

$$\|v_1\|_{W^{1,\infty}(\widetilde{D})} + \|v_2\|_{W^{2,p}(\widetilde{D})} \leqslant C\Lambda.$$

If $p > n$, then by Sobolev embedding we deduce that

$$\|v_2\|_{W^{1,\infty}(\tilde{D})} \leqslant C\Lambda$$

and (2.3) follows. If however $p \leqslant n$, we repeat the above process with a larger value of p; in fact, if $p < n$, then

$$\varphi \in L^q(\widetilde{D}), \quad \text{and } \|\varphi\|_{L^q(\tilde{D})} \leqslant C\Lambda, \quad \text{where } \frac{1}{q} = \frac{1}{p} - \frac{1}{n};$$

whereas if $p = n$, then

$$\varphi \in L^q(\widetilde{D}), \quad \text{and } \|\varphi\|_{L^q(\tilde{D})} \leqslant C\Lambda, \forall q < +\infty.$$

After a finite number of steps the proof of (2.3) is completed. □

We shall now specialize to 2-dimensional domains whose boundary has a singular point. The method of Miranda-Agmon does not extend to such domains, and, in fact, our estimates will also be quite different.

Let Ω_ω be a domain shown in Figure 1, consisting of two line segments $|\theta| = \omega/2$ ($\pi < \omega \leqslant 2\pi$), connected by an arc on the circle $|x| = 1$, and regularized around $|\theta| = \omega/2$, $|x| = 1$ so that $\partial\Omega_\omega \setminus \{0\} \in C^\infty$.

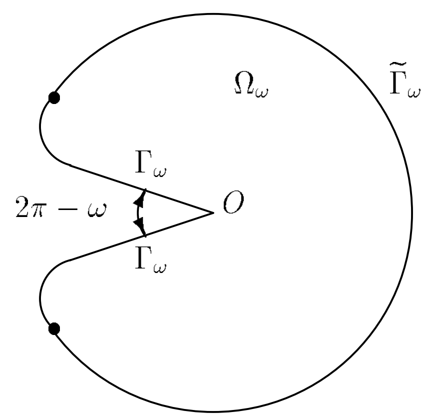

Fig. 1

Then $\partial\Omega_\omega = \Gamma_\omega \cup \widetilde{\Gamma}_\omega$, where $\widetilde{\Gamma}_\omega = \partial\Omega_\omega \setminus \Gamma_\omega = \partial\Omega_\omega \cap \{|x| = 1\}$ and Γ_ω consists of the two line segments $\theta = \pm\omega/2$ initiating at the origin with two small smooth arcs attached at each endpoint.

Theorem 2.2 *Suppose that* $\varphi \in H^2(\Omega_\omega)$,

$$\Delta^2\varphi = f \quad \text{in } \Omega_\omega, \tag{2.8}$$

$$\varphi = \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial\Omega_\omega \setminus \Gamma_\omega. \tag{2.9}$$

Then, for any $p > 2$, $\sigma > 0$, $\mu < 2/p$, there exists a constant $C = C_{p,\sigma,\mu} > 0$, depending only on the regularity of $\partial\Omega_\omega \cap \{|x| \geqslant 1/2\}$ (but not on the angle size ω), such that

$$\Big(\int_{\Omega_\omega} |x|^{-p\mu}|\varphi|^p dx\Big)^{1/p} \leqslant C\Big[\Big\{\int_{\Gamma_\omega}\Big(|x|^{-(1/2+2/p+\sigma)}|\varphi(x)|\Big)^{p/2} dS\Big\}^{2/p} + \int_{\Gamma_\omega} |x|^{-1/2}\Big|\frac{\partial\varphi(x)}{\partial n}\Big| dS + \int_{\Omega_\omega} |x|^{3/2}|f(x)|dx\Big]. \tag{2.10}$$

Proof. Define u to be the solution of the following problem:

$$u \in H^2(\Omega_\omega),$$

$$\Delta^2 u = g \quad \text{in } \Omega_\omega, \tag{2.11}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_\omega, \tag{2.12}$$

where g is any function in $L^2(\Omega_\omega)$. By integration by parts (cf. [14; equation (5)])

$$\int_{\Omega_\omega} E[u]dx = \int_{\Omega_\omega} ugdx \leqslant C\|u\|_{L^\infty(\Omega_\omega)}\|g\|_{L^1(\Omega_\omega)}$$

where $E[u] = \sum_{i,j=1}^{2}\Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2$, $1/q + 1/\widetilde{q} = 1$, $1 < q < 2$. By embedding,

$$\|u\|_{L^\infty(\Omega_\omega)} \leqslant C\Big(\int_{\Omega_\omega} E[u]dx\Big)^{1/2},$$

and hence

$$\int_{\Omega_\omega} E[u]dx \leqslant C\|g\|^2_{L^1(\Omega_\omega)}. \tag{2.13}$$

We shall now use the inequalities (40) and (47) of [14] (see also §9):

$$|u(x)|^2 \leqslant C|x|^{2+2\delta}\Big(\|g\|^2_{L^1(\Omega_\omega)} + \int_{\Omega_\omega} E[u]dx\Big),$$

$$|\nabla u(x)|^2 \leqslant C|x|^{2\delta}\Big(\|g\|^2_{L^1(\Omega_\omega)} + \int_{\Omega_\omega} E[u]dx\Big)$$

where $\delta = \delta(\widetilde{\omega})$ is the solution of

$$\sin^2(\widetilde{\omega}\delta) = \delta^2 \sin^2\widetilde{\omega}, \qquad 0 < \widetilde{\omega}\delta(\widetilde{\omega}) \leqslant \pi, \qquad \delta(\widetilde{\omega}) \geqslant \frac{1}{2},$$

and $\widetilde{\omega}$ is any constant such that $\omega \leqslant \widetilde{\omega}$ and $\pi < \widetilde{\omega} \leqslant 2\pi$; it is easy to verify that there exists a unique such $\delta = \delta(\widetilde{\omega})$. We take $\widetilde{\omega} = 2\pi$ so that $\delta(\widetilde{\omega}) = 1/2$. If we substitute (2.13) into these inequalities, we obtain

$$|u(x)|^2 \leqslant C|x|^3\|g\|^2_{L^1(\Omega_\omega)}, \tag{2.14}$$

$$|\nabla u(x)|^2 \leqslant C|x|\;\|g\|^2_{L^1(\Omega_\omega)}. \tag{2.15}$$

To obtain the estimates for the second and third order derivatives for u, we introduce the scaling

$$u_\varepsilon(x) = u(\varepsilon x) \quad \text{for } 1 \leqslant |x| \leqslant 4.$$

Then

$$\Delta^2 u_\varepsilon = \varepsilon^4 g(\varepsilon x) \quad \text{in } D = \{1 < |x| < 4,\ -\frac{\omega}{2} < \theta < \frac{\omega}{2}\}$$
$$u_\varepsilon = \frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{on } \partial D.$$

By interior-boundary $L^{\widetilde{p}}$ estimates,

$$\begin{aligned}\|u_\varepsilon\|_{W^{4,\widetilde{p}}(D_0)} &\leqslant C_{\widetilde{p}}\Big[\max_D |u_\varepsilon(x)| + \varepsilon^4\Big(\int_D |g|^{\widetilde{p}}(\varepsilon x)dx\Big)^{1/\widetilde{p}}\Big] \\ &\leqslant C_{\widetilde{p}}\Big(\varepsilon^{3/2}\|g\|_{L^1(\Omega_\omega)} + \varepsilon^{4-2/\widetilde{p}}\|g\|_{L^{\widetilde{p}}(\Omega_\omega\cap\{\varepsilon<|x|<4\varepsilon\})}\Big),\end{aligned}$$

where $D_0 = D \cap \{2 < |x| < 3\}$, $1/p + 1/\widetilde{p} = 1$. By embedding

$$\|u_\varepsilon\|_{W^{3,r}(D_0\cap\Gamma)} \leqslant C_{\widetilde{p}}\varepsilon^{3/2}\Big(\|g\|_{L^1(\Omega_\omega)} + \|\ |x|^{5/2-2/\widetilde{p}}g\|_{L^{\widetilde{p}}(\Omega_\omega\cap\{\varepsilon<|x|<4\varepsilon\})}\Big),$$
$$\|u_\varepsilon\|_{W^{2,\infty}(D_0\cap\Gamma)} \leqslant C_{\widetilde{p}}\varepsilon^{3/2}\Big(\|g\|_{L^1(\Omega_\omega)} + \|\ |x|^{5/2-2/\widetilde{p}}g\|_{L^{\widetilde{p}}(\Omega_\omega\cap\{\varepsilon<|x|<4\varepsilon\})}\Big),$$

where $\Gamma = \{\theta = \pm\omega/2\}$, $r = \widetilde{p}/(2-\widetilde{p}) = p/(p-2)$, and so $1/r + 1/r' = 1$ for $r' = p/2$. Rewriting this in terms of the original variables, we have

$$\Big(\int_{\{2\varepsilon<|x|<3\varepsilon\}\cap\Gamma_\omega} |D^3u(x)|^r dS\Big)^{1/r} \leqslant C\varepsilon^{-3/2+1/r}\Big(\|g\|_{L^1(\Omega_\omega)} + \|\ |x|^{5/2-2/\widetilde{p}}g\|_{L^{\widetilde{p}}(\Omega_\omega)}\Big),$$
$$\max_{\{2\varepsilon<|x|<3\varepsilon\}\cap\Gamma_\omega} |D^2u(x)| \leqslant C\varepsilon^{-1/2}\Big(\|g\|_{L^1(\Omega_\omega)} + \|\ |x|^{5/2-2/\widetilde{p}}g\|_{L^{\widetilde{p}}(\Omega_\omega)}\Big).$$

Setting $K = \Big(\|g\|_{L^1(\Omega_\omega)} + \|\ |x|^{5/2-2/\widetilde{p}}g\|_{L^{\widetilde{p}}(\Omega_\omega)}\Big)$, it then follows that

$$\int_{\{2\varepsilon<|x|<3\varepsilon\}\cap\Gamma_\omega\}} \Big(|x|^{3/2-1/r+\sigma}|D^3u(x)|\Big)^r dS \leqslant \Big(C\varepsilon^\sigma K\Big)^r, \tag{2.16}$$
$$|D^2u(x)| \leqslant C|x|^{-1/2}K. \tag{2.17}$$

Letting $\varepsilon_j = (3/2)^{-j}$ and summing over j, we conclude that

$$\Big(\int_{\{|x|<1/2\}\cap\Gamma_\omega} \Big(|x|^{3/2-1/r+\sigma}|D^3u(x)|\Big)^r dS\Big)^{1/r} \leqslant CK. \tag{2.18}$$

Inequality (2.18) is clearly valid if we integrate also over $\Gamma_\omega \setminus \{|x| < 1/2\}$, and similarly (2.17) is valid over all of Γ_ω.

We now multiply equation (2.11) by φ and integrate by parts to obtain

$$\int_{\Omega_\omega} \varphi \cdot g dx \ = \ \int_{\Gamma_\omega} \Big(\varphi\frac{\partial\Delta u}{\partial n} - \frac{\partial\varphi}{\partial n}\Delta u\Big)dS + \int_{\Omega_\omega} u \cdot f dx$$

$$
\begin{aligned}
&\leqslant \Big\{ \int_{\Gamma_\omega} \Big(|x|^{3/2-1/r+\sigma} |D^3 u(x)| \Big)^r dS \Big\}^{1/r} \Big\{ \int_{\Gamma_\omega} \Big(|x|^{-(3/2-1/r+\sigma)} |\varphi(x)| \Big)^{r'} dx \Big\}^{1/r'} \\
&\qquad + \sup \Big(|x|^{1/2} |D^2 u(x)| \Big) \int_{\Gamma_\omega} |x|^{-1/2} \Big| \frac{\partial \varphi(x)}{\partial n} \Big| dx \\
&\qquad + \sup \Big(|x|^{-3/2} |u(x)| \Big) \int_{\Omega_\omega} |x|^{3/2} |f(x)| dx \\
&\leqslant CK \Big[\Big\{ \int_{\Gamma_\omega} \Big(|x|^{-(1/2+2/p+\sigma)} |\varphi(x)| \Big)^{p/2} dS \Big\}^{2/p} \\
&\qquad + \int_{\Gamma_\omega} |x|^{-1/2} \Big| \frac{\partial \varphi(x)}{\partial n} \Big| dS + \int_{\Omega_\omega} |x|^{3/2} |f(x)| dx \Big]
\end{aligned}
$$

Since, by Hölder's inequality,

$$
K \;\leqslant\; C_\mu \| \, |x|^\mu g \|_{L^q(\Omega_\omega)}
$$

for any $\mu < 2/p$, the assertion (2.10) follows immediately by duality. □

Remark 2.1. Let $\lambda =$ right-hand side of (2.10). Then, by Theorem 2.2,

$$
\int_{\Omega_\omega} |x|^{-\mu p} |\varphi(x)|^p dx \leqslant C\lambda^p.
$$

Take $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$ as before. Then

$$
\int_{\{1<|x|<4\}} |\varphi_\varepsilon(x)|^p dx \leqslant \frac{1}{\varepsilon^2} \int_{\Omega_\omega \cap \{\varepsilon<|y|<4\varepsilon\}} |\varphi(y)|^p dy \leqslant C\lambda^p \varepsilon^{\mu p-2}.
$$

and, by Lemma 2.1, for $|x| = 2$,

$$
\begin{aligned}
|\varphi_\varepsilon(x)| + |\nabla \varphi_\varepsilon(x)| &\leqslant C \Big\{ \lambda \varepsilon^{\mu-2/p} + \max_{\Gamma_\omega \cap \{1<|x|<4\}} \Big[|\varphi_\varepsilon(x)| + |\nabla \varphi_\varepsilon(x)| \Big] \Big\} \\
&\leqslant C \Big\{ \lambda \varepsilon^{\mu-2/p} + \max_{\Gamma_\omega \cap \{\varepsilon<|y|<4\varepsilon\}} \Big[|\varphi(y)| + |y| |\nabla \varphi(y)| \Big] \Big\}.
\end{aligned}
$$

Rewriting this in terms of the original variables, we get

$$
|\varphi(x)| \leqslant C\lambda |x|^{\mu-2/p} + C \max_{\Gamma_\omega \cap \{|x|/2<|y|<2|x|\}} \Big[|\varphi(y)| + |y| |\nabla \varphi(y)| \Big], \tag{2.19}
$$

$$
|\nabla \varphi(x)| \leqslant C\lambda |x|^{\mu-1-2/p} + C \max_{\Gamma_\omega \cap \{|x|/2<|y|<2|x|\}} \Big[|y|^{-1} |\varphi(y)| + |\nabla \varphi(y)| \Big]. \tag{2.20}
$$

Later on we shall also need a maximum principle in a domain Ω^δ, shown in Figure 2; it is bounded by a circle $|x| = 1$, a circle $\Gamma_\delta : |x - (\delta, 0)| = \delta$, and a curve: $x_2 = f(x_1), -1 < x_1 < \delta$. We assume that $f(x_1)$ is in $C^{1+\alpha}$ for $-1 < x_1 < \delta$, and that $f(0) = 0$, $f'(0) = 0$.

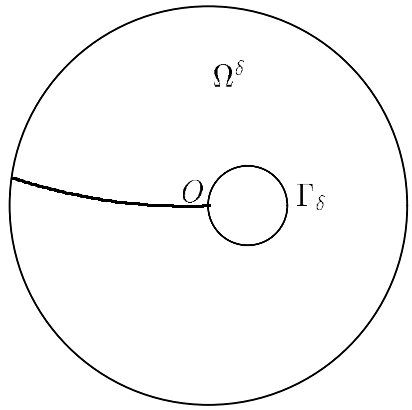

Fig. 2

Theorem 2.3 *Let the foregoing assumptions on Ω^δ, Γ_δ hold, and suppose that $\varphi \in H^2(\Omega^\delta)$,*

$$\Delta^2\varphi = f \quad in\ \Omega^\delta, \tag{2.21}$$

$$\varphi = \varphi_n = 0 \quad on\ \partial\Omega^\delta \setminus \Gamma_\delta. \tag{2.22}$$

Then, for any $p > 2$, $\sigma > 0$, $\mu < 2/p$, there exists a constant $C = C_{p,\sigma,\mu} > 0$ independent of δ such that

$$\Big(\int_{\Omega^\delta} |x|^{-p\mu}|\varphi|^p dx\Big)^{1/p} \leqslant C\Big[\Big\{\int_{\Gamma_\delta}\Big(|x|^{-(1/2+2/p+\sigma)}|\varphi(x)|\Big)^{p/2} dS\Big\}^{2/p} + \int_{\Gamma_\delta} |x|^{-1/2}\Big|\frac{\partial\varphi(x)}{\partial n}\Big| dS + \int_{\Omega^\delta} |x|^{3/2}|f(x)|dx\Big] \tag{2.23}$$

Proof. The proof is similar to that for Theorem 2.2. Denote by u the solution of the following problem

$$u \in H^2(\Omega^\delta),$$

$$\Delta^2 u = g \quad \text{in } \Omega^\delta, \qquad (p > 2) \tag{2.24}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta. \tag{2.25}$$

As before, we obtain

$$\|u\|_{L^\infty(\Omega^\delta)} + \Big(\int_{\Omega^\delta} E[u]dx\Big)^{1/2} \leqslant C\|g\|_{L^1(\Omega^\delta)}.$$

From (40) of [11],

$$|u(x)| \leqslant C|x|^{3/2}\|g\|_{L^1(\Omega^\delta)} \quad \text{in } \Omega^\delta, \tag{2.26}$$

where the constant C is independent of δ.

Now we proceed as in the derivation of (2.16), (2.17) to derive, for $\varepsilon < \delta$, the inequalities

$$\begin{aligned} &\int_{\{2\varepsilon<|x|<3\varepsilon\}\cap\Gamma_\delta\}} \Big(|x|^{3/2-1/r+\sigma}|D^3u(x)|\Big)^r dS \leqslant \Big(C\varepsilon^\sigma K\Big)^r, \\ &|D^2u(x)| \leqslant C\varepsilon^{-1/2}K \quad \text{ for } x \in \{2\varepsilon < |x| < 3\varepsilon\} \cap \Gamma_\delta, \end{aligned} \tag{2.27}$$

where K is defined as before as above but with Ω_ω replaced by Ω^δ. (The assumption $\varepsilon < \delta$ is necessary to ensure that the rescaled function satisfies the equation in a domain with uniformly smooth boundary.) The rest of the proof is the same as in Theorem 2.2. □

Finally, we prove a maximum principle for the domain $\Omega^{\delta,m}$, bounded by a circle $|x| = 1$, and $\Gamma^{\delta,m}$, where $m \geqslant 2\delta$, and

$$\Gamma^{\delta,m} = \Big\{(x_1,x_2) \;\Big|\; \frac{1}{\delta}(x_1,x_2) \in \Gamma^{1,m/\delta}\Big\},$$

and $\Gamma^{1,m/\delta}$ is a curve consisting of two semi-circles centered at $(1,0)$ and $(m/\delta,0)$ of radius 1, connected by two line segments, regularized near four points $(1,1)$, $(1,-1)$, $(m/\delta,1)$, $(m/\delta,-1)$ so that $\Gamma^{1,m/\delta} \in C^5$.

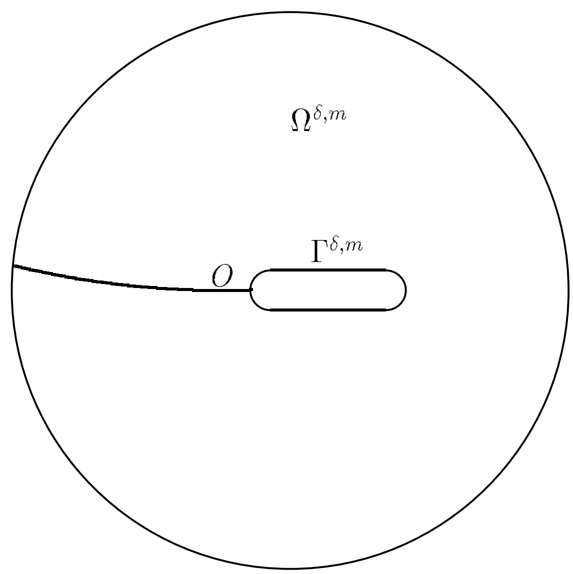

Fig. 3

We follow the proof of Theorem 2.2. First, it is clear that the estimates (2.16), (2.17) are still valid in the region $\{2\varepsilon < |x| < 3\varepsilon\}$ for $\varepsilon < \delta$. Next, for $|x| \geqslant 3\delta$, the argument of Theorem 2.2 leads to

$$\int_{\{2\varepsilon<d(x)<3\varepsilon\}\cap\Gamma^{\delta,m}} \Big(d(x)^{3/2-1/r+\sigma}|D^3u(x)|\Big)^r dS \leqslant \Big(C\varepsilon^\sigma \widetilde{K}\Big)^r, \tag{2.28}$$

$$|D^2u(x)| \leqslant Cd(x)^{-1/2}\widetilde{K} \quad \text{ for } |x| \geqslant 3\delta. \tag{2.29}$$

where

$$d(x)=\min(|x|,|x-(m,0)|),\qquad \delta\leqslant\varepsilon<m,$$

and $\widetilde{K}$ is defined as K but with $|x|^{5/2-2/q}$ replaced by $d(x)^{5/2-2/q}$ and with Ω_ω replaced by $\Omega^{\delta,m}$.

Now we can take $\varepsilon=\varepsilon_j$ as before to conclude:

Theorem 2.4 *Suppose that $\varphi\in H^2(\Omega^{\delta,m})$,*

$$\Delta^2\varphi=f\quad \textit{in } \Omega^{\delta,m}, \tag{2.30}$$

$$\varphi=\varphi_n=0\quad \textit{on } \partial\Omega^{\delta,m}\setminus\Gamma^{\delta,m}. \tag{2.31}$$

Then, for any $p>2$, $\sigma>0$ $\mu>0$, there exists a constant $C=C_{p,\sigma,\mu}>0$ independent of δ and m such that

$$\begin{aligned}\Big(\int_{\Omega^{\delta,m}} d(x)^{-p\mu}|\varphi|^p dx\Big)^{1/p}\leqslant C\Big[\Big\{\int_{\Gamma^{\delta,m}}\Big(d(x)^{-(1/2+2/p+\sigma)}|\varphi(x)|\Big)^{p/2}dS\Big\}^{2/p}\\ +\int_{\Gamma^{\delta,m}} d(x)^{-1/2}\Big|\frac{\partial\varphi(x)}{\partial n}\Big|dS+\int_{\Gamma^{\delta,m}} d(x)^{3/2}|f(x)|dx\Big]\end{aligned} \tag{2.32}$$

Remark 2.2. Remark 2.1 extends to both Theorems 2.3 and 2.4.

Remark 2.3. Maz'ya and Plameneveskii [17] use a different method to derive a maximum principle for a conical region: They first establish such a result in an infinite cylinder, and then, by a local Miranda-Agmon maximum principle (Lemma 10.1 in [17]), for a polycylinder. Finally, they map a conical region into a polycylinder. Our approach to derive a maximum principle with integral norms (e. g. Theorem 2.2) is much simpler, and our local Miranda-Agmon lemma (which is sharper than Lemma 10.1 in [17]) then yields pointwise estimates for $u,\nabla u$ as in (2.19), (2.20). We note that the Maz'ya-Plameneveskii estimates are in weighted sup norms, similar to (2.19), (2.20); however, for our purposes, the integral estimates will be more convenient. It is also important to note that whereas the conical region in [17] is assumed to have C^4 boundary, our method allows weaker regularity on the parts of the boundary with zero Dirichlet data. Thus in the case of Theorems 2.3, 2.4, our method requires only that the curve $x_2=f(x_1)$ is in $C^{1+\alpha}$.

3 The stress intensity factors

Throughout sections 3–6 we assume that

$$\varphi\in H^2(B_1\setminus\Gamma), \tag{3.1}$$

$$\Delta^2\varphi=0\quad \text{in } B_1\setminus\Gamma, \tag{3.2}$$

$$\varphi=\frac{\partial\varphi}{\partial n}=0\quad \text{from both sides of } \Gamma, \tag{3.3}$$

where B_1 is the unit disc $\{x_1^2 + x_2^2 < 1\}$ and

$$\Gamma = \{x_2 = f(x_1), \quad -x_* \leqslant x_1 \leqslant 0\} \tag{3.4}$$

is a curve contained in B_1 except for its end point $(-x_*, f(-x_*))$. We also assume that

$$f(0) = 0, \quad f'(0) = 0. \tag{3.5}$$

In this section and in section 4, we also assume that $f \in C^{1+\alpha}$, whereas in section 5 we shall require that $f \in C^{2+\alpha}$.

Theorem 3.1 *If $f \in C^{1+\alpha}[-x_*, 0]$, then (1.13) holds for any $\eta > 0$ such that $\alpha + \eta > 1/2$.*

We shall first prove a weaker result:

$$\varphi(x) = A_1 r^{3/2} B_1(\theta) + A_2 r^{3/2} B_2(\theta) + O(r^{3/2+\lambda}) \quad \text{for some } \lambda > 0. \tag{3.6}$$

Proof of (3.6). Define

$$\psi_\varepsilon(x) = \frac{\varphi(\varepsilon x)}{\varepsilon^{3/2}}, \qquad |x| < \frac{1}{\varepsilon}; \tag{3.7}$$

then

$$|\psi_\varepsilon(x)| \leqslant C|x|^{3/2} \quad \text{for } |x| < \frac{1}{\varepsilon}. \tag{3.8}$$

Under this change of variables, Γ is changed to

$$\Gamma_\varepsilon : \quad x_2 = f_\varepsilon(x_1) \equiv \frac{1}{\varepsilon} f(\varepsilon x_1), \tag{3.9}$$

where $f_\varepsilon \in C^{1+\alpha}$, and

$$f_\varepsilon(0) = f_\varepsilon'(0) = 0, \quad \|f_\varepsilon\|_{C^{1+\alpha}(-1 \leqslant x_1 \leqslant 0)} \leqslant C\varepsilon^\alpha. \tag{3.10}$$

Let $G = G_\varepsilon$ be a function defined on $\Omega_{2\pi}$ (Ω_ω is defined in the previous section; here we take $\omega = 2\pi$) as follows:

$$G \in H^2(\Omega_{2\pi}),$$

$$\Delta^2 G = 0 \quad \text{in } \Omega_{2\pi},$$

$$G = \frac{\partial G}{\partial n} = 0 \quad \text{on } \{\theta = \pm\pi\}, \tag{3.11}$$

$$G = \psi_\varepsilon, \quad \frac{\partial G}{\partial n} = \frac{\partial \psi}{\partial n} \quad \text{on rest of } \partial\Omega_{2\pi}. \tag{3.12}$$

Setting $\omega = 2\pi - 2C\varepsilon^\alpha$, we want to apply the Theorem 2.2 to $\psi_\varepsilon - G$ in the domain Ω_ω.

For $|x| \leqslant 1/2$, G has an expansion (see §1)

$$G = \sum_{k=1}^{\infty} r^{k/2+1} \widetilde{B}_k(\theta) \quad (r > 0, -\pi < \theta < \pi); \tag{3.13}$$

where both $\widetilde{B}_k$ and their derivatives are bounded. It follows that

$$|G|\Big|_{|\theta|=\pi-C\varepsilon^\alpha} \leqslant C\varepsilon^{2\alpha}|x|^{3/2},$$

$$|G_n|\Big|_{|\theta|=\pi-C\varepsilon^\alpha} \leqslant C\varepsilon^{\alpha}|x|^{1/2}.$$

Since $\psi \in C^{3/2}$ by [14], similar estimates are also valid for ψ:

$$|\psi|\Big|_{|\theta|=\pi-C\varepsilon^\alpha} \leqslant C\varepsilon^{3\alpha/2}|x|^{3/2},$$

$$|\nabla\psi|\Big|_{|\theta|=\pi-C\varepsilon^\alpha} \leqslant C\varepsilon^{\alpha/2}|x|^{1/2}.$$

It is also clear that a similar estimate is valid for $\psi_\varepsilon - G$ on two connecting small smooth arcs near $\{|\theta| = \pi - C\varepsilon^\alpha, \ \ |x| = 1\}$.

By Theorem 2.2, for any $p > 2$,

$$\|\psi_\varepsilon - G_\varepsilon\|_{L^p(B_1\cap\{-\pi+C\varepsilon^\alpha<\theta<\pi-C\varepsilon^\alpha\})} \leqslant C_p\varepsilon^{\alpha/2}.$$

Since also

$$|\psi_\varepsilon - G_\varepsilon| \leqslant |\psi_\varepsilon| + |G_\varepsilon| \leqslant C|f_\varepsilon|^{3/2} + C\varepsilon^{2\alpha}|x|^{3/2} \leqslant C\varepsilon^{\alpha/2}|x|^{3/2}$$

in $B_1 \cap \{|\theta \pm \pi| \leqslant C\varepsilon^\alpha\}$, we get

$$\|\psi_\varepsilon - G_\varepsilon\|_{L^p(B_1)} \leqslant C_p\varepsilon^{\alpha/2}. \tag{3.14}$$

Notice that $|\psi_\varepsilon - G_\varepsilon|$ is uniformly in $C^{3/2}$. By interpolation, for any $\beta < \alpha/2$, we can take p large enough such that

$$|\psi_\varepsilon(x) - G_\varepsilon(x)| \leqslant C\varepsilon^\beta \quad \text{ for } |x| \leqslant 1. \tag{3.15}$$

Rewriting this in terms of the original variables, we have

$$\Big|\frac{\varphi(x)}{\varepsilon^{3/2}} - G_\varepsilon\Big(\frac{x}{\varepsilon}\Big)\Big| \leqslant C\varepsilon^\beta \quad \text{ for } |x| \leqslant \varepsilon. \tag{3.16}$$

By (3.13),

$$G_\varepsilon(x) = r^{3/2}B_\varepsilon(\theta) + O(r^2) \quad \text{ for } r = \sqrt{x_1^2 + x_2^2} \leqslant \frac{1}{2}, \tag{3.17}$$

where $O(r^2)$ means a term which is bounded by Cr^2 with the constant C independent of ε. Thus

$$\Big|\varphi(x) - r^{3/2}B_\varepsilon(\theta)\Big| \leqslant C\varepsilon^{3/2+\beta} + C\varepsilon^{3/2}\Big(\frac{r}{\varepsilon}\Big)^2 \quad \text{ for } r \leqslant \frac{\varepsilon}{2}.$$

Hence

$$\Big|\frac{\varphi(x)}{r^{3/2}} - B_\varepsilon(\theta)\Big| \leqslant C\varepsilon^\beta\Big(\frac{\varepsilon}{r}\Big)^{3/2} + C\Big(\frac{r}{\varepsilon}\Big)^{1/2} \quad \text{ for } r \leqslant \frac{\varepsilon}{2}. \tag{3.18}$$

Setting $\eta = \beta/2$, we get

$$\Big|\frac{\varphi(x)}{r^{3/2}} - B_\varepsilon(\theta)\Big| \leqslant C\varepsilon^\beta(\varepsilon^{-\eta})^{3/2} + C\varepsilon^{\eta/2} \leqslant C\varepsilon^{\beta/4} \quad \text{ for } \varepsilon^{\eta+1} \leqslant r \leqslant 2^{\eta+1}\varepsilon^{\eta+1}. \tag{3.19}$$

Now take $\varepsilon_j = 2^{-j}$, $r = \varepsilon_{j+1}^{\eta+1}$. Then

$$\begin{aligned}\left|B_{\varepsilon_j}(\theta) - B_{\varepsilon_{j+1}}(\theta)\right| &\leqslant \left|\frac{\varphi(x)}{r^{3/2}} - B_{\varepsilon_j}(\theta)\right| + \left|B_{\varepsilon_{j+1}}(\theta) - \frac{\varphi(x)}{r^{3/2}}\right| \\ &\leqslant C2^{-j\beta/4} = C\varepsilon_j^{\beta/4}.\end{aligned}$$

It follows that the series

$$\sum_j \left|B_{\varepsilon_j}(\theta) - B_{\varepsilon_{j+1}}(\theta)\right|$$

is convergent, and

$$\sum_{j\geqslant k} \left|B_{\varepsilon_j}(\theta) - B_{\varepsilon_{j+1}}(\theta)\right| \leqslant C\varepsilon_k^{\beta/4}. \tag{3.20}$$

Setting

$$B(\theta) = \lim_{\varepsilon_j \to 0} B_{\varepsilon_j}(\theta),$$

we then have

$$\left|B(\theta) - B_{\varepsilon_j}(\theta)\right| \leqslant C\varepsilon_j^{\beta/4}, \tag{3.21}$$

so that, by (3.19),

$$\left|\frac{\varphi(x)}{r^{3/2}} - \mathbf{A}\cdot\mathbf{B}(\theta)\right| \leqslant Cr^{\beta/[4(\eta+1)]}, \tag{3.22}$$

where

$$\begin{aligned}&\mathbf{A}\cdot\mathbf{B}(\theta) = A_1B_1(\theta) + A_2B_2(\theta), \\ &\mathbf{A} = (A_1, A_2), \qquad \mathbf{B}(\theta) = (B_1(\theta), B_2(\theta)),\end{aligned}$$

and $B_1(\theta)$, $B_2(\theta)$ are defined in (1.12). This completes the proof of (3.6). □

Let C_1, C_2 be positive constants such that

$$\|\varphi\|_{H^2(B_1\setminus\Gamma)} \leqslant C_1, \quad \|f\|_{C^{1+\alpha}} \leqslant C_2, \quad 0 < \alpha < \frac{1}{2}. \tag{3.23}$$

In order to complete the proof of Theorem 3.1, it suffices to prove:

Lemma 3.2 *For any $\mu \in (0, \min(\alpha, \frac{1}{2}))$ there exists a constant $C = C_\mu$ depending only on C_1, C_2 such that*

$$|\varphi(x) - |x|^{3/2}(A_1B_1(\theta) + A_2B_2(\theta))| \leqslant Cr^{3/2+\mu} \quad \textit{in } B_1. \tag{3.24}$$

Proof. It is sufficient to show that (3.24) holds for all $r < \frac{1}{2}$. Setting

$$w(x) = \varphi(x) - A_1r^{3/2}B_1(\theta) - A_2r^{3/2}B_2(\theta),$$

we have, by (3.6),

$$|w(x)| \leqslant Cr^{3/2+\lambda}, \quad \text{for some } 0 < \lambda < \alpha. \tag{3.25}$$

The proof of (3.25) shows that C depends only on C_1, C_2. If $\mu \leqslant \lambda$, then there is nothing to prove. So we may assume that $\mu > \lambda$; consequently, for any $0 < \delta < 1$,

$$\frac{|w(x)|}{r^{3/2}(r^\mu + \delta^{\mu-\lambda} r^\lambda)} < C \tag{3.26}$$

for all $|x| < 1$. If C can be chosen to be independent of δ, as well as of C_1, C_2, then (3.24) follows by taking $\delta \to 0$ in (3.26). So it suffices to show that if such a C does not exist then we get a contradiction.

Assuming that such a C does not exists, there exist sequences f_n, w_n, δ_n and x_n such that

$$C_n = \sup_{|x|<1} \frac{|w_n(x)|}{r^{3/2}(r^\mu + \delta_n^{\mu-\lambda} r^\lambda)} = \frac{|w_n(x_n)|}{R_n^{3/2}(R_n^\mu + \delta_n^{\mu-\lambda} R_n^\lambda)} \to \infty,$$

if $n \to \infty$, where $R_n = |x_n| < 1$. In view of (3.25), we must then have

$$\delta_n \to 0, \qquad R_n \to 0.$$

Introduce a function $G_n(\xi)$ by

$$w_n(x) = C_n R_n^{3/2}(R_n^\mu + \delta_n^{\mu-\lambda} R_n^\lambda) G_n(\xi), \quad x = R_n \xi.$$

Then

$$G_n(\xi_n) = 1 \quad \text{where } \xi_n = \frac{x_n}{R_n}, \tag{3.27}$$

and

$$\begin{aligned} |G_n(\xi)| &\leqslant \frac{|x|^{3/2}(|x|^\mu + \delta_n^{\mu-\lambda}|x|^\lambda)}{R_n^{3/2}(R_n^\mu + \delta_n^{\mu-\lambda} R_n^\lambda)} \\ &\leqslant |\xi|^{3/2} \frac{R_n^{\mu-\lambda}|\xi|^\mu + \delta_n^{\mu-\lambda}|\xi|^\lambda}{R_n^{\mu-\lambda} + \delta_n^{\mu-\lambda}} \\ &\leqslant |\xi|^{3/2}(|\xi|^\mu + |\xi|^\lambda). \end{aligned}$$

As $n \to \infty$, the curves Γ_n, defined by $\xi_2 = f_n(\xi_1)$, converge in the ξ-plane to the ray

$$S_0 = \{(\xi_1, 0); \ -\infty < \xi_1 < 0\}$$

and, for a subsequence,

$$G_n(\xi) \to G(\xi) \tag{3.28}$$

uniformly in compact subsets of $\mathbb{R}^2 \subset \overline{S}_0$, and

$$\Delta^2 G = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{S}_0, \tag{3.29}$$

$$G = \frac{\partial G}{\partial \nu} = 0 \quad \text{from both sides of } S_0, \tag{3.30}$$

$$|G(\xi)| \leqslant |\xi|^{3/2}(|\xi|^\mu + |\xi|^\lambda) \quad \text{in } \mathbb{R}^2 \qquad (0 < \lambda < \mu < \min(\alpha, \frac{1}{2})). \tag{3.31}$$

To prove (3.30), we actually need to use sub-Schauder boundary estimates (cf. §9). We first apply Lemma 9.2 to the function $u(x) = \varepsilon^{-3/2}\varphi_n(\varepsilon x)$ in a ring $1/2 < |x| < 2$ to obtain, for any $\eta \in (0,1)$,

$$\frac{|\varphi_n(x)|}{|x|^{3/2}} \leqslant C\Big(\frac{d(x)}{|x|}\Big)^{1+\eta}, \qquad \frac{|\nabla\varphi_n(x)|}{|x|^{1/2}} \leqslant C\Big(\frac{d(x)}{|x|}\Big)^{\eta}, \tag{3.32}$$

where $d(x)$ is the distance to the Γ_n. Similar estimates are valid for $|x|^{3/2}[A_1B_1(\theta)+A_2B_2(\theta)]$ with $d(x)$ replaced by $|x_2|$. It follows that

$$\frac{|w_n(x)|}{|x|^{3/2}} \leqslant C\Big(\frac{d(x)+|x_2|}{|x|}\Big)^{1+\eta}, \qquad \frac{|\nabla w_n(x)|}{|x|^{1/2}} \leqslant C\Big(\frac{d(x)+|x_2|}{|x|}\Big)^{\eta}. \tag{3.33}$$

Since $\mu < \alpha$, we can choose η sufficiently close to 1 so that $\alpha\eta > \mu$. Rewriting (3.33) in terms of ξ, we then have,

$$|G_n(\xi)| \leqslant C_K \frac{R_n^{3/2} R_n^{\alpha(1+\eta)}|\xi|^{3/2}}{C_n R_n^{3/2}(R_n^\mu + \delta_n^{\mu-\lambda}R_n^\lambda)} \leqslant \frac{C_K}{C_n}|\xi|^{3/2} \quad \text{for } \xi \in \mathcal{C}_K, \tag{3.34}$$

$$|\nabla_\xi G_n(\xi)| \leqslant C_K \frac{R_n^{1/2} R_n^{\alpha\eta}|\xi|^{1/2}}{C_n R_n^{1/2}(R_n^\mu + \delta_n^{\mu-\lambda}R_n^\lambda)} \leqslant \frac{C_K}{C_n}|\xi|^{1/2} \quad \text{for } \xi \in \mathcal{C}_K, \tag{3.35}$$

where $\mathcal{C}_K = \{-K \leqslant \xi_1 \leqslant 0,\ \ |\xi_2| \leqslant -CR_n^\alpha \xi_1\}$ (the constant C is from (3.10)). Similarly

$$[\nabla_\xi G_n]_{C^\beta(\mathcal{C}_K)} \leqslant \frac{R_n^{1/2-\beta} R_n^{\alpha(\eta-\beta)}}{C_n R_n^{1/2-\beta}(R_n^\mu + \delta_n^{\mu-\lambda}R_n^\lambda)} \leqslant \frac{C_K}{C_n} \tag{3.36}$$

provided β is taken to be small enough. With these estimates, we can now apply Theorem 9.3 (ii) to $G_n(\xi)$ with the boundary S given by $|\xi_2| = -CR_n^\alpha\xi_1$ to obtain (3.30). We also have uniform convergence up to the boundary in (3.28), so that if $\xi_n \to e = (e_1, e_2)$, then

$$|G(e)| = 1. \tag{3.37}$$

We now invoke a Liouville theorem (Lemma 3.3 below) to deduce from (3.29)–(3.31) that $G(\xi) \equiv 0$, which is a contradiction to (3.37). □

Lemma 3.3 *If G is a function satisfying (3.29)–(3.31), then $G \equiv 0$.*

Proof. From the expansion

$$G(\xi) = \sum_{k=1}^{\infty} r^{k/2+1} B_k(\theta) \qquad (r = |\xi|) \tag{3.38}$$

near $\xi = 0$ and (3.31) (with $\lambda > 0$) it follows that $B_1 = 0$. Recalling that

$$r^2 B_2(\theta) = c\xi_2^2, \qquad c \text{ constant},$$

we introduce the function

$$H(\xi) = G(\xi) - c\xi_2^2.$$

Then

$$|H(\xi)| \leqslant C|\xi|^{5/2} \quad \text{near } \xi = 0.$$

The function

$$\Phi(\xi) = \frac{\partial H}{\partial \xi_1} = \frac{\partial G}{\partial \xi_1}$$

is a biharmonic in $\mathbb{R}^2 \setminus S_0$ and satisfies the same boundary conditions as in (3.30). Since Φ has an expansion similar to (3.38), we deduce that

$$|\nabla^j \Phi(\xi)| \leqslant C|\xi|^{5/2-1-j} \quad \text{near } \xi = 0. \tag{3.39}$$

Next, for $|\xi|$ large, we have

$$|G(\xi)| \leqslant 2|\xi|^{3/2+\mu}.$$

By scaling and applying elliptic boundary and interior estimates we deduce that

$$|\nabla^j \Phi(\xi)| \leqslant C|\xi|^{3/2+\mu-1-j} \quad \text{for } |\xi| \text{ large} \tag{3.40}$$

and $0 \leqslant j \leqslant 3$.

Set $D_R = \left\{ \frac{1}{R} < |\xi| < R \right\} \setminus S_0$. By integration by parts,

$$0 = \int_{D_R} \Phi \cdot \Delta^2 \Phi = \int_{D_R} |\Delta \Phi|^2 - I_R,$$

where I_R is a linear combination of integrals

$$A_{R,j} = \int_{\{|\xi|=1/R\}\setminus S_0} D^j\Phi \cdot D^{3-j}\Phi$$

$$B_{R,j} = \int_{\{|\xi|=R\}\setminus S_0} D^j\Phi \cdot D^{3-j}\Phi.$$

In view of (3.39), (3.40), we clearly have

$$|A_{R,j}| \leqslant \frac{C}{R}, \quad |B_{R,j}| \leqslant \frac{CR^{2\mu}}{R},$$

and, since $\mu < 1/2$, $I_R \to 0$ if $R \to \infty$. It follows that

$$\int_{\mathbb{R}^2\setminus S_0} |\Delta \Phi|^2 = 0,$$

so that $\Delta\Phi \equiv 0$. Since $\Phi = \partial\Phi/\partial\nu = 0$ on S_0, we get $\Phi \equiv 0$. This means that

$$\frac{\partial}{\partial \xi_1}(G - c\xi_2^2) = 0$$

and so $G = g(\xi_2)$, $g^{(4)} = \Delta^2 G = 0$. Hence G is a polynomial of degree $\leqslant 3$ in ξ_2, and by invoking (3.30), (3.31) we finally conclude that $G \equiv 0$. $\square$

4 A flatness lemma

Let the assumptions of Theorem 3.1 be satisfied and denote by $d(x)$ the distance from x to Γ. In this section we investigate the behavior of $\varphi(x)$ as x approaches the tip O while, at the same time, $d(x)/|x|$ tends to zero.

Theorem 4.1 *Under the assumptions of Theorem 3.1*

$$\frac{|\varphi(x)|}{|x|^{3/2}} \leqslant C\Big(|x|^{2\alpha} + \frac{d^2(x)}{|x|^2}\Big) \tag{4.1}$$

where C is a constant depending only on the C_1, C_2 in (3.23).

Remark 4.1. If we apply the sub-Schauder estimates (§9, Lemma 9.2) to $\varepsilon^{-3/2}\varphi(\varepsilon x)$ in $1/2 < |x| < 2$, where φ is as in Theorem 3.1, we get

$$\frac{|\varphi(x)|}{|x|^{3/2}} \leqslant C\Big(\frac{d(x)}{|x|}\Big)^{1+\eta} \quad \forall \eta > 0. \tag{4.2}$$

The estimate (4.1) is an improvement of (4.2) when

$$|x|^{\alpha+\eta'} < \frac{d(x)}{|x|} \ll 1 \quad \text{for some } \eta' > 0.$$

To prove Theorem 4.1, we shall establish a lemma which is of intrinsic interest.

Let

$$\Gamma = \{x_2 = f(x_1),\ -1 < x_1 < 1\},$$
$$D = \{(x_1,x_2);\ -1 < x_1 < 1,\ -1 < x_2 < f(x_1)\},$$

and let γ be a subarc of Γ:

$$\gamma = \{x_2 = f(x_1),\ -1/2 < x_1 < 1/2\}.$$

We assume a bound

$$\|f\|_{C^{1+\alpha}} \leqslant A \qquad (0 < \alpha < 1,\ \ A > 0), \tag{4.3}$$

and the "ε-flatness" condition:

$$|f(x_1)| < \varepsilon \quad (0 < \varepsilon < 1/2). \tag{4.4}$$

Denote by $d(x)$ the distance function

$$d(x) = \operatorname{dist}(x,\gamma), \quad x \in D.$$

Lemma 4.2 (Flatness Lemma) *If* $\psi \in H^2(D \setminus \Gamma)$, $\|\psi\|_{H^2(D\setminus\Gamma)} \leqslant C_1$,

$$\Delta^2 \psi = 0 \quad \text{in } D, \tag{4.5}$$

$$\psi = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \Gamma, \tag{4.6}$$

$$|\psi| < 1 \quad \text{in } D, \tag{4.7}$$

then

$$|\psi(x)| \leqslant C(d^2(x) + \varepsilon^2) \quad \text{in } D \tag{4.8}$$

where C *is a constant depending only on* A, α *and* C_1.

Remark 4.2. By sub-Schauder estimates (see Example 2 following Theorem 9.1)

$$|\psi(x)| \leqslant C d^{2-\eta}(x) \quad \forall \eta > 0;$$

(4.8) is an improvement when $d(x) \gg \varepsilon^{1+\eta'}$ for some $\eta' > 0$.

Proof. We assume that (4.8) is not valid and derive a contradiction. If (4.8) is not true then there exist sequences

$$\Gamma = \Gamma_n, \quad \gamma = \gamma_n, \quad D = D_n, \quad \varepsilon = \varepsilon_n, \quad \psi = \psi_n,$$

such that, with $d_n(x) = \text{dist}(x, \gamma_n)$,

$$C_n = \sup_{D_n} \frac{|\psi_n(x)|}{d_n^2(x) + \varepsilon_n^2} = \frac{|\psi_n(x_n)|}{d_n^2(x_n) + \varepsilon_n^2} \to \infty \quad \text{if } n \to \infty,$$

where $x_n \in D_n$; we necessarily have

$$d_n \equiv d_n(x_n) \to 0, \quad \varepsilon_n \to 0 \quad \text{if } n \to \infty$$

for otherwise the sequence C_n will remain bounded (by (4.7)). Denote by $\widetilde{x}_n$ the point on γ_n such that $d_n = |x_n - \widetilde{x}_n|$.

Introduce functions $G_n(\xi)$ by

$$\psi_n(x) = C_n(d_n^2 + \varepsilon_n^2) G_n(\xi) \quad \text{where } x - \widetilde{x}_n = d_n \xi.$$

Then

$$d_n(x) = d_n \widetilde{d}_n(\xi), \tag{4.9}$$

$$G_n(\xi_n) = 1, \quad \xi_n = \frac{x_n - \widetilde{x}_n}{d_n}, \quad \widetilde{d}_n(\xi_n) = 1, \tag{4.10}$$

where $\widetilde{d}_n(\xi)$ is the distance from ξ to the image $\widetilde{\gamma}_n$ of γ_n under the mapping $x - \widetilde{x}_n = d_n \xi$.

Clearly,

$$|G_n(\xi)| \leqslant \frac{d_n^2(x) + \varepsilon_n^2}{d_n^2 + \varepsilon_n^2} \leqslant \frac{d_n^2 \widetilde{d}_n^2(\xi) + \varepsilon_n^2}{d_n^2 + \varepsilon_n^2} \tag{4.11}$$

It is sufficient to consider the following two cases:

Case (1): $d_n/\varepsilon_n \to 0$ if $n \to \infty$.

Case (2): $d_n \geqslant c\varepsilon_n$ for some $c > 0$ for all large n.

In case (1), (4.11) clearly implies that

$$|G_n(\xi)| \leqslant \widetilde{d}_n^2(\xi)\Big(\frac{d_n}{\varepsilon_n}\Big)^2 + 1, \tag{4.12}$$

and in case (2), (4.11) implies that

$$|G_n(\xi)| \leqslant \frac{d_n^2(x)}{d_n^2} + 1 \leqslant |\xi|^2 + 1. \tag{4.13}$$

As $n \to \infty$, the curves $\widetilde{\gamma}_n$ in the ξ-plane converges to the line $\{\xi_2 = 0\}$ and, by (4.10),

$$G_n(\xi) \to G(\xi) \tag{4.14}$$

uniformly in compact subsets of $\{\xi_2 < 0\}$,

$$\Delta^2 G(\xi) = 0 \quad \text{in } \{\xi_2 < 0\}, \tag{4.15}$$

$$|G(\xi)| \leqslant 1 \quad \text{in } \{\xi_2 < 0\} \quad \text{in case (1)}, \tag{4.16}$$

$$|G(\xi)| \leqslant |\xi|^2 + 1 \quad \text{in } \{\xi_2 < 0\} \quad \text{in case (2)}. \tag{4.17}$$

By sub-Schauder estimates (Lemma 9.2) applied to $G_n(\xi)$ we deduce that

$$G = \frac{\partial G}{\partial \nu} = 0 \quad \text{on } \{\xi_2 = 0\}. \tag{4.18}$$

We also note that the convergence in (4.10) is uniform near the boundary so that, in particular, if

$$\xi_n = \frac{x_n - \widetilde{x}_n}{d_n} \to e \qquad (e = (e_1, e_2), e_2 \leqslant 0, |e| = 1),$$

then

$$|G(e)| = \lim |G_n(\xi_n)| = 1. \tag{4.19}$$

By a Liouville theorem (Lemma 4.3 below) we conclude from (4.15)–(4.17), (4.18) and (4.19) that

$$G(\xi) = K\xi_2^2, K \neq 0. \tag{4.20}$$

This contradicts (4.16) (case (1)).

We shall next derive a contradiction to (4.20) (case (2)). The proof depends on sharp estimates on the $\psi_n(x)$. Since,

$$|\psi_n(x)| \leqslant C_n(d_n^2(x) + \varepsilon_n^2) \quad \text{in } D_n$$

the flatness condition implies that

$$|\psi_n(x)| \leqslant CC_n\varepsilon_n^2 \quad \text{in } \{|x_1| < \frac{1}{2}, -4\varepsilon_n < x_2 < -2\varepsilon_n\}. \tag{4.21}$$

By interior elliptic estimates we then also have

$$|\nabla\psi_n(x)| \leqslant CC_n\varepsilon_n \quad \text{on } \{|x_1| < \frac{1}{4}, x_2 = -3\varepsilon_n\}. \tag{4.22}$$

We need to construct an auxiliary function. For this purpose we first consider the problem

$$\begin{aligned}
&\Delta^2\varphi = 0 \quad \text{in } \{x_2 > 0\},\\
&\varphi(x_1, 0) = f_0(x_1),\\
&\frac{\partial}{\partial x_2}\varphi(x_1, 0) = g_0(x_1),
\end{aligned}$$

where $f_0 \in C^{1+\alpha}_{loc}$, $g_0 \in C^{\alpha}_{loc}$.

If f_0, g_0 are uniformly bounded, then we can write a solution in the form

$$\varphi = P(f_0) + x_2\Big[P(g_0) - \frac{\partial}{\partial x_2}P(f_0)\Big] \tag{4.23}$$

where P is the Poisson kernel:

$$P(h)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2}{(x_1-\xi_1)^2 + x_2^2}h(\xi_1)d\xi_1.$$

Noting that

$$\frac{\partial}{\partial x_2}P(f_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x_1-\xi_1)^2 - x_2^2}{[(x_1-\xi_1)^2 + x_2^2]^2}f_0(\xi_1)d\xi_1,$$

we find that

$$|\varphi(x)| \leqslant C|f_0|_{L^\infty} + C|x_2|(|g_0|_{L^\infty} + |f_0|_{L^\infty}). \tag{4.24}$$

Let $\chi(x_1)$ be a C^2 function such that $\chi(x_1) = 1$ if $|x_1| < 1/8$, $\chi(x_1) = 0$ if $|x_1| > 1/4$. Introduce the functions

$$f_0 = \chi\psi_n, g_0 = \frac{\partial}{\partial x_2}(\chi\psi_n) \quad \text{on } x_2 = -3\varepsilon_n$$

and denote by Φ_n the biharmonic function in $\{x_2 < -3\varepsilon_n\}$ with Dirichlet data f_0, g_0 on $x_2 = -3\varepsilon_n$. Using a representation similar to (4.23) (with (x_1, x_2) replaced by $(x_1, -x_2-3\varepsilon_n)$) we get, from (4.24) and (4.21), (4.22), the estimate

$$|\Phi_n(x)| \leqslant CC_n(\varepsilon_n^2 + |x_2|\varepsilon_n) \quad \text{if } x_2 \leqslant -3\varepsilon_n. \tag{4.25}$$

Consider the function

$$\zeta_n = \psi_n - \Phi_n \quad \text{in } \Omega_n = \{|x_1| < \frac{1}{8}, -1 < x_2 < -3\varepsilon_n\}.$$

It satisfies

$$\begin{aligned}
&\Delta^2\zeta_n = 0 \quad \text{in } \Omega_n,\\
&\zeta_n = \frac{\partial\zeta_n}{\partial x_2} = 0 \quad \text{on } x_2 = -3\varepsilon_n,\ |x_1| < \frac{1}{8},
\end{aligned}$$

and

$$|\zeta_n(x)| \leqslant 1 + CC_n\varepsilon_n \equiv K_n \quad \text{in } \Omega_n,$$

by (4.25) and (4.7).

By standard elliptic regularity

$$\frac{1}{K_n}|\zeta_n(x_1, -3\varepsilon_n - Md_n)| \leqslant C(Md_n)^2 \quad \text{if } |x_1| < \frac{1}{16}$$

for any $M > 0$, and we shall later choose $M \gg 1$ (but $Md_n + 3\varepsilon_n < 1$). Combining this with (4.25), we get

$$|\psi_n(0, -3\varepsilon_n - Md_n)| \leqslant CC_n(\varepsilon_n^2 + d_n M\varepsilon_n) + (1 + CC_n\varepsilon_n)C(Md_n)^2. \tag{4.26}$$

On the other hand, by (4.10), (4.14), (4.20),

$$\psi_n(0, -3\varepsilon_n - Md_n) \geqslant C_n d_n^2 G_n(\widehat{\xi}_n) \sim C_n d_n^2 G(\widehat{\xi}_n) = KC_n d_n^2 |\widehat{\xi}_{n,2}|^2$$

where

$$(0, -\varepsilon_n - Md_n) - \widetilde{x}_n = d_n\widehat{\xi}_n = d_n(\widehat{\xi}_{n,1}, \widehat{\xi}_{n,2})$$

and $|\widehat{\xi}_{n,2}| \sim M$ if $M \gg 1$. Comparing this with (4.26) we get

$$KC_n d_n^2 M^2 \leqslant CC_n M\varepsilon_n^2 + C(1 + CC_n\varepsilon_n)M^2 d_n^2.$$

Since $d_n \geqslant c\varepsilon_n$ $(c > 0)$, choosing M to be large enough (say $KM > 2C$), we get a contradiction as we let $n \to \infty$. $\square$

Lemma 4.3 *If $G(\xi)$ is a function satisfying (4.15),(4.17), (4.18), and if $|G(\xi)| \leqslant C(1+|\xi|^2)$ in $\{\xi_2 < 0\}$, then*

$$G(\xi) \equiv K\xi_2^2 \tag{4.27}$$

where K is a constant.

Proof. The biharmonic function

$$H(\xi) = \frac{\partial^2}{\partial\xi_1^2}G(\xi)$$

satisfies the same boundary condition as G, and

$$|\nabla^j H(\xi)| \leqslant C|\xi|^{-j} \quad \text{if } |\xi| \text{ is large.}$$

By integration by parts,

$$\int_{\{|\xi|<R,\xi_2<0\}} H\Delta^2 H = \int_{\{|\xi|<R,\xi_2<0\}} (\Delta H)^2 + I_R$$

where $I_R \to 0$ if $R \to \infty$ (cf. the proof of Lemma 3.3). If follows that $\Delta H \equiv 0$ and, by unique continuation, $H \equiv 0$. Hence $G = g_1(\xi_2) + g_2(\xi_2)\xi_1$ and this implies (4.27). $\square$

Proof of Theorem 4.1. Applying the flatness lemma to $\varepsilon^{-3/2}\varphi(\varepsilon x)$ for $1/2 < |x| < 2$ and arbitrarily small ε, the assertion (4.1) easily follows. $\square$

5 Higher order expansion

In this section we assume that $f \in C^{2+\alpha}$ and obtain higher order expansion of φ.

Theorem 5.1 *Let φ be a solution of (3.1)–(3.5) and assume that $f \in C^{2+\alpha}[-x_*, 0]$ for some $\alpha > 0$. Then for any $0 < \eta < 1/2$ such that $\alpha + \eta > 1/2$, the expansion (1.14) holds.*

The assumptions of Theorem 5.1 imply that

$$\|\varphi\|_{H^2(B_1\setminus\Gamma)} \leqslant C_1, \quad \|f\|_{C^{2+\alpha}} \leqslant C_2.$$

For clarity we shall first prove a special case:

Lemma 5.2 *Under the assumptions of Theorem 5.1, for any $\eta \in (0, 1/2)$ there exists a constant C depending only on C_1, C_2 and η such that*

$$|\varphi(x) - r^{3/2}[A_1B_1(\theta) + A_2B_2(\theta)] - A_3r^2B_3(\theta)| \leqslant Cr^{2+\eta}. \tag{5.1}$$

In the sequel we shall need the following interpolation inequality:

$$\|\nabla u\|_{L^\infty} \leqslant C\Big(\|u\|_{L^\infty}^{\sigma/(1+\sigma)}\|\nabla u\|_{C^\sigma}^{1/(1+\sigma)} + \|u\|_{L^\infty}\Big) \quad (0 < \sigma < 1) \tag{5.2}$$

where the norms are taken in a bounded domain Ω. It suffices to prove (5.2) in dimension 1. For any $x \in \Omega$, let y, z belong to Ω such that

$$|y - x| = \varepsilon, \quad |z - x| = \varepsilon.$$

Then

$$\frac{u(y) - u(x)}{y - x} = u_x(\widetilde{x})$$

and

$$u_x(z) = u_x(z) - u_x(\widetilde{x}) + \frac{u(y) - u(x)}{y - x}.$$

Hence

$$\|u_x\|_{L^\infty} \leqslant \|u_x\|_{C^\sigma}\varepsilon^\sigma + \frac{2}{\varepsilon}\|u\|_{L^\infty}$$

and choosing

$$\varepsilon^{\sigma+1} = \|u\|_{L^\infty}/(\|u_x\|_{C^\sigma} + \|u\|_{L^\infty})$$

yields the assertion.

In the sequel we shall also use the interpolation inequality

$$\|\zeta\|_{C^\theta} \leqslant C\Big(\|\zeta\|_{L^\infty}^{(\beta-\theta)/\beta}[\zeta]_{C^\beta}^{\theta/\beta} + \|\zeta\|_{L^\infty}\Big) \quad (0 < \theta < \beta < 1), \tag{5.3}$$

which follows from

$$\frac{|\zeta(x) - \zeta(y)|}{|x - y|^\theta} = \frac{|\zeta(x) - \zeta(y)|}{|x - y|^\beta}|x - y|^{\beta-\theta} \leqslant [\zeta]_{C^\beta}\delta^{\beta-\theta} + \frac{2\|\zeta\|_{L^\infty}}{\delta^\theta}$$

by taking $\delta = (\|\zeta\|_{L^\infty}/\|\zeta\|_{C^\beta})^{1/\beta}$.

Proof of Lemma 5.2. The function

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^{3/2}}\varphi(\varepsilon x)$$

is biharmonic in $B_4(0)\setminus\Gamma_\varepsilon$, where

$$\Gamma_\varepsilon : x_2 = \frac{1}{\varepsilon}f(\varepsilon x_1) \equiv f_0(x_1)$$

is uniformly $C^{2,\alpha}$. By sub-Schauder estimates (cf. §9, Theorem 9.3),

$$|D^2\psi_\varepsilon| + [D^2\psi_\varepsilon]_\alpha \leqslant C \quad \text{for } 1 < |x| < 2.$$

Rewriting this inequality in terms of φ, we get

$$|D^2\varphi(x)| \leqslant C|x|^{-1/2}, \quad [D^2\varphi]_{C^\alpha[|x|<r<1]} \leqslant C|x|^{-1/2-\alpha} \tag{5.4}$$

where the constant C is independent of ε. Multiplying by a constant if necessary, we may assume without loss of generality that

$$|\psi_\varepsilon| \leqslant 1, \qquad |\nabla\psi_\varepsilon| \leqslant 1 \quad \text{for } |x| < 1,\ 0 < \varepsilon < 1.$$

Let $\varphi_0 = \psi_\varepsilon$ where ε will later on be chosen very small but fixed, and denote by G_0 the solution of

$$\begin{aligned}
&G \in H^2(B_1(0)\setminus\{\theta = \pm\pi\}),\\
&\Delta^2 G = 0 \quad \text{in } B_1(0)\setminus\{\theta=\pm\pi\},\\
&G = \frac{\partial G}{\partial n} = 0 \quad \text{on } \{\theta = \pm\pi\}, && (5.5)\\
&G = \varphi_0, \quad \frac{\partial G}{\partial n} = \frac{\partial\varphi_0}{\partial n} \quad \text{on rest of } \partial(B_1(0)\setminus\{\theta=\pm\pi\}). && (5.6)
\end{aligned}$$

It is clear that

$$|f_0''(x_1)| = \varepsilon|f''(\varepsilon x_1)| \leqslant \varepsilon,$$

and

$$|f_0(x_1)| \leqslant \frac{1}{2}\varepsilon|x_1|^2.$$

Hence, using (5.4),

$$|\varphi_0| + |G_0| \leqslant C|x|^{-1/2}\Big(\varepsilon|x_1|\Big)^2 \leqslant C\varepsilon^2|x|^{3/2} \quad \text{on } |\theta\pm\pi| \leqslant 2\varepsilon, \tag{5.7}$$

$$|\nabla\varphi_0| + |\nabla G_0| \leqslant C|x|^{-1/2}\Big(\varepsilon|x_1|\Big) \leqslant C\varepsilon|x|^{1/2} \quad \text{on } |\theta\pm\pi| \leqslant 2\varepsilon, \tag{5.8}$$

Applying Theorem 2.2, we get

$$\|\varphi_0 - G_0\|_{L^p(\Omega_{2\pi-2\varepsilon})} \leqslant C\varepsilon. \tag{5.9}$$

Notice that both φ_0 and G_0 are uniformly $C^{3/2}$. It is clear that if $\sigma < 1/4 - \eta/2$ (recall that $0 < \eta < 1/2$), then $[\nabla(\varphi_0 - G_0)]_{C^\sigma} \leqslant C$ for some universal constant C (Actually, $\varphi_0 - G_0$ is bounded also in the $C^{1/2}$ norm in this first step of the iteration; but we use the C^σ norm for the later iterations). Thus by interpolation, for any $\beta < 1$, if we choose p large enough we obtain

$$|\varphi_0 - G_0| \leqslant C\varepsilon^\beta \quad \text{for } |x| < 1. \tag{5.10}$$

By (5.2) with $u = \varphi_0 - G_0$, we also have

$$|\nabla(\varphi_0 - G_0)| \leqslant C|\varphi_0 - G_0|^{\sigma/(1+\sigma)}[\nabla(\varphi_0 - G_0)]_{C^\sigma}^{1/(1+\sigma)} + C\varepsilon^\beta \leqslant C\varepsilon^{\beta\sigma/(1+\sigma)} \quad \text{for } |x| < 1. \tag{5.11}$$

Clearly, there is a constant C and $P_0(x) = [r^{3/2}B_\varepsilon(\theta) + r^2\widetilde{B}_\varepsilon(\theta)]$ such that

$$|G_0 - P_0| \leqslant Cr^{5/2} \quad \text{for } r < 1/2, \tag{5.12}$$

$$|\nabla(G_0 - P_0)| \leqslant Cr^{3/2} \quad \text{for } r < 1/2. \tag{5.13}$$

It follows that

$$|\varphi_0 - P_0| \leqslant C\varepsilon^\beta + Cr^{5/2} \quad \text{for } r < 1/2,$$

$$|\nabla(\varphi_0 - P_0)| \leqslant C\varepsilon^{\beta\sigma/(1+\sigma)} + Cr^{3/2} \quad \text{for } r < 1/2.$$

We now fix small constants ε, λ such that

$$C\varepsilon^\beta + C(2\lambda)^{5/2} \leqslant \lambda^{2+\eta}, \qquad C\varepsilon^{\beta\sigma/(1+\sigma)} + C(2\lambda)^{3/2} \leqslant \lambda^{1+\eta};$$

we can actually take $\varepsilon = \lambda^\gamma$ for γ large enough and any λ sufficiently small. Since ε is now fixed, we shall simply write P_0 as

$$P_0 = r^{3/2}B^0(\theta) + r^2\widetilde{B}^0(\theta),$$

and we then have

$$|\varphi_0 - P_0| \leqslant \lambda^{2+\eta} \quad \text{for } |x| < 2\lambda, \tag{5.14}$$

$$|\nabla(\varphi_0 - P_0)| \leqslant \lambda^{1+\eta} \quad \text{for } |x| < 2\lambda. \tag{5.15}$$

Next, we define

$$\varphi_1 = \frac{1}{\lambda^{2+\eta}}\Big(\varphi_0(\lambda x) - P_0(\lambda x)\Big) \quad \text{for } r \leqslant 2. \tag{5.16}$$

Then

$$|\varphi_1| \leqslant 1, \qquad |\nabla\varphi_1| \leqslant 1 \quad \text{for } r \leqslant 2, \tag{5.17}$$

and, by (5.7), (5.8)

$$|\varphi_1| \leqslant C\lambda^{-2-\eta}|\lambda x|^{-1/2}(\varepsilon\lambda|\lambda x_1|)^2 \leqslant C\varepsilon^2\lambda^{3/2-\eta}|x|^{3/2} \quad \text{for } |\theta \pm \pi| < 2(\varepsilon\lambda), \tag{5.18}$$

$$|\nabla\varphi_1| \leqslant C\lambda^{-1-\eta}|\lambda x|^{-1/2}(\varepsilon\lambda|\lambda x_1|) \leqslant C\varepsilon\lambda^{1/2-\eta}|x|^{1/2} \quad \text{for } |\theta \pm \pi| < 2(\varepsilon\lambda). \tag{5.19}$$

Finally, since $|D^2\varphi_0| + |D^2 G_0| \leqslant C|x|^{-1/2}$,

$$[\nabla\varphi_1]_{C^\sigma(\{|\theta\pm\pi|<2(\varepsilon\lambda)\}\cap\{r<|x|\})} \leqslant C\lambda^{-1-\eta}|\lambda x|^{-1/2}(\varepsilon\lambda|\lambda x_1|)^{1-\sigma} \leqslant C\varepsilon^{1-\sigma}|x|^{1/2-\sigma} \tag{5.20}$$

since $\sigma < 1/4 - \eta/2$. We claim that

$$|\varphi_1(x)| \leqslant |x|^{1+\sigma} \quad \text{if } |x| < 2. \tag{5.21}$$

For clarity of exposition we shall postpone the proof until the end of this section, and in fact, establish a general result (namely Lemma 5.4) for which (5.21) follows as a special case (upon taking $f_j(x_1)$ to be parabolic curves $x_2 = \delta_j x_1^2$, with $-1 < \delta_1 < \delta_2 < 1$; notice that φ_1 is biharmonic outside the thin region enclosed by these two parabolic curves).

In view of (5.21), we can apply Theorem 9.3 (ii) to the function

$$\frac{\varphi_1(\tau x)}{\tau^{1+\sigma}}$$

which (by (5.21)) is bounded in $1/2 < |x| < 2$ and whose $C^{1+\sigma}$-norm is uniformly bounded in the sector $|\theta \pm \pi| \leqslant 2(\varepsilon\lambda)$ (by (5.19), (5.20)). We then have

$$[\nabla\varphi_1]_{C^\sigma(B_1)} \leqslant C \tag{5.22}$$

for some universal constant C. We can now proceed as above (with the same ε, λ) to derive

$$|\varphi_1 - P_1| \leqslant \lambda^{2+\eta} \quad \text{for } |x| < 2\lambda, \tag{5.23}$$

$$|\nabla(\varphi_1 - P_1)| \leqslant \lambda^{1+\eta} \quad \text{for } |x| < 2\lambda \tag{5.24}$$

for some $P_1(x) = r^{3/2}B^1(\theta) + r^2\widetilde{B}^1(\theta)$.

Proceeding by induction, we define

$$\begin{aligned}\varphi_{k+1}(x) &= \frac{1}{\lambda^{2+\eta}}\Big(\varphi_k(\lambda x) - P_k(\lambda x)\Big)\\ &= (\lambda^{-(2+\eta)})^{k+1}\varphi_0(\lambda^{k+1}x) - \sum_{j=0}^{k}(\lambda^{-(2+\eta)})^{j+1}P_{k-j}(\lambda^{j+1}x).\end{aligned}$$

Clearly,

$$\begin{aligned}|(\lambda^{-(2+\eta)})^{k+1}\varphi_0(\lambda^{k+1}x)| &\leqslant C\lambda^{-(2+\eta)(k+1)}|\lambda^{k+1}x|^{-1/2}\Big(\varepsilon\lambda^{k+1}|\lambda^{k+1}x|\Big)^2\\ &\leqslant C\varepsilon^2\lambda^{(3/2-\eta)k}|x|^{3/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1},\end{aligned}$$

and

$$\begin{aligned}\sum_{j=0}^{k}(\lambda^{-(2+\eta)})^{j+1}|P_{k-j}(\lambda^{j+1}x)| &\leqslant C\sum_{j=0}^{k}(\lambda^{-(2+\eta)})^{j+1}|\lambda^{j+1}x|^{-1/2}\Big(\varepsilon\lambda^{k+1}|\lambda^{j+1}x|\Big)^2\\ &\leqslant C\varepsilon^2\lambda^{(3/2-\eta)k}|x|^{3/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1}.\end{aligned}$$

It follows that

$$|\varphi_{k+1}(x)| \leqslant C\varepsilon^2\lambda^{(3/2-\eta)(k+1)}|x|^{3/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1}. \tag{5.25}$$

Similarly,

$$\begin{aligned}\left|\nabla\left[(\lambda^{-(2+\eta)})^{k+1}\varphi_0(\lambda^{k+1}x)\right]\right| &\leqslant C\lambda^{-(1+\eta)(k+1)}|\lambda^{k+1}x|^{-1/2}\Big(\varepsilon\lambda^{k+1}|\lambda^{k+1}x|\Big)\\ &\leqslant C\varepsilon\lambda^{(1/2-\eta)(k+1)}|x|^{1/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1},\end{aligned}$$

and

$$\begin{aligned}\sum_{j=0}^{k}(\lambda^{-(2+\eta)})^{j+1}\Big|\nabla\Big(P_{k-j}(\lambda^{j+1}x)\Big)\Big| &\leqslant C\sum_{j=0}^{k}(\lambda^{-(1+\eta)})^{j+1}|\lambda^{j+1}x|^{-1/2}\Big(\varepsilon\lambda^{k+1}|\lambda^{j+1}x|\Big)\\ &\leqslant C\varepsilon\lambda^{(1/2-\eta)k}|x|^{1/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1}.\end{aligned}$$

It follows that

$$|\nabla\varphi_{k+1}(x)| \leqslant C\varepsilon\lambda^{(1/2-\eta)(k+1)}|x|^{1/2} \quad \text{for } |\theta\pm\pi| < 2\varepsilon\lambda^{k+1}. \tag{5.26}$$

We can apply the same procedure to deduce that

$$[\nabla\varphi_{k+1}(x)]_{C^\sigma(\{|\theta\pm\pi|<2\varepsilon\lambda^{k+1}\}\cap\{r<|x|\})} \leqslant C\varepsilon^{1-\sigma}|x|^{1/2-\sigma}. \tag{5.27}$$

With the estimates (5.25) and (5.26) and (5.27) at hand (where the constants C are independent of ε, λ, k), the same procedure can be used to deduce that

$$|\varphi_{k+1} - P_{k+1}(x)| \leqslant \lambda^{2+\eta} \quad \text{for } |x| \leqslant 2\lambda, \tag{5.28}$$

$$|\nabla(\varphi_{k+1} - P_{k+1}(x))| \leqslant \lambda^{1+\eta} \quad \text{for } |x| \leqslant 2\lambda. \tag{5.29}$$

We can rewrite the inequality (5.28) in terms of the original variables:

$$|\varphi_0(x) - Q_k(x)| \leqslant (\lambda^k)^{2+\eta} \quad \text{for } |x| < 2\lambda^k,$$

where

$$Q_k(x) = \sum_{j=0}^{k}(\lambda^j)^{2+\eta}P_j\Big(\frac{x}{\lambda^j}\Big)$$

Since $P_j(x/\lambda^j)$ are bounded by $C|x/\lambda^j|^2 + C|x/\lambda^j|^{3/2}$, the series converges, and we let

$$Q(x) = \lim_{k\to\infty} Q_k(x) = \sum_{j=0}^{\infty}(\lambda^j)^{2+\eta}P_j\Big(\frac{x}{\lambda^j}\Big).$$

Since $Q_k(x)$ are of the form $r^{3/2}B^*(\theta) + r^2\widetilde{B}^*(\theta)$, the limit function is also of the same form, and we denote it by

$$Q(x) \equiv r^{3/2}B(\theta) + r^2\widetilde{B}(\theta).$$

We then have

$$|Q(x)-Q_k(x)| \leqslant \sum_{j=k+1}^{\infty} (\lambda^j)^{2+\eta}\Big[\Big|\frac{x}{\lambda^j}\Big|^{3/2}+\Big|\frac{x}{\lambda^j}\Big|^2\Big] \leqslant C\Big\{(\lambda^k)^{1/2+\eta}|x|^{3/2}+(\lambda^k)^{\eta}|x|^2\Big\}.$$

Therefore,

$$|\varphi_0(x)-Q(x)| \leqslant C(\lambda^k)^{2+\eta} \quad \text{for } |x|<\lambda^k. \tag{5.30}$$

For each $|x|<\lambda$, we choose k such that $\lambda^{k+1}<|x|\leqslant\lambda^k$. Then the above inequality implies that

$$|\varphi_0(x)-Q(x)| \leqslant C|x|^{2+\eta},$$

and the proof of Lemma 5.2 is complete. □

Remark 5.1. The above proof also shows that

$$|\nabla(\varphi_0(x)-Q(x))| \leqslant C|x|^{1+\eta}.$$

Remark 5.2. The iterative argument used in the proof of Lemma 5.2 is similar to the iterative argument used in the proof of Lemma 2.4 of [12].

Lemma 5.3 *Suppose G_j is the solution of the following system:*

$$G_j \in H^2(B_1(0)\setminus\Gamma_j),$$

$$\Delta^2 G_j = 0 \quad \textit{in } B_1(0)\setminus\Gamma_j,$$

$$G_j = \frac{\partial G_j}{\partial n} = 0 \quad \textit{on both sides of } \Gamma_j, \tag{5.31}$$

$$G_j = \varphi_j, \quad \frac{\partial G_j}{\partial n} = \frac{\partial \varphi_j}{\partial n} \quad \textit{on rest of } \partial(B_1(0)\setminus\Gamma_j), \tag{5.32}$$

where $\Gamma_j : x_2 = \frac{1}{2}f''(0)\lambda^j\varepsilon x_1^2,\ -2<x_1<0$, and

$$|f''(0)| \leqslant 1, \quad \|\varphi_j\|_{C^1(\overline{B}_1)} \leqslant 1. \tag{5.33}$$

Then, for any small $\delta>0$ there exist sufficiently small ε and λ such that the corresponding G_j satisfies:

$$|G_j(x)-P_j(x)| \leqslant \frac{1}{2}\lambda^{3-\delta} \quad \textit{for } |x|<2\lambda, \tag{5.34}$$

$$|\nabla G_j(x)-\nabla P_j(x)| \leqslant \frac{1}{2}\lambda^{2-\delta} \quad \textit{for } |x|<2\lambda, \tag{5.35}$$

where

$$P_j(x) = r^{3/2}\Big(A_1^jB_1(\theta)+A_2^jB_2(\theta)+A_2^j\varepsilon\mu_j r\dot{B}_2(\theta)\big)\Big)+r^2A_3^jB_3(\theta)$$
$$+r^{5/2}\Big(A_4^jB_4(\theta)+A_5^jB_5(\theta)\Big)-r^{5/2}\frac{3}{2}A_2^j\varepsilon\mu_jB_1(\theta),$$

for all $j\geqslant 0$, $0\leqslant\mu_j<1$.

Proof. Under the above assumptions it is clear that

$$|G_j| \leqslant C|x|^{3/2}, \quad |\nabla G_j| \leqslant C|x|^{1/2}, \quad [\nabla G_j]_{C^{1/2}(B_{1/2})} \leqslant C.$$

Let $\widetilde{G}_j$ be the solution of the following problem

$$\widetilde{G}_j \in H^2(B_{1/2}(0) \setminus \{\theta = \pm\pi\}),$$
$$\Delta^2 \widetilde{G}_j = 0 \quad \text{in } B_{1/2}(0) \setminus \{\theta = \pm\pi\},$$
$$\widetilde{G}_j = \frac{\partial \widetilde{G}_j}{\partial n} = 0 \quad \text{on } \{\theta = \pm\pi\}, \tag{5.36}$$
$$\widetilde{G}_j = G_j, \quad \frac{\partial \widetilde{G}_j}{\partial n} = \frac{\partial G_j}{\partial n} \quad \text{on rest of } \partial(B_{1/2}(0) \setminus \{\theta = \pm\pi\}). \tag{5.37}$$

By scaling and using $C^{2+\alpha}$ estimates as before, we get

$$|x|^{-3/2}\varepsilon^{-1}|G_j - \widetilde{G}_j| + |x|^{-1/2}|\nabla(G_j - \widetilde{G}_j)| \leqslant C\varepsilon \quad \text{ if } |\theta \pm \pi| \leqslant 2\varepsilon.$$

Then by maximum principle (Theorem 2.2),

$$\|G_j - \widetilde{G}_j\|_{L^p(B_{1/2})} \leqslant C\varepsilon.$$

For any $\beta < 1$, we use $C^{3/2}$ regularity and interpolation, and take p to be large enough to obtain

$$|G_j - \widetilde{G}_j| \leqslant C\varepsilon^{\beta}, \quad |\nabla(G_j - \widetilde{G}_j)| \leqslant C\varepsilon^{\beta/3}.$$

By (1.11), we have

$$|\widetilde{G}_j - P(x)| \leqslant Cr^3, \qquad |\nabla(\widetilde{G}_j - P(x))| \leqslant Cr^2,$$

where

$$P(x) = r^{3/2}B(\theta) + r^2\widetilde{B}(\theta) + r^{5/2}\widetilde{\widetilde{B}}(\theta).$$

Notice that the extra terms in $P_j(x)$ are of order ε. Choosing ε and λ such that

$$C\varepsilon^{\beta} + C(2\lambda)^3 < \frac{1}{4}\lambda^{3-\delta}, \quad C\varepsilon^{\beta/3} + C(2\lambda)^2 < \frac{1}{4}\lambda^{2-\delta}, \tag{5.38}$$

the proof is now complete. $\square$

Proof of Theorem 5.1. We shall modify the proof of Lemma 5.2 for C^2 expansion to obtain $C^{5/2}$ expansion. The P_j will be of the form

$$\begin{aligned} P_j(x) &= \Big\{r^{3/2}\Big(A_1^jB_1(\theta) + A_2^jB_2(\theta) + A_2^j\varepsilon\mu_j r\dot{B}_2(\theta)\Big)\Big\} \\ &\qquad + \Big\{r^2A_3^jB_3(\theta) + r^{5/2}\Big(A_4^jB_4(\theta) + A_5^jB_5(\theta)\Big) - r^{5/2}\frac{3}{2}A_2^j\varepsilon\mu_jB_1(\theta)\Big\}, \\ &\equiv P_j^1(x) + P_j^2(x) \end{aligned}$$

obtained from Lemma 5.3.

Notice that the terms $r^{5/2}\cos\frac{3}{2}\theta$ in P_j^1 and P_j^2 cancel out, and the term $r^{5/2}\cos\frac{1}{2}\theta$ (in P_j^1 and P_j^2) is biharmonic. Thus P_j still satisfies the biharmonic equation: $\Delta^2 P_j \equiv 0$.

We define φ_k inductively as

$$\begin{aligned}\varphi_{k+1} &= \frac{1}{\lambda^{3-\delta}}\Big(\varphi_k(\lambda x) - P_k(\lambda x)\Big)\\ &= (\lambda^{-(3-\delta)})^{k+1}\varphi_0(\lambda^{k+1}x) - \sum_{j=0}^{k}(\lambda^{-(3-\delta)})^{j+1}P_{k-j}(\lambda^{j+1}x),\end{aligned}$$

where μ_{k-j} are still to be determined.

Instead of two lines $\theta = \pm(\pi - 2\varepsilon\lambda^k)$, we now use the two C^∞ curves

$$\Gamma_\pm^k: \quad x_2 = \frac{1}{2}\varepsilon\lambda^k f''(0)x_1^2 \pm \varepsilon^{1+\alpha}\lambda^{(1+\alpha)k}x_1^2, \quad -2 < x_1 < 0,$$

which are $\varepsilon^{1+\alpha}\lambda^{(1+\alpha)k}x_1^2$ close to the original curve (instead of just $\varepsilon\lambda^k x_1$ close to the original curve). Then, using (5.4), we find that on $\Gamma_\pm^{k+1}$, as well as on any arc $|x|$ =const. in the thin region connecting Γ_+^{k+1} and Γ_-^{k+1},

$$|(\lambda^{-(3-\delta)})^{k+1}\varphi_0(\lambda^{k+1}x)| \leqslant C\varepsilon^{2(1+\alpha)}\lambda^{(k+1)(1/2+2\alpha+\delta)}|x|^{3/2}, \tag{5.39}$$

$$\left|\nabla\Big((\lambda^{-(3-\delta)})^{k+1}\varphi_0(\lambda^{k+1}x)\Big)\right| \leqslant C\varepsilon^{1+\alpha}\lambda^{(k+1)(\alpha+\delta-1/2)}|x|^{1/2}, \tag{5.40}$$

$$\left[\nabla\Big((\lambda^{-(3-\delta)})^{k+1}\varphi_0(\lambda^{k+1}x)\Big)\right]_{C^\sigma(\Gamma_\pm^{k+1}\cap\{|x|<R\})} \leqslant C\varepsilon^{(1+\alpha)(1-\sigma)}R^{1/2-\sigma}, \tag{5.41}$$

provided $0 < \delta < 1/2$, $\alpha + \delta > 1/2$, $0 < \sigma \leqslant \sigma_0$, where

$$\sigma_0 = \frac{\alpha + \delta - 1/2}{2 + \alpha};$$

cf. the proof of (5.18)–(5.20).

Next, we estimate the sum

$$\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}(\lambda^{j+1}x) = \sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^1(\lambda^{j+1}x) + \sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^2(\lambda^{j+1}x)$$

on the curves $\Gamma_\pm^{k+1}$ and in the thin region lying between them. The second derivatives of those terms involving r^2 and $r^{5/2}$ are bounded. Therefore, we can follow the calculations as in the case for C^2 expansion to conclude that on $\Gamma_\pm^{k+1}$ as well as any arc $|x|$ =const. which lies in the thin region lying between Γ_+^{k+1} and Γ_-^{k+1},

$$\left|\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^2(\lambda^{j+1}x)\right| \leqslant C\varepsilon^2\lambda^{k(1+\delta)}|x|^{3/2}, \tag{5.42}$$

$$\left|\nabla\Big(\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^2(\lambda^{j+1}x)\Big)\right| \leqslant C\varepsilon\lambda^{k\delta}|x|^{1/2}; \tag{5.43}$$

furthermore,

$$\begin{aligned}\left[\nabla\Big(\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^2(\lambda^{j+1}x)\Big)\right]_{C^\sigma(\Gamma_\pm^{k+1}\cap\{|x|<R\})} &\leqslant C\varepsilon^{(1-\sigma)}\lambda^{k(\delta-2\sigma)}R^{1/2-\sigma}\\ &\leqslant C\varepsilon^{(1-\sigma)}R^{1/2-\sigma},\end{aligned} \tag{5.44}$$

provided $\sigma \leqslant \delta/2$.

Next, we split P_k^1 into two parts.

$$\begin{aligned} P_k^1(x) &= r^{3/2}A_1^k B_1(\theta) + r^{3/2}A_2^k\Big(B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big) \\ &\equiv P_k^{11}(x)+P_k^{12}(x). \end{aligned}$$

Since

$$B_1(\theta)\Big|_{\theta=\pm\pi} = \frac{\partial}{\partial\theta}B_1(\theta)\Big|_{\theta=\pm\pi} = \frac{\partial^2}{\partial\theta^2}B_1(\theta)\Big|_{\theta=\pm\pi} = 0,$$

we have

$$B_1(\theta)=O(|\theta\pm\pi|^3), \quad \frac{\partial}{\partial\theta}B_1(\theta)=O(|\theta\pm\pi|^2).$$

Using the computation for C^2 expansion, we find that the estimates (5.42)–(5.44) are valid for $\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^{11}(\lambda^{j+1}x)$.

To estimate P_k^{12}, notice that $-x_1 = r+O(r^2)$ on the curve $x_2=\frac{1}{2}f''(0)x_1^2,\ x_1<0$. This curve can be rewritten in polar coordinates:

$$\theta = \pm\pi - \frac{1}{2}f''(0)r + O(r^2).$$

We take $\mu_j = \lambda^j f''(0)/2$ in order to make the curves $\Gamma_\pm^k \quad \varepsilon^{1+\alpha}\lambda^{(1+\alpha)k}r^2$ close to the curves $\theta=\pm\pi-\varepsilon\mu_k r$. Since $\dot{B}_2(\theta)=(\partial/\partial\theta)B_2(\theta)=\frac{3}{2}\cos\frac{3}{2}\theta+\frac{1}{2}\cos\frac{1}{2}\theta$, we have

$$B_2(\theta)\Big|_{\theta=\pm\pi} = \frac{\partial}{\partial\theta}B_2(\theta)\Big|_{\theta=\pm\pi} = \frac{\partial^3}{\partial\theta^3}B_2(\theta)\Big|_{\theta=\pm\pi} = 0, \quad \frac{\partial^2}{\partial\theta^2}B_2(\theta)\Big|_{\theta=\pm\pi} = \pm 2,$$

and

$$\begin{aligned} & B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big|_{\theta=\pm\pi} = 0, \\ & \frac{\partial}{\partial\theta}\Big(B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big)\Big|_{\theta=\pm\pi} = \pm 2\varepsilon\mu_k r, \\ & \frac{\partial^2}{\partial\theta^2}\Big(B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big)\Big|_{\theta=\pm\pi} = \pm 2, \\ & \Big|\frac{\partial^3}{\partial\theta^3}\Big(B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big)\Big|_{|\theta\pm\pi|<2\varepsilon\mu_k} \leqslant C\varepsilon\mu_k. \end{aligned}$$

Therefore

$$\begin{aligned} B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} &= \pm(2\varepsilon\mu_k r)(-\varepsilon\mu_k r)+\frac{1}{2}\cdot(\pm 2)\cdot(-\varepsilon\mu_k r)^2+O(|\varepsilon\mu_k r|^3) \\ &= O(|\varepsilon\mu_k r|^2), \\ \frac{\partial}{\partial\theta}\Big[B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big]\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} &= \pm 2\varepsilon\mu_k r \pm 2(-\varepsilon\mu_k r)+O(|\varepsilon\mu_k r|^2) \ = \ O(|\varepsilon\mu_k r|^2), \\ \frac{\partial}{\partial r}\Big[B_2(\theta)+\varepsilon\mu_k r\dot{B}_2(\theta)\Big]\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} &= \varepsilon\mu_k\dot{B}_2(\theta)\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} \ = \ \varepsilon\mu_k O(|\varepsilon\mu_k r|). \end{aligned}$$

Rewriting these estimates in terms of P_k^{12}, we have

$$|P_k^{12}(x)|\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} \leqslant C\varepsilon^2\mu_k^2 r^{7/2}, \tag{5.45}$$

$$|\nabla P_k^{12}(x)|\Big|_{\theta=\pm\pi-\varepsilon\mu_k r} \leqslant C\varepsilon^2\mu_k^2 r^{5/2}. \tag{5.46}$$

From the definition of P_k^{12} it is clear that

$$|D^2 P_k^{12}(x)| \leqslant C|x|^{-1/2} \quad \text{for } |x| < 2. \tag{5.47}$$

Recall that $\Gamma_\pm^k$ is $\varepsilon^{1+\alpha}\lambda^{(1+\alpha)k}r^2$ close to the curve $\theta = \pm\pi - \varepsilon\mu_k r$. Using (5.45)–(5.47), we can then derive the estimates (similarly to (5.39)–(5.41))

$$\begin{aligned}
&\Big|\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^{12}(\lambda^{j+1}x)\Big| \leqslant C\varepsilon^2\lambda^{(k+1)(1/2+\delta)}|x|^{3/2},\\
&\Big|\nabla\Big(\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^{12}(\lambda^{j+1}x)\Big)\Big| \leqslant C\varepsilon^{1+\alpha}\lambda^{(k+1)(\alpha+\delta-1/2)}|x|^{1/2},\\
&\Big[\nabla\Big(\sum(\lambda^{-(3-\delta)})^{j+1}P_{k-j}^{12}(\lambda^{j+1}x)\Big)\Big]_{C^\sigma(\Gamma_\pm^{k+1}\cap\{|x|<R\})} \leqslant C\varepsilon^{(1+\alpha)(1-\sigma)}R^{1/2-\sigma},
\end{aligned}$$

in the thin region bounded by $\Gamma_\pm^{k+1}$.

Combining all these estimates, we find that

$$|\varphi_{k+1}(x)|\Big|_{\Gamma_\pm^{k+1}} \leqslant C\varepsilon^2, \quad |\nabla\varphi_{k+1}(x)|\Big|_{\Gamma_\pm^{k+1}} \leqslant C\varepsilon, \quad [\nabla\varphi_{k+1}]_{C^\sigma(\Gamma_\pm^{k+1})} \leqslant C\varepsilon^{1-\sigma}. \tag{5.48}$$

By Theorem 2.2, we then have

$$\|\varphi_{k+1} - G_{k+1}\|_{L^p(B_1)} \leqslant C\varepsilon, \tag{5.49}$$

where G_{k+1} is given in Lemma 5.3. From this, we can argue in the same way as before (using Lemma 5.4 below in deriving Hölder estimates for $\nabla(\varphi_{k+1} - G_{k+1})$) to conclude that, for any $\beta < 1$,

$$\begin{aligned}
&|\varphi_{k+1} - G_{k+1}| \leqslant C\varepsilon^\beta,\\
&|\nabla(\varphi_{k+1} - G_{k+1})| \leqslant C\varepsilon^{\beta\sigma/(1+\sigma)}.
\end{aligned}$$

Combining this with Lemma 5.3, we conclude that

$$\begin{aligned}
&|\varphi_{k+1} - P_{k+1}| \leqslant \lambda^{3-\delta},\\
&|\nabla(\varphi_{k+1} - P_{k+1})| \leqslant \lambda^{2-\delta}.
\end{aligned}$$

We now proceed as in the proof of (5.28)–(5.30) (with $\lambda^{2+\eta}$ replaced by $\lambda^{3-\delta}$) to establish the estimate

$$\begin{aligned}
\varphi_0(x) \;=\; & r^{3/2}\Big(A_1B_1(\theta) + A_2B_2(\theta)\Big) + r^2A_3B_3(\theta)\\
& + r^{5/2}\Big[A_4B_4(\theta) + A_5B_5(\theta) - 2A_2\varepsilon f''(0)\cos\frac{1}{2}\theta\Big] + O(r^{3-\delta}),
\end{aligned}$$

which, when written in terms of the original variables, becomes

$$\varphi(x) = r^{3/2}\Big(A_1B_1(\theta)+A_2B_2(\theta)\Big)+r^2A_3B_3(\theta) + r^{5/2}\Big[A_4B_4(\theta)+A_5B_5(\theta)-2A_2f''(0)\cos\frac{1}{2}\theta\Big]+O(r^{3-\delta}),$$

with different coefficients A_3, A_4, A_5. This completes the proof of (1.14) (with $\eta=\delta$). □

Remark 5.3. Theorem 5.1 is new even if Γ is a parabola $x_2=x_1^2$.

Remark 5.4. If $f'(0)\neq 0$ then, in Theorem 5.1, the second derivative $f''(0)$ should be replaced by the curvature κ at 0 and $B_i(\theta)$ should be replaced by $B_i(\theta-\theta_0)$, where $\theta_0=f'(0)$.

Remark 5.5. As in the case of Remark 5.1, the proof of Theorem 5.1 shows that (1.14) can be differentiated, with

$$\nabla(O(r^{3-\eta}))=O(r^{2-\eta}).$$

The same remark applies to (1.13), with $\nabla(O(r^{2-\eta}))=O(r^{1-\eta})$ and the proof can be given by the method of §3 using (2.20).

The next lemma establishes (as a special case) the estimate (5.21) which was needed in the proof of Lemma 5.2.

Lemma 5.4 *Let $0<\alpha<1$, and let $f_j=f_j^\tau(x_1)$ $(j=1,2)$ be curves satisfying:*

$$\begin{gathered} f_j(0)=f_j'(0)=0, \quad [f_j']_{C^\alpha[-2,0]}\leqslant 1, \\ \tau x_1\leqslant f_1(x_1)\leqslant f_2(x_2)\leqslant -\tau x_1 \quad \text{for } -2<x_1<0, \end{gathered} \tag{5.50}$$

where $0<\tau<1$. Let $\varphi\in H^2(B_2\setminus\{f_1(x_1)\leqslant x_2\leqslant f_2(x_2),\ x_1<0\})$ satisfy

$$\Delta^2\varphi=0 \quad \text{in } B_2\setminus\{f_1(x_1)\leqslant x_2\leqslant f_2(x_2),\ x_1<0\}, \tag{5.51}$$

$$|\nabla\varphi|\leqslant 1 \quad \text{in } B_2, \tag{5.52}$$

$$\varphi(0)=0, \quad \nabla\varphi(0)=0, \tag{5.53}$$

$$[\nabla\varphi]_{C^\sigma(\{|x_2|\leqslant-\tau x_1\}\cap\{|x|<,x_1<0\})}\leqslant 1. \tag{5.54}$$

Then, for any $\sigma<1/2$,

$$|\varphi(x)|\leqslant C|x|^{1+\sigma}, \tag{5.55}$$

where the constant C depends only on σ and α, but not on the f_j and τ.

Proof. From (5.52) and (5.53) we deduce that

$$|\varphi(x)|\leqslant|x| \quad \text{in } B_2. \tag{5.56}$$

Next, (5.53) and (5.54) imply that

$$|\nabla\varphi(x)|\leqslant|x|^\sigma \quad \text{for } |x_2|\leqslant-\tau x_1,\ \ x_1<0 \ \ \text{and } |x|<2, \tag{5.57}$$

and therefore

$$|\varphi(x)| \leqslant |x|^{1+\sigma} \quad \text{for } |x_2| \leqslant -\tau x_1, \;\; x_1 < 0 \quad \text{and } |x| < 2. \tag{5.58}$$

Consider the function

$$Z_\delta(x) = \frac{\varphi(x)}{|x|^{1+\sigma} + \delta|x|} \qquad \forall \delta > 0. \tag{5.59}$$

By (5.56), $\sup\limits_{|x|<1} |Z_\delta(x)| < +\infty$ for any $\delta > 0$. We claim that

$$\sup_{|x|<1} |Z_\delta(x)| \leqslant C \tag{5.60}$$

for some constant C independent of δ, τ and f_j; once this is proved, we can then finish the proof of the lemma by letting $\delta \to 0$.

Suppose (5.60) is not true. Then there exist sequences $\varphi = \varphi_n$, $f_j = f_{nj}$ $(j = 1, 2)$, $\delta_n \to 0$, $x_n \to 0$, τ_n (τ_n may go to 0) such that

$$C_n = \sup_{|x|<1} |Z_{\delta_n}(x)| = |Z_{\delta_n}(x_n)| \to \infty.$$

Define $G_n(\xi)$ by:

$$\varphi_n(x) = C_n(R_n^{1+\sigma} + \delta_n R_n) G_n(\xi) \quad \text{where} \quad x = R_n \xi, \quad R_n = |x_n|.$$

Then

$$|G_n(\xi)| \leqslant \frac{|\varphi_n(x)|}{C_n(R_n^{1+\sigma} + \delta_n R_n)} \leqslant \frac{|\xi|^{1+\sigma} R_n^{1+\sigma} + \delta_n |\xi| R_n}{(R_n^{1+\sigma} + \delta_n R_n)} \leqslant |\xi|^{1+\sigma} + |\xi|,$$

and, from (5.58), (5.57) and (5.54) we also have

$$|G_n(\xi)| \leqslant \frac{R_n^{1+\sigma}}{C_n(R_n^{1+\sigma} + \delta_n R_n)} |\xi|^{1+\sigma} \leqslant \frac{1}{C_n} |\xi|^{1+\sigma} \qquad \text{for } |\xi_2| < -\tau_n \xi_1, |\xi| \leqslant \frac{2}{R_n},$$

$$|\nabla_\xi G_n(\xi)| \leqslant \frac{R_n^\sigma}{C_n(R_n^\sigma + \delta_n)} |\xi|^{1/2} \leqslant \frac{1}{C_n} |\xi|^{1/2} \qquad \text{for } |\xi_2| < -\tau_n \xi_1, |\xi| \leqslant \frac{2}{R_n},$$

and, for any $K > 0$,

$$[\nabla_\xi G_n]_{C^\sigma(\{|\xi_2| < -\tau_n \xi_1\} \cap \{|\xi| < K\})} \leqslant \frac{R_n^\sigma}{C_n(R_n^\sigma + \delta_n)} \leqslant \frac{1}{C_n}.$$

The curve $x_2 = f_j(x_1)$ $(j = 1, 2)$ under the change of variables $x \to \xi$ becomes

$$\xi_2 = \frac{1}{R_n} f_j(R_n \xi_1), \quad -\frac{2}{R_n} < \xi_1 < 0.$$

Under the assumptions of the lemma, we have, for any $K > 1$,

$$\left\{ \frac{1}{R_n} f_1(R_n \xi_1) \leqslant \xi_2 \leqslant \frac{1}{R_n} f_2(R_n \xi_1) \right\} \cap \{|\xi| < K\} \subset \{|\xi_2| \leqslant -R_n^\alpha K^\alpha \xi_1\},$$

$$\left\{ \frac{1}{R_n} f_1(R_n \xi_1) \leqslant \xi_2 \leqslant \frac{1}{R_n} f_2(R_n \xi_1) \right\} \subset \{|\xi_2| \leqslant -\tau_n \xi_1\}.$$

Therefore, for $\beta_n = \min(R_n^\alpha K^\alpha, \tau_n)$, we have

$$\left\{\frac{1}{R_n} f_1(R_n\xi_1) \leqslant \xi_2 \leqslant \frac{1}{R_n} f_2(R_n\xi_1)\right\} \cap \{|\xi| < K\} \subset \{|\xi_2| \leqslant -\beta_n \xi_1\}$$

Just as in the proof of Lemma 3.2, we can now apply Theorem 9.3 (ii) with boundary given by the rays $|\xi_2| = -\beta_n\xi_1$ ($|\xi| < K$) to conclude that (for a subsequence and any $K > 1$) $G_n(\xi) \to G(\xi)$, where $G(\xi)$ satisfies

$$\Delta^2 G = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{S}_0, \tag{5.61}$$

$$G = \frac{\partial G}{\partial \nu} = 0 \quad \text{from both sides of } S_0, \tag{5.62}$$

$$|G(\xi)| \leqslant |\xi|^{1+\sigma} + |\xi| \quad \text{in } \mathbb{R}^2, \tag{5.63}$$

$$|G(e)| = 1 \quad \text{where } e = \lim \frac{x_n}{R_n}, \tag{5.64}$$

and where $S_0 = \{(\xi_1, 0);\ \ \xi_1 < 0\}$. This is a contradiction to the Liouville theorem stated in the following lemma. □

Lemma 5.5 *If G satisfies (5.61)–(5.63), then $G(\xi) \equiv 0$.*

Proof. The proof is similar to that of Lemma 3.3. Let $\widetilde{G} \in H^2(B_1 \setminus S_0)$ be a solution of

$$\Delta^2 \widetilde{G} = 0 \quad \text{in } B_1 \setminus S_0,$$

$$\widetilde{G} = \frac{\partial \widetilde{G}}{\partial \nu} = 0 \quad \text{from both sides of } S_0,$$

$$\widetilde{G} = G, \quad \frac{\partial \widetilde{G}}{\partial \nu} = \frac{\partial G}{\partial \nu} \quad \text{on } \partial B_1.$$

Then $|\widetilde{G} - G| \leqslant C|\xi|$ and $|\nabla(\widetilde{G} - G)| \leqslant C$ near $\xi = 0$. Applying the maximum principle (Theorem 2.3) to $\widetilde{G} - G$ in the domain $B_1 \setminus (S_0 \cup B_\varepsilon)$ and then letting $\varepsilon \to 0$ we conclude that $\widetilde{G} - G \equiv 0$. Since $\widetilde{G}$ has an expansion at the origin, this expansion is valid also for $G(\xi)$:

$$G(\xi) = \sum_{k=1}^{\infty} r^{k/2+1} B_k(\theta) \qquad (r = |\xi|) \tag{5.65}$$

near $\xi = 0$. Introduce the function

$$H(\xi) = G(\xi) - r^{3/2} B_1(\theta) - r^2 B_2(\theta).$$

Then

$$|H(\xi)| \leqslant C|\xi|^{5/2} \quad \text{near } \xi = 0, \qquad |G(\xi) - r^{3/2} B_1(\theta)| \leqslant C|\xi|^{3/2} \quad \text{near } \xi = \infty.$$

Since $r^2 B_2(\theta) = c\xi_2^2$, $\partial/\partial\xi_1[r^2 B_2(\theta)] = 0$. The function

$$\Phi(\xi) = \frac{\partial H}{\partial \xi_1} = \frac{\partial G}{\partial \xi_1} - \frac{\partial}{\partial \xi_1}[r^{3/2} B_1(\theta)]$$

is biharmonic in $\mathbb{R}^2 \setminus S_0$ and satisfies the same zero boundary conditions. It follows that

$$|\nabla^j \Phi(\xi)| \leqslant C|\xi|^{5/2-1-j} \quad \text{near } \xi = 0, \quad j = 0,1,2,3,$$

and

$$|\nabla^j \Phi(\xi)| \leqslant C|\xi|^{3/2-1-j} \quad \text{near } \xi = \infty \quad j = 0,1,2,3.$$

Now we can follow the proof of Lemma 3.3 to conclude that $\Phi \equiv 0$, which immediately implies that $G \equiv 0$. □

Remark 5.7. The preceding Liouville theorem does not follow from a general theorem of Kondratév [13; Theorem 11] since one of the assumptions he makes,

$$\int_0^\infty \int_0^\infty r^\gamma |G|^2 r dr d\theta < \infty,$$

is not satisfies for any γ, in our case.

Remark 5.8. The proof of Lemma 5.2 can be extended to the case where $f \in C^{1+\alpha}$ to yield a different (although more complicated) proof of Theorem 3.1. In (5.16) we need to replace $\lambda^{2+\eta}$ by $\lambda^{1+\eta}$, and (5.7), (5.8) need to be modified by using the fact that φ_0 is $C^{3/2}$ and applying (5.2) to $\zeta = \nabla\varphi_1$ with $\theta = \sigma < 1/2$. Finally, (5.9) follows from Lemma 5.4 with $f_j(x_1) = \delta_j|x_1|^{1+\alpha}$, $-1 < \delta_2 < \delta_1 < 1$.

6 The crack propagation model

In this section we introduce a model of crack propagation. Let Ω be a domain in $\mathbb{R}^2$, representing a homogeneous elastic body. Let $u = (u_i)$, $\varepsilon = (\varepsilon_{ij})$ and $s = (\sigma_{ij})$ denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for homogeneous isotropic material consist of the constitutive law

$$\sigma_{ij} = \frac{E}{1+\nu}\Big(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\Big) \tag{6.1}$$

and the equilibrium conditions

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \tag{6.2}$$

provided there are no body forces. Here E is the Young modulus, ν is the Poisson ratio, and the strain-displacement relations are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad u_{i,j} = \partial_j u_i. \tag{6.3}$$

Suppose there is initially a crack in Ω, given by a non-intersecting curve Γ_0 with initial point on $\partial\Omega$ and terminal point (the "crack tip") $X_0 = (x_0, y_0)$ inside Ω. Under external forces the crack tip will generally propagate, and we shall denote it by $X(t)$. The crack propagation problem consists of finding the displacement u and path $X(t)$ such that

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega \setminus \Gamma(t) \tag{6.4}$$

where

$$\Gamma(t) = \Gamma_0 \cup \{X = X(s), 0 \leqslant s \leqslant t\}, \tag{6.5}$$

$$\sigma_{ij} n_j = 0 \quad \text{on } \Gamma^{\pm}(t) \quad (\text{no traction on } \Gamma^{\pm}(t)) \tag{6.6}$$

$\Gamma^{\pm}(t)$ means both sides of $\Gamma(t)$, (n_j) is the normal to the curve,

$$u_i = \beta_i \quad \text{on } \partial_1\Omega, \qquad \sigma_{ij} n_j = g_i \quad \text{on } \partial_2\Omega \tag{6.7}$$

where $\partial\Omega$ is a disjoint union of $\partial_1\Omega, \partial_2\Omega$, and an appropriate dynamical equation for $X(t)$. Based on [7] [10] [19] [20] [21], Friedman and Liu [9] introduced the following dynamics:

$$\dot{X}(t) = \frac{v(t)}{\gamma_0} \frac{v_\infty - v(t)}{v_\infty + v(t)} \mathbf{J}(X(t)) \tag{6.8}$$

where

$$v(t) = |\dot{X}(t)|, \quad X(0) = X_0. \tag{6.9}$$

Here γ_0, v_∞ are positive constants and $\mathbf{J}(X(t))$ is described in terms of the J-integral

$$\mathbf{J}_\gamma = \int_\gamma (W \cdot \vec{n} - \vec{s} \cdot Du) dl$$

where

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

is the strain energy density and

$$\vec{s} = (s_i) = (\sigma_{ij} n_j)$$

is the traction vector; γ is a curve in $\Omega \setminus \Gamma(t)$, initiating at Y and terminating at X, $\vec{n}$ is a normal to γ, and dl is the arc element. It is well known [11] that $\mathbf{J}_\gamma$ is independent of the path connecting Y to X. Denote by $S_\varepsilon(X)$ the circle with center X and radius ε, and set $\Lambda_\varepsilon(X(t)) = S_\varepsilon(X(t)) \cap \Omega$. By the path-independence property of the J-integral it follows that

$$\mathbf{J}(X(t)) \equiv \lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon(X(t))} (W \cdot \vec{n} - \vec{s} \cdot Du) dl \tag{6.10}$$

is well defined, and this is the function we use in (6.8); here $\vec{n}$ is the outward normal to the circles $S_\varepsilon(X(t))$. Taking the absolute value in (6.8) we get

$$1 = \frac{1}{\gamma_0} \frac{v_\infty - |\dot{X}(t)|}{v_\infty + |\dot{X}(t)|} |\mathbf{J}(X(t))|,$$

or

$$|\dot{X}(t))| = v_\infty \frac{(|\mathbf{J}(X(t))| - \gamma_0)_+}{|\mathbf{J}(X(t))| + \gamma_0}.$$

Hence (6.8) can also be written in the form

$$\dot{X}(t) = h(|\mathbf{J}(X(t))|) \mathbf{J}(X(t)) \tag{6.11}$$

where

$$h(s) = \frac{v_\infty}{s} \frac{(s-\gamma_0)_+}{s+\gamma_0}. \tag{6.12}$$

Note that the crack cannot propagate unless $|J(X(t))|$ is larger than γ_0. In particular, if $|J(X(0))| \leqslant \gamma_0$, then the crack does not propagate, and $\Gamma(t) \equiv \Gamma_0$. Hence in the sequel we shall always assume that $|J(X(0))| > \gamma_0$.

As in [9] we can express the stress components in terms of the stress function φ (which is determined up to an additive linear function):

$$\sigma_{11} = \frac{\partial^2 \varphi}{\partial y^2} = \varphi_{22},\ \sigma_{12} = -\frac{\partial^2 \varphi}{\partial x \partial y} = -\varphi_{12},\ \sigma_{22} = \frac{\partial^2 \varphi}{\partial x^2} = \varphi_{11}.$$

Then the system (6.1)–(6.6) becomes:

$$\Delta^2 \varphi = 0 \quad \text{in } \Omega \setminus \overline{\Gamma}(t), \tag{6.13}$$

$$\varphi = 0,\ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma(t), \text{ from both sides}, \tag{6.14}$$

$$\Gamma(t) = \Gamma_0 \cup \{X(s), 0 \leqslant s \leqslant t\}. \tag{6.15}$$

For definiteness we take boundary conditions (cf. [9])

$$\varphi = g,\ \frac{\partial \varphi}{\partial n} = h \quad \text{on } \partial\Omega. \tag{6.16}$$

We finally recall that

$$\mathbf{J}(X(t)) = (J_1(X(t)), J_2(X(t)))$$

can be computed in the form

$$J_i(X(t)) = \frac{1-\nu^2}{2E} \lim_{\varepsilon \to 0} \int_{\Lambda_\varepsilon(X(t))} \Big[(\Delta\varphi)^2 n_i - 2\vec{s} \cdot \vec{\Phi}_i \Big] dl, \tag{6.17}$$

where

$$\begin{aligned} &s_1 = \varphi_{22} n_1 - \varphi_{12} n_2, \quad s_2 = -\varphi_{12} n_1 + \varphi_{11} n_2, \\ &\vec{\Phi}_1 = (\Delta\varphi, (\Delta\varphi)^c),\ \vec{\Phi}_2 = (-(\Delta\varphi)^c, \Delta\varphi) \end{aligned} \tag{6.18}$$

and $(\Delta\varphi)^c$ is the harmonic conjugate of $\Delta\varphi$ determined up to an additive constant (the constant disappears in the limit in (6.17).

Definition. The crack problem, **Problem (C)**, is the problem of solving the system (6.13)–(6.18).

7 Reformulation of the crack propagation problem

In this section we reformulate the crack problem by first replacing the *dynamic formulation* (6.11) by a *geometric condition*, and then replacing the latter by the condition (1.17).

We assume that (φ, Γ) form a solution to problem (C) with Γ in $C^{1+\alpha}$, and write

$$\mathbf{J}(t) = (J_1(t), J_2(t)) = (J_1(X(t)), J_2(X(t)).$$

For simplicity we shall always assume that

$$f(0) = f'(0) = 0. \tag{7.1}$$

Lemma 7.1

$$\mathbf{J}(0) = \frac{\pi(1-\nu^2)}{2E}\Big(36A_1^2 + 4A_2^2, 24A_1A_2\Big), \tag{7.2}$$

Proof. Consider first the case of the tip $X(0)$. By Theorem 3.1,

$$\varphi(x) = A_1 r^{3/2}\Big(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\Big) + A_2 r^{3/2}\Big(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\Big) + G \equiv r^{3/2}\mathbf{A}\cdot\mathbf{B}(\theta) + G, \tag{7.3}$$

where

$$G = O(r^{3/2+\mu}) \qquad (\text{for any } 0 < \mu < \frac{1}{2}).$$

Since $\Delta^2 G = 0$, by interior elliptic estimates

$$|D^2 G| = O(r^{-1/2+\mu}).$$

Since $|\Delta\varphi| = O(r^{-1/2})$, we also have

$$|(\Delta\varphi)^c| = O(r^{-1/2}),$$

as can be seen by writing $(\Delta\varphi)^c$ as a line integral of $((\Delta\varphi)_y, -(\Delta\varphi)_x)$.

Similarly

$$|(\Delta G)^c| = O(r^{-1/2+\mu}).$$

From the above estimates we easily conclude that we can take $G \equiv 0$ in the calculation of the J-integral. Using complex variables $z = x_1 + ix_2 = re^{i\theta}$, we have

$$\begin{aligned}
\varphi &= A_1 r^{3/2}\Big(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\Big) + A_2 r^{3/2}\Big(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\Big),\\
&= Cz^{3/2} + \overline{C}\overline{z}^{3/2} + Dz\overline{z}^{1/2} + \overline{D}z^{1/2}\overline{z},
\end{aligned}$$

where

$$C = \frac{1}{2}(A_1 - iA_2), \quad D = \frac{1}{2}(3A_1 - iA_2).$$

Then (cf. [8; pp. 275–276])

$$\begin{aligned}
\Delta\varphi &= 6A_1 r^{-1/2}\cos\frac{1}{2}\theta + 2A_2 r^{-1/2}\sin\frac{1}{2}\theta \;=\; 2D\overline{z}^{-1/2} + 2\overline{D}z^{-1/2},\\
(\Delta\varphi)^c &= -6A_1 r^{-1/2}\sin\frac{1}{2}\theta + 2A_2 r^{-1/2}\cos\frac{1}{2}\theta \;=\; -2i\overline{D}z^{-1/2} + 2iD\overline{z}^{-1/2},\\
\frac{\partial}{\partial\theta}(\Delta\varphi) &= -3A_1 r^{-1/2}\sin\frac{1}{2}\theta + A_2 r^{-1/2}\cos\frac{1}{2}\theta \;=\; -i\overline{D}z^{-1/2} + iD\overline{z}^{-1/2},\\
\frac{\partial}{\partial\theta}(\Delta\varphi)^c &= -3A_1 r^{-1/2}\cos\frac{1}{2}\theta - A_2 r^{-1/2}\sin\frac{1}{2}\theta \;=\; -D\overline{z}^{-1/2} - \overline{D}z^{-1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_\Lambda (\Delta\varphi)^2 n_1 dl &= \int_\Lambda \Big[2D\overline{z}^{-1/2} + 2\overline{D}z^{-1/2}\Big]^2 \frac{1}{2}(e^{i\theta} + e^{-i\theta})dl \\
&= \int_{-\pi}^{\pi} \Big[2De^{i\theta/2} + 2\overline{D}e^{-i\theta/2}\Big]^2 \frac{1}{2}(e^{i\theta} + e^{-i\theta})d\theta \qquad (7.4) \\
&= 2\pi(2D^2 + 2\overline{D}^2) \;=\; 18\pi A_1^2 - 2\pi A_2^2.
\end{aligned}$$

where Λ is the circle $r = \varepsilon$, $-\pi < \theta < \pi$, traced counterclockwise. Similarly

$$\begin{aligned}
\int_\Lambda (\Delta\varphi)^2 n_2 dl &= \int_\Lambda \Big[2D\overline{z}^{-1/2} + 2\overline{D}z^{-1/2}\Big]^2 \frac{1}{2i}(e^{i\theta} + e^{-i\theta})dl \\
&= \int_{-\pi}^{\pi} \Big[2De^{i\theta/2} + 2\overline{D}e^{-i\theta/2}\Big]^2 \frac{1}{2i}(e^{i\theta} + e^{-i\theta})d\theta \qquad (7.5) \\
&= 2\pi i(2D^2 - 2\overline{D}^2) \;=\; 12\pi A_1 A_2.
\end{aligned}$$

To evaluate $\vec{s}$, we compute

$$\begin{aligned}
\vec{s} &= (\varphi_{22}n_1 - \varphi_{12}n_2, -\varphi_{12}n_1 + \varphi_{11}n_2), \\
&= \Big(\frac{\partial^2\varphi}{\partial x_2^2}\cos\theta - \frac{\partial^2\varphi}{\partial x_1\partial x_2}\sin\theta, -\frac{\partial^2\varphi}{\partial x_1\partial x_2}\cos\theta + \frac{\partial^2\varphi}{\partial x^2}\sin\theta\Big) \\
&= \Big(\frac{1}{r}\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_2}\Big), -\frac{1}{r}\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_1}\Big)\Big)
\end{aligned}$$

Next we compute $\partial\varphi/\partial x_2$ and $\partial\varphi/\partial x_1$.

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_1} &= \frac{\partial\varphi}{\partial z} + \frac{\partial\varphi}{\partial\overline{z}} \\
&= \frac{3}{2}\Big(Cz^{1/2} + \overline{C}\overline{z}^{1/2}\Big) + D\overline{z}^{1/2} + \overline{D}z^{1/2} + \frac{1}{2}\Big(\overline{D}z^{-1/2}\overline{z} + D\overline{z}^{-1/2}z\Big), \\
\frac{\partial\varphi}{\partial x_2} &= i\Big(\frac{\partial\varphi}{\partial z} - \frac{\partial\varphi}{\partial\overline{z}}\Big) \\
&= \frac{3}{2}i\Big(Cz^{1/2} - \overline{C}\overline{z}^{1/2}\Big) + i\Big(D\overline{z}^{1/2} - \overline{D}z^{1/2}\Big) + \frac{1}{2}i\Big(\overline{D}z^{-1/2}\overline{z} - D\overline{z}^{-1/2}z\Big).
\end{aligned}$$

By integration by parts (both $\partial\varphi/\partial x_1$ and $\partial\varphi/\partial x_2$ vanishes on $\theta = \pm\pi$) we then get

$$\begin{aligned}
\int_\Lambda \vec{s}\cdot\vec{\Phi}_1 dl &= \int_{-\pi}^{\pi} \Big(\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_2}\Big), -\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_1}\Big)\Big)\cdot(\Delta\varphi, (\Delta\varphi)^c)d\theta \\
&= \int_{-\pi}^{\pi} \Big(-\frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_1}\Big)\cdot\Big(\frac{\partial}{\partial\theta}\Delta\varphi, \frac{\partial}{\partial\theta}(\Delta\varphi)^c\Big)d\theta \qquad (7.6) \\
&= \int_{-\pi}^{\pi} \Big(-\frac{\partial\varphi}{\partial x_2}\frac{\partial}{\partial\theta}\Delta\varphi + \frac{\partial\varphi}{\partial x_1}\frac{\partial}{\partial\theta}(\Delta\varphi)^c\Big)d\theta \;\equiv\; K_1 + K_2,
\end{aligned}$$

Substituting the formulas obtained above into the various expression which appears in the integrand of K_1, we get

$$K_1 = -\int_{-\pi}^{\pi} \Big\{\frac{3}{2}i\Big(Ce^{i\theta/2} - \overline{C}e^{-i\theta/2}\Big) + i\Big(De^{-i\theta/2} - \overline{D}e^{i\theta/2}\Big)$$

$$
\begin{aligned}
&+\frac{1}{2}i\Big(\overline{D}e^{-i\,3\theta/2}-De^{i\,3\theta/2}\Big)\Big\}\cdot\Big(-i\overline{D}e^{-i\theta/2}+iDe^{i\theta/2}\Big)d\theta\\
=&\;\;2\pi\Big[-\frac{3}{2}C\overline{D}-\frac{3}{2}\overline{C}D+D^2+\overline{D}^2\Big].
\end{aligned}
$$

Similarly

$$
\begin{aligned}
K_2 \;=&\;\int_{-\pi}^{\pi}\Big\{\frac{3}{2}\Big(Ce^{i\theta/2}+\overline{C}e^{-i\theta/2}\Big)+\Big(De^{-i\theta/2}+\overline{D}e^{i\theta/2}\Big)\\
&+\frac{1}{2}\Big(\overline{D}e^{-i\,3\theta/2}+De^{i\,3\theta/2}\Big)\Big\}\cdot\Big(-De^{i\theta/2}-\overline{D}e^{-i\theta/2}\Big)d\theta\\
=&\;\;2\pi\Big[-\frac{3}{2}C\overline{D}-\frac{3}{2}\overline{C}D-D^2-\overline{D}^2\Big].
\end{aligned}
$$

It follows that

$$
K_1+K_2=6\pi(C\overline{D}+\overline{C}D)=-3\pi(3A_1^2+A_2^2),
$$

and, together with (7.4),

$$
J_1=\frac{\pi(1-\nu^2)}{2E}\Big[(18\pi A_1^2-2\pi A_2^2)-2(K_1+K_2)\Big]=\frac{\pi(1-\nu^2)}{2E}(36A_1^2+4A_2^2). \tag{7.7}
$$

In a similar way we compute

$$
\begin{aligned}
\int_\Lambda \vec{s}\cdot\vec{\Phi}_2dl=&\int_{-\pi}^{\pi}\Big(\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_2}\Big),-\frac{\partial}{\partial\theta}\Big(\frac{\partial\varphi}{\partial x_1}\Big)\Big)\cdot(-(\Delta\varphi)^c,\Delta\varphi)d\theta\\
=&\;\int_{-\pi}^{\pi}\Big(-\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_1}\Big)\cdot\Big(-\frac{\partial}{\partial\theta}(\Delta\varphi)^c,\frac{\partial}{\partial\theta}\Delta\varphi\Big)d\theta\\
=&\;\int_{-\pi}^{\pi}\Big(\frac{\partial\varphi}{\partial x_2}\frac{\partial}{\partial\theta}(\Delta\varphi)^c+\frac{\partial\varphi}{\partial x_1}\frac{\partial}{\partial\theta}\Delta\varphi\Big)d\theta\;\equiv\;\widetilde{K}_1+\widetilde{K}_2,
\end{aligned}
\tag{7.8}
$$

where

$$
\begin{aligned}
\widetilde{K}_1\;=&\;\int_{-\pi}^{\pi}\Big\{\frac{3}{2}i\Big(Ce^{i\theta/2}-\overline{C}e^{-i\theta/2}\Big)+i\Big(De^{-i\theta/2}-\overline{D}e^{i\theta/2}\Big)\\
&+\frac{1}{2}i\Big(\overline{D}e^{-i\,3\theta/2}-De^{i\,3\theta/2}\Big)\Big\}\cdot\Big(-De^{i\theta/2}-\overline{D}e^{-i\theta/2}\Big)d\theta\\
=&\;\;2\pi\Big[\frac{3}{2}i\overline{C}D-\frac{3}{2}iC\overline{D}-iD^2+i\overline{D}^2\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{K}_2\;=&\;\int_{-\pi}^{\pi}\Big\{\frac{3}{2}\Big(Ce^{i\theta/2}+\overline{C}e^{-i\theta/2}\Big)+\Big(De^{-i\theta/2}+\overline{D}e^{i\theta/2}\Big)\\
&+\frac{1}{2}\Big(\overline{D}e^{-i\,3\theta/2}+De^{i\,3\theta/2}\Big)\Big\}\cdot\Big(-i\overline{D}e^{-i\theta/2}+iDe^{i\theta/2}\Big)d\theta\\
=&\;\;2\pi\Big[-\frac{3}{2}iC\overline{D}+\frac{3}{2}i\overline{C}D-i\overline{D}^2+iD^2\Big].
\end{aligned}
$$

It follows that

$$
\widetilde{K}_1+\widetilde{K}_2=6\pi(i\overline{C}D-iC\overline{D})=-6\pi A_1A_2,
$$

and, together with (7.5),

$$J_2 = \frac{\pi(1-\nu^2)}{2E}\Big[12A_1A_2 - 2(\widetilde{K}_1 + \widetilde{K}_2)\Big] = \frac{\pi(1-\nu^2)}{2E}(24A_1A_2). \qquad \square \tag{7.9}$$

Remark 7.1. We denote by θ_P the angle from the positive x-axis to $\dot{X}(t)$. If we rotate the coordinate system by an angle θ_P, then the formula in Lemma 7.1 is valid in the new coordinate system. Therefore we expect the formula for $\mathbf{J} = (J_1, J_2)$ in the original coordinate system to be:

$$\begin{aligned} J_1 &= \frac{\pi(1-\nu^2)}{2E}\Big[(36A_1^2 + 4A_2^2)\cos\theta_P - 24A_1A_2\sin\theta_P\Big], \\ J_2 &= \frac{\pi(1-\nu^2)}{2E}\Big[(36A_1^2 + 4A_2^2)\sin\theta_P + 24A_1A_2\cos\theta_P\Big]. \end{aligned} \tag{7.10}$$

We shall now verify (7.10) directly. From the computations in Lemma 7.1, we get,

$$\mathbf{J} \;=\; \int \vec{n}\cdot(\Delta\varphi)^2 - 2\int\Big(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\Big)\begin{pmatrix} \frac{\partial}{\partial\theta}(\Delta\varphi)^c, & \frac{\partial}{\partial\theta}\Delta\varphi \\ -\frac{\partial}{\partial\theta}\Delta\varphi, & \frac{\partial}{\partial\theta}(\Delta\varphi)^c \end{pmatrix}.$$

Under the new coordinate system $x_1' = \cos\theta_P x_1 + \sin\theta_P x_2, x_2' = -\sin\theta_P x_1 + \cos\theta_P x_2$, we have

$$\begin{aligned} \vec{n} &= (\cos\theta, \sin\theta) \\ &= \Big(\cos(\theta-\theta_P)\cos\theta_P - \sin(\theta-\theta_P)\sin\theta_P, \cos(\theta-\theta_P)\sin\theta_P + \sin(\theta-\theta_P)\cos\theta_P\Big), \\ &= \Big(\cos(\theta-\theta_P), \sin(\theta-\theta_P)\Big)\begin{pmatrix} \cos\theta_P, & \sin\theta_P \\ -\sin\theta_P, & \cos\theta_P \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Big(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\Big) &= \Big(\frac{\partial\varphi}{\partial x_1'}\cos\theta_P - \frac{\partial\varphi}{\partial x_2'}\sin\theta_P, \frac{\partial\varphi}{\partial x_1'}\sin\theta_P + \frac{\partial\varphi}{\partial x_2'}\cos\theta_P\Big) \\ &= \Big(\frac{\partial\varphi}{\partial x_1'}, \frac{\partial\varphi}{\partial x_2'}\Big)\begin{pmatrix} \cos\theta_P, & \sin\theta_P \\ -\sin\theta_P, & \cos\theta_P \end{pmatrix} \end{aligned}$$

Clearly

$$\begin{aligned} &\begin{pmatrix} \cos\theta_P, & \sin\theta_P \\ -\sin\theta_P, & \cos\theta_P \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\theta}(\Delta\varphi)^c, & \frac{\partial}{\partial\theta}\Delta\varphi \\ -\frac{\partial}{\partial\theta}\Delta\varphi, & \frac{\partial}{\partial\theta}(\Delta\varphi)^c \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial\theta}(\Delta\varphi)^c, & \frac{\partial}{\partial\theta}\Delta\varphi \\ -\frac{\partial}{\partial\theta}\Delta\varphi, & \frac{\partial}{\partial\theta}(\Delta\varphi)^c \end{pmatrix}\begin{pmatrix} \cos\theta_P, & \sin\theta_P \\ -\sin\theta_P, & \cos\theta_P \end{pmatrix}, \end{aligned}$$

and the 2 matrix with the $\Delta\varphi$ is invariant under the above change of coordinates. Substituting these relations into $\mathbf{J}$, we get (7.10).

Set

$$\Gamma_s = \Gamma_0 \cup \{x_1 = f(x_2),\ 0 \leqslant x_2 \leqslant s\} \tag{7.11}$$

for any $0 \leqslant s \leqslant s_0$ and consider (6.13)–(6.16) with $\Omega \setminus \Gamma(t)$ replaced by $\Omega \setminus \Gamma_s$ and with (6.8) replaced by

$$f'(s) = \frac{J_2(s)}{J_1(s)} \tag{7.12}$$

where

$$\mathbf{J}(s) = \mathbf{J}((s, f(s)).$$

We shall refer to this problem as **Problem ($\mathbf{C}_0$)**.

Lemma 7.2 *Problems (C) and (C_0) are equivalent.*

Proof. For a solution $(\varphi, X(t))$ to problem (C), we have

$$\frac{\dot{X}_2(t)}{\dot{X}_1(t)} = \frac{J_2(X(t))}{J_1(X(t))}. \tag{7.13}$$

If we write $X_1(t) = s, X_2(t) = f(s)$, then

$$f'(s) = \frac{\dot{X}_2(t)}{\dot{X}_1(t)} = \frac{J_2((s, f(s))}{J_1((s, f(s))}$$

with $x_2 = f(s)$ defined by $f(s) = X_2(X_1^{-1}(s))$, which shows that $(\varphi, f(s))$ forms a solution to problem (C_0).

Conversely, let (φ, Γ_s) be a solution to problem (C_0) and define $X(t) = (X_1(t), X_2(t))$ by

$$\dot{X}(t) = h(|\mathbf{J}(t)|)\mathbf{J}(t) \tag{7.14}$$

where $\mathbf{J}(t) = \mathbf{J}(X(t))$. Writing $X_1(t) = s$ or $t = X_1^{-1}(s)$, we introduce a function $X_2 = \widehat{f}(s)$ by

$$\widehat{f}(s) = X_2(X_1^{-1}(s)).$$

Thus

$$\widehat{f}'(s) = \frac{\dot{X}_2(t)}{\dot{X}_1(t)} = \frac{J_2(s, f(s))}{J_1(s, f(s))}$$

which implies that $\widehat{f}'$ at $X_1(t)$ agrees with f' at s, i.e., $\widehat{f}$ and f define the same curve with different parameterizations. It follows that the $\mathbf{J}(t)$ in (7.14) is the J-integral for the tip $X(t)$ of the curve defined by $f(s)$, $0 \leqslant s \leqslant X_1(t)$; hence $(\varphi, X(t))$ is a solution to problem (C). $\square$

We proceed to consider problem (C_0), and denote by $A_1(s)$, $A_2(s)$, $\cdots$ the coefficients in the asymptotic expansion of the solution about the tip $X(s) = (x, f(s))$.

From Lemma 7.1, it is clear that

$$\frac{J_2}{J_1} = \frac{\tan\theta_P + g(A_2/A_1)}{1 - \tan\theta_P \, g(A_2/A_1)} \tag{7.15}$$

where

$$g(u) = \frac{6u}{9+u^2}, \qquad g'(u) = \frac{54-6u^2}{(9+u^2)^2}. \tag{7.16}$$

If the curve in problem (C_0) is given by $(s, f(s))$, then, at s,

$$\tan\theta_P = f'(s).$$

Substituting this into (7.15) we conclude that (7.12) is equivalent to

$$g\Big(\frac{A_2}{A_1}\Big) = 0$$

or, by (7.16),

$$\text{either } A_1 = 0 \quad \text{or } A_2 = 0.$$

We shall henceforce assume that

$$A_2(0) = 0. \tag{7.17}$$

Since $|\mathbf{J}(0)| > \gamma_0$, $A_1(0)$ is necessarily $\neq 0$ and by continuity (assuming that $|\mathbf{J}(s)| > \gamma_0$) we get that

$$A_2(s) = 0, \quad A_1(s) \neq 0. \tag{7.18}$$

In particular:

Theorem 7.3 *$(\varphi, f(s))$ is a solution to problem (C_0) if and only if*

$$A_2(s) \equiv 0. \tag{7.19}$$

Thus the crack problem is equivalent to the following:

Problem (C_1). Find a pair $(\varphi, f(s))$ such that φ satisfies (6.13)–(6.16) with $\Omega \setminus \Gamma$ replaced by $\Omega \setminus \Gamma_s$, Γ_s as in (7.11), with (7.12) replaced by (7.19).

Condition (7.19) implies that

$$\varphi = r_P^{3/2} A_1(s) B_1(\theta - \theta_P) + O(r_P^{3/2+\lambda}) \quad (0 < \lambda < 1/2)$$

in a neighborhood of the tip $P = (s, f(s))$. Consequently, as we approach P from $\Omega \setminus \Gamma_s$ along the tangent τ to Γ_s at P,

$$\varphi_{\tau\tau} \sim \frac{K}{r_P^{1/2}} \quad (K \neq 0), \quad \varphi_{\tau n} \to 0$$

where n is the direction normal to τ, or, in terms of the stress σ,

$$\sigma_{nn} \sim \frac{K}{r_P^{1/2}} \quad (K \neq 0), \quad \sigma_{\tau n} \to 0. \tag{7.20}$$

This local behavior is used by some authors (e.g. [7; p. 433] [4]) to model the propagation of cracks developed by traction (and commonly called mode I, or opening mode ([16; p. 24]).

Since, conversely, (7.20) implies (7.19), we have thus obtained a very interesting physical result:

Theorem 7.4 *In the modeling of the crack propagation problem, the conditions (7.13) and (7.20) are equivalent (assuming the crack is in $C^{1+\alpha}$).*

8 Remarks on problem (C_0)

The results of §§3, 5 can be used to study the regularity of the coefficients $A_i(s)$. As an example, we shall establish in this section the Hölder continuity of $\mathbf{A}(s) = (A_1(s), A_2(s))$. We assume that

$$\Gamma_s: \quad x_2 = f(s), -1 \leqslant s \leqslant \tau \ (\tau > 0) \tag{8.1}$$

is a $C^{1+\alpha}$ curve initiating on $\partial\Omega$ and contained in Ω, with

$$f(0) = 0, \quad f'(0) = 0, \tag{8.2}$$

and set $\Omega_s = \Omega \setminus \Gamma_s$. Let $\psi(x, s)$ be the solution of

$$\psi \in H^2(\Omega_s), \tag{8.3}$$

$$\Delta^2 \psi = 0 \quad \text{in } \Omega_s, \tag{8.4}$$

$$\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{from both sides of } \Gamma_s. \tag{8.5}$$

$$\psi = g, \frac{\partial \psi}{\partial n} = h \quad \text{on } \partial\Omega \tag{8.6}$$

where g, h are independent of s.

By Theorem 3.1, if $X(s) = (s, f(s))$, $0 \leqslant s \leqslant \tau$,

$$\psi(x, s) = |x - X(s)|^{3/2} \mathbf{A}(s) \cdot \mathbf{B}(\theta - \arctan f'(s)) + O(|x - X(s)|^{2-\eta}) \tag{8.7}$$

for any η such that $\alpha + \eta > 1/2$.

Set

$$w(x, s) = \psi(x, s) - \psi(x, 0) \quad \text{for } s > 0.$$

Lemma 8.1 *For any sufficiently large p,*

$$\Big(\int_{\Omega_0} |w(x, s)|^p dx \Big)^{1/p} \leqslant C_p s. \tag{8.8}$$

Proof. It is clear that, for any $s > 0$,

$$|\psi(x, s)| \leqslant C|x - (s, f(s))|^{3/2},$$
$$|\nabla_x \psi(x, s)| \leqslant C|x - (s, f(s))|^{1/2}.$$

It follows that

$$|w(x, s)| \leqslant C s^{3/2} \quad \text{for } |x - (s, 0)| \leqslant 2s,$$
$$|\nabla_x w(x, s)| \leqslant C s^{1/2} \quad \text{for } |x - (s, 0)| \leqslant 2s.$$

Applying Theorem 2.3 in the domain $\Omega_0 \cap \{|x - (s, 0)| > s\}$ with Γ_δ replaced by $\{|x - (s, 0)| = s\}$, we obtain the assertion (8.8). □

We shall use Lemma 8.1 and (8.7) to prove the following:

Theorem 8.2 *Let*

$$\|f\|_{C^1} \leqslant F_1, \quad [f']_{C^\alpha} \leqslant F_{1,\alpha}$$

for $0 < \alpha < 1/4$. *Then* $\mathbf{A}(s)$ *is Hölder continuous:*

$$|\mathbf{A}(X(s)) - \mathbf{A}(X(\widehat{s}))\Big| \leqslant C(s-\widehat{s})^{\widetilde{\mu}} \quad (0 \leqslant \widehat{s} < s \leqslant \tau) \tag{8.9}$$

for any $0 < \widetilde{\mu} < \alpha/4$ *where* C *depends only on* α, F_1, $F_{1,\alpha}$.

Proof. It suffices to take $\widehat{s} = 0$. Note that in (8.7),

$$|O(|x - X(s)|^{2-\eta})| \leqslant C|x - X(s)|^{2-\eta}$$

where C depends on F_1, $F_{1,\alpha}$ but is independent of s. It is also clear that $|X(s)| \leqslant Cs$. Therefore, for $\sqrt{s} \leqslant |x| \leqslant 2\sqrt{s}$,

$$\frac{w(x)}{|x|^{3/2}} = \mathbf{A}(X(0)) \cdot \mathbf{B}(\theta) - \mathbf{A}(X(s)) \cdot \mathbf{B}(\theta - \arctan f'(s)) + O(s^\mu), \quad \mu = \frac{1}{2}\Big(\frac{1}{2} - \eta\Big) \tag{8.10}$$

We substitute (8.10) into (8.8) and then integrate over the region $\{\sqrt{s} < |x| < 2\sqrt{s}\}$. By choosing p to be sufficiently large, we conclude that, for any $\widetilde{\mu} \in (0, \mu/2)$,

$$\Big(\int_{-\pi+Cs^{\alpha/2}}^{\pi-Cs^{\alpha/2}} |\mathbf{A}(X(s)) \cdot \mathbf{B}(\theta - \arctan f'(s)) - \mathbf{A}(0) \cdot \mathbf{B}(\theta)|^p d\theta\Big)^{1/p} \leqslant Cs^{\widetilde{\mu}}. \tag{8.11}$$

Since μ can be chosen arbitrarily close to (but smaller than) $\alpha/2$, $\widetilde{\mu}$ can be chosen arbitrarily close to (but smaller than) $\alpha/4$. Noting that $|\mathbf{B}(\theta - \arctan f'(s)) - \mathbf{B}(\theta)| \leqslant C[f']_\alpha s^\alpha$, (8.9) easily follows. $\square$

A simple approach to solving problem (C_1) is to introduce a family of curves

$$Y = \Big\{ f(s) \Big| \; f(0) = f'(0) = 0; [f']_{C^\alpha[0,s_0]} \leqslant M_{1,\alpha} \Big\}$$

where $0 < \alpha < 1/4$.

For any $\widetilde{f}(s) \in Y$, let $\widetilde{\varphi}(x,s)$ denote the solution of (6.13)–(6.16) with $\Omega \setminus \Gamma(t)$ replaced by $\widetilde{G}_s$, where $\widetilde{G}_s$ is defined as in (7.11) with f replaced by $\widetilde{f}$. Writing

$$\widetilde{\varphi}(x,s) = r_P^{3/2}\Big[\widetilde{A}_1(s)B_1(\theta - \theta_P) + A_2(s)B_2(\theta - \theta_P)\Big] + O(r_P^{3/2+\lambda})$$

(where $0 < \lambda < \alpha$), we introduce the functional

$$\mathcal{M}(\widetilde{f}) = \int_0^{s_0} \Big(\widetilde{A}_2(s)\Big)^2 ds, \tag{8.12}$$

and consider the minimization problem:

$$\min_{\widetilde{f} \in Y} \mathcal{M}(\widetilde{f}) = \mathcal{M}(f), \quad f \in Y. \tag{8.13}$$

Since the $\widetilde{A}_2(s)$ are uniformly Hölder continuous (by Theorem 8.2), a minimizing sequence $(\widetilde{f}_n, \widetilde{A}_{2,n})$ has a uniformly convergent subsequence to a limit (f, A_2). If the minimum in (8.13) is equal to zero, then $A_2 = 0$ and so f is a solution to problem (C_0). Thus we may view (8.13) as a relaxation of the crack propagation problem.

It is not clear how to prove that the minimum in (8.13) is equal to zero. In a future paper, currently under preparation, we shall use the results obtained in the previous sections in order to rewrite the condition $A_2(s) = 0$ as a relation between the curvature $\kappa(s)$ at $X(s)$ and leading coefficients in the expansions near $X(s)$ of $\psi(x,s)$ and its tangential derivative. This relation should enable us to establish the existence of a solution of problem (C_0).

9 Appendix: sub-Schauder estimates

Let Ω be a 2-dimensional bounded domain containing the origin, and $\Gamma_j \quad (j = 1, \cdots, m)$ be $C^{1+\alpha}$ arcs initiating at the origin and contained in Ω. Set $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$. For any small r denote by $\omega(r)$ the largest arc on the circle $|x| = r$ which is contained in $\Omega \setminus \Gamma$, and set $\widehat{\omega} = \inf_{0<r<r_0} \omega(r)$ for some small $r_0 > 0$, $\max(\pi, \widehat{\omega}) < \widetilde{\omega} \leqslant 2\pi$. It is easy to verify that there is a unique solution $\delta = \delta(\widetilde{\omega})$ of

$$\sin^2(\widetilde{\omega}\delta) = \delta^2 \sin^2 \widetilde{\omega} \quad \text{such that } 0 < \widetilde{\omega}\delta(\widetilde{\omega}) \leqslant \pi, \quad \delta(\widetilde{\omega}) \geqslant \frac{1}{2}.$$

Now let u be a solution of

$$\Delta^2 u = f \quad \text{ in } \Omega \setminus \Gamma, \quad u \in H^2(\Omega \setminus \Gamma), \tag{9.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{from both sides of } \Gamma, \tag{9.2}$$

where

$$\int_\Omega |f| \leqslant C_0. \tag{9.3}$$

Theorem 9.1 (Kondratév-Oleinik[14],[15]) *If $\|u\|_{H^2(\Omega\setminus\Gamma)} \leqslant C_1$, then the solution u belongs to $C^{1+\delta(\widetilde{\omega})}$ in r_0-neighborhood of the origin, and*

$$|u(x)| \leqslant C|x|^{1+\delta(\omega)}, \tag{9.4}$$

$$|\nabla u(x)| \leqslant C|x|^{\delta(\omega)}, \tag{9.5}$$

where the constant C depends only on C_0, C_1 and Ω.

Example 1. Γ consists of a single $C^{1+\alpha}$ arc with one endpoint at the origin. Theorem 9.1 is then valid with $\delta(\widetilde{\omega}) = 1/2$.

Example 2. Γ is a $C^{1+\alpha}$ curve passing through the origin (and $\Gamma = \Gamma_1 \cup \Gamma_2$). In this case, $\widetilde{\omega}$ can be taken arbitrarily close to π (if r_0 is small enough). Hence $\delta(\widetilde{\omega})$ can be taken arbitrarily close to 1 and, in particular, for any $\varepsilon > 0$,

$$\|u\|_{C^{2-\varepsilon}\{|x|<r_0\}} \leqslant C \tag{9.6}$$

if r_0 is small enough; C depends only on the $C^{1+\alpha}$ norm of Γ restricted to $\{|x| < r_0\}$ and on bounds on $\int_\Omega |f|$ and $\|u\|_{H^2(\Omega\setminus\Gamma)}$.

We shall establish a local version of this theorem whereby $\|u\|_{H^2(\Omega\setminus\Gamma)}$ is not assumed to be (uniformly) bounded by a constant C_1 but, instead, $\|u\|_{L^\infty(\Omega)}$ is (uniformly) bounded by a constant C_1. Let Ω is a 2-dimensional bounded domain, S an open $C^{1+\alpha}$ subarc of $\partial\Omega$ containing the origin in its interior, and

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u \in H^2(\Omega), \tag{9.7}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } S, \tag{9.8}$$

$$\int_\Omega |f| \leqslant C_0, \qquad |u| \leqslant C_1 \quad \text{in } \Omega. \tag{9.9}$$

Let Ω_0 be any subdomain of Ω such that $\overline{\Omega}_0 \subset \Omega \cup S$.

Lemma 9.2 *If (9.7)–(9.9) hold, then*

$$\|u\|_{H^2(\Omega_0)} \leqslant C_2 \tag{9.10}$$

where C_2 is a constant depending only on Ω_0, Ω, S, C_0 and C_1, and the estimates (9.4) (9.5) hold for any $\delta(\widetilde{\omega})$ arbitrarily close to 1.

Proof. For simplicity we may assume that the curve S is given by

$$S : x_2 = g(x_1), \quad -1 < x_1 < 1, \qquad g(0) = 0,$$

and $\Omega = \{x \in B_1;\ \ x_2 > g(x_1)\}$ and $\Omega_0 = B_{1/2} \cap \Omega$. For any $0 < r \leqslant 1/4$, let ζ be a cutoff function such that

$$\zeta = 1 \quad \text{for } |x| < 1 - 2r, \qquad \zeta = 0 \quad \text{for } |x| > 1 - r, \qquad |\nabla\zeta| \leqslant \frac{C}{r}, \quad |D^2\zeta| \leqslant \frac{C}{r^2}.$$

Since $u \in H^2(\Omega)$, Theorem 9.1 implies that for any $0 < \varepsilon < 1$, $u = O(d^{2-\varepsilon})$, $Du = O(d^{1-\varepsilon})$ (and then also $D^2u = O(d^{-\varepsilon})$ and $D^3 u = O(d^{-1-\varepsilon})$ for $|x| < 1 - r$ where $d = d(x) = \operatorname{dist}(x, S)$. We can therefore integrate by parts

$$\begin{aligned}
\int_{B_1} \zeta^4 u f &= \int_{B_1} \zeta^4 u \Delta^2 u = \int_{B_1} \Delta(\zeta^4 u)\Delta u \\
&= \int_{B_1} \zeta^4 |\Delta u|^2 + \int_{B_1} u\Delta u \Delta\zeta^4 + 2\int_{B_1} \Delta u \nabla u \cdot \nabla\zeta^4 \\
&= \int_{B_1} \zeta^4 |\Delta u|^2 + \int_{B_1} u\Delta u(12\zeta^2|\nabla\zeta|^2 + 4\zeta^3\Delta\zeta) + 2\int_{B_1} \Delta u \nabla u \cdot (4\zeta^3\nabla\zeta) \\
&\geqslant \frac{1}{2}\int_{B_1} \zeta^4|\Delta u|^2 - \int_{B_1} u^2(12|\nabla\zeta|^2 + 4\zeta\Delta\zeta)^2 - \int_{B_1} |\nabla u|^2 (8\zeta|\nabla\zeta|)^2.
\end{aligned}$$

Notice that

$$\begin{aligned}\int_{B_1}\zeta^4|\Delta u|^2 &= \sum_{i,j}\int_{B_1}\zeta^4 u_{ii}u_{jj}\\ &= \sum_{i,j}\int_{B_1}\zeta^4 u_{ij}u_{ij}+\sum_{i,j}\int_{B_1}u_{ii}u_j 4\zeta^3\zeta_j-\sum_{i,j}\int_{B_1}u_{ij}u_j4\zeta^3\zeta_i\\ &\geqslant \frac{1}{2}\int_{B_1}\zeta^4|D^2u|^2 - C\int_{B_1}|\nabla u|^2\zeta^2|\nabla\zeta|^2.\end{aligned}$$

Combining these two inequalities, and using also the fact that $|u|\leqslant C_1$, we obtain

$$\begin{aligned}\int_{B_1}\zeta^4|D^2u|^2 &\leqslant CC_1^2\int_{B_{1-r}\setminus B_{1-2r}}(12\zeta^2|\nabla\zeta|^2+4\zeta^3\Delta\zeta)+C\int_{B_{1-r}}|\nabla u|^2\zeta^2|\nabla\zeta|^2\\ &\leqslant \frac{CC_1^2}{r^3}+\frac{C^*}{r^2}\int_{B_{1-r}}|\nabla u|^2+C_1C_0.\end{aligned}$$

By embedding

$$\int_{B_{1-r}}|\nabla u|^2\leqslant\varepsilon\int_{B_{1-r}}|D^2u|^2+\frac{C}{\varepsilon}\int_{B_{1-r}}u^2,$$

where the constant C is independent of ε. Taking $\varepsilon = r/(32C^*)$, we get

$$\int_{B_{1-2r}}|D^2u|^2\leqslant\int_{B_1}\zeta^4|D^2u|^2\leqslant\frac{C^{**}(C_1^2+C_2^2)}{r^3}+\frac{1}{16}\int_{B_{1-r}}|D^2u|^2\quad\forall 0<r\leqslant\frac{1}{4},$$

which implies that

$$\int_{B_{1-r}}|D^2u|^2\leqslant\frac{16C^{**}(C_1^2+C_2^2)}{r^3}.$$

Taking $r=1/4$ the proof of (9.10) is complete. The conclusions (9.4), (9.5) now follows from Theorem 9.1 (example 2) with $\Gamma = S$. □

Remark 9.1. The proof of Lemma 9.2 extends to the case where S is replaced by Γ as defined at the beginning of this section; i. e., Theorem 9.1 is valid if the assumption $\|u\|_{H^2(\Omega\setminus\Gamma)}\leqslant C_1$ is replaced by the assumption $|u|\leqslant C_1$ in Ω.

The estimate (9.6) is a sub-Schauder estimates for $C^{1+\alpha}$ boundary. The next sub-Schauder estimates are for $C^{2+\alpha}$ boundary.

Theorem 9.3 *Let Ω be a bounded domain and let Ω_0 be a subdomain of Ω with $\overline{\Omega}_0\subset\Omega\cup S$ where S is a $C^{2+\alpha}$ subarc of $\partial\Omega$. Let u be a solution of*

$$\begin{aligned}&\Delta^2 u = f\quad \text{in } \Omega,\quad u\in H^2(\Omega),\\ &u=g,\quad \frac{\partial u}{\partial\nu}=h\quad \text{on } S.\end{aligned}$$

(i) ($C^{2+\alpha}$ estimate) If

$$\|g\|_{C^{2+\alpha}(S)}<\infty,\quad \|h\|_{C^{1+\alpha}(S)}<\infty,\quad \int_\Omega |f|^p dx<\infty\quad (p>\frac{1}{1-\alpha}),$$

then

$$\|u\|_{C^{2+\alpha}(\Omega_0)} \leqslant C\Big(\|g\|_{C^{2+\alpha}(S)} + \|h\|_{C^{1+\alpha}(S)} + \|f\|_{L^p(\Omega)} + c_0\|u\|_{L^\infty(\Omega)}\Big);$$

(ii) ($C^{1+\alpha}$ estimate) If

$$\|g\|_{C^{1+\alpha}(S)} < \infty, \quad \|h\|_{C^\alpha(S)} < \infty, \quad \int_\Omega |f|dx < \infty,$$

then

$$\|u\|_{C^{1+\alpha}(\Omega)} \leqslant C\Big(\|g\|_{C^{1+\alpha}(S)} + \|h\|_{C^\alpha(S)} + \int_\Omega |f|dx + c_0\|u\|_{L^\infty(\Omega)}\Big);$$

if $\Omega_0 = \Omega$ then the constant c_0 can be taken to be zero in both cases (i) and (ii).

If $S \in C^{4+\alpha}$ then the result is a consequence of [3; §9] (which is valid also for n-dimensional domains).

Proof. By subtracting the special solution (2.4), we may assume without loss of generality that $f(x) \equiv 0$. Let $y = \Phi(x)$ be the conformal mapping which flattens the boundary S. Under our assumptions, $\Phi \in C^{2+\alpha}$ up to the boundary.

Setting $w(y) = u(x)$, the equation $\Delta^2 u = 0$ becomes

$$\Delta_y[k(y)\Delta_y w(y)] = 0 \tag{9.11}$$

where $k(y) = |\nabla_x \Phi(x)|^2$ is in $C^{1+\alpha}$ up to the boundary. Now apply [3; §9] to immediately conclude both (i) and (ii). □

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