

# Symmetric Real-Valued Orthonormal Scaling Functions With Compact Support in $L_2(\mathbb{R}^s)$

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## Abstract

It is well known that there is no continuous real-valued orthonormal scaling function  $\phi \in L_2(\mathbb{R})$  such that  $\phi$  has compact support and  $\phi$  is symmetric about  $x = c$  for some  $c \in \frac{1}{2}\mathbb{Z}$ . In this note, we shall demonstrate a similar phenomenon for any dimension. More precisely, we prove that there is no *continuous* real-valued orthonormal scaling function  $\phi \in L_2(\mathbb{R}^s)$  such that  $\phi$  is compactly supported and  $\phi$  is symmetric about all the superplanes  $x_j = c_j, j = 1, \dots, s$  for some  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}$ . This result completely answers a question asked in [6]. Finally, we discuss some properties of real-valued compactly supported orthonormal scaling functions  $\phi \in L_2(\mathbb{R}^s)$  such that  $\phi$  is symmetric about a point  $x = c$  for some  $c \in \frac{1}{2}\mathbb{Z}^s$ .

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# 1 Introduction and Main Results

An orthonormal wavelet basis for  $L_2(\mathbb{R}^s)$  is derived from an orthonormal scaling function via a multiresolution analysis (see [1]). We say that a function  $\phi$  is an **orthonormal scaling function** if  $\phi \in L_2(\mathbb{R}^s)$  is a **refinable function**, i.e.,  $\phi$  satisfies the following refinement equation:

$$\phi = \sum_{k \in \mathbb{Z}^s} a_k \phi(2 \cdot -k), \quad (1.1)$$

such that the integer shifts of  $\phi$  consist of an orthonormal system, i.e.,

$$\int_{\mathbb{R}^s} \phi(t-k) \overline{\phi(t)} dt = \delta_{0k} \quad \forall k \in \mathbb{Z}^s, \quad (1.2)$$

where  $\delta$  denotes the **Dirac sequence** such that  $\delta_{00} = 1$  and  $\delta_{0k} = 0$  for all  $k \in \mathbb{Z}^s \setminus \{0\}$ .

In the refinement equation (1.1), the sequence  $a$  on  $\mathbb{Z}^s$  is assumed to be finitely supported and is called the **mask** for the refinable function  $\phi$ . When  $\sum_{k \in \mathbb{Z}^s} a_k = 2^s$ , it is known that there exists a unique distributional solution, denoted by  $\phi_a$  throughout the paper, to the refinement equation (1.1) with the normalized condition  $\widehat{\phi}_a(0) = 1$ . In this paper, we are mainly interested in real-valued refinable functions with compact support. Therefore, any mask  $a$  in this paper is assumed to be a finitely supported real-valued sequence on  $\mathbb{Z}^s$  such that  $\sum_{k \in \mathbb{Z}^s} a_k = 2^s$ .

A simple example of an orthonormal scaling function in  $L_2(\mathbb{R}^s)$  is the (tensor product) Haar scaling function  $\phi = \chi_{[0,1]^s}$ , which is the characteristic function of the unit cube  $[0, 1]^s$ . Note that the Haar scaling function  $\chi_{[0,1]^s}$  is symmetric about all the hyperplanes  $x_j = 1/2, j = 1, \dots, s$ , and  $\chi_{[0,1]^s}$  is not a continuous function.

Orthonormal wavelet bases for  $L_2(\mathbb{R})$  were first constructed by Daubechies in [1] from orthonormal scaling functions via multiresolution analyses. Due to many desirable properties of a wavelet basis such as sparse representations of functions and fast algorithms associated with it, orthonormal wavelet bases have proved to be very useful in many applications (see [1, 4, 6]). However, as proved by Daubechies in [1], up to an integer shift, the Haar scaling function  $\chi_{[0,1]}$  is the only real-valued orthonormal scaling function which is both compactly supported and symmetric about  $x = c$  for some  $c \in \frac{1}{2}\mathbb{Z}$ . This fact implies that there is no compactly supported real-valued symmetric orthonormal wavelet basis in dimension one which consists of continuous functions. However, on the other hand, in many applications symmetry (also called linear phase in the language of engineering) is a much desired property of a wavelet system. Symmetry of a wavelet system is claimed to produce less visual artifacts than non-linear phase wavelets and it helps to minimize phase distortion. For the importance about symmetry of a wavelet system, the reader is referred to [4, 6].

Since it is not possible to obtain continuous real-valued compactly supported symmetric orthonormal scaling function in *dimension one*, a lot of effort in the recent literature has been devoted to investigate the possibility of symmetric real-valued compactly supported orthonormal scaling functions with good smoothness in higher dimensions. Indeed, contrary to the case of dimension one, the following example was given by Kovačević and Vetterli in [4].

**Example 1.** The mask  $a$  on  $\mathbb{Z}^2$  is supported on  $[-2, 3]^2$  and is given by

$$\begin{bmatrix} \frac{3}{128} & -\frac{\sqrt{15}}{128} & \frac{3\sqrt{15}}{128} - \frac{15}{128} & -\frac{3\sqrt{15}}{128} - \frac{15}{128} & \frac{\sqrt{15}}{128} & \frac{3}{128} \\ \frac{\sqrt{15}}{128} & \frac{5}{128} & \frac{15}{128} - \frac{5\sqrt{15}}{128} & \frac{15}{128} + \frac{5\sqrt{15}}{128} & \frac{5}{128} & -\frac{\sqrt{15}}{128} \\ -\frac{3\sqrt{15}}{128} - \frac{15}{128} & \frac{15}{128} + \frac{5\sqrt{15}}{128} & \frac{15}{16} & \frac{15}{16} & \frac{15}{128} - \frac{5\sqrt{15}}{128} & \frac{3\sqrt{15}}{128} - \frac{15}{128} \\ \frac{3\sqrt{15}}{128} - \frac{15}{128} & \frac{15}{128} - \frac{5\sqrt{15}}{128} & \frac{15}{16} & \frac{15}{16} & \frac{15}{128} + \frac{5\sqrt{15}}{128} & -\frac{3\sqrt{15}}{128} - \frac{15}{128} \\ -\frac{\sqrt{15}}{128} & \frac{5}{128} & \frac{15}{128} + \frac{5\sqrt{15}}{128} & \frac{15}{128} - \frac{5\sqrt{15}}{128} & \frac{5}{128} & \frac{\sqrt{15}}{128} \\ \frac{3}{128} & \frac{\sqrt{15}}{128} & -\frac{3\sqrt{15}}{128} - \frac{15}{128} & \frac{3\sqrt{15}}{128} - \frac{15}{128} & -\frac{\sqrt{15}}{128} & \frac{3}{128} \end{bmatrix}.$$

Then it can be verified (see Section 2) that the refinable function  $\phi_a$  is indeed a continuous orthonormal scaling function in  $L_2(\mathbb{R}^2)$  ([4]) such that  $\phi$  is symmetric about the point  $x = (1/2, 1/2)$ , i.e.,  $\phi(1 - x_1, 1 - x_2) = \phi(x_1, x_2)$  for all  $(x_1, x_2) \in \mathbb{R}^2$ .

Using a cascade structure, systematic constructions were discussed in [4, 6] to obtain real-valued compactly supported orthonormal scaling functions  $\phi \in L_2(\mathbb{R}^2)$  such that  $\phi$  is symmetric about the point  $(1/2, 1/2)$ . However, as pointed out by Stanhill and Zeevi in [6], for some applications in high dimensions, stronger symmetry of a wavelet system may be required in order to facilitate symmetrical boundary conditions. For example, it may be desirable to have orthonormal scaling functions  $\phi \in L_2(\mathbb{R}^s)$  such that  $\phi$  is symmetric about all the hyperplanes  $x_j = c_j, j = 1, \dots, s$  for some  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}^s$ . The tensor product Haar scaling function  $\chi_{[0,1]^s}$  is one example of such symmetric orthonormal scaling functions since  $\chi_{[0,1]^s}$  is symmetric about all the hyperplanes  $x_j = 1/2, j = 1, \dots, s$ . Though up to an integer shift, the Haar scaling function  $\chi_{[0,1]}$  is the unique symmetric orthonormal scaling function in dimension one, there are many symmetric orthonormal scaling functions other than the tensor product Haar scaling function in dimension higher than one. An example of such symmetric orthonormal scaling function in  $L_2(\mathbb{R}^2)$  is as follows:

**Example 2.** The mask  $a$  on  $\mathbb{Z}^2$  is supported on  $[-1, 2]$  and is given by

$$\begin{bmatrix} -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix}.$$

Let  $\phi_a$  be the refinable function associated with the mask  $a$  in the refinement equation (1.1). Then we can verify (see Section 2) that  $\phi_a$  is an orthonormal scaling function such that  $\phi_a$  is symmetric about the lines  $x_j = 1/2, j = 1, 2$ .

However, both the Haar scaling function  $\chi_{[0,1]^s}$  and  $\phi_a$  in Example 2 are not continuous. Therefore, a natural question was raised in [6] (and also has been raised explicitly or implicitly elsewhere) as follows:

**Question:** Does there exist a *continuous* real-valued orthonormal scaling function  $\phi$  in  $L_2(\mathbb{R}^s)$  such that  $\phi$  is compactly supported and  $\phi$  is symmetric about all the superplanes  $x_j = c_j$ ,  $j = 1, \dots, s$  for some  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}^s$ ? Here the symmetry means that for all  $j = 1, \dots, s$ ,

$$\phi(x_1, \dots, x_{j-1}, 2c_j - x_j, x_{j+1}, \dots, x_s) = \phi(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_s) \quad \forall x_1, \dots, x_s \in \mathbb{R}.$$

In [6], the authors checked a particular family of orthonormal scaling functions in  $L_2(\mathbb{R}^2)$  with  $c_1 = c_2 = 1/2$ , i.e., the so-called order-factorable orthogonal wavelets with four-fold symmetry in [6]. They found that for that particular family in  $L_2(\mathbb{R}^2)$  the answer to the above question is no. Though, as pointed out in [6], such family of order-factorable wavelets does not include all the possible orthonormal scaling functions with the required four-fold symmetry, they conjectured that the answer to the above question is no in general in dimension two. It is the purpose of this paper to completely settle down the above question. Indeed we prove that the answer to the above question is no by obtaining a stronger result.

To state our main results in this paper, let us proceed to introduce some necessary definitions.

Given a mask  $a$  on  $\mathbb{Z}^s$ , we say that  $a$  satisfies the **sum rules** of order  $\ell$  if

$$\sum_{k \in \mathbb{Z}^s} a_{2k+j} (2k+j)^\mu = \sum_{k \in \mathbb{Z}^s} a_{2k} (2k)^\mu \quad \forall j \in \mathbb{Z}^s, \mu \in \mathbb{Z}_+^s, |\mu| < \ell, \quad (1.3)$$

where  $\mathbb{Z}_+^s := \{(\mu_1, \dots, \mu_s) \in \mathbb{Z}^s : \mu_j \geq 0, j = 1, \dots, s\}$ ,  $|\mu| = |\mu_1| + \dots + |\mu_s|$  and  $j^\mu := j_1^{\mu_1} \dots j_s^{\mu_s}$  for  $\mu = (\mu_1, \dots, \mu_s)$  and  $j = (j_1, \dots, j_s)$ . Let  $\phi$  be an orthonormal scaling function with the mask  $a$ . Then the wavelet basis derived from  $\phi$  has vanishing moments of order  $\ell$  if and only if  $a$  satisfies the sum rules of order  $\ell$ .

For any  $1 \leq p \leq \infty$  and  $0 < \alpha \leq 1$ , the **Lipschitz space**  $\text{Lip}(\alpha, L_p(\mathbb{R}^s))$  consists of those functions  $f$  in  $L_p(\mathbb{R}^s)$  for which

$$\|f(\cdot + t) - f\|_p \leq C \|t\|^\alpha \quad \forall t \in \mathbb{R}^s,$$

where  $C$  is a constant independent of  $t$ .

The smoothness of a function  $f \in L_p(\mathbb{R}^s)$  in the  $L_p$  norm is measured by its  $L_p$  **critical exponent**  $\nu_p(f)$  defined by

$$\nu_p(f) := \sup\{n + \alpha : \partial^k f \in \text{Lip}(\alpha, L_p(\mathbb{R}^s)) \quad \forall |k| = n, k \in \mathbb{Z}_+^s\},$$

where

$$\partial^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \dots \partial x_s^{k_s}}, \quad k = (k_1, \dots, k_s) \in \mathbb{Z}_+^s.$$

The main results in this paper are as follows:

**Theorem 1** *Let  $\phi \in L_2(\mathbb{R}^s)$  be a real-valued orthonormal scaling function with compact support such that  $\phi$  is symmetric about all the superplanes  $x_j = c_j$ ,  $j = 1, \dots, s$ , where  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}$ . Then  $c_1, \dots, c_s \in (\frac{1}{2} + \mathbb{Z})$ ,  $\phi$  is not a continuous function and the mask for  $\phi$  can satisfy the sum rules of order at most one. Moreover, if  $\phi \in L_p(\mathbb{R}^s)$  for some  $1 \leq p \leq \infty$ , then  $\nu_p(\phi) \leq 1/p$ .*

Note that the Haar scaling function  $\chi_{[0,1]^s}$  is symmetric about all the superplanes  $x_j = 1/2, j = 1, \dots, s$ . According to Theorem 1, the Haar scaling function achieves the best possible smoothness in the sense of any  $L_p$  norm since  $\nu_p(\chi_{[0,1]^s}) = 1/p$  for all  $1 \leq p \leq \infty$ .

Given a finitely supported sequence  $a$  on  $\mathbb{Z}^s$ , its **symbol**  $a(z_1, \dots, z_s)$  is defined as follows:

$$a(z_1, \dots, z_s) := \sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_s \in \mathbb{Z}} a_{(k_1, \dots, k_s)} z_1^{k_1} \cdots z_s^{k_s}, \quad z_1, \dots, z_s \in \mathbb{T}, \quad (1.4)$$

where  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ . Given a finitely supported mask  $a$  on  $\mathbb{Z}$ , it is a well known fact that  $a$  satisfies the sum rules of order  $\ell$  if and only if its symbol contains a special factor  $(1+z)^\ell$ , i.e.,  $a(z) = (1+z)^\ell c(z)$  for some finitely supported sequence  $c$  on  $\mathbb{Z}$ . Therefore, for higher dimensions, one may expect to require that the symbol of a finitely supported mask on  $\mathbb{Z}^s$  contains some factors  $(1+z_j), j = 1, \dots, s$ . However, we have the following result.

**Theorem 2** *Let  $\phi \in L_2(\mathbb{R}^s)$  be a real-valued orthonormal scaling function with compact support such that  $\phi$  is symmetric about a point  $x = c$  for some  $c \in \frac{1}{2}\mathbb{Z}^s$ , i.e.,  $\phi(2c - x) = \phi(x)$  for all  $x \in \mathbb{R}^s$ . Let  $a$  be the mask for the refinable function  $\phi$ . If  $a(z_1, \dots, z_s) = c(z_1, \dots, z_s) \prod_{j=2}^s (1+z_j)$  for all  $z_1, \dots, z_s \in \mathbb{T}$  for some finitely supported sequence  $c$  on  $\mathbb{Z}^s$ , then  $\phi$  is not a continuous function and the mask  $a$  for  $\phi$  can satisfy the sum rules of order at most one. Moreover, if  $\phi \in L_p(\mathbb{R}^s)$  for some  $1 \leq p \leq \infty$ , then  $\nu_p(\phi) \leq 1/p$ .*

Lai and Roach in [5] considered real-valued orthonormal scaling functions  $\phi$  in  $L_2(\mathbb{R}^2)$  such that  $\phi$  is supported on  $[0, 5]^2$ ,  $\phi$  is symmetric about  $x = (5/2, 5/2)$  and the symbol of its mask contains the factor  $(1+z_1)(1+z_2)$ . Theorem 2 demonstrates that all the examples of such bivariate orthonormal scaling functions constructed in [5] are discontinuous.

As for the orthonormal scaling functions which are symmetric about a point, we observe the following result which may be useful in designing smooth orthonormal scaling functions which are symmetric about a point.

**Theorem 3** *Let  $\phi \in L_2(\mathbb{R}^s)$  be an orthonormal scaling function with a finitely supported real-valued mask  $a$ . Suppose that there is a point  $c \in \frac{1}{2}\mathbb{Z}^s$  such that  $\phi(2c - x) = \phi(x)$  for all  $x \in \mathbb{R}^s$ . If  $a$  satisfies the sum rules of order  $\ell$ , then the following relation holds*

$$\sum_{k \in \mathbb{Z}^s} a_k k^\mu = 2^s c^\mu \quad \forall \mu \in \mathbb{Z}_+^s, |\mu| < \ell. \quad (1.5)$$

A **coiflet** of order  $\ell$  is derived from an orthonormal scaling function such that its mask  $a$  satisfies the sum rules of order  $\ell$  and  $\sum_{k \in \mathbb{Z}^s} a_k k^\mu = 2^s \delta_{0\mu}$  for all  $\mu \in \mathbb{Z}_+^s$  and  $|\mu| < \ell$ . It follows directly from the above Theorem 3 and Lemma 4 in Section 2 that there is no symmetric coiflet that can have order greater than one.

The structure of this paper is as follows. In Section 2, we shall recall some necessary results from [2] and prove several auxiliary results. In Section 3, we give a proof of Theorems 1 and 2. In Section 4, we shall prove Theorem 3.

## 2 Auxiliary Results

Given a mask  $a$ , in order to solve the refinement equation to obtain the refinable function  $\phi$  in the refinement equation (1.1), we start with an initial function  $\phi_0$  given by

$$\phi_0(x_1, \dots, x_s) = \prod_{j=1}^s h(x_j), \quad (x_1, \dots, x_s) \in \mathbb{R}^s,$$

where  $h(x) = \max\{1 - |x|, 0\}$ ,  $x \in \mathbb{R}$ . Then we consider an iteration scheme  $Q_a^n \phi_0$ ,  $n = 0, 1, 2, \dots$ , where  $Q_a$  is the linear operator on  $L_p(\mathbb{R}^s)$  ( $1 \leq p \leq \infty$ ) given by

$$Q_a f := \sum_{k \in \mathbb{Z}^s} a_k f(2 \cdot -k), \quad f \in L_p(\mathbb{R}^s).$$

The iteration scheme is called a **subdivision scheme** or **cascade algorithm** in the literature (see [1, 3]). If there exists a function  $f \in L_p(\mathbb{R}^s)$  such that  $\lim_{n \rightarrow \infty} \|Q_a^n \phi_0 - f\|_p = 0$ , then we say that the subdivision scheme associated with  $a$  converges in the  $L_p$  norm. If this is the case, then  $f$  must be the unique distributional solution  $\phi_a$  to the refinement equation (1.1). A characterization of  $L_p$  convergence of a subdivision scheme was given in [3].

Let  $\phi$  be an orthonormal scaling function in  $L_2(\mathbb{R}^s)$  with a finitely supported mask  $a$ . Then its mask  $a$  must satisfy the following **discrete orthogonal condition**

$$\sum_{k \in \mathbb{Z}^s} \overline{a_{k-2j}} a_k = 2^s \delta_{0j} \quad \forall j \in \mathbb{Z}^s. \quad (2.1)$$

Let  $a$  be a finitely supported mask on  $\mathbb{Z}^s$  and  $\phi_a$  be its associated refinable function in the refinement equation (1.1). Then it is known that the refinable function  $\phi_a$  is an orthonormal scaling function if and only if its mask  $a$  satisfies the discrete orthogonal condition in (2.1) and the subdivision scheme associated with  $a$  converges in the  $L_2$  norm. By [3, Theorem 4.3], it is easy to check that the subdivision scheme associated with the mask  $a$  in Example 1 converges in the  $L_2$  norm; therefore,  $\phi_a$  in Example 1 is an orthonormal scaling function since the mask  $a$  in Example 1 satisfies the discrete orthogonal condition. Moreover, by computation, we have  $\nu_2(\phi_a) = 0.943009$  for  $\phi_a$  in Example 1. Similarly, we can verify that the subdivision scheme associated with the mask  $a$  in Example 2 converges in the  $L_2$  norm; therefore,  $\phi_a$  in Example 2 is an orthonormal scaling function and  $\nu_2(\phi_a) = 0.125454$ .

Let  $\phi_a$  be a refinable function with a mask  $a$ . If  $\phi$  is symmetric about all the superplanes  $x_j = c_j$ ,  $j = 1, \dots, s$  or  $\phi$  is symmetric about the point  $(x_1, \dots, x_s) = (c_1, \dots, c_s)$  for some  $c_1, \dots, c_s \in \mathbb{R}$ , then from the refinement equation (1.1), it is necessary that  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}$ . From the refinement equation (1.1), it is easy to see that  $\phi_a$  is symmetric about the hyperplanes  $x_j = c_j$ ,  $j = 1, \dots, s$  for  $c_1, \dots, c_s \in \frac{1}{2}\mathbb{Z}$  if and only if its mask  $a$  is symmetric about the hyperplanes  $x_j = c_j$ ,  $j = 1, \dots, s$ . For any given  $k \in \mathbb{Z}^s$ , it is easy to observe that  $\phi(\cdot - k)$  satisfies the refinement equation with the mask  $a_{(\cdot - k)}$ . Therefore, without loss of generality, we only need to consider the case  $c_1, \dots, c_s \in \{0, 1/2\}$ .

**Lemma 4** *There exists no mask  $a$  on  $\mathbb{Z}^s$  such that  $\sum_{k \in \mathbb{Z}^s} a_k = 2^s$ ,  $a$  satisfies the discrete orthogonal condition in (2.1) and  $a$  is symmetric about the origin.*

**Proof:** Suppose that we do have such a mask  $a$  satisfying the requirements in the lemma. Define an order on the set  $\mathbb{Z}^s$  as follows: For  $j = (j_1, \dots, j_s)$  and  $k = (k_1, \dots, k_s) \in \mathbb{Z}^s$ , we say that  $j < k$  if  $j_1 = k_1, \dots, j_{\ell-1} = k_{\ell-1}$  but  $j_\ell < k_\ell$  for some  $1 \leq \ell \leq s$ . Since  $\sum_{k \in \mathbb{Z}^s} a_k = 2^s$  and the discrete orthogonal condition holds, it is easy to see that  $\{k \in \mathbb{Z}^s : a_k \neq 0\}$  can not be the empty set or  $\{0\}$ . Therefore, since  $a$  is symmetric about the origin, there exists a maximum element  $0 \neq m \in \mathbb{Z}^s$  such that  $a_m \neq 0$  but  $a_k = 0$  for all  $k > m$ . Putting  $j = m$  in (2.1), we have

$$0 = 2^s \delta_{0m} = \sum_{k \in \mathbb{Z}^s} \overline{a_{k-2m}} a_k = \sum_{k \leq m} \overline{a_{k-2m}} a_k = \sum_{k \leq m} \overline{a_{2m-k}} a_k = |a_m|^2 \neq 0,$$

where we used the fact that  $2m - k \geq 2m - m = m$  for  $k \leq m$ . This is a contradiction and we are done.  $\blacksquare$

If  $\phi$  is symmetric about  $x_j = c_j, j = 1, \dots, s$  for some  $c_1, \dots, c_s \in \{0, 1/2\}$ , then its mask is also symmetric about  $x_j = c_j, j = 1, \dots, s$ . Thus, to obtain an orthonormal scaling function with such symmetry, it is impossible that  $c_1 = \dots = c_s = 0$  by Lemma 4.

The following lemma is crucial in our proof of Theorem 1.

**Lemma 5** *Let  $a$  be a finitely supported mask on  $\mathbb{Z}^s$  such that  $a$  satisfies the discrete orthogonal condition in (2.1) and  $a$  is symmetric about  $x_1 = 1/2$ . Define a new sequence  $b$  on  $\mathbb{Z}^{s-1}$  by*

$$b_{(k_2, \dots, k_s)} = \frac{1}{2} \sum_{k_1 \in \mathbb{Z}} a_{(k_1, k_2, \dots, k_s)}, \quad k_2, \dots, k_s \in \mathbb{Z}. \quad (2.2)$$

Then

$$\sum_{k \in \mathbb{Z}^{s-1}} \overline{b_{k-2j}} b_k = 2^{s-1} \delta_{0j} \quad \forall j \in \mathbb{Z}^{s-1}. \quad (2.3)$$

Namely,  $b$  satisfies the discrete orthogonal condition.

**Proof:** Let us directly compute the left side of (2.3) using the definition of the sequence  $b$ .

$$\begin{aligned} \sum_{k \in \mathbb{Z}^{s-1}} \overline{b_{k-2j}} b_k &= \frac{1}{4} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \overline{a_{(m, k-2j)}} a_{(n, k)} \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} a_{(n, k)} \sum_{m \in \mathbb{Z}} \overline{a_{(n-m, k-2j)}} \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} a_{(n, k)} \left[ \sum_{m \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} + \sum_{m \in \mathbb{Z}} \overline{a_{(n+1-2m, k-2j)}} \right]. \end{aligned}$$

Using the symmetry of  $a$ , we have  $a_{(n+1-2m, k-2j)} = a_{(n+2(m-n), k-2j)}$ . Hence,

$$\begin{aligned} \sum_{k \in \mathbb{Z}^{s-1}} \overline{b_{k-2j}} b_k &= \frac{1}{4} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} a_{(n, k)} \left[ \sum_{m \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} + \sum_{m \in \mathbb{Z}} \overline{a_{(n+2(m-n), k-2j)}} \right] \\ &= \frac{1}{4} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} a_{(n, k)} \left[ \sum_{m \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} + \sum_{m \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} \right] \\ &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} a_{(n, k)}. \end{aligned}$$

Since  $a$  satisfies the discrete orthogonal condition in (2.1), we have

$$\sum_{k \in \mathbb{Z}^{s-1}} \sum_{n \in \mathbb{Z}} \overline{a_{(n-2m, k-2j)}} a_{(n, k)} = 2^s \delta_{0(m, j)} \quad \forall m \in \mathbb{Z}, j \in \mathbb{Z}^{s-1}.$$

Therefore,

$$\sum_{k \in \mathbb{Z}^{s-1}} \overline{b_{k-2j}} b_k = \frac{1}{2} \sum_{m \in \mathbb{Z}} 2^s \delta_{0(m, j)} = 2^{s-1} \delta_{0j}.$$

That is,  $b$  satisfies the discrete orthogonal condition. ■

Let  $\phi$  be a compactly supported function in  $L_p(\mathbb{R}^s)$  for some  $1 \leq p \leq \infty$ . We say that the shifts of  $\phi$  are **stable** if there exist two positive constants  $A$  and  $B$  such that

$$A \sum_{k \in \mathbb{Z}^s} |c_k|^p \leq \left\| \sum_{k \in \mathbb{Z}^s} c_k \phi(\cdot - k) \right\|_p^p \leq B \sum_{k \in \mathbb{Z}^s} |c_k|^p$$

for all finitely supported sequences  $\{c_k\}_{k \in \mathbb{Z}^s}$  on  $\mathbb{Z}^s$ .

The following result, which was given in Han [2], is crucial in our proof of Theorem 1.

**Theorem 6** ([2, Lemma 4,2]) *Let  $a$  be a finitely supported mask on  $\mathbb{Z}^s$ . Define a new sequence  $b$  on  $\mathbb{Z}$  as follows:*

$$b_{k_1} = 2^{1-s} \sum_{k_2 \in \mathbb{Z}} \cdots \sum_{k_s \in \mathbb{Z}} a_{(k_1, k_2, \dots, k_s)}, \quad k_1 \in \mathbb{Z}. \quad (2.4)$$

*For any  $1 \leq p \leq \infty$ , if the subdivision scheme associated with  $a$  converges in the  $L_p$  norm, then the subdivision scheme associated with the mask  $b$  also converges in the  $L_p$  norm. Let  $\phi_a$  and  $\phi_b$  be the refinable functions associated with the masks  $a$  and  $b$ , respectively. If  $\phi_a \in L_p(\mathbb{R}^s)$  and the shifts of  $\phi_a$  are stable, then  $\phi_b \in L_p(\mathbb{R})$  and  $\nu_p(\phi_a) \leq \nu_p(\phi_b)$ .*

By a similar argument as in the proof of the above Theorem 6 in [2], we have the following result.

**Theorem 7** *Let  $a$  be a finitely supported mask on  $\mathbb{Z}^s$ . Define a new sequence  $b$  on  $\mathbb{Z}^{s-1}$  as in (2.3). For any  $1 \leq p \leq \infty$ , if the subdivision scheme associated with  $a$  converges in the  $L_p$  norm, then the subdivision scheme associated with the mask  $b$  also converges in the  $L_p$  norm. Let  $\phi_a$  and  $\phi_b$  be the refinable functions associated with the masks  $a$  and  $b$ , respectively. If  $\phi_a \in L_p(\mathbb{R}^s)$  and the shifts of  $\phi_a$  are stable, then  $\phi_b \in L_p(\mathbb{R}^{s-1})$  and  $\nu_p(\phi_a) \leq \nu_p(\phi_b)$ .*

### 3 Proofs of Theorems 1 and 2

Since  $\phi$  is symmetric about  $x_j = c_j, j = 1, \dots, s$ , we observe that  $a$  is also symmetric about  $x_j = c_j, j = 1, \dots, s$ .

**Proof of Theorem 1.** We now prove Theorem 1 by induction on the dimension  $s$ . Without loss of generality, we may assume that  $c_1, \dots, c_s \in \{0, 1/2\}$  since otherwise we can consider another scaling function which is an integer shift of  $\phi$ . Obviously, Theorem 1 holds when  $s = 1$  since the Haar scaling function is the only real-valued compactly supported orthonormal scaling function which is symmetric about  $1/2$  (see [1]).

Suppose that we have verified Theorem 1 for the dimension  $s - 1$ . We now verify it for dimension  $s$ . By Lemma 4, it is not possible that  $c_1 = \dots = c_s = 0$ . Therefore, without loss of generality, we assume that  $c_1 = 1/2$ . Define a new sequence  $b$  on  $\mathbb{Z}^{s-1}$  as follows:

$$b_{(k_2, \dots, k_s)} = \frac{1}{2} \sum_{k_1 \in \mathbb{Z}} a_{(k_1, k_2, \dots, k_s)}, \quad k_2, \dots, k_s \in \mathbb{Z}.$$

Since the mask  $a$  is symmetric about  $x_j = c_j, j = 1, \dots, s$ , it is easy to see that the mask  $b$  is symmetric about  $x_j = c_{j+1}, j = 1, \dots, s - 1$ . Note that  $a$  satisfies the discrete orthogonal condition in (2.1) since  $\phi_a$  is an orthonormal scaling function. By Theorem 5, the new mask  $b$  also satisfies the discrete orthogonal condition. Since  $\phi_a$  is an orthonormal scaling function, the shifts of  $\phi$  are stable. Then by [3, Theorem 3.4], the subdivision scheme associated with  $a$  converges in the  $L_2$  norm. Hence, by Theorem 7, the subdivision scheme associated with  $b$  also converges in the  $L_2$  norm. Therefore,  $\phi_b$  is an orthonormal scaling function since  $b$  satisfies the discrete orthogonal condition. By induction hypothesis, we must have  $c_1 = c_2 = \dots = c_s = 1/2$ .

Suppose that  $\phi_a$  is a continuous function. Then by [3, Theorem 3.4], the subdivision scheme associated with  $a$  converges in the  $L_\infty$  norm. Hence, by Theorem 7 again, the subdivision scheme associated with  $b$  also converges in the  $L_\infty$  norm which implies that  $\phi_b$  is a continuous function. Thus,  $\phi_b$  is a continuous orthonormal scaling function in  $L_2(\mathbb{R}^{s-1})$  such that  $\phi_b$  is symmetric about  $x_j = c_{j+1}, j = 1, \dots, s - 1$ . This is a contradiction to our induction hypothesis. Therefore,  $\phi_a$  is not a continuous function.

If  $\phi_a \in L_p(\mathbb{R}^s)$  for some  $1 \leq p \leq \infty$ , then the shifts of  $\phi_a$  are stable since  $\phi$  is an orthonormal scaling function. By Theorem 7 again we have  $\nu_p(\phi_a) \leq \nu_p(\phi_b) \leq 1/p$  by the induction hypothesis.

By the definition of the sum rules, it is straightforward to check that if  $a$  satisfies the sum rules of order  $\ell$ , then  $b$  must satisfy the sum rules of order at least  $\ell$ . Hence, by induction

hypothesis again,  $a$  can satisfy the sum rules of order at most one.  $\blacksquare$

**Proof of Theorem 2.** Let  $b$  be the sequence on  $\mathbb{Z}$  as defined in (2.4). To complete the proof, by Theorem 6 and the proof of Theorem 1, it suffices to verify that  $b$  satisfies the discrete orthogonal condition.

Note that the discrete orthonormal condition in (2.1) can be rewritten in terms of the symbol of the sequence as follows:

$$\sum_{\varepsilon_1 \in \{-1, 1\}} \cdots \sum_{\varepsilon_s \in \{-1, 1\}} a(\varepsilon_1 z_1, \dots, \varepsilon_s z_s) \overline{a(\varepsilon_1 z_1, \dots, \varepsilon_s z_s)} = 4^s \quad \forall z_1, \dots, z_s \in \mathbb{T}. \quad (3.1)$$

On the other hand, we observe that  $b(z_1) = 2^{1-s} a(z_1, 1, \dots, 1)$ . From our assumption on  $a$ , we have  $a(z_1, \varepsilon_2, \dots, \varepsilon_s) = 0$  for all  $\varepsilon_2, \dots, \varepsilon_s \in \{-1, 1\}$  such that  $\varepsilon_j = -1$  for some  $j = 2, \dots, s$ . Setting  $z_2 = \dots = z_s = 1$  in (3.1), we have

$$a(z_1, 1, \dots, 1) \overline{a(z_1, 1, \dots, 1)} + a(-z_1, 1, \dots, 1) \overline{a(-z_1, 1, \dots, 1)} = 4^s.$$

In other words,  $b(z_1) \overline{b(z_1)} + b(-z_1) \overline{b(-z_1)} = 4$ ; therefore,  $b$  satisfies the discrete orthogonal condition.  $\blacksquare$

## 4 Proof of Theorem 3

In this section, based on a result in [2, Theorem 6.1], we shall prove Theorem 3.

Let  $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{Z}_+^s$  and  $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{Z}_+^s$ . We say that  $\nu \leq \mu$  if  $\nu_j \leq \mu_j$  for all  $j = 1, \dots, s$ . By  $\nu < \mu$  we mean  $\nu \leq \mu$  and  $\nu \neq \mu$ . Moreover,  $\mu! := \mu_1! \cdots \mu_s!$ .

An interesting result [2, Theorem 6.1] was employed in the CBC algorithm in [2] to construct biorthogonal wavelets with any preassigned order of vanishing moments. The following result is a special case of [2, Theorem 6.1].

**Theorem 8** *Let  $a$  be a finitely supported mask on  $\mathbb{Z}^s$ . If  $a$  satisfies the discrete orthogonal condition in (2.1) and  $a$  satisfies the sum rules of order  $\ell$ , then we have the following relation:*

$$\sum_{0 \leq \nu \leq \mu} (-1)^{|\mu-\nu|} \frac{\mu!}{\nu!(\mu-\nu)!} \overline{h_{\mu-\nu}} h_\nu = \delta_{0\mu} \quad \forall \mu \in \mathbb{Z}_+^s, |\mu| < \ell, \quad (4.1)$$

where  $h_\mu := 2^{-s} \sum_{k \in \mathbb{Z}^s} a_k k^\mu$ ,  $\mu \in \mathbb{Z}_+^s$ .

**Proof of Theorem 3.** Let  $h_\mu$  ( $\mu \in \mathbb{Z}_+^s$ ) be defined as in Theorem 8. Since  $a$  satisfies the discrete orthogonal condition and  $a$  satisfies the sum rules of order  $\ell$ , by Theorem 8, (4.1) holds. Note that the mask  $a$  is real-valued. We can rewrite (4.1) as follows:

$$(1 + (-1)^{|\mu|}) h_\mu = \delta_{0\mu} - \sum_{0 < \nu < \mu} (-1)^{|\mu-\nu|} \frac{\mu!}{\nu!(\mu-\nu)!} h_{\mu-\nu} h_\nu, \quad |\mu| < \ell. \quad (4.2)$$

On the other hand, since  $a_{2c-k} = a_k$  for all  $k \in \mathbb{Z}^s$ , we deduce that

$$\begin{aligned} h_\mu &= 2^{-s} \sum_{k \in \mathbb{Z}^s} a_k k^\mu = 2^{-s} \sum_{k \in \mathbb{Z}^s} a_{2c-k} k^\mu = 2^{-s} \sum_{k \in \mathbb{Z}^s} a_k (2c - k)^\mu \\ &= \sum_{0 \leq \nu \leq \mu} (-1)^{|\nu|} \frac{\mu!}{\nu!(\mu - \nu)!} (2c)^{\mu - \nu} 2^{-s} \sum_{k \in \mathbb{Z}^s} a_k k^\nu \\ &= \sum_{0 \leq \nu \leq \mu} (-1)^{|\nu|} \frac{\mu!}{\nu!(\mu - \nu)!} (2c)^{\mu - \nu} h_\nu. \end{aligned}$$

Therefore,

$$(1 - (-1)^{|\mu|}) h_\mu = \sum_{0 \leq \nu < \mu} (-1)^{|\nu|} \frac{\mu!}{\nu!(\mu - \nu)!} (2c)^{\mu - \nu} h_\nu, \quad \mu \in \mathbb{Z}_+^s. \quad (4.3)$$

In particular, from (4.2) and (4.3), we have

$$\begin{cases} h_0 = 1, \\ h_\mu = \frac{1}{2} \sum_{0 \leq \nu < \mu} (-1)^{|\nu|} \frac{\mu!}{\nu!(\mu - \nu)!} (2c)^{\mu - \nu} h_\nu, & 0 < |\mu| < \ell, |\mu| \text{ is odd}, \\ h_\mu = -\frac{1}{2} \sum_{0 < \nu < \mu} (-1)^{|\mu - \nu|} \frac{\mu!}{\nu!(\mu - \nu)!} h_{\mu - \nu} h_\nu, & 0 < |\mu| < \ell, |\mu| \text{ is even}. \end{cases} \quad (4.4)$$

It is straightforward to see that there is a unique solution to the above system of equations in (4.4). In fact,  $h_\mu$  ( $|\mu| < \ell$ ) can be recursively obtained from (4.4) and it is easy to check that  $h_\mu = c^\mu$  ( $|\mu| < \ell$ ) is the unique solution to the above system of equations in (4.4). Finally, it is easy to verify that the unique solution  $h_\mu = c^\mu$  ( $|\mu| < \ell$ ) indeed satisfies the equations in both (4.1) and (4.3).  $\blacksquare$

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