PERTURBATIONS OF GEODESIC FLOWS ON SURFACES OF CONSTANT NEGATIVE CURVATURE AND THEIR MIXING PROPERTIES

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PERTURBATIONS OF GEODESIC FLOWS ON SURFACES OF CONSTANT NEGATIVE CURVATURE AND THEIR MIXING PROPERTIES.

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Abstract. We consider one parameter analytic hamiltonian perturbations of the geodesic flows on surfaces of constant negative curvature. We find two different necessary and sufficient conditions for the canonical equivalence of the perturbed flows and the non perturbed ones. One condition says that the "Hamilton-Jacobi equation" (introduced in this work) for the conjugation problem should admit a solution as a formal power series (not necessarily convergent) in the perturbation parameter. The alternative condition is based on the identification of a complete set of invariants for the canonical conjugation problem. The relation with the similar problems arising in the KAM theory of the perturbations of quasi periodic hamiltonian motions is briefly discussed. As a byproduct of our analysis we obtain some results on the Livscic,Guillemin,Kazhdan equation and on the Fourier series for the SL(2,$\mathbb{R}$) group. We also prove that the analytic functions on the phase space for the geodesic flow of unit speed have a mixing property(with respect to the geodesic flow and to the invariant volume measure) which is exponential with a universal exponent, independent on the particular function, equal to the curvature of the surface divided by 2. This result is contrasted with the slow mixing rates that the same functions show under the horocyclic flow: in this case we find that the decay rate is the inverse of the time ("up to logarithms").

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1. The integrability problem and its invariants.

Integrable hamiltonian systems are important in Mechanics because they provide classes of systems whose dynamical behaviour is well understood and which can be used as a "reference behaviour" for systems close to integrable ones.

There are, however, other dynamical systems whose behaviour is well understood, altough very different from that of the integrable systems.

One should naturally think to use such systems, also, as a reference for the behaviour of other classes of mechanical systems. Therefore we shall extend the notion of integrability as follows.

Let Σ be an ℓ-dimensional analytic compact manifold and let $T^*\Sigma$ be the phase space for the hamiltonian flows on Σ .

As usual we shall denote a point in $T^*\Sigma$ by $(\underline{p},\underline{q})$, $\underline{q}$ being the coordinates of a point of Σ and $\underline{p}$ being a conjugate momentum in the same system of coordinates. Geometrically p are the coordinates of a cotangent vector.

An analytic hamiltonian system on $T^*\Sigma$ will be a pair (W,H) with $W\subset T^*\Sigma$ being the closure of an open set and with H being an analytic function on W [1] and, finally, such that for every $(\underline{p},\underline{q})\in W$ the solution $S_t(\underline{p}_o,\underline{q}_o)$ to the equations :

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}(\underline{p},\underline{q}) \ , \quad \dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}(\underline{p},\underline{q}) \qquad (\underline{p},\underline{q})\in W \qquad (1.1)$$

with initial datum $(\underline{p}_0,\underline{q}_0)\in W$, exists for all $t\in R$.

Then the following definition extends the well known notion of integrability :

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1) as usual f will be said analytic (or C^∞) on a closed set W if it is the restriction to W of a function analytic (or C^∞) in its vicinity.

DEFINITION 1 : Let $(W,H),(W',H')$ be two analytic hamiltonian systems on two compact analytic surfaces Σ and Σ'. We say that "(W',H') is (W,H)-integrable", or simply "H' is H-integrable" if there is a C^∞ canonical map $\mathcal{C}$ mapping W' onto W and an analytic function defined on $H(W)$, denoted F and such that $F'=(dF/dE) \neq 0$, and

$$H'(\mathcal{C}(\underline{p},\underline{q})) = F(H(\underline{p},\underline{q})) \qquad (\underline{p},\underline{q}) \in W \tag{1.2}$$

If $\mathcal{C}$ is also analytic we say that H' is analytically H-integrable.

The possibility that $\mathcal{C}$ is C^∞ but not necessarily analytic leaves us more flexibility in the formulation of the results that we are able to prove.

The above definition says, in other words, that H' is H-integrable if the flow generated by H on W is canonically conjugate, up to a time scale change given by $F'(H)$, to the flow generated by H' on W'.

For definiteness we recall that a map $\mathcal{C}$, $(\underline{p}',\underline{q}')= \mathcal{C}(\underline{p},\underline{q})$, of W onto W', is canonical if it is at least C^∞ with C^∞ inverse and, calling

$$G(\mathcal{C}) = \{(\underline{p},\underline{q},\underline{p}',\underline{q}') \mid (\underline{p},\underline{q}) \in W,\ (\underline{p}',\underline{q}') \in W',\ (\underline{p}',\underline{q}')= \mathcal{C}(\underline{p},\underline{q})\} \tag{1.3}$$

there is a $\psi \in C^\infty(G(\mathcal{C}))$ such that :

$$\underline{p}\cdot d\underline{q} = \underline{p}'\cdot d\underline{q}' + d\psi \tag{1.4}$$

A more precise name for such $\mathcal{C}$'s could be "action preserving global canonical transformations" : if λ is a closed curve in W and $\lambda' = \mathcal{C}(\lambda)$ the "actions" of λ and λ' are the same :

$$\oint_\lambda \underline{p}\cdot d\underline{q} = \oint_{\lambda'} \underline{p}'\cdot d\underline{q}' \tag{1.5}$$

More generally , in the literature one calls "locally canonical" a transformation $\mathcal{C}$ such that $d\underline{p}\wedge d\underline{q} = d\underline{p}'\wedge d\underline{q}'$. This is a more general notion which we do not use in this work. The name's reason lies in the fact that the last relation implies a relation like (1.4) in small neighborhoods of phase space (but Ψ needs not be "uniform", i.e. exist globally as a single valued function on $G(\mathcal{C})$) .

The simplest integrable systems are those which are part of a one parameter family of integrable systems.

DEFINITION 2 . Let $H_\varepsilon = H_0 + f_\varepsilon$ be an analytic function on $W\times[-\theta,\theta]$,$\theta > 0$, with $(\underline{p},\underline{q},\varepsilon)$ analytic in $W\times[-\theta,\theta]$ and divisible by ε .

We shall say that the family of hamiltonians H_ε , $\varepsilon \in [-\theta,\theta]$, is H_0-integrable if:

i) there is a family $\mathcal{C}_\varepsilon, \varepsilon \in [-\theta,\theta]$, of canonical maps of W into $T^*\Sigma$ such that $\mathcal{C}_\varepsilon(\underline{p},\underline{q})$ is C^∞ in (p,q,ε) $W\times[-\theta,\theta]$, analytic in ε and differs from the identity map by a quantity which is divisible by ε .

ii)

$$H_\varepsilon(\mathcal{C}_\varepsilon(\underline{p},\underline{q})) = F_\varepsilon(H_0(\underline{p},\underline{q})) \qquad (\underline{p},\underline{q}) \in W\times[-\theta,\theta] \tag{1.6}$$

with $F_\varepsilon(E)$ being analytic in (E,ε) on the set $\mathcal{E}\times[-\theta,\theta]$,$\mathcal{E} = \{E | E = H_0(\underline{p},\underline{q}), (\underline{p},\underline{q}) \in W\} \equiv H_0(W)$, and such that $F_\varepsilon(E) - E$ is divisible by ε .

Eq. (1.6) can be regarded, given H_ε , as an equation for $\mathcal{C}_\varepsilon$,F_ε and we shall call it the "Hamilton-Jacobi" equation for the integration of H_ε with respect to H_0 . Similarly we call (1.2) the "Hamilton-Jacobi" equation for the integration of H' with respect to H .

Families of integrable perturbations with respect to the system:

$$W = V\times T \quad , \quad V \subset R^\ell \ , \ H_0(\underline{A},\underline{\phi}) = \underline{\omega}_0 \cdot \underline{A} \tag{1.7}$$

where $(\underline{A},\underline{\phi})$ denotes a point on $V\times T^\ell$, T^ℓ being the ℓ-dimensional torus, $\underline{\omega}_0 \in R^\ell$, "harmonic oscillators", have been studied recently and enjoy remarkable properties. [9] .

The case when W is as in (1.7) and $H_0(\underline{A},\underline{\phi}) = h_0(\underline{A})$ is $\underline{\phi}$-independent is the problem studies by the well known KAM theory.

DEFINITION 3. If in definition 2 we replace the requirement on $\mathcal{C}_\varepsilon$ to be of class C^∞ with that of being analytic we obtain the notion of "analytically H_0-integrable" family of perturbations.

In this paper we analyse the case when H_0 is the hamiltonian for the geodesic flow on a riemanian surface of constant negative curvature, equal to -1 (say), and :

$$W = \{(\underline{p},\underline{q}) \mid H_0(\underline{p},\underline{q}) \in [1/2,3/2]\} \tag{1.8}$$

Our objective is to find some necessary and sufficient conditions for recognizing the H_0-integrability of a family of perturbations.

As in the classical case,arising in the KAM theory, it will be generally impossible to conjugate two hamiltonian flows. The obstacles may lie in the existence of "invariants", i.e. of quantities associated with the H_ε-flow that must assume values determined by H_0, just because the H_ε-flow and the H_0-flow are canonically conjugate.

We describe some of them here.

Suppose that, for simplicity, the H_0 and H_ε flows, for all $\varepsilon\in[-\theta,\theta]$, have only a denumerable family of closed orbits on each energy surface H_0 or $H_\varepsilon = E$, $E\in[1/2,3/4]$, and also that for all $\varepsilon\in[-\theta,\theta]$ such orbits stay pairwise distinct, i.e. they can be labeled by (n,E,ε),n= 1,2,..., $E\in[1/2,3/4]$, $\varepsilon\in[-\theta,\theta]$, in such a way that they depend continously (hence analytically) on (E,ε) at fixed n and are pairwise distinct for all n, at fixed (E,ε).

This assumption is verified if (W,H_0) is the hamiltonian for the geodesic flow on a surface of constant negative curvature : this is a simple consequence of the Anosov's structural stability theorem and of the isomorphism between geodesic flows of different energies (but,of course, on the same surface).

Then let :

$$T(E,n,\varepsilon) = \{\text{period of the orbit } (E,n,\varepsilon)\}$$

$$-\lambda_-(E,n,\varepsilon) = \lambda_+(E,n,\varepsilon) = \{\text{Lyapunov exponents of } (E,n,\varepsilon), -\lambda_- = \lambda_+ > 0\} \tag{1.9}$$

$$A(E,n,\varepsilon) = \{\text{action of } (E,n,\varepsilon)\} = \left\{\oint_{(E,n,\varepsilon)} \underline{p}\cdot d\underline{q}\right\}$$

If H_ε is H_0-integrable in the sense (1.6) it must be, clearly:

$$\frac{T(E,n,\varepsilon)}{T(E,m,\varepsilon)} = \frac{T(F_\varepsilon^{-1}(E),n,0)}{T(F_\varepsilon^{-1}(E),m,0)} = t_{nm} \tag{1.10}$$

$$\frac{\lambda_+(E,n,\varepsilon)}{\lambda_+(E,m,\varepsilon)} = \frac{\lambda_+(F_\varepsilon^{-1}(E),n,0)}{\lambda_+(F_\varepsilon^{-1}(E),m,0)} = \ell_{mn} \tag{1.11}$$

$$\frac{A(E,n,\varepsilon)}{A(E,m,\varepsilon)} = \frac{A(F_\varepsilon^{-1}(E),n,0)}{A(F_\varepsilon^{-1}(E),m,0)} = a_{mn} \tag{1.12}$$

where in the r.h.s. there is no (E,ε)-dependence because for $\varepsilon=0$ the flow is geodesic and the intermediate ratios do not depend on E.

Clearly the identities (1.10)./.(1.12) are necessary conditions for the integrability of the family H_ε .

It is easy to see that (1.10)./.(1.11) are not a complete set of invariants for the canonical integrability of a family of perturbations of the geodesic flow :see appendix E for an example.

However we shall prove :

PROPOSITION 1 : i) the numbers (1.12) are a complete set of invariants for the canonical conjugation ("H_0-integrability") problem on $W\times[-\overline{\theta},\overline{\theta}]$ if H_0 is the above described geodesic flow and $\overline{\theta}$ is small enough (given f_ε).
ii) furthermore the integrability in i) is in fact analytic in the sense of definition 3, whenever existing.

We shall call the (1.12) ,l.h.s., the "action invariants" and proposition 1 shows their completeness.

A related result that proposition 1 extends is the following .

Let g^0 be a riemanian metric for a surface of negative curvature, compact and smooth. Then the hamiltonian for the geodesic flow has the form:

$$H_0(\underline{p},\underline{q}) = \frac{1}{2} \sum_{i,j=1,2} (g^0(\underline{q})^{-1})_{ij} p_i p_j \tag{1.13}$$

and consider the following "geometric perturbations" :

$$f_\varepsilon(\underline{p},\underline{q}) = \frac{\varepsilon}{2} \sum_{i,j=1,2} s_{ij}(\underline{q},\varepsilon) p_i p_j \tag{1.14}$$

with s analytic in $(\underline{q},\varepsilon) \in \Sigma \times [-\theta,\theta]$: they correspond to changes in the metric $g^0 \to g_\varepsilon = ((g^0)^{-1} + \varepsilon s)^{-1}$.

Then one can ask when $H_\varepsilon = H_0 + f_\varepsilon$, which is still a geodesic flow, is H_0-integrable with $W = T^* \Sigma$ and $F_\varepsilon(E)=E$ and C_ε of the form :

$$C_\varepsilon(\underline{p},\underline{q}) = (J_\varepsilon(\underline{q})^{-1}\underline{p}, \hat{C}_\varepsilon(\underline{q})) \tag{1.15}$$

where $\hat{C}_\varepsilon$ is a diffeomorphism of Σ and J_ε is its jacobian matrix.

A complete set of invariants for this type of conjugation, "geometric conjugation" or "trivial conjugation", is in some cases known to be the set of the eigenvalues of the Laplace-Beltrami operator [2].

Our methods have on one hand some resemblance with those of [2], particularly in the use of the key result [3]: however we make strong use of the group theoretic structures provided by the assumption of constant curvature and obtain results which probably do not hold for the theory of perturbations of the geodesic flows on surfaces of non constant negative curvature (while [2] deals mainly with such general manifolds). But the main difference is that we do not deal with the geometrical conjugacy problem and consider, rather, the general action preserving canonical conjugacy using techniques developed in the context of the KAM theory. On the other hand propositions 1 and 2 (see §2) are, technically, an extension of a nice criterion of convergence for the Birkhoff series due to Russmann [12].

In sec. 2 we introduce a detailed description of the geodesic flow on a surface of constant negative curvature and describe another solution to the integrability problem (which, also, is our main result). In Sec. 6 we discuss some byproducts of our analysis: the evaluation of the mixing rates for analytic functions with respect to the unperturbed geodesic flow, (on a surface of constant negative curvature).

2. The flows on constant negative curvature surfaces. Good coordinates. Integrability and perturbation theory.

The surfaces of constant negative curvature are constructed as follows.

If $z = x+iy$, $x\in\mathbb{R}$, $y>0$, is regarded as a point in the upper half plane $\mathbb{C}_+$, the action of the group $PSL(2,\mathbb{R})$ on $\mathbb{C}_+$ is

$$z \to zg = \frac{az+c}{bz+d} \qquad \text{if} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

If Γ is a hyperbolic fuchsian group (i.e. a fuchsian group without parabolic or elliptic elements [7]) the (most general) compact analytic surface of constant negative curvature is :

$$\Sigma = \mathbb{C}_+/\Gamma \tag{2.2}$$

endowed with the $PSL(2,\mathbb{R})$-invariant metric $ds^2 = (dx^2+dy^2)/y^2$.

The surface Σ can be identified with a fundamental domain Σ_0 of Γ with "opposite sides" identified modulo Γ .

On Σ_0 the geodesic flow is described, by definition, by the hamiltonian :

$$H_0 = y^2(p_x^2+p_y^2)/2 \tag{2.3}$$

Any motion with non zero velocity will reach the $\partial\Sigma_0$ in a finite time t_0 at the point (x_0,y_0) with speed $(\dot{x}_0,\dot{y}_0)$: if γ is an element of Γ reflecting (x_0,y_0) to another element of $\partial\Sigma_0$, the motion will continue after t_0 by reappearing at $z_0' = z_0\gamma$, if $z_0=x_0+iy_0$, with velocity $\dot{z}_0' = \dot{x}_0' + i\dot{y}_0'$ given by :

$$\dot{z}_0' = \dot{z}_0\,(bz_0+d)^{-2} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

This is a somewhat complicate description of the geodesic flow.

There is, however a better representation : it is inspired by the paper [4]. Let :

$$j(z,g) = (bz+d) \tag{2.5}$$

It is easy to verify that the transformation C of $T^{*}\Sigma/\{H_0=0\}$ onto $\Gamma\backslash PGL(2,\mathbb{R})$:

$$C\ :\ (p_x,p_y,x,y)\ \leftrightarrow\ \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix} = g \tag{2.6}$$

defined by :

$$\begin{aligned} p_x+ip_y &= i(\det g)^2\overline{j(i,g^{-1})}^2/2 \\ x+iy &= ig^{-1} \end{aligned} \tag{2.7}$$

is such that (see appendix D) :

$$p_x dx + p_y dy = p_1 dq_1 + p_2 dq_2 - d(\det g)/2 \tag{2.8}$$

therefore it is canonical.

The map C transforms the hamiltonian (2.3) into (see appendix D):

$$H_0(g) = (\det g)^2/8 \tag{2.9}$$

Therefore if $f_\varepsilon(g)$ is a function of (g,ε) anlytic on $W\times[-\theta,\theta]$, divisible by ε , the hamiltonian equations for $H_\varepsilon = H_0+f_\varepsilon$ are in the new coordinates :

$$\dot{g} = -(\det g)g\sigma_z/4 + \sigma\frac{\partial f}{\partial g}(g)\,\sigma_x \tag{2.10}$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\sigma_x=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_z=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\left(\frac{\partial f}{\partial g}(g)\right)_{ij} \equiv \frac{\partial f}{\partial g_{ij}}(g)$, $ij=1,2$.

The Liouville measure on $T^{*}\Sigma/\{H_0=0\}$, realized as $\Gamma\backslash PSL(2,\mathbb{R})$, is just $dp_1dp_2dq_1dq_2$, i.e. it is the Haar measure of $PGL(2,\mathbb{R})$ considered as a measure on the homogeneous space $\Gamma\backslash PSL(2,\mathbb{R})$.

We do not discuss in more detail why the (2.9),(2.10) induce a flow on $\Gamma\backslash PGL(2,\mathbb{R})$: it is obvious that they do so on the whole $PGL(2,\mathbb{R})$ and the fact that it can be regarded as a flow on $PGL(2,\mathbb{R})$ stems, basically, from the observation that if $t \to g(t)$ is a solution to (2.10) also $t \to \gamma g(t)$ is such.

This observation can be used to say that the entries of $g \in PGL(2,\mathbb{R})$ provide a global system of coordinates on the phase space of the geodesic flow deprived of the points with zero velocity and provided one identifies the $g \in PGL(2,\mathbb{R})$ modulo Γ .

We are in a very similar situation to that arising when we represent a circle by the line (or T^ℓ by $\mathbb{R}^\ell$) modulo the action of the translations by integer multiples of 2π .

Then, in the same sense , the points $\phi \in \mathbb{R}$ (or $\underline{\phi} \in \mathbb{R}^{\ell}$) provide a global system of coordinates on the circle, where ϕ and $\phi + 2\pi$ are identified ("flat coordinates" or "angles"), (or on T^{ℓ}).

We shall use the above remarks quite extensively. The main reason of their usefulness is that they provide a simple characterization of the C^{∞}-canonical maps C defined on a set of the form $W = \{g \mid H_0(g) \in [1/2, 3/4]\}$.

Suppose that C is close to the identity and that it is defined on an open set W' larger than W, larger enough so that if $(\underline{p}', \underline{q}') = C(\underline{p}, \underline{q})$, $(\underline{p}, \underline{q}) \in W$, then $(\underline{p}', \underline{q}), (\underline{p}, \underline{q}') \in W'$.

Then C can be descibed by a generating function Φ of class C^{∞} on the set $\{(\underline{p}', \underline{q}) \mid (\underline{p}', \underline{q}) \in W', \exists\, (\underline{p}, \underline{q}') \in W' \text{ such that } (\underline{p}', \underline{q}') = C(\underline{p}, \underline{q})\} \equiv \widetilde{W}$ via the relation : $\forall\, (\underline{p}', \underline{q}) \in \widetilde{W}$, $(W \subset \widetilde{W} \subset W')$:

$$
\begin{aligned}
\underline{p} &= \underline{p}' + \frac{\partial \Phi(\underline{p}', \underline{q})}{\partial \underline{q}} \\
\underline{q}' &= \underline{q} + \frac{\partial \Phi(\underline{p}', \underline{q})}{\partial \underline{p}'}
\end{aligned}
\tag{2.11}
$$

which can be written in a better form in terms of matrices : let

$$
g = \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix}, \quad g' = \begin{pmatrix} p_1' & q_2' \\ -p_2' & q_1' \end{pmatrix}, \quad \hat{g} = \begin{pmatrix} p_1' & q_2 \\ -p_2' & q_1 \end{pmatrix}, \quad \hat{g}' = \begin{pmatrix} p_1 & q_2' \\ -p_2 & q_1' \end{pmatrix} \tag{2.12}
$$

then the (2.11) becomes :

$$
\hat{g}' = \hat{g} - \sigma \frac{\partial \Phi}{\partial g}(\hat{g})\, \sigma \tag{2.13}
$$

The whole point of the above discussion is the "globality of (2.13) on W " : it follows from the easily checked property that if $g' = C(g)$, $\gamma \in \Gamma$, then $\gamma g' = C(\gamma g)$ because $\gamma \hat{g}'$ and $\gamma \hat{g}$ solve (2.13) if $\hat{g}$, $\hat{g}'$ do.

What is slightly less clear is that viceversa any canonical transformation C close enough to the identity can be generated by a relation of the form (2.13).

This remarkable fact can be seen by observing that if C has the property :

$$\underline{p}\cdot d\underline{q} = \underline{p}'\cdot d\underline{q}' + d\,\Psi \tag{2.14}$$

with Ψ being a C^∞-function on the "graph of C" i.e. on the set of the $(\underline{p},\underline{q},\underline{p}',\underline{q}')$ such that $(\underline{p}',\underline{q}') = C(\underline{p},\underline{q})$, the (2.14) can be rewritten as :

$$\underline{p}\cdot d\underline{q} + \underline{q}'\cdot d\underline{p}' = d(\Psi + \underline{p}'\cdot\underline{q}') \tag{2.15}$$

where $\Psi + (\underline{p}'\cdot\underline{q}') = \Psi + \det g'$ is a C^∞ function on the graph of C because Ψ is such by assumption and $\det g'$ is single valued and C^∞ on $\Gamma\backslash PGL(2,\mathbb{R})$ and therefore it can be thought as a function on the graph of C .

Also the function, on the graph of C, defined by $(g,g') \to (\det\hat{g})$ is C^∞-smooth when C is so close to the identity that the relation $g' = C(g)$ can be put in the form $\hat{g}' = C(\hat{g})$, using the implicit functions theorem, and $C(\gamma g) = \gamma C(g)$ when g and γg belong to the boundary of a fundamental domain in PGL(2,R) with respect to the action of Γ .

Therefore the function $\Phi = \Psi + (\det g') - (\det\hat{g})$ is in C^∞ of the graph of C and, if C is close enough to the identity (in the C^1-sense at least), it can be regarded as a function of $\hat{g}$ on $\{\hat{g}\,|\,g\in W\}$: but the (2.15) becomes $\underline{p}\cdot d\underline{q} + \underline{q}'\cdot d\underline{p}' = d(\det\hat{g} + \Phi(\hat{g}))$, i.e. (2.14).

The situation is very similar to the one that arises in the change of coordinates in systems described in action angle variables $(\underline{A},\underline{\phi})\in V\times T^\ell$: in that case a change of coordinates $(\underline{A}',\underline{\phi}') = C(\underline{A},\underline{\phi})$ defined on a set of the form $V\times T^\ell$, $V\subset R^\ell$, T^ℓ = {ℓ-dimensional torus}is canonical (in our action preserving sense) if and only if it can be generated by a relation of the form :

$$\begin{aligned} \underline{A} &= \underline{A}' + \frac{\partial\Phi(\underline{A}',\underline{\phi})}{\partial\underline{\phi}} \\ \underline{\phi}' &= \underline{\phi} + \frac{\partial\Phi(\underline{A}',\underline{\phi})}{\partial\underline{A}'} \end{aligned} \tag{2.16}$$

and Φ is periodic in the $\underline{\phi}$'s.

The outcome of the above discussion is the fact that it allows us to replace the search of the solutions $\mathscr{E}_\varepsilon$, F_ε of (1.6) by the search of a function Φ , C^∞ on $W\times[-\bar{\theta},\bar{\theta}]$, $\bar{\theta}\le\theta$, analytic in ε :

$$\Phi(g,\varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k\,\Phi^{(k)}(g) \tag{2.17}$$

which generates C near W, if ε is small enough.

Analogously we set :

$$F_\varepsilon(E) = E + \sum_{k=1}^{\infty} \varepsilon^k F^{(k)}(E) \tag{2.18}$$

and rewrite (1.6) as :

$$H_\varepsilon(\underline{p}',\underline{q}+\frac{\partial\Phi}{\partial\underline{p}'}(\underline{p}',\underline{q})) = F_\varepsilon(H_0(\underline{p}'+\frac{\partial\Phi}{\partial\underline{q}}(\underline{p}',\underline{q}),\underline{q})) \tag{2.19}$$

Developing everythimg in powers of ε and denoting $\{.,.\}$ the Poisson bracket, one easily finds :

$$\{H_0,\Phi^{(1)}\}\ (\underline{p}',\underline{q}) = f^{(1)}(\underline{p}',\underline{q}) - F^{(1)}(H_0(\underline{p}',\underline{q}))$$

$$\begin{aligned}\{H_0,\Phi^{(2)}\}\ (\underline{p}',\underline{q}) &= f^{(2)}(\underline{p}',\underline{q}) - F^{(2)}(H_0(\underline{p}',\underline{q})) + \\ &+\frac{\partial f^{(1)}}{\partial\underline{p}'}(\underline{p}',\underline{q})\frac{\partial\Phi^{(1)}}{\partial q}(\underline{p}',\underline{q}) + \frac{dF^{(1)}}{dE}(H_0(\underline{p}',\underline{q}))\frac{\partial H_0}{\partial\underline{q}}(\underline{p}',\underline{q})\ \frac{\partial\Phi^{(1)}}{\partial\underline{p}'}(\underline{p}',\underline{q}) - \\ &- \frac{1}{2}\sum_{i,j=1}^{2}\{\frac{\partial^2 H_0}{\partial p_i'\partial p_j'}(\underline{p}',\underline{q})\ \frac{\partial\Phi^{(1)}}{\partial q_i}(\underline{p}',\underline{q})\ \frac{\partial\Phi^{(1)}}{\partial q_j}(\underline{p}',\underline{q}) - \\ &- \frac{\partial H_0}{\partial q_i\partial q_j}(\underline{p}',\underline{q})\ \frac{\partial\Phi^{(1)}}{\partial p_i'}(\underline{p}',\underline{q})\frac{\partial\Phi^{(1)}}{\partial p_j}(\underline{p}',\underline{q})\ \}\end{aligned} \tag{2.20}$$

$$\vdots$$

$$\{H_0,\ \Phi^{(k)}\}\ (\underline{p}',\underline{q}) = f^{(k)}(\underline{p}',\underline{q}) + \Delta^{(k)}(f^{(k-1)},\ldots,f^{(1)},\Phi^{(k-1)},\ldots,\Phi^{(1)},F^{(k-1)},\ldots,F^{(1)})$$

where $\Delta^{(k)}$ is a differential polynomial in its variables : very complicate, as the case k=2 already promises.

Let us study the first equation (2.20): it means that $\Phi^{(1)}$ is a function whose derivative along the H_0-flow is $f^{(1)}$ up to a constant $F^{(1)}(H_0)$.

Therefore integrating both sides with respect to the Liouville measure :

$$\mu_E(d\underline{p}d\underline{q}) = \text{const} \quad \delta(H_0(\underline{p},\underline{q})-E)d\underline{p}d\underline{q} \tag{2.21}$$

$$\text{const} = \left(\int\delta(H_0(\underline{p}',\underline{q}')-E)d\underline{p}'d\underline{q}' \right)^{-1} \tag{2.22}$$

one finds that it must be :

$$F^{(1)}(E) = \int f^{(1)}(\underline{p},\underline{q})\ \mu_E(d\underline{p}d\underline{q}) \equiv \overline{f}^{(1)}(E) \tag{2.23}$$

determining $F^{(1)}$ uniquely.

Then we shall say that the first order of perturbation theory is "well defined" if the equation :

$$\{H_0,\Phi^{(1)}\} = f^{(1)} - \overline{f}^{(1)}(H_0) \tag{2.24}$$

admits a unque $C^{\infty}(W)$-solution up to an (arbitrary) function of H_0 .

In this case $\Phi^{(1)}$ will be defined uniquely by imposing :

$$\int\mu_E(d\underline{p}d\underline{q})\ \Phi^{(1)}(\underline{p},\underline{q}) = 0 \tag{2.25}$$

Inductively we can say that the k-th order of perturbation theory is well defined if the equation :

$$\{H_0,\Phi^{(k)}\} = f^{(k)}-\overline{f}^{(k)} + \Delta^{(k)} - \overline{\Delta}^{(k)} \tag{2.26}$$

(with obvious notations) admits a unique solution $\Phi^{(k)}$ on W with $\int\mu_E(d\underline{p}d\underline{q})\ \Phi^{(k)}(\underline{p},\underline{q}) = 0$.

More generally one might like to say that a theory admits perturbation theory to order k if the (2.20) can be recursively solved up to order k by suitably choosing at each step the arbitrary function of H_0 which can be added to each $\Phi^{(k)}$. It is, however, not surprising that this notion is not really more general : a theory admits perturbation theory to order k in the weaker sense just proposed if and only if it admits perturbation theory to order k in the former stronger sense.

The simplest way to understand this is by remarking that the (1.6) admits obviously a lot of solutions if it just admits one : let in fact $(E,\varepsilon) \to R_\varepsilon(E)$ be defined and C^∞ on $W\times[-\theta,\theta]$, ε-analytic and divisible by ε, and consider the canonical transformation generated by $(\det \hat{g})+R_\varepsilon(H_0(\hat{g}))$:

$$\begin{array}{l} \underline{p} = \underline{p}' \ (1+R'_\varepsilon(H_0(\hat{g})) \\ \underline{q}' = \underline{q} \ \ (1+R'_\varepsilon(H_0(\hat{g})) \end{array} \quad \rightarrow \qquad \underline{p}\cdot\underline{q} \equiv \underline{p}'\cdot\underline{q}' \tag{2.27}$$

But (det g)= $\underline{p}\cdot\underline{q}$, $H_0(\underline{p},\underline{q}) = H_0(\underline{p}',\underline{q}')$: so that if (1.6) admits a solution $C_\varepsilon, F_\varepsilon$ then also $C_\varepsilon C^{(0)}_\varepsilon$, F_ε ,with $C^{(0)}_\varepsilon$ given by (2.27) is a solution.

This ambiguity of the solutions to (1.6) can be easily related to the ambiguity in the choice of $\Phi^{(k)}$ and used to prove that the k-th order of perturbation theory exists or does not exist independently of the arbitrary choices which one has to make in order to build it.

Therefore the following definition is very natural :

DEFINITION 4 : Let H_0 be the hamiltonian for the geodesic flow on a surface of constant negative curvature and let f_ε be analytic in $W\times[-\theta,\theta]$, $W=\{\underline{p},\underline{q} \mid H_0(\underline{p},\underline{q}) = E \in [1/2,3/2]\}$.

We say that f_ε "admits a finite perturbation theory" around H_0 if it admits the k-th order perturbation theory for all k=1,2,...

Of course "having a finite prerturbation theory" is a canonical invariant. This means that if C_ε is defined and analytic on $W\times[-\theta,\theta]$ and maps canonically W into $T^*\Sigma$ and differs from the identity map by an ε-divisible quantity, then defining $\hat{f}_\varepsilon$ by :

$$H_0(\ C_\varepsilon(\underline{p},\underline{q})) + f_\varepsilon(C_\varepsilon(\underline{p},\underline{q})) = H_0(\underline{p},\underline{q}) + \hat{f}_\varepsilon(\underline{p},\underline{q}) \tag{2.28}$$

on $W\times[-\theta',\theta']$, θ' small enough, then $\hat{f}_\varepsilon$ admits a finite perturbation theory around H_0.

The perturbation theory can be easily extended to cover the case when

$$H_\varepsilon = h_\varepsilon(H_0) + f_\varepsilon \tag{2.29}$$

where $h_\varepsilon(E)= E + \sum_{k=1}^{\infty}\varepsilon^k h^{(k)}(E)$ is analytic in $H_0(W)\times[-\theta,\theta]$ and one can give a similar definition of finiteness of the perturbation theory (canonically invariant in the same sense as above). We do not discuss the (trivial) details. One could reduce this case to the preceding by putting the ε -dependent part of h_ε into f_ε or, alternatively, repeat the arguments leading to (2.20).

It is then remarkable that :

PROPOSITION 2 : Let f_ε be analytic on $W\times[-\theta,\theta]$ and suppose that the perturbation theory for $H_\varepsilon = H_0 + f_\varepsilon$ is finite. Then :

i) the family H_ε is H_0-integrable for ε small enough.

ii) the theory of perturbations yields a convergent series for Φ_ε and F_ε for ε small enough and Φ_ε generates the integrating map C_ε .

iii)the family H_ε is analytically integrable(with respect to H_0) for ε small enough.

It is important to remark that the condition of finiteness of perturbation theory only, manifestly, involves the derivatives of f_ε at $\varepsilon = 0$. So does the constancy of the action invariants : in fact the closed periodic orbits depend analytically on ε and therefore their actions are also analytic in ε and therefore they can be determined by their derivatives at $\varepsilon=0$. Such derivatives , in turn, can be computed,(without having to solve the perturbed differential equations),in terms of the unperturbed motions, knowing only the derivatives at $\varepsilon=0$ of f_ε .

It will turn out that we shall prove first proposition 2 and then we shall show that the integrability conditions of the k-th order of perturbation theory are equivalent to the conditions that the Taylor coefficients of order 1,2,...,k , about $\varepsilon=0$, of the action invariants vanish identically in E, for all m,n (see (1.13)).

In §3 we prove only the statements i) of proposition 1 and 2, and statement ii), of proposition 2. Statement iii) of Proposition 2 is proven in Appendix G.

Other results are presented in #4,6: in #6 we discuss the relevance of our treatment of the Fourier analysis on $L_2(\Gamma\backslash PSL(2,R))$ for the analysis of the mixing properties of the geodesic and horocyclic flows on the surfaces of constant negative curvature.

In this paper we shall often bound the n-th derivative of a function, holomorphic in some variable w as it varies in some complex domain in C, by n! times the maximum of the function (in the given domain) divided by the distance of the point to the boundary of the domain raised to the n-th power : we call such an estimate a "dimensional estimate".

Usually our domains will be parametrized by parameters called ρ, ξ or θ and they will have the property that the distance between the boundaries of the domains parametrized by ρ',ξ',θ' and ρ,ξ ,θ is bounded below by one of the three numbers $(\rho -\rho')/2$,$(\xi - \xi')/2$, $(\theta- \theta')/2$. If $\rho'= \rho e^{-\sigma}$, as it will often be the case, $\xi'= \xi e^{-\delta}$, $\theta'= \theta e^{-\tau}$ the above numbers become $\hat{\rho}/2$, $\hat{\xi}/2$, $\hat{\theta}/2$ where , to shorten the notations, we set :

$$\hat{x} = (1-e^{-x}) \qquad \text{for } x> 0 \tag{2.30}$$

This last expression will often appear in our estimates .

3 Proof of proposition 2.

We shall regard f_ε as a function on PGL(2,$\mathbb{R}$) parametrized by ε and such that $f_\varepsilon(g) = f_\varepsilon(\gamma g)$, for all $\gamma \in \Gamma$ ("Γ-periodic function").

For convenience of notation we shall put $f_\varepsilon(g) = f_0(g,\varepsilon)$.

The analyticity of f_0 will be imposed by requiring that f_0 admits a holomorphic extension to a suitable complex vicinity of $W\times[-\theta,\theta]$ and, for convenience of notation we set $\theta = \theta_0$.

We shall look at $W = \{g \mid g \in \Gamma\backslash PGL(2,\mathbb{R}),\ H_0(g) \in [1/2,3/2]\}$ as consisting of the points :

$$g = \sqrt{D}\,\phi \tag{3.1}$$

with $D > 0$, $\phi \in \Gamma\backslash PSL(2,\mathbb{R}) \equiv T$; so we can write, see (2,11) :

$$W = \mathcal{D}\times T \qquad \text{with } \mathcal{D} = \{D \mid D \in R_+,\ D^2/8 \in [1/2,3/4]\} \tag{3.2}$$

We denote $\mathcal{D}(\rho_0)$, $H(\xi_0)$, $C(\theta_0)$, $T(\xi_0)$ the sets defined by :

$$\mathcal{D}(\rho_0) = \{D \mid D \in C,\ \exists\ D' \in \mathcal{D} \text{ such that } |D'-D| < \rho_0\}, \quad \rho_0 < 1 .$$

$$\tilde{H}(\xi_0) = \{g \mid g \in PSL(2,C), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ |d-1|\ ,\ |b|,|c| < \xi_0\} \qquad \xi_0 < 1 .$$

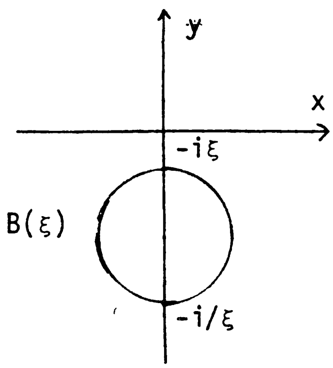
$$C(\theta_0) = \{\varepsilon \mid \varepsilon \in C,\ |\varepsilon| < \theta_0\}, \qquad \theta_0 < 1.$$

$$H(\xi_0) = \{g \mid g \in \tilde{H}(\xi_0/(1-\xi_0^2)^{\frac{1}{2}})\ ,\ C_+g \text{ is outside the circle } B(\xi_0)\} \tag{3.3}$$

$$B(\xi_0) = \{z \mid z \in C\ ,\ |z+i(\xi_0+\xi_0^{-1})/2| < (\xi_0^{-1}-\xi_0)/2\}$$

$$T(\xi_0) = TH(\xi_0)\ , \quad \tilde{T}(\xi_0) = T\tilde{H}(\xi_0) .$$

For later reference we draw the circle $B(\xi_0)$ in the z-plane:



For convenience we shall only consider small values for ξ_0 by assuming that $\xi_0 < 1/10$.

In terms of the above sets we can introduce several notions.

1) We say that a function f_0 on $W \times [-\theta_0,\theta_0]$ is (ρ_0,ξ_0,θ_0) - analytic if for all $\phi \in T \equiv \Gamma\backslash PSL(2,R)$ the function $f_0(\sqrt{D}\phi,\varepsilon)$ can be extended to a holomorphic function of (D,ϕ,ε) $\mathcal{D}(\rho_0) \times T(\xi_0) \times C(\theta_0)$ or, equivalently, if for all $\phi \in T$ the function

$$(U(h)f_0)(\sqrt{D},\phi,\varepsilon) = f(\sqrt{D}\phi h,\varepsilon)$$

can be holomorphically extended to $\mathcal{D}(\rho_0) \times H(\xi_0) \times C(\xi_0)$. If in the above definition we replace $T(\xi_0)$ by $T(\xi_0)$ we say that f_0 is strongly (ρ_0,ξ_0,θ_0) - analytic.

2) Similarly we can define the "ξ-analytic" or the "strongly ξ-analytic" functions on T as those f such that the function $f(\phi)$ admits a holomorphic extension to $T(\xi_0)$ or, equivalently, such that the function of h

$$h \to (U(h)f)(\phi) \equiv f(\phi h)$$

can be holomorphically extended to $H(\xi_0)$ or to $\widetilde{H}(\xi_0)$ (respectively).

3) And a function f on W, see (3.2), will be said (ρ_0,ξ_0)-analytic or strongly (ρ_0,ξ_0)-analytic if it is defined on W and if the function $(D,\phi) \to f(\sqrt{D}\phi)$ can be extended holomorphically to $\mathcal{D}(\rho_0) \times T(\xi_0)$ or $\mathcal{D}(\rho_0) \times \widetilde{T}(\xi_0)$ respectively; equivalently: if, the function $(D,h) \to f(\sqrt{D}\phi h)$ admits a holomorphic extension to $\mathcal{D}(\rho_0) \times H(\xi_0)$ or to $\mathcal{D}(\rho_0) \times \overline{H}(\xi_0)$ respectively.

4) If f is analytic on T in either of the above senses we shall set

$$\|f\|_{\xi} = \sup_{g \quad T(\xi)} |f(g)| \quad , \quad \|f\|_{2,\xi} = \sup_{h\in H(\xi)} \left(\int_T |f(\phi h)|^2 d\phi\right)^{1/2}$$

$$\widetilde{\|f\|}_{\xi} = \sup_{g \quad T(\xi)} |f(g)| \quad , \quad \widetilde{\|f\|}_{2,\xi} = \sup_{h\in H(\xi)} \left(\int_T |f(\phi h)|^2 d\phi\right)^{1/2} \tag{3.5}$$

It is conventient to regard (3.5) as defined for any function on T : whenever the function is not ξ-analytic in the sense necessary to make some of the r.h.s. of (3.5) meaningful we interpret it as being $+\infty$.

We shall found our proof on general results on the theory of the linearized Hamilton Jacobi equation (2.24)

Since the H_0-flow conserves the value of H_0 and the Poisson bracket is nothing else but the derivative along the H_0-flow (i.e. $\sqrt{D}\,\phi \to \sqrt{D}\phi\, e^{-D\sigma_z t/4}$, $t \in R$) the (2.24) can be written as

$$\frac{d}{dt}\, \Phi(\sqrt{D}\,\phi\, e^{-D\sigma_z t/4})\Big|_{t=0} = f(\sqrt{D}\phi) \tag{3.6}$$

The theory of (3.6) with f being (ρ,ξ)-analytic (Resp. strongly ξ-Analytic) will be reduced to the theory of the equation:

$$(L\Phi)(\phi) \equiv \frac{d}{dt}\, \Phi(\, e^{-\sigma_z t/2})\Big|_{t=0} = f(\phi) \tag{3.7}$$

with f being ξ-analytic on T. (Resp. strongly ξ-Analytic).

The first theorem on the theory of the equation (3.7) is the following [2],[3]:

PROPOSITION 3 : Consider the equation :

$$L\Phi = f \tag{3.8}$$

with $f \in C^1(T)$, $T \equiv \Gamma \backslash PSL(2,R)$.

Let p be a periodic orbit for the H_0-flow on T (corresponding to a closed geodesic on Σ). Suppose that for all such p and for all $\phi \in p$:

$$\int_0^{\tau(p)} f(\phi\, e^{-\sigma_z t/2})\, dt = 0 \tag{3.9}$$

where $\tau(p)$ is the period of p. Then :

i) there exists a unique $\Phi \in C^1(T)$ verifying (3.8) and such that $\int_T \Phi(\phi)d\phi = 0$.

ii) If $f \in C^\infty(T)$ then $\Phi \in C^\infty(T)$.

We shall need the following strengthened version of proposition 3

PROPOSITION 4: Let f be analytic on T and suppose that the equation (3.8) is soluble, i.e. (3.9) holds. There are three constants c, q, $\bar{\delta} > 0$, independent on f and such that for all ξ $(0,\xi_0)$:

$$\text{i)} \quad \|\Phi\|_{\xi e^{-\bar{\delta}}} \leq C\, \xi^{-q}\, \|f\|_{\xi} \tag{3.10}$$

$$\text{ii)} \quad \widetilde{\|\Phi\|}_{\xi e^{-\delta}} \leq C(\xi\hat{\delta})^{-q}\, \widetilde{\|\phi\|}_{\xi} \qquad \forall\delta > 0 \tag{3.11}$$

where $\hat{\delta} \equiv (1 - e^{-\delta})$ and the notation (3.5) is usual.

iii) Suppose that f depends also on n parameters $(x_1,\ldots,x_m) \in S \subset \mathbb{C}^n$ and that f is holomorphic on $T(\xi) \times S$. Then Φ also is holomorphic in $T(\xi e^{-\bar{\delta}}) \times S$ and verifies (3.10) for all $x \in S$.

iv) In the same situation met in iii) suppose that f is holomorphic on $\widetilde{T(\xi)} \times S$, then Φ also is holomorphic on $\widetilde{T(\xi)} \times S$ and verifies (3.11)

Of course i), iii) follow from the much stronger (3.11).

The statements i), ii) of proposition 2 depend only on the statements i), iii) of proposition 4 while the stronger result iii) of proposition 2 follows from ii) and iv) of proposition 4.

In this section we show how i), iii) of proposition 4 can be used to prove i), ii) of proposition 2. Actually we have written the proof in such a way that substituting everywhere the words "ξ-analytic" with "strongly ξ-analytic", "(ξ,ρ,θ) - analytic" with strongly (ξ,ρ,θ)-analytic" and changing everywhere the sets $H(\xi)$, $T(\xi)$ with $\widetilde{H(\xi)}$, $\widetilde{T(\xi)}$, one obtains the proof of iii) of proposition 2 from ii), iv) of proposition iv).

The proof of proposition 4, ii), iv), is much more intricate than that of proposition 4, i), so we provide independent proofs of i), iii) and of ii), iv) . The reader will easily find out the reason why the scheme of proof for i), iii) falls short of proving ii), iv) too: actually it motivates the interesting conjecture that ii), iv) could hold in the form obtained by deleting all the tildas where ever they appear. Furthermore it brings up some interesting properties of the Fourier transforms of analytic functions on T .

In sections 4, 5 we develop the proof of proposition 4, i), iii) and in appendix G we develop the proof of ii), iv) which also proves (independently) i), iii) again.

For simplicity of notation denote $W(\rho_0,\xi_0,\theta_0) = (\rho_0) \times T(\xi_0) \times C(\theta_0)$ and suppose:

$$0 < \xi_0 < \rho_0 < 1 \quad ; \quad \theta_0 < 1 \quad ; \quad \xi_0 < 1/10 \tag{3.12}$$

We shall consider hamiltonians of the form $H_0(g,\varepsilon)$:

$$H_0(g,\varepsilon) = h_0(H_0(g),\varepsilon) + f_0(g,\varepsilon) \tag{3.13}$$

where f_0 is divisible by ε and

$$h_0(E,\varepsilon) = E + \sum_{k=1}^{\infty} \varepsilon^k h^{(k)}(E) \tag{3.14}$$

is holomorphic as a function of $(D,\varepsilon) \in \mathcal{D}(\rho_0)\times C(\theta_0)$ and $E=D^2/8$. The (3.13) is slightly more general than H_0+f_0, which is the hamiltonian which we wish to really study.

We now ask under which conditions there is a canonical map C_ε analytic in ε and C^∞ on $W\times[-\theta_0,\theta_0]$, with ($C_\varepsilon$ - identity) divisible by ε, and such that :

$$h_0(H_0(C_\varepsilon(g)),\varepsilon) + f_0(C_\varepsilon(g),\varepsilon) = F_\varepsilon(H_0(g)) \tag{3.15}$$

for some F_ε which is C^∞ on $\mathcal{D}\times[-\theta_0,\theta_0]$ and analytic in ε with $F_\varepsilon(E)-E$ divisible by ε.

As already mentioned in #2 there is a perturbation theory for this problem : it is obtained from the one discussed at length in #2 by considering the family of hamiltonians $H_0+\hat{f}_0$ with $\hat{f}_0= f_0 + (h_0(H_0,\varepsilon)-H_0)$.

We shall suppose that the perturbation theory for (3.15) is finite and then we shall prove that C_ε and F_ε do indeed exist for ε small and are analytic in ε : so proposition 2 will be a particular case of this slightly more general case.

To proceed we introduce the following three sequences of positive numbers :

$$\rho_k = \rho_0 \exp- 5\sum_{j=1}^{k} \sigma_j \quad , \xi_k = \xi_0 \exp- 5\sum_{j=1}^{k} \delta_j$$
$$\theta_k = \theta_0 \exp- 5\sum_{j=1}^{k} \tau_j \tag{3.16}$$

with (see proposition 4 for the meaning of $\overline{\delta}$):

$$\sigma_k = \tau_k = (1+k^2)^{-1}$$

$$\delta_k = \begin{cases} \bar{\delta} & \text{if we wish to prove only i),ii) of prop. 2} \\ (1+k^2)^{-1} & \text{to prove iii) of proposition 2} \end{cases} \tag{3.17}$$

There will be no formal difference in the proof of i),ii) or of iii) in proposition 2 if one does not substitute the explicit expressions for δ_k :however many inequalities will be true only for the first choice of δ_k when we only suppose valid the i),iii) of proposition 4 , while they will be true also with the second choice if we suppose ii),iv) of proposition 4.

We shall use a recursive algorithm whose steps will be indexed by an integer n=1,2,...

The purpose of the algorithm is the construction of a sequence of canonical transformations parametrized by ε , $C^{(0)}$, $C^{(1)}$,... such that :

i) $C^{(n-1)}$ is holomorphic on $W(\rho_n,\xi_n,\theta_n)$ and, if we regard it as a map of $(\phi,\varepsilon) \to (\phi',\varepsilon')$ with $\varepsilon' \equiv \varepsilon$, it is :

$$C^{(n-1)}\, W(\rho_n,\xi_n,\theta_n) \subset W(\rho_{n-1},\xi_{n-1},\theta_{n-1}) \tag{3.18}$$

$$\text{ii)} \; \| C^{(n-1)} - \text{identity} \|_{\rho_n,\xi_n,\theta_n} \leq C\theta_0 e^{-n^2} \tag{3.19}$$

where $\| \; \|_{\rho,\xi,\theta}$ denotes the supremum of a function defined on $W(\rho,\xi,\theta)$.

Note that it immediately follows from (3.18),(3,19),(3.16) that the composition :

$$C_\varepsilon = \lim_{n\to\infty} C^{(0)} \ldots C^{(n-1)} \tag{3.20}$$

exists and is C^∞ on $W(\rho_\infty,\xi_\infty,\theta_\infty)$ and analytic in D,ε and even in ϕ if $\xi_\infty > 0$, i.e. if δ_k is given by the second of (3.17).

The map $C^{(n-1)}$ will be constructed inductively. $C^{(0)}$ is obtained by requiring that on $W(\rho_1, \xi_1, \theta_1)$:

$$h_0(H_0(C^{(0)}(g),\varepsilon) + f_0(C^{(0)}(g),\varepsilon) = h_1(H_0(g),\varepsilon) + f_1(g,\varepsilon) \tag{3.21}$$

with h_1 analytic on $\mathcal{D}(\rho_1)\times C(\theta_1) \equiv W(\rho_1,\theta_1)$ and differing from h_0 by a polynomial of order 1.

Then one builds successively $C^{(1)}, C^{(2)},\ldots$ by requiring that on $W(\rho_n,\xi_n,\theta_n)$:

$$h_{n-1}(H_0(C^{(n-1)}(g),\varepsilon) + f_{n-1}(C^{(n-1)}(g),\varepsilon) = h_n(H_0(g),\varepsilon)+ f_n(g,\varepsilon) \tag{3.22}$$

with f_n divisible by ε^{2^n} and $h_n - h_{n-1}$ being a polynomial in ε of order ε^{2^n-1} divisible by $\varepsilon^{2^{n-1}}$.

If we define $E_k,\varepsilon_k,\lambda_k$ to be such that :

$$\begin{aligned} E_k &> \sup_{\mathcal{D}(\rho_k)\times C(\theta_k)} \left|\frac{\partial h_k(E,\varepsilon)}{\partial E} - 1\right| \\ \varepsilon_k &> \sup_{W(\rho_k,\xi_k,\theta_k)} \left|\frac{\partial f_k(g)}{\partial g}\right| \\ \lambda_k &> \sup_{W(\rho_k,\xi_k,\theta_k)} |f_k| \end{aligned} \tag{3.23}$$

we shall also require that for ε_0 small enough and for suitable constants B,b it is , for all the considered choices of ρ_0,ξ_0,θ_0 :

$$E_k < B\varepsilon_0\ \xi_0^{-b} \quad ; \quad \varepsilon_k < (B\varepsilon_0\ \xi_0^{-b})^{(3/2)^k} \quad ; \quad \lambda_k < (B\varepsilon_0\ \xi_0^{-b})^{(3/2)^k} \tag{3.24}$$

It is clear that if (3.16),(3.18),(3.19),(3.24) hold the limit $\lim_{n\to\infty} h_n(E,\varepsilon)$ exists and is holomorphic on $\mathcal{D}(\rho_\infty)\times C(\theta_\infty)$ and :

$$H_\varepsilon(C_\varepsilon(g)) = h_\infty(H_0(g),\varepsilon) \tag{3.25}$$

It "remains" therefore to check that with the definitions (3.16), (3.17),(3.23) it is possible to define $C^{(0)}, C^{(1)},\ldots, h_1,h_2,\ldots,f_1,f_2,\ldots$ so that (3.18),(3.19),(3.24) hold for ε_0 small enough : in fact since f_0 is divisible by ε one can always reduce the value of ε_0 by redefining θ_0 (which does not appear explicitely in the (3.24)).

The remaining parts of the proof will be organized in several steps.

1) Definition of the generating function of $C^{(n)}$.

Assume inductively to have defined $C^{(0)}, C^{(1)}, \ldots, C^{(n-1)}, f_0, f_1,\ldots f_n,h_0,h_1,\ldots,h_n$ verifying (3.18),(3.19),(3.24) with the f_i's and the h_i's verifying the properties mentioned after (3.22).

Then $C^{(n)}$ will be defined via a generating function Φ .

The function Φ will be the solution to the equation :

$$\{H_0,\Phi\} = \left\{ \left[\frac{f_n(g,\varepsilon) - \bar{f}_n(H_0(g),\varepsilon)}{h_n'(H_0(g))}\right]\right\}^{[\leqslant 2^{n+1}-1]} \tag{3.26}$$

where $[\leqslant p]$ denotes the truncation of a Taylor series in ε to order p. The (3.26) arises from the requirement that :

$$h_n(C^{(n)}(g),\varepsilon) = h_{n+1}(H_0(g),\varepsilon) + O(\varepsilon^{2^{n+1}}) \tag{3.27}$$

$h_{n+1}-h_n$ = {polynomial in ε of order $2^{n+1}-1$ divisible by ε^{2^n} }

In fact the above requirement leads, via a simple calculation similar to the one needed to define the first order of perturbation theory (discussed in #2), to the requirement that :

$$h_n'(H_0(g),\varepsilon)\,\{H_0,\Phi\}(g) = f_n(g,\varepsilon) + G_n(H_0(g),\varepsilon) + O(\varepsilon^{2^{n+1}}) \tag{3.28}$$

with G_n being a polynomial of order $\varepsilon^{2^{n+1}-1}$ divisible by ε^{2^n}.
The (3.28) is equivalent to the (3.26).

The solubility of (3.26) has to be proved . It follows from the invariance of the finiteness of perturbation theory under canonical transformations (discussed in #2) that h_n+f_n admits a finite perturbation theory. However the perturbation theory for h_n+f_n yields a series for the generating function of the integrating map which starts from the order ε^{2^n} and it is easily seen by power counting that the sum of the orders between ε^{2^n} and $\varepsilon^{2^{n+1}-1}$ is a function Φ verifying (3.28) so that (3.28) and hence (3.26), which is equivalent to it, do have a solution.

Applying proposition 3, since (3.26) has the form (3.6) and it can be regarded as an equation of the form (3.8) parametrized by E (or D), we see that Φ can be bounded by :

$$||\Phi||_{\rho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}} < C(\xi_n\hat{\delta}_n)^{-q}\,\Big|\Big|\Big[\frac{f_n-\overline{f}_n}{h_n'(H_0,\varepsilon)}\Big]^{[\leq 2^{n+1}-1]}\Big|\Big|_{\rho_n,\xi_n,\theta_n e^{-\tau_n}} \tag{3.29}$$

and using dimensional bounds :

$$\begin{aligned}||\Phi||_{\rho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}} &< C(\xi_n\hat{\delta}_n)^{-q}\,\tau_n^{-1}\,\Big|\Big|\frac{f_n-\overline{f}_n}{h_n'(H_0,\ )}\Big|\Big|_{\rho_n,\xi_n,\theta_n} <\\ &< R\,C(\xi_n\hat{\delta}_n)^{-q}\,\tau_n^{-1}\,(1-E_n)^{-1}\Big|\Big|\frac{\partial f_n}{\partial g}\Big|\Big|_{\rho_n,\xi_n,\theta_n}\end{aligned} \tag{3.30}$$

where R is an estimate of the length of the maximal distance of two points in $W(\rho_0,\xi_0)$. Hence :

$$||\Phi||_{\rho_n,\xi_n e^{-\delta_n},\theta_n e^{-\tau_n}} < B_1 (\xi_n\hat{\delta}_n)^{-q}\hat{\tau}_n^{-1} \varepsilon_n \qquad \text{if } E_n < 1/2 \tag{3.31}$$

where B_1 is a suitable constant and we recall the notation $\hat{x}=(1-e^{-x})$, for $x>0$.

2) Definition of $C^{(n)}$.

Recallimg (2.11),(2.12) we define $C^{(n)}$ by inverting the (2.13) in the form :

$$\begin{aligned} g &= g' + \Delta(g') = C^{(n)}(g') \\ g' &= g + \Delta'(g) = \tilde{C}^{(n)}(g) \end{aligned} \tag{3.32}$$

the first being obtained by solving the second of (2.11) with respect to $\underline{q}$ and substituting in the first while the second of (3.32) is obtained by solving the first of (2.11) with respect to $\underline{p}'$ and substituting in the second (of (2.11)).

We want to have Δ, Δ' defined on a domain as large as possible, say $W(\rho_n e^{-3\sigma_n},\xi_n e^{-3\delta_n},\theta_n e^{-3\tau_n})$.

This can be reasonably well achieved by using an analytic implicit function theorem ,(see for instance [10] where a similar theorem is proved when $W(\rho',\xi',\theta')$ is replaced by a different multiply connected set).

It is, in any case, easy to see that inversions in (3.32) can be made under the (very strong, but dimensionally natural) conditions :

$$B_2' \left|\left|\frac{\partial\Phi}{\partial g}\right|\right| < \xi_n \hat{\delta}_n \quad ; \quad B_2'' \left|\left|\frac{\partial^2\Phi}{\partial g\,\partial g}\right|\right| < 1 \tag{3.33}$$

where $||.||$ denotes the supremum over, say, $W(\rho_n e^{-2\sigma_n},\xi_n e^{-2\delta_n},\theta_n e^{-2\tau_n})$ and B_2', B_2'' are suitably large positive numbers (note the role of (3.33) : the second condition guarantees the local analytic invertibility, while the first provides the globality of this local inversion of the (2.11)). Therefore, under the conditions (3.33) the functions Δ, Δ' can be defined on $W(\rho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$.

The conditions (3.33) can be implemented using (3.31) and dimensional estimates by the conditions (very generous) :

$$B_3(\xi_n \hat{\delta}_n)^{-q} \hat{\tau}_n^{-1} \varepsilon_n (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-2} < 1, \qquad E_n < 1/2 \tag{3.34}$$

where B_3 is a suitably large number . Here the inequality $\rho_n > \xi_n$ has been used to replace ρ_n^{-1} by ξ_n^{-1} and have it disappear.

Since Δ and Δ' are equal to some derivatives of Φ computed at suitable points (see (2.11) or (2.13)) we infer from (3.31), (3.34) that (by dimensional estimates and if $||.||$ denotes the sup on $W(\rho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n})$):

$$||\Delta||;||\Delta'|| < B_1(\xi_n \hat{\delta}_n)^{-q} \varepsilon_n \hat{\tau}_n^{-1} (\xi_n \hat{\sigma}_n \hat{\delta}_n)^{-1} < \frac{\xi_n \hat{\sigma}_n \hat{\delta}_n}{B_4} \tag{3.35}$$

where B_4 can be given any prefixed value provided we readjust the constant B_3 in (3.34) : for later use we imagine to fix the constants so that $e^{-4\delta_0} + \hat{\delta}_0/B_4 < e^{-3\delta_0}$.

So under the conditions (3.34) Δ , Δ' are defined on $W(\rho_n e^{-3\sigma_n}, \xi_n e^{-3\xi_n}, \theta_n e^{-3\tau_n})$ and by the last of (3.35) and by the choice of B_4 :

$$C^{(n)}, \tilde{C}^{(n)} : W(\rho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n}) \to W(\rho_n e^{-3\sigma_n}, \xi_n e^{-3\delta_n}, \theta_n e^{-3\tau_n}) \tag{3.36}$$

Here as well as above we keep changing the coefficients of ρ_n, ξ_n, θ_n simultaneously even when this is not really necessary (e.g. in all the above inequalities $\theta_n e^{-p\tau_n}$, $p > 1$, can be replaced simply by $\theta_n e^{-\tau_n}$): this is done to uniformize the notations.

Furthermore on $W(\rho_n e^{-4\sigma_n}, \xi_n e^{-4\xi_n}, \theta_n e^{-4\tau_n})$ the maps $C^{(n)}$, $\tilde{C}^{(n)}$ differ from the identity map by less than :

$$B_1(\xi_n \hat{\delta}_n)^{-q} \hat{\tau}_n^{-1} (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-1} \varepsilon_n \tag{3.37}$$

Finally a remarkable property of Δ, Δ' is related to the very definition of Φ as solution of (3.26),(3.27) : i.e. setting $H_0 \equiv H_0(g)$:

$$h_n'(H_0,\varepsilon)\{H_0,\Phi\}(g) = f_n^{[<2^{n+1}]}(g,\varepsilon) - \bar{f}_n^{[<2^{n+1}]}(H_0,\varepsilon) + r_n(g,\varepsilon) \tag{3.38}$$

$$r_n(g,\varepsilon) = -\left[\frac{f_n(g,\varepsilon) - \bar{f}_n(H_0,\varepsilon)}{h_n'(H_0,\varepsilon)} \right]^{[\geq 2^{n+1}]} h_n'(H_0,\varepsilon) + [f_n(g,\varepsilon) - \bar{f}_n(H_0,\varepsilon)]^{[\geq 2^{n+1}]} \tag{3.39}$$

(just multiply both sides of (3.26) by h_n').

Therefore, again by dimensional estimates,

$$||r_n||_{\rho_n e^{-\sigma_n},\ \xi_n e^{-\delta_n}, \theta_n e^{-\tau_n}} < B_5\, e^{-\delta_n 2^{n+1}}\, \hat{\tau}_n^{-1}\, \hat{\varepsilon}_n\, R \tag{3.40}$$

with B_5 being a suitably large number and R arises when, as in (3.30), we bound $f_n - \bar{f}_n$ by $\partial f_n/\partial g$ (expressing the first as a path integral of the latter).

The (3.38) could be expressed as relations involving Δ, Δ' by expressing $\partial\Phi/\partial g$ in terms of them : we do not need to do this explicitly.

3) Definition of h_{n+1}, f_{n+1}.

We shall naturally define :

$$\begin{aligned} h_{n+1}(E,\varepsilon) &= h_n(E,\varepsilon) + [\bar{f}_n(E,\varepsilon)]^{[<2^{n+1}]} \\ f_{n+1}(g',\varepsilon) &= f_n(g'+\Delta(g'),\varepsilon) + h_n(H_0(g'+\Delta(g'),\varepsilon) - h_{n+1}(H_0(g),\varepsilon) \end{aligned} \tag{3.41}$$

and we proceed to estimate $\varepsilon_{n+1}, E_{n+1}, \lambda_{n+1}$ on $W(\rho_{n+1}, \xi_{n+1}, \theta_{n+1})$.

The estimate, based again on dimensional considerations, is straightforward but quite technical and it is developed in detail in appendix A where the following bounds are proved : if (3.34) holds one can take E_{n+1}, ε_{n+1}, λ_{n+1} as :

$$
\begin{aligned}
E_{n+1} &= E_n + B_6\, \varepsilon_n\, \hat{\tau}_n^{-1} \\
\varepsilon_{n+1} &= B_7(\varepsilon_n^2 + \varepsilon_n \exp -\tau_n 2^{n+1})(\hat{\tau}_n\, \hat{\delta}_n\, \xi_n)^{-b_7} \\
\lambda_{n+1} &= B_8((\varepsilon_n^2 + \varepsilon_n \exp -\tau_n 2^{n+1})(\hat{\tau}_n\, \hat{\delta}_n\, \xi_n)^{-b_8} + \lambda_n\, \hat{\tau}_n^{-1} \exp -\tau_n 2^{n+1})
\end{aligned}
\tag{3.42}
$$

A simple inductive argument shows that there are B, b > 0 such that (3.24) (as well as (3.34) for all n) hold.

This completes the reduction of the proof of proposition 2 to that of propositon 4.

4 Fourier analysis on $L_2 \equiv L_2(\Gamma\backslash PSL(2,\mathbb{R}))$.

In this section we develop some tools for the proof of the statements i),iii) of proposition 4. Also the proof of ii),iv) will be based on the same tools.

We suppose the reader familiar with the chapter on SL(2,ℝ) of [5] as well as with the first chapter of [6] , where the theory of the unitary representations of SL(2,ℝ) and the related Fourier analysis is developed .

Let f be ξ-analytic (resp. strongly ξ-Analytic) on $T \equiv \Gamma\backslash PSL(2,\mathbb{R})$. It is easy to prove by using the Cauchy theorem that there is a constant C_1 such that, using #3 for the symbols and (2.30):

$$||f||_{2,\xi} < ||f||_{\xi} \quad , \; ||f||_{\xi e^{-\tau}} < C_1 \, \tau^{-3} \, ||f||_{2,\xi} \tag{4.1}$$

(resp. $\widetilde{||f||}_{2,\xi} < \widetilde{||f||}_{\xi} \quad , \quad \widetilde{||f||}_{\xi e^{-\tau}} < C_1 \, \tau^{-3} \, \widetilde{||f||}_{2,\xi}$)

(this is basically the well known procedure of bounding a holomorphic function by some integral norm).

Let us define the unitary representation U of PSL(2,ℝ) on $L_2(T) \equiv L_2$ induced by the action of PSL(2,R) on the homogeneous space $\Gamma\backslash PSL(2,\mathbb{R})$:

$$(U(g)f)(\phi) = f(\phi g) \tag{4.2}$$

here the scalar product considered in L_2 is $(f,f') = \int_T \overline{f(g)}\, f(g')dg$, with dg denoting the normalized Haar measure on T .

Let

$$L_2 = \bigoplus_{\alpha\in A} Y^{(\alpha)} \tag{4.3}$$

be a decomposition of L_2 into U-invariant, pairwise orthogonal subspaces on which U acts irreducibly: such a decomposition is always possible,[6].

We denote $U^{(a)}$ the restriction of U to $Y^{(a)}$ and we briefly recall some known facts about the above decomposition and a few of their developments that we shall need in the proof.

One associates with a hyperbolic fuchsian group $\Gamma \subset G_0 \equiv PSL(2,\mathbb{R})$ the following four classes of entities.

1) The automorphic forms of order n=1,3,5,... i.e. the functions which are holomorphic in the upper half plane $\mathbb{C}_+$ and such that : $\forall \gamma \in \Gamma$

$$\phi(z\gamma) = \phi(z)\, j(z,\gamma)^{n+1} \qquad \text{if } \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad j(z,\gamma)=(bz+d) \tag{4.4}$$

They form a n(g-1)-dimensional linear space, g being the genus of the compact surface associated with the given fuchsian group.

We shall, once and for all, choose a basis in the above linear space and we shall label its elements as $\phi^{(n,j,+)}$, $j = 1,\dots,(n-1)g$; we shall also suppose that $\phi^{(n,j,+)}$ are orthonormal with respect to a convenient scalar product:

$$\delta_{jk} = \int_T \overline{\phi^{(n,j,+)}(ig^{-1})}\,\phi^{(n,k,+)}(ig^{-1})\,|j(i,g^{-1})|^{-2(n+1)}\, dg \tag{4.5}$$

2) The antiautomorphic forms of order n=1,3,5,...: they are just the complex conjugates of the corresponding automorphic forms. We shall put:

$$\phi^{(n,j,-)} = \overline{\phi^{(n,j,+)}} \tag{4,6}$$

and (n,j,+),(n,j,-) will be often denoted by the symbol a.

3) The eigenfunctions of the Laplace-Beltrami operator relative to the eigenvalue $(1-u^2)/4 \in (0,+\infty)$. The normalization that we choose for the operator Δ on $L_2(\Sigma)$ is such that , in the ordinary cartesian coordinates on the upper half plane $\mathbb{C}_+$, it is

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{4.7}$$

The variable u will (therefore) only take finitely many values in (0,1) and countably many of the form is, $s \in (0,+\infty)$ (this is a consequence of the general properties of the spectrum of the Laplace operator on a compact riemanian surface).

The elements of a basis in the space of such eigenvectors will be labeled with an index $a=(u,j)$ where the first number fixes the eigenvalue and the second distinguishes the independent eigenvectors associated with the same eigenvalue $(1-u^2)/4$ whenever the latter is degenerate, otherwise $j=1$.

We fix once and for all the basis and we also suppose that it has the property :

$$\int_T \overline{\phi^{(u,j)}(ig^{-1})}\ \phi^{(u,k)}(ig^{-1})\ dg = \delta_{jk} \tag{4.8}$$

It follows from the general theory of the PSL(2,$\mathbb{R}$)-induced representations that the multiplicity of the eigenvalue $(1-u^2)/4$ cannot exceed the value $g|u^2|$, g being the genus of the surface.

We shall also denote (u,j) by a.

4) The function identically 1 on $\mathbb{C}_+$: it will be denoted $\phi^{(0)}$.

We shall denote by A the set of values that, according to the above classification, the index a can take plus, the index 0. Set $\bar{a}=(n,j,-)$ if $a=(n,j,+)$.

Then we can define the functions on T

$$\begin{aligned} E^{(a)}(g) &= \phi^{(a)}(ig^{-1}) && a \in A,\ a=0 \text{ or } a=(u,j) \\ E^{(a)}(g) &= \overline{E^{(\bar{a})}(g)} = \overline{\phi^{(a)}(ig^{-1})\ j(i,g^{-1})^{-n-1}} && a=(n,j,+) \end{aligned} \tag{4.9}$$

We shall define , for all a in A :

$$Y^{(a)} = \{\text{subspace of } L_2 \text{ spanned by } U(g)E^{(a)} \text{ as } g \text{ varies in } G_0\} \tag{4.10}$$

The "duality theorem" [6] on the induced representations says :

$$L_2 = \bigoplus_{a\in A} Y^{(a)} \tag{4.11}$$

and $Y^{(a)}$ is orthogonal to $Y^{(a')}$ if $a \neq a'$; furthermore U acts irreducibly on $Y^{(a)}$.

Each $Y^{(a)}$ can be realized in a "standard way" as a space of functions $\hat{Y}^{(a)}$ defined either on the line R or on the upper half plane $\mathbb{C}_+$ as follows :

i) if a=(n,j,+) then $Y^{(a)}$ can be realized as a subspace $\hat{Y}^{(a)} \subset L_2(\mathbb{C}_+, y^{n+1}\frac{dxdy}{y^2})$

consisting in the functions of (x,y) which are holomorphic in the variable $z=x+iy$.

If $\hat{a}=(n,j,-) = \bar{a}$ the space $Y^{(\hat{a})}$ can be realized as a subspace of $L_2(\mathbb{C}_+, y^{n-1}dxdy)$ containing the functions which are obtained from those in $\hat{Y}^{(a)}$ by complex cojugation (i.e. the antiholomorphic functions on $\mathbb{C}_+$), [6].

ii) if $a=(u,j)$, $u=is$, $s\in\mathbb{R}_+$, then $Y^{(a)}$ can be realized as $\hat{Y}^{(a)}=L_2(\mathbb{R},dx)$ while if $a=(s,j)$, $0<s<1$, the $Y^{(a)}$ can be realized as $\hat{Y}^{(a)}= L_2(\mathbb{R};s)$ where we denote by $L_2(\mathbb{R};s)$ the completion of $C_0^1(\mathbb{R})$ with respect to the scalar product :

$$(f,g) = \int \overline{f(x)}g(y)|x-y|^{-1+s}\, dxdy = \int \overline{\tilde{f}(p)}\tilde{g}(p)|p|^{1-s}\frac{dp}{p} \tag{4.12}$$

where $\tilde{f},\tilde{g}$ denote the ordinary naive Fourier transform,[6].

iii) finally the identification of $Y^{(a)}$ and $\hat{Y}^{(a)}$ and the realization of the restriction of U to $Y^{(a)}$ as a unitary representation $\hat{U}^{(a)}$ of G_0 acting on $\hat{Y}^{(a)}$ are implemented as follows. First one defines a irreducible representation $\hat{U}^{(a)}$ of G_0 on $\hat{Y}^{(a)}$, secondly one prescribes the representative $\hat{\phi}^{(a)}$ of $\phi^{(a)}$ in $\hat{Y}^{(a)}$ and finally one uses the cyclicity of the vector $\hat{\phi}^{(a)}$ under the action of $\hat{U}^{(a)}$ to associate with every vector of $\hat{Y}^{(a)}$ a (unique) vector in $Y^{(a)}$.

We set :

$$(\hat{U}^{(a)}(g)f)(z) = f(zg)\ j(z,g)^{-n-1} \qquad f \in \hat{Y}^{(a)},\ a=(n,j,+)$$
$$(\hat{U}^{(a)}(g)f)(z) = f(zg)\ \overline{j(z,g)^{-n-1}} \qquad f \in \hat{Y}^{(a)},\ a=(n,j,-) \qquad (4.13)$$
$$(\hat{U}^{(a)}(g)f)(x) = f(xg)\ (j(x,g)^2)^{-(1+u)/2} \qquad f \in \hat{Y}^{(a)},\ a=(u,j)$$

Since the space $\hat{Y}^{(a)}$ never depends on the index j in a we shall sometimes denote it as $\hat{Y}^{(n,+)}$ or $\hat{Y}^{(n,-)}$ or $\hat{Y}^{(u)}$, simply.

Then the following identification of $\phi^{(a)}$ as vectors $\hat{\phi}^{(a)} \in \hat{Y}^{(a)}$ is possible,[6] :

$$\phi^{(n,j,+)}(g) \leftrightarrow (z+i)^{-n-1}\ M_n \qquad (4.14)$$
$$\phi^{(n,j,-)}(g) \leftrightarrow (\bar{z}-i)^{-n-1}\ M_n$$

and

$$\phi^{(u,j)}(g) \leftrightarrow |x+i|^{-u-1}\ M_u \qquad (4.15)$$

where the constants M_a are the normalization constants ($M_n=(n\pi 2^{-2n})^{1/2}$, $M_s=(2^{s-1}(-s!))^{-1/2}$, $M_{is}=(\sqrt{\pi})^{-1}$).

The above associations (4.14),(4.15) and the irreducibility of $U^{(a)}$ allow tb uniquely associate with $f \in L_2$ "its Fourier transform $(\hat{f}^{(a)})_{a\in A}$ " where :

$$\hat{f}^{(a)} \in \hat{Y}^{(a)} \qquad (4.16)$$

is the function corresponding to the component $f^{(a)}$ of f on $Y^{(a)}$ via the basic relations (4.14),(4.15).

To discuss the remarkable properties of the Fourier transforms of functions on T which are ξ-analytic let us introduce the following notation :

$$||f||_{\xi} = \sup_{g\in G_0, h\in H(\xi)} |f(gh)| \tag{4.17}$$

(already used in #3). We also introduce the following notation :

$$\begin{aligned} F_f^{(n,j,+)}(z) &= (z+i)^{n+1}\, \hat{f}^{(n,j,+)}(z) && z\in \mathbb{C}_+ \\ F_f^{(n,j,-)}(z) &= (\bar{z}-i)^{n+1}\, \hat{f}^{(n,j,-)}(z) && z\in \mathbb{C}_+ \\ F_f^{(u,j)}(x) &= (x^2+1)^{(u+1)/2}\, \hat{f}^{(u,j)}(x) && x\in \mathbb{R} \end{aligned} \tag{4.18}$$

and for $z \in \mathbb{C}_+H(\xi)$, if $h=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in H(\xi)$ and $\tilde{a}=(n,j,+),(n,j,-)$:

$$A^{(\tilde{a})}(z,\xi) = \inf (n/4\pi)^{1/2} \left| \frac{(a\zeta+c)+i(b\zeta+d)}{\surd(\mathrm{Im}\ \zeta)} \right|^{n+1} \tag{4.19}$$

where the inf is taken over the $\zeta \in \mathbb{C}_+$, $h\in H(\xi)$, $\zeta h= z$; or for $z\in\mathbb{R}H(\xi)$, $\tilde{a}=(u,j)$, $\hat{\delta}= (1-e^{-\delta})$:

$$A^{(\tilde{a})}(z,\xi)= \inf C_3(\xi\hat{\delta})^{-1} \left| \left(\frac{(ax+c)^2+(bx+d)^2}{1+x^2} \right)^{(1+u)/2} \right| \tag{4.20}$$

where the inf is over all $\delta > 0$ and the pairs $x \in \mathbb{R}$, $h \in H(\xi e^{-\delta})$, $xh=z$, and the powers $(1+u)/2$ are defined by cutting the complex plane on the negative real axis (as we shall see the number in brackets never takes a negative value) and, finally, the constant C_3 has a value which will be conveniently fixed later.

Finally define

$$Q^{(a)}(\xi) = \begin{cases} \mathbb{C}_+H(\xi) & \text{if } a=(n,j,\pm) \\ \mathbb{R}H(\xi) & \text{if } a=(u,j) \end{cases} \tag{4.21}$$

and $Q^{(a)}$ will be called the "ξ-domain of the representation a" its form is simple :

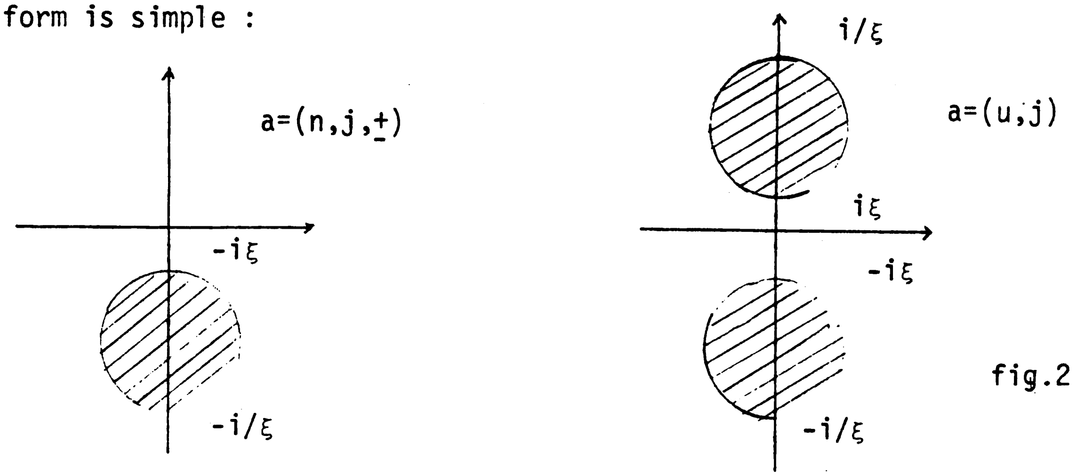


fig.2

i.e. $Q^{(a)}(\xi)$ is either $\mathbb{C}/B(\xi)$ or $\mathbb{C}/(-B(\xi)\cup B(\xi))$,see fig 1,and (3.3).

In appendix B we shall prove the following structural theorem on the Fourier transform of ξ-analytic functions.

LEMMA 1 : Let $f\in Y^{(a)}$, $a\in A$, $a\neq 0$.

i) The function f is ξ-analytic if and only if the functions $F_f^{(a)}$ can be holomorphically extended to the ξ-domain $Q^{(a)}(\xi)$ of the representation a.

ii) if f is ξ-analytic :

$$|F_f^{(a)}(z)| < A^{(a)}(z,\xi)\,||f||_\xi \tag{4.22}$$

iii) if f is ξ-analytic and there is $N>0$ such that :

$$|F_f^{(a)}(z)| < A^{(a)}(z,\xi)\,N \qquad z\in Q^{(a)}(\xi) \tag{4.23}$$

then

$$||f||_{\xi e^{-\delta}} < M_1\,(\xi\hat{\delta})^{-m_1}\,N\sqrt{|a|} \qquad \delta>0 \tag{4.24}$$

where $|a|=n$ if $a=(n,j,\pm)$, $|a|=|u|$ if $a=(u,j)$ and M_1,m_1 are independent on a,f, or ξ,δ.

The proof of the above lemma also provides the proof of :

LEMMA 2 : If $f \in L_2$ is ξ-analytic and depends on some complex parameters $\underline{w} \in W \subset \mathbb{C}^q$ and it is holomorphic in $(g,\underline{w}) \in T \times W$ then $F_f^{(a)}$ are holomorphic in $Q^{(a)}(\xi) \times W$ and (4.22) holds for each $\underline{w} \in W$. Viceversa if f is such that $f \in Y^{(a)}$ for all $\underline{w} \in W$ and $F_f^{(a)}$ is holomorphic on $Q^{(a)}(\xi) \times W$, then f is ξ-holomorphic in $T \times W$ and verifies (4.24) for each $\underline{w}$ for which $F_f^{(a)}$ verifies (4.23).

The ξ-analytic functions turn out, also, to have an "exponential decay" of their Fourier transform, expressed by the following lemma :

LEMMA 3 : If $f \in L_2$ is ξ-analytic :

$$||f^{(a)}||_{\xi e^{-\delta}} < M_2(\xi\hat{\delta})^{-m_2} e^{-v_2 \xi \hat{\delta} |a|} ||f||_\xi \qquad \delta > 0 \qquad (4.25)$$

where M_2, m_2, v_2 do not depend on f, ξ, δ, a .

Finally we shall need , for the proof of proposition 4, the following estimates for the functions $A^{(a)}(z,\xi)$.

Let, for $z \in \mathbb{R}H(\xi)$:

$$d(z,\xi) = \sup \{\delta |\ \delta > 0 \text{ such that } z \in \mathbb{R}H(\xi e^{-\delta}) \} \qquad (4.26)$$

then:

LEMMA 4 : there are constants $M_3, m_3, v_3 > 0$ such that

$$A^{(n,j,\pm)}(z,\xi) = \sqrt{(n/4\pi)}\ \left(|i+z|^2 \frac{1-\xi^2}{\xi} \frac{1}{(|z+i(\xi^{-1}+\xi)/2|^2 - ((\xi^{-1}-\xi)/2)^2)}\right)^{(n+1)/2} \qquad (4.27)$$

and for all $z \in \mathbb{R}H(\xi)$:

$$M_3^{-1}(\hat{d}(z,\xi))^{-m_3} e^{-v_3 \xi |u|} < A^{(u,j)}(z,\xi) < M_3(\hat{d}(z,\xi))^{-m_3} \qquad (4.28)$$

The lemmas 1./. 4 are proved in appendix B,C.

In #5 we shall show how proposition 4), i),iii)can be deduced from the theory of the Fourier transform developed in the above lemmata. The proof of ii),iv) can be done along the same lines but one needs a refinement of the bounds (4.28), see appendix G .

5 Proof of propositions 4 and 1.

Consider the equation (3,8) or (3,7). Since

$$\Phi(g e^{-\sigma_z t/2}) \equiv (U(e^{-\sigma_z t/2})\Phi)(g) \tag{5.1}$$

it is clear that (3.7) can be reduced by the reduction of the representation U : one just considers its projections on the spaces $Y^{(a)}$.

If $\hat{\Phi}^{(a)}$, $\hat{f}^{(a)}$ are the components of order a of the Fourier transforms of Φ,f and if one uses (4.13) one finds that the Fourier transforms obey the following equations :

$$\frac{d}{dt} \hat{\Phi}^{(a)}(e^{-t}z)\, e^{-(1+a)t/2} \Big|_{t=0} = \hat{f}^{(a)}(z) \tag{5.2}$$

where z varies in $\mathbb{C}_+$ or $\mathbb{R}$ according to the value of a and we have shortened the notations by writing (1+a) for (1+n) or (1+u) when a=(n,j,$\pm$) or, respectively, a=(u,j). More explicitly :

$$z \frac{d}{dz} \hat{\Phi}^{(a)}(z) + \frac{(1+a)}{2} \hat{\Phi}^{(a)}(z) = - \hat{f}^{(a)}(z) \tag{5.3}$$

The solution to (5.3) is, for a=(u,j) :

$$\hat{\Phi}^{(a)}(x) = - \int_0^x \left(\frac{y}{x}\right)^{(1+u)/2} \hat{f}^{(a)}(y)\, dy/y \;+\; K_a x^{-(1+u)/2} \tag{5.4}$$

and, for a=(n,j,+):

$$\hat{\Phi}^{(a)}(z) = - \int_0^z \left(\frac{\zeta}{z}\right)^{(1+n)/2} \hat{f}^{(a)}(\zeta)\, d\zeta/\zeta \;+\; K_a z^{-(1+n)/2} \tag{5.5}$$

and a similar expression holds for a=(n,j,-). K_a are arbitrary integration constants.

Recalling that $F_f^{(a)}$ are holomorphic functions in $Q^{(a)}(\xi)$ it is convenient to try to write (5.4),(5.5) in terms of $F_f^{(a)}$.

Since we assume that Φ exists it follows that $\hat{\Phi}^{(a)}$ must have some square integrability properties (i.e. $\hat{\Phi}^{(a)} \in \hat{Y}^{(a)}$) and the analyticity of $f^{(a)}$ together with the bounds in lemma 1 immediatly imply that in order that $\hat{\Phi}^{(a)}$ be square integrable it is necessary that

$$\int_0^{\pm\infty} |y|^{(1+u)/2} \hat{f}^{(u,j)}(y)\,dy/y = 0 \quad , \qquad K_{(u,j)}=0 \tag{5.6}$$

$$\int_0^{\infty} \zeta^{(1+n)/2}\, \hat{f}^{(n,j,+)}(\zeta)\, d\zeta/\zeta = 0 \quad , \qquad K_{(a,n,+)}=0$$

and similar conditions must hold for $a=(n,j,-)$.

In fact the $\mathcal{F}$-analyticity of f implies that $F_f^{(a)}$ are bounded at ∞ and therefore the $\hat{f}^{(a)}$ have simple decay properties at ∞ which show that the integrals (5.6) converge : since they provide the coefficient of the leading term in the decay of $\hat{\Phi}^{(a)}$ at ∞ and this leading term is not square integrable they must vanish.

Assuming (5.6) and defining the powers of the complex numbers by cutting the plane C so that arg $z \in (-\pi,\pi]$, say, (5.4), (5.5) can be written in a form which is suited to see the analyticity properties of $\Phi^{(a)}$. If Λ, Λ' are contours in $Q^{(a)}(\xi)$ linking 0 or ∞ with z and staying in the same quadrant as z itself (if z>0 it belongs to the first quadrant, if z <0 it belongs to the second) then , if $a=(u,j)$:

$$F_\Phi^{(a)}(z) = -\left((1+z^2)/z\right)^{(1+u)/2} \int_0^z \left(\zeta/(1+\zeta^2)\right)^{(1+u)/2} F_f^{(u,j)}(\zeta)\,d\zeta/\zeta$$

$$= \left((1+z^2)/z\right)^{(1+u)/2} \int_z^\infty \left(\zeta/(1+\zeta^2)\right)^{(1+u)/2} F_f^{(u,j)}(\zeta)\, d\zeta/\zeta \tag{5.7}$$

and if $a=(n,j,+)$:

$$F_\Phi^{(a)}(z) = -\left((i+z)^2/z\right)^{(1+n)/2} \int_0^z \left(\zeta/(i+\zeta)^2\right)^{(1+n)/2} F_f^{(n,j,+)}(\zeta)\,d\zeta/\zeta$$

$$= \left((i+z)^2/z\right)^{(1+n)/2} \int_z^\infty \left(\zeta/(i+\zeta)^2\right)^{(1+n)/2} F_f^{(n,j,+)}(\zeta)\, d\zeta/\zeta \tag{5.8}$$

where the integrals are along the paths Λ or Λ'; a similar expression holds for $a=(n,j,-)$.

It is easy to see that the (5.7),(5.8) define functions on $(C/[(-B(\xi)\cup B(\xi)\cup\{\text{real and imaginary axes}\}])$ or on$(C/[B(\xi))\cup\{\text{real and imaginary axes}\}])$, by analytically continuing their values on the real axis or on the first quadrant. However as a consequence of the (5.6) and of the boundedness at infinity of the F-functions, it is easily seen that there are no discontinuities in the values that the functions take at the two sides of the cuts : so the (5.7),(5.8) do actually define holomorphic functions in $Q^{(a)}(\xi)$,(i.e. their values on the real axis or on the first quadrant can be continued in a single valued way).

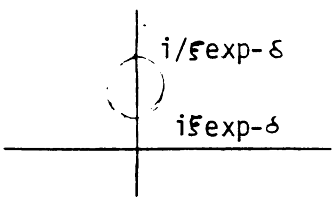
Therefore $\Phi^{(a)}$ are ξ-analytic for all a. The same argument can be applied to prove the strong ξ-analyticity of $\Phi^{(a)}$ if f is supposed strong ξ-analytic. However we need bounds on the size of $\Phi^{(a)}$ to conclude something about the analyticity of Φ.

Consider first $a=(u,j)$.

Let $z\in Q^{(a)}(\xi e^{-\delta})$ and observe that since $F_{\mp}^{(a)}$ is holomorphic in $Q^{(a)}(\xi)$ the value of $F_{\mp}^{(a)}(z)$ will be bounded by the maximum of $F_{\mp}^{(a)}(z)$ on the two circles forming the boundary of $Q^{(a)}(\xi)$, i.e. $\partial B(\xi e^{-\delta})\cup$ $\cup\partial B(\xi e^{-\delta})$.

Let z_0 be a point in, say, the upper circle.

We define a contour from $+i\infty$ to $i\xi e^{-\delta}$ going down the imaginary axis to $i\xi^{-1}e^{\delta}$ and then going around the circle $-B(\xi e^{-\delta})$ counterclockwise to $i\xi e^{-\delta}$: we call it Λ_∞ .



Define Λ_0 as the contour from 0 to $i\xi^{-1}e^{\delta}$ going up along the imaginary axis to $i\xi e^{-\delta}$ and then followimg counterclockwise the boundary of $-B(\xi e^{-\delta})$ up to $i\xi^{-1}e^{\delta}$.

There are two cases : either $z_0\in\Lambda_\infty$ or $z_0\in\Lambda_0$. In the first case we write (by (5.7)):

$$F_{\Phi}^{(a)}(z_0) = (\Lambda_\infty) \int_{z_0}^{\infty} ((1+z)^2/z)^{(1+u)/2} \, (\zeta/(1+\zeta^2))^{(1+u)/2} \; F_f^{(a)}(\zeta) \, d\zeta/\zeta \tag{5.10}$$

in the second case :

$$F_{\Phi}^{(a)}(z_0) = (\Lambda_0) \int_{0}^{z_0} \text{(same function as in (5.10))} \; d\zeta/\zeta \tag{5.11}$$

where the integrals are along the contours marked in parenthesis before the integral signs.

Along the integration paths the argument $\theta(\zeta)$ of $\zeta/(1+\zeta^2)$ relevant for the u dependent powers in (5.10),(5.11) is a monotonically decreasing function of ζ so that the u-dependent part verifies the inequality,

$$\left| ((1+z_0^2)/z_0)^{(1+u)/2} \, (\zeta/(1+\zeta^2)\,)^{(1+u)/2} \right| = \left| \frac{(1+z_0^2)\,\zeta}{z_0(1+\zeta^2)} \right|^{(1+\mathrm{Re}\ u)/2} .$$

$$\exp-(\mathrm{Im}\ u)(\,\theta(\zeta)-\theta(z_0))/2 \quad < \quad \left| \frac{1+z_0^2}{1+\zeta^2} \, \frac{\zeta}{z_0} \right|^{(1+\mathrm{Re}\ u)/2} \tag{5.12}$$

Therefore estimating the integrals :

$$\int \left| \frac{1+z_0^2}{1+\zeta^2} \; \frac{\zeta}{z_0} \right|^{(1+\mathrm{Re}\ u)/2} d|\zeta|/|\zeta| \qquad \text{along } \Lambda_0 \text{ or } \Lambda_\infty \tag{5.13}$$

by $M_4(\xi\hat{\delta}\,)^{-(1+\mathrm{Re} u)/2}$,we find, using (4.28) :

$$\left| F_{\Phi}^{(a)}(z_0) \right| < M_3(\,\xi\hat{\delta})^{-m_3} M_4(\hat{\delta}\,\xi)^{-(1+\mathrm{Re}\ u)/2} \; \| f^{(a)} \|_\xi \tag{5.14}$$

$$\left| F_{\Phi}^{(a)}(z\,) \right| < M_5 \,(\xi\hat{\delta}\,)^{-m_5} \, e^{v_3 \xi} \quad A^{(a)}(z, \xi\, e^{-\delta}) \, \| f^{(a)} \|_\xi \tag{5.15}$$

where the second inequality follows from the arbitrariness of δ provided $z \in RH(\,\xi\, e^{-\delta})$;and the constants are u-independent because Re $u \neq 0$ only for finitely many values of u.

The (5.15) implies by lemma 1 that

$$||\hat{\Phi}^{(a)}||_{\xi e^{-\delta}} < M_6(\xi\hat{\delta})^{-m_6} e^{v_6\xi|u|} ||f^{(a)}||_\xi \qquad (5.16)$$

where the constants M_6, v_6, m_6 are suitable numerical constants.

Consider now $a=(n,j,+)$.

We shall prove a better inequality (with respect to (5.15)):

$$|F_{\Phi}^{(a)}(z)| < M_7 (\log \xi^{-1}) A^{(a)}(z,\xi) ||f^{(a)}||_\xi \qquad (5.17)$$

which by lemma 1 implies :

$$|| \Phi^{(a)}||_{\xi e^{-\delta}} < M_8(\xi\hat{\delta})^{-m_8} ||f^{(a)}||_\xi \qquad \delta > 0 \qquad (5.18)$$

Then the (5.18) and (5.16), together with lemma 3 and the estimate on the multiplicities of the automorphic forms of given order and of the degeneracy of the eigenvalues of the Laplace-Beltrami operator, imply (3.10) with $\bar{\delta}$ chosen so that $v_2 \bar{\delta} < v_6 e^{-\bar{\delta}}$.

To derive (5.17) we note that it is equivalent, $(n+1)/2$ being an integer, to:

$$|\hat{\Phi}^{(a)}(z)| < M_7(\log\xi^{-1})||f^{(a)}||_\xi A'(z,a,\xi) \qquad (5.19)$$

having denoted $A'(z,a,\xi) = A^{(a)}(z,\xi)|(i+z)|^{(n+1)} = \{((1-\xi^2)/\xi[|z+(\xi^{-1}+\xi)/2|^2 - ((\xi^{-1}-\xi)/2)^2]\}^{(n+1)/2} (n/4\pi)^{1/2}$, and :

$$\hat{\Phi}^{(a)}(z) = -\int_0^z (\zeta/z)^{(n+1)/2} \hat{f}^{(a)}(\zeta)\, d\zeta/\zeta = \int_z^\infty (\zeta/z)^{(n+1)/2} \hat{f}^{(a)}(\zeta)\, d\zeta/\zeta \qquad (5.20)$$

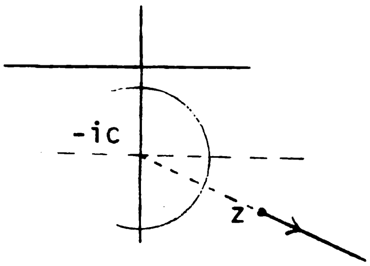
along contours in $Q^{(a)}(\xi)$.

Let $c=(\xi^{-1}+\xi)/2$, $r=(\xi^{-1}-\xi)/2$. We distinguish four cases .

Case 1) : $\text{Im } z \le -c$.

We choose, in (5.20), to integrate along the path :

$$\zeta(t) = (z+ic)\,t -ic \qquad t\in R_+ \tag{5.21}$$



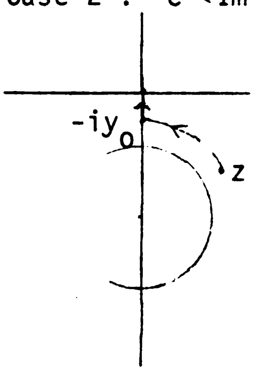
Setting $x=|z+ic|$ and using the bounds given by (4.22),(4.27) , and $(x^2t^2-r^2)^{-1} <$ $< (x^2-r^2)^{-1}\, t^{-2}$ and $|\zeta(t)/z| < t$, for all $t > 1$:

$$|\hat{\Phi}^{(a)}(z)| \le \left(\int_1^\infty |\zeta(t)/z|^{(n+1)/2} ((1-\xi^2)/\xi)^{(n+1)/2} (x^2t^2-r^2)^{-(n+1)/2}\, dt/t\right)\sqrt{n/4\pi}$$

$$\le ||f^{(a)}||_\xi \,((1-\xi^2)/\xi(x^2-r^2))^{-(n+1)/2} \left(\int_1^\infty t^{-(n+1)/2}\, dt/t\right)\sqrt{n/4\pi} \tag{5.22}$$

$$\le (2/n+1)\,||f^{(a)}||_\xi\, A'(z,a,\xi)$$

Case 2 : $-c < \text{Im } z$, $|z+ic| \le c$.

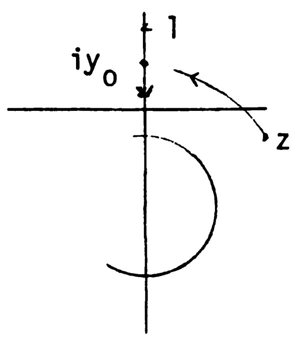


In this case we choose the path leading to the imaginary axis above -i along a circle with center in c and ending at $-iy_0$, then continue to zero along the imaginary axis.

Then the (4.22),(4.27) imply:

$$|\hat{\Phi}^{(a)}(z)| \le \pi A'(z,a,\xi)\,||f^{(a)}||_\xi + ||f||_\xi A'(z,a,\xi) \int_0^{y_0} (y/y_0)^{(n+1)/2} dy/y \le$$
$$\le (\pi + 2/(n+1))\, A'(z,a,\xi)\,||f^{(a)}||_\xi \tag{5.23}$$

Case 3 : $-c < \text{Im } z$, $c < |z+ic| \le c+1$



We move on a circle centered at -ic and leading upwards from z to the imaginary axis at iy_0 and then we go down to 0 along the imaginary axis.

As before the part contributed by the first piece of the contour to $\hat{\Phi}^{(a)}(z)$ will be bounded by $\pi A'(z,a,\xi)\ ||f^{(a)}||_{\xi}$.

The second part contributes :

$$\int_0^{y_0} (y/|z|)^{(n+1)/2}\, A'(iy,a,\xi)\,||f^{(a)}||_{\xi}\, dy/y \tag{5.24}$$

and using the inequality $y^{(n+1)/2}\, A'(iy,a,\xi) < y_0^{(n+1)/2}\, A'(iy_0,a,\xi)$, valid if $0 < y < y_0 < 1$, and $|z| > y_0$ and $A'(iy_0,a,\xi) = A'(z,a,\xi)$ we see that for all p between 0 and $(n+1)/2$:

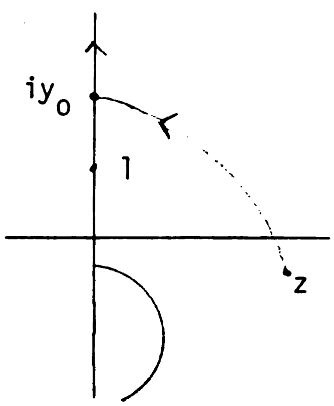
$$|\hat{\Phi}^{(a)}(z)| \leq ||f^{(a)}||_{\xi} A'(z,a,\xi) \left(\pi + \int_0^{y_0} \left(\frac{y}{y_0}\right)^{(n+1)/2} \frac{A'(iy,a,\xi)}{A'(iy_0,a,\xi)}\, dy/y \right)$$

$$\leq ||f^{(a)}||_{\xi}\, A'(z,a,\xi) \left(\pi + \int_0^{y_0} \left(\frac{y}{y_0}\, \frac{1+2cy_0+y_0^2}{1+2cy+y^2}\right)^p \frac{dy}{y}\right) \leq \tag{5.25}$$

$$\leq ||f^{(a)}||_{\xi}\, A'(z,a,\xi)\left(\pi + \min_{0<p<\frac{n+1}{2}} (2(1+c))^p \int_0^{y_0} \left(\frac{y}{y_0}\right)^p \frac{dy}{y} \right) \leq$$

$$\leq \quad ||f^{(a)}||_{\xi} A'(z,a,\xi)(\pi + \ e \log(2+2c))$$

Case 4 $|z+ic| > c+1$, $\operatorname{Im} z > -c$.



In this case we draw a path as in case 3 except that from iy_0 we proceed to $+i\infty$ along the imaginary axis. We find, using $A'(iy_0,a\ ,\xi) = A'(z,a,\xi)$ and the inequality $y^{(n+1)/2}\ A'(iy,a,\xi) < y_0^{(n+1)/2} A'(iy_0,a,\xi)$ if $1 < y_0 < y$, and using also the remark that in the same region of values of y :

$$y^{(n+1)/2}\ A'(iy,a,\xi)/\ y_0^{(n+1)/2}\ A'(iy_0,a,\xi) < \ 2(1+c)y_0/y \tag{5.26}$$

we find :

$$|\hat{\Phi}^{(a)}(z)| < ||f^{(a)}||_\xi \, A'(z.a,\xi) \, (\pi + \min_{0<p<\frac{n+1}{2}} \int_{y_0}^{\infty} \left(\frac{y(1+2cy_0+y_0^2)}{y_0(1+2cy+y^2)} \right)^p \frac{dy}{y}) <$$

$$< ||f^{(a)}||_\xi A'(z,a,\xi) \, (\pi + e \log 2(1+c)) \tag{5.28}$$

So that, collecting (5.28),(5.25),(5.23),(5.22), we obtain (5.18) and complete the proof of proposition 4.

We now prove proposition 1.

The action invariants are the same for h_0+f_0 and for h_n+f_n, of course.

Recalling that f_n is divisible by ε^{2^n} we see that the coefficients of the Taylor expansions in ε of the individual actions are explicitely computable up to order $O(\varepsilon^{2^{n+1}})$ by a simple calculation : the condition of constancy simply means that if (E,k,ε) denotes a closed periodic orbit of energy E for the hamiltonian H_ε then :

$$T(E,k,\varepsilon)^{-1} \int_{(E,k,\varepsilon)} \left[\frac{f_n(ge^{-\sigma_z(\det g)t})}{h_n'(E,\varepsilon)} \right] dt = \{\text{function of } E,\varepsilon\} + O(\varepsilon^{2^{n+1}}) \tag{5.29}$$

for $g \in (E,k,\varepsilon)$: i.e. the r.h.s of (2.19) does not depend on k up to terms of order 2^{n+1} in ε . Therefore the r.h.s. of (5.29) can be computed by letting k tend to ∞ and by using some general results [11] of ergodic theory which tell us that the limit must be the average of the function in square brackets of (5.29), up to terms of order 2^{n+1} ,i.e:

$$\int \left[\frac{f_n(g)}{h'(E,\varepsilon)} \right]^{[< 2^{n+1}]} \mu_E(dg) \tag{5.30}$$

Therefore the constancy of the action invariants can be written ,by (5.29),(5.30),as

$$\int \left[\frac{f_n(g)-\overline{f}_n(E)}{h'(E,\varepsilon)} \right]^{[< 2^{n+1}]} \mu_E(dg) = 0 \tag{5.31}$$

Therefore the constancy of the action invariants just yields the conditions necessary for the integrability of the equations defining the n-th step in the construction described in #3 to build the canonical transformation conjugating H_ε with a function of H_0, i.e. it permits, in the language of the proof of # 3 to define h_{n+1} , f_{n+1} in terms of h_n, f_n and, therefore, proposition 1 appears as a corollary of the proof of proposition 2.

6 Mixing rates for the geodesic and the horocyclic flows. Concluding remarks.

Consider the geodesic and the horocyclic flows on the unit cotangent bundle of our surface of constant negative curvature: in the preceding sections we have shown that the geodesic flow can be described as a flow on the space $T = \Gamma\backslash PSL(2,\mathbb{R})$ using the matrices $g_1(t) = \exp -\sigma_z t/2$; the flow is :

$$t : \ g \to g g_1(t) \qquad g \in T \tag{6.1}$$

Similarly the horocyclic flow can be described in T in terms of the matrices $g_0(t) = \exp t\sigma_+$, where $\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$: the flow is

$$t : \ g \to g g_0(t) \qquad g \in T \tag{6.2}$$

Let f be a ξ-analytic function on T with zero average with respect to the natural invariant measure dg . We want to study the quantity:

$$M(f,t;j) = \int_T \overline{f(g)}\, f(g g_j(t)) \ (dg) \qquad j = 0,1 \tag{6.3}$$

when $t \to \infty$.

By general results of ergodic theory it is known to tend to zero. We prove:

PROPOSITION 5 : There exist two functions $C(\xi), b(\xi)$ such that for all ξ-analytic functions f it is :

$$|M(f,t,j)| < \ ||f||_\xi^2 \, C(\xi). \quad \begin{matrix} t^{b(\xi)} \exp -t/2 & j=1 \\ (\log t)^{b(\xi)} \ t^{-1} & j=0 \end{matrix} \tag{6.4}$$

for all $t > 2$. The above bounds are optimal to leading order in the t-dependence.

Observation : note that even in the geodesic flow case this decay is not exp-(exponential in t) as it is the case in the toral automorphisms of the Anosov type. In other words the mixing is "pretty weak".

proof : let $f = \sum_{a \in A} f^{(a)}$. Clearly we have to study the functions $M(f^{(a)},t) = M(t)$ for each fixed $a \neq 0$.

We use the Fourier transform and the bounds (4.22), (4.28) to obtain when $a=(u,j)$, $u=is$:

$$\begin{aligned} M(t) &= \left|\int_{-\infty}^{+\infty} \overline{\hat{f}^{(a)}(x)}\ \hat{f}^{(a)}(xe^{-t})\ e^{-t(1+is)/2}\, dx\right| \\ &\le \left(M_3(\xi/2)^{-m_3}\right)^2 \,||f^{(a)}||^2_{\xi/2} \int_{-\infty}^{+\infty}(1+x^2)^{-1/2}(1+x^2e^{-2t})^{-1/2}\, e^{-t/2}\, dx \\ &\le M_3'\ \xi^{-m_3'}\ t\ e^{-t/2}\ \ ||f^{(a)}||_{\xi/2} \end{aligned} \tag{6.5}$$

A similar calculation yields the same results in the case of the discrete and of the supplememtary series.

Lemma 3,#4, is then used to preform the sum over the indices a, taking of course into account the multiplicity estimates given by the duality theorem, and one finds the first of (6.4).

The horocyclic flow can be studied in the same fashion and we leave the details to the reader.

The optimality statement relies on the consideration of special examples . Consider the function $E^{(a)}$ introduced in (4.9'); then using the (4.14) we can express in terms of elementary integrals the function $M(E^{(a)},t)$: then one studies such integrals explicitly. The slowest decay is given by the elements with $a=(is,j)$, i.e. by the functions associated with the principal series, and their M functions decay as $\exp -t/2$ times an oscillating function of t.

For instance :

$$M(E^{(is,j)},t) = \int_{-\infty}^{+\infty} \sqrt{(1+x^2)^{-(1-is)}}\sqrt{(1+x^2e^{-2t})^{-(1+is)}}\, e^{-(1+is)t/2}\, dx \tag{6.6}$$

A first concluding remark is that Proposition 4 can be regarded as a regularity theorem for the solutions of the (3.8), which is the equation studied by Livscic,Guillemin and Kazhdan (whose results become,in our case,the proposition 3).

A second remark is that the question that we have studied about the canonical conjugability, is the same question that, when asked for the perturbations of the integrable systems in the classical sense, leads to the KAM theory.

We see that the perturbations of the geodesic flows on surfaces of constant negative curvature are in some sense better than the ones of integrable systems. There are cases ("Birkhoff series") when the latter have a well defined perturbation series which, however, is known to be divergent. This phenomenon cannot happen in the cases that we have studied in this paper.

Appendix A : Estimates (3.42).

The estimate for E_{n+1} obviously follows from the first of (3.41).

To estimate ε_{n+1} we write (with $\Delta = \Delta(g')$) :

$$f_{n+1}(g',\varepsilon) \equiv h_n(H_0(g'+\Delta\ ),\varepsilon) - h_n(H_0(g'),\varepsilon) + f_n(g'+\Delta,\varepsilon) -$$

$$- [\overline{f}_n(H_0(g'),\varepsilon)]^{[<2^{n+1}]} = \text{(by the (3.38) =}$$

$$= h_n(H_0(g'+\Delta),\varepsilon) - h_n(H_0(g'),\varepsilon) - h_n'(H_0(g'),\varepsilon)\ \{H_0, \Phi\}(g) +$$

$$+ f_n(g'+\Delta\ ,\varepsilon) - f_n^{[<2^{n+1}]}(g',\varepsilon) + \overline{f}_n^{\,[<2^{n+1}]}(H_0(g'),\varepsilon) +$$

$$+ r_n(g',\varepsilon) - \overline{f}_n(H_0(g'),\varepsilon)^{[<2^{n+1}]} \qquad =$$

$$= \{\ h_n(H_0(g'+\Delta,\varepsilon)-h_n(H_0(g'),\varepsilon) - h_n'(H_0(g'),\varepsilon)\ \{H_0,\Phi\}\ (g')\ \} +$$

$$+ \{\ [f_k(g'+\Delta,\varepsilon)-f_k(g',\varepsilon)]^{[<2^{n+1}]*}\}+ \{\ f_n(g'+\Delta,\varepsilon)^{[<2^{n+1}]*}\} + \{r_n(g',\varepsilon)\}$$

where the $[.]^*$ denotes the truncation with respect to the powers of ε disregarding the ε-dependence of Δ itself; the last four functions in curly brackets will be respectively denoted $\tilde{f}$, f^{III}, f^{IV}, f^V so that :

$$f_{n+1}(g',\varepsilon) = \tilde{f} + f^{III} + f^{IV} + f^V$$

The function $\tilde{f}$ has to be elaborated a little more.

We introduce the following notations, temporarily : $g'=(\underline{p}',\underline{q}')$, $g=(\underline{p},\underline{q})$, $\underline{\Delta} = (\underline{\Delta}_1,\underline{\Delta}_2)$ so that $C^{(n)}$ becomes :

$$(\underline{p},\underline{q}) = (\underline{p}',\underline{q}') + (\ \underline{\Delta}_1,\underline{\Delta}_2) \tag{A.3}$$

Then,using $\underline{\Delta}_1(\underline{p}',\underline{q}')= \partial\Phi(\underline{p}',\underline{q})/\partial\underline{q}$, $\underline{\Delta}_2(\underline{p}',\underline{q}')=-\partial\Phi(\underline{p}',\underline{q})/\partial\underline{p}'$:

$$\{H_0,\Phi\}(\underline{p}',\underline{q}') = \left(\frac{\partial H_0}{\partial p'} \frac{\partial \Phi}{\partial \underline{q}'} - \frac{\partial H_0}{\partial \underline{q}'} \frac{\partial \Phi}{\partial \underline{p}'} \right) =$$

$$= \left(\frac{\partial H_0}{\partial p'} \underline{\Delta}_1 + \frac{\partial H_0}{\partial \underline{q}'} \underline{\Delta}_2 \right) + \left[\frac{\partial H_0}{\partial \underline{p}'} \left(\frac{\partial \Phi}{\partial \underline{q}'} - \frac{\partial \Phi}{\partial \underline{q}'} (\underline{p}',\underline{q}' + \underline{\Delta}_2) \right) - \right. \tag{A.4}$$

$$\left. - \frac{\partial H_0}{\partial \underline{q}'} \left(\frac{\partial \Phi}{\partial \underline{p}'} + \frac{\partial \Phi}{\partial \underline{p}'} (\underline{p}',\underline{q}'+\underline{\Delta}_2) \right) \right] \equiv (K_1)+[K_2]$$

where the functions whose arguments are not explicitly written are to be thought of as computed at $(\underline{p}',\underline{q}')$ and the function K_1, K_2 are introduced at the last step to shorten the notation (and they are naturally identified with the two main brackets in the intermediate term of (A.4)). So:

$$f(g',\varepsilon) = \{ h_n(H_0(\underline{p}'+\underline{\Delta}_1,\underline{q}'+ \underline{\Delta}_2),\varepsilon) - h_n(H_0(\underline{p}',\underline{q}'),\varepsilon) -$$

$$-h_n'(H_0(\underline{p}',\underline{q}'),\varepsilon)\ K_1\} + \{h_n'(H_0(\underline{p}',\underline{q}'),\varepsilon)\ K_2\} \equiv \{f^I\} + \{f^{II}\} \tag{A.5}$$

We now proceed to successively estimate f^I ./. f^V in $W(\rho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n})$.

By the (3.40) :

$$||f^V|| < A_1\, e^{-\tau_n 2^{n+1}}\, \varepsilon_n\, \hat{\tau}_n^{-1} \tag{A.6}$$

By a dimensional estimate :

$$||f^{IV}|| < A_2\, \hat{\tau}_n^{-1}\, \lambda_n\, e^{-\tau_n 2^{n+1}} \tag{A.7}$$

Again by a dimensional estimate, using also (3.35) :

$$||f^{III}|| < A_3'\, \varepsilon_n\, \hat{\tau}_n^{-1}\, (\hat{\delta}_n^{-1}\, \hat{\sigma}_n^{-1})||\Delta|| <$$

$$< A_3\, \varepsilon_n^2\, (\hat{\sigma}_n\, \hat{\tau}_n\, \hat{\delta}_n\, \xi_n)^{-a_3} \tag{A.8}$$

where A_3,a_3 are suitable constants and the exponents have been uniformized to simplify the notations.

Another dimensional estimate, plus (3.35),(3.31), show that :

$$||f^{II}|| < A_4' \, E_n B_1 (\xi_n \hat{\delta}_n)^{-q} \, \hat{\tau}_n^{-1} \, \varepsilon_n \, (\xi_n \hat{\delta}_n \hat{\tau}_n)^{-2} ||\Delta|| <$$
$$< A_4 \, \varepsilon_n^2 \, (\hat{\tau}_n \hat{\sigma}_n \hat{\delta}_n \xi_n)^{-a_4} \tag{A.8}$$

And , applying appropriatly the Taylor's formula to second order to exhibit the $||\Delta||^2$ factor :

$$||f^{I}|| < A_5' \, E_n (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-2} \, ||\Delta||^2 < \tag{A.9}$$

hence :

$$||f^{I}|| < A_5 (\xi_n \hat{\delta}_n \hat{\tau}_n \hat{\sigma}_n)^{-a_5} \, \varepsilon_n^2 \tag{A.9}$$

To estimate the derivatives we observe that the above estimates imply on $W(\rho_n e^{-4\sigma_n}, \xi_n e^{-4\delta_n}, \theta_n e^{-4\tau_n})$:

$$||f^{I}+f^{II}+f^{III}+f^{V}|| < A_6 (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-a_6} \, (\varepsilon_n^2 + \varepsilon_n e^{-\tau_n 2^{n+1}}) \tag{A.10}$$

we have on $W(\rho_n e^{-5\sigma_n}, \xi_n e^{-5\delta_n}, \theta_n e^{-5\tau_n})$:

$$\left|\frac{\partial (f^{I}+f^{II}+f^{III}+f^{V})}{\partial g}\right| < A_7 (\xi_n \hat{\delta}_n \hat{\sigma}_n)^{-a_7} \, (\varepsilon_n^2 + \varepsilon_n e^{-\tau_n 2^{n+1}}) \tag{A.11}$$

with $a_7 = a_6 + 1$. Also :

$$\frac{\partial f^{IV}}{\partial g'_{ij}}(g'+\Delta,\varepsilon) = \frac{\partial f^{IV}}{\partial g'_{ij}}(g'+\Delta,\varepsilon) + \sum_{h,k=1}^{\ell} \frac{\partial f^{IV}}{\partial g'_{hk}}(g'+\Delta,\varepsilon) \frac{\partial \Delta_{hk}}{\partial g'_{ij}} \tag{A.13}$$

so that, using (3.35) and the fact its r.h.s. is not larger than 1 :

$$\left|\left|\frac{\partial f^{IV}}{\partial g'}\right|\right| < A_8 \, \varepsilon_n e^{-\tau_n 2^{n+1}} \, \hat{\tau}_n^{-1} (1+1) \tag{A.14}$$

Clearly (A.14) and (A.11) prove the second of (3.42) while the third follows from (A.7) .

Appendix B : Proof of lemmas 1,2,4.

Consider first the case $a=(u,j)$. By assumption the functions $h \to \to (\phi, U(h)f)$ defined for $h \in \mathrm{Re}\, H(\xi)$ at given $f \in L_2$, can be holomorphically extended to $H(\xi)$ and, by Schwartz' inequality, verify:

$$|(\phi, U(h)f)| < \quad ||\phi||_2 \; ||f||_\xi \tag{B.1}$$

We choose ϕ to be an element $\phi_{x_0} \in Y^{(a)}$ such that for all $x_0 \in R$:

$$\hat{\phi}_{x_0}(x) = e^{-(x-x_0)} \chi(x-x_0) \qquad \text{if } u=is \tag{B.2}$$

where $\chi(x)=0$ if $x<0$ and $\chi(x)=1$ otherwise, or such that :

$$\int_{-\infty}^{+\infty} \hat{\phi}_{x_0}(y)|x-y-x_0|^{s-1}\, dy = e^{-(x-x_0)} \chi(x-x_0) \qquad \text{if } u=s \tag{B.3}$$

i.e. :

$$\hat{\phi}_{x_0}(y) = \frac{1}{2\pi}\int e^{-i(y-x_0)p} \frac{|p|^s \, dp}{(1-ip)(2\cos\pi s/2)((s-1)!))} \tag{B.4}$$

A simple calculation yields that the value of the L_2 norms in $Y^{(a)}$ of the above functions are

$$1/\sqrt{2} \qquad \text{if } u=is, \quad (4(\cos\pi s/2)^2((s-1)!))^{-1/2} \qquad \text{if } u=s \tag{B.5}$$

If $h \in \mathrm{Re}H(\xi)$, $h=\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}$, $\alpha=(1+bc)/d$, it is easy to check the identity :

$$\left(b\frac{\partial}{\partial d} + \alpha\frac{\partial}{\partial c} + 1\right)(\phi_{x_0}, U(h)f) \equiv \hat{f}^{(a)}(x_0 h)(j(x_0,h)^2)^{-(1+u)/2} \tag{B.6}$$

This proves that the r.h.s. , i.e. the function $(U(h)f^{(a)})(x_0)$ is a holomorphic function of h for $x_0 \in R$.

The (B.6) says more : in fact multiplying both sides by $(1+(x_0 h)^2)^{(1+u)/2}$ we see that :

$$F_f^{(a)}(x_0) = (\hat{U}^{(a)}(h)\hat{f}^{(a)}(x_0))(1+(x_0 h)^2)^{(1+u)/2}(j(x_0,h)^2)^{(1+u)/2} =$$

$$= (U(\hat{h})f^{(a)})(x_0)((\alpha x_0+c)^2+(bx_0+d)^2)^{(1+u)/2} \tag{B.7}$$

But since $\alpha\simeq 1, d\simeq 1, c\simeq 0, b\simeq 0$, recall that ξ is supposed small ($\xi<1/10$), it is easy to check that the quantity in the last bracket can be bounded as f varies in $H(\xi)$, by :

$$\begin{aligned} &(1+x_0)^2(1-m'\xi) < \mathrm{Re}\,((\alpha x_0+c)^2+(bx_0+d)^2) < (1+x_0)^2(1+m'\xi) \\ &\arg((x_0+c)^2+(bx_0+d)^2) < m''\xi \end{aligned} \tag{B.8}$$

for all real x_0 ; for simplicity we may and shall assume $\xi_0 < 1/2m'$.

This means that $F_f^{(a)}(x_0h)$ is holomorphic in h, i.e. in b,c,d,as h varies in $H(\xi)$, for all x_0 in R.

If $x>0$ is large enough the point xh varies in a neighborhood of ∞ covering it as h varies in $H(\xi)$: hence $F_f^{(a)}$ can be holomorphically extended from the positive real axis to a vicinity of ∞ . Also it happens that , under the same circumstances, xh covers a real neighborhood of ∞ as h varies in Re $H(\xi)$: therefore $F_f^{(a)}(-y)$, where y is large and positive, coincides with the analytic continuation of $F_f^{(a)}$ from the positive real axis through ∞ .

On the other hand the (B.7),(B.8) also show that $F_f^{(a)}(x)$ can be holomorphically extended to a strip around the whole real axis.

From the above considerations it follows that $F_f^{(a)}$ admits a holomorphic extension to the whole $RH(\xi)$, single valued in this multiply connected region.

We now use that $H(\xi e^{-\delta})$ has a boundary at a distance from the boundary of $H(\xi)$ which can be estimated to be no less than $\tilde{B}\xi\hat{\delta}$ if $\tilde{B}$ is a suitably small number, and $\hat{\delta}=(1-e^{-\delta})$. So (B.6) implies, by a dimensional estimate, the following bound:

$$|\hat{f}^{(a)}(xh)(j(x,h)^2)^{-(1+u)/2})| \equiv | F_f^{(a)}(xh)((\alpha x+c)^2+(bx+d)^2)^{-(1+u)/2} | \leq$$

$$\leq ||\phi_x||_2 ||f||_\xi \ (B\xi\hat{\delta})^{-1} \tag{B.9}$$

if B is a suitably large constant; the (B.9) holds for $h \in H(\xi e^{-\delta})$ and $x \in R$.

Let z be real and choose $h=\begin{pmatrix} \alpha & b \\ c & d \end{pmatrix} =(1-\xi^2)^{-1/2}\begin{pmatrix} 1 & \pm\xi \\ \pm\xi & 1 \end{pmatrix}$, $x=zh^{-1}$, so that the (B.9) gives :

$$| F_f^{(a)}(z) | < C_1 \ \xi^{-2-\mathrm{Re}\, u} \ ||f||_\xi \tag{B.10}$$

because $((\alpha x+c)^2+(bx+d)^2)^{-1} \equiv ((1+(xh)^2 \ j(x,h)^2)^{-1} \equiv ((1+z^2)j(z,h^{-1})^{-2})^{-1} \equiv$ $\equiv ((1+z^2)/((1+\xi|z|)^2 \ )^{-1}$ provided the arbitrary sign in h is appropriately chosen. The constant C_1 is bounded by sup $M'_{(u,j)}$ where M' are the constants in (B.5).

Consider now instead of f the function $U(h)f$, $h \in H(\xi e^{-\delta})$. Then $U(h)f$ is $\xi\hat{\delta}\, C_2$-analytic, if C_2 is a suitable small constant, for all $\delta > 0$ and therefore the (B.10) implies :

$$| F^{(a)}_{U(h)f}(x) | < C_3(\xi\hat{\delta})^{-2-\mathrm{Re}\, u} \ ||f||_\xi \tag{B.11}$$

or a suitably large C_3. Hence :

$$\begin{aligned} |\hat{f}(xh)(1+(xh)^2)^{(1+u)/2}| \ &< \ C_3 ||f||_\xi \ (\xi\hat{\delta})^{-2-\mathrm{Re}\, u} \\ &\left| \frac{((\alpha x+c)^2+(bx+d)^2)^{(1+u)/2}}{1+x^2} \right| \end{aligned} \tag{B.12}$$

so if z in $\mathbb{R}H(\xi)$:

$$|F_f^{(a)}(z)| < C_3 \ ||f||_\xi \ \xi^{-2-\mathrm{Re}\, u} \ \inf \ \hat{\delta}^{-2-\mathrm{Re}\, u} \left| \frac{((\alpha x+c)^2+(bx+d)^2}{1+x^2} \ )^{(1+u)/2} \right| \tag{B.13}$$

where the inf is taken over all the $\delta > 0$ such that $z \in \mathbb{R}H(\xi e^{-\delta})$ and over all the pairs $x \in \mathbb{R}$, $h \in H(\xi e^{-\delta})$ such that $xh=z$.

By (B.8) we see that :

$$A_\xi^{(a)}(z) > \quad C_3\ \xi^{-2-\mathrm{Re}\ u}\ e^{-m'\xi(\mathrm{Im}\ u)/2}(1-m'\xi)^{(1+\ \mathrm{Re}\ u)/2}\ \hat{d}(z,\xi)^{-2-\mathrm{Re} u} \tag{B.14}$$

To get an upper bound on $A_\xi^{(a)}$ we observe that if $z \in (\ \partial\mathbb{R}H(\xi e^{-\delta}) \cap C_+)$ then there exists $x_0 \in \mathbb{R}$ such that $x_0 h_0 = z$, with :

$$h_0 = (1- \quad \xi^2 e^{-2}\ )^{-1/2} \begin{pmatrix} 1 & i\xi e^{-\delta} \\ -i\xi e^{-\delta} & 1 \end{pmatrix} \tag{B.15}$$

and for this pair x_0, h_0 the number in the modulus sign of (B.13) is identically 1. Therefore we have also checked the (4.28), as a consequence of (B.14), (B.15), (recall here that u can take only finitely many real values so that the bound (4.28) has m_3 which can be chosen u-independent).

We now study the case a=(n,j,+).

In this case it follows from the holomorphy properties of the Fourier transforms (in the upper half plane) that for $\zeta \in C_+, h \in \mathrm{Re}H(\xi)$:

$$f(\zeta h) j(\zeta,h)^{-1-n} = (\ \phi_\zeta^{(a)}, U(h)f) \tag{B.16}$$

where

$$\hat{\phi}_\zeta^{(a)}(z) = \frac{(2i)^n}{-2\pi i}\ n\ (z- \bar{\zeta})^{-1-n} \qquad ||\phi^{(a)}||_2 = (\ \frac{n}{4\pi}\ \frac{1}{(\mathrm{Im}\ \zeta)^{1+n}}\ )^{1/2} \tag{B.17}$$

Therefore repeating an argument similar to the one given above to discuss the analyticity in the case of the representations with a=(u,j), we find from (3.10) that $\hat{U}^{(a)}(h)\hat{f}(z)$ can be holomorphically extended to $H(\xi)$ as a function of h at $z \in \mathbb{C}_+$ fixed and :

$$F_f^{(a)}(\zeta h) \;=\; \hat{U}^{(a)}(h)\hat{f}^{(a)}(\zeta)\;(\;(i+\zeta h)j(\zeta,h))^{1+n} \;= \\ \equiv (\hat{U}^{(a)}(h)\hat{f}^{(a)})(\zeta)\;(\;(\alpha\zeta+c)+i(b\zeta+d))^{1+n} \tag{B.18}$$

and this means that $F_f^{(a)}$ can be holomorphically extended to $C_+H(\zeta)$ (as well as $\hat{f}^{(a)}$ itself as, in these cases,(n+1)/2 is integer).

By arguments similar to the ones used in the cases a=(u,j) we find that the second of (B.17) and (B.18) imply :

$$|F_f^{(a)}(z)| \;\leq\; (n/4\pi)^{1/2}|\;(\;(i+z)^2 j(\zeta,h)^2/\mathrm{Im}\;\zeta)^{(n+1)/2}\;|\;||f||_\xi \tag{B.19}$$

for all $\zeta\in C_+$, $h\in H(\xi)$ such that $z=\zeta h$; this proves completely the first part of I) and ii) (since the discussion of the cases a=(n,j,-) can obviously be reduced to that of the cases a=(n,j,+)).

We now prove the second part of I) in lemma 1, i.e. that if $F_f^{(a)}$ can be holomorphically extended to $Q^{(a)}(\xi)$ then f is ξ-analytic.

If $F_f^{(a)}$ is holomorphic in $Q^{(a)}(\xi)$ then $F_f^{(a)}(z)$ is uniformly bounded in $Q^{(a)}(\xi e^{-\delta})$. Calling M_δ a bound for this function it follows that in the cases a=(u,j) (see (B.8)):

$$|\hat{U}^{(a)}\hat{f}^{(a)}(x)| \leq \quad M_\delta|(\;(\alpha x+c)^2+(bx+d)^2)^{-(1+u)/2}| \leq \quad M_\delta' \;\;(1+x^2)^{-(1+\mathrm{Re}\;u)/2} \tag{B.20}$$

if $h=\begin{pmatrix}\alpha & b\\ c & d\end{pmatrix}$, for a suitable M_δ' .

(B.20) shows, thanks to its uniformity in $h\in H(\xi e^{-\delta})$, that $\hat{U}^{(a)}(h)\hat{f}^{(a)}$ $\in \hat{Y}^{(a)}$ and for all $\phi\in\hat{Y}^{(a)}$ the function $h\to(\;\phi,\hat{U}^{(a)}(h)\hat{f}^{(a)})$ is holomorphic in $H(\xi e^{-\delta})$. Therfore f itself is holomorphic on $TH(\xi e^{-\delta})$, $\forall\;\delta>0$.*

An identical argument can be given in the cases $a=(n,j,\pm)$. So i) of lemma 1 is completely proved and it remains to prove iii).

We discuss now the proof of iii) in the case a=(u,j), the others being very similar .

Observe that (4.23),(4.28) imply that $U^{(a)}(h)f^{(a)}(x)$ is holomorphic in $h\in H(\xi)$ and, for all $h\in H(\xi e^{-\delta})$:

*) here we use a general property explicitly stated in lemma 5 below.

$$|U^{(a)}f^{(a)}(x)| < NC_3\xi^{-2-\mathrm{Re}\,u}\,\delta^{-2-\mathrm{Re}\,u}\,(1+x^2)^{-(1+\mathrm{Re}u)/2} \qquad \text{(B.21)}$$

so that :

$$||U^{(a)}(h)f^{(a)}||_2 < NC_4(\xi\delta)^{-2-\mathrm{Re}\,u} \qquad \text{for all } h\in H(\xi e^{-\delta}) \qquad \text{(B.22)}$$

which by (4.1) implies (4,24) (in this case $\sqrt{|a|}$ can be replaced by 1).

The cases $a=(n,j,\pm)$ are discussed in a identical way and the role of the (4.28)is now played by the (4.27.) which can be checked as follows.

Note that the h-image of $\mathbb{C}_+$ is the complement of a circle with center and radius R, and if $z=\zeta h$, $\zeta\in\mathbb{C}_+$:

$$|(\mathrm{Im}\,\zeta)j(\zeta,h)^{-2}| = ((|z-c|^2 - R^2)/2R) \qquad \text{(B.23)}$$

which can be seen by direct calculation or by suitable geometric arguments.

Furthermore the matrix :

$$h = (1-\xi^2)^{-1/2}\begin{pmatrix} 1 & i\xi \\ -i\xi & 1 \end{pmatrix} \qquad \text{(B.24)}$$

is on the boundary of $H(\xi)$ and maps $\mathbb{C}_+$ onto the complement of the circle $B(\xi)$ with :

$$c=(\xi^{-1}+\xi)/2 \equiv c_0 \quad , \qquad R=(\xi^{-1}-\xi)/2 \equiv R_0 \qquad \text{(B.25)}$$

Finally a circle G with center c and radius R which contains a circle with center c' and radius R' is such that for all z outside G :

$$(|z-c'|^2 - R'^2)/2R' \leq (|z-c|^2 - R^2)/2R \qquad \text{(B.26)}$$

as it is easily seen (e.g. remark that one can always suppose that G is the unit circle centered at the origin and that G' is a smaller circle centered at the point (0,c') and with radius R' such that $c'+R'\leq 1$, $c'>0$; then the above relation can be checked by simple considerations).

The same inequality holds if G and G' are two disjoint circles and z is inside G : this can be shown as in the preceding case or, alternatively, by noting that the inequality (B.26) is invariant with respect to the operation of inversion with respect to the circle G.

The above inequalities are exactly saying that the infimum over $H(\xi)$ is obtained by considering $h=h_\xi$ and this yields (4.27).

So lemma 1 and lemma 4 are completely proved.

The direct part of lemma 2 follows from the explicit formulae (B.6),(B.16) and from the uniform boundedness of $F_f^{(a)}$ in every compact subset of $Q^{(a)}(\xi)xW$.

The converse statement,appearing in lemma 2, follows from the argument after (B.20), guaranteeing that $h \to U(h)f$ regarded as a L_2-valued function defined on Re $H(\xi)$ can be extended holomorphically to $H(\xi)$, combined with the :

LEMMA 5 : Let $f \in L_2$ and let $h \to U(h)f$, regarded as a L_2-valued function defined for $h \in \mathrm{Re}\, H(\xi)$, be holomorphically extendible to a L_2-valued function on $H(\xi)$: then f is ξ-analytic.

If f depends parametrically on $\underline{w} \in W \subset \mathbb{C}^q$ and the L_2-valued function $(h,\underline{w}) \to U(h)f$, defined on $(\mathrm{Re}\, H(\xi))xW$, is holomorphically extendible to $H(\xi)xW$ then the function $(h,\underline{w}) \to f(gh)$ is also holomorphically extendible to $H(\xi)xW$ for all g.

In other words ξ-analyticity of f and holomorphy on $H(\xi)$ of the L_2-valued function $U(h)f$ are equivalent properties (note that the ξ-analyticity obviously implies that $h \to U(h)f$ is holomorphic as a vector valued function).

The above lemma has a well known"elementary" analogue in the case when $T= \Gamma\backslash PSL(2,\mathbb{R})$ is replaced by an m-dimensional torus $T^m= \mathbb{R}^m/Z^m$ and $PGL(2,\mathbb{R})$ by $\mathbb{R}^m$: in the latter case the proof, however, does not really use the commutativity of $\mathbb{R}^m$ and relies only on the Cauchy's theorem on holomorphic functions. Therefore the proof relative to the commutative case can be extended to yield a proof of lemma 5 or of an even more general version of it for groups of matrices.

Appendix C : Proof of lemma 3.

with the matrix notations of # 3,4 let $E(\underline{t}) = \exp(t_1\sigma + t_2\sigma_x + t_3\sigma_z)$. Let $\mathcal{D}$ be the Casimir operator on $L_2(T)$, for $U(.)$, given by [6]:

$$\mathcal{D}f(g) = -\frac{1}{4}\left(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_3^2}\right) f(g\,E(\underline{t}))\Big|_{\underline{t}=\underline{0}} \tag{C.1}$$

Observe that $|t_j^{(r)}| < \xi\hat{\delta}/4q$ implies $E(\underline{t}^{(q)}). \;\; . E(\underline{t}^{(1)}) \in H(\xi\delta/2)$ and that $H(\xi e^{-\delta}) \cdot H(\xi\hat{\delta}/2) \subset H(\xi)$: so that the function $U(h)f(g\, E(\underline{t}^{(q)}) .. E(\underline{t}^{(1)})$ is holomorphic in the $\underline{t}$ variables for $|t_j^{(r)}| < \xi\hat{\delta}/4q$. Therefore by a dimensional estimate and by the Schwartz inequality :

$$|(\phi, \mathcal{D}^q U(h)f)| < 4^{-q}\, 3^q\, 2^q ((4q/\xi\hat{\delta})^2)^q\, ||\phi||_2\, ||f||_\xi \tag{C.2}$$

i.e. for all $h \in H(\xi e^{-\delta})$:

$$||\mathcal{D}^q U(h)f\,||_2 < (6q/\xi\hat{\delta})^{2q} ||f||_\xi \tag{C.3}$$

But by orthogonality and by the fact that $\mathcal{D}$ has on $Y^{(a)}$ the constant value $V(a)=(1-u^2)/4$, if $a=(u,j)$, or $(1-n^2)/4$ if $a=(n.j,\pm)$:

$$V(a)^q\, ||U(h)f^{(a)}||_2 \equiv ||\mathcal{D}^q U(h)f^{(a)}||_2 < ||\mathcal{D}^q U(h)f||_2 < (6q/\xi\hat{\delta})^{2q} ||f||_\xi \tag{C.4}$$

so that for all $h \in H(\xi e^{-\delta})$:

$$||U(h)f^{(a)}||_2 < (6q/\xi\hat{\delta}\sqrt{V(a)}\,)^{2q} ||f||_\xi \tag{C.4}$$

which implies **lemma** 3 by optimizing this inequality on q and, finally, using (4.1).

Appendix D : A canonical map.

We rewrite (2.9), recalling that $g = \begin{pmatrix} p_1 & q_2 \\ -p_2 & q_1 \end{pmatrix}$:

$$p_x+ip_y = \frac{1}{2}\, i(p_1+iq_2)^2 \equiv p \qquad\qquad (D.1)$$
$$x+iy = \frac{p_2+iq_1}{p_1-iq_2} \equiv q$$

and since $p_x dx+p_y dy = \mathrm{Re}\, p\overline{dq}$ we find :

$$p_x dx+p_y dy = p_1 dq_1+p_2 dq_2 - \frac{1}{2} d(\det g) \qquad (D.2)$$

and

$$H_0(g) = \frac{1}{2}\, y^2(p_x^2+p_y^2) = \frac{1}{2}\,(\mathrm{Im}\, q)^2\, p^2 = \frac{1}{8}\,(\det g)^2 \qquad (D.3)$$

Note that the function $\det g$ is single valued on $\Gamma\backslash PGL(2,R)$, hence on the graph of the map (D.1).

Appendix E : An example of non global canonical map and the non sufficiency of the period and of the Lyapunov invariants.

Let $C_\varepsilon(\underline{p},\underline{q}) = (p',q')$ be defined as follows : put $p=p_x+ip_y$, $z=x+iy$, $p'=p'_x+ip'_y$ and let ϕ be an automorphic form of order 1, verifying (4.4) , then:

$$C_\varepsilon(p,z) = (p',z') \qquad \begin{aligned} p' &= p+\varepsilon\overline{\phi(z)} \\ q' &= q \end{aligned} \qquad (E.1)$$

Then $\mathrm{Re}\, p\overline{dz} = \mathrm{Re}\, p'\overline{dz'} + \varepsilon \mathrm{Re}\, \overline{\phi(z)dz}$: since ϕ is holomorphic the last differential form is closed (i.e. C is locally canonical) and since $\phi(z)$ is covariant, (4.4), while z is contravariant $(dz' = j(z,\gamma)^{-2}dz$ for all $\gamma \in \Gamma)$ the form $\phi(z)dz$ is defined on the whole Σ and (E.1) is well defined as a map of $T^*\Sigma$ into itself, i.e. C_ε is globally defined though not globally canonical.

The non globality of the canonical character follows from the fact that $\phi(z)dz$ is not an integrable form on Σ (because there are no non singular automorphic functions on Σ [7]) : hence for some closed curve on Σ we have $\oint \phi(z)dz \neq 0$ and we may suppose that we change ϕ by a suitable phase factor (constant on Σ) so that the value of the integral is actually positive. With such a choice (E.1) does not preserve the actions of closed orbits in $T^{*}\Sigma$.

Consider the hamiltonian:

$$H_0(\,C_\varepsilon^{-1}(\underline{p},\underline{q})) \equiv H_\varepsilon(\underline{p},\underline{q}) \tag{E.2}$$

Clearly H_ε cannot be conjugated with H_0 by a global canonical map : however it is conjugated to it by $(\underline{p}.\underline{q})= C_\varepsilon(\underline{p}',\underline{q}')$. This suffices for the conservation of the period and of the Lyapunov invariants.

Appendix F : hint to (B.6).

(B.6) is obtained as a consequence of the following formal identities:

$$\begin{aligned}(\phi_{x_0},U(h)f)= &\int_{x_0}^{\infty} e^{-(x-x_0)} f(xh)\ |j(x,h)|^{-(1+u)}\,dx = \\ \equiv &\int_{x_0 h}^{\infty h} e^{-(yh^{-1}-x_0)}\ f(y)\ |j(y,h^{-1})|^{\,(1+u)-2}\,dy\end{aligned} \tag{F.1}$$

Then one uses the following relations valid if $\alpha =(1+bc)/d$:

$$\begin{array}{lll}
1) & (b\ \partial/\partial d +\alpha\ \partial/\partial c)j(y,h^{-1}) \equiv 0 & \text{for all } y\\
2) & (b\ \partial/\partial d +\alpha\ \partial/\partial c\ + 1\ ) \exp-(yh^{-1}) \equiv & \text{for all } y\\
3) & (b\ \partial/\partial d +\alpha\ \partial/\partial c\ )(x_0 h) = j(x,h)^{-2} = j(y,h^{-1})^2 & \text{for } y=xh\\
4) & (b\ \partial/\partial d +\alpha\ \partial/\partial c\ )\infty h \equiv 0 &
\end{array} \tag{F.2}$$

Appendix G. Proof of i), iv) in proposition 4.

We shall deal throughout this section wth the strongly ξ-analytic functions although some of the results hold also for the ξ-analytic functions (as we shall mention).

The basic properties of their Fourier transforms are discussed exactly as in Section 4 with the few obvious changes that we list below.

Modify lemma 1 by replacing $\|f\|_\xi$ by $\|\tilde{f}\|_\xi$ and the words "ξ-analytic" by "strongly ξ-analytic" and the domains $Q^{(a)}(\xi)$ by $\tilde{Q}^{(a)}(\xi)$

$$Q^{(a)}(\xi) = \begin{array}{ll} C_+\tilde{H}(\xi) & a = (n,j,\pm) \\ \mathbb{R}\,\tilde{H}(\xi) & a = (u,j) \end{array} \tag{G.1}$$

and finally replace $A^{(a)}(Z,\xi)$ by the expressions $A^{(a)}(Z,\xi)$ obtained by (4.19), (4.20) with the infima over the $\zeta \in \mathbb{C}_+, h \in \tilde{H}(\xi)$, $\zeta h = Z$ or over $\delta > 0$, $x \in \mathbb{R}$, $h \in H(\xi e^{-\delta})$ (i.e. replace everywhere $H(\xi)$ by $\tilde{H}(\xi)$).
In this way we obtain a lemma which we shall call lemma 1' which is also true.

Changing lemmas 2,3 according to the same prescriptions as the ones used above leads to lemmas which we refer as lemma 2' , lemma 3' which, also, are true. The proofs of lemmas 1', 2', 3' are obtained from those of lemmas 1, 2, 3 by just replacing everywhere the words ξ-analytic by strongly ξ-analytic and the domain $H(\xi)$ by $\tilde{H}(\xi)$.

The lemma 4 also has an analogue: however since the scheme of proof used to prove proposition 4, i), iii), failed to give also the proof of ii), iv) because of the weakness of lemma 4 we do not mention the analogue to lemma 4.

We shall rather base the proof of the ii), iv) of proposition 4 on the following alternative necessary and sufficient condition of strong ξ-analyticity.

LEMMA 6: Let $f \in Y^{(a)}$, $a \in A$, $a \neq 0$. There exist three constants D, D' , q independent on f and a such that if f is strongly ξ-analytic then

i) the functions $(U(h)f)(z)$ defined by (4.13), for the real and z in the appropriate domain can be extended holomorphically in h , at fixed z, to $\tilde{H}(\xi)$.

ii) if $a = (u,j)$ and if $0 < \delta < \xi$, $h \in \tilde{H}(\xi - \delta)$

$$|(U(h)\hat{f})(x)| \leq D\delta^{-1} \|\tilde{f}\|_\xi \tag{G.2}$$

$$|U(h)\hat{f}(x)| \leq D\delta^{-1} (1 + \delta|x|)^{-1 - \mathrm{Re}u} \|\tilde{f}\|_\xi \tag{G.3}$$

iii) if $a = (n,j,\pm)$ then for all $\delta \in (0,\xi)$, $h \in H(\xi - \delta)$:

$$|U(h)\hat{f}(z)| \leq \sqrt{\frac{n}{4\pi}} \frac{D}{(\mathrm{Im}\ z)^{\frac{1+n}{2}}} \|\tilde{f}\|_\xi \tag{G.4}$$

$$|U(h)\hat{f}(z)| \leq \sqrt{\frac{n}{4\pi}} \frac{D}{(\mathrm{Im}\ z)^{\frac{n-1}{2}} \delta(1 + \delta^2|z|^2)} \|\tilde{f}\|_\xi \tag{G.5}$$

Vice versa if $f \in L_2$, $f \in Y^{(a)}$ and $(U(h)\hat{f})(z)$ can be extended holomorphically to $h \in \tilde{H}(\xi)$ and if the (F.2) - (F.5) hold with $\|\tilde{f}\|_\xi$ replaced by some N then f is strongly ξ-analytic and

$$\|\tilde{f}\|_{\xi - \delta} \leq D'\delta^{-q}N \tag{G.6}$$

Identical results hold if strongly ξ-analytic functions are replaced by ξ-analytic functions and if $\|\tilde{f}\|_\xi$ are replaced by $\|f\|_\xi$ and $\widetilde{H(\xi)}$ by $H(\xi)$.

The proof of lemma 6 is implicit in the proof of appendix B.

In fact i) follows from (B.6) for $a = (u,j)$ (see comment following (B.6)) and from (B.16) for $a = (n,j,\pm)$: of course the substitutions of $\|f\|_\xi$ by $\|\tilde{f}\|_\xi$ and $H(\xi)$ by $\widetilde{H(\xi)}$ or $A^{(a)}(z,\xi)$ by $A^{(a)}(z,\xi)$ etc. have to be made where met in the text preceding such formulas.

The bound (G.2) follows from (B.6) (with the appropriate insertions of $\sim$, as above) by a dimensional estimate of the l.h.s and by (4.1) and (B.5).

The bound (G.4) follows from (B.16) by a dimensional estimate (again one should appropriately insert them in the obvious places) and by (4.1), (B.17).

The bound (G.3) follows from (G.2) by noting that $U(h)f$ is, for all $h \in \tilde{H}(\xi - \delta)$, still strongly $\frac{\delta}{2}$ -analytic so that for $h \in \tilde{H}(\frac{\delta}{2})$ real

$$|U(h')(U(h)\hat{f})(x')| \equiv |U(h)\hat{f}(x'h')| \; |(j(x',h')^2)^{-(1+n)/2}| <$$
$$< D(\frac{\delta}{2})^{-1} \|\tilde{f}\|_\xi$$

i.e.

$$|U(h)\hat{f}(x)| < 2D\delta^{-1} \|\tilde{f}\|_\xi \quad \inf \; |j(x',h')|^{-(1+n)/2}$$

where the infimum is over all the $x' \in R$, $h' \in \mathrm{Re}\tilde{H}(\delta/2)$, $x'h' = x$. Use now $j(x',h') = j(x,h'^{-1})$ and pick $h' = (1 - (\delta/2)^2)^{-1/2}\begin{pmatrix} 1 & \pm\delta/2 \\ \mp\delta/2 & 1 \end{pmatrix}$ to obtain an upper bound to the above infimum, proportional to $(1 + \delta|x|)^{-1}$. Then redefine D to have it equal in (F.2), (F.3) .

Similarly one argues to infer the bound (G.5) from (G.4). Proceeding as above one finds

$$|U(h)\hat{f}(z)| < \sqrt{\frac{n}{4\pi}}\; D\|\tilde{f}\|_{\xi}\; \inf \frac{1}{(\operatorname{Im} z')^{(n+1)/2}}\; \frac{1}{|j(z,h'^{-1})|^{1+n}}$$

where the infimum is over the $h' \in \tilde{H}(\frac{\delta}{2})$, $z' \in \mathbb{C}_+$, $z'h' = z$ and the infimum is estimated by the choice

$$h' = (1 - (\tfrac{\delta}{2})^2)^{-1/2} \begin{pmatrix} 1 & -i\delta/2 \\ i\delta/2 & 1 \end{pmatrix}$$

noting that $\operatorname{Im} z' > \delta/2$, so that

$$|U(h)\hat{f}(z)| < \sqrt{\frac{n}{4\pi}}\; D\|\tilde{f}\|_{\xi}\; (\tfrac{\delta}{2})^{-(n+1)/2}\; |1 - i\tfrac{\delta}{2} z|^{-(1+n)/2} \qquad \text{(G.7)}$$

Then (G.5) follows by raising the (G.4) to the power $(n-1)/(n+1)$ and the (G.7) to the power $2/(n+1)$ and multiplying them together.

Again one has to redefine D in order to have it identical in (G.2) - (G.5). The statement iii) is a simple consequence of the calculation of the $Y^{(a)}$ norms of the r.h.s of (G.3), (G.5) with $\|\tilde{f}\|_{\xi}$ replaced by N, and of the remark that such norms can be bounded essentially independently of a.

We now proceed to the proof of the following Lemma 7 which will immediately imply ii), iv) of proposition 4.

In fact from the estimates (G.14), (G.15) below and using lemma 6, iii), we obtain estimates on $\|\widetilde{\Phi^{(a)}}\|_{\xi e^{-\delta}}$: such estimates can be combined with lemma 3' , 2' and lead to proposition 4, ii), iv) (in the same way as (5.16), (5.18) combined with lemmas 3, 2 yielded proposition 4, i), iii).

LEMMA 7: Let $0 < 5\delta < \xi < \xi_0$.

i) Let $f \quad Y^{(a)}$, $a = (u,j)$, be strongly ξ - analytic and $\|\tilde{f}\|_\xi = 1$. Suppose that the equation $\mathcal{L}\Phi = f$ admits a solution $\Phi \in Y^{(a)}$, then for all $h \in \tilde{H}(\xi - 4\delta)$

$$|(U(h)\hat{\Phi})(x)| \leq C\delta^{-7/2} \qquad \forall\, x \quad R \tag{G.8}$$

and for all $h \in \tilde{H}(\xi - 5\delta)$:

$$|(U(h)\hat{\Phi})(x)| \leq C\delta^{-7/2}(1 + \delta|x|)^{-1-\mathrm{Re}u} \qquad \forall x \in \mathbb{R} \tag{G.9}$$

ii) Let $f \in Y^{(a)}$, $a = (n,j,\pm)$, and suppose that $\Phi = f$ has a solution $\Phi \in Y^{(a)}$ then for all $h \quad H(\xi - 4\delta)$:

$$|U(h)\Phi(z)| \leq \sqrt{\frac{n}{4\pi}}\; C(\mathrm{Im}\, z)^{-(n+1)/2}\,\delta^{-2} \qquad \forall z \in \mathbb{C}^+ \tag{G.10}$$

and for all $h \in \tilde{H}(\xi - 5\delta)$

$$|U(h)\hat{\Phi}(z)| \leq \sqrt{\frac{n}{4\pi}}\; C(\mathrm{Im}\, z)^{-(n-1)/2}\delta^{-3}(1 + \delta^2|z|^2)^{-1} \tag{G.11}$$

for all $z \in \mathbb{C}^+$.

Remark: as in the proof of lemma 6 the (G.9) , (G.11) are consequence of (G.8), (G.10), up to a redefinition of the constants. So it suffices to prove (G.8), (G.10).

In the proof we shall set $\tilde{H}(\xi) \equiv H_\xi$ for simplicity of notation, and we shall examine separately the cases $a = (u,j)$, $u = is$ or $a = (u,j)$, $u = s$ or $a = (n,j,\pm)$. Each of these cases will be further subdivided in several subcases. After the analysis of the first few cases the philosophy of the proof should become clear.

Lemma 7 will be an easy consequence of the following Lemmas.

LEMMA 8

i) Let f be a strongly ξ analytic function such that $f \quad Y^{(u,j)}$, $u \in \mathbb{R}$ or $u \in i\,\mathbb{R}$ and satisfy condition 5.6. Then the equation $\mathcal{L}\Phi = f$ has a unique solution Φ such that if $0 < 4\delta < \xi < 0.1$, then for all $x \in \mathbb{R}$, and all $h \in \tilde{H}_\xi - 4\delta$, $U(h)\Phi(x)$ is well defined, holomorphic in h, and satisfies

$$|U(h)\,\hat{\Phi}(x)| < c'\,\delta^{-7/2}\,\|\tilde{f}\|_\xi$$

ii) Let f be a strongly ξ analytic function such that $f \in Y^{(n,j,\pm)}$ and satisfy condition 5.6. Then the equation $\mathcal{L}\Phi = f$ has a unique solution Φ such that if $0 < 4\delta < \xi < 0.1$, then for all $z \in \mathbb{C}_+$, and all $h \in \tilde{H}_{\xi - 4\delta}$, $U(h)\Phi(z)$ is well defined, holomorphic in h, and satisfies

$$|U(h)\hat{\Phi}(z)| \leq \frac{c'\quad\delta^{-2}}{(\operatorname{Im} z)^{(n+1)/2}}\quad\|\tilde{f}\|_\xi\;\sqrt{\frac{n}{4\pi}}\quad.$$

Lemma 9

i) Assume f is strongly ξ analytic, and $f \in Y^{(u,j)}$. Assume also that

$$|U(h)\hat{f}(x)| \leq 1 \quad \forall h \in \tilde{H}_\xi\,, \quad \forall x \in \mathbb{R}\,.$$

Then

$$|U(h)\hat{f}(x)| \leq \frac{5}{2}\;\frac{1}{(1+\delta|x|)^{1+\operatorname{Re} u}}$$

$\forall h \in \tilde{H}_{\xi - \delta}$ and $\forall x \in R$.

ii) Assume f is strongly ξ analytic, and $f \in Y^{(n,j,\pm)}$. Assume also that

$$|U(h)f(\lambda)| \leq \frac{1}{(\mathrm{Im}\ \lambda)^{(n+1)/2}} \qquad \forall h \in \tilde{H}_{\xi}\ ,\ \ \forall \lambda \in C_+ .$$

Then

$$|U(h)f(\lambda)| \leq \frac{8\ \delta^{-1}}{(\mathrm{Im}\ \lambda)^{(n-1)/2}\ [(1 + \delta \mathrm{Im}\ \lambda)^2 + \delta^2(\mathrm{Re}\ \lambda)^2]}$$

$$\forall h \in \tilde{H}_{\xi - \delta}\ ,\ \forall \lambda \in C_+ .$$

Note that the proof of lemma 9, which we give below for completeness is, essentially, a repetition of the proof of lemma 6 above.

Proof.

Proof of i). Let $g_\kappa = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$, and $h \in \tilde{H}_{\xi - \delta}$. It is easy to verify that if $|\kappa| \leq \dfrac{\delta(1 - \xi)}{1 + \xi^2}$, then $g_\kappa h \in \tilde{H}_\xi$. Therefore for κ real

$$|U(h)\hat{f}(x)| = |U(g_\kappa^{-1})\ U\ (g_\kappa h)\hat{f}(x)|$$

$$= |U(g_\kappa h)\hat{f}(x g_\kappa^{-1})|\ \ |j(x, g_\kappa^{-1})|^{1 + \mathrm{Re} u}$$

We chose $\kappa = -(\mathrm{sign}\ x)\ \delta\ (1 - \xi)(1 + \xi^2)^{-1}$, and we obtain

$$|U(h)\hat{f}(x)| \leq |j(x, g_\kappa^{-1})|^{-1 - \mathrm{Re} u} = (1 + \delta\, \frac{1 - \xi}{1 + \xi^2}\ |x|)^{-1\ - \mathrm{Re} u}$$

We now observe that $\dfrac{1 - \xi}{1 + \xi^2} > \dfrac{4}{5}$ and the result follows.

Proof of ii). Let $g^{-1} = \frac{1}{\sqrt{1-\delta^2/4}}\begin{pmatrix} 1 & -i\delta/2 \\ i\delta/2 & 1 \end{pmatrix}$, it is easy to verify that if $h \in \tilde{H}_{\xi-\delta}$, then $gh \in \tilde{H}_{\xi}$. We have $U(h)\hat{f}(\lambda) = u(g^{-1})\, U(gh)\hat{f}(\lambda) = j(\lambda,g^{-1})^{-(n+1)}\, U(gh)\hat{f}(\lambda g^{-1})$. Therefore

$$|U(h)\hat{f}(\lambda)| \leq |\mathrm{Im}(\lambda g^{-1})|^{-k/2} \quad |j(\lambda,g^{-1})|^{-k} , \qquad k = n+1$$

since g^{-1} maps $\mathbb{C}_+$ into $\mathbb{C}_+ + i\delta/2$. If $\lambda = x+iy$, we obtain

$$|U(h)\hat{f}(\lambda)| \leq [2\delta^{-1}[(1+\delta y/2)^2 + \delta^2 x^2/4]]^{-k/2} .$$

Multiplying the bound of the hypothesis raised to the power $1 - \frac{2}{k}$, and the above bound raised to the power $2/k$, we obtain the result.

Q.E.D.

It is obvious that Lemma 7 follows from Lemma 8 and Lemma 9.

It is easy to derive from formula 5.4, 5.5 and 5.6 that if $f \in Y^a$, $U(h)\hat{\Phi}(z)$ is given by

$$\begin{aligned} U(h_0)\Phi(z_0) &= -P(z_0,h_0)^{-k/2}\int_0^{z_0h_0} \zeta^{k/2-1} f(\zeta)d\zeta \\ &= -P(z_0,h_0)^{-k/2}\int_{z_0h_0}^{+\infty} \zeta^{k/2-1} f(\zeta)d\zeta \end{aligned} \qquad \text{(G12)}$$

where h_0 is real, $P(z_0,h_0) = (a_0z_0 + c_0)(b_0z_0 + d_0)$ if $h_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, $z_0 \in \mathbb{R}$, $k = n+1$ if $a = (u,j)$, and $z_0 \in \mathbb{C}_+$, $k = n+1$ if $a = (n,j,\pm)$. This formula together with the analyticity of $\hat{f}$ allow $U(h)\hat{\Phi}$ to be analytically continued in h in a small complex neighborhood of the identity.

In the case of the principal and supplementary series, the non uniformity of the function $\omega \to \omega^{k/2}$ requires some precautions. Along the paths to be used, ζ will always be of the form $\zeta = zh$ where $z \in \mathbb{C}_+$ (in the case of the principal and supplementary series $z \in \mathbb{R}$), and $h \in \widetilde{H}_\xi$, so that the integrand becomes $P(z,h)^{k/2}\, U(h)\hat{f}(z)\zeta^{-1}$. The rule is that, at the start of the path, $P(z,h) = P(z_0,h_0)$ and the determination of $P(z,h)^{k/2}$ to be chosen is $P(z_0,h_0)^{k/2}$. Along the path it is then determined by analytic continuation. We adopt the following terminology: let $\circledR = \mathbb{C}_+$ in the case of the discrete series, $\circledR = \mathbb{R}$ in the case of the principal or supplementary series. We call "path of type P_0 (resp. P_∞)" a continuous path in the complex plane, starting from some $\zeta_0 = z_0h_0$, $z_0 \in \circledR$, $h_0 \in \widetilde{H}_{\xi - \delta}$, ending at 0 (resp. ∞), and such that

a) Each ζ on the path is of the form zh, $z \in \circledR$, $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{H}_{\xi - \delta}$, varying continuously along the path with $P(z,h) = P(z_0,h_0)$ at the start of the path

b) $ax + c$ never vanishes along the path except, if the path is of type P_0, at the end $\zeta = 0$. $(bz + d)$ never vanishes along the path except, if the path is of type P_∞ at the end $\zeta = \infty$.

c) There is a continuous arc $\{\zeta_u\}$ joining ζ_0 to some $\zeta_1 = z_1h_1$, $z_1 \in \circledR$, h_1 real in $\widetilde{H}_{\xi - \delta}$ and a corresponding continuous family of paths starting at ζ_u (and ending always at 0 or always at ∞) satisfying the conditions a) and b) and ending with a real path. (This condition will be non-trivial only in the case of the principal and supplementary series).

Along any such paths, there is an analytic continuation of f. If the corresponding integrals along all paths of the family (of condition c) are absolutely convergent, they define an analytic continuation of $\hat{\Phi}$ along the arc $\{\zeta_u\}$ and, consequently, an analytic continuation of $U(h)\hat{\Phi}(z)$ to the point (h_0, z_0).

We denote

$$h_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{pmatrix} 1 & \mu_0 \\ \sigma_0 & 1 + \sigma_0\mu_0 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$$

with $\mu_0 = a_0 b_0$, $\sigma_0 = c_0 a_0^{-1}$. In proving the bound for $U(h_0)\hat{\Phi}(z_0)$, it is enough to prove the bound for $\sigma_0 = i\eta_0$, η_0 R. Indeed, let h_1 be given by

$$h_1 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} h_0 , \quad t \text{ real, then}$$

$$|U(h_0)\hat{\Phi}(z)| = |U(h_0)\hat{\Phi}(z + t)|$$

and we have obviously $\mathrm{Im}(z + t) = \mathrm{Im}(z)$. In order to prove 8,i) for the principal series, we shall prove first the following result.

<u>Lemma 10</u>. <u>Let</u> $f \in \gamma^{(is,j)}$ <u>be strongly</u> ξ <u>analytic and satisfy</u> 5.6. <u>Then, for all</u> $x_0 \in \mathbb{R}$, $U(h_0)\hat{\Phi}(x_0)$ <u>is holomorphic in</u> h_0 <u>in</u> $\tilde{H}_{\xi - 3\delta}$ <u>and if</u> $|\mu_0| = |a_0 b_0| \geq \delta/4$, <u>satisfies</u>

$$|U(h_0)\hat{\Phi}(x_0)| \leq c(\xi)\sigma^{-7/2} \|\tilde{f}\|_{\xi} .$$

Let us first derive 8i), for the principal series from Lemma 10. Assume h_0 belongs to $\tilde{H}_{\xi - 4\delta}$, $|b_0| < \xi - 4\delta$, $|c_0| < \xi - 4\delta$. If $|\mu_0| \geq \delta/4$, the assertion of Lemma 8 i) , is explicitly in Lemma 9. If $|\mu_0| < \delta/4$, let

$$h_\nu = \begin{pmatrix} 1 & \mu_0 + \nu \\ \sigma_0 & 1 + (\mu_0 + \nu)\sigma_0 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} .$$

If $|\nu| \leq \delta/2$ then $h_\nu \in \tilde{H}_{\xi - 3\delta}$. By Lemma 10, $U(h_\nu)\hat{\Phi}(x_0)$ is holomorphic in ν in a neighborhood of the disk $|\nu| \leq \delta/2$ and by the maximum principle, it satisfies

$$|U(h_\nu)\hat{\Phi}(x_0)| < c(\xi)\|\tilde{f}\|_\xi \ \delta^{-7/2}$$

since the edge of the disk satisfies

$$|\mu_0 + \nu| \geq |\nu| - |\mu_0| > \frac{\delta}{4} .$$

Proof of Lemma 10.

Note that it follows from our assumptions that $U(h)\hat{f}(x)$ is simultaneously analytic in h and x . Given $\zeta_0 = x_0 h_0$, any path of the type P_0 or P_∞ originating at ζ_0 such that the corresponding integrals are absolutely convergent, defines an analytic continuation of $U(h)\hat{\Phi}(x)$ to the point (h_0, x_0) . To settle the possibility of non-uniformity of such continuations, we shall proceed as follows. The domain of analyticity asserted by Lemma 10 will be divided into several open sets. In each of these we first prove the analyticity, then find integration paths that yield the required bounds. The integrals in (G12) can be rewritten

$$U(h_0)\hat{\Phi}(x_0) = \int_{\zeta_0}^{(0 \text{ or } \infty)} \left[\frac{P(x,h)}{p(x_0,h_0)}\right]^{k/2} U(h)\hat{f}(x)\frac{d\zeta}{\zeta} \qquad \text{(G13)}$$

where $x = \zeta h$ along the chosen path and $k = 1 + is$. The quantity $[P(x,h)/P(x_0,h_0)]^{k/2}$ will be defined by analytic continuation along the path starting with the value 1. We denote $x + \sigma = re^{i\phi}$, $x_0 + \sigma_0 = r_0 e^{i\phi_0}$, $\mu = \rho e^{i\theta}$, $\mu_0 = \rho_0 e^{i\theta_0}$, $\theta_0 + \phi_0 = \alpha$, $\theta + \phi = \beta$, $1 + \mu(x + \sigma) = me^{i\psi}$, $1 + \mu_0(x_0 + \sigma_0) = m_0 e^{i\psi_0}$. The condition $h_0 \in \tilde{H}_{\xi - 3\delta}$ implies

$$|a_0 - 1| < \xi - 3\delta \ , \ |a_0||\sigma_0| < \xi - 3\delta \ , \ |a_0|^{-1}\rho_0 < \xi - 3\delta$$
$$\rho_0 < (\xi - 3\delta)(1 + \xi) \quad (< 2\xi) \ .$$

An easy consequence of Lemma 9 is that for $h \in \tilde{H}_{\xi - \delta}$, and $x \in \mathbb{R}$, we have

$$|U(h)\hat{f}(x)| \leq \sqrt{2}\,\delta^{-1} \tag{G14}$$

$$|U(h)\hat{f}(x)| \leq \frac{5\delta^{-1}}{\sqrt{2}}(1 + \delta|x|)^{-1} \tag{G15}$$

(we have assumed $\|\tilde{f}\|_\xi = 1$) .

I) <u>Case $|a_0|r_0 < \xi - \delta$.</u>

a) <u>Analyticity</u> We use a path given by $\zeta = xh$, $x = tx_0$, $\mu = \mu_0$, $\sigma = t\sigma_0$ with $0 \leq t \leq 1$. This gives

$$U(h_0)\phi(x_0) = \int_0^1 \left[\frac{t(1 + t\mu_0(x_0 + \sigma_0))}{1 + \mu_0(x_0 + \sigma_0)}\right]^{k/2} \frac{dt}{t[1 + \mu_0(x_0 + \sigma_0)]}\, U(h)\hat{f}(x) \ .$$

This equation holds for real h_0 and extends analytically in $\{h_0 \in \tilde{H}_{\xi - \delta}, \ |a_0||x_0 + \sigma_0| < \xi - \delta\}$. If $\alpha = 0$ or π, $P(x,h)$ is positive and

$$|U(h_0)\hat{\Phi}(x_0)| \leq \int_0^1 \frac{\sqrt{2}}{\delta} \;[(1 - r_0\rho_0)t(1 - t\rho_0 r_0)]^{-1/2}\, dt$$

$$\leq \frac{2\sqrt{2}}{\delta(1 - 2\xi^2)}$$

b) Bound. Assume $\alpha \neq 0 , \pi$. A first path is given by $\rho = \rho_0$, $r = r_0$, $x = r_0 \cos \phi$, $\sigma = ir_0 \sin \phi$, $\beta = \theta + \phi$ increases to π if α $(0,\pi)$, or decreases to $-\pi$ if α $(-\pi,0)$ while ϕ varies so that the argument of $P(x,h)$ remains constant. This path produces a contribution bounded by $\dfrac{2\pi \sqrt{2}}{\delta(1 - 2\xi^2)}$. At the end of it we are in the case $\alpha = \pi$ and we can apply the previous estimate since the final value of $|[P(r_0\cos\phi,h)]^{k/2}|$ is inferior to its initial value. We finally obtain

$$|U(h_0)\hat{\Phi}(x_0)| \leq \frac{2(1 + \pi)\sqrt{2}}{\delta(1 - 2\xi^2)} .$$

II. Case $|a_0|r_0 > \xi - 3\delta$. This implies $|r_0\cos\phi_0| > 0$. We treat the case $r_0\cos \phi_0 > 0$, the case $r_0 \cos \phi_0 < 0$ being similar. If $1 + \mu_0(x_0 + \sigma_0) \neq 0$ we define a path of type P_∞ parametrized by $t \in [1,\infty]$ by $\sigma = i\eta_0$, $x = \sqrt{t^2 r_0^2 - \eta_0^2}$ (hence $r = tr_0$) , $\rho = \dfrac{\rho_0}{t}$, $\theta = \alpha - \phi$. Defining ϕ , $\phi_0 \in [-\frac{\pi}{2} , \frac{\pi}{2}]$ this implies that ϕ varies monotonically from ϕ_0 to 0 , $\zeta = te^{i(\phi - \phi_0)}\zeta_0$ and $P(x,h) = te^{i(\phi - \phi_0)}P(x_0,h_0)$. We obtain

$$U(h_0)\hat{\Phi}(x_0) = \int_1^\infty [te^{i(\phi - \phi_0)}]^{(1 + is)/2} \frac{r_0 dt \;\; e^{-i\phi_0}}{te^{i(\phi - \phi_0)} \sqrt{r_0^2 - \eta_0^2/t^2}}\, U(h)\hat{f}(x) \qquad \text{(G16)}$$

Using (G15), we obtain

$$|U(h_0)\hat{\Phi}(x_0)| \leqslant \int_1^\infty e^{-s/2(\phi-\phi_0)} \frac{5\,dt}{\sqrt{2}\delta \cos\phi_0 \sqrt{t}\,(1+\delta r_0 \cos\phi_0 t)} \tag{G17}$$

For fixed x_0 , a_0 and σ_0 , the left hand side of (G16) defines a function $I(h_0,x_0)$ of μ_0 holomorphic in the whole disk $|a_0||\mu_0| < \xi - 3\delta$ including the points if any where $1 + \mu_0(x_0 + \sigma_0) = 0$, and C^∞ in all variables, when σ_0 and μ_0 vary at fixed x_0 , a_0 , avoiding points where $1 + \mu_0(x_0 + \sigma_0) = 0$, I defines a holomorphic function. Hence, by standard properties of analytic functions of several complex variables, I is holomorphic in h_0 in the required domain, coincides with $U(h_0)\hat{\Phi}(x_0)$ at real h_0 and is also analytic in x_0 . If $\phi_0 < 0$, $\phi - \phi_0$ increases along the path and we deduce

$$|I(h_0,x_0)| \leqslant 5\pi/(2\delta^{3/2} \cos\phi_0 \sqrt{r_0 \cos\phi_0}) \quad .$$

We can assume $|x_0| > \frac{\delta}{4}$ since $|x_0| < \frac{\delta}{2}$ implies $|a_0||x_0 + i\sigma_0| < \xi - \delta$ which falls in case I .

<u>Subcase II.1</u> $x_0\eta_0 \leqslant 0$. From (G17) we derive , $(\sigma_0 = i\eta_0)$

$$|U(h_0\Phi(x_0))| < \frac{5\pi r_0}{\sqrt{2}(\delta\ x_0)^{3/2}} < \frac{5\pi(x_0+\xi)}{\sqrt{2}(\delta\ x_0)^{3/2}} < \frac{5\sqrt{2}\ \pi\xi}{\delta^3} \quad .$$

<u>Subcase II.2</u> $r_0\rho_0 \geqslant 1$

We use a path of type P_∞ parametrized by t [0,1] with $x = x_0$, $\sigma = \sigma_0$, $m = tm_0$, $\psi = \psi_0$. Along the path μ follows a straight segment contained in $|\mu| < \rho_o$ so that h remains in $H_{\xi - \delta}$. The integral to be computed is

$$\int_0^1 \frac{dt}{t}\, t^{(1 + is)/2}\, U(h) f(x_o)$$

This is holomorphic in h_o in the indicated domain, coincides with $U(h_o)\hat{\Phi}(x_o)$ for real h_o and is bounded by $2\sqrt{2}\,\delta^{-1}$. It is also analytic in x_o. To connect the point (h_o, x_o) by a continuous arc $\{(h_u, x_u)\}$ satisfying the same conditions $(h_u \in \tilde{H}_{\xi - 3\delta}, r_u\rho_u \geq 1, |a_u| r_u > \xi - 3\delta)$ to a real point, we vary the angles ϕ_o and θ_o so that $|\sin\phi_o|$ decreases to 0 and $|\theta_o|$ decreases to 0, while x_u varies so that $\mathrm{Re}\,\sigma$ remains constant.

<u>Subcase II.3</u> $r_o\rho_o \leq 1$, $m_o < \delta$. (hence $r_o\rho_o > 1 - \delta$)

In this case, $r_o \cos\phi_o = x_o + R_e\sigma_o$ has the same sign as x_o which we take to be positive, the other case being similar. The path of type P_∞ is defined by $\rho = \rho_o$, $\phi = \phi_o$, $\sigma = ir\sin\phi_o$, $x = r\cos\phi_o$, $re^{i\beta}$ varies so that ψ remains fixed and m decrease from m_o to 0. Along the path, r has at most one minimum (r^2 is a convex function of m) and we have $r \leq \frac{1}{\rho_o}$. Moreover, $|a_o||n| < \frac{|a_o||n_o|}{1 - \delta} < \xi - \delta$ so that h remains in $\tilde{H}_{\xi - \delta}$. The integral to be computed is bounded by

$$\left|\int_{r_1}^{r_o} \left(\frac{rm}{r_o m_o}\right)^{1/2} \frac{\sqrt{2}}{\delta}\,\frac{dr}{r}\right| + \left|\int_{r_1}^{1/\rho_o} \left(\frac{rm}{r_o m_o}\right)^{1/2} \frac{\sqrt{2}}{\delta}\,\frac{dr}{r}\right|$$

$$+ \left|\int_0^{m_o} \left(\frac{rm}{r_o m_o}\right)^{1/2} \frac{\sqrt{2}}{\delta}\,\frac{dm}{m}\right| < \frac{6\sqrt{2}}{\delta(1 - \delta)}$$

To end the proof of this subcase we construct an arc joining any point satisfying the preceding condition to a real point while keeping the condition fulfilled. For fixed $x_0 > 0$ the set $D_{x_0,a_0} = \{\mu_0,\sigma_0 : r_0\rho_0 \leq 1 , m_0 < \delta , |a_0\sigma_0| < \xi - 3\delta , |a_0^{-1}\mu_0| < \xi - 3\delta , |a_0|r_0 > \xi - 3\delta\}$ does not depend on the argument of a_0. Therefore we start by moving a_0 along $|a_0| = \text{cte}$ until it reaches $|a_0|$. We now move σ and μ keeping $|\mu| = |\mu_0|$ and $r = |x_0 + \sigma| = r_0$. The center of $\{\sigma | \, |x_0 + \sigma| < r_0\}$ is outside of the disk $|\sigma| \leq |\sigma_0|$ since $x_0 > 2\xi$. Hence there is an arc of this circle passing through σ_0, contained in the disk $|\sigma| \leq |\sigma_0|$ and intersecting the real axis at σ_1. We let σ follow this arc from σ_0 to σ_1, while the argument of μ is varied so that $\mu(x_0 + \sigma)$ remains constant. Along this arc $(\mu,\sigma) \in D_{x_0,a_0}$. Then we let $\text{Im}\,\mu \to 0$, $\text{Re}\mu$ and $\sigma = \sigma_1$ being kept fixed.

Subcase II.4

$\sin\phi_0 \cos\phi_0 > 0$, $0 \leq \alpha \leq \pi$, $\rho_0 r_0 < 1$, $m_0 > \frac{\delta}{2}$, $|a_0|\, r_0 > \xi - 3\delta$.
We can assume $0 \leq \phi_0 \leq \frac{\pi}{2}$ since the other case is similar. We start by a path such that $\sigma = i\eta_0$, $|\mu| = \rho_0$, $\theta + \phi = \alpha$,are fixed. We have $\psi \in [0,\pi/2]$. The argument of $P(x,h)$ is $\phi + \psi$, and

$$r \frac{d}{dx} (\phi + \psi) = - \frac{\eta_0}{r} + \frac{\rho_0 \, x \sin\alpha}{1 + \rho_0^2 r^2 + 2\rho_0 r \cos\alpha} ,$$

and $r \frac{d}{dx} (\phi + \psi)$ is an increasing function of x if $1 - \rho_0^2 r^2 \geq 0$. We distinguish two subcases.

Subcase II.4.A

$\frac{d}{dx}(\phi + \psi) \geq 0$ at $x = r_0 \cos \phi_0$.

We choose a path where x increases from $r_0 \cos \phi_0$ to $x_1 = \sqrt{\frac{1}{\rho_0^2} - \eta_0^2}$ so that r increases from r_0 to $\frac{1}{\rho_0}$. Along this path, $\phi + \psi$ increases. The contribution of this path is bounded by

$$\frac{\sqrt{2}}{\delta} \int_{r_0}^{1/\rho_0} \left| \frac{r(1 + \rho_0 r e^{i\alpha})}{r_0(1 + \rho_0 r_0 e^{i\alpha})} \right|^{1/2} \left|\frac{d\phi}{dr}\right| \; dr$$

$$+ \frac{\sqrt{2}}{\delta} \int_{r_0}^{1/\rho_0} \frac{dr}{|r_0(1 + \rho_0 r_0 e^{i\alpha}) r (1 + \rho_0 r e^{i\alpha})|^{1/2}}$$

We now assume that $\rho_0 \geq \frac{\delta}{4}$. The first integral is bounded by $4\pi \sqrt{2}\, \delta^{-2}[(1 + \xi)/(\xi - 3\delta)]^{1/2}$ since $m_0 > \delta/2$. If $\cos \alpha > -1/2$, the second integral is bounded by $8\sqrt{2}\, \delta^{-3/2}[(1 + \xi)/(\xi - 3\delta)]^{1/2}$. If $\cos \alpha \leq -1/2$, the second integral is bounded by $32\delta^{-5/2}[(1 + \xi)/(\xi - 3\delta)]^{-1}$. At the end of this path we have $x = x_1$, $h = h_1 \in \tilde{H}_{\xi - 3\delta}$ and $|P(x_1,h_1| \leq \frac{2}{\rho_0} \leq \frac{4(1 + \xi)}{\rho_0 \delta(\xi - 3\delta)} |P(x_0,h_0)|$, while $\operatorname{Arg} P(x_1,h_1) \geq \operatorname{Arg} P(x_0,h_0)$. Therefore, finishing the integration as in subcase II.2, we obtain, using $\rho_0 \geq \frac{\delta}{4}$, the bound $33\delta^{-5/2}(1 + \xi)/(\xi - 3\delta)$. We now show how to let a group element

$$h' = \begin{pmatrix} 1 & \mu' \\ \sigma' & 1 + \mu'\sigma' \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$$

follow a continuous arc from h_0 to a real value while continuing to satisfy all the conditions of this subcase, notably, with x_0 fixed

$$-\frac{n'}{r'} + \frac{\rho' \sin\alpha' r' \cos\phi'}{(1 + \rho'^2 r'^2 + 2\rho' r' \cos\alpha')} \geq 0 \,.$$

The circle $\{\sigma' : |x_0 + \sigma'| = r_0\}$, passing through σ_0 and $\overline{\sigma_0}$ intersects the real axis at $\sigma_1 = r_0 - x_0$ inside the disk $\{\sigma' : |\sigma'| \leq |\sigma_0|\}$ since $|\sigma_0| = |(x_0 + \sigma_0) - x_0| \geq |r_0 - x_0|$. Since, by assumption $0 < r_0 \cos\phi_0 = x_0 + \mathrm{Re}\sigma_0 \leq r_0$, $\sigma_1 \geq \mathrm{Re}\,\sigma_0$. We let σ' follow this arc of circle from σ_0 to σ_1 . Denoting $x_0 + \sigma' = r_0 e^{i\phi'}$, ϕ' decreases from ϕ_0 to 0 , hence $\sin\phi' = \frac{n'}{r'}$ decreases from $\sin\phi_0$ to 0 while $r_0 \cos\phi'$ increases. At the same time we keep $\rho' = \rho_0$ and $\alpha' = \alpha$ by varying the argument of μ' . Then we let α' vary so that $\cos\alpha'$ increases from $\cos\alpha$ to 1 while $\sin\alpha'$ remains ≥ 0 .

<u>Subcase II.4B</u>

$$\frac{d}{dx}(\phi + \psi) \leq 0 \quad \text{at} \quad x = r_0 \cos\phi_0 \,.$$

We let x decrease from $r_0 \cos\phi_0$ to 0 . Along this path $\phi + \psi$ increases as x decreases. Again the integral has two contributions bounded by:

$$\frac{\sqrt{2}}{\delta} \int_{n_0}^{r_0} \left| \frac{r(1 + \rho_0 r\, e^{i\alpha})}{r_0(1 + \rho_0 r_0\, e^{i\alpha})} \right|^{1/2} \left| \frac{d\phi}{dr} \right| \, dr$$

which is bounded by $4\pi \sqrt{2}\, \delta^{-3/2}$, and

$$\frac{\sqrt{2}}{\delta} \int_{n_0}^{r_0} \frac{dr}{|r_0(1 + \rho_0 r_0 e^{i\alpha}) r(1 + \rho_0 r e^{i\alpha})|^{1/2}} \tag{G18}$$

If $\cos\alpha \geq 0$ the contribution of (G18) is bounded by $2\sqrt{2}\,\delta^{-1}$. If $\cos\alpha < 0$, let $r_1 = (1 - \delta/2)/\rho_0|\cos\alpha|$. If $r_0 < r_1$, the contribution of (G18) is bounded by $4\sqrt{2}\,\delta^{-2}$. Suppose now $\cos\alpha < 0$, and $r_0 > r_1$. We divide the integration range in (G18) into $[n_0,r_1] \cup [r_1,r_0]$. The contribution of the first part is again bounded by $4\sqrt{2}\,\delta^{-2}$. That of $[r_1,r_0]$ is bounded by $2\sqrt{2}\,\delta^{-1}(1 - \delta/2)^{-1}$. At the end of the path, $\zeta = 0h_1$, $h_1 \in \widetilde{H}_{\xi - 3\delta}$, and

$$|[P(0,h_1)]^{(1 + is)/2}| \leq \frac{2}{\sqrt{\delta}}\ |P(x_0,h_0)^{(1 + is)/2}|$$

We finish the integration as in case I and we obtain, the subcase II.4.B a contribution bounded by $5\sqrt{2}\,\delta^{-2}$. We now have to describe how to let h' follow a continuous arc from h_0 to a real value while continuing to satisfy all the conditions of this subcase (in particular $\frac{d}{dx}(\phi + \psi) \leq 0$). We first fix $\sigma' = \sigma_0$, hence $r' = r_0$ and $\phi' = \phi_0$. If $\cos\alpha \leq 0$, we first vary $\rho' e^{i\alpha'}$ inside the disk $|\rho' e^{i\alpha'}| \leq \rho_0$ so that $\rho'\sin\alpha'$ remains constant and $\rho'\cos\alpha'$ increases from $\rho_0\cos\alpha$ to 0. If $\cos\alpha \geq 0$ we omit the preceding step and proceed to the next: ρ' is kept fixed, α' decreases so that $\sin\alpha'$ decreases to 0 and $\cos\alpha'$ increases to 1. All conditions remain satisfied in this process. Finally we let σ' move from σ_0 to σ_1 as in subcase II.4.A, while varying the argument θ' of μ' so that $\alpha' = \phi' + \theta'$ remains equal to 0.

Subcase II.5

$$\sin\phi_0\cos\phi_0 > 0\ ,\ -\pi < \alpha < 0\ ,\ \rho_0 r_0 < 1\ ,\ |a_0| r_0 > \xi - 3\delta\ ,\ m_0 > \delta/2\ .$$

This case will be reduced to the case $\alpha = 0$ or π. The path is defined by $x = x_0$, $\sigma = \sigma_0$, $m = m_0$, $\theta = \theta_0$, ψ increases from its initial value

$\psi_0 \in [-\frac{\pi}{2}, 0]$ to 0. This path gives a contribution bounded by $\frac{\pi}{2}\frac{\sqrt{2}}{\delta}$. The remainder of the integral to be computed is treated as in subcase II.4.b. To find an arc from a point h_0 satisfying the above conditions to the reals, it suffices to let h_0 follow the path described above to the end, then to let σ_0 go to $\sigma_1 = r_0 - x_0$ as in subcase II.4.A while $\alpha = \theta + \phi$ is kept fixed.

With cases I and II it has been shown that, for fixed x_0, $U(h_0)\hat{\Phi}(x_0)$ has a univorm analytic continuation is each of the three open sets

$$\Delta_I(x_0) = \{h_0 \in \tilde{H}_{\xi - 3\delta} : |a_0| r_0 < \xi - \delta\} ,$$

$$\Delta_{II}^{\pm}(x_0) = \{h_0 \in \tilde{H}_{\xi - 3\delta} : |a_0| r_0 > \xi - 3\delta , \pm r_0 \cos\phi_0 < 0\} ,$$

with $\Delta_{II}^{\pm}(x_0) \cap \Delta_{II}^{-}(x_0) \neq \emptyset$. To prove that these analytic continuation match, we show that if, e.g $h_0 \in \Delta_{II}^{+}(x_0) \cap \Delta_I(x_0)$ it can be connected to the reals by a continuous arc not leaving this intersection: first we vary a_0 at constant $|a_0|$ until it reaches $|a_0|$. Then we vary continuously the argument θ of $\mu = \rho_0 e^{i\theta}$, keeping $|\mu|$ constant, until $\theta = 0$. Finally we move σ from σ_0 to $\sigma_1 = x_0 - r_0$ (as explained at the end of subcase II.4.A) keeping $|x_0 + \sigma| = r_0$ constant.

Thus $U(h_0)\hat{\Phi}(x_0)$ extends to a holomorphic function of h_0 in $\tilde{H}_{\xi - 3\delta}$ and, if $\rho_0 \geq \frac{\delta}{4}$, it is easy to see that this bound for subcase II.4.A majorizes the bounds found for all other subcases. It is itself majorized by $\frac{80}{\delta^{7/2}}$. This concludes the proof of Lemma 10.

We now continue the proof of Lemma 8 by looking at the supplementary series. An easy consequence of Lemma 9 is that if $\|\tilde{f}\|_{\xi} = 1$, and $f \in \gamma^{(s,j)}$, there is a positive constant B which does not depend of f such that

$$|U(h)f(x)| < \frac{B}{(1 + \delta|x|)^{s+1}} < B \tag{G19}$$

for all $x \in \mathbb{R}$, and $h \in \tilde{H}_{\xi-\delta}$. We shall now define and estimate $U(h)\hat{\Phi}(x)$. We shall use the notations introduced in the proof of Lemma 9. There are two cases.

<u>Case I $|a_0|r_0 < \xi - \delta$.</u>

We use a path $x = tx_0$, $\sigma = t\sigma_0$, $\mu = \mu_0$, $t \in [0,1]$. As in the case of the principal series,

$$U(h_0)\hat{\Phi}(x_0) = \int_0^1 \left[\frac{t(1 + t\mu_0(x_0 + \sigma_0))}{1 + \mu_0(x_0 + \sigma_0)}\right]^{\frac{s+1}{2}} \frac{U(h)f(x)}{t[1 + t\mu_0(x_0 + \sigma_0)]} \, dt \, .$$

Therefore $|U(h_0)\hat{\Phi}(x_0)| < 2B(1 - \xi^2)^{-1}(1 + 2\xi^2)$ since $|1 + t\mu_0(x_0 + \sigma_0)| > 1 - \xi^2$, and using (G19).

<u>Case II $|a_0|r_0 > \xi - 3\delta$.</u>

As in the case of the principal serie, we note that this implies $|r_0 \cos \phi_0| > 0$ and we consider the case $\cos \phi_0 > 0$ (the other case being similar). We use the same contour as in the case of the principal serie. We have

$$U(h_0)\Phi(x_0) = \int_1^{+\infty} \left[te^{i(\phi - \phi_0)}\right]^{\frac{1+s}{2}} \frac{r_0 e^{-i\phi_0}\, dt}{te^{i(\phi - \phi_0)}\sqrt{r_0^2 - \eta_0^2/t^2}}\, U(h)\hat{f}(x) ,$$

hence using (G19),

$$|U(h_0)\Phi(x_0)| \leq B \int_1^{\infty} \frac{r_0}{x_0} \frac{t^{(s-1)/2}\, dt}{[1 + t\delta x_0]^{s+1}}$$

$$\leq \pi B r_0 x_0^{-1} (\delta x_0)^{-(1+s)/2}$$

As explained in the case of the principal series, (See Case II), it suffices to consider the case $|x_0| > \frac{\delta}{4}$, and using $r_0 \leq x_0 + \xi$, we obtain

$$|U(h)\hat{\Phi}(x_0)| \leq 16\pi B(1 + \xi)\delta^{-3} .$$

This ends the proof of part i) in Lemma 8.

We now show part ii). From Lemma 9 it follows that if $f \quad \gamma^{(n,j,\pm)}$, and if $\|\tilde{f}\|_{\xi} = \sqrt{\frac{4\pi}{n}}$, we have

$$|U(h)\hat{f}(\lambda)| \leq (\mathrm{Im}\,\lambda)^{-k/2} \tag{G20}$$

and

$$|U(h)\hat{f}(\lambda)| \leq 8[\delta(\mathrm{Im}\,\lambda)^{k/2-1}[(1 + \delta\mathrm{Im}\,\lambda)^2 + (\delta\,\mathrm{Re}\,\lambda)^2]]^{-1} \tag{G21}$$

for any $\lambda \quad \mathbb{C}_+$, and $h \in \tilde{H}_{\xi - \delta}$.

We adopt the following notations.

$$h_0 = \begin{pmatrix} 1 & \mu_0 \\ i\eta_0 & 1 + i\mu_0\eta_0 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$$

(it is sufficient to consider the case $\eta_0 \in \mathbb{R}$),

$$\lambda_0 + i\eta_0 = r_0 e^{i\phi_0} \quad , \ \mu_0 = \rho_0 e^{i\theta_0} \quad , \quad \phi_0 + \theta_0 = \alpha \ .$$

The integration variable will always be of the form λh , $\lambda \in \mathbb{C}_+$, and h of the form

$$h = \begin{pmatrix} 1 & \mu \\ i\eta & 1 + i\mu\eta \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$$

and we denote

$$\lambda + i\eta = re^{i\phi} \ , \ \mu = \rho e^{i\theta} \quad .$$

There are two cases.

I <u>Case $\eta_0 < 0$</u>

Our first path is defined by $\eta = \eta_0$, $r = r_0$, $\rho = \rho_0$, ϕ varies in such a way that $\sin\phi$ increases from $\sin\phi_0$ to 1 , and θ is varied so that $\theta + \phi = \alpha$ remains constant. Along this path, the imaginary part of λ increases. This path gives a contribution bounded by $\frac{\pi}{2} (\mathrm{Im}\ \lambda_0)^{-k/2}$. At the end of the path we have $\phi = \frac{\pi}{2}$, and let h_1 and $\lambda_1 = i(r_0 - \eta_0)$ be the values assumed by h and λ . Notice that $|P(\lambda_1,h_1)| = |P(\lambda_0,h_0)|$. We now have to treat the case $\phi_0 = \pi/2$, $\lambda_0 = i(r_0 - \eta_0) = iy_0$. We define a path by $r = r_0$, $\rho = \rho_0$, $\phi = \phi_0 = \pi/2$, and θ varies in such a way that $\sin\theta$ increases from θ_0 to 1. Along this path, $|1 + i\rho_0 r_0 e^{i\theta}|^2$ decreases, and the contribution is bounded by $\rho_0 r_0 \,\Delta\theta\, y_0^{-k/2}\, |1 + i\rho_0 r_0 e^{i\theta_0}|^{-1}$ where $\Delta\theta = |\frac{\pi}{2} - \theta_0|$ or $|\theta_0 + 3\,\frac{\pi}{2}|$. Since $\Delta\theta \leq \pi|i - e^{i\theta_0}| = \pi|1 + ie^{i\theta_0}|$, and

$$\rho_0 r_0 |1 + ie^{i\theta_0}|\;|1 + i\rho_0 r_0 e^{i\theta_0}|^{-1} \leq 16 ,$$

The contribution of this path is bounded by $16\,\pi\, y_0^{-k/2}$. Along this path, $|P(\lambda,h)|$ has decreased, while $\operatorname{Im}\lambda$ remained constant. Therefore it is now enough to treat the special case $\lambda_0 = iy_0 = i(v_0 - \eta_0)$, $v_0 > 0$, $\mu_0 = i\rho_0$, $\rho_0 > 0$. We use a path where $h = h_0$ is fixed and $\lambda = iy$ varies. Let $v = y + \eta_0$, $v \geq 0$, we have to estimate

$$U(h_0)\hat{\Phi}(iy_0) = \int_{v_0}^{0 \text{ or } 1/\rho_0} [v(1 - \rho_0 v)]^{k/2-1}[v_0(1 - \rho_0 v_0)]^{-k/2} dv\; U(h) f(i(v - \eta_0))$$

Using (G21) this is bounded in modulus by

$$\int_{v_0}^{0 \text{ or } 1/\rho_0} 8\delta^{-1}[v_0(1-\rho_0 v_0)]^{-k/2}|v(1-\rho_0 v)/(v-\eta_0)|^{k/2-1}|1+\delta(v-\eta_0)|^{-2}\; dv$$

Let γ be defined by

$$\gamma = \eta_0 + \sqrt{\eta_0^2 - \eta_0\rho_0^{-1}} \quad .$$

There are three cases.

a) <u>If $0 \leq v_0 \leq \gamma$</u>, we integrate from 0 to v_0. The function $v(1 - \rho_0 v)/(v - \eta_0)$ is positive and increasing, and using $\gamma < \frac{1}{2\rho_0}$, we have $1 - \rho_0 v_0 \geq 1/2$, we obtain a contribution bounded by $16\delta^{-2} y_0^{-k/2}$.

b) $\gamma < v_0 < \rho_0^{-1}$. In this case we integrate from v_0 to $\frac{1}{\rho_0}$. $v(1 - \rho_0 v)(v - \eta_0)$ is positive and decreasing. The contribution is therefore bounded by

$$8\delta^{-1}v_0^{-1} \; |1 - \rho_0 v_0|^{-1} \; |v_0 - \eta_0|^{1-k/2} \int_{v_0}^{1/\rho_0} \frac{dv}{[1 + \delta(v - \eta_0)]^2}$$

which is less than $32\,\delta^{-2}\, y_0^{-k/2}$ since $\gamma > -\eta_0$.

c) If $v_0 > \rho_0^{-1}$, we integrate from ρ_0^{-1} to v_0 . Using the bound $|U(h)\hat{f}(i(v - \eta))| < (v - \eta_0)^{-k/2}$, the contribution of this path is bounded by $2y_0^{-k/2}$.

II. Case $\eta_0 > 0$.

In this case we have

$$(U(h_0)\hat{\Phi})(\lambda_0) = (U(h_1)\hat{\Phi})(\lambda_0 + i\eta_0)$$

where

$$h_1 = \begin{pmatrix} 1 & \mu_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix} \in \widetilde{H}_{\xi} - \delta .$$

Therefore applying case 1, we obtain

$$|U(h_0)\hat{\Phi}(\lambda_0)| < 33\delta^{-2}(y_0 + \eta_0)^{-k/2} < 33\delta^{-2}\, y_0^{-k/2} .$$

This completes the proof of Lemma 8.

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