On Model for Quantum Friction II. Fermi's Golden Rule and Dynamics at Positive Temperatures

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Abstract

We investigate the dynamics of an N-level atom linearly coupled to a field of massless bosons at positive temperature. We use complex deformation technique to develop a time-dependent perturbation theory for the model, investigate spectral properties of the Hamiltonian of the system and calculate the lifetime of resonances.

1 Introduction

Let $\mathcal{A}$ be a quantum mechanical N-level system with energy operator H_A on the Hilbert space $\mathcal{H}_A = C^N$. We assume that the self-adjoint matrix H_A has

N simple eigenvalues $\{E_j : 1 \leq j \leq N\}$, listed in increasing order, and let ψ_j be the corresponding eigenvectors. We will colloquially refer to $\mathcal{A}$ as an atom or a small system.

Let $\mathcal{B}$ be an infinite heat bath. In this paper $\mathcal{B}$ will be an infinite free Bose gas at inverse temperature $\beta = 1/kT$ without Bose-Einstein condensate. The system $\mathcal{B}$ is described (see e.g. [BR], [D], [D1]) by a triple $\{\mathcal{H}_B, \Omega_B, H_B\}$, where $\mathcal{H}_B$ is a Hilbert space, H_B is a self-adjoint operator on $\mathcal{H}_B$ and Ω_B is a unit vector in $\mathcal{H}_B$. There is a representation of CCR on $\mathcal{H}_B$,

$$W_B(f) = \exp(i\varphi_B(f)), \qquad f,\ f/\sqrt{\omega} \in L^2(R^3), \tag{1.1}$$

where $\varphi_B(f)$ are field operators, satisfying

$$(\Omega_B, W_B(f)\Omega_B) = \exp\left[-\frac{\|f\|^2}{4} - \frac{1}{2}\int_{R^3} |f(k)|^2 \rho(k) dk\right], \tag{1.2}$$

$$\exp(itH_B) W_B(f) \exp(-itH_B) = W_B(\exp(it\omega)f). \tag{1.3}$$

The function $\omega(k)$ is the energy of a boson with momentum $k \in R^3$. The function $\rho(k)$ is the momentum density of bosons and is given by

$$\rho(k) = \frac{1}{\exp(\beta\omega(k)) - 1}. \tag{1.4}$$

We are interested in a physically realistic case when bosons have zero mass, namely when $\omega(k) = |k|$.

Let us suppose that systems $\mathcal{A}$ and $\mathcal{B}$, isolated at time $t = 0$, start interacting. One then expects that the temperature of the small system will start to change and approach the value $1/\beta$. Since $\mathcal{B}$ is an infinite system, its temperature does not change and the thermal equilibrium is achieved when both systems are at temperature $1/\beta$. Roughly speaking, this series of papers is devoted to study of this approach to the thermal equilibrium.

The representation of CCR satisfying properties (1.1)-(1.3) is usually constructed using the GNS procedure. We prefer to work in the explicit representation constructed by Araki and Woods [AW]. This representation is central in our approach.

The configuration space of a single boson is R^3 and its energy is $\omega(k) = |k|$ (we will always work in the momentum representation). The single particle Hilbert space is $L^2(R^3)$. Let $\mathcal{H}_f$ be the symmetric Fock space constructed from $L^2(R^3)$, Ω be the vacuum, $H_{bos} = d\Gamma(\omega)$ the energy operator. Denote by $a(f)$, $a^*(f)$ the annihilation and creation operators (see [RS] for definitions, $[a(f)]^* = a^*(f)$) and let

$$\varphi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f)), \qquad f \in L^2(R^3), \tag{1.5}$$

be the field operators.

In the Araki-Woods representation, the triple $\{\mathcal{H}_B, \Omega_B, H_B\}$ is given by

$$\mathcal{H}_B = \mathcal{H}_f \otimes \mathcal{H}_f, \qquad \Omega_B = \Omega \otimes \Omega, \qquad H_B = H_{bos} \otimes I - I \otimes H_{bos}. \tag{1.6}$$

The annihilation and creation operators are given by

$$a_B(f) = a((1+\rho)^{1/2} f) \otimes I + I \otimes a^*(\rho^{1/2} \overline{f}),$$

$$a_B^*(f) = a^*((1+\rho)^{1/2} f) \otimes I + I \otimes a(\rho^{1/2} \overline{f}),$$

and the field operators are defined as

$$\varphi_B(f) = \frac{1}{\sqrt{2}}(a_B(f) + a_B^*(f)). \tag{1.7}$$

In the sequel, whenever it is clear within the context, we will write A for $A \otimes I$ or $I \otimes A$.

When the thermal bath is at zero-temperature, the following formalism is used to describe system $\mathcal{A} + \mathcal{B}$. The Hilbert space of the system is $\mathcal{H}_A \otimes \mathcal{H}_f$, and the Hamiltonian is given by

$$\tilde{H}_\lambda = H_A \otimes I + \lambda Q \otimes \varphi(f) + I \otimes H_{bos} = H_A + \lambda \tilde{H}_I + H_{bos}. \tag{1.8}$$

There, Q is a self-adjoint matrix on $\mathcal{H}_A$, $f \in L^2(R^3)$ and $\lambda \in R$. In the sequel we will refer to f as the form factor and λ as the friction constant.

If $f/\sqrt{\omega} \in L^2(R^3)$ then $\tilde{H}_I$ is infinitesimally small with respect to $\tilde{H}_0$ and the operator $\tilde{H}_\lambda$ is essentially self-adjoint on $\mathcal{H}_A \otimes D(H_{bos})$. The choice of the Hamiltonian is motivated by the dipole approximation in non-relativistic QED. The extensively studied spin-boson Hamiltonian also has the form (1.8).

When the heat bath is at positive temperature, the Hilbert space of the joint system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the Hamiltonian is formally given by

$$H_\lambda = H_A \otimes I + \lambda Q \otimes \varphi_B(f) + I \otimes H_B = H_A + \lambda H_I + H_B, \tag{1.9}$$

see [D], [D1], [PU], [H], [BR]. In Section 3 we will prove that if $f/\sqrt{\omega}$, $f\omega \in L^2(R^3)$ then H_λ is essentially self-adjoint on $\mathcal{H}_A \otimes D(H_{bos}) \otimes D(H_{bos})$. However, H_I is not a relatively bounded perturbation of H_0. Note that at zero-temperature ($\beta = \infty$) the operator H_λ decouples and acts trivially on the second Fock space. The effective Hamiltonian acts on $\mathcal{H}_A \otimes \mathcal{H}_f$ and has the form (1.8). Thus, the zero-temperature model can be realized as a (strong resolvent) limit of positive temperature models, as expected.

The goal of this paper is to develop the time-dependent perturbation theory for the model (1.9). In the sequel we will briefly outline the physical content of the theory. It will be further discussed in the third and fourth paper in the series.

The time-dependent perturbation theory was developed by Dirac in 1920's [DI], and further refined by Weisskopf and Wigner in [W]. For the other developments we refer the reader to [HEI] and [SC]. Dirac have used the theory to study the emission and absorption of light by matter, and to derive Einstein A-B law from the first principles of quantum mechanics. Weisskopf and Wigner gave an improved solution of the equations of the perturbation theory, computed radiative lifetimes of the excited states of an atom and showed that the theory accounts for the observed width of the spectral lines emitted by an atom.

The Hamiltonian H_0 has the following spectrum: $\sigma_{ac}(H_0) = R$, $\sigma_{sc}(H_0) =$

$\emptyset$, $\sigma_{pp}(H_0) = \{E_j\}$. The eigenfunction associated to E_j is $\psi_j \otimes \Omega_B$. Let

$$b_j(t) = |(\psi_j \otimes \Omega_B, \exp(-itH_\lambda)\psi_j \otimes \Omega_B)|^2, \tag{1.10}$$

be the survival probability of the state $\psi_j \otimes \Omega_B$. The usual "textbook" derivation of radiative lifetimes starts with the relation

$$b_j(t) = \exp(-\Gamma_j(\lambda)t). \tag{1.11}$$

The inverse radiative lifetime of the state $\psi_j \otimes \Omega_B$, $\Gamma_j(\lambda)$, is related to to the width of the spectral lines by the uncertainty relation for time and energy. An application of the formal perturbation theory yields an expansion $\Gamma_j(\lambda) = \lambda^2\Gamma_j + O(\lambda^3)$ where Γ_j is given by

$$\Gamma_j = \sum_{\substack{k=1\\k\neq j}}^{N} \Gamma_{jk}, \tag{1.12}$$

$$\Gamma_{jk} = \pi Q_{kj}(E_j - E_k)^2 \cdot \left| \frac{\exp(\beta(E_j - E_k))}{\exp(\beta(E_j - E_k)) - 1} \right| \cdot \int_{S^2} |f(|E_j - E_k|, \theta, \phi)|^2 dS.$$

There, $Q_{kj} = |(Q\psi_k, \psi_j)|^2$, $f(r, \theta, \phi)$ is the polar representation of the function f, S^2 is the unit sphere in R^3 and dS its surface measure.

The second order perturbation theory takes into account only the processes in which one quanta of radiation is either emitted or absorbed. It follows from Dirac's theory that if $E_k < E_j$ then $\lambda^2\Gamma_{jk}$ is the probability per unit time that an atom will emit a photon of frequency $\nu = (E_j - E_k)/2\pi$ and make a transition from the energy level j to the level k. If $E_k > E_j$ then $\lambda^2\Gamma_{jk}$ is the probability per unit time that an atom will absorb a photon of frequency $\nu = (E_k - E_j)/2\pi$ and make a transition $k \to j$. For historical reasons (see e.g. [H], page 52) the terms Γ_j are often referred to as Fermi's Golden Rule. Note that at zero-temperature $\Gamma_{jk} = 0$ if $E_j < E_k$. The coefficient $\lambda^2\Gamma_j$ is the total transition probability per unit time from the level j. Let now p_j be the probability that system $\mathcal{A}$ is in the pure state $|\psi_j\rangle\langle\psi_j|$.

If the system $\mathcal{A}+\mathcal{B}$ is in thermal equilibrium, the detailed balance equation has to hold:

$$p_j\Gamma_j = \sum_{k\neq j} p_k\Gamma_{kj}.$$

One solution of the above system is $p_j = \exp(-\beta E_j)/\sum_k \exp(-\beta E_k)$, and it is the only solution if $\Gamma_{jk} > 0$ for all j, k [D1]. Therefore, an atom in thermal equilibrium with a blackbody radiation is in the Gibbs state, as expected.

The time-dependent perturbation theory, used in the above formal argument, resisted a general mathematical formulation for over forty years. Among the partly successful work on the subject, the most notable involve the master equation techniques [D], [D1], [D2], [HA], [PR]. This method has been discussed in [JP] and we will discuss it further in the latter papers in the series. Concerning the "usual" derivation of (1.11)-(1.12), note that the relation (1.11) cannot hold at zero-temperature for all times, since spectrum of $\tilde{H}_\lambda$ is bounded from below. Even at positive temperature, it can hold only as an approximation, and to quote [SI], "it is often discussed fact in the physics literature that the usual "textbook derivation" of the time-dependent series is internally inconsistent and there is not universal agreement among physicists concerning either the higher order terms in the series or the precise quantity which is being approximated".

The foundations of the time-dependent perturbation theory for n-body non-relativistic quantum systems, as well as the precise mathematical definition of resonance, were given in [SI]. We refer the reader to [SI] and [RS] for a list of references concerning earlier work on the subject. The notions introduced in [SI] have a natural extension to non-relativistic QED. The time-dependent perturbation series is supposed to describe what happens with the eigenvalues E_j of H_0 embedded in the continuum after the perturbation H_I is "turned on". It is expected that these eigenvalues will "dissolve", namely there is an $\epsilon > 0$ and $\eta > 0$ so that for $0 < |\lambda| < \epsilon$, H_λ has no eigenvalues in

$(E_j - \eta, E_j + \eta)$. Let γ be a contour enclosing $\sigma(H_\lambda)$. The formula

$$(\Psi, \exp(-itH_\lambda)\Psi) = \int_\gamma \exp(-itz)(\Psi, (H_\lambda - z)^{-1}\Psi)dz, \tag{1.13}$$

relates radiative lifetime of the state Ψ to the poles of the function

$$(\Psi, (H_\lambda - z)^{-1}\Psi). \tag{1.14}$$

Following [SI], we formulate the following program for the analysis of the spectrum in the interval $(E_j - \eta, E_j + \eta)$, and for the rigorous derivation of (1.11). If the coupling function f is regular enough then for a dense set of states the functions (1.14) should have a meromorphic continuation from the upper half plane onto the region $O = \{z : |z - E_j| < \eta\}$, and be regular on O except for a simple pole at $E_j(\lambda)$. This pole should not depend on the choice of Ψ. Let $\Gamma_j(\lambda) = -2Im(E_j(\lambda))$. If $\Gamma_j(\lambda) > 0$ then H_λ has only purely absolutely continuous spectrum on $(E_j - \eta, E_j + \eta)$. The function $\Gamma_j(\lambda)$ is expected to be analytic in λ for $|\lambda| < \epsilon$, and the first non-trivial coefficient in its expansion should be given by (1.12). One then can attempt to derive a formula for the decay of $b_j(t)$ using the relation (1.13). This formula, in the first approximation, should be given by (1.11).

For the zero-temperature model with massive bosons, this program was carried in part in [JP] and [OY]. The physically important case when bosons have zero mass, however, was beyond reach except in some special cases [A]. The difficulty, usually called an infrared divergence, is related to the fact that there are vectors Ψ in domain of $\tilde{H}_\lambda$ for which $(\Psi, N\Psi) = \infty$, where N is the number operator. For many years no method could be designed to avoid this difficulty. Recently, V. Bach, J. Fröhlich and I.M. Sigal [BFS] have developed a sophisticated renormalization algorithm to address this problem. We refer reader to [HS] for an exposition of their results.

In this paper, the program presented above is carried out for the positive temperature model (1.9).

At the end, we note that an application of the formal scattering theory relates expressions (1.11)-(1.12) to the experimental results [M]. Therefore, an important problem is the development of the scattering theory for the model (1.9). The method developed here yields some partial understanding of the scattering processes: the perturbative analysis of the resonance scattering and calculation of the energy distributions of photons emitted and absorbed in transitions will be subject of the fourth paper in the series. The investigation of the long time behavior of the interacting system $\mathcal{A}+\mathcal{B}$, and in particular of the stability of the equilibrium states, is based on the fusion of algebraic and spectral methods and will be subject of the third paper in the series [JP1].

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2 Statements of Results

We begin with

Proposition 2.1 *If* $f/\sqrt{\omega}, f\omega \in L^2(R^3)$ *then* H_λ *is essentially self-adjoint on* $\mathcal{H}_A \otimes D(H_{bos}) \otimes D(H_{bos})$ *for any* $\lambda \in R$.

Before stating our results, we introduce some additional notation. Let $H^2(\delta;\mathcal{H}) = H^2(\delta)$ be the Hardy class of the strip $\mathcal{S}(\delta) = \{z : |Im(z)| < \delta\}$ with values in some Hilbert space $\mathcal{H}$. The Hilbert space $H^2(\delta)$ consists of all functions $f : \mathcal{S}(\delta) \to \mathcal{H}$ which are analytic in the interior of $\mathcal{S}(\delta)$ and such that

$$(f,f)_{H^2(\delta)} = \|f\|^2_{H^2(\delta)} = \sup_{|a|<\delta} \int_{-\infty}^{\infty} \|f(x+ia)\|^2_{\mathcal{H}} dx < \infty.$$

Let (r,θ,ϕ) be polar coordinates on R^3. Given a function f on R^3 let $\tilde{f}$ be the function defined on $R \times S^2$ as

$$\tilde{f}(r,\theta,\phi) = \begin{cases} r^{1/2} \cdot f(r,\theta,\phi) & \text{if } r \geq 0 \\ -|r|^{1/2} \cdot \overline{f}(|r|,\theta,\phi) & \text{otherwise.} \end{cases} \tag{2.15}$$

Our central technical hypothesis is

(H1) The form factor f in (1.8) is such that $\tilde{f} \in H^2(\delta; L^2(S^2, dS))$ for some $\delta > 0$.

The hypothesis (H1) implies that $f/\sqrt{\omega} \in L^2(R^3)$. It is satisfied if, for example, $f(k) = \sqrt{|k|}\exp(-|k|^2)$. We may assume without loss of generality that $\delta < \pi/\beta$, see Section 3 for details.

To avoid some technical complications, we set an additional hypothesis:

(H2) The eigenvalues of H_A are simple and $f\omega \in L^2(R^3)$.

Theorem 2.2 *Suppose that (H1) and (H2) are satisfied. Then for each $0 < \eta < \delta$ there exists $\Lambda(\eta) > 0$ and a dense set of vectors $\mathcal{E} \subset \mathcal{H}$, independent of η, so that for $|\lambda| < \Lambda(\eta)$ the matrix elements*

$$(\Phi, (H_\lambda - z)^{-1}\Psi), \qquad \Phi, \Psi \in \mathcal{E}, \tag{2.16}$$

have a meromorphic continuation from the upper half plane onto the region $O = \{z : Im(z) > -\eta\}$. The functions (2.16) are analytic on O except for N simple poles (independent of Φ, Ψ) at $E_j(\lambda)$. The functions $E_j(\lambda)$ are analytic for $|\lambda| < \Lambda(\eta)$ and

$$E_j(\lambda) = E_j + \lambda^2 a_j^{(2)} + O(\lambda^3).$$

Furthermore, if Γ_j *is given by (1.12) then* $Im(a_2^{(j)}) = -\Gamma_j/2$.

Remark: Vectors $\psi \otimes \Omega_B$ belong to $\mathcal{E}$ for any $\psi \in \mathcal{H}_A$.

Theorem 2.2 and Proposition 4.1 in [CFKS] yield

Corollary 2.3 *Suppose that (H1) and (H2) are satisfied, and that* $\Gamma_k > 0$ *for* $1 \leq k \leq N$. *Then there is* $\Lambda > 0$ *so that for* $0 < |\lambda| < \Lambda$ *the spectrum of* H_λ *is purely absolutely continuous and fills the real axis.*

We now turn to the dynamical aspects of the system.

Theorem 2.4 *Suppose that (H1) and (H2) are satisfied. Then for each* $0 < \eta < \delta$ *there is* $\Lambda(\eta) > 0$ *and a dense set of vectors* $\mathcal{E} \subset \mathcal{H}$, *independent of* η, *so that for* $0 < |\lambda| < \Lambda(\eta)$ *and* $\Phi, \Psi \in \mathcal{E}$ *we have an expansion as* $t \to \infty$,

$$(\Phi, \exp(-itH_\lambda)\Psi) = \sum_{j=1}^{N} C_j(\Phi, \Psi, \lambda) \exp(-itE_j(\lambda)) + O(\exp(-\eta t)).$$

The constants $C_j(\Phi, \Psi, \lambda)$ *are computed in terms of the projections on resonance eigenfunctions.*

Let $b_j(t)$ be given by (1.10), and let $\Gamma_j(\lambda) = -2Im(E_j(\lambda))$.

Corollary 2.5 *Under the conditions of Corrolary 2.3, there is* $\Lambda > 0$ *and* $\epsilon > 0$ *so that for* $|\lambda| < \Lambda$

$$b_j(t) = \exp(-\Gamma_j(\lambda)t) + O(\lambda^2 \exp(-\epsilon\lambda^2 t)).$$

Remark 1: It follows from our arguments that the constant $\Lambda(\eta)$ in Theorem 2.2 is proportional to $1/\beta$. Thus, we cannot use a limiting argument to analyze the zero-temperature case.

Remark 2: The hypothesis (H1) is chosen so that it covers physically important examples in which $f(k) \sim \sqrt{|k|}$ for small k. From the discussion in Section 3 one can deduce other variants of (H1). For example, if $h : R \to C$,

$|h(r)| = 1$, is a measurable function, and if the function $h(r)\tilde{f}(r, \theta, \phi)$ belongs to $H^2(\delta)$, then all the results hold. The configuration space of the bosons can be any R^d, and the fact that $\omega(k) = |k|$ is of no particular importance. Let $\omega(k) = g(|k|)$ be a rotationally invariant function. Assume that $g(0) = 0$ and that $g(r)$ is a strictly increasing, unbounded, differentiable function on R^+. Denote by h its inverse. If the form factor f is a real-valued function and the function

$$\tilde{f}(r, \theta, \phi) = \frac{r}{|r|^{3/2}} h'(|r|) f(h(|r|), \theta, \phi)$$

belongs to $H^2(\delta; L^2(S^2, dS))$ for some $\delta > 0$, then all the results hold.

Remark 3: The hypothesis (H2) can be relaxed at the expense of a number of additional technical complications. The method of [JP1] can be used for the analysis of degenerate eigenvalues.

Remark 4: All the results hold if the system $\mathcal{B}$ is an infinite free Fermi gas.

Remark 5: The Theorem 2.2 and Corollary 2.3 could be extended to infinite dimensional setting. It is likely that the same is true for the Theorem 2.4 and Corollary 2.5. Let $\mathcal{H}_A = L^2(R^3)$, $H_A = -\Delta + V(x)$, and assume that $V(x) \to \infty$ as $|x| \to \infty$, and that Q is an operator of multiplication by a Schwartz class function. If $\inf_{i \neq j} |E_i - E_j| > 0$, then the Theorem 2.2 and Corrolary 2.3 hold. If $V(x) \to 0$ as $|x| \to \infty$ and $V \in L^\infty(R^3) + L^2(R^3)$, one can prove a (local in energy) version of the Theorem 2.2 for a simple eigenvalue $E_j \in \sigma_d(H_A)$.

3 Preliminaries

Let $\mathcal{T}(\mathcal{H})$ denote the symmetric Fock space constructed on the Hilbert space $\mathcal{H}$. Let F be the vectors space of finite particle vectors in $\mathcal{T}(\mathcal{H})$, and let N be the number operator. We will make use of the following well-known result [A], [RS].

Proposition 3.1 *Let $\mathcal{H} = \mathcal{T}(L^2(M, d\mu))$. Let $\omega(k)$ be a positive measurable function on M and let $H_0 = d\Gamma(\omega)$.*
a) If $f \in L^2(M)$ then for $\Psi \in F$

$$\|a^{\#}(f)\Psi\| \leq \|f\| \cdot \|(N+I)^{1/2}\Psi\|,$$

where $a^{\#}(f)$ represents either $a(f)$ or $a^*(f)$.
b) If f, $f/\sqrt{\omega} \in L^2(M)$ then for $\Psi \in F$

$$\|a^{\#}(f)\Psi\| \leq (\|f/\sqrt{\omega}\| + \|f\|) \cdot (\|H_0^{1/2}\Psi\| + \|\Psi\|).$$

In particular, $\varphi(f)$ is infinitesimally small with respect to H_0.

Proof of Proposition 2.1: We will use the commutator theorem (Theorem X.37 in [RS]). Let $\hat{N} = d\Gamma(\omega) \otimes I + I \otimes d\Gamma(\omega) + I \otimes I$. The operator $\hat{N}$ is essentially self-adjoint on $D = \mathcal{H}_A \otimes D(H_{bos}) \otimes D(H_{bos})$, and it remains to show that for any $\Psi \in D$ and some $d > 0$

$$\|H_\lambda \Psi\| \leq d\|\hat{N}\Psi\|, \qquad |(H_\lambda\Psi, \hat{N}\Psi) - (\hat{N}\Psi, H_\lambda\Psi)| \leq d\|\hat{N}^{1/2}\Psi\|^2. \quad (3.17)$$

Since $i[\hat{N}, \varphi_B(f)] = \varphi_B(if\omega)$, the estimates (3.17) follow from Proposition 3.1. □

The rest of this section is devoted to the construction of a new representation of the bath Hilbert space. This new representation is central in our approach.

For the proof of the following theorem we refer the reader to [BSZ].

Theorem 3.2 *For any two Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, there is a unitary mapping*

$$U : \mathcal{T}(\mathcal{H}_1) \otimes \mathcal{T}(\mathcal{H}_2) \rightarrow \mathcal{T}(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

so that for any two unitary mappings U_1 and U_2 and any two vectors $\phi \in \mathcal{H}_1$, $\psi \in \mathcal{H}_2$,

$$U[\Gamma(U_1) \otimes \Gamma(U_2)]U^{-1} = \Gamma(U_1 \oplus U_2),$$

$$U[\varphi(\phi)\otimes I + I\otimes\varphi(\psi)]U^{-1} = \varphi(\phi\oplus\psi).$$

Furthermore, if Ω is the vacuum on $\mathcal{T}(\mathcal{H}_1\oplus\mathcal{H}_2)$, and Ω_1, Ω_2 are vacua on $\mathcal{T}(\mathcal{H}_1)$, $\mathcal{T}(\mathcal{H}_2)$, then $U(\Omega_1\otimes\Omega_2) = \Omega$.

It follows from the theorem that there exist a unitary mapping

$$U : \mathcal{H}_B \to \mathcal{T}(L^2(R^3)\oplus L^2(R^3)) \tag{3.18}$$

so that

$$U\exp(itH_B)U^{-1} = \Gamma(\exp(it\omega)\oplus\exp(-it\omega)),$$

$$U\varphi_B(f)U^{-1} = \varphi((1+\rho)^{1/2}f\oplus\rho^{1/2}\overline{f}).$$

If (r,θ,ϕ) are the polar coordinates on R^3 let W be the unitary map defined by

$$W : L^2(R^3)\oplus L^2(R^3) \to L^2(R\times S^2, dr\otimes dS) \tag{3.19}$$

be an unitary equivalence defined as

$$W(f\oplus g) = \begin{cases} r\cdot f(r,\theta,\phi) & \text{if } r\geq 0 \\ r\cdot g(|r|,\theta,\phi) & \text{otherwise.}\end{cases}$$

It is easy to show that

$$W(\exp(it\omega)f\oplus\exp(-it\omega)g) = \exp(itr)W(f\oplus g),$$

$$W((1+\rho)^{1/2}f\oplus\rho^{1/2}\overline{f}) = f_\beta,$$

where

$$f_\beta(r,\theta,\phi) = \left[\frac{r\exp(\beta r)}{\exp(\beta r)-1}\right]^{1/2}\tilde{f}(r,\theta,\phi),$$

and $\tilde{f}$ is given by (2.15). Note that if $\tilde{f}\in H^2(\delta)$ for some $\delta > \pi/\beta$, then $f_\beta\in H^2(\pi/\beta-\epsilon)$ for any $0<\epsilon<\pi/\beta$ and that $f_\beta\notin H^2(\pi/\beta+\epsilon)$ for any $\epsilon>0$. Thus, without loss of generality we may assume in the hypothesis (H1) that $\delta<\pi/\beta$.

In the sequel we will identify the spaces $L^2(R\times S^2)$, $L^2(R)\otimes L^2(S^2)$, $L^2(R;L^2(S^2))$, and denote all of them by $\mathcal{H}_s$. Denote by S the operator of multiplication by r on $\mathcal{H}_s$ and let

$$\hat{H}_\lambda = H_A + \lambda Q \otimes \varphi(f_\beta) + d\Gamma(S). \tag{3.20}$$

Let $\mathcal{D} = D(N)\cap D(d\Gamma(S))$. The operator $\hat{H}_\lambda$ is well-defined, symmetric operator on the domain $\mathcal{D}$.

Lemma 3.3 *If $f/\sqrt{\omega}$, $f\omega \in L^2(R^3)$, then the operator $\hat{H}_\lambda$ is essentially self-adjoint on $\mathcal{D}$.*

Proof: Use the commutator theorem, as in the proof of Proposition 2.1, with $\hat{N} = d\Gamma(|S|) + I$. □

We now come to the central point of the construction. If W is given by (3.19) and U by (3.18), then

$$\Gamma(W)U : \mathcal{H}_B \to \mathcal{T}(\mathcal{H}_s),$$

$$[\Gamma(W)U]\varphi_B(f)[\Gamma(W)U]^{-1} = \varphi(f_\beta),$$

$$[\Gamma(W)U]\exp(itH_B)[\Gamma(W)U]^{-1} = \Gamma(\exp(itS)),$$

and the Trotter product formula yields that H_λ is unitarily equivalent to $\hat{H}_\lambda$. We summarize the above discussion in

Theorem 3.4 *There exist an unitary mapping $\hat{U}:\mathcal{H}\to\hat{\mathcal{H}}$ so that $\hat{U}D(H_\lambda) = D(\hat{H}_\lambda)$ and*

$$\hat{U}H_\lambda\hat{U}^{-1}\Psi = \hat{H}_\lambda\Psi \qquad for\ \ \Psi\in D(\hat{H}_\lambda).$$

In the sequel we set

$$\mathcal{H} = \hat{\mathcal{H}}, \qquad H_\lambda = \hat{H}_\lambda,$$

and work solely with Hamiltonian (3.20).

We would like to add a few comments concerning the above construction.

The simplest and most widely used complex-deformation technique is based on the Aguilar-Combes theory and the group of dilation operators, [AC], [BC], [RS], [SI]. The investigation of the zero-temperature model has been, so far, based on the second-quantization of the dilation group. This approach has been used in [OY], [JP], as well as in a recent work of Bach, Fröhlich and Sigal [BFS]. In the massless case the infrared problem reflects itself in the fact that the eigenvalues $\{E_i\}$ are not uncovered after the Hamiltonian $\tilde{H}_0$ is dilated, and regular perturbation theory cannot be directly applied. Since $\tilde{H}_I$ is not a relatively compact perturbation of $\tilde{H}_0$, it is difficult to analyze the spectrum of $\tilde{H}_\lambda$ in the regions around E_i's, and to show that functions (1.14) have a meromorphic continuation on the second Riemann sheet.

The resolution of the problem in the positive temperature case is based on the replacement of dilation analyticity technique with the second-quantized translation analyticity method. The latter one originated in the study of resonances of an atom in a homogeneous electric field (see [AH], [HE] for an example). The formal connection between the two problems becomes transparent in the representation (3.20). The complex deformation shifts the essential spectrum into the lower half-plane and uncovers the eigenvalues $\{E_i\}$. However, the approach is not free of difficulties. The domain of the transformed Hamiltonian is not the same as of the original one, and the main bulk of the technical work below will center around resolution of this difficulty.

4 Spectral Deformations and Fermi's Golden Rule

Throughout this section we assume that hypothesis (H1)-(H2) hold.

Let $u(a)$ be the translation unitary group on $\mathcal{H}_s$,

$$u(a)f = u(a)f(r,\theta,\phi) = f(r+a,\theta,\phi) = f^a.$$

If $U(a) = \Gamma(u(a))$ is the second quantization of $u(a)$ we get that

$$\varphi(u(a)f_\beta) = \varphi(f_\beta^a) = U(a)\varphi(f_\beta)U(-a),$$

$$U(a)d\Gamma(S)U(-a) = d\Gamma(S) + aN.$$

Thus (we write $U(a)$ for $I \otimes U(a)$)

$$\begin{aligned} H_\lambda(a) &= U(a)H_\lambda U(-a) = H_A + \lambda Q \otimes \varphi(f_\beta^a) + d\Gamma(S) + aN \\ &= H_A + \lambda H_I(a) + d\Gamma(S) + aN. \end{aligned}$$

For $a \in \mathcal{S}(\delta)$ let

$$\varphi_a(f_\beta^a) = \frac{1}{\sqrt{2}}\left(a(f_\beta^{\overline{a}}) + a^*(f_\beta^a)\right), \qquad H_I(a) = Q \otimes \varphi_a(f_\beta^a).$$

Recall that the domain $\mathcal{D}$ is defined as

$$\mathcal{D} = D(N) \cap D(d\Gamma(S)) = D(N) \cap D(H_0).$$

The operator $H_\lambda(a)$ is closable on $\mathcal{D}$ for each $(\lambda, a) \in C \times \mathcal{S}(\delta)$ and we denote by the same letter its closed extension.

Let $\mathcal{S}^\pm(\delta) = \{z : 0 < \pm Im(z) < \delta\}$. The following proposition summarizes some basic properties of the family of operators $H_\lambda(a)$.

Proposition 4.1 *a) For $a \in \mathcal{S}(\delta)$ and $\Psi \in \mathcal{D}$,*

$$\|H_0(a)\Psi\|^2 = \|H_0(Re(a))\Psi\|^2 + |Im(a)|^2 \cdot \|N\Psi\|^2.$$

b) For $a \in \mathcal{S}^\pm(\delta)$, $D(H_0(a)) = \mathcal{D}$.

c)

$$\sigma(H_0(a)) = (\cup_{n>0}\{na + t : t \in R\}) \cup \{E_1, E_2, \ldots, E_N\}.$$

d) For $a \in \mathcal{S}^\pm(\delta)$, $H_I(a)$ is infinitesimally small with respect to $H_0(a)$.

e) For $(\lambda, a) \in C \times \mathcal{S}^\pm(\delta)$, $D(H_\lambda(a)) = \mathcal{D}$.

f) For $(\lambda, a) \in C \times \mathcal{S}(\delta)$, $[H_\lambda(a)]^* = H_{\overline{\lambda}}(\overline{a})$.

Proof: Simple calculation yields a) and a) $\Rightarrow$ b). Decoupling $H_0(a)$ along the n-particle subspaces, one establishes c). Since d) $\Rightarrow$ e) $\Rightarrow$ f), it remains to establish d). We argue as follows: There is a constant $C > 0$ such that for any $0 < \epsilon < 1$ and $\Psi \in \mathcal{D}$,

$$\begin{aligned} \|Q \otimes \varphi_a(f_\beta^a)\Psi\|^2 &= (\Psi, Q^2 \otimes \varphi_a(f_\beta^a)^* \varphi_a(f_\beta^a)\Psi) \\ &\leq \frac{\epsilon}{2}\|[\varphi_a(f_\beta^a)^*\varphi_a(f_\beta^a)]^2\Psi\| + \frac{1}{2\epsilon}\|Q^2\Psi\|^2 \\ &\leq C\left(\epsilon\|N\Psi\|^2 + \frac{1}{\epsilon}\|\Psi\|^2\right). \end{aligned} \tag{4.21}$$

In deriving (4.21) we have used Proposition 3.1. The part a) combined with (4.21) yields that for any $0 < \epsilon < 1$ there is $C_\epsilon > 0$ so that

$$\|H_I(a)\Psi\| \leq \frac{\epsilon}{|Im(a)|}\|H_0(a)\Psi\| + C_\epsilon\|\Psi\|,$$

and the part d) follows. □

We will need the following technical result.

Lemma 4.2 *a) If $a \in \mathcal{S}(\delta)$ and $\gamma = dist(a, \partial\mathcal{S})$ then*

$$\|f_\beta^a - f_\beta^{Re(a)}\|_{L^2(R;L^2(S^2))} \leq \frac{|Im(a)|}{\gamma} \cdot \|f_\beta\|_{H^2(\delta;L^2(S^2))}. \tag{4.22}$$

b) For any $\epsilon > 0$ and $\Psi \in D(N)$ we have that

$$|Im(\Psi, H_I(a)\Psi)| \leq \frac{1}{\sqrt{2}}\left[\epsilon \cdot \left[\frac{|Im(a)| \cdot \|f_\beta\|_{H^2(\delta)}}{\gamma}\right]^2 (\Psi, (N+1)\Psi) + \frac{1}{\epsilon}\|Q\Psi\|^2\right]. \tag{4.23}$$

c) If $a_1, a_2 \in \mathcal{S}(\delta - \epsilon)$ then there is $C_\epsilon > 0$ so that

$$\|f_\beta^{a_1} - f_\beta^{a_2}\|_{L^2(R;L^2(S^2))} \leq C_\epsilon\|f_\beta\|_{H^2(\delta;L^2(S^2))}\,|a_1 - a_2|.$$

d) If $\Psi \in D(N)$ then $H_I(a)\Psi$ is an analytic vector-valued function on the strip $\mathcal{S}(\delta)$.

Proof: a) Let $0 < \eta < \delta - |Im(a)|$ and let

$$H(x) = f_\beta^a(x) - f_\beta^{Re(a)}(x) = f_\beta(x+a) - f_\beta(x + Re(a)).$$

Applying the Cauchy integral formula along the contour $C = \{z : |Im(z)| = |Im(a)| + \eta\}$ we get that

$$|H(x)| \leq \frac{|Im(a)|}{2\pi} \int_R \frac{|f_\beta(t + i(|Im(a)| + \eta))| + |f_\beta(t - i(|Im(a)| + \eta))|}{(t - x - Re(a))^2 + \eta^2} dt.$$

The Cauchy-Schwartz inequality yields that

$$|H(x)|^2 \leq \left|\frac{Im(a)}{2\pi}\right|^2 \cdot \frac{2\pi}{\eta} \cdot A(x), \tag{4.24}$$

where

$$A(x) = \int_R \frac{|f_\beta(t + i(|Im(a)| + \eta))|^2 + |f_\beta(t - i(|Im(a)| + \eta))|^2}{(t - x - Re(a))^2 + \eta^2} dt.$$

The relation (4.22) follows from (4.24) and the Fubini theorem.

b) Note first that

$$Im(\Psi, H_I(a)\Psi) = Im(\Psi, [H_I(a) - H_I(Rea)]\Psi).$$

Now for $\Psi \in D(N)$ we have that

$$|(\Psi, Q \otimes a(f_\beta^a - f_\beta^{Re(a)})\Psi)| \leq \frac{\epsilon}{2} \|f_\beta^a - f_\beta^{Re(a)}\|^2_{L^2(R)} (\Psi, (N+1)\Psi) + \frac{1}{2\epsilon}\|Q\Psi\|^2.$$

Since an analogous estimate holds for $|(\Psi, a^*(f_\beta^a - f_\beta^{Re(a)})\Psi)|$, the statement follows from the part a).
The proof of the part c) follows the lines of the proof of the part a).
d) Note that f_β^a and $\overline{f_\beta}^{\overline{a}}$, as functions of the a-variable, belong to $H^2(\delta)$, and that $[f'_\beta]^a$ and $[\overline{f'_\beta}]^{\overline{a}}$ belong to $H^2(\delta - \epsilon)$ for any $0 < \epsilon < \delta$, see e.g. [JS]. Let

$$H'_I(a) = \frac{1}{\sqrt{2}} Q \otimes [a([f'_\beta]^{\overline{a}}) + a^*([f'_\beta]^a)].$$

If $\Psi \in D(N)$ and $a, a + \Delta a \in \mathcal{S}(\delta)$, then

$$\|[(H_I(a+\Delta a) - H_I(a))/\Delta a]\Psi - H_I'(a)\Psi\| \leq \|Q\| \cdot \|(N+1)^{1/2}\Psi\| \cdot h(a, \Delta a),$$

where

$$h(a, \Delta a) = \|(f_\beta^{a+\Delta a} - f_\beta^a)/\Delta a - [f_\beta']^a\|_{L^2(R;L^2(S^2))} +$$
$$+ \|(\overline{f_\beta}^{\overline{a+\Delta a}} - \overline{f_\beta}^{\overline{a}})/\overline{\Delta a} - [\overline{f_\beta'}]^{\overline{a}}\|_{L^2(R;L^2(S^2))}.$$

Arguing as in the proof of the part a) we get that $\lim_{\Delta a \to 0} h(a, \Delta a) = 0$, and the part d) follows. □

We will make use of the following simple facts (see e.g [K]). If T is a closed operator with domain $D(T)$, let

$$\Theta(T) = \{(\phi, T\phi) : \phi \in D(T),\ \|\phi\| = 1\}$$

be its numerical range. Let $N(T)$ be the closure of $\Theta(T)$.

Lemma 4.3 *Let T be a closed operator such that $D(T) = D(T^*)$. Then $\sigma(T) \subset N(T)$, and for $z \in C \setminus N(T)$,*

$$\|(T-z)^{-1}\| \leq [dist\{z, N(T)\}]^{-1}. \tag{4.25}$$

If T_1, T_2 are two closed operators on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ such that $D(T_i) = D(T_i^*)$, then $N(T_1 \oplus T_2)$ is contained in the closed convex hull of the set $N(T_1) \cup N(T_2)$.

Proof: Let $\phi \in D(T)$, $\|\phi\| = 1$. Then

$$\|(T-z)\phi\| \geq |z - (\phi, T\phi)|, \tag{4.26}$$

and so if $z \notin N(T)$ then $T - z$ is one-one. Arguing similarly, if $\overline{z} \notin N(T^*)$ then $(T-z)^*$ is one-one, and so $T - z$ is onto. Since $N(T^*) = \overline{N(T)}$, we conclude that if $z \notin N(T)$ then $T - z$ has a bounded inverse. The estimate (4.25) follows from (4.26). □

We utilize the above results in the proof of

Proposition 4.4 *Let γ be as in Lemma 4.2. For any $(\lambda, a) \in C \times \mathcal{S}^-(\delta)$ there is a finite constant $D(\lambda)$, which depends on λ only, so that*

$$\sigma(H_\lambda(a)) \subset \{z : Im(z) \leq D(\lambda)/\gamma\}. \tag{4.27}$$

Furthermore, if $Im(z) > D(\lambda)/\gamma$ then

$$\|(H_\lambda(a) - z)^{-1}\| \leq [Im(z) - D(\lambda)/\gamma]^{-1}.$$

A similar result holds for $a \in \mathcal{S}^+(\delta)$.

Proof: Since $D(H_\lambda(a)) = D([H_\lambda(a)]^*)$, it suffices to show that $N(H_\lambda(a))$ is a subset of the set $\{z : Im(z) \leq D(\lambda)/\gamma\}$. Let P_1, P_2 be two orthogonal projections on $\mathcal{H}$ given by

$$P_1 = I \otimes (\Omega, \cdot)\Omega, \qquad P_2 = 1 - P_1. \tag{4.28}$$

If $\mathcal{H}^{(n)}$ is the n-th particle subspace of $\mathcal{T}(\mathcal{H}_s)$ then $RanP_1 = \mathcal{H}_A \otimes \mathcal{H}^{(0)}$ and $RanP_2 = \bigoplus_{n>0} \mathcal{H}_A \otimes \mathcal{H}^{(n)}$. Let

$$D_2 = \{\Psi : \Psi \in RanP_2 \cap D(H_\lambda(a)), \|\Psi\| = 1\}.$$

Since $P_1 H_\lambda(a) P_1 = P_1 H_A P_1$, and since $P_1 H_\lambda(a) P_2$ and $P_2 H_\lambda(a) P_1$ are bounded operators with $O(\lambda)$ norms, to establish the statement it suffices to show that

$$\Theta(P_2 H_\lambda(a) P_2) = \{(\Psi, H_\lambda(a)\Psi) : \Psi \in D_2\} \subset \{z : Im(z) \leq D(\lambda)/\gamma\}.$$

Clearly

$$\Theta(P_2 H_\lambda P_2) \subset \{z : Im(z) \leq \lambda Im(\Psi, H_I(a)\Psi) + Im(a)(\Psi, N\Psi), \ \Psi \in D_2\}.$$

Choose ϵ in (4.23) so that

$$\chi(\epsilon, \lambda) = \frac{\epsilon|\lambda|\delta \cdot \|f_\beta\|^2_{H^2(\delta)}}{\sqrt{2}\gamma^2} < 1.$$

Since $(\Psi, N\Psi) \geq 1$ for $\Psi \in D_2$ we have that

$$N(P_2 H_\lambda(a) P_2) \subset \{z : Im(z) \leq Im(a)(1-\chi(\epsilon,\lambda)) + |\lambda| \frac{\|Q\|^2}{\epsilon\sqrt{2}} + \delta\chi(\epsilon,\lambda)\},$$

and the statement follows. □

For the latter use, we note that if we take $\epsilon = 1$ in the part b) of Lemma 4.2 and if

$$\chi(\lambda) = \frac{|\lambda|\delta \cdot \|f_\beta\|^2_{H^2(\delta)}}{\sqrt{2}\gamma^2} < 1 \tag{4.29}$$

then

$$N(P_2 H_\lambda(a) P_2) \subset \{z : Im(z) < Im(a)(1-\chi(\lambda)) + |\lambda| \frac{\|Q\|^2}{\sqrt{2}} + \delta\chi(\lambda)\}. \tag{4.30}$$

An immediate consequence of the above discussion is

Proposition 4.5 *On the regions $C \times \mathcal{S}^{\pm}(\delta)$, $H_\lambda(a)$ is an analytic family of type A in each variable separately.*

We denote by $B(z,r)$ the open ball of radius r centered at z and by $\rho(T)$ the resolvent set of a closed operator T. Let $P(x) = \{z : Im(z) > x\}$.

Lemma 4.6 *Let $a \in \mathcal{S}^-(\delta)$ and $0 < \eta < |Im(a)|$ be fixed. Then there is $\Lambda = \Lambda(\eta, a) > 0$ so that for $0 < |\lambda| < \Lambda$ the operator $H_\lambda(a)$ has only discrete spectrum in the region $P(-\eta)$ which consist of exactly N simple eigenvalues $E_j(\lambda)$. These eigenvalues are analytic in λ for $|\lambda| < \Lambda$ and independent of a.*

Proof: Let P_1, P_2 be projections given by (4.28), and let

$$H^{(1)} = P_1 H_\lambda(a) P_1 + P_2 H_\lambda(a) P_2, \qquad H^{(2)} = P_1 H_\lambda(a) P_2 + P_2 H_\lambda(a) P_1.$$

Let $\xi = (|Im(a)| - \eta)/2$. First, let us choose λ so that (recall (4.29) and (4.30))

$$Im(a)(1-\chi(\lambda)) + |\lambda| \frac{\|Q\|^2}{\sqrt{2}} + \delta\chi(\lambda) < Im(a) - \frac{\xi}{2}, \tag{4.31}$$

and consequently that

$$N(P_2 H_\lambda(a) P_2) \subset C \setminus P(-\eta - \xi/2).$$

The operator $H^{(1)}$ has only discrete spectrum in the region $P(-\eta-\xi/2)$ and

$$\sigma_d(H^{(1)}) \cap P(-\eta - \xi/2) = \sigma(H_A).$$

The estimate (4.25) yields that if $z \in P(-\eta - \xi/2)$ then

$$\|(H^{(1)} - z)^{-1}\| \leq [dist\{z, \sigma(H_A)\}]^{-1} + \frac{1}{Im(z) + \eta + \xi/2}.$$

If $z \in \rho(H^{(1)})$ and $\|H^{(2)}(H^{(1)} - z)^{-1}\| < 1$ then $z \in \rho(H_\lambda(a))$ and

$$(H_\lambda(a) - z)^{-1} = (H^{(1)} - z)^{-1}[1 + H^{(2)}(H^{(1)} - z)^{-1}]^{-1}.$$

Let $L = 2\sup_{a \in \mathcal{S}(\delta)} \|P_1 H_I(a) P_2\|$ and let

$$\alpha = \min\{\min_{i \neq j} |E_i - E_j|, \eta\}.$$

Note that $\|H^{(2)}\| \leq L|\lambda|$. Thus, if in addition to (4.31) we have that

$$|\lambda| < \frac{1}{4L} \cdot \left[\frac{1}{\alpha} + \frac{1}{\xi}\right]^{-1}, \tag{4.32}$$

then the spectrum of $H_\lambda(a)$ within $P(-\eta-\xi/4)$ is contained in the $\cup_j B(E_j, \alpha/4)$, and for

$$z \in P(-\eta) \setminus (\cup_j B(E_j, \alpha/2)), \tag{4.33}$$

we have an estimate

$$\|(H_\lambda(a) - z)^{-1}\| \leq D_1. \tag{4.34}$$

The constant D_1 can be explicitly evaluated in terms of α and ξ. We conclude that for any z satisfying (4.33), $(H_\lambda(a)-z)^{-1}$ is an analytic bounded-operator valued function in λ-variable, as long as (4.31) and (4.32) hold. From this point one can follow line by line the proof of Kato-Rellich theorem (Theorem XII.8 in [RS]) to establish the statement of the lemma, except for the independence of the eigenvalues $E_j(\lambda, a)$ on the parameter a. Let a_0 be fixed value of the parameter a for which the above discussion hold, and let λ be fixed so that (4.31) and (4.32) hold. Since $H_\lambda(a)$ is an analytic family in the a-variable, we have that $E_j(\lambda, a)$ is an analytic function in a-variable in some narrow strip centered at $Im(a_0)$ (so that (4.31) and (4.32) still hold). Since $H_\lambda(a)$ and $H_\lambda(a')$ are unitarily related if $a - a'$ is real, $E_j(\lambda, a)$ is independent of a along the lines $Im(a) = const$, and so is a constant in the a-variable. □

The functions $E_j(\lambda)$ can be expanded in λ,

$$E_j(\lambda) = E_j + \lambda a_j^{(1)} + \lambda^2 a_j^{(2)} + \dots .$$

We proceed to calculate the coefficients $a_j^{(1)}$, $a_j^{(2)}$ following an argument in [RS], [SI]. If Γ_j is given by (1.12) we have

Proposition 4.7 *$a_j^{(1)} = 0$ and $Im(a_j^{(2)}) = -\Gamma_j/2$.*

Proof: Let a, α be as in Lemma 4.6, and let $\Psi_j = \psi_j \otimes \Omega$. Since $(\Psi_j, H_I(a)\Psi_j) = 0$, $a_j^{(1)} = 0$. The coefficient $a_j^{(2)}$ is given by

$$a_j^{(2)} = \frac{1}{2\pi i} \oint_{|E_j - z| = \alpha/2} (\Psi_j, [H_I(a)(H_0(a) - z)^{-1}]^2 \Psi_j) dz. \tag{4.35}$$

Rewrite (4.35) as

$$a_j^{(2)} = \frac{1}{2\pi i} \oint_{|E_j - z| = \alpha/2} (\Psi_j, H_I(a)(H_0(a) - z)^{-1} H_I(a)\Psi_j) \frac{1}{E_j - z} dz.$$

Let

$$F(a, z) = (\Psi_j, H_I(a)(H_0(a) - z)^{-1} H_I(a)\Psi_j).$$

The function $F(a, z)$ is analytic on $P(-\eta)$ in the z-variable, and by the Cauchy integral theorem

$$a_j^{(2)} = -\lim_{\epsilon \to 0} F(a, E_j + i\epsilon).$$

Let us show that $F(a, E_j + i\epsilon) = F(0, E_j + i\epsilon)$. For fixed z, $F(a, z)$ is independent of the a-variable since it is analytic in a as long as $Im(a) < 0$, and manifestly independent of a on the lines $Im(a) = c$. Thus,

$$F(a, E_j + i\epsilon) = \lim_{a \to 0} F(a, E_j + i\epsilon) = F(0, E_j + i\epsilon)$$

An explicit calculation yields that

$$F(0, E_j + i\epsilon) = \frac{1}{2} \sum_{i=1}^{N} |(Q\psi_i, \psi_j)|^2 \int_{S^2} dS \int_R \frac{|f_\beta(r, \theta, \phi)|^2}{r - (E_j - E_i) - i\epsilon} dr$$

The result follows from the formula

$$\lim_{\epsilon \to 0} \frac{1}{x - x_0 - i\epsilon} = -i\pi\delta(x - x_0) + PV \frac{1}{x - x_0}. \ \Box$$

The above argument yields an expression for $Re(a_j^{(2)})$, the Lamb shift of the j-th energy level.

Let $E \subset \mathcal{T}(\mathcal{H}_s)$ be the set of entire vectors for the group $U(a)$. We recall that E consists of all vectors Ψ such that the vector-valued function $U(a)\Psi$ extends to an entire function on C. Let $\tilde{E}$ be a subset of $E \cap F$ so that $\Omega \in \tilde{E}$ and that the n-particle component $\Psi^{(n)}$ of any $\Psi \in \tilde{E}$ satisfies the estimate

$$|\Psi^{(n)}(r_1 + a, \ldots, r_n + a)| \leq \frac{C_{Im(a),N}}{1 + |\sum_i (r_i + a)^2|^{N/2}}, \tag{4.36}$$

for any $N > 0$ and $a \in C$. Let $\mathcal{E} = \mathcal{H}_A \otimes \tilde{E}$. It follows from the Paley-Wiener theorem that $\mathcal{E}$ is a dense set of vectors in $\mathcal{H}$. The estimate (4.36) will be used in the proof of Theorem 2.4.

Proof of Theorem 2.2: Let $0 < \eta < \delta$ be fixed and let $\xi = (\delta - \eta)/2$. For $\Phi, \Psi \in \mathcal{H}$, let

$$f(a) = (\Phi, (H_\lambda(a) - z)^{-1}\Psi),$$

and let $D(\lambda)$ be the constant from Proposition 4.4. If z is such that

$$Im(z) > 1 + 2D(\lambda)/(\delta - \eta),$$

then it follows from Propositions 4.4 and 4.5 that for $a \in \mathcal{S}^-(\eta+\xi)$

$$\|(H_\lambda(a)-z)^{-1}\| \leq 1, \tag{4.37}$$

and that $f(a)$ is an analytic function on the region $\mathcal{S}^-(\eta+\xi)$ satisfying

$$|f(a)| \leq \|\Phi\| \cdot \|\Psi\|. \tag{4.38}$$

Let $a_0 \in R$ be fixed and let $a_n \in \mathcal{S}^-(\eta+\xi)$ be a sequence such that $\lim_{n\to\infty} a_n = a_0$. Since $U(a_0)\mathcal{D} = \mathcal{D}$, $H_\lambda(a_0)$ is essentially self-adjoint on $\mathcal{D} = D(N) \cap D(d\Gamma(S))$ (recall Lemma 3.3), and so

$$R = \{(H_\lambda(a_0)-z)\tilde{\Phi} \, : \tilde{\Phi} \in \mathcal{D}\}$$

is a dense set in $\mathcal{H}$. If $\Psi = (H_\lambda(a_0)-z)\tilde{\Phi}$ then the resolvent identity, Proposition 3.1 and estimate (4.37) yield that

$$|f(a_n)-f(a_0)| \leq |\lambda| \cdot \|\Phi\| \cdot \|[H_I(a_n)-H_I(a_0)]\tilde{\Phi}\| \leq$$

$$\leq |\lambda| \cdot \|Q\| \cdot \|\Phi\| \cdot \|(N+1)^{1/2}\tilde{\Phi}\| \cdot \left[\|f_\beta^{\overline{a}_n} - f_\beta^{a_0}\| + \|f_\beta^{a_n} - f_\beta^{a_0}\|\right].$$

It follows from the part b) of Lemma 4.2 that for $\Psi \in R$, $\lim_{n\to\infty} f(a_n) = f(a_0)$. Since R is dense in $\mathcal{H}$, the estimate (4.38) yields that $f(a)$ is a continuous function on $\mathcal{S}^-(\eta+\xi) \cup R$ for any $\Phi, \Psi \in \mathcal{H}$. For $\Phi, \Psi \in \mathcal{E}$ and $a \in \mathcal{S}^-(\eta+\xi) \cup R$ let

$$h(a) = (U(\overline{a})\Phi, (H_\lambda(a)-z)^{-1}U(a)\Psi).$$

The function h is analytic on $\mathcal{S}^-(\eta+\xi)$, and independent of a along the lines $Im(a) = c$. Therefore, h is a constant on $\mathcal{S}^-(\eta+\xi)$. If a_0 and a_n are as before then

$$|h(a_n)-h(a_0)| \leq \|U(\overline{a}_n - a_0)\Phi\| \cdot \|U(a_n)\Psi\| + \|U(a_n-a_0)\Psi\| + |f(a_n)-f(a_0)|,$$

and so h is continuous on $\mathcal{S}^-(\eta+\xi) \cup R$. We conclude that for $\Phi, \Psi \in \mathcal{E}$ the relation

$$(\Phi, (H_\lambda - z)^{-1}\Psi) = (U(\overline{a})\Phi, (H_\lambda(a)-z)^{-1}U(a)\Psi), \tag{4.39}$$

holds for $-\eta - \xi < Im(a) \leq 0$. The statement of the theorem follows from Lemma 4.6 and Proposition 4.7 □

Proof of Theorem 2.4: Let $\Phi, \Psi \in \mathcal{E}$. Choose Λ and a so that for $|\lambda| < \Lambda$ and ϵ small enough, the function

$$f(z) = (\Phi, (H_\lambda - z)^{-1}\Psi)$$

has a meromorphic continuation from the upper half-plane onto the region $\{z : Im(z) > -\eta - \epsilon\}$ given by

$$f(z) = (U(\overline{a})\Phi, (H_\lambda(a) - z)^{-1} U(a)\Psi),$$

and that (recall (4.34)) for $Im(z) \geq -\eta$ and $|Re(z)|$ large enough

$$\|(H_\lambda(a) - z)^{-1}\| \leq D. \tag{4.40}$$

Denote $\Phi_{\overline{a}} = U(\overline{a})\Phi$, $\Psi_a = U(a)\Psi$. The resonances $E_j(\lambda)$ are the only possible singular points of the function f, and the following relation holds:

$$\lim_{x\to\pm\infty} \sup_{-\eta\leq y\leq\epsilon} |f(x+iy)| = 0. \tag{4.41}$$

To establish (4.41), note that the Lebesgue dominated convergence theorem yields that for any $\Phi \in \mathcal{E}$

$$\lim_{x\to\pm\infty} \sup_{-\eta\leq y\leq\epsilon} \|(H_0(\overline{a}) - x + iy)^{-1}\Phi_{\overline{a}}\| = 0,$$

$$\lim_{x\to\pm\infty} \sup_{-\eta\leq y\leq\epsilon} \|H_I(\overline{a})(H_0(\overline{a}) - x + iy)^{-1}\Phi_{\overline{a}}\| = 0.$$

The relation (4.41) follows from the resolvent identity and the estimate (4.40). Let

$$C_1 = \{z : Im(z) = \epsilon\}, \quad C_2 = \{z : Im(z) = -\eta\}.$$

The relation (4.40) yields that the Cauchy integral theorem can be applied along the infinite contour $C_1 \cup C_2$;

$$\int_{C_1} \exp(-itz) f(z) dz = \sum_{j=1}^{N} \exp(-itE_j(\lambda))(\Phi_{\overline{a}}, P_j(\lambda, a)\Psi_a) +$$

$$+\int_{C_2} \exp(-itz)(\Phi_{\overline{a}}, (H_\lambda(a) - z)^{-1}\Psi_a)dz. \quad (4.42)$$

There, $P_j(\lambda, a)$ are projections on the resonance eigenfunctions. To finish the proof, we have to estimate the integral term in (4.42). Note that for $t > 0$

$$\int_{C_2} \exp(-itz)(\Phi_{\overline{a}}, (H_0 - z)^{-1}\Psi_a)dz = 0,$$

$$\int_{C_2} \exp(-itz)(\Phi_{\overline{a}}, (H_0 - z)^{-1}[\lambda H_I(a) + aN](H_0 - z)^{-1}\Psi_a) = 0.$$

Let

$$A(z, a) = \|[(\lambda H_I(\overline{a}) + \overline{a}N)(H_0 - \overline{z})^{-1}]^2\Phi_{\overline{a}}\|.$$

The estimate (4.36) yields that

$$A(x - i\eta, a) = O([1 + |x|]^{-2}).$$

Applying twice the resolvent identity we get that

$$\begin{aligned} \left|\int_{C_2} \exp(-itz)f(z)dz\right| &\leq D\|\Psi_{\overline{a}}\| \cdot \left|\int_{C_2} \exp(-itz)A(z, a)dz\right| \\ &= O(\exp(-\eta t)). \end{aligned}$$

and the theorem follows. The additional observation

$$(\psi_i \otimes \Omega, P_j(\lambda, a)\psi_i \otimes \Omega) = \delta_{ij} + O(\lambda^2)$$

yields the Corollary 2.5. □

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