Harmonic maps from 2-torus to 2-sphere and its heat flow

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Abstract

Harmonic maps are the critical points of the Dirichlet energy functional for maps between two Riemannian manifolds. In this thesis, we study the corresponding heat flows (the negative L^2 gradient flows) from 2-torus to 2-sphere. In particular, we investigate the stability problem for minimizing harmonic maps within any given homotopy class.

We show the stability result for the lineraized equation at a fixed steady state under a proper choice of parametrization for the perturbation term. We also consider a *finite dimensional model problem* of the gradient flow problem and prove its stability. Up to first order, it is analogous to the nonlinear problem but with the contracting part being finite-dimensional as well.

Contents

Chapter 1

Introduction

Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimension m and n, respectively, and assume N is compact. The Dirichlet energy density of a C^1-map $u\colon M \to N$ is defined to be the trace of the pullback of the metric of N by u. It is denoted by

$$e(u) := \mathrm{tr}_M u^*(h)\,.$$

The total energy of u is defined by the integral of the energy density over M, denoted by

$$\mathcal{E}(u) := \frac{1}{2}\int_M e(u)\,dV_g\,.$$

We call a C^1-map u from M to N a *harmonic map* if it is a critical point of the energy functional $\mathcal{E}(u)$.

Let $U\colon M \times (-\epsilon, \epsilon) \to N$ be a 1-parameter family of variations of u with $u(t) := U(\cdot, t)$ and $u(0) = u$. Denote by $X := U_*\left(\frac{\partial}{\partial t}\right)\big|_{t=0}$ the variational field of u.

If $u \in C^2(M, N)$ is a harmonic map, the Euler-Lagrange equation for the energy functional is given by

$$0 = \left.\frac{\partial \mathcal{E}(u(t))}{\partial t}\right|_{t=0} = -\int_M (X, \tau(u))_N\,,$$

where $(\cdot,\cdot)_N$ is the Riemannian metric on N, and $\tau(u)\colon M \to f^*(TN)$ is a section of the pull-back of the tangent bundle of N to M called the *tension field* of u, given by

$$\tau(u) := \mathrm{tr}(\nabla du)\,,$$

where ∇ is the pull-back covariant derivative in the bundle $T^*M \otimes u^*TN$. Since X is arbitrary, the harmonic map equation reads

$$\tau(u) = 0\,.$$

We now express the above-mentioned quantities in local coordinates.

Let $\{x^1, x^2, \cdots, x^m\}$ and $\{u^1, u^2, \cdots, u^n\}$ be local coordinate systems of M and N near $x_0 \in M$ and $u_0 = u(x_0) \in N$, and let $g = g_{ij}dx^i dx^j$ and $h = h_{\alpha\beta}du^\alpha du^\beta$ be the Riemannian metrics of M and N in local coordinates. Write $u\colon M \to N$ in local coordinates as $u = (u^1, u^2, \cdots, u^n)$. The energy density in terms of coordinates of M and N is

$$\begin{aligned} e(u) &= g^{ij}u^*h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g^{ij}h\left(du\left(\frac{\partial}{\partial x^i}\right), du\left(\frac{\partial}{\partial x^j}\right)\right) \\ &= g^{ij}h\left(\frac{\partial u^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha}, \frac{\partial u^\beta}{\partial x^j}\frac{\partial}{\partial u^\beta}\right) = g^{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}h_{\alpha\beta}\,. \end{aligned}$$

Let $u(t) = (u^1(t), u^2(t), \cdots, u^n(t))$ be a 1-parameter family of variations in local coordinates with $u(0) = u$, and let $v^\alpha = \frac{\partial u^\alpha}{\partial t}\big|_{t=0}$ be the corresponding variation field, for $1 \le \alpha \le n$. In these coordinates, the Euler-Lagrange equation is

$$0 = \frac{\partial \mathcal{E}(u(t))}{\partial t}\bigg|_{t=0} = -\int_M v^\alpha\left(\Delta_g u^\beta + g^{ij}\frac{\partial u^\gamma}{\partial x^i}\frac{\partial u^\delta}{\partial x^j}\Gamma^\beta_{\gamma\delta}\right)h_{\alpha\beta}\,,$$

where

$$\Delta_g = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{|g|}\frac{\partial}{\partial x^j}\right)$$

is the Laplace-Beltrami operator on M, with $|g| := \det(g_{ij})$, and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}h^{\alpha\delta}\left(\frac{\partial h_{\gamma\delta}}{\partial u^\beta} + \frac{\partial h_{\beta\delta}}{\partial u^\gamma} - \frac{\partial h_{\beta\gamma}}{\partial u^\delta}\right)$$

are the Christoffel symbols of N. Therefore the harmonic map equations for C^2-map from M to N are given locally by

$$0 = \tau^\alpha(u) := \Delta_g u^\alpha + g^{ij} \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\gamma}{\partial x^j} \Gamma^\alpha_{\beta\gamma} \text{ , for all } 1 \leq \alpha \leq n \,. \tag{1.1}$$

where $\tau^\alpha(u)$ represents the tension field $\tau(u)$ in local coordinates.

By Nash's embedding theorem, we may assume that (N^n, h) is isometrically embedded in $\mathbb{R}^L$ for some L. Since N is a smooth compact submanifold, we can find a *tubular neighborhood*

$$N_\delta := \left\{ u \in \mathbb{R}^L \,\middle|\, d(u, N) := \inf_{z \in N} |u - z| < \delta \right\} ,$$

and the smooth *nearest point projection map* $\Pi_N \colon N_\delta \to N$ satisfying

$$\Pi_N(u) \in N \text{ and } |u - \Pi_N(u)| = d(u, N) \text{ for } u \in N \,.$$

Notice that for $u \in N$, the map $P(u) := \nabla \Pi_N(u) \colon \mathbb{R}^L \to T_u N$ is a projection map , and

$$A(u) := \nabla P(u) \colon T_u N \otimes T_u N \to (T_u N)^\perp$$

is the second fundamental form of $N \subset \mathbb{R}^L$. Let $\{\nu_{n+1}(u), \cdots, \nu_L(u)\}$ be a local orthonormal frame of the normal bundle $(T_u N)^\perp$. The harmonic map equation (1.1) from the extrinsic point of view is given by

$$\Delta_g u + \sum_{n+1 \leq \alpha \leq L} g^{ij} A^\alpha(u) \left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right) \nu_\alpha(u) = 0 \,,$$

where $A^\alpha := \nabla \nu_\alpha$ is the second fundamental form of N in the normal direction ν_α.

In particular, when the domain is the Euclidean space $M = \mathbb{R}^m$ and the target is the unit sphere $N = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, a map $u \in C^2(\mathbb{R}^m, \mathbb{S}^n)$ is a harmonic map if and only if

$$\Delta u + |\nabla u|^2 u = 0 \text{ in } \mathbb{R}^m \,.$$

A basic problem in geometric topology is to find harmonic maps representing a given homotopy class. Moreover, given any homotopy class $\alpha \in [M, N]$, we want to know under what conditions can the minimal energy:

$$c_\alpha := \inf \{\mathcal{E}(u) \,|\, u \in C^\infty(M, N), [u] = \alpha \in [M, N]\}$$

be achieved. Since the topological properties are not preserved under the weak convergence in $W^{1,2}(M, N)$ in general, the direct method may not work.

For example, consider the 1-parameter family of maps u_λ from the extended complex plane $\hat{\mathbb{C}} \cong \mathbb{S}^2$ (defined on the next page) to itself:

$$u_\lambda(z) := \lambda z \text{ for } \lambda > 0.$$

For each $\lambda > 0$, we have $\mathcal{E}(u_\lambda) = 4\pi$ and $\deg(u_\lambda) = 1$. However, $u_\lambda \to (0, 0, 1)^T \in \mathbb{S}^2$ weakly in $W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$. Thus neither the topological degree nor the Dirichlet energy are preserved under the limiting process.

To overcome such difficulties, Eells-Sampson [4] first proposed the study of the corresponding evolution problem. More precisely, they considered the negative L^2-gradient flow

$$\frac{\partial u}{\partial t} = \tau(u) \text{ on } M \times [0, \infty)$$

with initial and boundary data $u = u_0$ at $t = 0$ and on $\partial M \times [0, \infty)$. As the energy is decreasing in time, one can hope that if the limiting steady state u_∞ exists and is smooth, such a continuous deformation $u(\cdot, t)$ of u_0 preserves the homotopy class, and is a critical point of $\mathcal{E}$ within the homotopy class.

In a series of papers [7], [8], [9], [10], [11], Gustafson, Tsai et al. considered the map evolution problem: $u(\cdot, t) \colon \mathbb{R}^2 \to \mathbb{S}^2$ for the *Landau-Lifshitz equations*:

$$\frac{\partial u}{\partial t} = a(\Delta u + |\nabla u|^2 u) + bu \times \Delta u$$

with $a \geq 0$, $b \in \mathbb{R}$. Notice that $a = 1$, $b = 0$ corresponds to the harmonic map heat flow from $\mathbb{R}^2$ to $\mathbb{S}^2$. Under additional symmetry assumptions, different stability results can be obtained. For example, for any fixed integer m, consider an *m-equivariant map* $u \colon \mathbb{R}^2 \to \mathbb{S}^2$ of the form $u = u(r, \theta) = e^{m\theta R} v(r)$, where (r, θ) is the polar coordinate system on $\mathbb{R}^2$, $v \colon [0, \infty) \to \mathbb{S}^2$, and R is the matrix generating rotations around the

u_3-axis.

Denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the *extended complex plane*. Consider the stereographic projection $\pi \colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ and its inverse $\pi^{-1} \colon \hat{\mathbb{C}} \to \mathbb{S}^2$ given by:

$$\pi \colon \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \frac{u_1 + iu_2}{1 - u_3} \ , \ \pi^{-1} \colon z \mapsto \frac{1}{1+|z|^2} \begin{bmatrix} 2\mathrm{Re}(z) \\ 2\mathrm{Im}(z) \\ |z|^2 - 1 \end{bmatrix} ,$$

where π maps the north pole $N := (0,0,1)$ to ∞. For any C^1-map $u \colon \mathbb{R}^2 \to \mathbb{S}^2$, let $\deg(u)$ be the Brouwer degree of $u \circ \pi \colon \mathbb{S}^2 \to \mathbb{S}^2$ (i.e., defined by integrating the pull-back of $u \circ \pi$ of the volume form on $\mathbb{S}^2$.)

The Dirichlet energy has a lower bound: $\mathcal{E}(u) \geq 4\pi|\deg(u)|$. In the context of m-equivariant maps, denote

$$\Sigma_m := \left\{ u : \mathbb{R}^2 \to \mathbb{S}^2 \,\middle|\, u = u(r,\theta) = e^{m\theta R} v(r) \ , \ \mathcal{E}(u) < \infty \ , \ v(0) = \hat{k} \ , \ v(\infty) = -\hat{k} \right\} ,$$

where $\hat{k} = (0,0,1)^T \in \mathbb{S}^2$. For any $u \in \Sigma_m$, the Dirichlet energy is bounded below: $\mathcal{E}(u) \geq \mathcal{E}_{min} = 4\pi|m|$, where the minimum is attained by a 2-parameter family $\mathcal{O}_m := \left\{ e^{m\theta R} h^{s,\alpha}(r) \,|\, s > 0 \ , \ \alpha \in \mathbb{R} \right\}$ of harmonic maps within Σ_m.

Guan-Gustafson-Tsai [8] considered initial maps with energy near the minimum $\mathcal{E}_{min} = 4\pi|m|$. That is, assume the initial data is

$$u_0 \in \Sigma_m \ , \ \mathcal{E}(u_0) = 4\pi|m| + \delta_0^2 \ , \ \delta_0 \ll 1.$$

Let $u(t) \in C([0,T), \Sigma_m)$ be the solution corresponding to the initial data u_0 (Σ_m topologized with the energy $\dot{H}^1$ norm). For $|m| \geq 4$, it was shown that the maximal existence time $T = T(u_0)$ can be taken to be infinity. Moreover, there exists continuous parameters $s = s(t)$ and $\alpha = \alpha(t)$ on $[0, \infty)$ which converges as $t \to \infty$, such that $\|\nabla \left(u(x,t) - e^{m\theta R} h^{s(t),\alpha(t)}(r) \right) \|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \lesssim \delta_0$.

Inspired by the aforementioned *asymptotic stability* result for harmonic map heat flow from $\mathbb{R}^2$ to $\mathbb{S}^2$, we would like to obtain similar stability results for the compact manifold of flat 2-tori $\mathbb{T}^2$.

Notice that since $\mathbb{T}^2$ is compact, connected, and oriented, by Hopf Degree Theorem, two smooth maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ are homotopic if and only if they have the same topological degree.

For fixed non-zero complex numbers ω_1 and ω_2 which are linearly independent over $\mathbb{R}$, let

$$\Lambda := \{n\omega_1 + m\omega_2 \,|\, n, m \in \mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ generated by ω_1 and ω_2, and let $\mathbb{T}^2 := \mathbb{C}/\Lambda$ be the *flat torus* defined as the quotient space of $\mathbb{C}$ over the lattice Λ.

The prototypical harmonic map from $\mathbb{T}^2$ to $\mathbb{S}^2$ is $\pi^{-1} \circ \wp$, where $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ is the stereographic projection:

$$\pi\colon (u_1, u_2, u_3)^T \mapsto \frac{u_1 + iu_2}{1 - u_3},$$

and $\wp\colon \mathbb{T}^2 \to \hat{\mathbb{C}}$ is the Weierstrass $\wp$ function defined by:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right],$$

where $\Lambda^* := \Lambda \backslash \{0\}$. Being a double cover, the Brouwer degree of $\pi^{-1} \circ \wp\colon \mathbb{T}^2 \to \mathbb{S}^2$ is two.

In fact, all elliptic functions from $\mathbb{T}^2$ to $\mathbb{C}$ can be written as a rational function of $\wp$ and $\wp'$. This enables one to classify all conformal mappings from $\mathbb{T}^2$ to $\mathbb{S}^2$ by topological degrees.

While harmonic maps on $\mathbb{T}^2$ may not be conformal maps in general (an example was constructed by Hitchin [14] in the case of maps to the 3-sphere $\mathbb{S}^3$), we only consider energy-minimizing ones, which are all in the class of conformal mappings.

For example, the collection of degree-2 conformal maps (with Dirichlet energy $\mathcal{E}_{min} = 8\pi$) are given by:

$$\mathcal{M}_2^8 = \left\{ \pi^{-1} \circ \frac{a\wp(\cdot - z_0) + b}{c\wp(\cdot - z_0) + d} \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C}), z_0 \in \mathbb{C} \right\},$$

a manifold of (real) dimension $d(2) = 8$.

More generally, for any $m \geq 2$, the collection of degree-m conformal maps from $\mathbb{T}^2$ to $\mathbb{S}^2$, denoted by $\mathcal{M}_m^d$, is a d-dimensional manifold for some finite $d = d(m) < \infty$. We conjecture that if the initial condition u_0 from $\mathbb{T}^2$ to $\mathbb{S}^2$ of topological degree m is close enough to a fixed steady state $\bar{u} \in \mathcal{M}_m^d$ in some suitable norm space, then the harmonic map heat flow:

$$u_t = \Delta u + |\nabla u|^2 u \ , \ u|_{t=0} = u_0,$$

can have its maximal existence time taken to be infinity, and there exists another harmonic map $\bar{\bar{u}} \in \mathcal{M}_m^d$ of degree m such that $u(\cdot, t) \to \bar{\bar{u}}$ as $t \to \infty$ in a suitable function space.

Conjecture 1. *Given any integer $m \geq 2$, let $\mathcal{M}_m^d$ be the d-dimensional manifold of conformal maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ of degree m. For any $\bar{u} \in \mathcal{M}_m^d$, there exists $\delta_0 = \delta_0(\bar{u}) > 0$ such that if $u_0 \in H^2(\mathbb{T}^2, \mathbb{S}^2)$ is of degree m (hence homotopic to $\bar{u}$) with*

$$\|u_0 - \bar{u}\|_{H^2(\mathbb{T}^2, \mathbb{R}^3)} \leq \delta_0^2$$

and

$$4\pi m \leq \mathcal{E}(u_0) < 4\pi m + \delta_0^2,$$

then for the harmonic map heat flow:

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \\ u|_{t=0} &= u_0 \end{aligned}$$

(a) The maximal existence time $T = T(u_0)$ can be taken to be infinity.

(b) There exists $\bar{\bar{u}} \in \mathcal{M}_m^d$ such that

$$u(\cdot, t) \to \bar{\bar{u}} \text{ in } H^2(\mathbb{T}^2, \mathbb{S}^2) \text{ as } t \to \infty.$$

The main difference between the domains $\mathbb{T}^2$ and $\mathbb{R}^2$ is that the the compactness of the torus $\mathbb{T}^2$ implies the existence of a smallest non-zero eigenvalue $\lambda_1 > 0$ for the second variation $Q_{\bar{u}}$ of the energy functional $\mathcal{E}(u)$, and therefore all perturbation ξ in $(\ker(Q_{\bar{u}}))^{\perp}$ (the perpendicular space of co-dimension d) will decay exponentially like $e^{-\lambda_1 t}$ in the proper function space.

In other words, the compactness of $\mathbb{T}^2$ guarantees a *gap* in the spectrum of $Q_{\bar{u}}$, which is in contrast to the non-compact manifold $\mathbb{R}^2$ for which an arbitrary slow decay of the $(\ker(Q_{\bar{u}}))^\perp$ part may occur.

To illustrate the situation, let $\bar{u}$ be a fixed degree-m conformal map from $\mathbb{T}^2$ to $\mathbb{S}^2$. Since $\Delta\bar{u} + |\nabla\bar{u}|^2\bar{u} = 0$, the corresponding linearized map near $\bar{u}$ reads:

$$L_{\bar{u}}[v] = \Delta v + |\nabla\bar{u}|^2 v + 2(\nabla\bar{u}\cdot\nabla v)\bar{u}.$$

A natural parametrization for functions u of degree m near $\bar{u}$ $(\in \mathcal{M}_m^d)$ is given by

$$u = \frac{\bar{u}+\xi}{|\bar{u}+\xi|}\ , \text{ where } \xi : \mathbb{T}^2 \to \mathbb{R}^3 \text{ satisfies } \bar{u}\cdot\xi = 0 \text{ point-wise in } \mathbb{T}^2.$$

The harmonic map heat equation in terms of ξ is given by:

$$\begin{aligned}\left(I_3 - \frac{(\bar{u}+\xi)\otimes\xi}{1+|\xi|^2}\right)\xi_t = \Delta\xi + \frac{|\nabla\bar{u}|^2(-|\xi|^2\bar{u}+\xi) + (-\Delta\xi\cdot\xi + 2\nabla\bar{u}\cdot\nabla\xi)(\bar{u}+\xi)}{1+|\xi|^2} \\ - \frac{1}{1+|\xi|^2}\nabla(\bar{u}+\xi)\cdot\nabla|\xi|^2 + \frac{|\nabla|\xi|^2|^2}{2(1+|\xi|^2)^2}(\bar{u}+\xi).\end{aligned} \tag{1.2}$$

The following theorem is the main result of this thesis concerning *the stability for the linearization of the* ξ*-equation at the steady state* $\bar{u}$.

Theorem 1. *Let* $\{\phi_i\colon 1\le i\le d\}$ *be a* L^2*-orthonormal basis of* $\ker(L_{\bar{u}})$*. Consider the linearized equation*

$$\xi_t = \Delta\xi + |\nabla\bar{u}|^2\xi + 2(\nabla\bar{u}\cdot\nabla\xi)\bar{u} =: L_{\bar{u}}[\xi].$$

For any initial condition decomposed as:

$$\xi|_{t=0} = \sum_{i=1}^{d} c_i\phi_i(z) + \eta(z) \in H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2),$$

where $c_i\in\mathbb{R}$ *for each* $1\le i\le d$ *, and* $\eta\in(\ker(L_{\bar{u}}))^\perp$ *in* $L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$*, we have*

$$\xi(z,t) \to \sum_{i=1}^{d} c_i\phi_i(z)\ (\in \ker(L_{\bar{u}}))\ \text{ in } L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)\ \text{ as } t\to\infty.$$

The aforementioned assertion is infinite-dimensional in essence. Namely, the function space $H^2(\mathbb{T}^2, \mathbb{S}^2)$ and its linearized tangent space $H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$ at $\bar{u}$ are infinite-dimensional.

While theorem 1 demonstrates the stability for the lineraized equation, the infinite-dimensional nature of $H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$ presents greater difficulties in obtaining the desired *stability* results for the non-linear equation (1.2).

To that end, we consider the following *finite-dimensional model* of the gradient flow problem in the following proposition. Up to first order, it is analogous to the non-linear problem but with the *contracting* part being finite-dimensional:

Proposition 1. *For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$, let $A = A(x) \in C^1(\mathbb{R}^d, \mathbb{R}^{n\times n})$ and define*

$$f = f(x, y) := \frac{1}{2}\left(A(x) \cdot y, y\right)_{\mathbb{R}^n}.$$

Denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm of $n \times n$ matrices. Assume

$$c_1|\eta|^2 \le (A(x)\cdot\eta, \eta)_{\mathbb{R}^n} \le c_2|\eta|^2 \text{ for all } x \in \mathbb{R}^d \text{ and } \eta \in \mathbb{R}^n \text{ , for some } 0 < c_1 < c_2 < \infty;$$

and

$$\max_{1\le i\le d}\left\|\frac{\partial A}{\partial x_i}(x)\right\|_{HS} \le K(1+|x|) \text{ for all } x \in \mathbb{R}^d \text{ , for some } K > 0.$$

Consider the gradient flow:

$$\begin{aligned} \frac{dx_i}{dt} &= -\nabla_{x_i} f = -\frac{1}{2}\left(\frac{\partial A}{\partial x_i}(x)\cdot y, y\right)_{\mathbb{R}^n} \text{ , for each } 1 \le i \le d, \\ \frac{dy}{dt} &= -\nabla_y f = -A(x)\cdot y. \end{aligned}$$

For any initial condition $(x(0), y(0)) = (x_0, y_0)$, the maximal time of existence is equal to infinity, and

$$x(t) \to \bar{x} \text{ , } y(t) \to 0 \text{ as } t \to \infty \text{ for some } \bar{x} \in \mathbb{R}^d.$$

In chapter 2, we survey some known results regarding harmonic maps and its heat flow, including the short time existence result of Eells-Sampson [4], the global-in-time smooth solutions for target manifolds with non-positive sectional curvature [4], the blow-up analysis in dimension 2 by Sacks-Uhlenbeck [21], the existence of global weak solutions with finitely many singular points by Struwe [24], and the finite time blow-up example of Chang-Ding-Ye [2]. More details can be found in books like [18] and [22].

In chapter 3, we present the main ideas of *asymptotic stability* result for harmonic map heat flow from $\mathbb{R}^2$ to $\mathbb{S}^2$ due to Gustafson, Tsai et al. [7], [8], [9], [10], [11].

In chapter 4, we introduce some basic properties of elliptic functions and formulate our main conjecture (Conjecture 1) regarding harmonic map heat flow from $\mathbb{T}^2$ to $\mathbb{S}^2$.

In chapter 5, we proved the main result of this thesis (Theorem 1) concerning the stability result for the lineraized ξ-equation.

In chapter 6, we present the proof of the stability result for the analogous finite-dimensional model of proposition 1.

Chapter 2

Survey of Known Results

2.1 Harmonic map heat flow

A fundamental problem in geometric topology is to find harmonic maps between two Riemannian manifolds (M, g) and (N, h) representing any given homotopy class $\alpha \in [M, N]$. For example, a basic question one can ask is: can

$$c_\alpha := \inf \{\mathcal{E}(u) \,|\, u \in C^\infty(M, N), [u] = \alpha \in [M, N]\}$$

be achieved?

The answer is negative in general. For example,

Proposition 2. *(Eells-Wood [3]) Let $M = \mathbb{T}^2$ and $N = \mathbb{S}^2$ be the flat 2-torus and the standard 2-sphere. There does not exist any harmonic map in the homotopy class $\alpha \in [\mathbb{T}^2, \mathbb{S}^2]$, the collection of continuous maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ of degree 1.*

Also, for (M, g) with non-empty boundary ∂M, the following proposition shows that if we consider harmonic maps representing relative homotopy classes, then there may be no solutions.

Proposition 3. *Let $m \geq 3$ and $M = \mathbb{B}^m$ be the unit ball in $\mathbb{R}^m$. If $u \in C^\infty(\overline{\mathbb{B}^m}, N)$ is a harmonic map such that $u|_{\partial \mathbb{B}^m}$ =constant, then u=constant.*

Eells-Sampson [4] proposed to study the corresponding *evolution problem* for the homotopy problem. More precisely, for any $u_0 \in C^\infty(M, N)$, let $A(u)$ denotes the second fundamental form of N isometrically embedded in some $\mathbb{R}^L$. Consider the evolution equation for $u : M \times [0, \infty) \to N$:

$$\frac{\partial u}{\partial t} - \Delta_g u = A(u)(\nabla u, \nabla u) \text{ in } M \times (0, \infty) \tag{2.1}$$

$$u|_{t=0} = u_0 \tag{2.2}$$

The key idea is that eq. (2.1) is the negative L^2-gradient flow of the Dirichlet energy $\mathcal{E}$, under which the energy is decreasing and hence one may hope to find harmonic maps (critical points of $\mathcal{E}$) along the flow. Notice that any homotopy class is preserved by a continuous deformation $u(\cdot, t)$ of u_0.

2.2 Local existence

The existence of local-in-time smooth solutions was first established by Eells and Sampson:

Theorem 2. *(Eells-Sampson [4]) Assume (M, g) and $(N, h) \subset \mathbb{R}^L$ are compact manifolds without boundaries. For any $u_0 \in C^\infty(M, N)$, there exists a maximal existence time $T^* = T^*(M, N, u_0) \leq \infty$ such that eq. (2.1) and eq. (2.2) admits a unique smooth solution $u \in C^\infty(M \times [0, T^*), N)$. Moreover, if $T^* < \infty$, then*

$$\lim_{t \to T^*} \|\nabla u(\cdot, t)\|_{C^0(M)} = +\infty.$$

Proof. To simplify the proof, we assume $M = \mathbb{R}^m$ with the standard Euclidean metric. Recall that the fundamental solution to the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) G(x, t) = 0 \text{ for } x \in \mathbb{R}^m, t > 0,$$

$$\lim_{t \to 0+} G(x, t) = \delta_0(x) \text{ for } x \in \mathbb{R}^m$$

is given by

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x|^2}{4t}} \text{ for } x \in \mathbb{R}^m, t > 0.$$

Let $u\colon \mathbb{R}^m \times [0,T) \to \mathbb{R}^L$ be the solution to the Cauchy problem

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = f \text{ for } x \in \mathbb{R}^m, t \in (0,T),$$
$$u = u_0 \text{ for } x \in \mathbb{R}^m.$$

The representation formula gives

$$u(x,t) = \int_{\mathbb{R}^m} G(x-y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^m} G(x-y,t-s)f(y,s)\,dyds.$$

Recall the following properties regarding linear parabolic equations [16], [17]:

- If $u_0 \in C(\mathbb{R}^m, \mathbb{R}^L)$ and $f \in L^\infty(\mathbb{R}^m \times [0,T], \mathbb{R}^L)$, then
  (i) $u \in C(\mathbb{R}^m \times [0,T], \mathbb{R}^L)$, and
  (ii) $u(\cdot,t) \in C^{1,\alpha}(\mathbb{R}^m)$ uniformly in $t \in [\epsilon, T]$, for each $\epsilon > 0$.
- If $f(\cdot,t) \in C^\alpha(\mathbb{R}^m)$ uniformly in $t \in [\epsilon,T]$ for some $\alpha \in (0,1)$ and $\epsilon > 0$, then
  (i) u_t, ∇u, $\nabla^2 u \in C(\mathbb{R}^m \times [\epsilon,T], \mathbb{R}^L)$.

If $u \in C^\infty(\mathbb{R}^m \times [0,T), N)$ solves the harmonic map heat equation (2.1) and (2.2), then the representation formula implies

$$u(x,t) = v_0(x,t) + \int_0^t \int_{\mathbb{R}^m} G(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s)\,dyds,$$

where

$$v_0(x,t) := \int_{\mathbb{R}^m} G(x-y,t)u_0(y)\,dy.$$

Now for small $\epsilon > 0$, consider the function space

$$X_\epsilon := \left\{u \in C^0(\mathbb{R}^m \times [0,\epsilon], \mathbb{R}^L) \,|\, u(\cdot,0) = u_0, u(\cdot,t) \in C^1(\mathbb{R}^m) \text{ for any } 0 \le t \le \epsilon\right\},$$

endowed with the norm

$$\|u\|_{X_\epsilon} := \|u\|_{C^0(\mathbb{R}^m \times [0,\epsilon])} + \sup_{0 \le t \le \epsilon} \|\nabla u(\cdot,t)\|_{C^0(\mathbb{R}^m)}.$$

Let

$$B_\delta := \{u \in X_\epsilon \mid \|u - v_0\|_{X_\epsilon} \le \delta\}$$

be the ball in X_ϵ with radius δ centered at v_0, and define the map $T\colon X_\epsilon \to X_\epsilon$ by

$$(Tu)(x,t) = v_0(x,t) + \int_0^t \int_{\mathbb{R}^m} G(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s)\,dyds$$

for $(x,t) \in \mathbb{R}^m \times [0,\epsilon)$. By the representation formula, $u \in C^\infty(\mathbb{R}^m \times [0,\epsilon], N)$ solves eq. (2.1) and eq. (2.2) if and only if $Tu = u$.

- For $u \in B_\delta$, $\|u\|_{X_\epsilon} \le \|u - v_0\|_{X_\epsilon} + \|v_0\|_{X_\epsilon} \le \delta + \|v_0\|_{X_\epsilon}$ is uniformly bounded. So for any $(x,t) \in \mathbb{R}^m \times [0,\epsilon)$, we have

$$\begin{aligned} |Tu - v_0|(x,t) &\lesssim \int_0^t \int_{\mathbb{R}^m} G(x-y,s-t)|\nabla u|^2(y,s)\,dyds \\ &\lesssim \sup_{0\le t\le\epsilon} \|\nabla u\|^2_{C^0(\mathbb{R}^m)} \int_0^\epsilon \int_{\mathbb{R}^m} G(x-y,t-s)\,dyds \lesssim \epsilon \end{aligned}$$

  and

$$\begin{aligned} |\nabla(Tu - v_0)|(x,t) &\lesssim \int_0^t \int_{\mathbb{R}^m} |\nabla G|(x-y,s-t)|\nabla u|^2(y,s)\,dyds \\ &\lesssim \sup_{0\le t\le\epsilon} \|\nabla u\|^2_{C^0(\mathbb{R}^m)} \int_0^\epsilon \int_{\mathbb{R}^m} |\nabla G|(x-y,t-s)\,dyds \lesssim \epsilon. \end{aligned}$$

  Therefore for any $\delta > 0$, there exists $\epsilon > 0$ such that

$$T\colon B_\delta \to B_\delta.$$

- For any u, $v \in B_\delta$ and fixed $(x,t) \in \mathbb{R}^m \times [0,\epsilon)$, we have

$$\begin{aligned} |Tu - Tv|(x,t) &\le \int_0^t \int_{\mathbb{R}^m} G(x-y,t-s)|A(u)(\nabla u,\nabla u) - A(v)(\nabla v,\nabla v)|(y,s)\,dyds \\ &\lesssim \int_0^\epsilon \int_{\mathbb{R}^m} G(x-y,t-s)\left\{|\nabla u|^2|u-v|\right\}(y,s)\,dyds \\ &+ \int_0^\epsilon \int_{\mathbb{R}^m} G(x-y,t-s)\left\{(|\nabla u| + |\nabla v|)|\nabla(u-v)|\right\}(y,s)\,dyds \\ &\lesssim \epsilon\|u-v\|_{X_\epsilon} \end{aligned}$$

and

$$\begin{aligned}
&|Tu - Tv|(x,t) \\
&\leq \int_0^t \int_{\mathbb{R}^m} |\nabla G|(x-y,t-s)|A(u)(\nabla u,\nabla u) - A(v)(\nabla v,\nabla v)|(y,s)\,dyds \\
&\lesssim \int_0^\epsilon \int_{\mathbb{R}^m} |\nabla G|(x-y,t-s)\left\{|\nabla u|^2|u-v|\right\}(y,s)\,dyds \\
&+ \int_0^\epsilon \int_{\mathbb{R}^m} |\nabla G|(x-y,t-s)\left\{(|\nabla u|+|\nabla v|)|\nabla(u-v)|\right\}(y,s)\,dyds \\
&\lesssim \epsilon\|u-v\|_{X_\epsilon}
\end{aligned}$$

Thus for any $\delta > 0$, there exists $\epsilon > 0$ and $\beta \in (0,1)$ such that

$$\|Tu - Tv\|_{X_\epsilon} \leq \beta\|u-v\|_{X_\epsilon} \text{ for any } u, v \in B_\delta.$$

By contraction mapping principle, (i) and (ii) implies the existence of a unique $u \in B_\delta$ such that $u = Tu$, which mean $u\colon \mathbb{R}^m \times [0,\epsilon) \to \mathbb{R}^L$ solves eq. (2.1) and eq. (2.2).

To show that $u(\mathbb{R}^m \times [0,\epsilon)) \subset N$, let $\Pi_N\colon N_\delta \to N$ be the smooth nearest point projection from the tubular neighborhood N_δ of N in $\mathbb{R}^L$, and define

$$\rho(u) := |\Pi_N(u) - u|^2\colon \mathbb{R}^m \times [0,\epsilon) \to \mathbb{R}.$$

Direct calculations gives

$$\begin{aligned}
\rho(u)(\cdot,0) &= 0 \text{ in } \mathbb{R}^m, \\
\left(\frac{\partial}{\partial t} - \Delta\right)\rho(u) &= -|\nabla(\Pi_N(u) - u)|^2 \leq 0 \text{ in } \mathbb{R}^m \times [0,\epsilon).
\end{aligned}$$

Hence by maximum principle we have $\rho(u) \equiv 0$, which means $u = \Pi_N(u) \in N$.

Finally, as a quasi-linear parabolic system, if

$$\lim_{t\to T} \|\nabla u(\cdot,t)\|_{L^\infty(M)} < \infty,$$

then u can be extended to a smooth solution beyond T [11]. Therefore there exists a maximal time $T^* \in (0, \infty]$ such that $u \in C^\infty(M \times [0, T^*), N)$, and

$$\lim_{t \to T^*} \|\nabla u(\cdot, t)\|_{C^0(M)} = +\infty$$

if $T^* < +\infty$. □

Remark 1. The $C^0(M)$ norm characterization of the first finite singular time T^* is not optimal in higher dimensions. Wang [26] showed that for $m \geq 3$, the first finite singular time can be characterized by

$$\limsup_{t \to T^*} \|\nabla u(\cdot, t)\|_{L^m(M)} = +\infty.$$

2.3 Global existence under $K^N \leq 0$: Eells-Sampson

When the sectional curvature K^N of the target manifold N is non-positive, Eells and Sampson proved that the existence of global-in-time solutions are in fact attainable.

Theorem 3. *([4]) Assume (M, g) is compact without boundary and the sectional curvature K^N of (N, h) is non-positive. For any $u_0 \in C^\infty(M, N)$, the Cauchy problem eq.* (2.1) *and eq.* (2.2) *admits a unique, smooth solution $u \in C^\infty(M \times [0, \infty), N)$. Moreover, we can choose an increasing sequence of times $t_k \to \infty$ such that*

$$u(\cdot, t_k) \to u_\infty \text{ as } k \to \infty \text{ in } C^2(M, N)$$

for some harmonic map $u_\infty \in C^\infty(M, N)$.

Proof. For fixed $T \in (0, \infty]$, assume $u \in C^\infty(M \times [0, T), N)$ solves eq. (2.1). Since $A(u)(\nabla u, \nabla u)$ is perpendicular to $T_u N$, if we take the inner product of eq. (2.1) with u_t and integrate over M, we will get:

$$\int_M |u_t|^2 \, dV_g + \frac{d}{dt}\mathcal{E}(u(t)) = 0,$$

which implies the *energy equality*:

$$\mathcal{E}(u(t)) + \int_0^t \int_M |u_t|^2 \, dV_g dt = \mathcal{E}(u_0) \text{ , for each } t \in [0, T). \tag{2.3}$$

- If $u \in C^\infty(M \times [0,T), N)$, the Bochner identity for the energy density $e(u)$ reads:

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) e(u) = -|\nabla du|^2 - Ric^M(\nabla u, \nabla u) + R^N(\nabla u, \nabla u, \nabla u, \nabla u).$$

In particular, when $K^N \leq 0$, there exists a constant $C > 0$, depending only on the Ricci curvature Ric^M of M, such that

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) e(u) \leq Ce(u).$$

- Let $\mathcal{L} := \partial_t - \Delta$ be the heat operator on M, and let i_M be the injectivity radius of M. For $z_0 = (x_0, t_0) \in M \times (0, \infty)$ and $R \in (0, \min\{i_M, \sqrt{t_0}\})$, denote by

$$P_R(z_0) := \left\{z = (x,t) \in M \times (0,\infty) \,|\, |x - x_0| \leq R, t_0 - R^2 \leq t \leq t_0\right\}$$

the parabolic cylinder around z_0. Recall Moser's Harnack inequality for subsolutions of the heat equation [17], [19]: If $v \in C^\infty(P_R(z_0))$ is non-negative and satisfies

$$\mathcal{L}v \leq Cv \text{ in } P_R(z_0)$$

for some $C > 0$, then there exists $C_1 > 0$ such that

$$v(z_0) \leq C_1 R^{-(m+2)} \int_{P_R(z_0)} v \, dz.$$

- Apply Moser's Harnack inequality to the energy density $e(u)$ and use the energy equality eq. (2.3), we have

$$\begin{aligned} e(u)(z_0) &\lesssim R^{-(m+2)} \int_{P_R(z_0)} e(u)\, dz \leq R^{-(m+2)} \int_{t_0 - R^2}^{t_0} \mathcal{E}(u(t))\, dt \\ &\leq R^{-(m+2)} \cdot R^2 \cdot \mathcal{E}\left(u(t_0 - R^2)\right) \leq R^{-m} \mathcal{E}(u_0), \end{aligned}$$

for any $z_0 = (x_0, t_0) \in M \times (0, \infty)$ and small $R > 0$. Therefore $|\nabla u|$ is uniformly bounded on $M \times [0,T]$, and thus by higher regularities of linear parabolic equations [17], we conclude that

$$u \in C^{\infty}(M \times [0, \infty), N).$$

- Direct calculations have

$$\frac{\partial}{\partial t}|u_t|^2 = \Delta_g |u_t|^2 - |\nabla u_t|^2 + R^N(u)(\nabla u, u_t, \nabla u, u_t),$$

which implies

$$\left(\frac{\partial}{\partial t} - \Delta_g\right)|u_t|^2 = -|\nabla u_t|^2 + R^N(u)(\nabla u, u_t, \nabla u, u_t) \leq 0,$$

when $K^N \leq 0$. Hence again by Moser's Harnack inequality, for any $\alpha \in (0,1)$ and $t > 2$, we have

$$\|u_t\|_{C^\alpha(M\times[t-1,t])} \leq C(\alpha)\|u_t\|_{L^2(M\times[t-2,t])}.$$

On the other hand, since

$$\int_0^t \int_M |u_t|^2 \, dV_g dt \leq \mathcal{E}(u_0) < \infty \text{ for any } t > 0,$$

we have

$$\lim_{t\to\infty} \int_{t-2}^t \int_M |u_t|^2 \, dV_g dt = 0.$$

Therefore

$$\|u_t\|_{C^\alpha(M\times[t-1,t])} \leq C(\alpha)\|u_t\|_{L^2(M\times[t-2,t])} \to 0 \text{ as } t \to \infty,$$

which implies that we can choose a sequence $t_k \to \infty$ such that

$$u_t(\cdot, t_k) \to 0 \text{ in } C^\alpha(M, N) \text{ and } u(\cdot, t_k) \to u_\infty \text{ in } C^2(M, N)$$

for some $u_\infty \in C^2(M, N)$. Moreover, since $u_\infty \in C^2(M, N)$ solves

$$\Delta_g u_\infty + A(u_\infty)(\nabla u_\infty, \nabla u_\infty) = 0,$$

u_∞ is in fact a smooth harmonic map.

□

Remark 2. Using the second variational formula for energy, it can be shown that the harmonic map $u_\infty \in C^\infty(M, N)$ constructed above is in fact energy-minimizing within its homotopy class, provided $K^N \leq 0$. Hartman [13] used this fact to prove that the harmonic map u_∞ is independent of the choice of the sequence $t_k \to \infty$.

Remark 3. A similar result was obtained by Hamilton [12] for the initial and boundary value problem when $\partial M \neq \emptyset$. More precisely, assume $K^N \leq 0$, then given any $\phi \in C^\infty(\bar{M}, N)$, consider the initial-boundary value problem for $u\colon \bar{M} \times [0, \infty) \to N$:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta_g u &= A(u)(\nabla u, \nabla u) \text{ in } M \times (0, \infty), \\ u|_{t=0} &= \phi \text{ on } M, \\ u(x, t) &= \phi(x) \text{ for } x \in \partial M, t > 0. \end{aligned}$$

There exists a unique smooth solution $u \in C^\infty(\bar{M} \times [0, \infty), N)$.

2.4 Blow-up analysis in dimension 2: Sacks-Uhlenbeck

When M is two-dimensional, the Dirichlet energy is conformally invariant. It is important to understand the limiting behavior of sequences of solutions to harmonic maps and their evolution problems in dimension two. Sacks-Uhlenbeck [21] developed the powerful blow up technique to study harmonic maps in dimension two. They discovered that the failure of strong convergence of solutions of harmonic maps stems from the energy concentration at finitely many points. In this section we introduce some basic ideas from Sacks-Uhlenbeck [21] and record some important lemmas without giving proofs.

Let (M^2, g) be a compact Riemannian surface with or without boundary and let $N \subset \mathbb{R}^L$ be a compact Riemannian manifold without boundary. For $\alpha \geq 1$, consider the Sobolev space

$$W^{1,2\alpha}(M, N) := \left\{ u : M \to N \,|\, \nabla u \in L^{2\alpha}(M, \mathbb{R}^{2L}) \right\},$$

and define

$$E_\alpha(u) := \int_M (1 + |\nabla u|^2)^\alpha \, dV_g \text{ for } u \in W^{1,2\alpha}(M, N).$$

E_α can be viewed as sub-critical approximations of $E_1(u) = \mathrm{Vol}(M) + \mathcal{E}(u)$, whose critical points are harmonic maps. More precisely, for $\alpha > 1$, since M is two-dimensional, we have $W^{1,2\alpha}(M, N) \subseteq C^{1-\frac{1}{\alpha}}(M, N)$ by Sobolev embedding theorem. Therefore, $W^{1,2\alpha}(M, N)$ is a C^2 separable Banach manifold. Since

$$E_\alpha : W^{1,2\alpha}(M, N) \to \mathbb{R}^+$$

satisfied the Palais-Smale condition [20], the direct method in calculus of variations yield:

Lemma 1. *Given $\alpha > 1$ and $\phi \in C^\infty(M, N)$, there exists $u_\alpha \in C^\infty(M, N)$ in the same homotopy class as ϕ such that*

$$E_\alpha(u_\alpha) = \inf \left\{ E_\alpha(v) \,|\, v \in W^{1,2\alpha}(M, N), [v] = [\phi] \right\}.$$

Moreover, u_α solves the 2α-harmonic map equation:

$$div \left((1 + |\nabla u_\alpha|^2)^{\alpha - 1} \nabla u_\alpha \right) = (1 + |\nabla u_\alpha|^2)^{\alpha - 1} A(u_\alpha)(\nabla u_\alpha, \nabla u_\alpha).$$

Since all estimates here are local, we may assume $M \subset \mathbb{R}^2$. The followings two interior estimates can be attained:

Lemma 2. *Let $B := \mathbb{B}^2$ be the unit disc in $\mathbb{R}^2$ and let $u \colon B \to N$ be a critical point of E_α. For fixed $p \in (1, \infty)$, if $\alpha - 1$ is small enough depending on p, then for any smaller disc $B' \subset B$, there exists a constant $C = C(p, B', B, \|\nabla u\|_{L^4(B)})$ such that*

$$\|\nabla u\|_{L^p(B)} \leq C \|\nabla u\|_{L^4(B)}.$$

Lemma 3. *Let $B := \mathbb{B}^2$ be the unit disc in $\mathbb{R}^2$. There exists $\epsilon_0 = \epsilon_0(N) > 0$ and $\alpha_0 > 1$ such that if $1 \leq \alpha < \alpha_0$, $u \in C^\infty(B, N)$ is a critical point of E_α, and*

$$E_\alpha(u, B) \leq \epsilon_0^2,$$

then for any $p \in (1,\infty)$ and smaller disc $B' \subset B$, there exists a constant $C = C(p, B', B)$ such that

$$\|\nabla u\|_{W^{1,p}(B')} \leq C\|\nabla u\|_{L^2(B)}.$$

In particular, we have

$$\sup_{B'} |\nabla u| \lesssim \|\nabla u\|_{L^2(B)}.$$

Another crucial ingredient in the analysis concerns the removability of isolated singularities of harmonic maps.

Lemma 4. *Let $B := \mathbb{B}^2$ be the unit disc in $\mathbb{R}^2$. If $u \in C^\infty(B\backslash\{0\}, N)$ is a harmonic map and $E_1(u)(= Vol(B) + \mathcal{E}(u)) < \infty$, then $u \in C^\infty(B, N)$.*

Combined with the local estimates, Sacks and Uhlenbeck proved:

Theorem 4. *(Sacks-Uhlenbeck [21]) Let $u_{\alpha_k} \in C^\infty(M, N)$ be critical points of E_{α_k} for a sequence $\alpha_k > 1$, where $\alpha_k \searrow 1$ as $k \to \infty$. Assume*

$$E_{\alpha_k}(u_{\alpha_k}) \leq E_0$$

for all k for some constant E_0. If $u_{\alpha_k} \to u$ weakly in $W^{1,2}(M, N)$ as $k \to \infty$, then up to a subsequence, there exists finitely many points $\{x_1, \cdots, x_l\} \subset M$ such that

$$u_{\alpha_k} \to u \text{ in } C^2_{loc}(M\backslash\{x_1, \cdots, x_l\}, N) \text{ as } k \to \infty,$$

where l depends on E_0, M, and N, for some smooth harmonic map $u \in C^\infty(M, N)$.

Proof. Define Radon measures

$$\mu_{\alpha_k} := (1 + |\nabla u_{\alpha_k}|^2)^{\alpha_k} dx \text{ on } M.$$

Since $\mu_{\alpha_k}(M) = E_{\alpha_k}(u_{\alpha_k}) \leq E_0$, up to a subsequence, there exists a non-negative Radon measure μ on M such that

$$\mu_{\alpha_k} \to \mu \text{ as } k \to \infty,$$

as convergence of Radon measures on M. Let $\epsilon_0 := \epsilon_0(N) > 0$ as in the previous lemma on interior estimates, then it is easy to see that there exists a non-negative integer

$l \leq E_0/\epsilon_0^2$ and a finite set

$$\Sigma := \{x_1, \cdots, x_l\} \subset M$$

such that

$$\inf_{r>0} \mu(B_r(x_i)) \geq \epsilon_0^2$$

for each $1 \leq i \leq l$; and for any $x_0 \in M \backslash \Sigma$, there exists $r_0 > 0$ such that $\mu(B_{r_0}(x_0)) < \epsilon_0^2$. Hence for $\alpha_k - 1$ small enough,

$$\mu_{\alpha_k}(B_{r_0}(x_0)) \leq \epsilon_0^2,$$

and thus by the interior estimates, there exists $C = C(n, \epsilon_0)$ such that

$$\|u_{\alpha_k}\|_{C^n(B_{\frac{r_0}{2}}(x_0))} \leq C(n, \epsilon_0)$$

for any $n \geq 1$. Thus after choosing a subsequence if necessary, we have $u_{\alpha_k} \to u$ in $C^2(B_{\frac{r_0}{2}}(x_0), N)$ for some u. Since $x_0 \in M \backslash \Sigma$ is arbitrary, we have

$$u_{\alpha_k} \to u \text{ in } C^2_{loc}(M \backslash \Sigma, N),$$

and hence $u \in C^\infty(M \backslash \Sigma, N)$ is a harmonic map. Then by lemma 4 on removable singularities, we have $u \in C^\infty(M, N)$. □

$\Sigma := \{x_1, \cdots, x_l\}$ in theorem 4 are called the set of *singularity points* of u.

Definition. A non-trivial smooth harmonic map $\omega\colon \mathbb{S}^2 \to N$ is called a *bubble*.

If $\Sigma \neq \emptyset$, the local behavior of u near points of Σ can be characterized in terms of bubbles.

Theorem 5. *Assume $u_\alpha\colon M \to N$ is a sequence of critical points of E_α such that*

$$u_\alpha \to u \textit{ in } C^2(M \backslash \{x_1, \cdots, x_l\}, N) \textit{ but not in } C^2(M \backslash \{x_2, \cdots, x_l\}, N) \textit{ as } \alpha \to 1,$$

then there exists a bubble $\omega\colon \mathbb{S}^2 \to N$ such that

$$\omega(\mathbb{S}^2) \subset \cap_{r>0} \left(\cap_{\alpha \to 1} \cup_{\beta \leq \alpha} u_\beta(B_r(x_1)) \right),$$

where $B_r(x_1)$ is the geodesic ball of radius r centered at $x_1 \in M$. Moreover,

$$\mathcal{E}(u) + \mathcal{E}(\omega) \leq \limsup_{\alpha \to 1} \mathcal{E}(u_\alpha).$$

2.5 Almost smooth heat flows in dimension 2: Struwe

Struwe [24] developed techniques similar to that of Sacks-Uhlenbeck [21] in the context of harmonic map heat flow in dimension two. The existence of a global weak solution of the harmonic map heat flow with finitely many singular points from a Riemannian surface to a compact Riemannian manifold was established. We will record here the main results in Struwe [24].

Theorem 6. *(Struwe [24]) Let M^2 be a compact Riemannian surface without boundary and $N \subset \mathbb{R}^L$ be a compact Riemannian manifold without boundary. For any initial condition $u_0 \in W^{1,2}(M, N)$,*

1. *There exists a global weak solution $u\colon M \times [0, \infty) \to N$ of eq. (2.1) and eq. (2.2) satisfying the energy decay estimate:*

$$\mathcal{E}(u(T)) + \int_0^T \int_M |u_t|^2 \, dV_g \, dt \leq \mathcal{E}(u_0) \text{ for all } T > 0,$$

   which, in particular, implies the energy inequality:

$$\mathcal{E}(u(T)) \leq \mathcal{E}(u_0) \text{ for all } T > 0.$$

2. *There exists an integer $K \geq 0$, depending on M, N, and $\mathcal{E}(u_0)$, such that*

$$u \in C^\infty(M \times (0, \infty) \backslash \{(x_k, t_k)\}_{k=1}^K, N) \text{ for some } (x_k, t_k) \in M \times (0, \infty).$$

   That is, the solution is smooth outside finitely many singular points $\{(x_k, t_k)\}_{k=1}^K$. Moreover, the solution is unique in this class.

3. *At each singularity (x_k, t_k), a harmonic sphere $\omega_k\colon \mathbb{S}^2 \to N$ bubbles off; i.e., at each singular point (x_k, t_k) for $1 \leq k \leq K$, there exists a sequence $(x_k^j, t_k^j)_{j \in \mathbb{N}}$ and*

scales $(r_k^j)_{j\in\mathbb{N}}$ *satisfying*

$$x_k^j \to x_k, t_k^j \nearrow t_k, r_k^j \searrow 0 \text{ as } j \to \infty,$$

such that

$$u_k^j(x) := u\left(\exp_{x_k^j}(r_k^j x), t_k^j\right) : B_{1/r_k^j}(\subset \mathbb{R}^2) \to N$$

converges to ω_k *in* $W^{2,2}_{loc}(\mathbb{R}^2, N)$.

4. *Lastly, there exists a sequence of times* $t_k \nearrow \infty$ *such that* $u(\cdot, t_k)$ *converges weakly to a smooth harmonic map* $u_\infty \colon M \to N$ *in* $W^{1,2}(M,N)$ *as* $k \to \infty$. *Moreover, the convergence is strong away from finitely many points* $\{x_p^\infty\}_{p=1}^I$.

5. *The number* K *and* I *defined above are bounded above by a constant depending on* $\mathcal{E}(u_0)$ *and* N *only. In fact,* $K + I \le \frac{\mathcal{E}(u_0)}{\epsilon_0^2}$, *where*

$$\epsilon_0^2 := \inf\{\mathcal{E}(\omega) \,|\, \omega : \mathbb{S}^2 \to N \text{ is a bubble }\}(> 0)$$

only depends on N.

Remark 4. Chang [15] extended the result to harmonic map heat flow from any Riemannian surface M with boundary ∂M under the Dirichlet boundary condition, with solution smooth near the boundary.

Remark 5. Struwe [23] applies the same method later to recover similar results to those of Sacks and Uhlenbeck.

Remark 6. Freire [5] [6] showed the uniqueness of weak solutions to the harmonic map heat flow in dimension two in the class where the energy $\mathcal{E}(u(t))$ is non-increasing in t. On the other hand, Topping [25] constructed weak solutions that are different from Struwe's solution by attaching *reverse bubbles* so that the energy $\mathcal{E}(u(t))$ increase by a jump of 4π each time a bubble is attached.

2.6 Finite time blow-up: Chang-Ding-Ye

Without the curvature assumption on the target manifold N, the short time smooth solution may develop singularities in finite time. The first such example in the case of

2-dimensional domain was presented by Chang-Ding-Ye [2].

Let $M = B := \mathbb{B}^2$ be the unit ball in $\mathbb{R}^2$. Consider *equivariant maps*

$$u_0 \colon B \to \mathbb{S}^2$$

of the form:

$$u_0(r,\theta) = \begin{bmatrix} \cos\theta \sin\phi_0(r) \\ \sin\theta \sin\phi_0(r) \\ \cos\phi_0(r) \end{bmatrix},$$

where $\phi_0 \in C^1([0,1])$ with $\phi_0(0) = 0$ such that $u_0(0,\theta) = (0,0,1)^T \in \mathbb{S}^2$. Let

$$u \colon B \times [0,T) \to \mathbb{S}^2$$

be the corresponding unique weak solution in the sense of Struwe, which is smooth up to a maximal time T. By uniqueness, u is also equivariant of the form:

$$u(r,\theta,t) = \begin{bmatrix} \cos\theta \sin\phi(r,t) \\ \sin\theta \sin\phi(r,t) \\ \cos\phi(r,t) \end{bmatrix} \text{ for } (r,\theta,t) \in [0,1] \times [0,2\pi] \times [0,T).$$

By direct calculations, it is easy to show that the harmonic map heat equation with initial and boundary conditions in this context can be written as

$$\begin{gathered} \phi_t = \phi_{rr} + \frac{1}{r}\phi_r - \frac{\sin 2\phi}{2r^2} \text{ for } 0 \le r \le 1, 0 < t < T, \\ \phi(r,0) = \phi_0(r) \text{ for } 0 \le r \le 1, \\ \phi(0,t) = \phi_0(0) = 0 \text{ for } 0 \le t < T, \\ \phi(1,t) = \phi_0(1) := b \text{ for } 0 \le t < T. \end{gathered}$$

Also, the Dirichlet energy of u in terms of ϕ is given by:

$$\mathcal{E}(u) = \frac{1}{2}\int_B |\nabla u|^2 \, dxdy = \pi \int_0^1 \left(\phi_r^2 + \frac{\sin^2\phi}{r^2} \right) rdr.$$

Remark 7. In the energy class: $\mathcal{E}(u) < \infty$, we have $\int_0^1 \phi_r^2 \, rdr < \infty$. Therefore $\phi(\cdot,t)$

is locally Holder continuous on $(0,1]$, and the only possible singularity happens at the origin.

Remark 8. Let

$$\theta(r) := 2\arctan r = \arccos\left(\frac{1-r^2}{1+r^2}\right),$$

which satisfies the harmonic map equation:

$$\theta_{rr} + \frac{1}{r}\theta_r - \frac{\sin 2\phi}{2r^2} = 0$$

and $\theta(0) = 0$. It is easy to show that the 1-parameter family of functions:

$$\theta_\lambda(r) := \theta\left(\frac{r}{\lambda}\right) \text{ for } \lambda > 0$$

are all harmonic maps satisfying the one-sided boundary condition $\theta_\lambda(0) = 0$.

Theorem 7. *(Chang-Ding-Ye [2]) Under the above assumptions, if $|b| > \pi$, then the solution ϕ blows up in finite time.*

Remark 9. When $|b| \leq \pi$, Chang-Ding [1] showed the existence of a global smooth solution. Thus theorem 7 is indeed optimal.

The key ingredients needed in proving theorem 7 are the comparison principle and the construction of a sub-solution which blows up in finite time. By a sub-solution, we mean a function $f = f(r,t)$ defined for $0 \leq r \leq 1$ and $0 \leq t \leq T_0$ for some $T_0 < \infty$, satisfying:

$$\begin{aligned}
&f_t \leq f_{rr} + \frac{1}{r}f_r - \frac{\sin 2f}{2r^2} \text{ on } (0,1)\times(0,T_0),\\
&f(0,t) := f_0(0) = 0,\\
&f(1,t) < b = \phi(1,t) \text{ for } 0 \leq t < T_0.
\end{aligned}$$

To construct a sub-solution, we consider adding a perturbation to the steady-state solution $\theta_\lambda(r)$.

Lemma 5. *Let $0 < \nu < 1$ and let σ, λ_0 and μ be positive constants. Define*

$$\lambda(t) := \left(\lambda_0^{1-\nu} - \sigma(1-\nu)t\right)^{\frac{1}{1-\nu}},$$

which is the unique solution to $\lambda' = -\sigma\lambda^\nu$, $\lambda(0) = \lambda_0$, *with a finite blow-up time*

$$T_\lambda := \sup\{t > 0 \,|\, \lambda(t) > 0\} = \frac{\lambda_0^{1-\nu}}{\sigma(1-\nu)}.$$

Consider the following function:

$$f(r,t) := \theta_{\lambda(t)}(r) + \theta_\mu(r^{1+\nu}) \text{ for } (r,t) \in [0,1] \times [0, T_\lambda],$$

we have:

(a) $f \in C^\infty([0,1] \times [0, T_\lambda))$.

(b) $\lim_{r \to 0^+} f(r,t) = 0$ *for each* $t \in [0, T_\lambda)$.

(c) $\lim_{r \to 0^+} f(r, T_\lambda) = \pi$.

(d) *There exists* $\bar{\mu} > 0$ *such that for each* $\mu > \bar{\mu}$, *there exists* $\bar{\sigma} = \bar{\sigma}(\mu, \nu)$ *such that* $\sigma < \bar{\sigma}$ *implies*
$$f_t \leq f_{rr} + \frac{1}{r} f_r - \frac{\sin 2f}{2r^2} \text{ on } (0,1) \times (0, T_\lambda).$$

Proof. Part (a), (b), and (c) follows from definition. For part (d),

- Because
$$\theta_{rr} + \frac{1}{r}\theta_r = \frac{\sin 2\theta}{2r^2},$$
direct calculation for $\Theta(r) := \theta(r^{1+\nu})$ gives

$$\begin{aligned}
\left(\Theta_{rr} + \frac{1}{r}\Theta_r\right)(r) &= \left[(1+\nu)^2 r^{2\nu}\theta_{rr} + \nu(1+\nu) r^{\nu-1}\theta_r\right](r^{1+\nu}) + \frac{1}{r}(1+\nu) r^\nu \theta_r(r^{1+\nu}) \\
&= (1+\nu)^2 r^{2\nu}\left[\theta_{rr} + \frac{1}{r^{1+\nu}}\theta_r\right](r^{1+\nu}) \\
&= (1+\nu)^2 r^{2\nu}\left(\frac{\sin 2\theta(r^{1+\nu})}{2r^{2(1+\nu)}}\right) = (1+\nu)^2\left[\frac{\sin 2\Theta}{2r^2}\right](r)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(f_{rr} + \frac{1}{r} f_r - \frac{\sin 2f}{2r^2}\right)(r,t) \\
&= \frac{\sin 2\theta_\lambda(r)}{2r^2} + (1+\nu)^2 \frac{\sin 2\theta_\mu(r^{1+\nu})}{2r^2} - \frac{\sin 2f}{2r^2} \\
&= \frac{1}{2r^2}\left\{\sin 2\theta_\lambda(r) + (1+\nu)^2 \sin 2\theta_\mu(r^{1+\nu}) - \sin\left(2\theta_\lambda(r) + 2\theta_\mu(r^{1+\nu})\right)\right\} \\
&= \frac{1}{r^2} \sin\theta_\mu(r^{1+\nu})\left\{(1+\nu)^2 \cos\theta_\mu(r^{1+\nu}) - \cos\left(2\theta_\lambda(r) + \theta_\mu(r^{1+\nu})\right)\right\}.
\end{aligned}$$

Combined with

$$f_t(r,t) = -\frac{r\lambda'}{\lambda^2} \cdot \frac{2}{1+\frac{r^2}{\lambda^2}} = \frac{\sigma r}{\lambda^{2-\nu}} \cdot \frac{2}{1+\frac{r^2}{\lambda^2}} = \frac{2\sigma r \lambda^\nu}{\lambda^2 + r^2},$$

it is enough to show that

$$\frac{2\sigma r\lambda^\nu}{\lambda^2 + r^2} \le \frac{1}{r^2} \sin\theta_\mu(r^{1+\nu})\left[(1+\nu)^2 \cos\theta_\mu(r^{1+\nu}) - \cos\left(2\theta_\lambda(r) + \theta_\mu(r^{1+\nu})\right)\right]. \tag{2.4}$$

- We first choose $\mu = \mu(\nu) \gg 1$ such that

$$\cos\theta_\mu(r^{1+\nu}) \ge \cos\theta_\mu(1) = \frac{\mu^2 - 1}{\mu^2 + 1} \ge \frac{1}{1+\nu}$$

for $0 \le r \le 1$. Moreover, since

$$\theta_\mu(r^{1+\nu}) = \theta\left(\frac{r^{1+\nu}}{\mu}\right) = \arccos\left(\frac{\mu^2 - r^{2+2\nu}}{\mu^2 + r^{2+2\nu}}\right),$$

we have

$$\sin\theta_\mu(r^{1+\nu}) = \frac{2\mu r^{1+\nu}}{\mu^2 + r^{2+2\nu})} \ge \frac{2\mu r^{1+\nu}}{\mu^2 + 1}$$

for $0 \le r \le 1$. Thus for such μ, we have

$$\text{RHS of (2.4)} \ge \frac{1}{r^2} \sin\theta_\mu(r^{1+\nu})[(1+\nu) - 1] = \frac{\nu}{r^2} \sin\theta_\mu(r^{1+\nu}) \ge \frac{2\mu\nu}{\mu^2+1} \cdot \frac{1}{r^{1-\nu}}.$$

Hence it is enough to show that for $0 \leq r \leq 1$, we have:

$$\frac{2\sigma r\lambda^\nu}{\lambda^2 + r^2} \leq \frac{2\mu\nu}{\mu^2+1} \cdot \frac{1}{r^{1-\nu}},$$

or equivalently,

$$\frac{\left(\frac{r}{\lambda}\right)^{2-\nu}}{1 + \left(\frac{r}{\lambda}\right)^2} \leq \frac{\mu\nu}{\sigma(\mu^2+1)},$$

which is true for all $(r, \lambda) \in (0,1) \times (0, \infty)$ as long as $\sigma > 0$ is chosen small enough. This completes the proof of part (d).

□

Next, we prove the comparison principle.

Lemma 6. *Let* ϕ_1 , $\phi_2 \in BC([0,1]\times[0,T]) \cap C^2((0,1],(0,T))$ *be sub-solution and super-solution, respectively. That is, assume*

$$\phi_{1,t} \leq \phi_{1,rr} + \frac{1}{r}\phi_{1,r} - \frac{\sin 2\phi_1}{2r^2} \text{ and}$$
$$\phi_{2,t} \geq \phi_{2,rr} + \frac{1}{r}\phi_{2,r} - \frac{\sin 2\phi_2}{2r^2}$$

on $(0,1) \times (0, T_0)$, *with* $\phi_1(0,t) = \phi_2(0,t) = 0$. *If*

$$\phi_1 \leq \phi_2 \text{ on } \{t = 0\} \cup \{r = 1\},$$

then

$$\phi_1(r,t) \leq \phi_2(r,t) \ , \forall (r,t) \in [0,1] \times [0,T] \,. \tag{2.5}$$

Proof. Define $\phi := \phi_2 - \phi_1$, then $\phi \geq 0$ on $\{t=0\} \cup \{r=0\} \cup \{r=1\}$.

- By assumptions, we have

$$\begin{aligned}\phi_t &\geq \phi_{rr} + \frac{1}{r}\phi_r - \frac{1}{2r^2}(\sin 2\phi_2 - \sin 2\phi_1) \\ &= \phi_{rr} + \frac{1}{r}\phi_r - \frac{1}{r^2}\cos(\phi_2+\phi_1)\sin(\phi_2-\phi_1) \\ &= \phi_{rr} + \frac{1}{r}\phi_r + p\phi,\end{aligned}$$

where

$$p = p(r,t) := -\frac{1}{r^2}\cos(\phi_2+\phi_1)\frac{\sin(\phi_2-\phi_1)}{\phi_2-\phi_1}.$$

- For fixed $T_1 \in (0,T)$, because $\phi_1(0,t) = \phi_2(0,t) \equiv 0$, by continuity, there exists $0 < \delta = \delta(T_1) \le 1$ such that

$$-\frac{\pi}{8} \le \phi_1, \phi_2 \le \frac{\pi}{8} \text{ on } [0,\delta]\times[0,T_1],$$

which implies $p < 0$ on $[0,\delta]\times[0,T_1]$. Combined with $|p| \le \frac{1}{r^2}$, we get

$$p(r,t) \le K \text{ on } [0,1]\times[0,T_1],$$

where $K := 1/\delta^2$.

- For fixed $T_2 \in (T_1, T)$ and $\epsilon > 0$, define

$$\begin{aligned} v &= v(r,t) := e^{-(K+1)t}\phi(r,t), \\ \Gamma &= \Gamma(r,t) := \frac{1}{T_2-t}e^{\frac{r^2}{4(T_2-t)}}\text{ , and} \\ w^\epsilon &= w^\epsilon(r,t) := v(r,t) + \epsilon\Gamma(r,t). \end{aligned}$$

By direct calculations, we have

$$v_t \ge v_{rr} + \frac{1}{r}v_r + (p(r,t)-K-1)v \text{ and } \Gamma_t = \Gamma_{rr} + \frac{1}{r}\Gamma_r,$$

which implies

$$w_t^\epsilon \ge w_{rr}^\epsilon + \frac{1}{r}w_r^\epsilon + (p(r,t)-K-1)(w^\epsilon - \epsilon\Gamma)\,. \tag{2.6}$$

- Now suppose $\inf_{[0,1]\times[0,T_1]} w^\epsilon < 0$. Since

$$w^\epsilon(0,t) = e^{-(K+1)t}\phi(0,t) + \epsilon\Gamma(0,t) \ge 0,$$

there exists $(r_0, t_0) \in (0,1] \times (0, T_1]$ such that $w^\epsilon(r_0, t_0) = \inf_{[0,1]\times[0,T_1]} w^\epsilon < 0$. Hence

$$w_t^\epsilon(r_0, t_0) \leq 0,\ w_r^\epsilon(r_0, t_0) = 0, \text{ and } w_{rr}^\epsilon(r_0, t_0) \geq 0. \tag{2.7}$$

But $p(r_0, t_0) - K - 1 < 0$, so (2.6) contradicts with (2.7). Therefore $w^\epsilon \geq 0$ on $[0,1] \times [0, T_1]$.

- Let $\epsilon \to 0$, we recover $v \geq 0$, which is equivalent to $\phi \geq 0$, on $[0,1] \times [0, T_1]$. Now let $T_1 \to T$, we get

$$\phi \geq 0 \text{ on } [0,1] \times [0, T].$$

□

Proof. (of Theorem 7)

Let $f = f(r,t)$ be the sub-solution constructed in the lemma. It can be shown that when $|b| = |\phi(1,t)| > \pi$, f can be adjusted such that the additional assumption on the initial condition:

$$\phi(r, 0) \geq f(r, 0) \text{ on } [0,1]\,, \tag{2.8}$$

is satisfied. By the Comparison Principle, we have

$$\phi(r,t) \geq f(r,t) \text{ on } [0,1] \times [0, T_0].$$

On the other hand, since the first derivative of f blows up at the origin in finite time:

$$f_r(0,t) \to \infty \text{ as } t \to T_0,$$

there exists some (possibly earlier) blow-up time $T \leq T_0$ such that

$$\lim_{t\to T} \phi_r(0,t) = \infty,$$

which implies

$$\lim_{t\to T} \|\nabla u(\cdot, t)\|_{L_x^\infty} = \infty,$$

which completes the proof. □

Chapter 3

Harmonic Maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ and its Heat Flow

Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimension m and n, respectively, and assume N is compact. By Nash's embedding theorem, we may assume that (N^n, h) is isometrically embedded in $\mathbb{R}^L$ for some L. Since N is a smooth compact submanifold, we can find a *tubular neighborhood*

$$N_\delta := \left\{ u \in \mathbb{R}^L \,\middle|\, d(u, N) := \inf_{z \in N} |u - z| < \delta \right\} ,$$

and the smooth *nearest point projection map* $\Pi_N \colon N_\delta \to N$ satisfying

$$\Pi_N(u) \in N \text{ and } |u - \Pi_N(u)| = d(u, N) \text{ for } u \in N \,.$$

Notice that for $u \in N$, the map $P(u) := \nabla \Pi_N(u) \colon \mathbb{R}^L \to T_u N$ is a projection map, and

$$A(u) := \nabla P(u) \colon T_u N \otimes T_u N \to (T_u N)^\perp$$

is the second fundamental form of $N \subset \mathbb{R}^L$. Let $\{\nu_{n+1}(u), \cdots, \nu_L(u)\}$ be a local orthonormal frame of the normal bundle $(T_u N)^\perp$. The harmonic map equation from the extrinsic point of view is given by

$$\Delta_g u + \sum_{n+1 \le \alpha \le L} g^{ij} A^\alpha(u) \left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j} \right) \nu_\alpha(u) = 0\,,$$

where $A^\alpha := \nabla \nu_\alpha$ is the second fundamental form of N in the normal direction ν_α.

From this chapter onward, we will restrict our attention to the situation where the target manifold is the 2-sphere $N = \mathbb{S}^2 \subset \mathbb{R}^3$. The *extrinsic* description of harmonic map in this case is given by

$$\Delta u + |\nabla u|^2 u = 0 \text{ in } M, \tag{3.1}$$

and the corresponding harmonic map heat flow is

$$u_t = \Delta u + |\nabla u|^2 u. \tag{3.2}$$

We record here the second variational of energy for harmonic maps into n-sphere.

Proposition 4. *Let $\bar{u} \in C^2(M, \mathbb{S}^n)$ be a harmonic map and let $\xi \in C_0^2(M, \mathbb{R}^{n+1})$, then*

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \mathcal{E}\left(\frac{\bar{u} + \epsilon\xi}{|\bar{u} + \epsilon\xi|} \right) = \int_M \left(|\nabla \hat{\xi}|^2 - |\nabla \bar{u}|^2 |\hat{\xi}|^2 \right) dV_g,$$

where $\hat{\xi} := P^{\bar{u}}(\xi) := \xi - (\xi \cdot \bar{u})\bar{u}$ is the projection of ξ onto the tangent space $T_{\bar{u}}\mathbb{S}^n$.

Proof. Denote by

$$u_\epsilon := \frac{\bar{u} + \epsilon\xi}{|\bar{u} + \epsilon\xi|}$$

the 1-parameter variation of $\bar{u}$. Direct calculations give

$$\left.\frac{du_\epsilon}{d\epsilon}\right|_{\epsilon=0} = \xi - (\xi \cdot \bar{u})\bar{u} = \hat{\xi}$$

and

$$\left.\frac{d^2 u_\epsilon}{d\epsilon^2}\right|_{\epsilon=0} = 3(\xi \cdot \bar{u})^2 \bar{u} - |\xi|^2 \bar{u} - 2(\xi \cdot \bar{u})\xi.$$

Therefore

$$
\begin{aligned}
\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} \mathcal{E}\left(\frac{\bar{u}+\epsilon\xi}{|\bar{u}+\epsilon\xi|}\right) &= \frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0}\left(\frac{1}{2}\int_M |\nabla u_\epsilon|^2 dV_g\right) \\
&= \int_M \left(\left|\nabla\left(\frac{du_\epsilon}{d\epsilon}\bigg|_{\epsilon=0}\right)\right|^2 + \left(\nabla\bar{u}\cdot\nabla\left(\frac{d^2u_\epsilon}{d\epsilon^2}\bigg|_{\epsilon=0}\right)\right)\right) dV_g \\
&= \int_M \left(|\nabla\hat{\xi}|^2 - \Delta_g\bar{u}\cdot\left(3(\xi\cdot\bar{u})^2\bar{u} - |\xi|^2\bar{u} - 2(\xi\cdot\bar{u})\xi\right)\right) dV_g \\
&= \int_M \left(|\nabla\hat{\xi}|^2 + |\nabla\bar{u}|^2\bar{u}\cdot\left(3(\xi\cdot\bar{u})^2\bar{u} - |\xi|^2\bar{u} - 2(\xi\cdot\bar{u})\xi\right)\right) dV_g \\
&= \int_M \left(|\nabla\hat{\xi}|^2 - |\nabla\bar{u}|^2\cdot(|\xi|^2 - (\xi\cdot\bar{u})^2)\right) dV_g \\
&= \int \left(|\nabla\hat{\xi}|^2 - |\nabla\bar{u}|^2|\hat{\xi}|^2\right) dV_g.
\end{aligned}
$$

□

3.1 Riemann sphere and rational maps

Let $\hat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ be the *extended complex plane*, where the arithmetic on $\mathbb{C}$ is extended to $\hat{\mathbb{C}}$ by assuming

$$\infty + a = a + \infty = \infty \text{ for } a \in \mathbb{C} \text{ , and}$$

$$b\cdot\infty = \infty\cdot b = \infty \text{ for } 0 \neq b \in \mathbb{C}.$$

We attach the point ∞ by requiring that every sequence $z_i \in \mathbb{C}$ for $i \geq 1$ with $|z_i|$ diverging to infinity converges to ∞. Under this assumption, it is easy to see that every sequence in $\hat{\mathbb{C}}$ has a convergent subsequence. The set $\hat{\mathbb{C}}$ can be viewed as a coordinate system of the 2-sphere $\mathbb{S}^2$ through *stereographic projection*. More precisely, consider $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ and its inverse $\pi^{-1}\colon \hat{\mathbb{C}} \to \mathbb{S}^2$ given by:

$$\pi\colon \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \frac{u_1 + iu_2}{1-u_3} \text{ , } \pi^{-1}\colon z \mapsto \frac{1}{1+|z|^2}\begin{bmatrix} 2\mathrm{Re}(z) \\ 2\mathrm{Im}(z) \\ |z|^2 - 1 \end{bmatrix} ,$$

where π maps the north pole $N := (0,0,1)^T$ to ∞. The continuity of π at every point other than N is evident from the formula. To see that π is continuous at N, observe that if U approaches N in $\mathbb{S}^2$, then u_3 approaches $+1$ from below. This implies that $|\pi(U)|$ tends to infinity, which implies $\pi(U)$ tends to ∞ in $\hat{\mathbb{C}}$.

The set $\hat{\mathbb{C}}$ with the aforementioned arithmetic and convergence of sequences is called the *Riemann sphere*. The notion of continuity of maps can be extended to maps from $\hat{\mathbb{C}}$ to itself. More precisely, for a function $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, we say f is continuous at $z \in \hat{\mathbb{C}}$ if every sequence that converges to z is mapped by f to a sequence that converges to $f(z)$. When $f(\infty) = \infty$, the continuity of f at ∞ is equivalent to the continuity of the function $g(z) := \frac{1}{f(\frac{1}{z})}$ at $z = 0$. When $f(\infty) = a \neq \infty$, the continuity of f at ∞ is equivalent to the continuity of $g(z) := f(\frac{1}{z})$ at $z = 0$. When $f(a) = \infty$ for some $a \neq \infty$, the continuity of f at a is equivalent to the continuity of $g(z) := \frac{1}{f(z)}$ at $z = a$.

We can now extend the concept of holomorphic functions to the Riemann sphere $\hat{\mathbb{C}}$:

Definition. Assume $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is continuous and $a \in \hat{\mathbb{C}}$.

1. If $a = f(a) = \infty$, we say f is holomorphic at a if $g(z) := \frac{1}{f(\frac{1}{z})}$ is holomorphic at $z = 0$.

2. If $a = \infty$ and $f(a) \neq \infty$, we say f is holomorphic at a if $g(z) := f(\frac{1}{z})$ is holomorphic at $z = 0$.

3. If $a \neq \infty$ and $f(a) = \infty$, we say f is holomorphic at a if $g(z) := \frac{1}{f(z)}$ is holomorphic at $z = a$.

The following proposition classifies all holomorphic maps from $\hat{\mathbb{C}}$ to itself.

Proposition 5. *Assume $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic map. Then there exists polynomial $P(z)$ and $Q(z)$ with no common factors such that*

$$f(z) = \frac{P(z)}{Q(z)}.$$

3.2 Topological lower bound on energy

In this section, we consider harmonic map heat flow from the Euclidean space $M = \mathbb{R}^2$ to the 2-sphere $N = \mathbb{S}^2 \subset \mathbb{R}^3$. The harmonic map equation and its heat flow are given by

$$\Delta u + |\nabla u|^2 u = 0 \text{ and } u_t = \Delta u + |\nabla u|^2 u. \tag{3.3}$$

Notice that since $|u|^2 = 1$, we have $0 = \nabla |u|^2 = 2u \cdot \nabla u$. Therefore $0 = \nabla \cdot (u \cdot \nabla u) = u \cdot \Delta u + |\nabla u|^2$, which implies $|\nabla u|^2 = -u \cdot \Delta u$. Therefore eq. (3.3) can be rewritten geometrically as

$$P^u(\Delta u) = 0 \text{ and } u_t = P^u(\Delta u),$$

where $P^u \colon \mathbb{R}^3 \to T_u\mathbb{S}^2$ is the projection operator onto then tangent space

$$T_u\mathbb{S}^2 := \{\phi \in \mathbb{R}^3 \mid \phi \cdot u = 0\}$$

given by

$$P^u(\xi) := \xi - (\xi \cdot u)u.$$

For an arbitrary C^1 map $u \colon \mathbb{R}^2 \to \mathbb{S}^2$ for which the limit at infinity exists; namely, if

$$u(\infty) := \lim_{|x| \to \infty} u(x) \in \mathbb{S}^2$$

exists, we define the *degree* of $u \colon \mathbb{R}^2 \to \mathbb{S}^2$ to be the *Brouwer degree* of the composed map $u \circ \pi \colon \mathbb{S}^2 \to \mathbb{S}^2$, where $\pi \colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ is the stereographic projection defined in the previous section.

$$\deg(u) := \deg(u \circ \pi).$$

Under such definition, it can be shown that the degree of u can be computed by the formula:

$$\deg(u) := \frac{1}{4\pi} \int_{\mathbb{R}^2} u_x \cdot (u \times u_y)\, dxdy = -\frac{1}{4\pi} \int_{\mathbb{R}^2} u_y \cdot (u \times u_x)\, dxdy,$$

where we used the facts that $|u| = 1$ and thus u_x and u_y are perpendicular to u.

On the other hand, since the Dirichlet energy is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2\, dxdy = \frac{1}{2} \int_{\mathbb{R}^2} \sum_{j=1}^{3} \left(\left| \frac{\partial u_j}{\partial x} \right|^2 + \left| \frac{\partial u_j}{\partial y} \right|^2 \right) dxdy,$$

the energy and the degree of u are related by:

$$\begin{aligned}\mathcal{E}(u) - 4\pi\deg(u) &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|u_x|^2 + |u_y|^2 - 2u_x\cdot(u\times u_y)\big)\ dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|u_x|^2 + |u\times u_y|^2 - 2u_x\cdot(u\times u_y)\big)\ dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2} |u_x - u\times u_y|^2\, dxdy \geq 0,\end{aligned}$$

where the equality is achieved if and only if

$$u_x = u \times u_y. \tag{3.4}$$

Now let $f = f_1 + if_2 := \pi\circ u\colon \mathbb{R}^2 \to \hat{\mathbb{C}}$ be the composition of u with the stereographic projection $\pi : \mathbb{S}^2 \to \hat{\mathbb{C}}$. Thus

$$u = \pi^{-1}\circ f = \frac{1}{1+|f|^2}\begin{bmatrix} 2f_1 \\ 2f_2 \\ |f|^2 - 1\end{bmatrix}.$$

Direct computations show that eq. (3.4) is equivalent to

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \text{ and } \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x},$$

which means f satisfies the Cauchy-Riemann equation. Therefore $f = \pi\circ u\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic, and thus a rational function by proposition 5.

Similarly, we have

$$\begin{aligned}\mathcal{E}(u) + 4\pi\deg(u) &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|u_x|^2 + |u_y|^2 - 2u_y\cdot(u\times u_x)\big)\ dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \big(|u\times u_x|^2 + |u_y|^2 - 2u_y\cdot(u\times u_x)\big)\ dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^2} |u_y - u\times u_x|^2\, dxdy \geq 0,\end{aligned}$$

where the equality is achieved if and only if

$$u_y = u \times u_x,$$

which corresponds to $f := \pi \circ u$ being anti-holomorphic.

In summary, we have

$$\mathcal{E}(u) \geq 4\pi|\deg(u)|, \tag{3.5}$$

where the equality happens if and only if

$$u = \pi^{-1} \circ R,$$

for some rational function $R = \frac{P}{Q}$ in either $z = x + iy$ or $\bar{z} = x - iy$.

Moreover, if

$$R(z) = \frac{P(z)}{Q(z)}$$

for some polynomials $P(z)$ and $Q(z)$ with no common factors, then

$$\deg(\pi^{-1} \circ R) = \max(\deg(P), \deg(Q)).$$

3.3 Asymptotic stability in the equivariant class

Given a C^1 map $u_0 : \mathbb{R}^2 \to \mathbb{S}^2$ of finite Dirichlet energy $\mathcal{E}(u_0) < \infty$, we define a weak solution to the harmonic map heat flow in the sense of Struwe [24] and Freire [6]:

Definition. Let $Z := \{u : \mathbb{R}^2 \to \mathbb{R}^3 \,|\, \nabla u \in L^2(\mathbb{R}^2)\ ,\ |u| = 1 \text{ a.e.}\}$, a function $u = u(x,t)$ is a weak solution of the harmonic map heat flow eq. (3.2) on the interval $I = [0,T]$ if

1. $u \in L^\infty(I, Z) \cap C_{weak}(I, Z)$.
2. $u(0) = u_0$.
3. $\int_I \int_{\mathbb{R}^2} \{u \cdot \phi_t - \nabla u \cdot \nabla \phi + |\nabla u|^2 u \cdot \phi\}\ dxdt = 0$, $\forall \phi \in C_0^\infty(\mathbb{R}^2 \times I, \mathbb{R}^3)$.

As was mentioned before, the weak solution is unique under the assumption that the Dirichlet energy $\mathcal{E}(u(t))$ is non-increasing in t. Moreover, formally taking the dot product of eq. (3.2) with u and integrating in space and time yields the basic energy identity:

$$\mathcal{E}(u(t)) + \int_0^t \int_{\mathbb{R}^2} |u_t|^2 = \mathcal{E}(u_0). \tag{3.6}$$

With the gradient flow structure eq. (3.6) of the harmonic map heat flow and the topological lower bound on energy eq. (3.5) in the previous section, it is reasonable to expect that if the Dirichlet energy of the initial map is close to its topological lower bound, then the corresponding heat flow exists globally in time, and converges to a fixed harmonic map. More precisely, for a given C^1 map

$$u_0 \colon \mathbb{R}^2 \to \mathbb{S}^2$$

of finite Dirichlet energy which is extendable to the Riemann sphere $\hat{\mathbb{C}}$ with $\deg(u_0) = m$, if

$$0 \leq \mathcal{E}(u_0) - 4\pi|m| \ll 1,$$

then we hope the maximal existence time $T = T(u_0)$ can be taken to be infinity, and the corresponding solution $u(\cdot, t)$ converges to some fixed steady state (harmonic map) of degree m.

Unfortunately, the general situation of our flow equations is too big a challenge to handle directly. A good starting point is to assume additional symmetry conditions. Guan-Gustafson-Tsai [8] considered *m-equivariant maps* with finite energy. Namely, they considered

$$\Sigma_m := \left\{ u : \mathbb{R}^2 \to \mathbb{S}^2 \,\middle|\, u = u(r,\theta) = e^{m\theta R} v(r) \ , \ \mathcal{E}(u) < \infty \ , \ v(0) = -\hat{k} \ , \ v(\infty) = \hat{k} \right\} ,$$

where m is any integer, (r, θ) is the polar coordinates on $\mathbb{R}^2$, $\hat{k} = (0,0,1)^T$, $v = (v_1, v_2, v_3)^T \colon [0, \infty) \to \mathbb{S}^2$ is a radial function, and R is the matrix which generates rotations around the u_3-axis:

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ , \ e^{\alpha R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \ ,$$

and hence

$$u = u(r,\theta) = e^{m\theta R}v(r) = \begin{bmatrix} \cos(m\theta)v_1(r) - \sin(m\theta)v_2(r) \\ \sin(m\theta)v_1(r) + \cos(m\theta)v_2(r) \\ v_3(r) \end{bmatrix}.$$

The energy density in terms of polar coordinates (r,θ) is given by

$$|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2}|u_\theta|^2 = |v_r|^2 + \frac{m^2}{r^2}|Rv|^2\,, \tag{3.7}$$

which implies that the total energy is

$$\mathcal{E}(u) = \pi \int_0^\infty \left(|v_r|^2 + \frac{m^2}{r^2}|Rv|^2 \right) r dr.$$

It is easy to check by direct calculations that m-equivariant maps have degree m.

For $u \in \Sigma_m$, the energy $\mathcal{E}(u)$ can be decomposed as

$$\mathcal{E}(u) = \pi \int_0^\infty \left| v_r - \frac{|m|}{r} v \times Rv \right|^2 r dr + \mathcal{E}_{min},$$

with

$$\mathcal{E}_{min} := 2\pi \int_0^\infty v_r \cdot \left(\frac{|m|}{r} v \times Rv \right) r dr = 2\pi|m| \int_0^\infty v_{3,r}\, dr = 4\pi|m|.$$

This topological lower bound is clearly saturated if and only if

$$v_r = \frac{|m|}{r} v \times Rv,$$

which is attained precisely by the 2-parameter family of harmonic maps

$$\mathcal{O}_m := \left\{ e^{m\theta R} h^{s,\alpha}(r) \,|\, s > 0 \;,\; \alpha \in \mathbb{R} \right\},$$

where

$$h^{s,\alpha}(r) := e^{\alpha R} h\left(\frac{r}{s}\right) \;,\; h(r) = \begin{bmatrix} h_1(r) \\ 0 \\ h_3(r) \end{bmatrix} \;,\; h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}} \;,\; h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}.$$

The rotation parameter α is determined up to shifts of 2π. Notice that $\mathcal{O}_m$ is the orbit of a single harmonic map $e^{m\theta R}h(r)$ under the symmetries of scaling (s) and rotation (α).

Remark 10. By uniqueness of weak solution, the equivariant class is preserved by the harmonic map heat flow. In other words, if $u_0 \in \Sigma_m$, then the corresponding solution $u(\cdot, t) \in \Sigma_m$ for any positive time t.

Remark 11. The 2-parameter family $\mathcal{O}_m$ corresponds to monomials Az^m with non-zero complex number A under stereographic projection. That is, composed with $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$, we have

$$\{\pi \circ u \,|\, u \in \mathcal{O}_m\} = \left\{ f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \,|\, f(z) = Az^m, 0 \neq A \in \mathbb{C} \right\}.$$

Guan-Gustafson-Tsai [8] showed, among other things, the stability results concerning m-equivariant maps with energy closed to the minimal energy $\mathcal{E}_{min} = 4\pi|m|$. More precisely, given initial data

$$u_o \in \Sigma_m \ , \ \mathcal{E}(u_0) = 4\pi|m| + \epsilon_0^2 \ , \ \epsilon_0 \ll 1,$$

let

$$u(t) \in C([0,T), \Sigma_m)$$

be the corresponding unique solution to the initial map u_0 (Σ_m equipped with the $\dot{H}^1$ norm.) They proved the following *asymptotic stability* result:

Theorem 8. *(Guan-Gustafson-Tsai [8]) For $|m| \geq 4$, there exists a constant $\delta_1 = \delta_1(m) > 0$ such that if $u_0 \in \Sigma_m$ and $\delta_0^2 := \mathcal{E}(u_0) - 4\pi|m| \leq \delta_1^2$, then:*

(a) The maximal existence time $T = T(u_0)$ can be taken to be infinity.

(b) There exist $s = s(t)$ and $\alpha = \alpha(t) \in C([0,\infty), \mathbb{R}^+)$ such that

$$\|\nabla \left(u(x,t) - e^{m\theta R} h^{s(t),\alpha(t)}(r) \right) \|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \lesssim \delta_0.$$

(c) There exists $s_\infty > 0$ and $\alpha_\infty \in \mathbb{R}$ such that

$$(s(t), \alpha(t)) \to (s_\infty, \alpha_\infty) \ \text{as} \ t \to \infty.$$

We describe here the strategies to show such *asymptotic stability* result:

- The first step is to decompose the map $u(x,t)$ into a nearby harmonic map (finite-dimensional part) and a deviation from the harmonic map family (infinite-dimensional part). Consider

$$v(r) = e^{\alpha(t)R}\left[h(\rho) + \xi(\rho,t)\right] \ , \text{ where } \rho := \frac{r}{s(t)}.$$

  The choice of the parameters $s(t)$ and $\alpha(t)$ is important, and will be addressed later. Consider the following orthonormal basis of $T_{h(\rho)}\mathbb{S}^2$:

$$\hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \text{ and } h(\rho) \times \hat{j} = \begin{bmatrix} -h_3(\rho) \\ 0 \\ h_1(\rho) \end{bmatrix}.$$

  We further split the perturbation term $\xi(\rho,t)$ into tangent and normal components to the sphere:

$$\xi(\rho,t) = z_1(\rho,t)\hat{j} + z_2(\rho,t)h(\rho) \times \hat{j} + \gamma(\rho,t)h(\rho).$$

  Define

$$z(\rho,t) := z_1(\rho,t) + iz_2(\rho,t).$$

  Since $|u| \equiv 1$, we have $\gamma = \mathcal{O}(|z|^2)$ if $|\xi| \ll 1$. Notice that the complex function $z(\rho,t)$ and the parameters $s(t)$, $\alpha(t)$ give a full description of the original solution $u(x,t)$.

- By plugging in the decomposition above into the harmonic map heat flow equation, it can be shown that $z(\rho,t)$ satisfies a non-linear equation of the form

$$s^2 z_t = -Nz + (ims\dot{s} - s^2\dot{\alpha})h_1 + F,$$

  where F is a non-linear term depending on z, $\dot{s}$, and $\dot{\alpha}$, and N is the self-adjoint operator

$$N = -\partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \frac{m^2}{\rho^2}(1 - 2h_1^2(\rho)) = L_0^* L_0 \ , \text{ where}$$

$$L_0 := \partial_\rho + \frac{m}{\rho}h_3 = h_1\partial_\rho\frac{1}{h_1}\ , \text{ and } L_0^* = -\frac{1}{\rho h_1}\partial_\rho \rho h_1$$

is the adjoint of L_0 on $L^2(\rho d\rho)$.

- We next address the issue of how to choose $s(t)$ and $\alpha(t)$. Suppose for a moment that $s(t) \equiv 1$ and $\alpha(t) \equiv 0$. Since $\ker N = \ker L_0$ is 1-dimensional and spanned by h_1, the linearized equation for z reads:

$$z_t = -Nz,$$

which admits the constant-in-time non-trivial solution $z(\rho, t) \equiv h_1(\rho)$. Since we would like $z(\rho, t)$ to decay in time, we choose $s(t)$ and $\alpha(t)$ to satisfy the *orthogonality condition*:

$$(z, h_1)_{L^2} = \int_0^\infty z(\rho)h_1(\rho)\,\rho d\rho \equiv 0.$$

Such a condition defines a co-dimension-1 subspace of L^2, and is invariant under the linearized equation $z_t = -Nz$. Moreover, we have

$$(z_t, h_1)_{L^2} = 0\ , \text{ and } (Nz, h_1)_{L^2} = (L_0 z, L_0 h_1)_{L^2} = 0.$$

Therefore by taking the L^2-inner-product of h_1 with equation original z equation, we have

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim |(h_1, F)_{L^2}|.$$

We can then use the point-wise estimate of the non-linear term F to show

$$|s\dot{s}| + |s^2\dot{\alpha}| \lesssim \left\|\frac{z_\rho}{\rho}\right\|_{L^2}^2 + \left\|\frac{z}{\rho^2}\right\|_{L^2}^2. \tag{3.8}$$

- Next, we consider the vector

$$w(r, t) := v_r(r, t) - \frac{|m|}{r}v(r, t) \times Rv(r, t) \in T_{v(r,t)}\mathbb{S}^2,$$

which quantifies the excess of energy:

$$\|w\|_{L^2}^2 = \frac{\mathcal{E}(u) - 4\pi|m|}{\pi} \ll 1.$$

We then find an orthonormal frame $\{e, v \times e\}$ of $T_v\mathbb{S}^2$ such that the complex function $q = q(r,t)$ defined through

$$w(r,t) = q_1(r,t)e + q_2(r,t)v \times e \ , \ q(r,t) := q_1(r,t) + iq_2(r,t),$$

satisfies:

$$q_t = q_{rr} + \frac{1}{r}q_r - \left[\frac{(1-mv_3)^2}{r^2} + \frac{mv_{3,r}}{r}\right] q - qN_0(q)\,, \tag{3.9}$$

where N_0 is a non-local non-linear operator of higher order terms.

- The next crucial step is to relate q and z. More precisely, we want to estimate z by q in suitable norms. Under the orthogonality condition $(z, h_1)_{L^2} = 0$, it can be shown that when $|z|$ is small, we have

  (i) If $|m| \geq 3$ and $2 \leq p < \infty$, then

  $$\|z_\rho\|_{L^p} + \|\frac{z}{\rho}\|_{L^p} \lesssim s^{1-2/p}\|q\|_{L^p}.$$

  (ii) If $|m| \geq 4$, then

  $$\|\frac{z_\rho}{\rho}\|_{L^2} + \left\|\frac{z}{\rho^2}\right\|_{L^2} \lesssim s\left\|\frac{q}{r}\right\|_{L^2}. \tag{3.10}$$

  Notice in particular that (3.8) and (3.10) implies

  $$|s^{-1}\dot{s}| + |\dot{\alpha}| \lesssim \left\|\frac{q}{r}\right\|^2_{L^2}$$

  when $|m| \geq 4$.

- The remaining task it to obtain estimates for $q(r,t)$. Substitute the decomposition for v into equation (3.9) and dropping the non-linear terms, we arrive at

  $$q_t = Hq \ , \text{ where } H := -\Delta_r + V(r) := -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1 + m^2 - 2mh_3(r)}{r^2}.$$

It turns out that e^{-tH} defines a contraction semi-group on $L^2(\mathbb{R}^2)$ with the following estimates similar to the ones for the heat semi-group $e^{t\Delta}$: Assume $|m| \geq 2$, $\frac{1}{r} + \frac{1}{p} = \frac{1}{\tilde{r}} + \frac{1}{\tilde{p}} = \frac{1}{2}$, and $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1$. Let $\phi(\cdot)$ and $f(\cdot, t)$ be radial functions on $\mathbb{R}^2$, then

(i)

$$\|e^{-tH}\phi\|_{L_t^r L_x^p} + \left\| \int_0^t e^{-(t-s)H} f(s)\, ds \right\|_{L_t^r L_x^p} \lesssim \|\phi\|_{L^2} + \|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}.$$

(ii)

$$\begin{aligned} &\left\| \frac{1}{|x|} e^{-tH}\phi \right\|_{L_t^2 L_x^2} + \| \left(e^{-tH}\phi\right)_{,r} \|_{L_t^2 L_x^2} + \left\| \frac{1}{|x|} \int_0^t e^{-(t-s)H} f(s)\, ds \right\|_{L_t^2 L_x^2} \\ &+ \left\| \left(\int_0^t e^{-(t-s)H} f(s)\, ds \right)_{,r} \right\|_{L_t^2 L_x^2} \lesssim \|\phi\|_{L^2} + \|f\|_{L_t^{\tilde{r}'} L_x^{\tilde{p}'}}. \end{aligned}$$

- Finally, the method of variation of constants gives the integral representation of q:

$$q(\cdot, t) = e^{-tH} q_0 + \int_0^t e^{-(t-s)H} G(q(\cdot, s))\, ds.$$

We then use the semi-group estimates (i) and (ii) to control the inhomogeneous term G from the q-equation to obtain:

$$\begin{aligned} &\|z_\rho\|^2_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} + \left\| \frac{z}{\rho} \right\|^2_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} + \|s^{-1}\dot{s}\|_{L_t^1} + \|\dot{\alpha}\|_{L_t^1} \\ \lesssim &\|q\|^2_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} + \left\| \frac{q}{r} \right\|^2_{L_t^2 L_x^2} + \|q_r\|^2_{L_t^2 L_x^2} \\ \lesssim &\|q_0\|^2_{L_x^2} \approx \mathcal{E}(u_0) - 4\pi|m| \ll 1. \end{aligned}$$

Therefore $s(t)$ and $\alpha(t)$ converges as $t \to \infty$, and

$$\begin{aligned} &\|\nabla \left(u(x,t) - e^{m\theta R} h^{s(t),\alpha(t)}(r) \right) \|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \approx \|\nabla \left(e^{(m\theta + \alpha(t))R} z(\rho, t) \right) \|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \\ \approx &\|z_\rho\|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} + \left\| \frac{z}{\rho} \right\|_{L_t^2 L_x^\infty \cap L_t^\infty L_x^2} \lesssim \sqrt{\mathcal{E}(u_0) - 4\pi|m|}. \end{aligned}$$

Chapter 4

Elliptic Functions; Harmonic Maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ and its Heat Flow

Composed with the stereographic projection $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$, the classical Weierstrass elliptic function $\wp$ is a canonical conformal (and harmonic) map from $\mathbb{T}^2$ to $\mathbb{S}^2$. As a double cover, the topological degree of $\wp$ is two, which minimizes the Dirichlet energy in this homotopy class.

With the result of Tsai et al. from the previous chapter in mind, we would like to study harmonic map heat flow from $\mathbb{T}^2$ to $\mathbb{S}^2$ with initial data $u_0\colon \mathbb{T}^2 \to \mathbb{S}^2$ of degree 2 with energy closed to $\mathcal{E}_{min} = 8\pi$. We conjecture that a similar result holds true. Namely, the solutions are regular, global-in-time, and converge to a finite dimensional manifold consisting of suitable conformal transformations of $\wp$.

4.1 Weierstrass elliptic functions

Let ω_1 and ω_2 be two non-zero complex numbers which are linearly independent over $\mathbb{R}$. Let

$$\Lambda := \{n\omega_1 + m\omega_2 \,|\, n, m \in \mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ generated by ω_1 and ω_2. Associated to the lattice Λ is the **fundamental parallelogram** defined by

$$P_0 := \{z = a\omega_1 + b\omega_2 \in \mathbb{C} \,|\, 0 \leq a, b < 1\}.$$

We call a function $f\colon \mathbb{C} \to \mathbb{C}$ **doubly periodic** if

$$f(z + \omega_1) = f(z) \text{ and } f(z + \omega_2) = f(z)$$

for all $z \in \mathbb{C}$. Successive applications of periodicity conditions yield

$$f(z + \omega) = f(z) \text{ for all } z \in \mathbb{C} \text{ and } \omega \in \Lambda,$$

and thus we can think of f as defined on the *flat torus* $\mathbb{T}^2 := \mathbb{C}/\Lambda$, the quotient space of $\mathbb{C}$ over the lattice Λ.

In this chapter, we consider C^1 maps from the 2-torus $\mathbb{T}^2$ to the 2-sphere $\mathbb{S}^2$, whose topological structure is uniquely determined by their degrees. As was mentioned in section 2.1, Eells-Wood [3] showed that there does not exist any harmonic map in the homotopy class of continuous maps from $M = \mathbb{T}^2$ to $N = \mathbb{S}^2$ of degree 1.

Proposition 6. *An entire (i.e., holomorphic on all of $\mathbb{C}$) doubly periodic function is constant.*

Proof. Since the function is completely determined by its values on P_o and the closure of P_0 is compact, the function is bounded on $\mathbb{C}$. Hence by Liouville's theorem, the function is constant. □

Definition. A non-constant doubly periodic meromorphic function is called an **elliptic function**.

Since a meromorphic function can have only finitely many zeros and poles in any bounded domain, we see that an elliptic function will have only finitely many zeros and poles in the fundamental parallelogram P_0. As usual, we count poles and zeros with multiplicities.

Proposition 7. *The total number of poles of an elliptic function f in P_0 is always at least 2. In other words, f cannot have only one simple pole.*

Proof. Firstly, since the number of poles of f is finite, we can assume f has no poles on the boundary ∂P_0 by considering $P := P_0 + h$ for some small $h \in \mathbb{C}$ if necessary. Now suppose f has no poles on ∂P_0, we have

$$\int_{\partial P_0} f(z)\, dz = 2\pi i \sum \text{Res} f$$

by the residue theorem. On the other hand, since f is doubly periodic,

$$\int_0^{\omega_1} f(z)\, dz = \int_{\omega_2}^{\omega_1+\omega_2} f(z)\, dz \text{ and } \int_0^{\omega_2} f(z)\, dz = \int_{\omega_1}^{\omega_1+\omega_2} f(z)\, dz,$$

and therefore

$$\int_{\partial P_0} f(z)\, dz = \int_0^{\omega_1} f(z)\, dz + \int_{\omega_1}^{\omega_1+\omega_2} f(z) dz + \int_{\omega_1+\omega_2}^{\omega_2} f(z) dz + \int_{\omega_2}^{0} f(z) dz = 0.$$

Hence $\Sigma \text{Res} f = 0$, and therefore f must have at least two poles in P_0. □

Proposition 8. *The number of poles and the number of zeros of an elliptic function are the same.*

Proof. Let $\mathcal{N}_z$ and $\mathcal{N}_p$ be the number of zeros and poles of f in the fundamental parallelogram P_0 respectively, counting multiplicities. Recall that

$$\int_{\partial P_0} \frac{f'(z)}{f(z)}\, dz = 2\pi i(\mathcal{N}_z - \mathcal{N}_p)$$

by the argument principle. On the other hand, due to the doubly periodicity of f, $\int_{\partial P_0}(f'/f) = 0$. Hence $\mathcal{N}_z = \mathcal{N}_p$. □

We call the number of poles (counted according to their multiplicities) of an elliptic function its **order**, which is equal to its degree.

To construct an elliptic function of order two, it is natural to consider

$$\sum_{\omega \in \Lambda} \frac{1}{(z+\omega)^2}.$$

However, this series does not converge absolutely. The simplest approach to overcome the non-absolute convergence of the series is to consider instead the following series:

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right],$$

where Λ^* denote the lattice minus the origin, that is, $\Lambda^* := \Lambda\backslash\{0\}$. The series converges absolutely, and defined a meromorphic function.

Definition. $\wp(z)$ is called the **Weierstrass $\wp$ function**.

Proposition 9. *$\wp$ is an elliptic function with periods ω_1 and ω_2, and double poles at the lattice points.*

Proof. It remains to show that $\wp$ is doubly periodic with the correct periods. First notice that the $\wp'$ can be derived from term-wise differentiation. That is,

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^3}.$$

Therefore $\wp'(z+\omega_1) = \wp'(z+\omega_2) = \wp'(z)$, and hence $\wp'$ is doubly periodic with the desired periods. Hence

$$\wp(z+\omega_1) = \wp(z) + a \text{ and } \wp(z+\omega_2) = \wp(z) + b$$

for some constants $a, b \in \mathbb{C}$. On the other hand, since $\wp$ is an even function by definition, $\wp(-\frac{\omega_1}{2}) = \wp(\frac{\omega_1}{2})$ and $\wp(-\frac{\omega_2}{2}) = \wp(\frac{\omega_2}{2})$. Therefore $a = b = 0$, and hence $\wp$ is indeed doubly periodic with periods ω_1 and ω_2. □

Proposition 10. *$\wp'$ is also elliptic, with triple pole at 0 and simple zeros at $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1+\omega_2}{2}$.*

Proof. Near $z = 0$, we have

$$\wp'(z) = \frac{-2}{z^3} + \cdots,$$

hence $\wp'$ has a triple pole at 0. It is clear that 0 is the only pole of $\wp'$ in P_0. On the other hand, since $\wp$ is even, $\wp'$ is odd. Thus combined with the double periodicity of $\wp'$, we have

$$-\wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(-\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_1}{2}\right),$$

therefore $\wp'\left(\frac{\omega_1}{2}\right) = 0$. Similarly, $\wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1+\omega_2}{2}\right) = 0$. Since the number of poles equal to the number of zeros, $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1+\omega_2}{2}$ are the three (simple) zeros of $\wp'$. □

Proposition 11. *$\wp$ and $\wp'$ satisfied the functional equation:*

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

where $e_1 := \wp(\frac{\omega_1}{2})$, $e_2 := \wp(\frac{\omega_2}{2})$, and $e_3 := \wp(\frac{\omega_1+\omega_2}{2})$.

Proof. First notice that since $\frac{\omega_1}{2}$ is a zero for both $\wp - e_1$ and its derivative $\wp'$, $\frac{\omega_1}{2}$ is indeed a double zero of $\wp - e_1$. Similarly, $\frac{\omega_2}{2}$ and $\frac{\omega_1+\omega_2}{2}$ are double zeros of $\wp - e_2$ and $\wp - e_3$, respectively. Let

$$F(z) := 4(\wp - e_1)(\wp - e_2)(\wp - e_3),$$

which has poles of order 6 at 0, and zeros of multiplicity 2 at $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1+\omega_2}{2}$. On the other hand, $(\wp')^2$ also has poles of order 6 at 0 and double zeros at $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1+\omega_2}{2}$. Therefore $G := \frac{(\wp')^2}{F}$ is holomorphic and still doubly periodic, which implies G is a constant. Since

$$\wp(z) = \frac{1}{z^2} + \cdots \text{ and } \wp'(z) = \frac{-2}{z^3} + \cdots$$

near $z = 0$, we have $G = 1$. □

Proposition 12. *Every even elliptic function F with periods ω_1 and ω_2 is a rational function of $\wp$.*

Proof. If F has a zero or pole at the origin, then it must be of even order because F is an even function. Therefore, there exists an integer m such that $F\wp^m$ has no zero or pole at the lattice point. Thus we may assume that F has no zero or pole on Λ. Since F is even, if a is a zero of F, then so is $-a$. Therefore we can list all zeros of F as

$$\mathcal{A}_z = \{a_1, -a_1, \cdots, a_m, -a_m\},$$

counted with multiplicities. Similarly, we can list all poles of F as

$$\mathcal{A}_p = \{b_1, -b_1, \cdots, b_m, -b_m\}.$$

Now consider

$$G(z) := \frac{[\wp(z) - \wp(a_1)] \cdots [\wp(z) - \wp(a_m)]}{[\wp(z) - \wp(b_1)] \cdots [\wp(z) - \wp(b_m)]},$$

which is doubly periodic with the exact same zeros and poles of F. Therefore $\frac{F}{G}$ is holomorphic, and thus constant. This completes the proof. □

Proposition 13. *Every elliptic function f with periods ω_1 and ω_2 is a rational function of $\wp$ and $\wp'$.*

Proof. Decompose $f = f_{even} + f_{odd}$, the even and odd parts of f. Since both f_{even} and $\frac{f_{odd}}{\wp'}$ are even elliptic functions, we have

$$f = f_{even} + f_{odd} = f_{even} + \frac{f_{odd}}{\wp'}\wp' = R_1(\wp) + R_2(\wp)\wp'$$

for some rational functions R_1 and R_2. □

4.2 Topological lower bound on energy

Recall the stereographic projection $\pi \colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ and its inverse $\pi^{-1} \colon \hat{\mathbb{C}} \to \mathbb{S}^2$ given by:

$$\pi \colon \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \frac{u_1 + iu_2}{1 - u_3} \ , \ \pi^{-1} \colon z \mapsto \frac{1}{1+|z|^2} \begin{bmatrix} 2\mathrm{Re}(z) \\ 2\mathrm{Im}(z) \\ |z|^2 - 1 \end{bmatrix},$$

where $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane. Also recall that for a C^1 map $u \colon \mathbb{T}^2 \to \mathbb{S}^2$, the Dirichlet energy is given by

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2 \, dxdy = \frac{1}{2}\int_{\mathbb{T}^2} \sum_{j=1}^{3} \left(\left|\frac{\partial u_j}{\partial x}\right|^2 + \left|\frac{\partial u_j}{\partial y}\right|^2 \right) dxdy.$$

As in chapter 3, the *Brouwer degree* of $u \colon \mathbb{T}^2 \to \mathbb{S}^2$ can be computed through the formula

$$\deg(u) := \frac{1}{4\pi}\int_{\mathbb{T}^2} u_x \cdot (u \times u_y) \, dxdy = -\frac{1}{4\pi}\int_{\mathbb{T}^2} u_y \cdot (u \times u_x) \, dxdy.$$

Let $f = f_1 + if_2 := \pi \circ u \colon \mathbb{T}^2 \to \hat{\mathbb{C}}$ be the composition of u with the stereographic projection $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$. As before, we can show that

$$\mathcal{E}(u) \geq 4\pi|\deg(u)|,$$

and the equality occurs if and only if either f or its complex conjugate $\bar{f}$ is holomorphic from $\mathbb{T}^2$ to $\hat{\mathbb{C}}$. That is,

$$u\colon \mathbb{T}^2 \to \mathbb{S}^2 \text{ is harmonic}$$

if and only if

$$\text{either } f = f_1 + if_2 := \pi \circ u\colon \mathbb{T}^2 \to \hat{\mathbb{C}} \text{ or } \bar{f} = f_1 - if_2\colon \mathbb{T}^2 \to \hat{\mathbb{C}} \text{ is elliptic.}$$

Moreover, the energy and degree in terms of f are given by

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2\, dxdy = 2\int_{\mathbb{T}^2} \frac{|\nabla f|^2}{(1+|f|^2)^2}\, dxdy = 2\int_{\mathbb{T}^2} \frac{(f_{1,x}^2 + f_{1,y}^2 + f_{2,x}^2 + f_{2,y}^2)}{(1+|f|^2)^2}\, dxdy,$$

$$\deg(u) = \frac{1}{4\pi}\int_{\mathbb{T}^2} u_x \cdot (u \times u_y)\, dxdy = \frac{1}{\pi}\int_{\mathbb{T}^2} \frac{(f_{1,x}f_{2,y} - f_{1,y}f_{2,x})}{(1+|f|^2)^2}\, dxdy,$$

and hence

$$\mathcal{E}(u) - 4\pi\deg(u) = 2\int_{\mathbb{T}^2} \frac{(f_{1,x} - f_{2,y})^2 + (f_{1,y} + f_{2,x})^2}{(1+|f|^2)^2}\, dxdy \geq 0,$$

$$\mathcal{E}(u) + 4\pi\deg(u) = 2\int_{\mathbb{T}^2} \frac{(f_{1,x} + f_{2,y})^2 + (f_{1,y} - f_{2,x})^2}{(1+|f|^2)^2}\, dxdy \geq 0.$$

4.3 Conjecture

From the previous section, we know that given any integer $m \geq 2$, the collection of conformal maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ of degree m is a finite-dimensional manifold of dimension $d = d(m)$. For example, the collection of degree-2 conformal maps (with Dirichlet energy $\mathcal{E}_{min} = 8\pi$) are given by:

$$\mathcal{M}_2^8 = \left\{ \pi^{-1} \circ \frac{a\wp(\cdot - z_0) + b}{c\wp(\cdot - z_0) + d} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C}), z_0 \in \mathbb{C} \right\},$$

a manifold of (real) dimension $d(2) = 8$.

We will restate here our main conjecture (**Conjecture 1**) from chapter 1:

Given any integer $m \geq 2$, let $\mathcal{M}_m^d$ be the d-dimensional manifold of conformal maps from $\mathbb{T}^2$ to $\mathbb{S}^2$ of degree m. For any $\bar{u} \in \mathcal{M}_m^d$, there exists $\delta_0 = \delta_0(\bar{u}) > 0$ such that if $u_0 \in H^2(\mathbb{T}^2, \mathbb{S}^2)$ is of degree m (hence homotopic to $\bar{u}$) with

$$\|u_0 - \bar{u}\|_{H^2(\mathbb{T}^2,\mathbb{R}^3)} \leq \delta_0^2$$

and

$$4\pi m \leq \mathcal{E}(u_0) < 4\pi m + \delta_0^2,$$

then for the harmonic map heat flow:

$$\begin{aligned} u_t &= \Delta u + |\nabla u|^2 u \\ u|_{t=0} &= u_0 \end{aligned}$$

(a) The maximal existence time $T = T(u_0)$ can be taken to be infinity.

(b) There exists $\bar{\bar{u}} \in \mathcal{M}_m^d$ such that

$$u(\cdot, t) \to \bar{\bar{u}} \text{ in } H^2(\mathbb{T}^2, \mathbb{S}^2) \text{ as } t \to \infty.$$

Chapter 5

Partial Results (Linearized Equation)

5.1 Harmonic map heat flow in projected coordinates

Let $u = u(z,t) \colon \mathbb{T}^2 \times \mathbb{R}^+ \to \mathbb{S}^2$ be a solution of the harmonic map heat flow near a fixed steady state (harmonic map) $\bar{u} = \bar{u}(z) \colon \mathbb{T}^2 \to \mathbb{S}^2$ of positive degree m (hence $\mathcal{E}(\bar{u}) = \mathcal{E}_{min} = 4\pi m$). That is,

$$
\begin{aligned}
&\Delta\bar{u} + |\nabla\bar{u}|^2\bar{u} = 0 \text{ in } \mathbb{T}^2,\\
&u_t = \Delta u + |\nabla u|^2 u \text{ in } \mathbb{T}^2 \times (0,T),\\
&u|_{t=0} = u_0 \text{ in } \mathbb{T}^2,\\
&\|u_0 - \bar{u}\|_{H^2(\mathbb{T}^2,\mathbb{R}^3)} \le \delta_0^2 \text{ , and}\\
&0 \le \mathcal{E}(u_0) - \mathcal{E}_{min} \ll 1
\end{aligned}
$$

for some $0 < \delta_0 = \delta_0(m) \ll 1$. Denote by

$$H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$$

the collection of H^2 functions from $\mathbb{T}^2$ to $\mathbb{R}^3$ that are perpendicular to $\bar{u}$ point-wise in $\mathbb{T}^2$. It is natural to rewrite the equation of u in the non-linear space $H^2(\mathbb{T}^2, \mathbb{S}^2)$ as an equation in the linear space $H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$.

Proposition 14. *Decompose u as*

$$u = \frac{\bar{u} + \xi}{|\bar{u} + \xi|} \ , \ \text{where } \bar{u} \cdot \xi = 0 \text{ point-wise,}$$

for some $\xi \in H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$, then

(a)

$$\left(I_3 - \frac{(\bar{u} + \xi) \otimes \xi}{1 + |\xi|^2}\right) \xi_t = \Delta \xi + \frac{|\nabla \bar{u}|^2(-|\xi|^2 \bar{u} + \xi) + (-\Delta \xi \cdot \xi + 2\nabla \bar{u} \cdot \nabla \xi)(\bar{u} + \xi)}{1 + |\xi|^2} - \frac{1}{1 + |\xi|^2} \nabla(\bar{u} + \xi) \cdot \nabla |\xi|^2 + \frac{|\nabla |\xi|^2|^2}{2(1 + |\xi|^2)^2}(\bar{u} + \xi). \tag{5.1}$$

(b) The linearization of eq. (5.1) *at the steady state $\bar{u}$ reads*

$$\xi_t = \Delta \xi + |\nabla \bar{u}|^2 \xi + 2(\nabla \bar{u} \cdot \nabla \xi) \bar{u}, \tag{5.2}$$

for $\xi \colon \mathbb{T}^2 \times \mathbb{R}^+ \to \mathbb{R}^3$ such that $\bar{u} \cdot \xi = 0$ point-wise.

Remark 12. Notice that for each $w \in \mathbb{S}^2$,

$$\left| w^T \frac{(\bar{u} + \xi) \otimes \xi}{1 + |\xi|^2} w \right| = \frac{|w \cdot (\bar{u} + \xi)| \cdot |w \cdot \xi|}{1 + |\xi|^2} \leq \frac{|\bar{u} + \xi| \cdot |\xi|}{1 + |\xi|^2} = \frac{|\xi|}{\sqrt{1 + |\xi|^2}} < 1,$$

so the matrix $\left(I_3 - \frac{(\bar{u}+\xi)\otimes\xi}{1+|\xi|^2}\right)$ is invertible.

Remark 13. For the linearized equation (5.2), direct computations yield:

$$\begin{aligned}
(\bar{u} \cdot \xi)_t &= \bar{u} \cdot \xi_t = \bar{u} \cdot \left(\Delta \xi + |\nabla \bar{u}|^2 \xi + 2(\nabla \bar{u} \cdot \nabla \xi)\bar{u}\right) \\
&= \bar{u} \cdot \Delta \xi + |\nabla \bar{u}|^2 \bar{u} \cdot \xi + 2(\nabla \bar{u} \cdot \nabla \xi) \\
&= \Delta(\bar{u} \cdot \xi) + \left(\Delta \bar{u} + |\nabla \bar{u}|^2 \bar{u}\right) \cdot \xi = \Delta(\bar{u} \cdot \xi).
\end{aligned}$$

If $\bar{u} \cdot \xi = 0$ point-wise at time $t = 0$, then by the maximum principle for heat equation (on periodic domain $\mathbb{T}^2$), we have $\bar{u} \cdot \xi = 0$ on $\mathbb{T}^2 \times \mathbb{R}^+$. In other words, the perpendicular condition $\bar{u} \cdot \xi = 0$ is preserved by eq. (5.2).

Proof. (a) 1. Let $u = \frac{v}{|v|}$, where $v := \bar{u} + \xi$, then

$$u_t = \frac{v_t}{|v|} - \frac{|v|_{,t}}{|v|^2}v = \frac{v_t}{|v|} - \frac{v \cdot v_t}{|v|^3}v,\ \nabla u = \frac{\nabla v}{|v|} - \frac{v \cdot \nabla v}{|v|^3}v,$$

and

$$\begin{aligned} u_{xx} &= \frac{v_{xx}}{|v|} - \frac{|v|_{,x}}{|v|^2}v_x + \left(-\frac{|v_x|^2 + v \cdot v_{xx}}{|v|^3} + \frac{3(v \cdot v_x)|v|_{,x}}{|v|^4}\right)v - \frac{v \cdot v_x}{|v|^3}v_x \\ &= \frac{v_{xx}}{|v|} - \frac{2(v \cdot v_x)v_x}{|v|^3} - \frac{|v_x|^2}{|v|^3}v - \frac{v \cdot v_{xx}}{|v|^3}v + \frac{3||v|^2|_{,x}^2}{4|v|^5}v. \end{aligned}$$

Hence

$$\Delta u = \frac{\Delta v}{|v|} - \frac{2\left((v \cdot v_x)v_x + (v \cdot v_y)v_y\right)}{|v|^3} - \frac{|\nabla v|^2}{|v|^3}v - \frac{v \cdot \Delta v}{|v|^3}v + \frac{3|\nabla |v|^2|^2}{4|v|^5}v,$$

and therefore

$$\begin{aligned} \frac{v_t}{|v|} - \frac{v \cdot v_t}{|v|^3}v = u_t &= \Delta u + |\nabla u|^2 u \qquad (5.3) \\ &= \frac{\Delta v}{|v|} - \frac{2\left((v \cdot v_x)v_x + (v \cdot v_y)v_y\right)}{|v|^3} - \frac{|\nabla v|^2}{|v|^3}v - \frac{v \cdot \Delta v}{|v|^3}v + \frac{3|\nabla |v|^2|^2}{4|v|^5}v \\ &+ \left(\frac{|\nabla u|^2}{|v|^2} - 2\frac{(v \cdot \nabla v)^2}{|v|^4} + \frac{(v \cdot \nabla v)^2}{|v|^4}\right)\frac{v}{|v|} \\ &= \frac{\Delta v}{|v|} - \frac{2\left((v \cdot v_x)v_x + (v \cdot v_y)v_y\right)}{|v|^3} - \frac{v \cdot \Delta v}{|v|^3}v + \frac{|\nabla |v|^2|^2}{2|v|^5}v. \end{aligned}$$

2. Since $|\bar{u}| = 1$, $\bar{u} \cdot \xi = 0$, and $v = \bar{u} + \xi$, we have

$$\begin{gathered} v_t = \bar{u}_t + \xi_t = \xi_t,\ \Delta v = \Delta\bar{u} + \Delta\xi = \Delta\xi - |\nabla\bar{u}|^2\bar{u},\ \text{ and hence} \\ v \cdot \Delta v = (\bar{u} + \xi) \cdot (\Delta\xi - |\nabla\bar{u}|^2\bar{u}) = (\bar{u} + \xi) \cdot \Delta\xi - |\nabla\bar{u}|^2. \end{gathered}$$

Also, $\bar{u} \cdot \xi = 0$ implies

$$0 = \Delta\bar{u}\cdot\xi + 2\nabla\bar{u}\cdot\nabla\xi + \bar{u}\cdot\Delta\xi = -|\nabla\bar{u}|^2\bar{u}\cdot\xi + 2\nabla\bar{u}\cdot\nabla\xi + \bar{u}\cdot\Delta\xi = 2\nabla\bar{u}\cdot\nabla\xi + \bar{u}\cdot\Delta\xi.$$

On the other hand, $|v|^2 = |\bar{u}|^2 + 2\bar{u} \cdot \xi + |\xi|^2 = 1 + |\xi|^2$, which implies

$v \cdot v_t = \xi \cdot \xi_t$, $\nabla |v|^2 = \nabla |\xi|^2$, and hence

$$2((v \cdot v_x)v_x + (v \cdot v_y)v_y) = |\xi|^2_{,x} v_x + |\xi|^2_{,y} v_y = \nabla v \cdot \nabla |\xi|^2 = \nabla(\bar{u} + \xi) \cdot \nabla |\xi|^2.$$

Therefore eq. (5.3) is equivalent to

$$\begin{aligned}
\xi_t - \frac{\xi \cdot \xi_t}{1 + |\xi|^2}(\bar{u} + \xi) &= v_t - \frac{v \cdot v_t}{|v|^2} v \\
&= \Delta v - \frac{2\left((v \cdot v_x)v_x + (v \cdot v_y)v_y\right)}{|v|^2} - \frac{v \cdot \Delta v}{|v|^2} v + \frac{|\nabla |v|^2|^2}{2|v|^4} v \\
&= \Delta \xi - |\nabla \bar{u}|^2 \bar{u} - \frac{\nabla(\bar{u} + \xi) \cdot \nabla |\xi|^2}{1 + |\xi|^2} \\
&+ \frac{|\nabla \bar{u}|^2 - \Delta \xi \cdot \xi - \bar{u} \cdot \Delta \xi}{1 + |\xi|^2}(\bar{u} + \xi) + \frac{|\nabla |\xi|^2|^2}{2(1 + |\xi|^2)^2}(\bar{u} + \xi) \\
&= \Delta \xi + \frac{|\nabla \bar{u}|^2(-|\xi|^2 \bar{u} + \xi) + (-\Delta \xi \cdot \xi + 2\nabla \bar{u} \cdot \nabla \xi)(\bar{u} + \xi)}{1 + |\xi|^2} \\
&- \frac{1}{1 + |\xi|^2} \nabla(\bar{u} + \xi) \cdot \nabla |\xi|^2 + \frac{|\nabla |\xi|^2|^2}{2(1 + |\xi|^2)^2}(\bar{u} + \xi).
\end{aligned}$$

(b) Substitute $\epsilon \xi$ for ξ near $\epsilon = 0$, then eq. (5.1) is equivalent to

$$\begin{aligned}
&\epsilon \xi_t - \epsilon^2 \frac{(\bar{u} + \epsilon \xi) \otimes \xi}{1 + \epsilon^2 |\xi|^2} \xi_t \\
&= \epsilon \Delta \xi + \frac{|\nabla \bar{u}|^2(-\epsilon^2 |\xi|^2 \bar{u} + \epsilon \xi) + (-\epsilon^2 \Delta \xi \cdot \xi + 2\epsilon \nabla \bar{u} \cdot \nabla \xi)(\bar{u} + \epsilon \xi)}{1 + \epsilon^2 |\xi|^2} \\
&- \epsilon^2 \frac{1}{1 + \epsilon^2 |\xi|^2} \nabla(\bar{u} + \epsilon \xi) \cdot \nabla |\xi|^2 + \epsilon^4 \frac{|\nabla |\xi|^2|^2}{2(1 + \epsilon^2 |\xi|^2)^2}(\bar{u} + \epsilon \xi),
\end{aligned}$$

which implies

$$\epsilon \xi_t + \mathcal{O}(\epsilon^2) = \epsilon \left(\Delta \xi + |\nabla \bar{u}|^2 \xi + 2(\nabla \bar{u} \cdot \nabla \xi) \bar{u} \right) + \mathcal{O}(\epsilon^2).$$

Therefore $\xi_t + \mathcal{O}(\epsilon) = \Delta \xi + |\nabla \bar{u}|^2 \xi + 2(\nabla \bar{u} \cdot \nabla \xi) \bar{u} + \mathcal{O}(\epsilon)$, and hence the linearization of eq. (5.1) at the steady state $\bar{u}$ is given by

$$\xi_t = \Delta\xi + |\nabla\bar{u}|^2\xi + 2(\nabla\bar{u}\cdot\nabla\xi)\bar{u},$$

under the constraint $\bar{u}\cdot\xi \equiv 0$.

□

Remark 14. A faster way to derive the linearized equation (5.2) is by examining the second variation of the Dirichlet energy. More precisely, since $\xi \in T_{\bar{u}}\mathbb{S}^2$, by proposition 4 and integration by parts, we have

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \mathcal{E}\left(\frac{\bar{u}+\epsilon\xi}{|\bar{u}+\epsilon\xi|}\right) = \int_{\mathbb{T}^2}\left(|\nabla\xi|^2 - |\nabla\bar{u}|^2|\xi|^2\right)\,dxdy, = -\int_{\mathbb{T}^2}\left(\Delta\xi + |\nabla\bar{u}|^2\xi\right)\cdot\xi\,dxdy.$$

Since $\bar{u}\cdot\xi = 0$ pointwise, the linearized ξ-equation is given by

$$\xi_t = \Delta\xi + |\nabla\bar{u}|^2\xi + M(x)\bar{u} \tag{5.4}$$

for some function $M = M(x)$. We can then take the inner product of eq. (5.4) with $\bar{u}$ to find $0 = (\bar{u}\cdot\xi)_t = \bar{u}\cdot\xi_t = \bar{u}\cdot\Delta\xi + 0 + M \implies M = -\bar{u}\cdot\Delta\xi = 2(\nabla\bar{u}\cdot\nabla\xi)$.

5.2 Stability for the linearized equation

The linearized equation (5.2) at $\bar{u}$ is

$$\xi_t = L_{\bar{u}}[\xi] := \Delta\xi + |\nabla\bar{u}|^2\xi + 2(\nabla\bar{u}\cdot\nabla\xi)\bar{u}, \tag{5.5}$$

where $L_{\bar{u}}$ is the linearization at the steady-state $\bar{u}$ in the original equation. That is, for $\Delta\bar{u} + |\nabla\bar{u}|^2\bar{u} = 0$, the corresponding linearized map near $\bar{u}$ reads:

$$L_{\bar{u}}[v] = \Delta v + |\nabla\bar{u}|^2 v + 2(\nabla\bar{u}\cdot\nabla v)\bar{u}.$$

Recall that by proposition 4, the second variation of the Dirichlet energy

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{T}^2}|\nabla u|^2\,dxdy$$

with target manifold $\mathbb{S}^2$ is given by:

$$Q_{\bar{u}}[\xi] := \frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} \mathcal{E}\left(\frac{\bar{u}+\epsilon\xi}{|\bar{u}+\epsilon\xi|}\right) = \int_{\mathbb{T}^2} \left(|\nabla\xi|^2 - |\nabla\bar{u}|^2|\xi|^2\right) dxdy,$$

if $\xi \in H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$.

Proposition 15. $\ker(L_{\bar{u}}) = \ker(Q_{\bar{u}})$.

Proof. Since harmonic maps are critical points of the Dirichlet energy within homotopy class (maps of degree m), clearly $\ker(Q_{\bar{u}}) \subseteq \ker(L_{\bar{u}})$. On the other hand, if $\xi \in \ker(L_{\bar{u}})$, by integration by parts,

$$\begin{aligned}
\int_{\mathbb{T}^2} \left(|\nabla\xi|^2 - |\nabla\bar{u}|^2|\xi|^2\right) dxdy &= -\int_{\mathbb{T}^2} \xi\cdot\Delta\xi \, dxdy - \int_{\mathbb{T}^2} |\nabla\bar{u}|^2|\xi|^2 \, dxdy \\
&= \int_{\mathbb{T}^2} \xi\cdot\left(|\nabla\bar{u}|^2\xi + 2(\nabla\bar{u}\cdot\nabla\xi)\bar{u}\right) dxdy - \int_{\mathbb{T}^2} |\nabla\bar{u}|^2|\xi|^2 \, dxdy \\
&= 2\int_{\mathbb{T}^2} (\nabla\bar{u}\cdot\nabla\xi)(\xi\cdot\bar{u}) \, dxdy = 0,
\end{aligned}$$

which implies $\xi \in \ker(Q_{\bar{u}})$. (The last equality holds because $\xi\cdot\bar{u} = 0$ point-wise in $\mathbb{T}^2$.) Therefore $\ker(L_{\bar{u}}) \subseteq \ker(Q_{\bar{u}})$. Hence $\ker(L_{\bar{u}}) = \ker(Q_{\bar{u}})$. □

Proposition 16. *For any* $\xi \in H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$ *in* $\ker(L_{\bar{u}})$, *there exists a 1-parameter family* $U\colon \mathbb{T}^2 \times (-\epsilon,\epsilon) \to \mathbb{S}^2$ *of holomorphic maps from* $\mathbb{T}^2$ *to* $\mathbb{S}^2$ *with* $U(\cdot,0) = \bar{u}$ *such that* $\frac{d}{d\epsilon}\big|_{\epsilon=0} U(\cdot,\epsilon) = \xi$.

Proof. Recall the stereographic projection $\pi\colon \mathbb{S}^2 \to \hat{\mathbb{C}}$ and its inverse $\pi^{-1}\colon \hat{\mathbb{C}} \to \mathbb{S}^2$ given by:

$$\pi\colon \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mapsto \frac{u_1 + iu_2}{1-u_3}\ , \ \pi^{-1}\colon f = f_1 + if_2 \mapsto \frac{2}{1+|f|^2}\begin{bmatrix} f_1 \\ f_2 \\ \frac{|f|^2-1}{2} \end{bmatrix},$$

where $\hat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ is the extended complex plane. Let $\bar{u} = (u_1,u_2,u_3)^T$ be the fixed harmonic map and denote by $f = f_1 + if_2 := \pi\circ\bar{u}\colon \mathbb{T}^2 \to \hat{\mathbb{C}}$. To prove the proposition, it is enough to show that if f is holomorphic, then for any $\xi \in \ker(L_{\bar{u}})$, its pushforward $\eta := d\pi_{\bar{u}}(\xi)\colon \mathbb{T}^2 \to T_f\hat{\mathbb{C}} \cong \mathbb{R}^2$ is also holomorphic.

- For any smooth map $u\colon \mathbb{T}^2 \to \mathbb{S}^2$, let $\phi = \phi_1 + i\phi_2 := \pi \circ u\colon \mathbb{T}^2 \to \mathbb{C}$. By direct calculations, the Dirichlet energy and the Brouwer degree of the map u in terms of ϕ are given by:

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{T}^2} |\nabla u|^2\, dxdy = 2\int_{\mathbb{T}^2} \frac{|\nabla\phi|^2}{(1+|\phi|^2)^2}\, dxdy,$$

$$4\pi \deg(u) = \int_{\mathbb{T}^2} u_x \cdot (u \times u_y)\, dxdy = 4\int_{\mathbb{T}^2} \frac{J(\phi)}{(1+|\phi|^2)^2}\, dxdy,$$

where $J(\phi) := \phi_{1,x}\phi_{2,y} - \phi_{1,y}\phi_{2,x}$ is the Jacobian of the map ϕ.

- In terms of $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, we have $|\nabla\phi|^2 = 2(|\phi_z|^2 + |\phi_{\bar{z}}|^2)$ and $J(\phi) = |\phi_z|^2 - |\phi_{\bar{z}}|^2$. Therefore

$$\mathcal{E}(u) = \int_{\mathbb{T}^2} \frac{4}{(1+|\phi|^2)^2}\left(|\phi_z|^2 + |\phi_{\bar{z}}|^2\right)\, dxdy,$$

$$\mathcal{J}(u) := 4\pi\deg(u) = \int_{\mathbb{T}^2} \frac{4}{(1+|\phi|^2)^2}\left(|\phi_z|^2 - |\phi_{\bar{z}}|^2\right)\, dxdy.$$

- Now let $\eta = \eta_1 + i\eta_2\colon \mathbb{T}^2 \to \mathbb{C}$ be arbitrary and consider the 1-parameter family of maps $\phi = \phi^\epsilon := f + \epsilon\eta\colon \mathbb{T}^2 \to \mathbb{C}$ for $|\epsilon| << 1$. Since $u^\epsilon := \pi^{-1} \circ \phi^\epsilon\colon \mathbb{T}^2 \to \mathbb{S}^2$ is homotopic to $\bar{u} = \pi^{-1} \circ f\colon \mathbb{T}^2 \to \mathbb{S}^2$, $\mathcal{J}(u^\epsilon) = 4\pi\deg(\bar{u}) = 4\pi m$ is constant in ϵ. Therefore the second variation of $\mathcal{E}(u^\epsilon)$ is the same as that of

$$\mathcal{K}(u^\epsilon) := \mathcal{E}(u^\epsilon) - \mathcal{J}(u^\epsilon) = \int_{\mathbb{T}^2} \frac{8}{(1+|\phi^\epsilon|^2)^2}|\phi^\epsilon_{\bar{z}}|^2\, dxdy = \int_{\mathbb{T}^2} \frac{8}{(1+|\phi^\epsilon|^2)^2}\phi^\epsilon_{\bar{z}} \cdot \overline{\phi^\epsilon_{\bar{z}}}\, dxdy.$$

- Since $\phi^\epsilon = f + \epsilon\eta$ and f is holomorphic, we have $\phi^\epsilon_{\bar{z}}\big|_{\epsilon=0} = \overline{\phi^\epsilon_{\bar{z}}}\big|_{\epsilon=0} = 0$. Therefore

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\mathcal{K}(u^\epsilon)\bigg|_{\epsilon=0} &= 2\int_{\mathbb{T}^2} \frac{8}{(1+|\phi^\epsilon|^2)^2}\bigg|_{\epsilon=0}\left(\frac{d}{d\epsilon}\phi^\epsilon_{\bar{z}}\bigg|_{\epsilon=0} \cdot \frac{d}{d\epsilon}\overline{\phi^\epsilon_{\bar{z}}}\bigg|_{\epsilon=0}\right) dxdy \\ &= \int_{\mathbb{T}^2} \frac{16}{(1+|f|^2)^2}\eta_{\bar{z}} \cdot \overline{\eta_{\bar{z}}}\, dxdy = 16\int_{\mathbb{T}^2} \frac{|\eta_{\bar{z}}|^2}{(1+|f|^2)^2}\, dxdy \geq 0.\end{aligned}$$

The equality happens if and only if $\eta_{\bar{z}} \equiv 0$, which means η is holomorphic.

□

We are now ready to prove our main theorem (**Theorem 1**) from chapter 1:

Let $\{\phi_i\colon 1 \le i \le d\}$ *be a* L^2*-orthonormal basis of* $\ker(L_{\bar{u}})$*. Consider the linearized equation*

$$\xi_t = \Delta\xi + |\nabla\bar{u}|^2\xi + 2(\nabla\bar{u}\cdot\nabla\xi)\bar{u} =: L_{\bar{u}}[\xi].$$

For any initial condition decomposed as:

$$\xi|_{t=0} = \sum_{i=1}^{d} c_i\phi_i(z) + \eta(z) \in H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2),$$

where $c_i \in \mathbb{R}$ *for each* $1 \le i \le d$ *, and* $\eta \in (\ker(L_{\bar{u}}))^{\perp}$ *in* $L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$*, we have*

$$\xi(z,t) \to \sum_{i=1}^{d} c_i\phi_i(z)\,(\in \ker(L_{\bar{u}}))\ \textit{ in } L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)\ \textit{ as } t\to\infty.$$

Proof. Since $\mathbb{T}^2$ is periodic, by integration by parts, we have

$$Q_{\bar{u}}[\xi] = -\int_{\mathbb{T}^2} L_{\bar{u}}[\xi]\cdot\xi\,dxdy = (-L_{\bar{u}}[\xi],\xi)_{L^2}.$$

On the compact manifold $\mathbb{T}^2$, from the general theory of compact self-adjoint operators, the spectrum of $-L_{\bar{u}}$ with respect to L^2-norm is discrete and consists of only real eigenvalues with finite multiplicity. Moreover, the corresponding eigenspaces are mutually perpendicular, and forms an orthogonal decomposition of $L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$.

Since $\bar{u} \in \mathcal{M}_m^d$, from the two previous propositions, the quadratic form $Q_{\bar{u}}$ is positive-semi-definite with a d-dimensional kernel $\ker(L_{\bar{u}}) = \ker(Q_{\bar{u}})$ obtained by perturbing within $\mathcal{M}_m^d$: the collection of steady-states (conformal maps) of degree m from $\mathbb{T}^2$ to $\mathbb{S}^2$.

Therefore the spectrum of $-L_{\bar{u}}$ consists of eigenvalues $\{\lambda_k\}_{k\ge 0}$, where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$ and $\lim_{k\to\infty}\lambda_k = \infty$. Denote by $E(\lambda_k)$ the corresponding finite-dimensional eigenspaces, then

$$L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2) = \bigoplus_{k\ge 0} E(\lambda_k)\ ,\ \text{and}\ (-L_{\bar{u}})\big|_{E(\lambda_k)} = \lambda_k Id_{E(\lambda_k)}.$$

For any initial condition $\xi|_{t=0} = \sum_{i=1}^{d} c_i\phi_i(z) + \eta(z)$, decompose $\eta \in (\ker(L_{\bar{u}}))^{\perp}$ as $\eta(z) = \sum_{k\geq 1} \eta_k(z)$, where $\eta_k \in E(\lambda_k)$. The solution to the parabolic equation

$$\xi_t = L_{\bar{u}}[\xi]$$

is given by

$$\xi(z,t) = \sum_{i=1}^{d} c_i\phi_i(z) + \sum_{k\geq 1} e^{-\lambda_k t}\eta_k(z).$$

Therefore

$$\begin{aligned}\left\|\xi(z,t) - \sum_{i=1}^{d} c_i\phi_i(z)\right\|_{L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)} &= \sum_{k\geq 1} e^{-\lambda_k t}\|\eta_k\|_{L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)} \\ &\leq e^{-\lambda_1 t}\sum_{k\geq 1}\|\eta_k\|_{L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)} = e^{-\lambda_1 t}\|\eta\|_{L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)},\end{aligned}$$

and hence

$$\xi(z,t) \to \sum_{i=1}^{d} c_i\phi_i(z)\,(\in \ker(L_{\bar{u}})) \text{ in } L^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2) \text{ as } t \to \infty.$$

□

Chapter 6

Finite Dimensional Model Problem

The assertions in Conjecture 1 are infinite-dimensional in nature. Namely, the function space $H^2(\mathbb{T}^2, \mathbb{S}^2)$ and its linearized tangent space $H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$ near $\bar{u}$ are infinite-dimensional; where $H^2(\mathbb{T}^2, T_{\bar{u}}\mathbb{S}^2)$ denote the collection of H^2 functions from $\mathbb{T}^2$ to $\mathbb{R}^3$ that are perpendicular to $\bar{u}$ point-wise in $\mathbb{T}^2$.

In the simpler situation where the *contracting* part is also finite-dimensional, the situation in Conjecture 1 is equivalent to the *finite-dimensional model* of the gradient flow problem in the following proposition, up to first order.

We will prove here our main proposition (**Proposition 1**) from chapter 1:

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$, let $A = A(x) \in C^1(\mathbb{R}^d, \mathbb{R}^{n\times n})$ and define

$$f = f(x, y) := \frac{1}{2} \left(A(x) \cdot y, y\right)_{\mathbb{R}^n} .$$

Denote by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm of $n \times n$ matrices. Assume

$c_1|\eta|^2 \leq (A(x) \cdot \eta, \eta)_{\mathbb{R}^n} \leq c_2|\eta|^2$ *for all $x \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^n$, for some $0 < c_1 < c_2 < \infty$;*

and

$$\max_{1\leq i\leq d} \left\| \frac{\partial A}{\partial x_i}(x) \right\|_{HS} \leq K(1 + |x|) \text{ for all } x \in \mathbb{R}^d \text{ , for some } K > 0.$$

Consider the gradient flow:

$$\frac{dx_i}{dt} = -\nabla_{x_i} f = -\frac{1}{2}\left(\frac{\partial A}{\partial x_i}(x) \cdot y, y\right)_{\mathbb{R}^n} \text{ , for each } 1 \leq i \leq d,$$
$$\frac{dy}{dt} = -\nabla_y f = -A(x) \cdot y.$$

For any initial condition $(x(0), y(0)) = (x_0, y_0)$, the maximal time of existence is equal to infinity, and

$$x(t) \to \bar{x} \text{ , } y(t) \to 0 \text{ as } t \to \infty \text{ for some } \bar{x} \in \mathbb{R}^d.$$

Proof. 1. Since $\frac{d}{dt}\|y(t)\|_2^2 = 2(y'(t), y(t))_{\mathbb{R}^n} = -2(A(x(t))\cdot y(t), y(t))_{\mathbb{R}^n} \leq -2c_1\|y(t)\|_2^2$, we have $\frac{d}{dt}\left(e^{2c_1 t}\|y(t)\|_2^2\right) \leq 0$. Therefore

$$\|y(t)\|_2 \leq e^{-c_1 t}\|y(0)\|_2 = e^{-c_1 t}\|y_0\|_2 \to 0 \text{ as } t \to \infty.$$

2. For each $1 \leq i \leq d$, because $\|y(t)\|_2 \leq e^{-c_1 t}\|y_0\|_2$ from above,

$$|x_i'(t)| = \left|-\frac{1}{2}\left(\frac{\partial A}{\partial x_i}(x(t)) \cdot y(t), y(t)\right)_{\mathbb{R}^n}\right|$$
$$\leq \frac{K}{2}\left(1 + |x(t)|\right)\|y(t)\|_2^2 \leq \frac{K\|y_0\|_2^2}{2} e^{-2c_1 t}\left(1 + |x(t)|\right),$$

which implies

$$\left|\frac{d}{dt}\ln\left(1 + |x(t)|\right)\right| = \frac{\left|\frac{d}{dt}|x(t)|\right|}{1 + |x(t)|} \leq \frac{\sum_{i=1}^d |x_i'(t)|}{1 + |x(t)|} \leq \frac{dK\|y_0\|_2^2}{2} e^{-2c_1 t}.$$

Therefore

$$\left|\int_0^\infty \frac{d}{dt}\ln\left(1 + |x(t)|\right) dt\right| \leq \int_0^\infty \left|\frac{d}{dt}\ln\left(1 + |x(t)|\right)\right| dt \leq \frac{dK\|y_0\|_2^2}{4c_1} < \infty,$$

and hence $x(t) \to \bar{x}$ as $t \to \infty$ for some $\bar{x} \in \mathbb{R}^d$.

□

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