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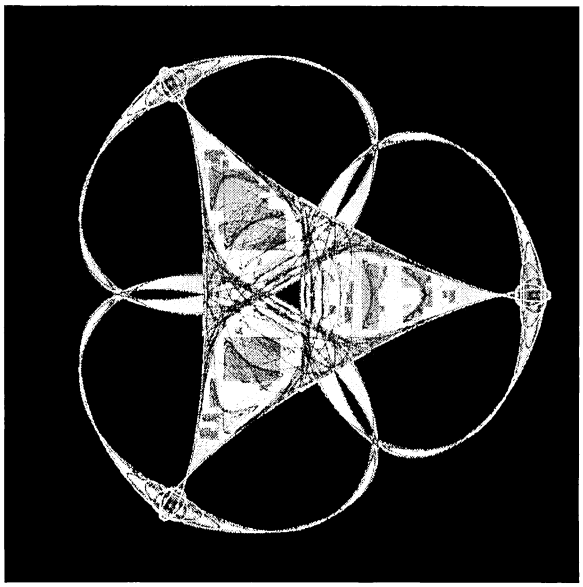
IMA Preprint Series # 1646

October 1999



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A NOTE ON THE PAPER OF ANDREWS

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The present paper embodies extension of the work of Andrews on the generating functions for two important classes of generalized frobenius representations.

Key words:- Frobenius partitions, generating functions, triple products, Infinite products, theta functions.

1. INTRODUCTION

Decomposition of basic hypergeometric series into even and odd parts and their special cases havebeen discussed by Madhekar and Chamle[3], sums of certain basic bilateral hypergeometric series, Some of which contain a quadratic factor $q^{n(n-1)/2}$, have been obtained by singh[4]. Analytic auxiliary functional generalization usable in combinatorial interpretations of generalized q-identities of the Rogers -Ramanujan type has also been given by singh[5],While dealing with bibasic hypergeometric transformations.In a memoir,Andrews[1] has defined and studied generalized Frobenius partitions,involving similar sequences needed in connection with a study of Baxter's[2] solution of the Hard hexagon model. Andrews[1] has shown that the generating functions for two impotant classes of generalized frobenius partitions, Viz, $C\phi_k$ (q) and ϕ_k (q),are in fact multidimensional theta functions. He has also given infinite product representations for both these functions when k = 1, 2, 3, and has remarked that "after the above results the expressions quickly become long and messy." His remark being a problem, still being unsolved, has made us to extend the work of Andrews on these two generating functions by deriving infinite product representations for them for the case k = 4 and indicate the nature of the results for arbitrarily large k.

2. NOTATIONS AND GENERAL FORMULATION

We shall use the following notations,

$$\text{For} \quad |\alpha|<1, \qquad |\beta|<1, \qquad |q|<1,$$

$$\text{let} \quad \pi(\alpha;\beta) = \prod_{n=1}^{\infty} (1-\alpha\beta^{n-1}), \qquad \pi(\alpha) = \pi(\alpha;\alpha),$$

$$p(q) = \frac{\pi(q^2)\,\pi(q^6)\,\pi(q^{48})}{\{\pi(q)\}^4}$$

$$G(\alpha;\beta) = \pi(-\alpha;\beta)\,\pi\left(\frac{-\beta}{\alpha};\beta\right), \quad k_z(\alpha) = G(z^3\alpha;q^2),$$

$$\text{and} \qquad H_s(z,\alpha) = G(z\alpha;q^6)\,G(z^{-4}\alpha^s;q^{48}).$$

further for $k \geq 2$ let

$$\Psi_k(z,q) = \frac{1}{\{\pi(\alpha)\}^k} \sum_{m_1,m_2,\ldots,m_{k-1}=-\infty}^{\infty} z^{P(m_1,m_2,\ldots,m_{k-1})}\, q^{Q(m_1,m_2,\ldots,m_{k-1})} \qquad \ldots\ldots\ldots\ldots(2.1)$$

where $p(m_1,m_2,\ldots,m_{k-1}) = (k-1)m_1 + (k-2)m_2 + \cdots + 2m_{k-2} + m_{k-1}$

$$Q(m_1,m_2,\ldots,m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1\leq i<j\leq k-1} m_i m_j$$

an empty sum being taken to be 0

In particular,

$$C\phi_k(q) = \Psi_k(1;q), \qquad \phi_k(q) = \Psi_k(\xi;q),$$

$$\text{Where} \qquad \xi = \exp\{2\pi i/(k+1)\}$$

Thus, in This notation, the two results of Andrews for k=3 mentioned above can be stated as follows; for z=1 and z=q

$$\Psi_3(z,q) = \frac{\pi(q^2)\,\pi(q^6)}{\{\pi(q)\}3}\left[G(z^2q,q^2)\,G(q^3,q^6) + zqG(q^2z^2,q^6)\,G(q^6,q^6)\right] \qquad \ldots\ldots\ldots(2.2)$$

3. STATEMENT AND PROOF OF THE THEOREM

We shall now prove the following theorem for $|q| < 1$,

$$\Psi_4(z,q) = P(q)\left[k_z(q)\{H_8(z,q^3) + z^{-1}q^3H_8(z,q^5) + z^{-2}q^{12}H_8(z,q^7)\}\right.$$

$$+q\,k_z(q^2)\{zH_8(z,q^4)+q^6H_8(z,q^6)+z^{-1}q^{18}H_8(z,q^8)+z^2H_4(z,q^6)+zq^4H_5(z,q^8)$$

$$\left.+q^{14}H_{\frac{28}{5}}(z,q^{10})\}+q^3K_z(q^3)\{z^3H_{\frac{32}{7}}(z,q^7)+z^2q^7H_{\frac{48}{9}}(z,q^9)+zq^{20}H_{\frac{64}{11}}(z,q^{11})\}\right] \ldots\ldots\ldots(3.1)$$

<u>Proof.</u> Let $S(z,q) = \{\pi(q)\}^4 \Psi_4(z,q)$, where $\Psi_4(z,q)$ is defined by definition (1). Then

$$S(z,q) = \sum_{m_1, m_2, m_3 = -\alpha}^{\alpha} z^{3m_1+2m_2+m_3} q^{m_1^2+m_2^2+m_3^2+m_1m_2+m_1m_3+m_2m_3}$$

Breaking the sum for m_2 into two parts, one for even m_2 and the other for odd m_2 and replacing m_1 by $m_1 - m_2$ in both the latter sums, we get

$$S(z,q) = \sum_{m_1, m_2, m_3 = -\alpha}^{\alpha} z^{3m_1+m_2+m_3} q^{m_1^2+3m_2^2+m_3^2+m_1m_3+m_2m_3} \left[1+z^2q^{m_1+3m_2+m_3+1} \right]$$

$$= s_1 + s_2, \text{say.} \qquad \ldots\ldots(3.2)$$

Again splitting the sum for m_3 in each of the two sums s_1 and s_2 according to the parity of m_3, and then replacing m_1 by $m_1 - m_3$ in all the four resulting sums,

We get

$$S(z,q) = \sum_{m_1, m_2, m_3 = -\alpha}^{\alpha} z^{3m_1+m_2-m_3} q^{m_1^2+3m_2^2+3m_3^2+2m_2m_3} \left[1+zq^{m_1+m_2+3m_3+1} + z^2q^{m_1+3m_2+m_3+1} + z^2q^{2m_1+4m_2+4m_3+3}\right]$$

$$= s_1' + s_1'' + s_2' + s_2'', \text{say.} \qquad \ldots\ldots(3.3)$$

splitting yet again the sum for m_3 in each of the last four sums modulo 3 (i.e. according to the residue classes of m_3 modulo 3) and replacing m_2 by $m_2 - m_3$ in all the twelve resulting sums, we get

$$S(z,q) = \sum_{m_1, m_2, m_3 = -\alpha}^{\alpha} z^{3m_1+m_2-4m_3} q^{m_1^2+3m_2^2+24m_3^2} [1+z^{-1}q^{2m_2+16m_3+1}+z^{-2}q^{4m_2+32m_3+12}+zq^{m_1+m_2+8m_3+1}+q^{m_1+3m_2+24m_3+7}$$

$$+z^{-1}q^{m_1+5m_2+40m_3+19}+z^2q^{m_1+3m_2+1}+zq^{m_1+5m_2+16m_3+5}+q^{m_1+7m_2+32m_3+15}$$

$$+z^3q^{2m_1+4m_2+8m_3+3}+z^2q^{2m_1+6m_2+24m_3+10}+zq^{2m_1+8m_2+40m_3+23}$$

$$= \sum_{j=1}^{3} (s_{1j}' + s_{1j}'' + s_{2j}' + s_{2j}''), \text{say.} \qquad \ldots\ldots(3.4)$$

Each of these twelve multiple sums now decomposes into a product of three simple sums for m_1, m_2, m_3. which can all be summed directly by using Jacobi's wellknown triple - product identity (1,eq. 3.1).

$$\pi(-zq;q)\,\pi(-z^{-1};q)\,\pi(q) = \sum_{m=-\alpha}^{\alpha} z^m q^{m(m-1)/2}, \qquad |q|<1, \text{and this easily leads to (3.1)} \qquad \ldots\ldots(3.5)$$

<u>4. SPECIAL CASES</u>

As special cases of our theorem, we get the following representations for the functions

$C\phi_4(q)$ and $\phi_4(q)$ of Andrew's.

Corollary1. For $|q| < 1$,

$$C\phi_4(q) = P(q)\,[K_1(q)\{H_8(1,q^3)+q^3H_8(1,q^5)+q^{12}H_8(1,q^7)\}$$

$$+qK_1(q^2)\{H_8(1,q^4)+q^6H_8(1,q^6)+q^{18}H_8(1,q^8)+H_4(1,q^6)+q^4H_5(1,q^8)+q^{14}H_{\frac{28}{5}}(1,q^{10})\}$$

$$+q^3K_1(q^3)\{H_{\frac{32}{7}}(1,q^7)+q^7H_{\frac{48}{9}}(1,q^9)+q^{20}H_{\frac{64}{11}}(1,q^{11})\}] \qquad \text{.........................(4.1)}$$

Corollary 2. for $|q| <1$,

$$\phi_4(q) = P(q)\,[K_\xi(q)\{\xi H_8(\xi,q^3)+\xi^{-1}q^3H_8(\xi,q^5)+\xi^{-2}q^{12}H_8(\xi,q^7)\}$$

$$+qK_\xi(q^2)\{\xi H_8(\xi,q^4)+q^6H_8(\xi,q^6)+\xi^{-1}q^{18}H_8(\xi,q^8,\}+\xi^2H_4(\xi,q^6)+\xi q^4H_5(\xi,q^8)$$

$$+q^{14}H_{\frac{28}{5}}(\xi,q^{10})\}+K_\xi(q^3)\{\xi^3H_{\frac{32}{7}}(\xi,q^7)+\xi^2q^7H_{\frac{48}{9}}(\xi,q^9)+\xi q^{20}H_{\frac{64}{11}}(\xi,q^{11})\}]. \qquad \text{...................(4.2)}$$

5.CONCLUSION

We shall now briefly indicate the nature of similar results which can be obtained for $C\phi_k(q)$ and $\phi_k(q)$ for arbitrarily large vaiues of K, An infinte product represention for $\Psi_k(z,q)$ may be obtained by alternately decomposing, for i=j, j+1,.....,k-1,the sums for m_i modulo j and replacing m_{j-1} by $m_{j-1}-m_i$ for j=2,3,..............., k-1. For example, for k=5, denoting decomposition of an m_i sum modulo j by $m_i(j)$ and replacement of m_{j-1} by $m_{j-1}-m_i$ by $m_{j-1}\to m_{j-1}-m_i$, We have the following decomposition replacement sequence : $m_2(2), m_1\to m_1-m_2$; $m_3(2), m_1\to m_1-m_3$; $m_4(2), m_1\to m_1-m_4$; $m_3(3)$, $m_2\to m_2-m_3$; $m_4(3), m_2\to m_2-m_4$; $m_4(4), m_3\to m_3-m_4$.

It is easy to see that, for general k, this process decomposes $\Psi_k(z,q)$ into $\prod_{n=1}^{k-1}(n!)$ summands .Each of these summands can be expressed as a 3 (k-1) product with the help of (3.5). In this way, one can obtain a representation for $\Psi_k(z,q)$, and hence for $c\,\phi_k(q)$ and $\phi_k(q)$ as well, involving 3 (k-1) $\prod_{n=1}^{k-1}(n!)$ infinite products. For example, for K=5, we have 3x4x288 = 3456 infinite products appearing in such a representation.

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