# On large deviations of empirical measures for stationary Gaussian processes

Włodzimierz Bryc *
Department of Mathematics
University of Cincinnati
Cincinnati, OH 45 221
bryc@uc.edu

Amir Dembo†
Deparment of Mathematics
and Department of Statistics
Stanford University
Stanford, CA 94 305
amir@playfair.stanford.edu

December 30, 1993

**Abstract**

We show that the Large Deviation Principle with respect to the weak topology holds for the empirical measure of any stationary continuous time Gaussian process with continuous vanishing at infinity spectral density. We also point out that Large Deviation Principle might fail in both continuous and discrete time if the spectral density is discontinuous.

## 1 Introduction

Large deviation principle (LDP) of empirical process for Gaussian random sequences with continuous spectral density were analyzed by several authors: Donsker and Varadhan [8] give LDP with explicit rate function for $\mathbb{R}$-valued $\mathbb{Z}$-indexed processes with continuous spectral density $f(s)$ such that $\int_{-\pi}^{\pi} \log f(s)ds > -\infty$; Steinberg and Zeitouni [13, Theorem 1] extend the LDP to $\mathbb{R}$-valued $\mathbb{Z}^d$-indexed processes with continuous spectral density $f(s)$ such that $\inf_s f(s) > 0$; Baxter and Jain [1, Theorem 4.25] prove LDP for $\mathbb{R}^d$-valued $\mathbb{Z}$-indexed processes with continuous spectral density matrix $F(s)$ such that $F(-\pi) = F(\pi)$. LDP of empirical process for a continuous time Gaussian process LDP is established by hypercontractivity methods in [5, Section 5] for $\mathbb{R}^d$-valued $\mathbb{R}$-indexed processes with differentiable spectral density matrix that satisfies certain additional assumptions. Several authors studied LDP for (certain) additive functionals of Gaussian processes/sequences/fields, see [2, 4, 7].

In this paper we analyze the large deviation principle (LDP) of empirical measures for stationary Gaussian processes. For simplicity, we consider only one-dimensional index set, real processes, and limit our attention to empirical measures only. We extend the continuous time LDP that follows from [5] to continuous spectral densities that vanish at infinity. We also show that both this result and the discrete time result from [1] cannot be extended to all bounded spectral densities. This should be contrasted with LDP of quadratic additive functionals, as analyzed in [4], where bounded spectral densities suffice.

We use the following notation.

*Partially supported by C.P. Taft Memorial Fund
†Partially supported by NSF DMS92-09712 grant and by a US-ISRAEL BSF grant.
**Key Words:** large deviations, empirical measure, Gaussian processes.
**AMS (1991) Subject Classification:** 60F10

Let $\Sigma$ be a Polish space and $M_1(\Sigma)$ denote the space of (Borel) probability measures on $\Sigma$ equipped with the weak topology, i.e., topology generated by the collection

$$\{\nu \in M_1(\Sigma) \ : \ |\int_\Sigma F d\nu - x| < \delta\} \ ,$$

where $x \in \mathbb{R}$, $\delta > 0$ and $F \in C_b(\Sigma, \mathbb{R})$ – the vector space of all bounded, real-valued, continuous functions on $\Sigma$. It is well known that $M_1(\Sigma)$ is a Polish space with the metric $\beta(\mu,\nu) = \sup |\int F d\mu - \int F d\nu|$, where the supremum is taken over all bounded Lipshitz functions $F$ of Lipschitz constant at most 1 with $\|F\|_\infty \leq 1$, see [10, Sections 11.3 and 11.5].

By $C_0(\mathbb{R})$ we denote the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ that vanish at $\pm\infty$. We shall use the fact that if $f \in C_0(\mathbb{R})$ then $f$ is bounded and uniformly continuous.

Recall the following.

**Definition 1** *A family of random variables $\{Y_T\}_{T>0}$ taking values in a topological space $\mathcal{X}$ equipped with the Borel $\sigma$-field $\mathcal{B}$ satisfies the Large Deviation Principle, if there is a lower semicontinuous rate function $I : \mathcal{X} \to [0,\infty]$, with compact level sets $I^{-1}([0,a])$ for all $a > 0$, and such that for all $\Gamma \in \mathcal{B}$,*

$$-\inf_{x\in\Gamma^o} I(x) \leq \liminf_{T\to\infty} \frac{1}{T}\log P(Y_T \in \Gamma) \leq \limsup_{T\to\infty} \frac{1}{T}\log P(Y_T \in \Gamma) \leq -\inf_{x\in\overline{\Gamma}} I(x) \ ,$$

*where $\overline{\Gamma}$ denotes the closure of $\Gamma$, $\Gamma^o$ the interior of $\Gamma$, and the infimum of a function over an empty set is interpreted as $\infty$.*

*We shall say that* weak *LDP holds, if the upper bound holds for pre-compact $\Gamma \in \mathcal{B}$ only and the sets $I^{-1}([0,a])$ are only required to be closed.*

Clearly, it is enough to verify the LDP on sub-sequences. With this in mind we give the following.

**Definition 2** *A family of random variables $\{Y_T\}_{T>0}$ is exponentially tight*[1]*, if for every sequence $T_n \to \infty$ and each $M > 0$ there is a pre-compact $K \in \mathcal{B}$ such that*

$$\limsup_{n\to\infty} \frac{1}{T_n}\log P(Y_{T_n} \notin K) \leq -M.$$

It is well known that weak LDP for exponentially tight sequence implies LDP, see [6]; for continuous index $T$ this holds true also with the definition of exponential tightness as given above.

Recall that the spectral density of a continuous time real process $X_t$ is an integrable even non-negative function $f$ such that $EX_0X_t = \int_{-\infty}^{\infty} e^{its} f(s)ds$. The spectral density of a discrete time process $\{X_j\}$ is a periodic (with the period $2\pi$) even non-negative integrable function $f$ such that $EX_0X_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iks} f(s)ds$.

In this note we limit our attention to Gaussian stationary processes that posses spectral density. In particular, by continuity of the covariance such a process is mean-square continuous, hence it has a measurable modification (see [9, Chapter II, Theorem 2.6]). We assume hereafter that we are dealing with such a modification.

For a continuous time $\Sigma$-valued measurable processes $\{X_t\}_{t\geq 0}$ on a complete probability space we define empirical measures by

$$L_T = \frac{1}{T}\int_0^T \delta_{X_t} dt \qquad \text{(Pettis integral)}, \tag{1}$$

where $\delta_y$ denote the probability measure degenerate at $y \in \Sigma$. Notice that $L_T$ is a well defined $M_1(\Sigma)$-valued random variable; for the details, see Appendix A.

[1] For a continuous index $T$, this is not the standard definition. However, it effectively replaces the usual one in the large deviation proofs.

For a sequence $\{X_j\}_{j=0}^{\infty}$ of random variables which take values in $\Sigma$ we consider the empirical measures

$$L_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{X_j} \ .$$

Since the discrete time case embeds into continuous case by the piecewise constant mapping $t \mapsto X_{[t]+1}$, in general statements below we consider continuous time only.

# 2 Results

The following is a special case of [1, Theorem 4.25]. A short self-contained proof is given below for completeness.

> **Theorem A** *If $\{X_k\}$ is a real stationary Gaussian sequence such that $X_k - m$ has continuous spectral density, then the empirical measures $L_n$ satisfy the LDP in $M_1(\mathbb{R})$ with a convex rate function.*

Our next result is a continuous time analog of Theorem A; it partially answers [1, Section 6: Question (d)]. The main improvement is that we do not require differentiability of the spectral density as assumed in [5]. We consider LDP of empirical measures, but we do not anticipate any difficulties with the extension to empirical process level, provided product topology is used in the trajectories $\mathbb{R}^{[0,\infty)}$.

As in discrete time, the LDP rate function is not easily identifiable, c.f., [1, Section 6 (a) & (c)].

**Theorem 2.1** *If $\{X_t\}$ is a real measurable stationary Gaussian process such that $X_t - m$ has spectral density $f \in C_0(\mathbb{R})$, then empirical measures $L_T$ satisfy the LDP in $M_1(\mathbb{R})$ with the convex rate function.*

Theorem 2.1 implies the LDP for the $p$-th moment averages, completing [4, Theorem 2.1]. (We omit the proof which is based on Lemma 3.1(ii) and [4, Theorem 2.1].)

**Corollary 2.1** *If $\{X_t\}$ has spectral density in $C_0(\mathbb{R})$ then for $0 \le p \le 2$ the averages $\frac{1}{T}\int_0^T |X_t|^p dt$ satisfy the LDP with a convex rate function.*

In [4], the LDP for quadratic functionals of Gaussian processes is established under the sole assumption of boundness of the spectral density. This raises the question of generalizing Theorem A and Theorem 2.1 to a larger class of spectral densities. In Theorem 2.2 below we show that [4, Theorems 2.1 and 2.2] cannot be extended from quadratic to all bounded continuous additive functionals. In particular, Theorem A does not hold for all bounded spectral densities, and Theorem 2.1 does not hold for all bounded, compactly supported spectral densities.

The case of piecewise continuous spectral densities is left unresolved. In fact, it is not known whether [4, Corollary 2.2] holds for $p = 1$ and piecewise continuous spectral densities with finite number of discontinuities and left/right limits; this class of spectral densities occurs in electrical engineering literature. Our method of proof is not applicable to this case, see Remark 4.1.

**Theorem 2.2** *(i) Suppose $\{X_j\}$ is a real centered stationary Gaussian sequence with strictly positive and bounded spectral density $f(s) = 2 + \sin\log|s|, -\pi \le s \le \pi$. Then there is a bounded continuous function $F : \mathbb{R} \to \mathbb{R}$ such that the LDP for $\frac{1}{n}\sum_{j=1}^{n} F(X_j)$ fails.*
*(ii) Suppose $\{X_t\}$ is a real centered stationary Gaussian process with bounded and compactly supported spectral density $g(s) = f(s)\frac{(s/2)^2}{\sin^2(s/2)}1_{|s|\le\pi}$, with $f(s)$ the spectral density from part (i). Then there is a bounded continuous function $F : \mathbb{R} \to \mathbb{R}$ such that the LDP for $\frac{1}{T}\int_0^T F(X_t)dt$ fails.*

By contraction principle we get.

**Corollary 2.2** *(i) There is a real stationary centered Gaussian sequence $\{X_j\}$ with bounded and strictly positive spectral density whose empirical measures $L_n = \frac{1}{n}\sum_{j=1}^n \delta_{X_j}$ do not satisfy the LDP in $M_1(\mathbb{R})$.*
*(ii) There is a real stationary centered Gaussian process $\{X_t\}$ with bounded and compactly supported spectral density whose empirical measures $L_T = \frac{1}{T}\int_0^T \delta_{X_t} dt$ do not satisfy the LDP in $M_1(\mathbb{R})$.*

Since the spectral density of Theorem 2.2 satisfies $3 - f(s) \geq 0$ we also get the following.

**Corollary 2.3** *There are independent real stationary centered Gaussian processes $\{X_j\}, \{Y_j\}$ such that the empirical measures of $\{X_j + Y_j\}$ satisfy the LDP (being i.i.d.), but the LDP for empirical measures of $\{X_j\}$ fails.*

In our proofs we shall use the criterion for exponential tightness, which follows from the proof of [14, Lemma 8.7].

**Theorem 2.3** *Suppose $\{X_t\}$ is a measurable $\mathbb{R}^d$-valued process such that for some $q, \alpha > 0$ we have*

$$\limsup_{T\to\infty} T^{-1} \log E \exp \alpha \int_0^T \|X_t\|^q dt < \infty. \tag{2}$$

*Then empirical measures $L_T$ are exponentially tight.*

In [4] we show that condition (2) with $q = 2$ is satisfied for stationary Gaussian processes with bounded spectral densities. Combining this with [12, Theorem 1] we get the following completion to Corollary 2.2.

**Corollary 2.4** *If $X_t$ (respectively, $X_j$) is a stationary Gaussian process (sequence) with bounded spectral density, then from any sequence of empirical measures $L_{T_n}$ one can select a (deterministic) subsequence that satisfies LDP.*

# 3 Proofs

## 3.1 General Lemmas

The following definition is a specification of [6, Section 4.2.2].

**Definition 3** *$L_T^M$ are exponentially good approximations of $L_T$ if for every $\delta > 0$*

$$\lim_{M\to\infty} \limsup_{T\to\infty} T^{-1} \log P(\beta(L_T^M, L_T) > \delta) = -\infty. \tag{3}$$

Our proof of Theorem 2.1 is based on the following approximation lemma, similar to [4, Lemma 4.8] and [1, Theorem 4.9], who give discrete time versions. We state Lemma 3.1 in more generality than what is needed below.

Let $\mathbb{E}$ be a separable Banach space with the norm $\|\cdot\|$.

**Lemma 3.1** *Suppose $X_t = Y_t^M + Z_t^M$ are $\mathbb{E}$-valued and such that for each $\theta > 0$*

$$\lim_{M\to\infty} \limsup_{T\to\infty} T^{-1} \log E \left( \exp(\theta \int_0^T \|Z_t^M\| dt) \right) = 0. \tag{4}$$

*(i) If for all $M$ the empirical measures $L_T^M$ for $Y_t^M$ satisfy the LDP in $M_1(\mathbb{E})$ with the rate function $I_M(\cdot)$, then the empirical measures for $X_t$ satisfy weak LDP in $M_1(\mathbb{E})$ with a rate function $I(\cdot)$. Moreover, if for all $M$ the rate functions $I_M(\cdot)$ are convex, then $I(\cdot)$ is convex.*
*(ii) If for all $M$ the averages of $Y_t^M$ satisfy the LDP then the averages of $X_t$ satisfy the weak LDP.*

**Remark 3.1** *In case (i) one may replace $\|Z_t^M\|$ in (4) by $\|Z_t^M\| \wedge 1$.*

**Proof.** We prove only part (i); for a proof in discrete time - compare [1]. The proof of (ii) is very similar (also yielding convexity of rate function if all rate functions for averages of $Y_t^M$ are convex).

Notice that (4) implies that $L_T^M$ are exponentially good approximations of $L_T$ in $M_1(\mathbb{E})$. Indeed, for $f$ of Lipschitz constant at most 1 and bounded by 1 we have $|f(X_t) - f(Y_t^M)| \leq 2\|Z_t^M\| \wedge 1$, implying that

$$\beta(L_T^M, L_T) \leq 2T^{-1} \int_0^T (\|Z_t^M\| \wedge 1) dt.$$

Thus exponentially good approximation follows from (4) by Chebyshev inequality, taking first $T \to \infty$, then $M \to \infty$, and finally $\theta \to \infty$.

By [6, Theorem 4.2.16(a)] the weak LDP of $L_T$ follows with the rate function $I(\cdot)$ given by [6, (4.2.17)]. Suppose now that all rate functions $I_M(\cdot)$ are convex. To prove that then $I(\cdot)$ is also convex we use the alternative expression

$$I(\mu) = \sup_{\delta > 0} \limsup_{M \to \infty} \inf_{\nu \in B(\mu, \delta)} I_M(\nu), \tag{5}$$

given in the last line of the proof of [6, Theorem 4.2.16 (a)].

Fix $\mu, \nu \in M_1(\mathbb{E})$. If $\beta(\mu, \mu_1) < \delta$ and $\beta(\nu, \nu_1) < \delta$ then clearly $\beta\left(\frac{1}{2}(\mu + \nu), \frac{1}{2}(\mu_1 + \nu_1)\right) < \delta$. Therefore from (5)

$$I\left(\frac{1}{2}(\mu + \nu)\right) \leq \sup_{\delta > 0} \limsup_{M \to \infty} \inf_{\nu_1 \in B(\nu, \delta),\ \mu_1 \in B(\mu, \delta)} I_M\left(\frac{1}{2}(\nu_1 + \mu_1)\right) \leq \frac{1}{2}\left(I(\mu) + I(\nu)\right).$$

□

The following comparison lemma is of interest.

**Lemma 3.2** *If $Z_t$ and $V_t$ are stationary centered Gaussian processes (sequences) with spectral densities $h, g$ such that*

$$h(s) \leq g(s) \quad \forall s$$

*then for bounded measurable $a(t) \geq 0$ and $1 \leq q \leq 2$*

$$E \exp \int_0^T a(t)|Z_t|^q dt \leq E \exp \int_0^T a(t)|V_t|^q dt \tag{6}$$

**Proof.** Take an independent of $\{Z_t\}$ stationary centered Gaussian $\{Y_t\}$ with spectral density $g(s) - h(s)$. Then $V_t = Y_t + Z_t$ and by independence $Z_t = E\{V_t | \sigma(Z_t, t \in \mathbb{R})\}$. Inequality (6) follows by Jensen's inequality and the convexity of the map $V. \mapsto \exp(\int_0^T a(t)|V_t|^q dt)$. □

Our main estimate is as follows.

**Lemma 3.3** *If the spectral density of a (centered) Gaussian process (sequence) is bounded, $f(s) \leq M$, then for every $\theta > 0$*

$$\limsup_{T \to \infty} T^{-1} \log E \exp \theta \int_0^T |X_t| dt \leq 26 M \theta^2 + \frac{E X_0^2}{13 M}. \tag{7}$$

*(In discrete time put $X_t = X_{[t]+1}$.)*

**Proof.** We prove only continuous time version. For the discrete time (and multivariate) version, one needs only to use [4, Theorem 2.2] instead of [4, Theorem 2.1].

Notice that for every $A > 0$ we have

$$\int_0^T |X_t| dt \le \sqrt{T} \left( \int_0^T |X_t|^2 dt \right)^{1/2} \le AT + \frac{1}{A} \int_0^T |X_t|^2 dt.$$

Using this with $A = 26M\theta$ we get by [4, Theorem 2.1]

$$\limsup_{T\to\infty} \frac{1}{T} \log E \exp \theta \int_0^T |X_t| dt \le 26M\theta^2 - \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \left( 1 - \frac{2\pi}{13M} f(s) \right) ds.$$

Since for $0 < x < \frac{2\pi}{13}$ we have $-\log(1-x) = x + x^2/2 + x^3/3 \ldots < 2x$, the inequality follows. □

**Proof of Theorem A.** Without loss of generality we assume $m = 0$. Exponential tightness follows from Lemma 3.2 and Theorem 2.3, used with $q = 1$; see also Remark 4.2.

Given i.i.d. N(0,1) sequence $\gamma_k$, write $X_t = \sum_{k=-\infty}^{\infty} a_k \gamma_{k+t}$ (which holds, if spectral density exists), and let

$$Y_t^M = \sum_{|k|<M} a_k \left( 1 - \frac{|k|}{M} \right) \gamma_{k+t}.$$

Since the spectral density is continuous, $Z_t^M = X_t - Y_t^M$ is of spectral density $g_M(s) \to 0$ uniformly in $s$ (Fejér's theorem). By Remark 4.2 (and Hölder's inequality), this establishes (4).

Since $Y_t^M$ is finitely dependent, and for finitely dependent sequences the LDP holds (see e.g., [6, Section 6.4.2]), Lemma 3.1(i) ends the proof. □

## 3.2 Proof of Theorem 2.1

The proof of Theorem 2.1 proceedes in the same pattern as the discrete one. First, without loss of generality we assume $m = 0$.

By (7) and Theorem 2.3, the sequence $\{L_T\}_{T>0}$ is exponentially tight in $M_1(\mathbb{R})$. Therefore it is enough to prove the weak LDP.

We shall use Lemma 3.1 and a continuous time variant[2] of the approximation scheme from [8]. Let $f^2(s)$ be the spectral density of $\{X_t\}$ with $f(s) \ge 0$. Write $X_t = \int_{-\infty}^{\infty} e^{its} f(s) dW_s$ (the spectral representation; see [9, Chapter XI, (8.2)]) and define

$$Y_t = \int_{-\infty}^{\infty} e^{its} h(s) dW_s$$

where

$$h(s) = h_M(s) = \int_{-\infty}^{\infty} f(s-u) \frac{1 - \cos Mu}{\pi M u^2} du. \tag{8}$$

Clearly, $h = f \star v$ where $f \ge 0$ is in $\mathbb{L}_2$ and $v \ge 0$ is in $\mathbb{L}_1$; by Young's inequality $h \in \mathbb{L}_2$, and $Y_t$ is well defined. The Fourier transform $H$ of $h(s)$ is $H(t) = (1 - \frac{|t|}{M}) F(t)$ for $|t| < M$, where $F$ is the Fourier transform of $f$; $H(t) = 0$ for $|T| > M$. Put $Z_t^M = X_t - Y_t$.

We first show that (4) holds. To this end observe that

$$h_M(s) \to f(s) \tag{9}$$

[2] Direct discretization of time runs into technical difficulties even for processes with continuous trajectories.

uniformly in $s$. Indeed, let $\epsilon > 0$. Since $f$ is uniformly continuous, one can find $\delta > 0$ such that if $|u| < \delta$ then $\sup_s |f(s-u) - f(s)| < \epsilon$. Therefore

$$\sup_s |h_M(s) - f(s)| \le \sup_s \int_{-\infty}^{\infty} |f(s-u) - f(s)| \frac{1 - \cos Mu}{\pi M u^2} du$$

$$\le \epsilon + \frac{8}{\pi M} \|f\|_\infty \int_\delta^\infty \frac{du}{u^2} = \epsilon + \frac{8C}{\pi M \delta}.$$

Taking $M \to \infty$, this proves (9).

Since the spectral density $g_M$ of $Z_t^M$ equals $g_M(s) = (f(s) - h_M(s))^2$, by (9) it converges to 0 uniformly in $s$.

We also need to check that

$$EZ_0^2 = \int_{-\infty}^{\infty} g_M(s) ds \to 0$$

as $M \to \infty$. Indeed, by Plancherel's identity

$$\int_{-\infty}^{\infty} g_M(s) ds = (2\pi)^{-1} \int_{-\infty}^{\infty} |F(t) - H(t)|^2 dt = (2\pi)^{-1} \int_{-\infty}^{\infty} \left( \frac{|t|}{M} \wedge 1 \right)^2 |F(t)|^2 dt.$$

For all $\epsilon > 0$ we have

$$\int_{-\infty}^{\infty} \left( \frac{|t|}{M} \wedge 1 \right)^2 |F(t)|^2 dt \le \epsilon^2 \int_{-\infty}^{\infty} |F(t)|^2 dt + \int_{|t| > \epsilon M} |F(t)|^2 dt.$$

Since $\int |F(t)|^2 = 2\pi \int f^2(s) ds < \infty$, it follows by taking first $M \to \infty$ and then $\epsilon \to 0$ that $\int g_M(s) ds \to 0$ as $M \to \infty$.

Therefore, given $\epsilon > 0$ by (7) for all $M$ large enough (so that $\|g_M\|_\infty < \epsilon$) we have

$$\limsup_{T \to \infty} T^{-1} \log E(\exp(\theta \int_0^T \|Z_t^M\| dt)) \le 26\epsilon\theta^2 + E\left(Z_0^M\right)^2 / (13\epsilon).$$

Since $\epsilon > 0$ is arbitrary, taking the limit as $M \to \infty$ proves (4).

Now notice that $\{Y_t\}$ is $2M$-dependent. Indeed, its spectral density is $h^2(s)$. Therefore $EY_0Y_t = \int_{-\infty}^{\infty} e^{its} h^2(s) ds$ is the convolution of the Fourier transform $H$ of $h(s)$ with itself (evaluated at $t$), c.f. [15, (2.33) page 253]. Since $H(t) = 0$ for $|T| > M$, its convolution $H \star H$ vanishes at $|t| > 2M$ proving $2M$-dependence.

Empirical measures for a finitely-dependent process $Y_t = Y_t^M$ satisfy the LDP with the convex rate function, see [5]. By Lemma 3.1(i) and exponential tightness, the LDP of empirical measures for $X_t$ follows and the rate function is convex. □

## 3.3 Proof of Theorem 2.2

Let $S_n = \sum_{j=1}^n X_j$; analoguously, in continuous time let $S_T = \int_0^T X_t dt$. We shall prove Theorem 2.2 by contradiction using the following.

**Lemma 3.4** *(i) If the spectral density is bounded and for every bounded continuous $F$ the sequence $\frac{1}{n} \sum_{j=1}^n F(X_j)$ satisfies the LDP, then $\frac{1}{n} S_n$ satisfies the LDP.*
*(ii) If the spectral density is bounded and for every bounded continuous $F$ the sequence $\frac{1}{T} \int_0^T F(X_t) dt$ satisfies the LDP, then $\frac{1}{T} S_T$ satisfies the LDP.*

**Proof.** Let $Y_j^M = X_j I_{|X_j|\le M} + M I_{|X_j|>M} \text{sign } X_j$. Since $Y_j^M = F(X_j)$ with bounded continuous $F$, for every $M$ the sequence $\frac{1}{n}\sum_{j=1}^n Y_j^M$ satisfies the LDP. Moreover, for $\theta > 0$

$$E \exp\theta \sum_{j=1}^n |Y_j^M - X_j| \le E \exp\theta \sum_{j=1}^n |X_j| I_{|X_j|>M} \le E \exp \frac{\theta}{M} \sum_{j=1}^n |X_j|^2.$$

By [4, Theorem 2.2], condition (4) is satisfied. The LDP for $\frac{1}{n}S_n$ follows from Lemma 3.1(ii) since $\frac{1}{n}S_n$ is exponentially tight by (7).

The proof of part (ii) procedes analoguously, and is omitted. □

**Proof of Theorem 2.2.** We first prove part (i). Since $f(s)$ under consideration is bounded, by Lemma 3.4, it is enough to show that the LDP fails for the arithmetic means of the sequence. However, since $S_n$ is Gaussian and (7) holds, it is easy to see that the LDP for $\frac{1}{n}S_n$ holds if and only if

$$\sigma_n = \frac{1}{n} Var(S_n) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} \frac{\sin^2(ns/2)}{\sin^2(s/2)} f(s) ds \tag{10}$$

converges as $n \to \infty$. It is known, see [11, Theorem C1] (see also [15, Chapter XI, Theorem 2-26(ii)]) that (10) converges if and only if

$$\lim_{t\to 0} \frac{1}{t} \int_0^t (f(u) + f(-u)) du \tag{11}$$

exists. It is easy to check that for our choice of $f$ and $t > 0$ we have $\int_0^t (f(u) + f(-u)) du = 4t + t(\sin\log t - \cos\log t)$ so that the limit (11) does not exist.

To prove part (ii), notice that since $x/\sin x$ is monotone hence bounded on $[-\pi/2, \pi/2]$, $g(s)$ is bounded and compactly supported. Following the line of proof of part (i), we check that

$$\sigma_T = \frac{1}{T} Var(S_T) = \int_{-\infty}^{\infty} \frac{\sin^2(Ts/2)}{T(s/2)^2} g(s) ds = \frac{1}{T} \int_{-\pi}^{\pi} \frac{\sin^2(Ts/2)}{\sin^2(s/2)} f(s) ds$$

does not converge as $T \to \infty$ by the previous reasoning. □

# 4 Remarks

**Remark 4.1** *The proof of Theorem A and of Theorem 2.1 uses exponentially good approximations (see [6]). In connection with this method, one can verify that the mapping*

$$B[-\pi, \pi] \ni f \mapsto \limsup_{n\to\infty} \frac{1}{n} \log E(\exp(\sum_{j=1}^n |X_j| \wedge 1)) \in \mathbb{R}$$

*from bounded spectral densities, is not continuous with respect to* $\mathbb{L}_q$ *norms,* $q < \infty$.

*Indeed, Theorem 2.2 gives* $\{X_k\}$ *of bounded spectral density for which empirical measures fail LDP. On the other hand, the spectral density of* $\{X_k\}$ *can be approximated in* $\mathbb{L}_q$ *norm by a continuous one (vanishing in the neighborhood of the origin). Hence the empirical measures for the corresponding sequence* $\{Y_k^M\}_k$ *have LDP by Theorem A. The continuity (applied to* $Z_k^M = X_k - Y_k^M$*) would imply (by Lemma 3.1(i), Theorem 2.3 and Lemma 3.3) the LDP of empirical measures for* $\{X_k\}$*, a contradiction.*

**Remark 4.2** *For $d = 1$ and discrete time, using Lemma 3.2 with i.i.d. sequence $V_j$ of spectral density $g(s) = M$, it is easy to see that (7) can be replaced by the non-asymptotic estimate*

$$\frac{1}{n} \log E \exp \theta \sum_{j=1}^{n} |X_j|^q \leq \log E \exp \theta |V_1|^q \leq b(q)(\theta^2 M^q)^{1/(2-q)} + \log a(q)$$

*for all $1 \leq q < 2$, all $\theta$, and some universal finite constants $a(q)$, $b(q)$. In particular, $a(1) = 2$ and $b(1) = 1/2$.*

**Acknowledgement** Part of this research was done while the first named author (WB) was visiting the Institute of Applied Mathematics at the University of Minnesota. The authors would like to thank: Naresh Jain for preprint [1] and the discussion that lead to Theorem 2.1, and Corollary 2.3; T. Y. Lee for a comment that motivated shifting $X_t$ to $X_t - m$; Timo Seppalainen, W. Smoleński, and O. Zeitouni for helpful discussions.

# A Appendix: Empirical measures in continuous time

Here we show that in the continuous time the empirical measure of a measurable process is well defined as the Pettis integral. Let $\mathcal{P}_a(\Sigma)$ denote the space of all finitely additive regular non-negative set-functions of unit total mass, on the field generated by the closed subsets of Polish space $(\Sigma, d)$; $\mathcal{P}_a(\Sigma)$ is equipped with weak topology, i.e., topology generated by the collection

$$\{\nu \in \mathcal{P}_a(\Sigma) \ : \ |\int_\Sigma F d\nu - x| < \delta\} \ ,$$

where $x \in \mathbb{R}$, $\delta > 0$ and $F \in C_b(\Sigma, \mathbb{R})$.

Fix $T > 0$ and let $\lambda$ be normalized Lebesgue measure on $[0, T]$. Given a measurable function $x(t) = X_t(\omega)$ (with $\omega \in \Omega$ fixed), the mapping $\Phi : F \mapsto \int F(x(t))\lambda(dt)$ is a continuous, linear, and non-negative functional on $C_b(\Sigma, \mathbb{R})$ such that $\Phi(1) = 1$. Therefore (explicit reference is [3, page 54, Theorem 1]) $\Phi(F) = \int F(x) d\mu(x)$ for some $\mu \in \mathcal{P}_a(\Sigma)$. Since for $F_n \downarrow 0$ pointwise, $\Phi(F_n) \to 0$ by dominated convergence, it follows that $\mu$ has a (unique) countably additive extension $\tilde{\mu}$ on $\mathcal{B}_\Sigma$. Empirical measure $L_T : \Omega \to M_1(\Sigma)$ is defined now by $L_T(\omega) = \tilde{\mu}$, if $t \mapsto X_t(\omega)$ is measurable and say $\delta_{x_0}$ otherwise. By Fubini's theorem $\omega \to \int F dL_T$ is measurable. This shows that $\Omega \ni \omega \mapsto L_T(\omega) \in M_1(\Sigma)$ is weakly-measurable; since $M_1(\Sigma)$ is separable, the latter is the same as measurability.

# References

[1] Baxter, J. R. & Jain, N. C. An approximation condition for large deviations and an application to stationary sequences, preprint, 1993.

[2] Benitz, G. R. & Bucklew, J. A. Large Deviation rate calculation for nonlinear detectors in Gaussian noise, IEEE IT 36 (1990), 358–371.

[3] Bergstrom, H. *Weak convergence of measures*, Acad. Press, New York 1982.

[4] Bryc, W. & Dembo, A. Large Deviations for quadratic functionals of Gaussian processes (expanded version) IMA Preprint Series # 1179, October 1993.

[5] Chiyonobu, T. & Kusuoka, S. The large deviation principle for hypermixing processes, Probab. Theory Relat. Fields 78, (1988) 627–649.

[6] Dembo, A. & Zeitouni, O. *Large Deviations Techniques and Applications.* Jones and Bartlett, Boston, MA, 1993.

[7] Deuschel, J. Invariance principle and empirical mean large deviations of the critical Ornstein-Uhlenbeck process, Ann. Probab. 17 (1989), 74–90.

[8] Donsker, M. D. & Varadhan, S. R. S. Large deviations for Stationary Gaussian Processes, Commun. Math. Phys. 97 (1985), 187–210.

[9] Doob, J. *Stochastic Processes*, Wiley, New York 1953.

[10] Dudley, R. M. *Real analysis and probability*, Wadsworth & Brooks/Cole, 1989.

[11] Hardy, G. H. & Littlewood, J. E. Solution of the Cesàro summability problem for power series and Fourier series, Math. Zeitschr. 23 (1924), 67–94.

[12] Puhalskii, A. On functional principle of large deviations, New Trends In Probab. and Statist. (Ed: V. Sazonov & T. Shervashiodze), 1991, 198–218.

[13] Steinberg, Y. & Zeitouni, O. On tests for normality, IEEE Transactions on Information Theory 38 (1992), 1779–1787.

[14] Stroock, D. W. *An introduction to the theory of large deviations* Springer, New York 1984.

[15] Zygmund, A. *Trigonometric Series*, Vol II, Cambridge University Press 1959, Cambridge.